Exact Scenario Simulation for Selected Multi-dimensional Stochastic Processes

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Abstract. Accurate scenario simulation methods for solutions of multi-dimensional stochastic differential equations find application in stochastic analysis, the statistics of stochastic processes and many other areas, for instance, in finance. They have been playing a crucial role as standard models in various areas and dominate often the communication and thinking in a particular field of application, even that they may be too simple for more advanced tasks. Various discrete time simulation methods have been developed over the years. However, the simulation of solutions of some stochastic differential equations can be problematic due to systematic errors and numerical instabilities. Therefore, it is valuable to identify multi-dimensional stochastic differential equations with solutions that can be simulated exactly. This avoids several of the theoretical and practical problems encountered by those simulation methods that use discrete time approximations. This paper provides a survey of methods for the exact simulation of paths of some multi-dimensional solutions of stochastic differential equations including Ornstein-Uhlenbeck, square root, squared Bessel, Wishart and Lévy type processes.

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1 Introduction

Accurate scenario simulation of solutions of stochastic differential equations (SDEs) is widely applicable in stochastic analysis and its areas of application, for instance, in finance or filtering, see Kallianpur (1980). Monographs in the direction of stochastic numerical methods include Kloeden & Platen (1999), Kloeden, Platen & Schurz (2003), Milstein (1995), Jäckel (2002) and Glasserman (2004). Many simulation techniques have been developed over the years. However, some SDEs can be problematic in terms of simulation. Therefore, it is necessary to understand and avoid the problems that may arise during the simulation of solutions of such SDEs. For illustration, let us consider a family of SDEs of the form

$$dX_t = a(X_t)dt + \sqrt{2X_t}dW_t. \tag{1.1}$$

Note that the diffusion coefficient function $f(x) = \sqrt{2x}$ is non-Lipschitz. Its derivative becomes infinite as x tends to 0. The standard convergence theorems derived in the above mentioned literature do not easily cover such case. It is, therefore, of interest to identify approximate simulation methods for various nonlinear types of SDEs and also for multi-dimensional SDEs. This paper emphasizes the fact that the problem of non-Lipschitz continuous coefficients is circumvented for SDEs where we can simulate exact solutions. For instance, for squared Bessel processes of integer dimension, see Revuz & Yor (1999), we will explain how to simulate exact solutions. Based on results in Craddock & Platen (2004), exact solutions can be simulated by sampling from the explicitly available transition density of some nonlinear SDEs where the drift function $a(\cdot)$ in (1.1) takes a particular form. These then include also squared Bessel processes of noninteger dimension.

Another problem with the simulation of SDEs may be the lack of sufficient numerical stability of the chosen scheme. Numerical stability is understood as the ability of a scheme to control the propagation of initial and roundoff errors. This ability may get lost for some parameter ranges when using some given scheme. There is a wide range of literature which deals with the issue of numerical stability. In particular, implicit or predictor-corrector methods are used to control the propagation of errors. We refer here to papers by Talay (1982), Klauder & Petersen (1985), Milstein (1988), Hernandez & Spigler (1993), Saito & Mitsui (1993), Kloeden & Platen (1992), Milstein, Platen & Schurz (1998), Higham (2000), Higham, Mao & Yuan (2007), Alcock & Burrage (2006), Bruti-Liberati & Platen (2008) and Platen & Shi (2008). The issue of numerical stability can be circumvented when it is possible to simulate exact solutions.

Moreover, for theoretically strictly positive processes it is often not acceptable to use discrete time simulation methods that may generate negative values. This problem, however, can be in some cases solved by a transformation of the initial SDE, via the Itô formula, to a process which lives on the entire real axis. This is, in particular, useful for geometric Brownian motion, the standard asset dynamics under the Black-Scholes model in finance. Here one can take the logarithm to obtain a linearly transformed Wiener process. One may try such an approach to transform the square root process of the form

$$dX_t = \kappa(\theta - X_t)dt + \sigma\sqrt{X_t}dW_t, \tag{1.2}$$

where $t \in \Re^+$. This process remains strictly positive for dimension $\delta = \frac{4\kappa\theta}{\sigma^2} > 2$. Suppose that we simulate for $\delta > 2$ the process $Y_t = \sqrt{X_t}$ using a standard explicit numerical scheme such as the Euler scheme, see Kloeden & Platen (1999). The SDE of the corresponding stochastic process $Y = \{Y_t = \sqrt{X_t}, t \geq 0\}$ has then additive noise and is given by

$$dY_t = \left(\frac{\kappa\theta - \sigma^2/4}{2Y_t} - \frac{\kappa}{2}Y_t\right)dt + \frac{\sigma}{2}dW_t. \tag{1.3}$$

Theoretically, by squaring the resulting trajectory of Y we should obtain an approximate trajectory of the square root process X. However, note that the drift coefficient $\frac{\kappa\theta-\sigma^2/4}{2y} - \frac{\kappa}{2}y$ is non-Lipschitz and may almost explode for small y. Even that we have additive noise, this feature will most likely produce simulation problems for trajectories near zero. This kind of problem becomes dominant for small dimension $\delta < 2$ of the square root process. In such case it would be very valuable to have an exact solution to avoid this kind of problem.

Beyond the Wiener process with its direct transformations, including the geometric Brownian motion and the Ornstein-Uhlenbeck process, the family of square root and squared Bessel processes are probably the most frequently used diffusion models in applications. In general, it is a challenging task to obtain efficiently a reasonably accurate trajectory of a square root process using simulation, as is documented in an increasing literature on this topic. This literature includes the use of the balanced implicit method introduced in Milstein, Platen & Schurz (1998), the adaptive Milstein scheme of Kahl (2004), the balanced Milstein method of Kahl & Schurz (2005) and Alcock & Burrage (2006). Additionally, various other methods have been designed to approximate the square root process. Here we refer to Deelstra & Delbaen (1998), Diop (2003), Bossy & Diop (2004), Berkaoui, Bossy & Diop (2005), Alfonsi (2005), Broadie & Kaya (2006), Lord, Koekkoek & van Dijk (2006), Smith (2007) and Andersen (2008). The current paper contributes to this literature by studying also the simulation of multi-dimensional square root and squared Bessel processes.

In various areas of application one has to model vectors or even matrices of evolving dependent stochastic quantities. This is typically the case, for instance, when modeling asset prices in financial markets. Also in hidden Markov chain filtering vectors of unconditional probabilities have to be numerically evaluated. The above mentioned numerical problems can arise in a complex manner when simulating the trajectories of multi-dimensional models. For instance, different time scales in the dynamics of certain components can create stiff SDEs, see Hairer & Wanner (1996) and Kloeden & Platen (1999), which are almost impossible to handle by standard schemes. This makes it also worthwhile to identify classes of multi-dimensional SDEs with exact solutions or almost exact approximations.

The current paper surveys and develops exact simulation methods for solutions of several classes of multi-dimensional SDEs, aiming to avoid the numerical problems mentioned above. In Section 2 we show how the systematic application of the Itô formula extends the family of SDEs for which exact solutions can be generated via simulation. Furthermore, in Sections 3-6 we discuss the exact simulation of selected matrix valued stochastic processes. These include: the matrix Ornstein-Uhlenbeck process, the Wishart process, the matrix affine processes and the matrix Lévy processes. We conclude with Section 7, where we discuss sampling from available transition densities of multi-dimensional stochastic processes. In this section we also describe alternatives related to the simulation of solutions of multi-dimensional SDEs.

2 Multi-dimensional Itô Formula

We will start our discussion by recalling some basic facts including the application of the multi-dimensional Itô formula. Given some family of explicitly solvable multi-dimensional SDEs, one can obtain by application of the multi-dimensional Itô formula another family of explicitly solvable multi-dimensional SDEs. This results in a wide range of multi-dimensional SDEs that can be simulated exactly. In this section we illustrate this property by simulating a 2-dimensional Black-Scholes model, the standard asset price model in finance.

Vector of Independent Wiener Processes

Let us consider an m-dimensional Wiener process $\boldsymbol{W} = \{\boldsymbol{W}_t = (W_t^1, \dots, W_t^m)^\top, t \in [0, \infty)\}$. We assume that the components of this vector stochastic process $\boldsymbol{W}$, are independent. The increments of the Wiener processes $W_t^j - W_s^j$ for $j \in \{1, 2, \dots, m\}$, $t \geq 0$ and $s \leq t$ are then independent Gaussian random variables with mean zero and variance equal to $t - s$. Therefore, one obtains the vector increments of the standard m-dimensional Wiener process $\boldsymbol{W}_t - \boldsymbol{W}_s \sim \mathcal{N}_d(\boldsymbol{0}, (t-s)\boldsymbol{I})$ as a vector of zero mean independent Gaussian random variables with variance $t-s$. $\boldsymbol{I}$ denotes here the identity matrix. We obtain for the values of the trajectory of the standard m-dimensional Wiener process at the discretization times $t_i = i\Delta, i \in \{0, 1, 2, \dots\}$, with $\Delta > 0$ the following iterative formula

$$\begin{aligned} \boldsymbol{W}_0 &= \boldsymbol{0} \\ \boldsymbol{W}_{t_{i+1}} &= \boldsymbol{W}_{t_i} + \sqrt{\Delta}\boldsymbol{N}_{i+1}, \end{aligned} \tag{2.1}$$

where $\boldsymbol{N}_{i+1} \sim \mathcal{N}_m(\boldsymbol{0}, \boldsymbol{I})$ is an independent standard Gaussian random vector and $\boldsymbol{0}$ denotes the corresponding vector of zeros.

Multi-dimensional Itô Formula

Let be given the m-dimensional Wiener process $\boldsymbol{W} = \{\boldsymbol{W}_t = (W_t^1, \ldots, W_t^m)^\top, t \in [0, \infty)\}$, a d-dimensional drift coefficient vector function $\boldsymbol{a} : [0,T] \times \Re^d \to \Re^d$ and a $d \times m$-matrix diffusion coefficient function $\boldsymbol{b} : [0,T] \times \Re^d \to \Re^{d \times m}$. In this framework we assume that we have already a family of explicitly solvable d-dimensional SDEs given as

$$d\boldsymbol{X}_t = \boldsymbol{a}(t, \boldsymbol{X}_t)dt + \boldsymbol{b}(t, \boldsymbol{X}_t)d\boldsymbol{W}_t, \tag{2.2}$$

for $t \in [0, \infty)$, $\boldsymbol{X}_0 \in \Re^d$. This means that the kth component of (2.2) equals

$$dX_t^k = a^k(t, \boldsymbol{X}_t)dt + \sum_{j=1}^{m} b^{k,j}(t, \boldsymbol{X}_t)dW_t^j. \tag{2.3}$$

For a sufficiently smooth vector function $\boldsymbol{U} : [0,T] \times \Re^d \to \Re^k$ of the solution $\boldsymbol{X}_t$ of (2.2) we obtain a k-dimensional process

$$\boldsymbol{Y}_t = \boldsymbol{U}(t, \boldsymbol{X}_t). \tag{2.4}$$

The expression for its pth component, resulting from the application of the Itô formula, satisfies the SDE

$$\begin{aligned} dY_t^p &= \left(\frac{\partial U^p}{\partial t} + \sum_{i=1}^{d} a^i \frac{\partial U^p}{\partial x^i} + \frac{1}{2} \sum_{i,j=1}^{d} \sum_{l=1}^{m} b^{i,l} b^{j,l} \frac{\partial^2 U^p}{\partial x_i \partial x_j} \right) dt \\ &\quad + \sum_{l=1}^{m} \sum_{i=1}^{d} b^{i,l} \frac{\partial U^p}{\partial x_i} dW_t^l, \end{aligned} \tag{2.5}$$

for $p \in \{1, 2, \ldots, k\}$, where the terms on the right-hand side of (2.5) are evaluated at $(t, \boldsymbol{X}_t)$. It is a trivial but very valuable observation that also the paths of the solution of the SDE (2.5) can be exactly simulated since $\boldsymbol{X}_{t_i}$ can be obtained exactly at all discretization points and, by (2.4), $\boldsymbol{Y}_{t_i}$ is simply a function of $\boldsymbol{X}_{t_i}$.

Vector of Correlated Wiener Processes

Let us now define a d-dimensional continuous process $\tilde{\boldsymbol{W}} = \{\tilde{\boldsymbol{W}}_t = (\tilde{W}_t^1, \tilde{W}_t^2, \ldots, \tilde{W}_t^d)^\top,\ t \in [0, \infty)\}$ such that its components $\tilde{W}_t^1, \tilde{W}_t^2, \ldots, \tilde{W}_t^d$ are transformed scalar Wiener processes. In vector notation, such a d-dimensional transformed Wiener process can be expressed by the linear transform

$$\tilde{\boldsymbol{W}}_t = \boldsymbol{a}t + \boldsymbol{B}\boldsymbol{W}_t, \tag{2.6}$$

where $\boldsymbol{a} = (a_1, a_2, \ldots, a_d)^\top$ is a d-dimensional vector, $\boldsymbol{B}$ is a $d \times m$-matrix and $\boldsymbol{W} = \{\boldsymbol{W}_t = (W_t^1, W_t^2, \ldots, W_t^m)^\top, t \in [0, \infty)\}$ is an m-dimensional standard Wiener process. By the application of the multi-dimensional Itô formula one obtains

$$d\tilde{W}_t^k = a_k dt + \sum_{i=1}^{m} b_{k,i} dW_t^i, \tag{2.7}$$

for $k \in \{1, 2, \dots, d\}$. This means that $\tilde{W}_t^k$, $k \in \{1, 2, \dots, d\}$, is constructed as a linear combination of components of the vector $\boldsymbol{W}_t$ plus some trend.

From the properties of Gaussian random variables, the following relation results

$$\begin{aligned} \tilde{\boldsymbol{W}}_0 &= \mathbf{0}, \\ \tilde{\boldsymbol{W}}_{t_{i+1}} &= \tilde{\boldsymbol{W}}_{t_i} + \boldsymbol{a}\Delta + \sqrt{\Delta}\tilde{\boldsymbol{N}}_{i+1}, \end{aligned} \tag{2.8}$$

for $t_i = i\Delta$, $i \in \{0, 1, \dots\}$ with $\Delta > 0$. For each $i \in \{0, 1, 2, \dots\}$ the random vector $\tilde{\boldsymbol{N}}_{i+1} \sim \mathcal{N}_d(\mathbf{0}, \boldsymbol{\Sigma})$ is here a d-dimensional Gaussian vector with the correlation matrix $\boldsymbol{\Sigma} = \boldsymbol{B}\boldsymbol{B}^\top$.

Multi-dimensional Geometric Brownian Motions

Now, we describe multi-dimensional geometric Brownian motions that yield the Black-Scholes model. This model emerges when taking the exponent of the linearly transformed Wiener process. Denote by $\boldsymbol{S}_t$ a diagonal matrix with jth diagonal element S_t^j, $j \in \{1, 2, \dots, d\}$, representing the jth asset price at time $t \in [0, \infty)$. Then the SDE for the jth Black-Scholes asset price S_t^j is defined by

$$dS_t^j = S_t^j \left(a_t^j dt + \sum_{k=1}^{d} b_t^{j,k} dW_t^k \right) \tag{2.9}$$

for $t \in [0, \infty)$ and $j \in \{1, 2, \dots, d\}$. Here $W^k, k \in \{1, 2, \dots, d\}$, denotes an independent standard Wiener process. Note that the Zakai equation for the Wonham filter is of a similar form, see Kallianpur (1980). To represent the above SDE in matrix form we introduce the diagonal matrix $\boldsymbol{A}_t = [A_t^{i,j}]_{i,j=1}^d$ with

$$A_t^{i,j} = \begin{cases} a_t^j & \text{for } i = j \\ 0 & \text{otherwise} \end{cases} \tag{2.10}$$

and diagonal matrix $\boldsymbol{B}_t^k = [B_t^{k,i,j}]_{i,j=1}^d$ with

$$B_t^{k,i,j} = \begin{cases} b_t^{j,k} & \text{for } i = j \\ 0 & \text{otherwise} \end{cases} \tag{2.11}$$

for $k, i, j \in \{1, 2, \dots, d\}$ and $t \in [0, \infty)$. If all these diagonal matrices commute in the sense

$$\boldsymbol{A}\boldsymbol{B}^l = \boldsymbol{B}^l\boldsymbol{A} \quad \text{and} \quad \boldsymbol{B}^l\boldsymbol{B}^k = \boldsymbol{B}^k\boldsymbol{B}^l \tag{2.12}$$

for all $k, l \in \{1, 2, \dots, m\}$, then we can write the SDE (2.9) as matrix SDE

$$d\boldsymbol{S}_t = \boldsymbol{A}_t\boldsymbol{S}_t dt + \sum_{k=1}^{d} \boldsymbol{B}_t^k \boldsymbol{S}_t dW_t^k \tag{2.13}$$

for $t \in [0, \infty)$. Consequently, we obtain for the jth asset price the explicit solution

$$S_t^j = S_0^j \exp\left\{ \int_0^t \left(a_s^j - \frac{1}{2}\sum_{k=1}^{d} (b_t^{j,k})^2 \right) ds + \sum_{k=1}^{d} \int_0^t b_s^{j,k} dW_s^k \right\} \tag{2.14}$$

for $t \in [0, \infty)$ and $j \in \{1, 2, \ldots, d\}$. When taking the above exponential elementwise, the explicit solution of (2.9) can be expressed as the following exponential

$$\boldsymbol{S}_t = \boldsymbol{S}_0 \exp\left\{\int_0^t \left(\boldsymbol{A}_s - \frac{1}{2}\sum_{k=1}^{d}\left(\boldsymbol{B}_s^k\right)^2\right) ds + \sum_{k=1}^{d}\int_0^t \boldsymbol{B}_s^k dW_s^k\right\} \tag{2.15}$$

for $t \geq 0$. Additionally, if the appreciation rates and volatilities are piecewise constant, then we can simulate exact solutions. The main advantage of the multi-dimensional Black-Scholes model, which also made it so popular, is that it provides an explicit solution for the market dynamics and allows a range of explicit formulas.

Before we consider later more complicated SDEs let us give a simple example for a two-dimensional Black-Scholes model with

$$\boldsymbol{B}^1 = \begin{pmatrix} b_1 & 0 \\ 0 & b_2\varrho \end{pmatrix} \quad \text{and} \quad \boldsymbol{B}^2 = \begin{pmatrix} 0 & 0 \\ 0 & b_2\sqrt{1-\varrho^2} \end{pmatrix}. \tag{2.16}$$

Here we obtain the following exact solution

$$S_t^1 = S_0^1 \exp\left\{\left(a_1 - \frac{1}{2}b_1^2\right)t + b_1 W_t^1\right\}, \tag{2.17}$$

$$S_t^2 = S_0^2 \exp\left\{\left(a_2 - \frac{1}{2}b_2^2\right)t + b_2\left(\varrho W_t^1 + \sqrt{1-\varrho^2}W_t^2\right)\right\}, \tag{2.18}$$

for $t \in [0, \infty)$. The trajectory of this two-dimensional model is illustrated in Fig. 2.1 for the parameter choice $S_0^1 = S_0^2 = 1$, $a_1 = a_2 = 0.1$, $b_1 = b_2 = 0.2$ and $\varrho = 0.8$.

3 Matrix Ornstein-Uhlenbeck Processes

In this section we will show how to simulate matrices of Ornstein-Uhlenbeck (OU)-processes. This can be performed by using matrices of time changed Wiener processes. Therefore, we will first introduce matrix Wiener processes and show how to simulate time changed matrix Wiener processes.

Matrix Wiener Processes

Let us define a $d \times m$ standard matrix Wiener process $\boldsymbol{W} = \{\boldsymbol{W}_t = [W_t^{i,j}]_{i,j=1}^{d,m}, t \in [0, \infty)\}$. This matrix stochastic process can be obtained by the following construction

$$\begin{aligned} \boldsymbol{W}_0 &= \boldsymbol{0} \\ \boldsymbol{W}_{t_{i+1}} &= \boldsymbol{W}_{t_i} + \sqrt{\Delta}\boldsymbol{N}_{i+1}, \end{aligned} \tag{3.1}$$

for the times $t_i = i\Delta$, $i = \{0, 1, \ldots\}$ with $\Delta > 0$ and $d \times m$-matrix $\boldsymbol{0}$ of zero elements. Here $\boldsymbol{N}_{i+1} \sim \mathcal{N}_{d\times m}(\boldsymbol{0}, \boldsymbol{I}_m \otimes \boldsymbol{I}_d)$ is a matrix of zero mean Gaussian

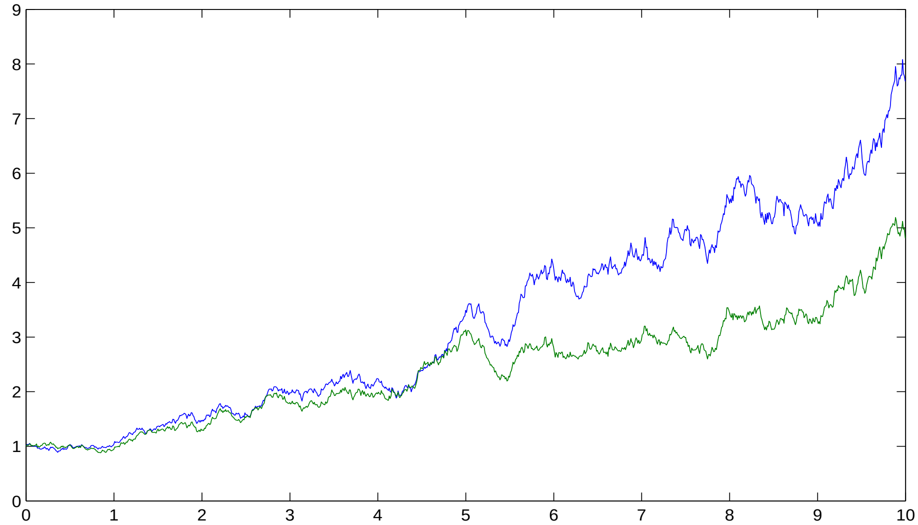

Figure 2.1: Trajectory of a two-dimensional Black-Scholes model with parameters $S_0^1 = S_0^2 = 1$, $a_1 = a_2 = 0.1$, $b_1 = b_2 = 0.2$ and $\varrho = 0.8$

distributed random variables. The covariance matrix $\boldsymbol{I}_m \otimes \boldsymbol{I}_d$ is an $m \times m$ block matrix with $d \times d$ block matrices as its elements, that is,

$$\boldsymbol{I}_m \otimes \boldsymbol{I}_d = \begin{pmatrix} \boldsymbol{I}_d & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I}_d & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \boldsymbol{I}_d \end{pmatrix}. \tag{3.2}$$

Here $\boldsymbol{I}_d$ denotes the $d \times d$ identity matrix. Moreover, similar to the vector case, we are able to define a transformed matrix Wiener process $\tilde{\boldsymbol{W}} = \{\tilde{\boldsymbol{W}}_t = [\tilde{W}_t^{i,j}]_{i,j=1}^{d,m}, t \in [0, \infty)\}$ using the above matrix Wiener process $\boldsymbol{W}$ as follows

$$\tilde{\boldsymbol{W}}_t = \boldsymbol{M}t + \boldsymbol{\Sigma}_1 \boldsymbol{W}_t \boldsymbol{\Sigma}_2^\top, \tag{3.3}$$

where $\boldsymbol{M}$ is a $d \times m$ matrix and $\boldsymbol{\Sigma}_1$ and $\boldsymbol{\Sigma}_2$ are nonsingular $d \times d$ and $m \times m$ matrices, respectively. Values of such a matrix stochastic process can be obtained at the discrete times $t_i = i\Delta$ by the following recursive computation

$$\begin{aligned} \tilde{\boldsymbol{W}}_0 &= \mathbf{0} \\ \tilde{\boldsymbol{W}}_{t_{i+1}} &= \tilde{\boldsymbol{W}}_{t_i} + \boldsymbol{M}\Delta + \sqrt{\Delta}\tilde{\boldsymbol{N}}_{i+1}, \end{aligned} \tag{3.4}$$

for $i \in \{0, 1, \dots\}$ and $\tilde{\boldsymbol{N}}_{i+1} \sim \mathcal{N}_{d\times m}(\mathbf{0}, \boldsymbol{\Sigma}_2 \otimes \boldsymbol{\Sigma}_1)$. Here, the covariance matrix

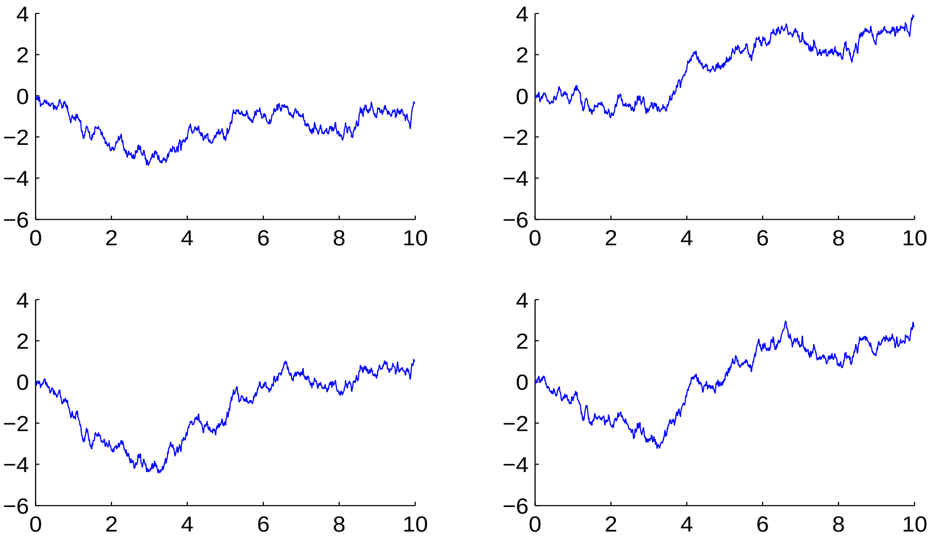

Figure 3.1: 2×2 matrix Wiener process with both correlated rows and columns

$\boldsymbol{\Sigma}_2 \otimes \boldsymbol{\Sigma}_1$ is an $m \times m$ block matrix of the form

$$\boldsymbol{\Sigma}_2 \otimes \boldsymbol{\Sigma}_1 = \begin{pmatrix} \sigma^2_{1,1}\boldsymbol{\Sigma}_1 & \sigma^2_{1,2}\boldsymbol{\Sigma}_1 & \dots & \sigma^2_{1,m}\boldsymbol{\Sigma}_1 \\ \sigma^2_{2,1}\boldsymbol{\Sigma}_1 & \sigma^2_{2,2}\boldsymbol{\Sigma}_1 & \dots & \sigma^2_{2,m}\boldsymbol{\Sigma}_1 \\ \vdots & \vdots & \vdots & \vdots \\ \sigma^2_{m,1}\boldsymbol{\Sigma}_1 & \sigma^2_{m,2}\boldsymbol{\Sigma}_1 & \dots & \sigma^2_{m,m}\boldsymbol{\Sigma}_1 \end{pmatrix}, \tag{3.5}$$

where $\boldsymbol{\Sigma}_1 = [\sigma^1_{i,j}]^d_{i,j}$ and $\boldsymbol{\Sigma}_2 = [\sigma^2_{i,j}]^m_{i,j}$.

In Fig. 3.1 we illustrate a 2×2 matrix transformed Wiener process $\tilde{\boldsymbol{W}}$ for $\varrho = 0.8$, which was obtained from the standard 2×2 matrix Wiener process $\boldsymbol{W}$ by the following transformation

$$\tilde{\boldsymbol{W}}_t = \boldsymbol{\Sigma}_1 \boldsymbol{W}_t \boldsymbol{\Sigma}_2^\top, \tag{3.6}$$

where

$$\boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_2 = \begin{pmatrix} 1 & 0 \\ \varrho & \sqrt{1-\varrho^2} \end{pmatrix}. \tag{3.7}$$

We note in Fig. 3.1 the correlation effect on the trajectories on both the elements of the columns and rows of such a 2×2 matrix valued transformed Wiener process.

Time Changed Wiener Processes

Instead of multiplying the time by some constant to scale the fluctuations of the Wiener paths, one can introduce time dependent scaling by a, so called, time change $\varphi = \{\varphi(t), t \geq 0\}$. Let us now consider a vector of time changed standard independent Wiener processes $\boldsymbol{W} = \{\boldsymbol{W}_{\varphi(t)} = (W^1_{\varphi(t)}, \dots, W^m_{\varphi(t)})^\top, t \in [0, \infty)\}$. Given the time discretization $t_i = i\Delta$, $i \in \{0, 1, 2, \dots\}$, with time step size $\Delta > 0$

we obtain this time changed standard Wiener process at discretization times by the following iterative formula

$$\begin{aligned}\boldsymbol{W}_{\varphi(0)} &= \mathbf{0} \\ \boldsymbol{W}_{\varphi(t_{i+1})} &= \boldsymbol{W}_{\varphi(t_i)} + \sqrt{\varphi(t_{i+1}) - \varphi(t_i)}\boldsymbol{N}_{i+1},\end{aligned} \tag{3.8}$$

where the vector $\boldsymbol{N}_{i+1} \sim \mathcal{N}_m(\mathbf{0}, \boldsymbol{I})$ is formed by independent standard Gaussian random variables. Here $\boldsymbol{I}$ is the $m \times m$ identity matrix. Obviously, it is possible to apply different time changes to different elements of the vector $\boldsymbol{W}$. For instance, let us define

$$\varphi_j(t) = \frac{b_j^2}{2c_j}(e^{2c_j t} - 1) \tag{3.9}$$

for $t \in [0, \infty)$, $b_j > 0, c_j > 0$ and $j \in \{1, 2, \ldots, m\}$. Then the Gaussian elements of the vector $\boldsymbol{W}_{\varphi(t)}$ can be defined such that

$$W^j_{\varphi_j(t_{i+1})} - W^j_{\varphi_j(t_i)} \sim \mathcal{N}(0, \varphi_j(t_{i+1}) - \varphi_j(t_i)), \tag{3.10}$$

where $W^j_{\varphi_j(0)} = 0$, $j \in \{1, 2, \ldots, m\}$ and $i \in \{0, 1, \ldots\}$.

In order to obtain a time changed vector Wiener process, whose elements are correlated time changed Wiener processes, it is sufficient to define a new vector $\tilde{\boldsymbol{W}} = \{\tilde{\boldsymbol{W}}_{\varphi(t)} = (\tilde{W}^1_{\varphi(t)}, \ldots, \tilde{W}^d_{\varphi(t)})^\top, t \in [0, \infty)\}$ by the following transformation

$$\tilde{\boldsymbol{W}}_{\varphi(t)} = \boldsymbol{B}\boldsymbol{W}_{\varphi(t)}, \tag{3.11}$$

where $\boldsymbol{B}$ is a $d \times m$-matrix of coefficients and $\boldsymbol{W} = \{\boldsymbol{W}_{\varphi(t)} = (W^1_{\varphi(t)}, \ldots, W^m_{\varphi(t)})^\top$, $t \in [0, \infty)\}$ is an m-dimensional time changed Wiener process with independent components as in (3.10).

Additionally, let us define a $d \times m$ standard time changed matrix Wiener process $\boldsymbol{W} = \{\boldsymbol{W}_{\varphi(t)} = [W^{j,k}_{\varphi(t)}]^{d,m}_{j,k=1}, t \in [0, \infty)\}$. Here, the independent elements of the matrix $\boldsymbol{W}_{\varphi(t)}$ are such that

$$W^{j,k}_{\varphi_{j,k}(t_{i+1})} - W^{j,k}_{\varphi_{j,k}(t_i)} \sim \mathcal{N}\left(0, \varphi_{j,k}(t_{i+1}) - \varphi_{j,k}(t_i)\right), \tag{3.12}$$

where $W^{k,j}_{\varphi_{k,j}(0)} = 0$, $t_i = i\Delta$, $i \in \{0, 1, \ldots\}$ and $j \in \{1, 2, \ldots, d\}$, $k \in \{1, 2, \ldots, m\}$. For instance, we may define the (j, k)th time transformation by

$$\varphi_{j,k}(t) = \frac{b_{j,k}^2}{2c_{j,k}}\left(e^{2c_{j,k}t} - 1\right) \tag{3.13}$$

for $t \in [0, \infty)$, $b_{j,k} > 0, c_{j,k} > 0$, and $j \in \{1, 2, \ldots, d\}$, $k \in \{1, 2, \ldots, m\}$. In order to obtain a time changed matrix Wiener process with correlated elements we can use the formula (3.3).

In Fig. 3.2 we display a time changed matrix Wiener process for $d = m = 2$ with the covariance matrix $\boldsymbol{I} \otimes \boldsymbol{\Sigma}_1$, where $\boldsymbol{\Sigma}_1$ is as in (3.7), $\varrho = 0.8$ and the parameters

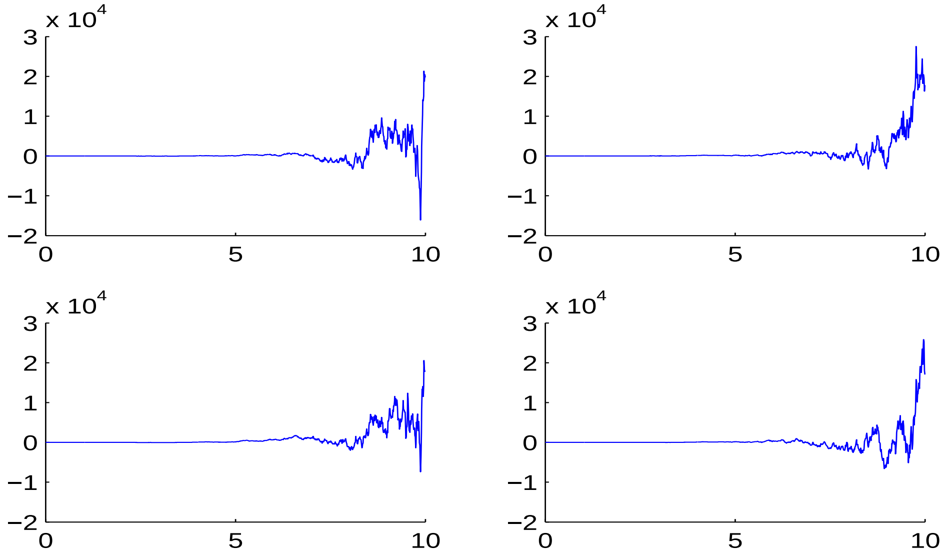

Figure 3.2: Matrix valued time changed Wiener process

in the time change equal $b_{j,k} = \sqrt{2}$ and $c_{j,k} = 1$ for $j, k \in \{1, 2\}$. That is, the same time change is applied to each of the elements of this matrix Wiener process. More precisely, we construct $\tilde{\boldsymbol{W}}$ by the relation

$$\tilde{\boldsymbol{W}}_{\varphi(t)} = \boldsymbol{\Sigma}_1 \boldsymbol{W}_{\varphi(t)} \boldsymbol{I}. \tag{3.14}$$

In this case we obtain a time changed matrix Wiener process $\tilde{\boldsymbol{W}}$ whose rows have independent elements, while its columns have dependent elements.

Multi-dimensional Ornstein-Uhlenbeck (OU)-processes

Let us now consider vector and matrix valued OU-processes. We will here construct multi-dimensional OU-processes as time changed and scaled multi-dimensional Wiener processes. Note that given the following two functions

$$s_t = \exp\{-ct\} \quad \text{and} \quad \varphi(t) = \frac{b^2}{2c}(e^{2ct} - 1) \tag{3.15}$$

for $t \in [0, \infty)$, $b, c > 0$, a scalar OU-process $Y = \{Y_t, t \in [0, \infty)\}$ can be represented in terms of a time changed and scaled scalar Wiener process, that is

$$Y_t = s_t W_{\varphi(t)}, \tag{3.16}$$

where $W = \{W_\varphi, \varphi \geq 0\}$ is a standard Wiener process in φ-time. By Itô's formula we obtain

$$\begin{aligned} dY_t &= W_{\varphi(t)} ds_t + s_t dW_{\varphi(t)} = -\frac{Y_t}{s_t} c s_t dt + s_t \frac{b}{s_t} d\tilde{W}_t \\ &= -cY_t dt + b d\tilde{W}_t, \end{aligned} \tag{3.17}$$

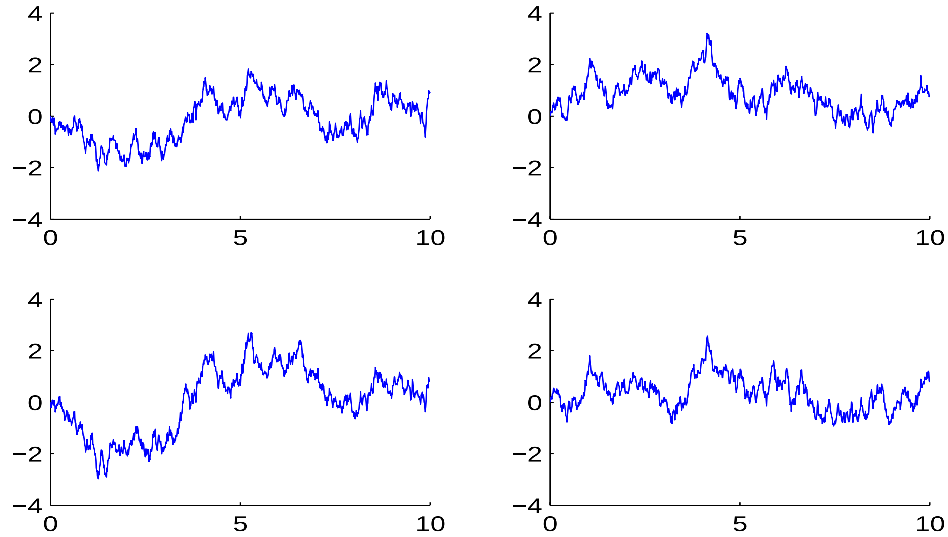

Figure 3.3: Matrix valued Ornstein-Uhlenbeck process

where $dW_{\varphi(t)} = \frac{b}{s_t} d\tilde{W}_t$, with $\tilde{W}$ denoting a standard Wiener process in t-time.

It is straightforward to obtain a vector OU-process by

$$\boldsymbol{Y}_t = s_t \boldsymbol{W}_{\varphi(t)}, \tag{3.18}$$

that is,

$$Y_t^j = s_t^j W_{\varphi_j(t)}^j \tag{3.19}$$

for $j \in \{1, 2, \ldots, d\}$ and $t \geq 0$. The generalization to a matrix OU-process is obvious. The construction of this process starts by forming a time changed $d \times m$ matrix Wiener process and is then scaling each element of this matrix by a function $s_t^{j,k}$ for $j \in \{1, 2, \ldots, d\}$ and $k \in \{1, 2, \ldots, m\}$. Hence, the elements of such a matrix can be expressed by the relation

$$Y_t^{j,k} = s_t^{j,k} W_{\varphi_{j,k}(t)}^{j,k} \tag{3.20}$$

for $j \in \{1, 2, \ldots, d\}$ and $k \in \{1, 2, \ldots, m\}$.

We illustrate in Fig. 3.3 the matrix OU-process, obtained from the time changed matrix Wiener process in Fig. 3.2, by the use of formula (3.20). Since, the time changed matrix Wiener process has correlated rows and independent columns, the OU-process in Fig. 3.3 shares this feature.

Multi-dimensional Geometric Ornstein-Uhlenbeck Processes

The Itô formula provides a general tool to generate a world of exact solutions of SDEs based on functions of the solutions of those SDEs we have already considered. As an example, let us generate explicit solutions for a geometric OU-process.

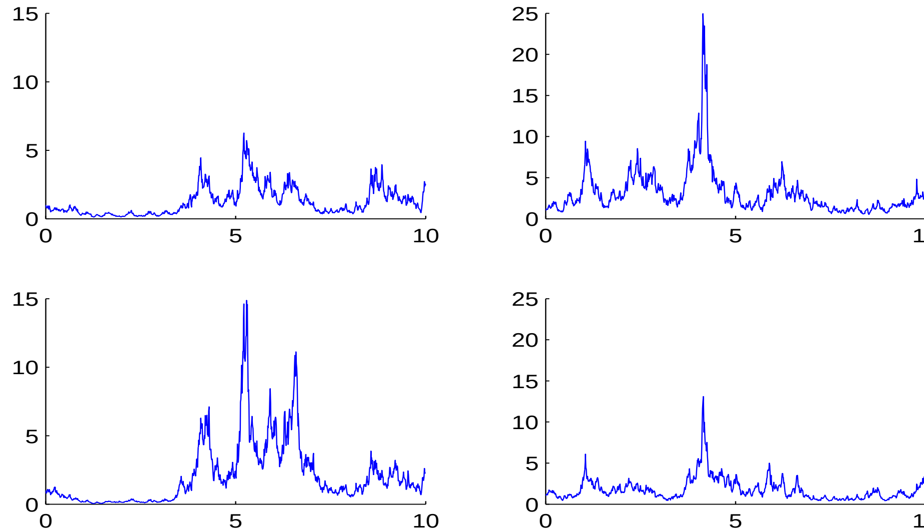

Figure 3.4: Matrix valued geometric OU-process

Here each element of a matrix valued OU-process is simply exponentiated. More precisely, when denoting by $\boldsymbol{X}_t = [X_t^{j,k}]_{j,k=1}^{d,m}$ a $d \times m$ matrix OU-process value and by $\boldsymbol{Y}_t = [Y_t^{j,k}]_{j,k=1}^{d,m}$ the corresponding $d \times m$ matrix geometric OU-process value at time t, then we obtain the elements of the matrix $\boldsymbol{Y}_t$ by

$$Y_t^{j,k} = \exp\{X_t^{j,k}\} \tag{3.21}$$

for $t \in [0, \infty)$.

In Fig. 3.4 we illustrate a 2×2 matrix geometric OU-process obtained from the matrix OU-process in Fig. 3.3 by application of (3.21) to each of its elements. More complex applications of the Itô formula for generating exact solutions will be considered in the next section.

4 Wishart Processes

In this section we will discuss the exact simulation of Wishart processes, see Bru (1991). These are matrix valued stochastic processes whose one-dimensional case is generating squared Bessel processes. Therefore, we will start by describing the exact simulation of a squared Bessel process, which later will be generalized to its matrix equivalent, the Wishart process.

Squared Bessel Processes

A squared Bessel process (BESQ$_x^\delta$) $X = \{X_\varphi,\ \varphi \in [\varphi_0, \infty)\}$, $\varphi_0 \geq 0$, of dimension $\delta \geq 0$ and with initial value $x > 0$, see Revuz & Yor (1999), is a fundamental stochastic process which appears in various ways, for instance, in financial modeling. This process can be described by the SDE

$$dX_\varphi = \delta\, d\varphi + 2\sqrt{|X_\varphi|}\, dW_\varphi \tag{4.1}$$

for $\varphi \in [\varphi_0, \infty)$ with $X_{\varphi_0} = x \geq 0$, where $W = \{W_\varphi,\ \varphi \in [\varphi_0, \infty)\}$ is a standard Wiener process starting at the initial φ-time, $\varphi = \varphi_0$, $\delta > 0$. This means, for $\varphi \in [\varphi_0, \infty)$ one has as increment of the quadratic variation of W the difference

$$[W]_\varphi - [W]_{\varphi_0} = \varphi - \varphi_0$$

for all $\varphi \in [\varphi_0, \infty)$. Furthermore, if we fix the behavior of X_φ at the boundary zero as reflection, then the absolute sign under the square root in (4.1) can be removed, and X_φ remains nonnegative and has a unique strong solution, see Revuz & Yor (1999).

The solution of the above SDE can be simulated exactly for the case when the dimension of this process is an integer, that is $\delta \in \{1, 2, \dots\}$. More precisely, for $\delta \in \{1, 2, \dots\}$ and $x \geq 0$ the dynamics of a BESQ$_x^\delta$ process X can be expressed as the sum of the squares of δ independent Wiener processes $W^1, W^2, \dots, W^\delta$ in φ-time, which start at time $\varphi = \varphi_0$ in $w^1 \in \Re, w^2 \in \Re, \dots w^\delta \in \Re$, respectively, such that $x = \sum_{k=1}^{\delta}(w^k)^2$. We can now construct the solution of (4.1) as

$$X_\varphi = \sum_{k=1}^{\delta}(w^k + W_\varphi^k)^2 \tag{4.2}$$

for $\varphi \in [\varphi_0, \infty)$. Applying the Itô formula we obtain

$$dX_\varphi = \delta d\varphi + 2\sum_{k=1}^{\delta}(w^k + W_\varphi^k)dW_\varphi^k \tag{4.3}$$

for $\varphi \in [\varphi_0, \infty)$ with $X_0 = \sum_{k=1}^{\delta}(w^k)^2 = x$. Furthermore, by setting

$$dW_\varphi = |X_\varphi|^{-\frac{1}{2}}\sum_{k=1}^{\delta}(w^k + W_\varphi^k)dW_\varphi^k \tag{4.4}$$

we obtain the SDE (4.1). Note that we have for W_φ the quadratic variation

$$[W]_\varphi = \int_{\varphi_0}^{\varphi}\frac{1}{X_s}\sum_{k=1}^{\delta}(w^k + W_s^k)^2 ds = \varphi - \varphi_0. \tag{4.5}$$

Hence, by the Lévy theorem the process W_φ is a Wiener process in φ-time.

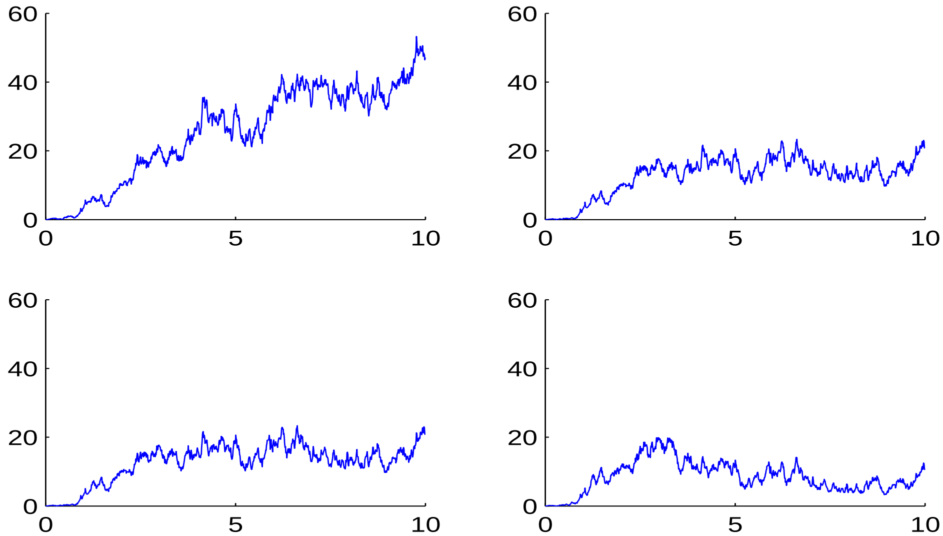

Figure 4.1: Wishart process

Wishart Process

The matrix generalization of a squared Bessel process is a Wishart process, see Bru (1991). The $m \times m$ matrix valued Wishart process with dimension $\delta \in \{1, 2, \ldots\}$ is the matrix process $\boldsymbol{S} = \{\boldsymbol{S}_t, t \geq 0\}$ with

$$\boldsymbol{S}_t = \boldsymbol{W}_t^\top \boldsymbol{W}_t \tag{4.6}$$

for $t \in \Re^+$ and initial matrix $\boldsymbol{s}_0 = \boldsymbol{W}_0^\top \boldsymbol{W}_0$. Here $\boldsymbol{W}_t$ is the value at time $t \geq 0$ of a $\delta \times m$ matrix Wiener process. Itô calculus applied to the relation (4.6) results in the following SDE

$$d\boldsymbol{S}_t = \delta \boldsymbol{I} dt + d\boldsymbol{W}_t^\top \boldsymbol{W}_t + \boldsymbol{W}_t^\top d\boldsymbol{W}_t, \tag{4.7}$$

where $\boldsymbol{I}$ is the $m \times m$ identity matrix. It can be shown that $\tilde{\boldsymbol{W}}_t$ expressed by

$$d\tilde{\boldsymbol{W}}_t = \left(\sqrt{\boldsymbol{S}_t}\right)^{-1} \boldsymbol{W}_t^\top d\boldsymbol{W}_t \tag{4.8}$$

is an $m \times m$ matrix Wiener process. Here $\sqrt{\boldsymbol{S}_t}$ represents the symmetric positive square root of $\boldsymbol{S}_t$, while $\left(\sqrt{\boldsymbol{S}_t}\right)^{-1}$ is the inverse of the matrix $\sqrt{\boldsymbol{S}_t}$. Note also that

$$\begin{aligned} d\tilde{\boldsymbol{W}}_t^\top &= d\boldsymbol{W}_t^\top \boldsymbol{W}_t \left(\left(\sqrt{\boldsymbol{S}_t}\right)^{-1}\right)^\top = d\boldsymbol{W}_t^\top \boldsymbol{W}_t \left(\left(\sqrt{\boldsymbol{S}_t}\right)^\top\right)^{-1} \\ &= d\boldsymbol{W}_t^\top \boldsymbol{W}_t \left(\sqrt{\boldsymbol{S}_t^\top}\right)^{-1} = d\boldsymbol{W}_t^\top \boldsymbol{W}_t \left(\sqrt{\boldsymbol{S}_t}\right)^{-1}, \end{aligned} \tag{4.9}$$

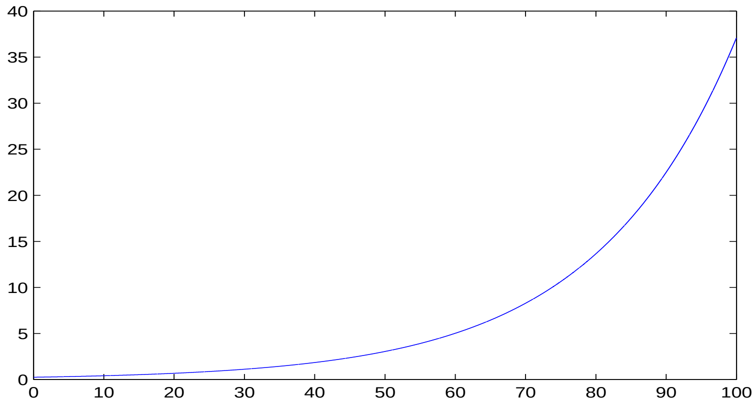

Figure 5.1: Time $\varphi(t)$ against time t

since $\boldsymbol{S}_t$ is a symmetric matrix. Therefore, (4.7) can be rewritten in the following form

$$d\boldsymbol{S}_t = \delta \boldsymbol{I} dt + \sqrt{\boldsymbol{S}_t} d\tilde{\boldsymbol{W}}_t + d\tilde{\boldsymbol{W}}_t^\top \sqrt{\boldsymbol{S}_t} \tag{4.10}$$

for $t \in \Re^+$.

In Fig. 4.1 we plot a 2×2 Wishart process of dimension $\delta = 2$. The matrix Wiener process in this example was obtained by assuming the covariance matrix $\boldsymbol{I} \otimes \boldsymbol{\Sigma}_1$, where $\boldsymbol{\Sigma}_1$ is as in (3.7), with $\varrho = 0.8$.

5 Affine Matrix Processes

Another group of matrix valued stochastic processes that can be simulated exactly is a matrix of affine processes. This family of stochastic processes has as its special case a matrix of square root (SR) processes, which are directly linked to Wishart processes. They also can be obtained from matrices of OU-processes. These two methods of exact simulation are described below.

SR-Processes Generated via OU-Processes

Let us first consider $\delta \in \{1, 2, \dots\}$ standard OU-processes, that is

$$dX_t^i = -cX_t^i dt + b dW_t^i \tag{5.1}$$

for $t \in [0, \infty)$, with $X_0^i = x_0$, $c, b \in \Re$ and independent standard Wiener processes W^i for $i \in \{1, 2, \dots, n\}$. The square of such an OU-process has the Itô differential

$$d(X_t^i)^2 = (b^2 - 2c(X_t^i)^2) + 2bX_t^i dW_t^i, \tag{5.2}$$

for $t \in [0, \infty)$ and $i \in \{1, 2, \dots, \delta\}$. Furthermore, we can form the sum of the δ

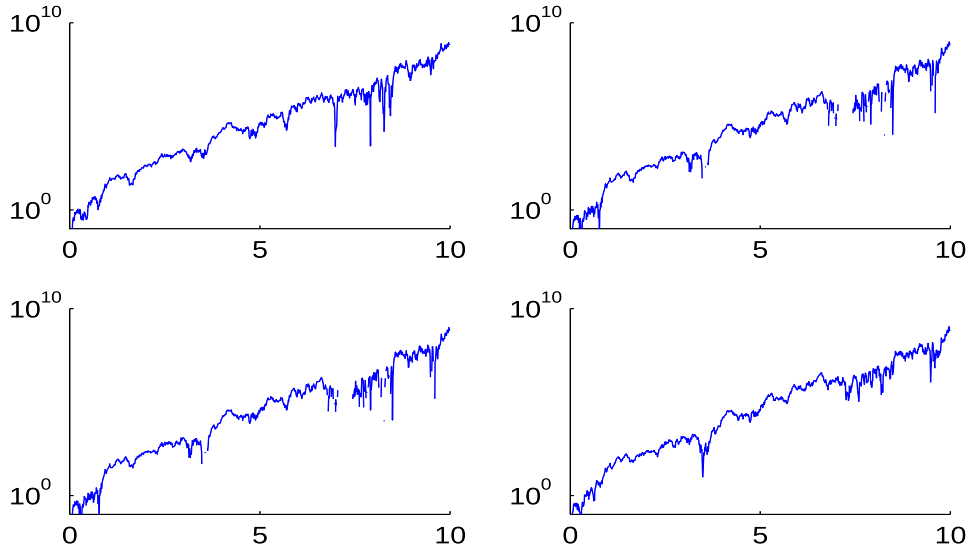

Figure 5.2: Time changed Wishart process in log-scale

squared OU-processes, that is,

$$Y_t = \sum_{i=1}^{\delta} (X_t^i)^2 \tag{5.3}$$

for $t \in [0, \infty)$. The SDE for Y_t turns out to be

$$dY_t = \sum_{i=1}^{\delta} \left(b^2 - 2c(X_t^i)^2\right) dt + 2b \sum_{i=1}^{\delta} X_t^i dW_t^i \tag{5.4}$$

for $t \in [0, \infty)$. In order to simplify the above SDE we introduce another Wiener process $\bar{W} = \{\bar{W}_t, t \in [0, \infty)\}$ defined as

$$\bar{W}_t = \int_0^t d\bar{W}_s = \sum_{i=1}^{\delta} \int_0^t \frac{X_s^i}{\sqrt{Y_s}} dW_s^i \tag{5.5}$$

for $t \in [0, \infty)$. It can be shown that the quadratic variation of $\bar{W}$ equals

$$[\bar{W}]_t = \int_0^t \sum_{i=1}^{n} \frac{(X_s^i)^2}{Y_s} ds = t. \tag{5.6}$$

Hence, by the Lévy theorem we see that $\bar{W}$ is a standard Wiener process. Therefore, we obtain an equivalent SDE for the square root process Y in the form

$$dY_t = (\delta b^2 - 2cY_t) dt + 2b\sqrt{Y_t} d\bar{W}_t \tag{5.7}$$

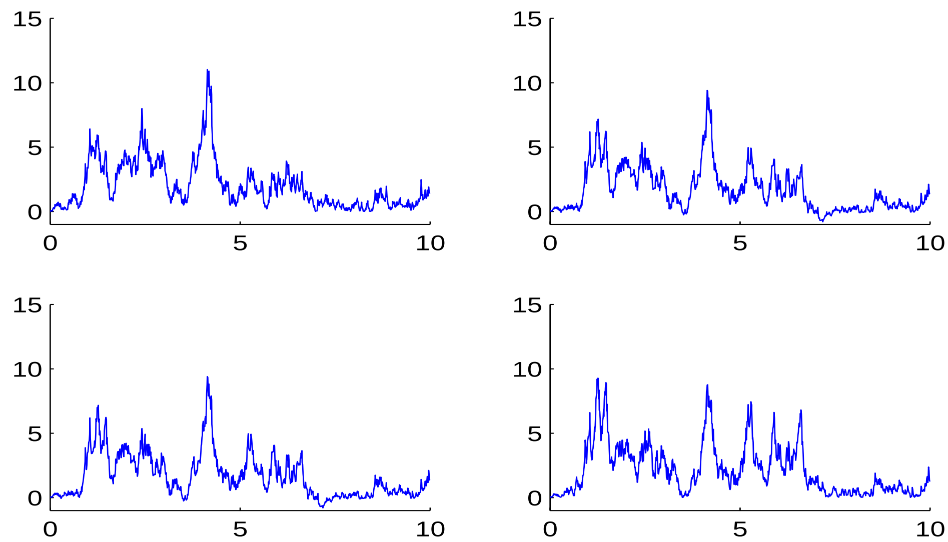

Figure 5.3: Matrix valued square root process

for $t \in [0, \infty)$ with $Y_0 = \delta(x_0)^2$. Note that this process is an SR-process of dimension $\delta \in \{1, 2, \dots\}$. It is well-known that for $\delta = 1$ the value Y_t can reach zero and is reflected at this boundary. For $\delta \in \{2, 3, \dots\}$ the process never reaches zero for $x_0 > 0$.

Matrix Valued Squares of OU-Processes

Kendall (1989) and Bru (1991) studied the matrix generalization for squares of OU-processes. Denote by $\boldsymbol{X}_t$ a $\delta \times m$ matrix solution of the SDE

$$d\boldsymbol{X}_t = -c\boldsymbol{X}_t dt + bd\boldsymbol{W}_t, \tag{5.8}$$

for $t \geq 0$, with $\boldsymbol{X}_0 = \boldsymbol{x}_0$. Here $\boldsymbol{W}_t$ is a $\delta \times m$ matrix Wiener process and $\boldsymbol{x}_0$ is a $\delta \times m$ deterministic initial matrix; $b, c \in \Re$. By setting

$$\boldsymbol{S}_t = \boldsymbol{X}_t^\top \boldsymbol{X}_t, \quad \boldsymbol{s}_0 = \boldsymbol{x}_0^\top \boldsymbol{x}_0 \tag{5.9}$$

and denoting $d\tilde{\boldsymbol{W}}_t = \sqrt{\boldsymbol{S}_t^{-1}} \boldsymbol{X}_t^\top d\boldsymbol{W}_t$ we obtain an $m \times m$ matrix SR process $\boldsymbol{S}$ of dimension $\delta = \{1, 2, \dots\}$. Note that the elements of $\tilde{\boldsymbol{W}}_t$ can be correlated. Then $\boldsymbol{S}_t$ solves the SDE

$$d\boldsymbol{S}_t = (\delta b^2 \boldsymbol{I} - 2c\boldsymbol{S}_t)dt + b(\sqrt{\boldsymbol{S}_t} d\tilde{\boldsymbol{W}}_t + d\tilde{\boldsymbol{W}}_t^\top \sqrt{\boldsymbol{S}_t}) \tag{5.10}$$

for $t \geq 0$, $\boldsymbol{S}_0 = \boldsymbol{s}_0$. Here $\boldsymbol{S}_t$ corresponds to a continuous-time process of stochastic, symmetric, positive definite matrices, while $\sqrt{\boldsymbol{S}_t}$ is the positive symmetric square root of the matrix $\boldsymbol{S}_t$, see Gouriéroux & Sufana (2004). Furthermore, $\boldsymbol{S}_t^{-1}$ is the inverse of the symmetric positive definite $m \times m$ matrix $\boldsymbol{S}_t$ and $\sqrt{\boldsymbol{S}_t^{-1}}$ its square root.

Note that for $m = 1$ the transform (5.9) simplifies to equation (5.3). We illustrate in Fig. 5.3 the matrix SR-process obtained from the matrix OU-process from Fig. 3.3. Note that not all elements of such a matrix remain always positive. The elements $S^{1,2}$ and $S^{2,1}$ are identical and, in general, not positive. Most importantly, the diagonal elements $S^{1,1}$ and $S^{2,2}$ are correlated SR-processes, which are always positive.

SR-processes Generated via Squared Bessel Processes

Using squared Bessel processes one can derive SR-processes by certain transformations. For this reason let $c : [0, \infty) \to \Re$ and $b : [0, \infty) \to \Re$ be given deterministic functions of time. We introduce the exponential

$$s_t = s_0 \exp\left\{ \int_0^t c_u \, du \right\} \tag{5.11}$$

and the φ-time

$$\varphi(t) = \varphi(0) + \frac{1}{4} \int_0^t \frac{b_u^2}{s_u} \, du \tag{5.12}$$

for $t \in [0, \infty)$ and $s_0 > 0$. Note that we have an explicit representation for the function $\varphi(t)$ in the case of constant parameters $b_t = \bar{b} \neq 0$ and $c_t = \bar{c} \neq 0$, where

$$\varphi(t) = \varphi(0) + \frac{\bar{b}^2}{4 \bar{c} s_0} (1 - \exp\{-\bar{c} t\}) \tag{5.13}$$

for $t \in [0, \infty)$ and $s_0 > 0$. Furthermore, if $\varphi(0) = -\frac{\bar{b}^2}{4 \bar{c} s_0}$, then this function simply equals

$$\varphi(t) = -\frac{\bar{b}^2}{4 \bar{c} s_0} \exp\{-\bar{c} t\} \tag{5.14}$$

for $t \in [0, \infty)$, $s_0 > 0$, $\bar{b} \neq 0$ and $\bar{c} \neq 0$. We show the function $\varphi(t)$ in Fig. 5.1 for the choice of $\bar{b} = 1$, $\bar{c} = -0.05$, $s_0 = 20$ and $\varphi(0) = -\frac{\bar{b}_t^2}{4 \bar{c} s_0} = 0.25$.

Given a squared Bessel process X of dimension $\delta > 0$, using our previous notation, we introduce the SR-process $Y = \{Y_t, t \geq 0\}$ of dimension $\delta > 0$ via the relation

$$Y_t = s_t \, X_{\varphi(t)} \tag{5.15}$$

indexed by time $t \geq 0$, see also Delbaen & Shirakawa (1997).

Furthermore, by (4.1), (5.15), (5.11) and (5.12) and the Itô formula we can express (5.15) in terms of the SDE

$$dY_t = \left(\frac{\delta}{4} \, b_t^2 + c_t \, Y_t \right) dt + b_t \sqrt{Y_t} \, dU_t \tag{5.16}$$

for $t \in [0, \infty)$, $Y_0 = s_0 X_{\varphi(0)}$ and

$$dU_t = \sqrt{\frac{4 s_t}{b_t^2}} dW_{\varphi(t)}.$$

Note that U_t forms by the Lévy theorem a Wiener process, since

$$[U]_t = \int_0^t \frac{4s_z}{b_z^2} d\varphi(z) = t. \tag{5.17}$$

The same time-change formula can be applied in the more general matrix case. Given the Wishart process $\boldsymbol{X}$ it can be shown that the matrix square root process can be obtained from the Wishart process by the following transformation

$$\boldsymbol{Y}_t = s_t \boldsymbol{X}_{\varphi(t)}, \tag{5.18}$$

where s_t and $\varphi(t)$ are as in (5.11) and (5.12), respectively. By (4.10), (5.18), (5.11) and (5.12) and the Itô formula we can express (5.18) in terms of the matrix SDE

$$d\boldsymbol{Y}_t = \left(\frac{\delta}{4} b_t^2 \boldsymbol{I} + c_t \boldsymbol{Y}_t\right) dt + \frac{b_t}{2}\left(\sqrt{\boldsymbol{Y}_t} d\boldsymbol{U}_t + d\boldsymbol{U}_t^\top \sqrt{\boldsymbol{Y}_t}\right) \tag{5.19}$$

for $t \in [0, \infty)$, $\boldsymbol{Y}_0 = s_0 \boldsymbol{X}_{\varphi(0)}$ and where $d\boldsymbol{U}_t = \sqrt{\frac{4s_t}{b_t^2}} d\boldsymbol{W}_{\varphi(t)}$ is the stochastic differential of a matrix Wiener process.

In Fig. 5.2 we display the trajectory of the elements of a 2×2 matrix time changed Wishart process $\boldsymbol{X}_{\varphi(t)}$ in log-scale. Here the off-diagonal elements do not show any value for the time periods when the argument of the logarithm becomes negative. Such periods do not arise for the diagonal elements which are of main interest. We now construct a trajectory of a 2×2 matrix SR-process obtained as time changed Wishart process by the use of formula (5.18). Note that this matrix SR-process is identical to the matrix SR-process in Fig. 5.3 obtained via squares of OU-processes. In Fig. 5.3 we see that the off-diagonal elements have near the time $t = 7$ indeed negative values.

Multi-dimensional Affine Processes

Let us now transform further the above obtained multi-dimensional SR-process in order to obtain multi-dimensional affine processes, see Duffie & Kan (1994). These processes have affine, that is linear drift and linear squared diffusion coefficients. In order to obtain members of this class of multi-dimensional processes we can simply shift the multi-dimensional SR-process by a nonnegative, differentiable function of time $a : [0, \infty) \to [0, \infty)$, defined through its derivative

$$a'_t = \frac{da_t}{dt} \tag{5.20}$$

for $t \in [0, \infty)$ with $a_0 \in [0, \infty)$. More precisely, we define the process $\boldsymbol{R} = \{\boldsymbol{R}_t, t \in [0, \infty)\}$ such that

$$\boldsymbol{R}_t = \boldsymbol{Y}_t + a_t \boldsymbol{I} \tag{5.21}$$

for $t \in [0, \infty)$. It is also possible to obtain more general affine processes by shifting the matrix valued SR-process by a matrix $\boldsymbol{A}_t$ of nonnegative differentiable functions of the type (5.20), that is

$$\boldsymbol{R}_t = \boldsymbol{Y}_t + \boldsymbol{A}_t \tag{5.22}$$

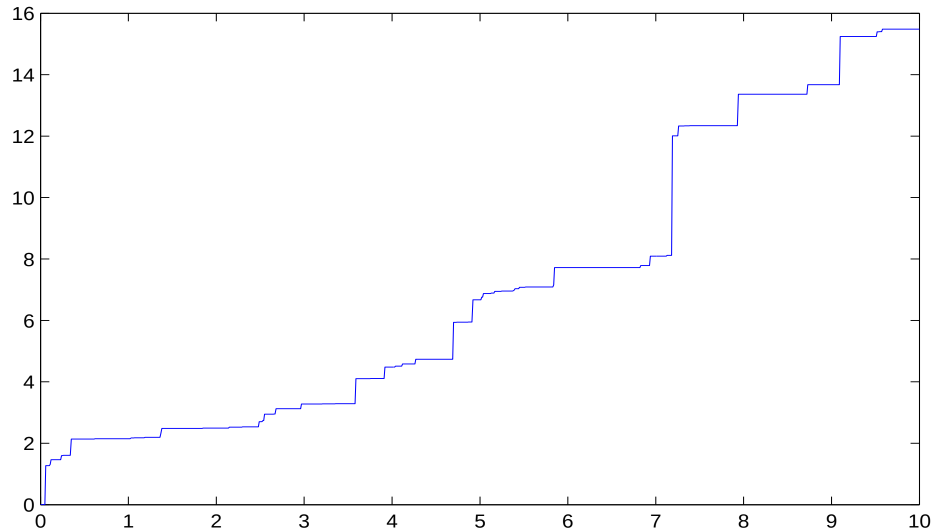

Figure 6.1: Gamma process

for $t \in [0, \infty)$. In this case $\boldsymbol{R}_t$ solves the following matrix SDE

$$d\boldsymbol{R}_t = \left(\frac{\delta}{4} b_t^2 \boldsymbol{I} + \boldsymbol{A}_t' - c_t \boldsymbol{A}_t + c_t \boldsymbol{R}_t\right) dt + \frac{b_t}{2}\left(\sqrt{\boldsymbol{R}_t - \boldsymbol{A}_t} d\tilde{\boldsymbol{W}}_t + d\tilde{\boldsymbol{W}}_t^\top \sqrt{\boldsymbol{R}_t - \boldsymbol{A}_t}\right), \tag{5.23}$$

for $t \in [0, \infty)$. Here $\boldsymbol{A}_t'$ denotes the matrix of the derivatives of the type (5.20) for the shifts of each element. Obviously, we applied here the Itô formula to the equation (5.22).

6 Matrix Lévy Processes

We considered so far the exact simulation of solutions of multi-dimensional SDEs driven by vector or matrix Wiener processes. The simulation methods described can be, however, adapted also to multi-dimensional SDEs when these are driven by more general vector or matrix valued Lévy processes. In principle, one can substitute the Wiener processes by some Lévy processes.

Since Lévy processes have independent stationary increments, it is possible to construct paths of a wide range of d-dimensional Lévy processes $\boldsymbol{L} = \{\boldsymbol{L}_t, t \geq 0\}$ at given discretization times $t_i = i\Delta, i \in \{0, 1, 2, \ldots\}$, with fixed time step size $\Delta > 0$. The distribution of the Lévy increments $\boldsymbol{L}_{t_{i+1}} - \boldsymbol{L}_{t_i}$, however, must be infinitely divisible for the process $\boldsymbol{L}$ to be the transition distribution of a Lévy process. One example of a family of infinitely divisible distributions is the generalized hyperbolic (GH) distribution, see for instance McNeil, Frey &

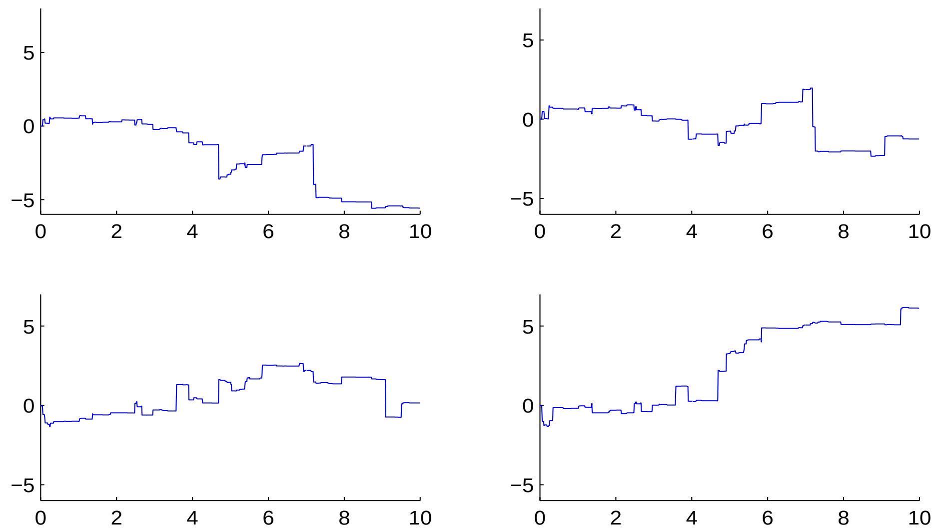

Figure 6.2: Matrix VG-process

Embrechts (2005). This family of distributions yields variance gamma (VG) and normal inverse Gaussian (NIG) processes as special cases.

Simulation of the d-dimensional VG and NIG processes results from their representation as subordinated vector Wiener processes with drift. That is,

$$\boldsymbol{L}_t = \boldsymbol{a} V_t + \boldsymbol{B}\boldsymbol{W}_{V_t}, \tag{6.1}$$

for $t \in [0,\infty)$. Here $\boldsymbol{a} = (a_1, a_2, \ldots, a_d)^\top$ is a d-dimensional vector, $\boldsymbol{B}$ is a $d \times m$-matrix and $\boldsymbol{W} = \{\boldsymbol{W}_{V_t} = (W^1_{V_t}, W^2_{V_t}, \ldots, W^m_{V_t})^\top, t \in [0,\infty)\}$ is a standard m-dimensional vector Wiener process. When V_t is the gamma process or the inverse Gaussian process, we obtain, respectively, the d-dimensional VG process and the NIG process.

We also define the $d \times m$ matrix VG and NIG processes by

$$\boldsymbol{L}_t = \boldsymbol{M} V_t + \boldsymbol{\Sigma}_1 \boldsymbol{W}_{V_t} \boldsymbol{\Sigma}_2, \tag{6.2}$$

where $\boldsymbol{M}$ is a $d \times m$ matrix and $\boldsymbol{\Sigma}_1$ and $\boldsymbol{\Sigma}_2$ are nonsingular $d \times d$ and $m \times m$ matrices, respectively. Here $\boldsymbol{W} = \{\boldsymbol{W}_{V_t} = [W^{i,j}_{V_t}]^{d,m}_{i,j=1}, t \in [0,\infty)\}$ is a standard $d \times m$ matrix Wiener process.

Processes of type (6.1) and (6.2) possess a number of useful properties because they are conditionally Gaussian. In particular, if one knows how to simulate the increments of the subordinator V, the values of $\boldsymbol{L}$ in (6.2) can be obtained at the discrete times $t_i = i\Delta$ by the following recursive computation

$$\begin{aligned} \boldsymbol{L}_0 &= \boldsymbol{0} \\ \boldsymbol{L}_{t_{i+1}} &= \boldsymbol{L}_{t_i} + \boldsymbol{M}\Delta V_{i+1} + \sqrt{\Delta V_{i+1}}\tilde{\boldsymbol{N}}_{i+1}, \end{aligned} \tag{6.3}$$

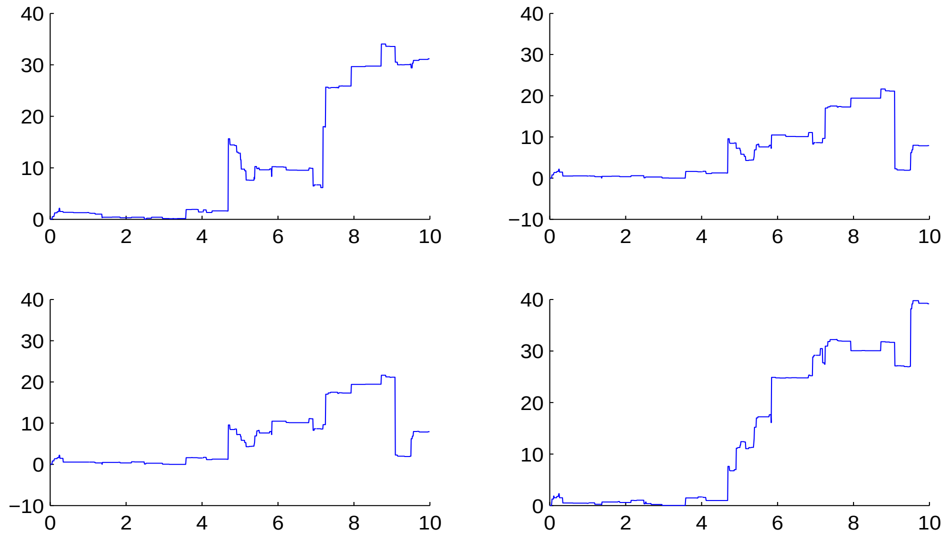

Figure 6.3: Wishart process driven by the matrix VG-process in Fig. 6.2

for $i \in \{0, 1, \dots, \}$ and $\tilde{\boldsymbol{N}}_{i+1} \sim \mathcal{N}_{d\times m}(\boldsymbol{0}, \boldsymbol{\Sigma}_2 \otimes \boldsymbol{\Sigma}_1)$. Here the covariance matrix $\boldsymbol{\Sigma}_1 \otimes \boldsymbol{\Sigma}_2$ is as in (3.5).

The VG process is obtained by (6.3), where $\Delta V_{i+1} \sim \kappa \mathrm{Ga}(\frac{\Delta}{\kappa}, 1)$ are gamma random variables for $i \in \{1, 2, \dots\}$, while the NIG process is obtained by (6.3), where $\Delta V_{i+1} \sim \mathrm{IGaussian}(\frac{\Delta^2}{\kappa}, \Delta)$ are inverse Gaussian random variables for $i \in \{1, 2, \dots\}$. Here the parameter κ is the variance of the subordinator V. See also Cont & Tankov (2004) who describe exact simulation of scalar VG and NIG processes. This includes algorithms for generators of gamma and inverse Gaussian random variables.

Since we can simulate the paths of such driving Lévy processes exactly it is possible to simulate solutions for the type of the above introduced SDEs when driven by Lévy noise. For instance, let us consider a Wishart process of dimension δ driven by a VG-process. That is, we consider the multi-dimensional SDE of the form

$$d\boldsymbol{S}_t = \delta \boldsymbol{I} dt + \sqrt{\boldsymbol{S}_t} d\boldsymbol{L}_t + d\boldsymbol{L}_t^\top \sqrt{\boldsymbol{S}_t} \tag{6.4}$$

for $t \in \Re^+$. In order to simulate this Wishart process, which may be driven by the VG-process $\boldsymbol{L}$, we first need to simulate a $\delta \times m$ matrix VG-process. Afterwards we obtain the $m \times m$ VG-Wishart process of dimension $\delta \in \{1, 2, \dots\}$ by setting $\boldsymbol{S}_t = \boldsymbol{L}_t^\top \boldsymbol{L}_t$, for $t \in \Re^+$.

In Fig. 6.1 we show a trajectory of a gamma process, which is always nondecreasing. Here we have chosen $\kappa = 1$. Moreover, in Fig. 6.2 we display a 2×2 VG matrix process, with parameters $\boldsymbol{M} = \boldsymbol{0}$ and the covariance matrix $\boldsymbol{I} \otimes \boldsymbol{\Sigma}_1$, where $\boldsymbol{\Sigma}_1$ is as in (3.7) with $\varrho = 0.8$. The subordinator is here chosen to be the gamma process illustrated in Fig 6.1. Additionally, in Fig. 6.3 we display the corresponding trajectory of the resulting 2×2 Wishart process of dimension $\delta = 2$.

The subordination methodology can be widely applied to generate trajectories of other matrix Lévy processes, for instance, matrix Lévy OU-processes and matrix Lévy-affine processes.

7 Direct Sampling

It remains to emphasize that advanced software packages, as Matlab and Mathematica, provide routines that generate a range of random variables with various distributions that are needed in the above described simulations. For some multi-dimensional distribution functions it is possible by using such software to sample corresponding vector random variables. These are directly available for computation and should have exactly the requested multivariate transition distribution function. For instance, the increment of the multi-dimensional Wiener process can be simulated from available Gaussian random vectors

$$\boldsymbol{X}_{t+\Delta} - \boldsymbol{X}_t \sim \mathcal{N}_d(\mathbf{0}, \boldsymbol{\Sigma}\Delta), \tag{7.1}$$

where $\mathcal{N}_d$ denotes a d-dimensional Gaussian distribution with mean vector $\mathbf{0}$ and covariance matrix $\boldsymbol{\Sigma}\Delta$. Of course, one can build these random vectors also from independent Gaussian random variables. Similarly, the value at time $t + \Delta$ of the standard d-dimensional OU-process can be obtained by using the Gaussian random vector

$$\boldsymbol{X}_{t+\Delta} \sim \mathcal{N}_d(\boldsymbol{X}_t e^{-\Delta}, \boldsymbol{\Sigma}(1 - e^{-2\Delta})). \tag{7.2}$$

The $m \times m$ Wishart process value $\boldsymbol{X}_{t+\Delta}$ can be simulated from the noncentral Wishart distribution W_m with δ degrees of freedom, covariance matrix $\boldsymbol{\Sigma}\Delta$ and noncentrality matrix $\boldsymbol{\Sigma}^{-1}\boldsymbol{X}_t\Delta^{-1}$, when random variables of the kind

$$\boldsymbol{X}_{t+\Delta} \sim W_m(\delta, \boldsymbol{\Sigma}\Delta, \boldsymbol{\Sigma}^{-1}\boldsymbol{X}_t\Delta^{-1}) \tag{7.3}$$

are directly available. For details on how to sample from the noncentral Wishart distribution we refer to Gleser (1976).

Also the increments of some Lévy processes, which are constructed from a GH distribution, can be obtained exactly by using available vector random variables

$$\boldsymbol{X}_{t+\Delta} - \boldsymbol{X}_t \sim GH_d(\lambda, \chi, \psi, \mu, \Sigma, \gamma) \tag{7.4}$$

for fixed $\Delta > 0$.

Finally, we refer to Kloeden & Platen (1999) for a list of different, mostly scalar specific examples of explicitly solvable multi-dimensional SDEs. These dynamics can be further generalized in the above indicated directions, using time changes and applying time changed Lévy processes instead of Wiener processes. For instance, the Wiener process driving the Black-Scholes model can be easily generalized by subordination to multi-dimensional Lévy processes, yielding exponential Lévy asset price models, similar to those described in Barndorff-Nielsen & Shephard (2001), Geman, Madan & Yor (2001) and Eberlein (2002).

The above presented exact and almost exact simulation methods for multi-dimensional SDEs lead to accurate scenario simulations that are reliable over long periods of time. This is important for various applications, for instance, the pricing of insurance contracts. Typically arising numerical stability problems are simply avoided by exact simulation. Finally, we remark that there is no major problem to introduce further jump effects into the considered type of dynamics via a jump-adapted time discretization, see Platen (1982), which enlarges significantly the class of processes that allow exact simulation.

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