G-AUTOMATA, COUNTER LANGUAGES AND THE CHOMSKY HIERARCHY

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ABSTRACT. We consider how the languages of G-automata compare with other formal language classes. We prove that if the word problem of G is accepted by a machine in the class $\mathcal{M}$ then the language of any G-automaton is in the class $\mathcal{M}$. It follows that the so called *counter languages* (languages of $\mathbb{Z}^n$-automata) are context-sensitive, and further that counter languages are indexed if and only if the word problem for $\mathbb{Z}^n$ is indexed.

1. Introduction

In this article we compare the languages of G-automata, which include the set of *counter languages*, with the formal language classes of context-sensitive, indexed, context-free and regular. We prove in Theorem 5 that if the word problem of G is accepted by a machine in the class $\mathcal{M}$ then the language of any G-automaton is in the class $\mathcal{M}$. It follows that the counter languages are context-sensitive. Moreover it follows that counter languages are indexed if and only if the word problem for $\mathbb{Z}^n$ is indexed.

The article is organized as follows. In Section 2 we define G-automata, linear-bounded automata, nested-stack, stack, and pushdown automata, and the word problem for a finitely generated group. In Section 3 we prove the main theorem, and give the corollary that counter languages are indexed if and only if the word problem for $\mathbb{Z}^n$ is indexed for all n.

2. Definitions

If G is a group with generating set $\mathcal{G}$, we say two words u, v are equal in the group, or $u =_G v$, if they represent the same group element. We say u and v are identical if the are equal in the free monoid, that is, they are equal in $\mathcal{G}^*$.

Definition 1 (Word problem)**.** The *word problem* of a group G with respect to a finite generating set $\mathcal{G}$ is the set of words $\{w \mid w =_G 1\}$. Note that

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the word problem for a finitely generated group is a language over a finite alphabet.

Definition 2 (G-automaton)**.** Let G be a group and Σ a finite set. A (non-deterministic) *G-automaton* A_G over Σ is a finite directed graph with a distinguished *start vertex* q_0, some distinguished *accept vertices*, and with edges labeled by $(\Sigma^{\pm 1} \cup \{\epsilon\}) \times G$. If p is a path in A_G, the element of $(\Sigma^{\pm 1})$ which is the first component of the label of p is denoted by $w(p)$, and the element of G which is the second component of the label of p is denoted $g(p)$. If p is the empty path, $g(p)$ is the identity element of G and $w(p)$ is the empty word. A_G is said to *accept* a word $w \in (\Sigma^{\pm 1})$ if there is a path p from the start vertex to some accept vertex such that $w(p) = w$ and $g(p) =_G 1$.

Definition 3 (Counter language)**.** A language is *k-counter* if it is accepted by some $\mathbb{Z}^k$-automaton. We call the (standard) generators of $\mathbb{Z}^k$ *counters*. A language is *counter* if it is k-counter for some $k \geq 1$.

These definitions are due to Mitrana and Striebe [9]. Note that in these counter automata, the values of the counters is not accessible until the final accept/fail state. For this reason they are sometimes called *blind*. Elston and Ostheimer [1] proved that the word problem of G is deterministic counter with an extra "inverse" property if and only if G is virtually abelian. Recently Kambites [8] has shown that the inverse property restriction can be removed from this theorem.

It is easy to see that the word problem for G is accepted by a G-automaton.

Lemma 4. *The word problem for a finitely generated group G is accepted by a deterministic G-automaton.*

Proof: Construct a G-automaton with one state and a directed loop labeled by (g, g) for each generator g. The state is both start and accept. A word in the generators is accepted by this automaton if and only if it represents the identity, by definition. □

Recall the definitions of the formal language classes of recursively enumerable, decidable, context-sensitive, indexed, stack, context-free, and regular. Each of these can be defined as the languages of some type of restricted Turing machine as follows.

Consider a machine consisting of finite *alphabet* Σ, a finite *tape alphabet* Γ, a *finite state control* and an infinite *work tape*, which operates as follows. The finite state control is a finite graph with a specified *start node*, some specified *accept nodes*, and edges labeled by an alphabet letter and an instruction for the work tape. The instructions in general are are the form `read, write, move left, move right` and one reads/writes letters from Γ on the tape. The tape starts out blank.

One inputs a finite string in Σ^* one letter at a time, read from left to right. For each letter $x \in \Sigma$, the finite state control performs some instructions on the work tape corresponding to an edge who label is $(x, \texttt{instructions})$,

and moves to the target node of the edge. One starts at the start node, and *accepts* the string if there is some path from the start node to an accept node labeled by the letters of the string. The *language* of a machine is the set of strings in Σ^* which the machine accepts. If the finite state control includes edges of the form $(\epsilon, \texttt{instructions})$ then one can work on the work tape without reading input, and if a machine has such edges, or two edges with the same letter in the first coordinate of the edge label, then such machines (and their languages) are called *non-deterministic*.

If we allow no further restrictions on how this machine operates, then we have a *Turing machine*. By placing increasingly strict restrictions on the machine, we obtain a hierarchy of languages corresponding to the machines. If the Turing machine halts on accepted strings then the language is *recursively enumerable* or *r.e.*, and if it halts both accepted and rejected strings the language is *decidable*. If we restrict the number of squares of tape that can be used to be a constant multiple of the length of the input string, then we obtain a *linear bounded automaton*, and the languages of these are called *context-sensitive*. If we make the tape act as a *nested stack* (see [4]) then the language of such a machine is called *indexed*.

If the tape is a stack (first in last out) where the pointer may read but not write on any square, then the machine and its languages are called *stack*. If the tape is a stack such that the pointer can only read the top square, we have a *pushdown automaton*, the languages of which are *context-free*. Finally, if we remove the tape altogether we are left with just the finite state control, which we call a *finite state automaton*, languages of which are *regular*.

For more precise definitions see [4] for nested stack automata, [5] for stack, [12] for regular and context-free and [7] for these plus linear bounded automata. Two good survey articles are [11] and [3].

3. Main theorem

Theorem 5. *Let $\mathcal{M}$ be a formal language class: (regular, context-free, stack, indexed, context-sensitive, decidable, r.e.) and let G be a finitely generated group. The word problem for G is in $\mathcal{M}$ if and only if the language of every G-automaton is in $\mathcal{M}$.*

Proof: By Lemma 4 the word problem of G is accepted by a G-automaton, so one direction is done.

Let L be a language over an alphabet Σ accepted by a G-automaton P. Fix a finite generating set $\mathcal{G}$ for G which includes all elements of G that are the second coordinate of an edge label in P. Let N be an $\mathcal{M}$-automaton which accepts the word problem for G with respect to this generating set.

Construct a machine M from the class $\mathcal{M}$ to accept L as follows. The states of M are of the form (p_i, q_j) where p_i is a state of P and q_i is a state of N. The start state is (p_S, q_S), and accept states are (p_A, q_A) where p_S, p_A, q_S, q_A are start and accept states in P and N.

The transitions are defined as follows. For each edge (x, g) in P from p to p', and for each edge $(g, \texttt{instruction})$ in N from q to q', add an edge from (p, q) to (p', q') in M labeled $(x, \texttt{instruction})$.

Then M accepts a string in Σ^* if there is a path in M corresponding to paths in P from the start state to an accept state such that the labels of the second coordinate of the edges give a word w in $\mathcal{G}^*$, and a path in N from a start state to an accept state labeled by w which evaluates to the identity of G and respects the tape instructions.

Note that M is deterministic if both N and P are deterministic and no state in P has two outgoing edges with the same group element in the first coordinate of the edge labels. □

It is easy to see that if G is a finite group, then the construction describes a finite state automaton (we don't need any tape). Herbst [6] showed that the word problem for a group is regular if and only if the group is finite. If G is virtually free then Muller and Schupp [10] proved that G has a context-free word problem, so the language of every G-automaton for G virtually free is context-free.

Since the word problem for G virtually abelian is context-sensitive, we get the following corollary.

Corollary 6. *Counter languages are context-sensitive.*

More generally, Gersten, Holt and Riley [2, Corollary B.2] have shown that finitely generated nilpotent groups of class c have context-sensitive word problems, so the language of every G-automaton for G finitely generated nilpotent is context-sensitive.

Figure 1 shows how counter languages fit into the hierarchy.

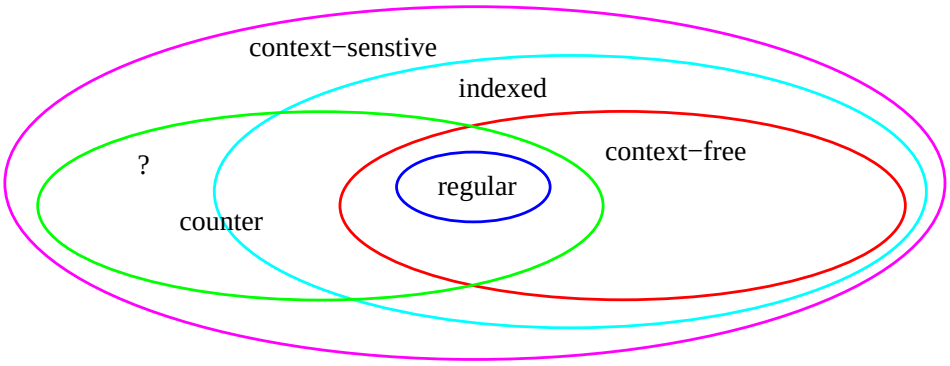

FIGURE 1. How counter languages fit into the hierarchy

Corollary 7. *The word problem for $\mathbb{Z}^n$ is indexed for all $n \geq 1$ if and only if every counter language is indexed.*

Proof: By Theorem 5 if the word problem for $\mathbb{Z}^n$ is indexed then the language of every $\mathbb{Z}^n$-automaton is indexed, proving one direction.

Since the word problem of $\mathbb{Z}^n$ is an n-counter language then if it is not indexed then not every counter language is indexed. □

By Gilman and Shapiro, if the word problem of G is accepted by a nested stack automaton with some extra restrictions, then it is virtually free. We still do not know whether word problem of $\mathbb{Z}^n$ is accepted by a nested stack automaton without extra restrictions.

Conjecture 8. The word problem for $\mathbb{Z}^2$ is not indexed.

To motivate this conjecture and suggest a possible proof strategy, define $L_n(y, z)$ to be the set of words in $\{y, z\}^*$ with n ys and n zs, and consider the language $L = \{x^n w_n \mid n \in \mathbb{N}, w_n \in L_n(y, z)\}$. This language is the word problem $\mathbb{Z}^2$ with respect to the generators a, b, a^{-1}, b^{-1}, intersected with the regular language $\{(ab)^n w_n \mid w_n \in L_n(a^{-1}, b^{-1})\}$. So if one could prove L is not indexed then one would prove the conjecture.

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