Asymmetry of Technical Analysis and Market Price Volatility

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Abstract

Within the framework of heterogeneity and bounded rationality, this paper models the financial market as an interaction of two types of boundedly rational investors—fundamentalists and chartists. The model is characterised by a nonlinear dynamical system. Applying stability and bifurcation theory, we examine the dynamics of the market price and market behaviour. It is found that investors' behaviour, in particular the asymmetric beliefs of the chartists and the interaction of the two types of investors play a very important role in explaining the market volatility. Numerical simulations of the corresponding stochastic model demonstrate that the model is able to generate the stylised facts, including volatility clustering, high kurtosis, and the long-range dependence of asset return observed in financial markets.

Keywords: price fluctuation, bounded rationality, asymmetrical beliefs, stability, bifurcation

JEL: C62, D53, E32, E44, G12

1 Introduction

Volatility is an inherent characteristic of financial markets. It provides investors opportunities for obtaining high excess returns which are however associated with high risk in general. With the development and diversity of financial markets, it becomes more important and interesting to understand market volatility and its underlying market mechanism.

The traditional financial theory developed since the 1950s is based on the assumptions of the rational expectation and the representative agent. Based on the assumptions, asset prices are the outcome of the market interaction of utility maximising agents who use rational expectations when forming their expectations about future market outcomes. Preferences of agents are assumed to satisfy conditions that enable the mass of investors to be considered as a single representative agent. Since agents are rationally impounding all relevant information into their trading decisions, the movement of prices is assumed to be perfectly random and hence exhibit the so-called random walk behaviour. Namely, market prices do not change if there is no new information, see Fama (1965). These are the classical rational expectation and efficient market hypothesis. This view is important in empirical finance, see Sargent (1993). It is also the basis of the stochastic price mechanisms assumed in many of the key theoretical models in finance, such as the optimal portfolio theory that has developed out of the work of Markowitz (1952) and Merton (1971), the static and dynamic capital asset pricing model of Sharpe (1964), Lintner (1965), Mossin (1966) and Merton (1973), and models for the pricing of contingent claims beginning with the work of Black and Scholes (1973). The empirical work represented by Fama (1976) provided favourable evidence in earlier empirical studies to support the market efficiency and random walk hypothesis and financial experts and economists have used the theory to explain observed market prices.

There appears a range of abnormal phenomena in financial markets, however, which cannot be explained within the framework of the traditional financial theory since the 1980s. They include equity premium puzzle (the anomalously higher historical real returns of stocks over government bonds), excess volatility (excess volatility of stock price cannot be explained by rational fundamental value adjustment based on random news), volatility clustering (large changes tend to be followed by large changes, of either sign, and small changes tend to be followed by small changes), long memory (returns per se are unpredictable but their absolute and squared values have significant autocorrelation with a hyperbolic decay structure), popularity of technical analysis (such as simple moving average, momentum strategy and percentage retracement) and herd behaviour (the tendency for individuals to mimic the rational or irrational actions of a larger group). In addition, the return distribution displays characteristics which are different from normal distribution underlying the traditional finance theory, such as skewness, fat tail and higher kurtosis. Certainly the 1987 crash has been well researched but so far there is no satisfactory explanation of what was the “news event” that triggered such a large market movement. This led to some questioning of the basic tenets of the rational expectation and efficient market hypothesis, though they have played very important roles in the earlier development of finance theory. Due to incomplete and asymmetrical information in financial markets, investors are very often not able to make perfectly rational decisions. In addition, market price is the outcome of interaction of different investors in the market and these heterogeneity and interaction are not characterised in the traditional representative agent financial theory.

In 1957, Simon proposed a hypothesis of bounded rationality. Since then, this hypothesis has been developed further, in particular, in the 1990s. "Bounded rationality" means that investors in the real world can at best be boundedly rational when making decision under uncertainty, have limited computational power, information and wealth, and are heterogeneous with respect to risk-preferences and the way in which they form expectations. Heterogeneity was proposed in Zeeman (1974), Beja and Goldman (1980) and De Long, Shleifer, Summers and Waldmann (1990) by considering markets that are characterised by investors with heterogeneous beliefs and different psychology behaviour. Within the framework of bounded rationality and heterogeneous markets, the interaction of investors and the change of market prices can be described by nonlinear dynamical systems. Based on this new framework, Day and Huang (1990) and Chiarella (1992) find that, even without new information, interaction among investors with heterogeneous beliefs in markets can generate rich market price behaviour and even market bubbles and crashes. It shows that it is due to the internal mechanism, rather than external disturbances, that can produce complex market fluctuation, which offers a new explanation to the market crash in 1987. The stability and bifurcation theory of nonlinear dynamical systems provides a mathematical tool to study market complexity within the new framework, leading to a broad application to the studies in finance and economics over the past twenty years, see the survey paper of Chang (2006). In particular, within the framework of bounded rationality, Brock and Hommes (1997, 1998) propose a concept of adaptively rational equilibrium, called BH model hereafter. A key aspect of the model is the expectation feedback. Namely, agents adapt their beliefs over time by choosing different expectations and investment strategies based upon their past performance; at the same time, their beliefs influence the market price and hence the performance of different trading strategies. Applying the theory of nonlinear dynamical systems, they find that the rational selection of investors can be characterised by a switching intensity of investors among different strategies based on their performance. When the switching intensity increases, for example turnover ratio is high, market prices fluctuate with high volatility and even become chaotic. This phenomenon underlines a fact that individual's rational decision can lead to randomness and complexity of the market. The findings in Brock and Hommes provide a theoretical explanation to the change of the trading rule used in Chinese stock markets from $T + 0$ to $T + 1$ to reduce the switching intensity of investors and stabilise the market. Later, following the framework of BH model, Chiarella and He (2002, 2003) study the effect of risk averse coefficients and investment strategies of investors in different market scenarios on market prices. Hommes, Huang and Wang (2005) demonstrate the robustness of BH model. Chiarella, He and Hommes (2006) extend BH model to study the dynamic behaviour of moving average rules and explain various market price phenomena including temporary bubbles, sudden market crashes, and price switching between different levels. Gaunersdorfer, Hommes and Wagener (2007) try to provide an explanation in the generating mechanism on the volatility clustering based on BH model. Using empirical data, Boswijk, Hommes and Manzan (2007) and Gaunersdorfer and Hommes (2007) test the explanation power of BH model to real markets by applying some statistic methods. Recently, He and Li (2007*b*, 2008) use a simpler market fraction model to describe the relationship between the phenomena and statistic characteristics observed in real markets and the dynamic behaviour of the underlying deterministic model, showing that the model is able to generate many stylised facts (especially long memory) which is difficult to explain in the traditional finance theory. They provide an explanation successfully on the generating mechanism from the viewpoint of dynamics. We refer the reader to the survey paper of Hommes (2006) for the development in this literature. In another survey paper, Chiarella, Dieci and He (2009) provide a comprehensive summary about how to use the method of nonlinear dynamical systems to study the market price changes within the framework of heterogeneous belief and bounded rationality. In addition, there is a great contribution in artificial financial markets to explore complex phenomena in real markets, for example, Lettau (1997), LeBaron, Arthur and Palmer (1999) and Chen and Yeh (2001). We refer the reader to the surveys of LeBaron (2006) and Chen (2007) for the development in this literature. All the literature has demonstrates that the models are able to describe some stylised phenomena observed in financial markets, such as volatility persistence and fat tail, and provide insight into the important role of investors' heterogeneity and bounded rationality in explaining market volatility. This demonstrates that nonlinear dynamical system as a mathematical tool provides a great potential to the study of this new framework. More recently, the method has been extended to the study in exchange rate markets and macroeconomics, see, for example Chiarella, He and Zheng (2009). In this paper, we seek to use the framework of heterogeneous beliefs and bounded rationality to study the impact of asymmetric belief in technical analysis on market volatility.

In the current literature of heterogeneous agent models, it is usually assumed that heterogeneous beliefs of agents are either linear or nonlinear but symmetric, meaning that investors have symmetric beliefs to changes of market prices in different market situations. However, in real markets, investors can have different views on the market condition and react differently to ups and downs of market prices. In other words, there exists an asymmetric effect in financial markets and investors can have asymmetric views on price signals in different market situations. For example, Veronesi (1999) shows that in equilibrium, investors' willingness to hedge against changes in their own "uncertainty" on the true state makes stock price overreact to bad news in good times and underreact to good news in bad times. Lu and Xu (2004) find that the asymmetric market reaction in the Chinese stock markets is different from other countries. Using EGARCH model, they test asymmetric characteristics of the Chinese stock markets in bullish/bearish states and explain the "Matthew Effect" from aspects of anticipation, structure, psychology and trading mechanism. Based on the data from the Shanghai Stock Exchange, He and Li (2007*a*) analyse the asymmetric characteristics of return and volatility in different market trends and find that there is obvious difference of mean-reverting between bull and bear markets and rational expectation hypothesis does not hold. Hirshleifer (2001) gives a nice survey on investor psychology and asset pricing.

The questions are then how to model the asymmetric views of investors and what is the impact of the asymmetric reaction on market price volatility. An understanding of these questions hopes to help us to explain complex phenomena in financial markets. In this paper, within the framework of BH model and Chiarella, He and Hommes (2006), we introduce an asymmetric belief into one of commonly used technical strategies, namely percentage retracement, and examine the impact of the asymmetric beliefs of the chartists on the market complexity, in particular, the market price volatility by using the tools of nonlinear dynamical systems and statistics. By analysing the corresponding deterministic model, we find that the asymmetric beliefs of the chartists to market price signals play an important role in the deviation of market price from its fundamental value. In addition, the adaptive behaviour of investors to update their strategies based on their performance results in herding behaviour and enforces market volatility. This implies that the bounded rational behaviour of investors is an important source of market volatility. Therefore, market volatility is the intrinsic characteristics of the development of financial markets.

The paper is organised as follows. Section 2 sets up an asset pricing model of the fundamentalists and chartists who have asymmetric beliefs about market. Mathematically, the model is described by a high-dimensional stochastic nonlinear dynamical system. To help our understanding, we focus more on the economic intuition of the analysis of the model. Following the standard approach in nonlinear dynamical systems, Section 3 applies the stability and bifurcation theory to analyse the corresponding deterministic model without random perturbations and to examine the stability and complexity of the model. This analysis provides a theoretic foundation for the analysis of the original stochastic model. In Section 4, we introduce market noise and assume that the fundamental price follows a stochastic process. By numerical simulations, we explore the impact of the asymmetric beliefs and the random perturbations on market price. Further economic discussion and a conclusion are included in Section 5. To focus on the economic intuition, rather than the technical details, of the model, we leave the mathematical proofs of the theoretic results in the appendix. We hope this paper can provide a new framework and approach for the Chinese researchers to study qualitatively and quantitatively financial market behaviour and help us to improve our understanding of market operation and to manage the risk, especially in the currently globally financial crisis.

2 Model

Consider a market with one risky asset (for instance stock, index or managed fund) and one risk free asset. Let P_t denote the price (ex dividend) per share of the risky asset at time t, and let y_t be the stochastic dividend process of the risky asset. The risk free asset is perfectly elastically supplied at return r. In our paper, we assume the risky asset price is determined by a market maker who adjusts the market price based on the imbalance between demand and supply. That is, when demand exceeds supply, the price goes up and otherwise the price goes down. Different from the rational expectation and representative agent hypothesis in the traditional theory, we assume that investors have heterogeneous beliefs and they are bounded rational. The heterogeneous beliefs are described by different expectations about future price, while the bounded rationality is captured by different trading strategies based on different beliefs and adaptive switching among different strategies based on their performance. Heterogeneous expectations have been found in empirical studies, especially in exchange rate markets, and have been used widely in modelling financial market behaviour. In line with Chiarella, He and Hommes (2006), we group different types of investors by their beliefs and the risky asset price can be set period by period upon the aggregate excess demand D_t, which is given by

$$D_t = \sum_{q\in Q} n_{q,t} D_t^q,$$

where Q represents the set of types of investors, $D_t^q (q \in Q)$ is the excess demand of type q trader at time t, $n_{q,t}$ is the market fraction of type q traders at time t. The number of investors can be arbitrary but the fractions of all types of investors must satisfy $\sum_{q\in Q} n_{q,t} = 1$. Here it is necessary to point out that it is usually assumed in traditional models that the market maker adjusts the risky asset price in absolute amount based on the aggregate excess demand, that is $P_{t+1} - P_t = \mathcal{P}(D_t)$. Under this mechanism, the market price can be negative; also, when there is a same excess demand for the risky asset with different fundamental prices, the stock price evolution is the same, which is not realistic. In our model, we adopt the relative price, instead of the absolute price, adjustment mechanism of the market maker. Namely,

$$\frac{P_{t+1} - P_t}{P_t} = \mathcal{S}(D_t) = \mathcal{S}\left(\sum_{q\in Q} n_{q,t} D_t^q\right), \tag{1}$$

where $\mathcal{S}(x)$ is a monotonically increasing function, which models the function of price adjustment by the market maker in market maker scenario. A consideration of wealth constraint and the stylising role of the market maker implies that the function $\mathcal{S}(x)$ is usually assumed to be an S-shaped function, see Figure 1. For simplicity, we take $\mathcal{S}(x) = k\tanh(hx)$, where $0 < k < 1$, $h > 0$. Therefore, when $D_t > 0$, there exists an excess demand in the market which pushes the price up, namely $P_{t+1} > P_t$. When $D_t < 0$, there exists an excess supply in the market, pushing the price down $P_{t+1} < P_t$.

In addition, under the assumption of $\mathcal{S}(x) = k\tanh(hx)$,

$$\left|\frac{P_{t+1} - P_t}{P_t}\right| < k.$$

Hence, k represents the maximum up and low limits of the risky asset price, like the 10% up and low price limits used in the Chinese stock markets. The parameter $\mu = \mathcal{S}'(0) = kh$ measures the adjustment speed near zero excess demand, which captures the future price change when the excess demand is close to zero. For an economic explanation of the S-shaped function, we refer the reader to Chiarella (1992) and Chiarella, Dieci and Gardini (2002).

Insert Figure 1 here

In financial markets, investment strategies capturing different price expectations are used widely, among which, two types of investors are especially common, fundamentalists who use fundamental analysis and chartists who use various price indices of the future price trend. For simplicity, we assume throughout this paper that there are only two types of traders, fundamentalists and chartists, that is $Q = \{fundamentalists,\ chartists\}$ (in brief $Q = \{f, c\}$). Their trading strategies and excess demand functions are specified in the following discussion.

Fundamentalists–The fundamentalists are assumed to know the fundamental information of the risky asset. They use the fundamental analysis and believe that the market price can deviate from its fundamental value, but in long run, it will converge to the neighborhood of the fundamental value. Therefore, the fundamentalists focus on the long-run change of the risky asset price. We assume that the fundamentalists know the fundamental price of the risky asset and their excess demand depends on the deviation of the market price from the fundamental value. At time t, without constraint in wealth, their excess demand D_t^f is assumed to be proportional to the difference between the fundamental price F_t and the market price P_t, namely

$$D_t^f = \alpha(F_t - P_t),$$

where $\alpha > 0$ is the reaction coefficient of the fundamentalists, which represents the mean-reverting beliefs of the fundamentalists. When $F_t > P_t$, the fundamentalists believe that the risky asset is undervalued and so they take a long position. Contrarily, the price is regarded as overvaluation and so they take a short position. Moreover, the more price deviation, the more excess demand. In general, from the viewpoint of the market price containing the information of the fundamental, the fundamentalists play an active role in stabilising markets. However, when their adjustment speed (α) is too high, they can make the market unstable. The phenomena of the stability and instability induced by the fundamentalists' over-adjustment will be discussed in detail.

Chartists–The chartists believe that the charting signals from past prices contain some information about the future change of the price of the risky asset. They prefer cheaper thumb strategies, such as simple moving average, percentage retracement, momentum, Fibonacci studies, and other popular technical methods in financial markets, see Murphy (2004). In our paper, we mainly seek to analyse the effect of percentage retracement (PR). A retracement is a countertrend move. The retracement is based on the thought that prices will reverse or "retrace" a portion of the previous movement before resuming their underlying trend in the original direction. For example, 33%, 50% and 66% retracement rates are well known as illustrated in Figure 2. In particular, when the chartists believe 50% retracement, when the short down-adjustment of the market price during the rising price trend does not cross the line of 50% PR, the chartists believe that the price down-trend has not been formed, the market price is just in a temporary adjustment and it will go up after the adjustment and they hence take a long position on the risky asset; otherwise they believe that the price down-trend is doomed and the rising price phase is over, so they take a short position on the risky asset, see Figure 2(a). Similarly, the price up-adjustment case is shown in Figure 2(b). It shows that the chartists focus on the short-term change of the risky asset price in making their investment decision. This short-term investment behaviour is one of the sources of market instability, especially when there is an instantaneous big shock in the market.

Insert Figure 2 here

The percentage retracement strategy of the chartists can therefore be expressed by the weighted historical prices, that is,

$$\widehat{P}_t = (1-\omega)P_{t-1} + \omega P_{t-2},$$

which captures the expected price of the chartists, where $\omega \in [0,1]$ represents the coefficient of the retracement strategy. In particular, if $\omega = 50\%$, $\widehat{P}_t$ is the simple moving average of length 2. $\omega = 33\%$ and 66% respectively represent the 33% and 66% retracement strategies. A trading signal to trade for the chartists is defined as the difference[1] δ_t between the current price P_t and the expectation price $\widehat{P}_t$, that is $\delta_t = P_t - \widehat{P}_t$. When $\delta_t > 0$, the chartists believe that the price is in a rising trend and they hence take a long position; otherwise, they take a short position. Mathematically, this strategy of the chartists can be written as

$$D_t^c = g(\delta_t).$$

Usually g is a monotonically increasing function of trading signals. Considering a wealth constraint for the chartists, g is usually assumed to be an S-shaped function, for example $g(x) = \mathbb{u}\tanh(\mathbb{v}x)(\mathbb{u} > 0,\ \mathbb{v} > 0)$. Note that under the assumption, $g(\cdot)$ is a symmetric and monotonically increasing function and $\lim_{x\to\pm\infty} |g(x)| = \mathbb{u} < \infty$. It means that the chartists' views to positive and negative trading signals are symmetric and the chartists are cautious but not very cautious when the price difference δ_t is large (either positive or negative). In contrast, empirical analysis in Veronesi (1999) and Lu and Xu (2004) show that investors have asymmetric views to market information in financial markets. In addition, when the market price is far away from the level indicated by the price trend, the chartists become cautious. In this paper, we consider the chartists who have different views on the market, for instance, they prefer buying/selling to selling/buying in bullish/bearish markets facing the same price deviation δ_t; they buy/sell more/smaller in bullish markets compared with bearish markets. They are cautious for large price rise or fall. These behaviour assumptions imply that g satisfies the following general properties, see Figure 3: there exist $x_1^* < 0,\ x_2^* > 0$ such that

$$g(0) = 0,\ xg(x) > 0 \text{ for } x \neq 0, \tag{2a}$$

$$g'(x) > 0 \text{ for } x_1^* < x < x_2^*; \quad g'(x) < 0 \text{ for } x < x_1^* \text{ or } x > x_2^*, \tag{2b}$$

$$\lim_{x \to x_1^*} g(x) = g_l < +\infty, \quad \lim_{x \to x_2^*} g(x) = g_u > -\infty. \tag{2c}$$

Insert Figure 3 here

In this paper, we take $g(x) = \frac{ax}{1+bx+c^2x^2}$ where $a > 0,\ c > 0,\ b \in (-2c, 2c)$. It can be verified that g satisfies the conditions in (2). In particular, the choice of the function has the following implications.

(i) $x_1^* = -\frac{1}{c},\ x_2^* = \frac{1}{c},\ g(-\frac{1}{c}) = -\frac{a}{2c-b}, g(\frac{1}{c}) = \frac{a}{2c+b}$. Condition (2a) means that when the price has no trend (that is $\delta_t = 0$), the chartists do nothing. However when there is positive (negative) trend in the risky asset price, they will talk a long (short) position which is limited because of the wealth constraint. The maximum (minimum) excess demand achieves at $|\delta_t| = 1/c$ and the amount depends on the parameters a, b, c.

(ii) The parameter c measures the confident level of the chartists to extrapolate the price trend. When the price deviation $|\delta_t|$ is below the confident level $\frac{1}{c}$, the chartists are confident. The larger the deviation of the price is, the more the excess demand of the chartists is. However, if the price deviation $|\delta_t|$ is beyond the confident level $\frac{1}{c}$, the chartists are cautious in making an investment decision: the larger the deviation of the price is, the smaller the excess demand of the chartists is. Thus, $\pm\frac{1}{c}$ captures the taking-profit and stop-losing levels of the chartists.

(iii) The parameter $a = g'(0) > 0$ measures the chartists' extrapolation when x is small. For small (large) values of a, the chartists' demand is very insensitive (sensitive) to small changes in x, which may characterise the unwillingness (willingness) of the chartists to get into the market when changes in x are small.

(iv) The parameter b measures the asymmetry of the chartists' response to changes of x. For $b = 0$, the chartists' long and short positions are symmetric with respect to the change of x. However, for $b \neq 0$, the position of the chartists for positive and negative information is not symmetric. This means that the chartists have different views towards the market. The phenomenon is called *the asymmetric effect.* In particular, when $b < 0$, we have $\frac{a}{2c+b} > \frac{a}{2c-b}$ and the chartists have a bullish view on the market. Otherwise, when $b > 0$, $\frac{a}{2c+b} < \frac{a}{2c-b}$ and the chartists have a bearish view on the market. We call b *the bull-bear coefficient*, in brief B-coefficient[2].

(v) The function satisfies $\lim_{x \to \pm\infty} g(x) = 0$, meaning that when the price difference x between the current price and the expectation price trend of the chartists is getting far away, the chartists become cautious and reduce their demand.

We have quantitatively introduced the heterogeneous beliefs of the investors and their rational investment strategies guided by their beliefs. Now we turn to bounded rationality of the investors based on the adaptive behaviour of the investors.

Fitness Measure and Fraction Evolution–At time t, the fitness measure $\pi_{q,t}(q = f,\ c)$ of type q traders can be defined as their realised net profit:

$$\pi_{q,t} = (P_t + y_t - RP_{t-1})D^q_{t-1} - C_q,$$

where $R = 1 + r$, $C_q \geq 0$ is the transaction cost of type q trader. Since fundamental information is expensive, the cost for the fundamentalists is more than that for the chartists, implying that $C = C_f - C_c \geq 0$. More generally, one can introduce accumulated performance measure by considering a weighted average of the net realised profit and past performance as follows:

$$U_{q,t} = \pi_{q,t} + \eta U_{q,t-1}, \tag{3}$$

where the parameter $\eta \in [0, 1)$ represents the memory of the accumulated fitness function.

Similarly to Brock and Hommes (1997, 1998) and Chiarella, He and Hommes (2006), the market fraction $n_{q,t}$ follows the discrete choice probability

$$n_{q,t} = \frac{e^{\beta U_{q,t}}}{N_t},$$

where $N_t = \sum_{q \in Q} e^{\beta U_{q,t}}$ and the parameter $\beta(\geq 0)$ is the switching intensity measuring how fast the fractions of different type agents in the market switch each other, based on the performance measure. In particular, for $\beta = \infty$, all the investors will choose the optimal strategy in each period. For a more extensive discussion on the discrete choice model, we refer to Manski and McFadden (1981) .

Summarising the above analysis, we can get the following nonlinear dynamical system:

$$\begin{cases} P_{t+1} = P_t\big[1 + \mathcal{S}(n_{f,t}D_t^f + n_{c,t}D_t^c)\big], \\ n_{f,t} = \dfrac{e^{\beta U_{f,t}}}{e^{\beta U_{f,t}} + e^{\beta U_{c,t}}}, \quad n_{c,t} = 1 - n_{f,t}, \\ U_{q,t} = (P_t + y_t - RP_{t-1})D_{t-1}^q - C_q + \eta U_{q,t-1},\ q \in \{f, c\}, \\ D_t^f = \alpha(F_t - P_t), \quad D_t^c = g\big(P_t - (1-\omega)P_{t-1} - \omega P_{t-2}\big). \end{cases} \tag{4}$$

Let $U_t = U_{f,t} - U_{c,t} = (\pi_{f,t} - \pi_{c,t}) - C + \eta U_{t-1}$. Then system (4) can be rewritten into

$$\begin{cases} P_{t+1} = P_t\left[1 + \mathcal{S}\left(\dfrac{e^{\beta U_t}}{e^{\beta U_t}+1}\alpha(F_t - P_t) + \dfrac{1}{e^{\beta U_t}+1}g(P_t - (1-\omega)P_{t-1} - \omega P_{t-2})\right)\right], \\ U_{t+1} = (P_{t+1} + y_{t+1} - RP_t)\Big[\alpha(F_t - P_t) - g\big(P_t - (1-\omega)P_{t-1} - \omega P_{t-2}\big)\Big] - C + \eta U_t. \end{cases} \tag{5}$$

When the fundamental price F_t and the dividend y_t are random, (5) is a stochastic dynamical system. Given the challenges in the theory of stochastic dynamical systems, we firstly assume F_t and y_t are deterministic and focus on the effect of investment behaviour of the fundamentalists and chartists on the market price by using theory of deterministic dynamic systems. We then analyse the original system (5) through stochastic simulations.

3 Dynamical Behaviour of Deterministic Model

In this section, we assume that the fundamental price is deterministic, that is $F_t \equiv F^* > 0$ and $y_t = F_t r \equiv F^* r$, denoted by $\bar{y}$. Then system (5) can be written into a deterministic dynamical system

$$\begin{cases} P_{t+1} = P_t\left[1 + \mathcal{S}\left(\dfrac{e^{\beta U_t}}{e^{\beta U_t}+1}\alpha(F^* - P_t) + \dfrac{1}{e^{\beta U_t}+1}g\big(P_t - (1-\omega)P_{t-1} - \omega P_{t-2}\big)\right)\right], \\ U_{t+1} = (P_{t+1} + \bar{y} - RP_t)\Big[\alpha(F^* - P_t) - g\big(P_t - (1-\omega)P_{t-1} - \omega P_{t-2}\big)\Big] - C + \eta U_t. \end{cases} \tag{6}$$

The system is equivalent to a 4-dimensional nonlinear difference system. The dynamic study of the system includes the existence and uniqueness, stability of equilibria, and the complex phenomena induced from instability. This is the standard method in the stability and bifurcation theory of nonlinear dynamical systems. The theory has been well developed and used widely in many fields. In the past twenty years, the theory has been applied into many aspects of economic study by providing a new approach to study complex phenomena of financial markets. In this section, we apply the theory to study the system (6) in order to understand the impact of heterogeneous beliefs and bounded rationality on the market price.

The stability of the market plays a very important role in the development of financial markets. The stability of market equilibria of (6) can be described by the following proposition.

Proposition 1 (Existence and Stability of Equilibria) *Denote* $U^* = -\frac{C}{1-\eta}$, $n_f^* = \frac{e^{\beta U^*}}{e^{\beta U^*}+1}$, $n_c^* = 1 - n_f^*$ *and* $\bar{\alpha} := F^* \mu n_f^* \alpha$, $\bar{a} := F^* \mu n_c^* a$. *Then for the deterministic dynamical system (6),*

(1) *there exist two steady states* $(P_1^*, U_1^*) = (0, \frac{\alpha F^* \bar{y} - C}{1-\eta})$, *and* $(P_2^*, U_2^*) = (F^*, U^*)$,

(2) (P_1^*, U_1^*) *is unstable for all parameters* $(\bar{a}, \bar{\alpha}, \omega)$,

(3) (P_2^*, U_2^*) *is locally asymptotically stable(LAS) for* $(\bar{a}, \bar{\alpha}, \omega) \in D$, *where*

$$D = \left\{ (\bar{a}, \bar{\alpha}, \omega) : 0 < \omega \leq 1,\ 0 < \bar{a} < \frac{1}{\omega},\ \max\left\{0,\ \frac{(\omega \bar{a} + 1)\big((1+\omega)\bar{a} - 1\big)}{\bar{a}\omega}\right\} < \bar{\alpha} < 2 + 2\bar{a}(1-\omega); \right.$$
$$\left. \omega = 0,\ 0 < \bar{a} < 1,\ 0 < \bar{\alpha} < 2 + 2\bar{a} \right\}.$$

Insert Figure 4 here

Proposition 1 indicates that the system (6) has two steady states. The price of the first steady state (P_1^*, U_1^*) is zero. This steady state is always unstable. This implies that an asset with zero price has no value and such asset does not exist in financial markets. In addition, it also implies that an under-valued asset will recover its value. This phenomenon is consistent with what we observe in financial markets. The price of the second steady state (P_2^*, U_2^*) is the fundamental value F^* of the risky asset and we call (P_2^*, U_2^*) *the fundamental steady state*. The stability of the fundamental steady state is determined by the parameters ω, $\bar{a} = F^* \mu n_c^* a$ and $\bar{\alpha} = F^* \mu n_f^* \alpha$, see Figure 4, where $(\bar{a}, \bar{\alpha})$ are the adjusted reaction intensities (or called extrapolation rates) of the fundamentalists and chartists scaled by the fundamental price F^*, the speed of the price adjustment μ, and the market fractions, n_f and n_c. Based on the stable region of the fundamental steady state in Figure 4(b), we can see that, when the parameters $(\bar{a}, \bar{\alpha})$ are close to the diagonal, the fundamental steady state is stable; when ω is smaller, $(\bar{a}, \bar{\alpha})$ need to be closer to the diagonal in order to maintain the stability of the fundamental steady state. This indicates that the stability of the fundamental steady state can be maintained when the behaviour between the fundamentalists and chartists are balanced. Thus, when the beliefs of the two types of investors are very different, especially when the chartists put more weight to the last price when forming their expectation, the stability of the fundamental price is hardly maintained and the market price will be driven away from its fundamental value. In addition, if the up and low limit (k) of the risky asset price is large, the adjustment parameter (μ) of the price increases and moreover, $\bar{a}$ and $\bar{\alpha}$ become larger so that $(\bar{a}, \bar{\alpha})$ leave the stability region of the fundamental steady state and (P_2^*, U_2^*) becomes unstable. Hence, an increase in price up and low limit increases the possibility of the market price deviation from its fundamental value. Thus, by controlling this limit, the market can be stabilised. This observation supports the policy of limiting daily price change by up to 10% implemented in the Chinese stock markets since 1996.

In addition, as an adjusting coefficient, the fundamental price also affects the stability of the fundamental steady state. That implies that asset with different fundamental prices can have very different dynamical behaviour, which, to some extent, explains the diversity of market prices.

The parameter β measures the switching intensity between investment strategies by investors. The bigger β is, the more the investors prefer to the strategy with better performance. When $\beta = 0$, the fundamentalists and chartists hold 50% of the market fraction respectively, namely $n_f = n_c \equiv \frac{1}{2}$. This means that, when there is no switching among different strategies, there is no evolution of the fractions between the fundamentalists and chartists. When $\beta > 0$, the market fractions of the fundamentalists and chartists at the fundamental steady state are respectively given by

$$n_f^* = \frac{1}{e^{\frac{\beta C}{1-\eta}} + 1}, \quad n_c^* = \frac{e^{\frac{\beta C}{1-\eta}}}{e^{\frac{\beta C}{1-\eta}} + 1}.$$

Thus, whenever the fundamental price is stable, the chartists always hold a positive market fraction and will not be driven out of the market. When it costs more for the fundamentalists than for the chartists, that is $C > 0$, the chartists will hold more fraction than the fundamentalists, that is $n_c^* > 0.5 > n_f^* > 0$, when the market is stable. In particular, if the relative cost between the fundamentalists and chartists becomes higher so that $C \to \infty$, then the market is totally controlled by the chartists, that is $n_c^* \to 1$. The fundamentalists and chartists have the same proportions $n_c^* = n_f^* = 0.5$ only when they pay the same cost $C = 0$. At the steady state, both forecasting rules yield exactly the same performance and hence the traders prefer a cheaper strategy. We call this phenomenon *the cost effect*. Note that the cost effect is comparative in terms of $C = C_f - C_c$. If the costs of all investors increase but the comparative cost stays the same, then the dynamics of (6) does not change.

In summary, the behaviour of the fundamentalists and chartists plays an important role in market evolution. When the activities of the fundamentalists and chartists are balanced each other so that $(\bar{\alpha}, \bar{a}, \omega) \in D$, the fundamental steady state is stable. Otherwise, the market price of the risky asset deviates from its fundamental value. Proposition 1 plots a quantitative picture for this phenomenon.

The stability of financial markets might be desirable, especially in the current global financial crisis; however, the instability of the market is the most of the market circumstances. Thus, it is more important to study the market instability and complex phenomena resulted in. In order to provide some insight into the impact of the investors' behaviour on the risky asset price, in particular when the market becomes unstable, we focus on the role of the extrapolation rates of the fundamentalists and chartists in the following discussion, in particular, the effect of the extrapolation rate (a) and B-coefficient (b) of the chartists while the effect of the fundamentalists is referred to Zheng (2007).

By assuming that the chartists hold a percentage retracement strategy with a constant weight $\omega \in (0, 1)$, we now study the effect of the adjusted extrapolation rate ($\bar{a}$)

and the asymmetric belief coefficient (b) of the chartists on the market. The stability of the fundamental steady state depends on the location of the eigenvalues of the characteristic polynomial of the steady state. When all the eigenvalues are located inside the unit circle, the fundamental steady state is locally asymptotically stable; otherwise, it becomes unstable. The change from the stability to instability of the fundamental steady state is determined by the movement of the eigenvalues from the inside to outside of the unit circle. When the adjusted extrapolation rate of the fundamentalists $\bar{\alpha}$ satisfies $0 < \bar{\alpha} < \frac{2}{\omega}$ and the adjusted extrapolation rate of the chartists increases to $\bar{a}^*$, there exists a pair of complex conjugate eigenvalues on the unit circle for the corresponding characteristic equation, where

$$\bar{a}^* = \frac{\bar{\alpha}\omega - 1 + \sqrt{(\bar{\alpha}\omega - 1)^2 + 4(\omega + \omega^2)}}{2(\omega + \omega^2)}.$$

Therefore, at $\bar{a} = \bar{a}^*$, the stability of the fundamental steady state is destroyed, leading to periodic fluctuations or more complex behaviour in (6). This phenomenon is described by different types of bifurcations in the framework of nonlinear dynamical systems. In our model, the instability of the fundamental steady state can be induced by either Neimark-Sacker or Chenciner bifurcations. A Neimark-Sacker bifurcation occurs when changing parameter of the system leads to a change in the stability of the fixed point, resulting in a (quasi-)periodic cycle around the fixed point. In this case the market price will fluctuate (quasi-)periodically around its fundamental value. A Chenciner bifurcation occurs in the following situation. Near the bifurcation values of the parameters, a locally stable fixed point is companied by a locally stable (quasi-)periodic cycle around it, and these two attractors are separated by an unstable (quasi-)periodic cycle. As the parameters change, the unstable (quasi-)periodic cycle merges together with the fixed point so that the fixed point becomes unstable and there remains a locally stable (quasi-)periodic cycle. The appearance of a Neimark-Sacker or Chenciner bifurcation depends on the first and second Lyapunov coefficients, see Kuznetsov (2004), which are denoted by $\ell_1(\bar{a}, b)$ and $\ell_2(\bar{a}, b)$, respectively. More precisely, we have the following result.

Proposition 2 (Generalised Neimark-Sacker Bifurcation) *For fixed $\omega \in (0, 1)$, we choose $\bar{\alpha} \in (0, \frac{2}{\omega})$ and assume $(1 + \omega)\bar{a}^* + 1 - \bar{\alpha} \neq 2\cos(\frac{2\pi}{q})$, where $q = 1, \ldots, 6$.*

(1) If for any b, $\ell_1(\bar{a}^*, b) \neq 0$, then the fundamental steady state undergoes a Neimark-Sacker bifurcation at $\bar{a} = \bar{a}^*$.

(2) If there exists b^* such that at $b = b^*$, $\ell_1(\bar{a}^*, b^*) = 0$, $\frac{\partial \ell_1(\bar{a}, b)}{\partial b}\Big|_{\bar{a}=\bar{a}^*, b=b^*} \neq 0$ and $\ell_2(\bar{a}^*, b^*) \neq 0$, then the fundamental steady state undergoes a Chenciner bifurcation at $(\bar{a}, b) = (\bar{a}^*, b^*)$.

Proposition 2 depicts the characteristics of the market price when the fundamental steady state loses its stability through different bifurcations. In order to understand the dynamical characteristics when the fundamental steady state loses its stability, it is necessary to calculate the first and second Lyaponov coefficients of (6) at the fundamental steady state. By the theorem of central manifold and the normal form theory of generalised Neimark-Sacker bifurcation, see Kuznetsov (2004), the first and second Lyapunov coefficients can be calculated by using the Maple program. For convenience in calculation, we select $\omega = 0.5$, $F^* = 100$, $k = 0.1$, $h = 0.1$, $\alpha = 1.2$, $\beta = 1$, $\eta = 0.2$, $R = 1.001$, $C = 0.1$ and choose a as the bifurcation parameter, then the Neimark-Sacker bifurcation point occurs at $a^* \approx 1.4514$ which is independent of the choice of b and c. For fixed $c = 1.2$, the type of bifurcations is determined by the first Lyapunov coefficient satisfying a quadratic function of $b \in (-2c, 2c)$

$$\ell_1 = 0.1898b^2 - 0.0019b - 0.4683. \tag{7}$$

When $b = b_1^* \approx -1.5660$ or $b = b_2^* \approx 1.5759$, $\ell_1 = 0$, $\frac{\partial \ell_1}{\partial b}\big|_{b=b_{1,2}^*} \neq 0$ and $\ell_2(b_1^*) \approx -0.2736$, $\ell_2(b_2^*) \approx -0.2721$. Therefore, when $a = a^*$ and $b = b_{1,2}^*$, the fundamental steady state undergoes a Chenciner bifurcation. That is,

(i) when $b_1^* < b < b_2^*$, if $a < a^*$, the fundamental steady state is stable; at $a = a^*$, the fundamental steady state undergoes a supercritical Neimark-Sacker bifurcation with $\ell_1 < 0$; however if $a > a^*$, the fundamental steady state is unstable and a stable invariant circle appears around the fundamental steady state;

(ii) when $b < b_1^*$ or $b > b_2^*$, if $a < a^*$, the fundamental steady state is stable; if a increases to a^*, there exist a stable fundamental steady state and a stable invariant circle which are separated by a unstable invariant circle. At $a = a^*$, the fundamental steady state undergoes a subcritical Neimark-Sacker bifurcation with $\ell_1 > 0$. For $a > a^*$, the instable invariant circle submerges into the fundamental steady state, the fundamental steady state loses its stability, and there exists a stable invariant circle which is far away from the fundamental steady state.

Equation (7) clearly indicates the impact of the asymmetric beliefs of the chartists on the market price. The change of B-coefficient b affects the first Lyapunov coefficient and the dynamics of the whole system (6). In particular, when the beliefs are symmetric ($b = 0$), we have $\ell_1 < 0$, meaning that the resulting Neimark-Sacker is supercritical. In other words, when the extrapolation rate of the chartists increases, the price of the risky asset fluctuates around its fundamental value and is characterised by a stable invariant circle. However, when the chartists have asymmetric beliefs (so that $b \neq 0$), changes of the intensity of the asymmetric beliefs of the chartists, measured by the B-coefficient, lead to changing signal of the first Lyapunov coefficient so that ℓ_1 becomes positive. In this case, the Neimark-Sacker bifurcation becomes subcritical, meaning that there exist two attractors, one is the stable fundamental steady state and one is a stable invariant circle. Under this circumstance, the price evolution of the system (6) depends on the initial state. When the initial price is close to its fundamental value, the risky asset price converges to the fundamental value asymptotically; however, when the initial price is far away from its fundamental value, the risky asset price fluctuates wildly around the fundamental value. This indicates that the asymmetric beliefs of the chartists is one of the important sources for the complexity of market price.

Insert Figure 5 here

Figures 5 and 6 illustrate that when the extrapolation rate of the chartists to the price trend increases, different types of Neimark-Sacker bifurcations occur. In particularly, Figure 5 displays the bifurcation plot and phase plot when the beliefs of the chartists are symmetrical, $b = 0$, to the up and down price signals. We demonstrate that when the extrapolation rate of the chartists a increases to $a^* \approx 1.4514$, the fundamental steady state loses its stability and a stable invariant circle around the fundamental steady state appears and a supercritical Neimark-Sacker bifurcation occurs with the corresponding first Lyapunov coefficient of $\ell_1 = -0.4683 < 0$. Consequently, the market price oscillates in the neighborhood of the fundamental price.

Insert Figure 6 here

However, when the chartists' beliefs are bullish, for instance $b = -1.6$, at the bifurcation point $a = a^*$, the first Lyapunov coefficient of the fundamental steady state is positive, $\ell_1 = 0.0205 > 0$. This implies that, when the extrapolation rate of the chartists a crosses the Neimark-Sacker bifurcation point a^*, the fundamental steady state undergoes a subcritical Neimark-Sacker bifurcation, leading to the coexistence of two attractors. More precisely, when the activity of the chartists is very weak so that $a = 1.42 < a^*$, the fundamental steady state is stable, see Figure 6(a). When the activity of the chartists is relatively weak so that $a = 1.451 \lessapprox a^*$, then there exist a stable fundamental steady state and a stable invariant circle which are separated by an unstable invariant circle, as shown in Figure 6(c). From Figure 6(e), we can see that if the initial price is far away from its fundamental value, the risky asset price converges to the stable invariant circle and generates high fluctuations; otherwise, the price converges to its fundamental value. To sum up, the price behaviour of the risky asset depends on the initial states of the system.

There is a strikingly simple economic intuition on the coexistence of a stable fundamental steady state and a stable invariant circle in our simple evolutionary model when the chartists have strong bullish beliefs on the market but the extrapolation rate of the chartists is relatively weakly. When the extrapolation rate of the chartists is weak, the fundamental steady state will be locally stable, since the extrapolation from the chartists for prices near the fundamental steady state is not strongly enough for prices to diverge and the price will therefore converge to the fundamental steady state. However, when the chartists' extrapolate rate is weak, an upward price trend far away from the fundamental price will be reenforced, making the price to deviate even further away from the fundamental price due to the strong bullish beliefs of the chartists and a large long position they have. Hence, such a diverging upward price trend is accelerated by the demand of the chartists until the deviation reaches their threshold confidence level at which the chartists become cautious. Once the chartists become less confident, they reduce their long position. At the same time, since the price is also far away from the fundamental, hence the fundamentalists buy the risky asset at a lower price and sell it at a higher price. Therefore, the positions held by the fundamentalists and the chartists are opposite, which slows down the upward price trend and eventually the price trend is reversed and moves towards the fundamental price. This results in a situation that the fundamentalists and chartists both are taking the short position, pushing the price down so that the price will cross the fundamental price and drop further. However, because of the strong bullish beliefs of the chartists, the downward price trend is slowed down by relative small short position taken by the chartists. This, together with the pressure of the long position taken by the fundamentalists, will reverse the price trend and push the price upwards. By taking a long position along the price trend, the performance of the chartists becomes better. The adaptive switching mechanism then makes the market fraction of the chartists increase, resulting in the price overshooting the fundamental price. This completes a full cycle around the locally stable fundamental steady state, as illustrated in Figure 7(a). In the whole process, the asymmetric beliefs of the chartists and their extrapolation to the market signal play very important roles.

Insert Figure 7 here

If the extrapolation rate of the chartists increases beyond the bifurcation point a^* so that $a = 1.46 > a^*$, then the fundamental steady state loses its stability, resulting in the disappearance of the unstable invariant circle which is then submerged into the fundamental steady state, together with the stable invariant circle, as shown in Figure 6(g). In this case, even when the initial price is close to the fundamental value, the market price will move away from the fundamental price and fluctuate along the stable invariant circle. Hence, the overreaction of the chartists increases the fluctuation amplitude of the market price. In addition, when a Neimark-Sacker or Chenciner bifurcation happens with the change of some parameters, the size of the invariant circle generated from the bifurcation will be enlarged with the increase of the parameters. Hence, as a policy or regulation issues, if the bifurcation parameters, such as the reaction intensity of the chartists to the price deviation, decrease when the price is in the uptrend along a larger invariant circle, the trajectory of the price will move from the larger invariant circle to a smaller one. Although the uptrend may not be changed immediately, the limitation of the maximum price deviation from the fundamental steady state becomes smaller and hence the market risk is reduced.

Similarly, when the chartists have the bearish beliefs so that $b = 1.6$, a subcritical Neimark-Sacker bifurcation occurs at $a = a^*$ with the corresponding Lyapunov coefficient of $\ell_1 = 0.0145 > 0$, as illustrated in Figure 6(right panel) and Figure 7(b). The difference from the bullish belief case is that the price is below its fundamental value most of time, see Figure 6(f).

In this section, the complex phenomena induced by the asymmetric beliefs of the chartists to market signals is examined when the fundamental steady state is unstable. The application of the bifurcation theory uncovers the intrinsic characteristics of the complex phenomena. The theoretic analysis in this section provides a guide to the following numerical simulations of the stochastic model.

4 Stochastic Simulations

In Section 3, under the assumption of the constant fundamental steady state, we analyse the complex dynamical behaviour of the system (6) with the change of parameters. However, in financial markets, the fundamental price usually is not constant but random, which is affected by international financial markets. In addition, some uncertain factors, such as the arrival of new events or new policies, can also influence the market. Therefore, the randomness is an important element of financial markets. In this section we explore the effect of random perturbations on the model by using stochastic simulations.

Consider the system

$$\begin{cases} P_{t+1} &= P_t\Big[1+\mathcal{S}\big(\frac{e^{\beta U_t}}{e^{\beta U_t+1}}\alpha(F_t-P_t)+\frac{1}{e^{\beta U_t+1}}g_t(P_t-(1-\omega)P_{t-1}-\omega P_{t-2})\big)\Big](1+m_t) \\ U_{t+1} &= (P_{t+1}+y_{t+1}-RP_t)\big[\alpha(F_t-P_t)-g_t(P_t-(1-\omega)P_{t-1}-\omega P_{t-2})\big]-C+\eta U_t, \end{cases}$$

where $\{F_t\}$ is a random fundamental price process, $\{y_t\}$ is a random dividend process, $\{m_t\}$ represents a market noise process, and $g_t(x) = ax/(1+b_tx+c^2x^2)$. Here $b_t \in (-2c, 2c)$ is a time-varying parameter to capture the change of the beliefs of the chartists. Based on the analysis in Section 3, we know that the system (6) displays different dynamical characteristics with the change of b. Especially, when $b = 0$, the reaction of the chartists to the signal of the market price is symmetric and, when the fundamental steady state is unstable, a small invariant circle around the fundamental steady state appears. However when $b \neq 0$, the chartists have asymmetric beliefs about the market price. When $b > 0$, the chartists have the bearish beliefs so they react strongly when there is downward price trend, leading to larger drop of the price. When $b < 0$, the influence of the chartists on the market is other way around. The chartists may change their beliefs from time to time. To capture this change, we consider that the chartists change their beliefs over time and assume

$$b_t = b_{t-1} * (N_t = 0) + \zeta_t * (N_t > 0), \tag{8}$$

where N_t is the signal of the belief changing of the chartists. When $N_t = 0$, the chartists keep their former beliefs so that $b_t = b_{t-1}$; when $N_t > 0$, the chartists seek to change their beliefs to ζ_t. For simplicity, we assume N_t is a Poisson process with a jump intensity

λ and ζ_t is a Markovian chain with three states $\{b_-, 0, b_+\}$ where $b_-(< 0)$ and $b_+(> 0)$ correspond, respectively, to bullish and bearish beliefs. The question is then how the chartists change their beliefs. In general, the state transformation of ζ_t can be determined by an endogenous variable such as utility functions. As the first step towards the future research, we assume that the transformation probability of ζ_t is given by a constant matrix P.

In addition, assume that the logarithm of the fundamental price is given by a random walk process, satisfying $\log F_{t+1} = \log F_t + \frac{\sigma_F}{\sqrt{K}}\varepsilon_t$, $F_0 = F^*$, where $\sigma_F \geq 0$ measures the volatility of the fundamental return in K days, $\{\varepsilon_t\}$ is independent and identically-distributed, $\varepsilon_t \sim \mathcal{N}(0, 1)$. Hence, $E(\log F_t) = \log F^*$. Note that this special structure of the fundamental price implies that the change of the logarithmic fundamental price is stationary. For the market noise $\{m_t\}$, we assume $m_t = e^{\sigma_m \xi_t} - 1$, where $\sigma_m \geq 0$, $\{\xi_t\}$ is independent and identically-distributed, $\xi_t \sim \mathcal{N}(0, 1)$ and $\{\xi_t\}$ is independent with $\{\varepsilon_t\}$. Then $E\big(\log(1 + m_t)\big) = 0$. Define the changes of logarithmic prices as returns which is then expressed by the market noise as

$$r_{t+1} = \log \frac{P_{t+1}}{P_t} = r_t^* + \sigma_m \xi_t,$$

where $r_t^* = \log\Big[1 + \mathcal{S}\big(\frac{e^{\beta U_t}}{e^{\beta U_t}+1}\alpha(F_t - P_t) + \frac{1}{e^{\beta U_t}+1}g_t(P_t - (1-\omega)P_{t-1} - \omega P_{t-2})\big)\Big]$.

Similarly to the deterministic model, we select $\omega = 0.5$, $F^* = 100$, $k = 0.1$, $h = 0.1$, $\alpha = 1.2$, $a = 1.46$, $c = 1.2$, $\beta = 1$, $\eta = 0.2$, $R = 1.001$, $C = 0.1$. Let the annual volatility of the fundamental return be $\sigma_F = 0.25$ and $K = 250$ be the number of trading days per year. The daily volatility of the market noise is fixed at $\sigma_M = 0.008$. The intensity of the Poisson process is $\lambda = 0.05$. The three states of the Markovian process ζ_t are given by $\{b_-, 0, b_+\} = \{-1.5, 0, 1.5\}$ and the corresponding transformation probability matrix is defined by

$$\begin{array}{cc} & \begin{array}{ccc} b_- & \quad 0 \quad & b_+ \end{array} \\ \mathbf{P} = & \begin{pmatrix} 0.85 & 0.05 & 0.1 \\ 0.25 & 0.5 & 0.25 \\ 0.1 & 0.05 & 0.85 \end{pmatrix} \begin{array}{c} b_- \\ 0 \\ b_+ \end{array} \end{array}.$$

Insert Figure 8 here

Under the random perturbations, the stochastic model shows the deviation of the market price from its fundamental value, see Figure 8(a), and volatility clustering illustrated in Figure 8(c). Also there is no significant autocorrelation among the returns of the risky asset price, see Figure 8(f), but the absolute returns and squared returns have significantly positive autocorrelations which decrease with the increase of time lags, see

Figures 8(g)(h). In addition, the density of the return has high peak and fat tail with kurtosis of 5.0394, see Figures 8(d)(e).

We now provide some intuitions on how the model is able to generate the deviation from the fundamental value and volatility clustering. In fact, based on the dynamical analysis of the corresponding deterministic system (6), we see that the bifurcation point of (6) depends on the value of the fundamental price F^*. Hence, when the logarithmic fundamental price follows a random walk process, the fundamental steady state switches between stability and instability. Furthermore, the daily return of the risky asset oscillates irregularly between two different levels underlined by the two attractors of the deterministic model. This makes the absolute and squared returns significantly positive autocorrelative. But because of the existence of the market noise, the daily returns per se is unforeseen. Thus, the volatility clustering observed in the real market is replicated. Also, since the chartists can hold the asymmetric, either bullish or bearish, beliefs, the market price can deviate from its fundamental level upwards or downwards. As indicated in the analysis of the corresponding deterministic system, the bullish beliefs can force the risky asset price upward away from its fundamental value; in contrast, the bearish beliefs can push the price down away from its fundamental value. Figure 9(a) shows the deviation of the market price from its fundamental value and Figure 9(b) plots the corresponding time series of B-coefficient.

Insert Figure 9 here

Note that in our model, investors adjust their investment strategy based on the performance of the different trading strategies in the previous one period. Figure 10 illustrates the price trajectory and the time series of the performance difference and the market fractions of the fundamentalists and chartists. From Figure 10, we can see that if one strategy, either the fundamental analysis or the technical analysis, performs better, then investors will switch to the better performed strategy. For example, at time around the 2330th or 2367th time point, the fundamental analysis makes a better performance and hence the market fraction of the fundamentalists increases while the population of the chartists drops dramatically. However, at around the 2360th time point, the market appears bullish so the technical analysis is more effective and the chartists perform better, leading the agents to switch to the chartists. This phenomenon is called herd behaviour, which strengthens the fluctuation of the market. The herd behaviour in the Chinese markets has been studied by some authors, for example, Guo and Wu (2005) prove the existence of the herd behaviour in the Chinese stock markets by mixed asset pricing model. In addition, Figure 10 also indicates that the market is dominated by the fundamentalists most of time. Given the advantage of the fundamentalists knowing the fundamental value, the fundamentalists perform better when there are no obvious fluctuations in the market such as at the 2367th time point. Nevertheless, the chartists perform better than the fundamentalists only when there are persistent up- or downtrends in the market, like around the 2360th time point. Moreover, if we consider the accumulative performance measure defined by letting $\eta = 1$ in (3), then the fundamentalists accumulate much more profits than the chartists, as shown in Figure 11, while the accumulated profits of the chartists are associated with some fluctuations and uncertainty. This demonstrates the different roles of the fundamental analysis in stablising the market and the technical analysis in activising the market.

Insert Figure 10 here

Insert Figure 11 here

5 Conclusion

Different from the perfectly rational expectation and representative agent hypothesis in the traditional financial theory, this paper uses heterogeneous beliefs and bounded rationality as basic building blocks to construct a model of asset pricing with two types of traders, fundamentalists and chartists. By analysing the dynamical characteristics and stochastic simulations, we study the impact of the investors' behaviour on the volatility of a risky asset price and provide a new approach to study the volatility of the market price by using the theory of nonlinear dynamical systems. Different from the literature, we induce the asymmetric beliefs of the chartists to the market signal to describe their bullish or bearish views, which lead to the coexistence of two attractors, the stable fundamental steady state and a stable invariant circle around the fundamental steady state. The stochastic simulations based on the theoretic analysis is able to generate the stylised facts observed in financial markets. This paper, on the one hand, describes the intrinsic characteristics of market volatility determined by the bounded rationality of investors. On the other hand, it shows that it is important to control the intensity of the asymmetric beliefs of investors in order to reduce the large fluctuation of the price. Therefore, a consideration of strengthening the understanding of investors to the fundamentals, regulating the speculative behaviour like momentum strategy, and limiting the intensity of the asymmetric reaction of investors, is very useful to reduce the deviation amplitude of the market price from its fundamental value and the possibility of large fluctuations. By the stability analysis, we find that price limitation mechanism is very useful to control the fluctuation amplitude of the market price, which supports the policy of setting price limits to stabilise the market. In addition, encouraging institution investors and reducing the cost of fundamental analysis are useful to increase the market fraction of the fundamentalists, to improve the rationality of investment and to avoid large market fluctuation. Based on our analysis, it is very helpful to avoid the large price fluctuation by reducing the speed of changing strategies among investors, regulating trading rules, introducing policies like $T + 1$ trading rule and encouraging long-term investments for the benefit of taxes.

Appendix

Proof of Proposition 1: (1)A steady state should satisfy

$$\left\{\begin{array}{lcl} P &=& P\Big[1+\mathcal{S}\big(\frac{e^{\beta U}}{e^{\beta U}+1}\alpha(F^*-P)+\frac{1}{e^{\beta U}+1}g(0)\big)\Big] \\ U &=& (P+\bar{y}-RP)\big[\alpha(F^*-P)-g(0)\big]-C+\eta U. \end{array}\right. \tag{A1}$$

It is obvious that $(P_1^*, U_1^*) = (0, \frac{\alpha F^* \bar{y}-C}{1-\eta})$ is a solution of (A1). However, if $P \neq 0$, then$\frac{e^{\beta U}}{e^{\beta U}+1}\alpha(F^*-P)=0$. Furthermore, $P=F^*$ and $U=U^*(=\frac{-C}{1-\eta})$ are regarded as the second solution of (A1), denoted as (P_2^*, U_2^*).

(2) The characteristic polynomial of the steady state $(P_1^*, U_1^*) = (0, \frac{\alpha F^*\bar{y}-C}{1-\eta})$ is $\Gamma_1(\lambda)=\lambda^2(\lambda-\eta)(\lambda-1-\mathcal{S}(\frac{e^{\beta U_1^*}}{e^{\beta U_1^*}+1}\alpha F^*))$. By $\frac{e^{\beta U_1^*}}{e^{\beta U_1^*}+1}\alpha F^*>0$, $1+\mathcal{S}(\frac{e^{\beta U_1^*}}{e^{\beta U_1^*}+1}\alpha F^*)>1$. Thus, for all parameters, (P_1^*, U_1^*) is unstable.

(3) The characteristic polynomial of the steady state $(P_2^*, U_2^*) = (F^*, U^*)$ is $\Gamma_2(\lambda) = (\lambda-\eta)(\lambda^3+c_1\lambda^2+c_2\lambda+c_3)$, where $c_1=-1+\bar{\alpha}-\bar{a}$, $c_2=\bar{a}(1-\omega)$, $c_3=\bar{a}\omega$. Hence

$$\begin{aligned} \pi_1 &= 1+c_1+c_2+c_3=\bar{\alpha}, \\ \pi_2 &= 1-c_1+c_2-c_3=2-\bar{\alpha}+\bar{a}(2-2\omega), \\ \pi_3 &= 1-c_2+c_1c_3-c_3^2=1-\bar{a}+\bar{\alpha}\bar{a}\omega-\bar{a}^2(\omega+\omega^2). \end{aligned}$$

By Jury Test, (F^*, U^*) is locally asymptotically stable if and only if $\pi_i>0$ $(i=1,2,3)$ and $\bar{a}\omega<1$. Therefore, for $\omega\neq 0$ and $\omega=0$, the local stable region of (F^*, U^*) can be expressed respectively by

$$D(\bar{a},\bar{\alpha}) := \left\{(\bar{a},\bar{\alpha}) : 0<\bar{a}<\frac{1}{\omega},\ \max\{0,\ \frac{(\omega\bar{a}+1)((1+\omega)\bar{a}-1)}{\bar{a}\omega}\}<\bar{\alpha}<2+2\bar{a}(1-\omega)\right\},$$

$$D_0(\bar{a},\bar{\alpha}) := \{(\bar{a},\bar{\alpha}) : 0<\bar{a}<1,\ 0<\bar{\alpha}<2+2\bar{a}\}. \qquad \blacksquare$$

Proof of Proposition 2: By Jury Test and Theorem of Generalised Neimark-Sacker Bifurcation (cf. Kuznetsov, 2004), the result is obvious. ■

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Notes

[1] If the chartists also have the information about the fundamentals, especially taking the fundamental price into account of their investment strategy, then we will have a different system from (6), leading to different dynamics. We refer to Brock and Hommes (1998), Boswijk, Hommes and Manzan (2007) and Hommes (2006) for further discussion along these lines.

[2] Since there is only one type of chartists, all chartists in the market have the same beliefs about the market trend: all chartists think the market is either bullish or bearish. In general, different chartists can have different beliefs and adopt different strategies, meaning that different chartists can have different B-coefficients. For more general cases, B-coefficient can rely on the market price. We leave these to future work.

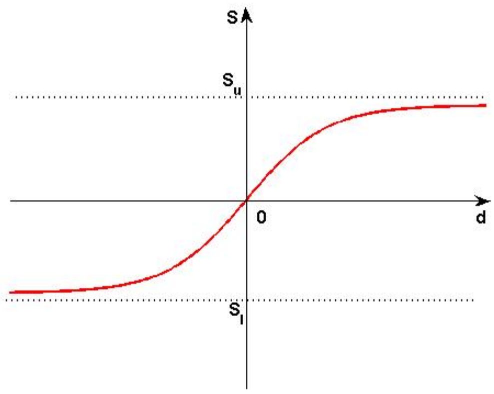
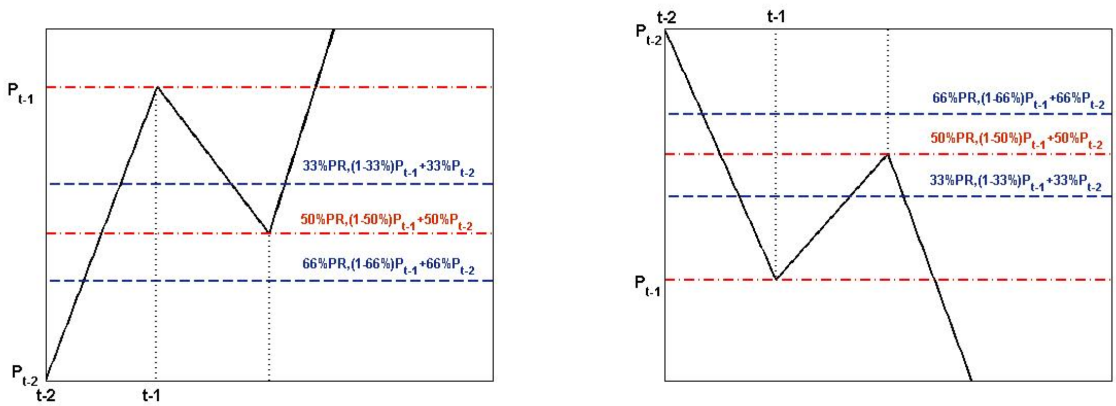


Figure 1: $\mathcal{S}$-function



(a) Down-adjustment

(b) Up-adjustment

Figure 2: Percentage Retracement (PR)

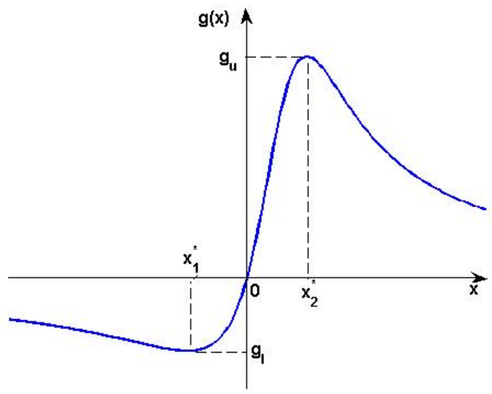
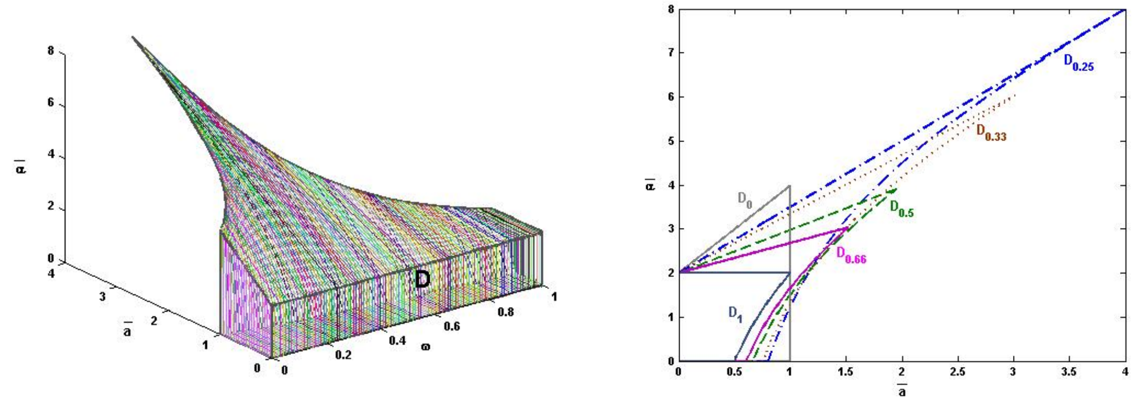
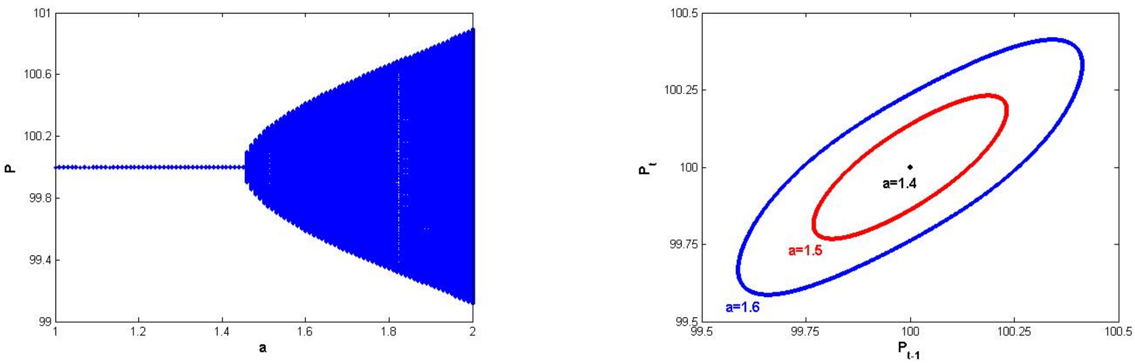


Figure 3: g-function



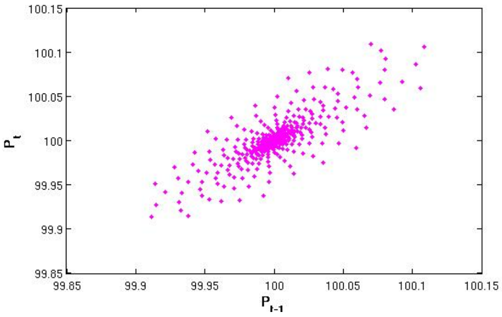
(a) (b)

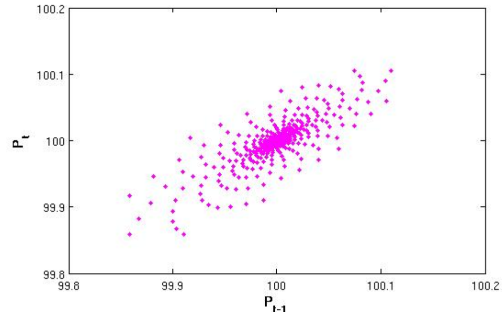
Figure 4: The stable region of the steady state (P_2^*, U_2^*) in the parameter space of $(\bar{a}, \bar{\alpha}, \omega)$ (a) and D_{ω^*} is denoted as the projection of D onto the plane $\omega = \omega^*$ (b).



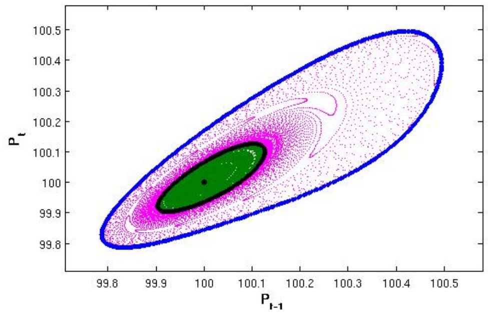
(a) bifurcation plot (b) phase plot

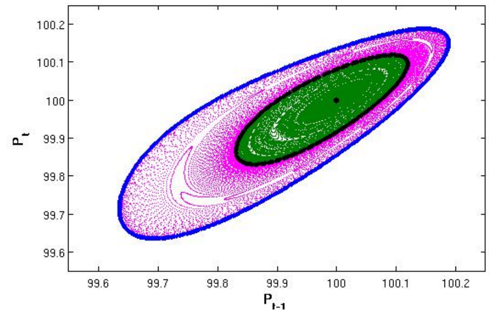
Figure 5: Fix $\omega = 0.5$, $F^* = 100$, $k = 0.1$, $h = 0.1$, $\alpha = 1.2$, $b = 0$, $c = 1.2$, $\beta = 1$, $\eta = 0.2$, $R = 1.001$, $C = 0.1$, consider the symmetric belief case, that is $b = 0$, and regard the reaction coefficient of the chartists a as the bifurcation parameter. When a increases, the fundamental steady state loses its stability at $a \approx 1.4514$ and a stable invariant circle appears whose first Lyapunov coefficient is $\ell_1 = -0.4683$.



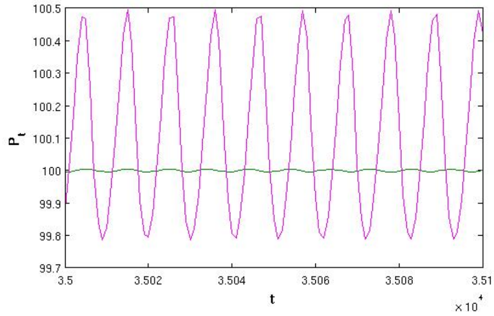


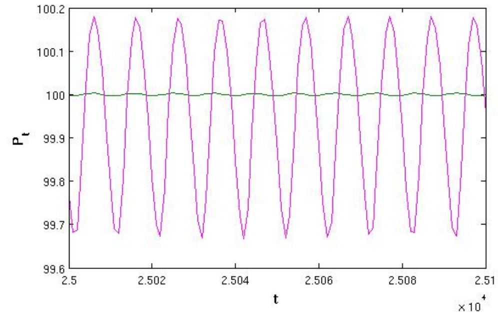
(a) $b < 0$, $a = 1.42$ (b) $b > 0$, $a = 1.42$



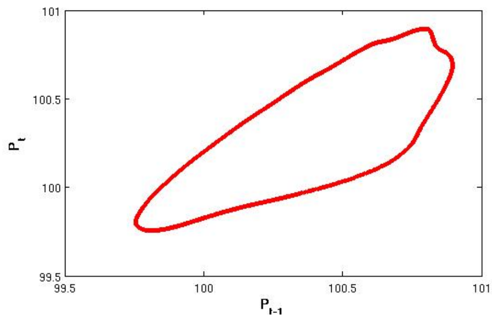


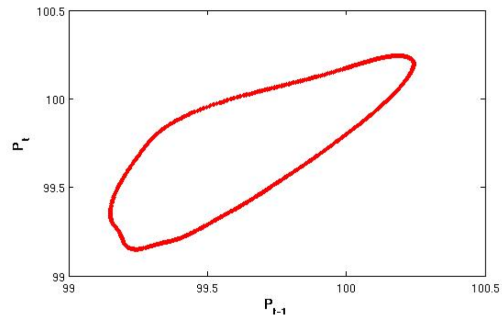
(c) $b < 0$, $a = 1.451$ (d) $b > 0$, $a = 1.451$





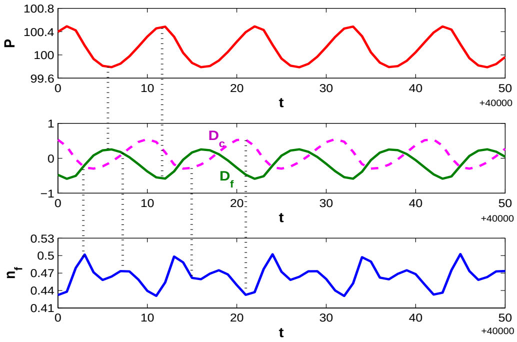
(e) $b < 0$, $a = 1.451$, Time Series (f) $b > 0$, $a = 1.451$, Time Series



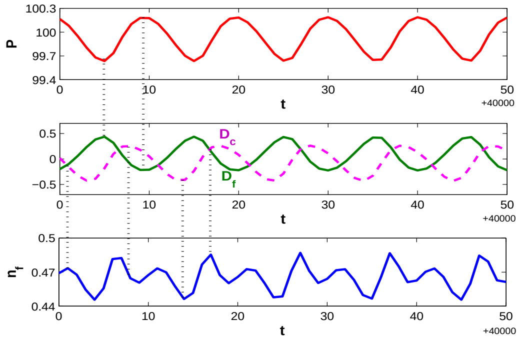


(g) $b < 0$, $a = 1.46$ (h) $b > 0$, $a = 1.46$

Figure 6: Fix $\omega = 0.5$, $F^* = 100$, $k = 0.1$, $h = 0.1$, $\alpha = 1.2$, $c = 1.2$, $\beta = 1$, $\eta = 0.2$, $R = 1.001$, $C = 0.1$ and regard the reaction coefficient of the chartists a as the bifurcation parameter. When $b = -1.6 < 0$ (left panel) and $b = 1.6 > 0$ (right panel), at $a \approx 1.4514$, the subcritical bifurcations occur and the stable fundamental steady state with a stable invariant circle coexists (c), (d).

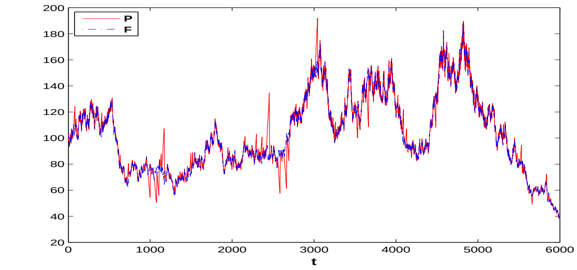


(a) $b < 0$, $a = 1.451$

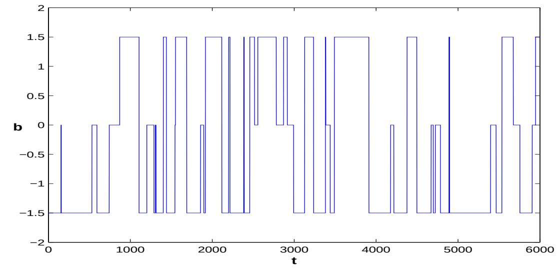


(b) $b > 0$, $a = 1.451$

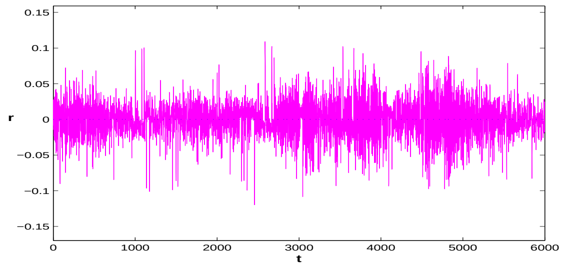
Figure 7: When the coexistence of two attractors appears, the process of the price, excess demands of the fundamentalists and chartists and market fractions corresponding to the stable invariant circle are shown.



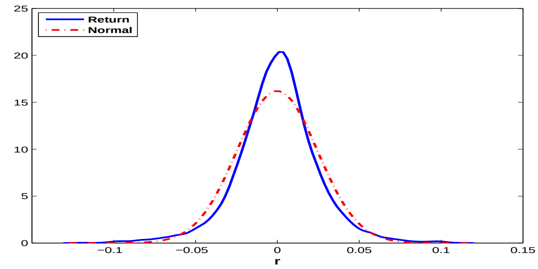
(a) Risky asset price and its fundamental price



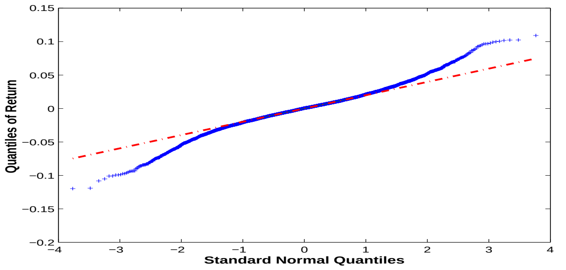
(b) B-coefficient



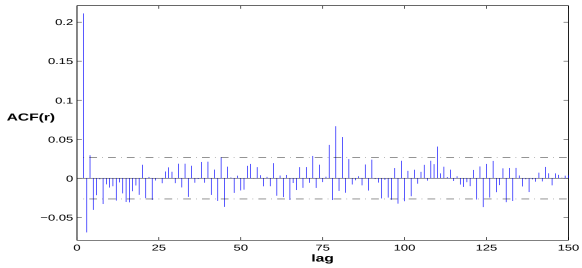
(c) Return of the risky asset price (r)



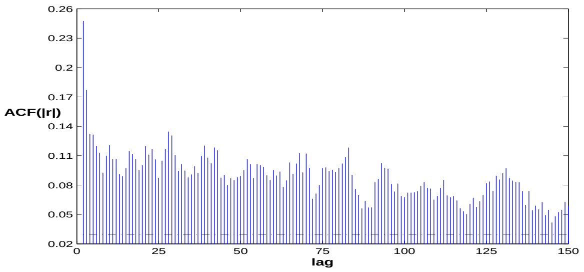
(d) Density of the return



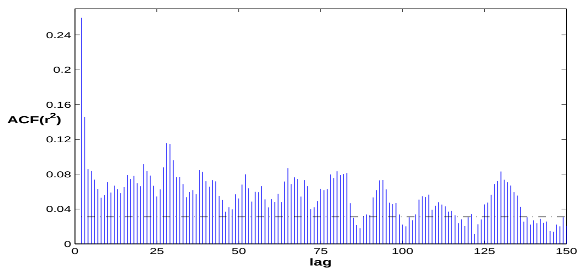
(e) QQ-plot of the return



(f) Autocorrelation of the return $ACF(r)$

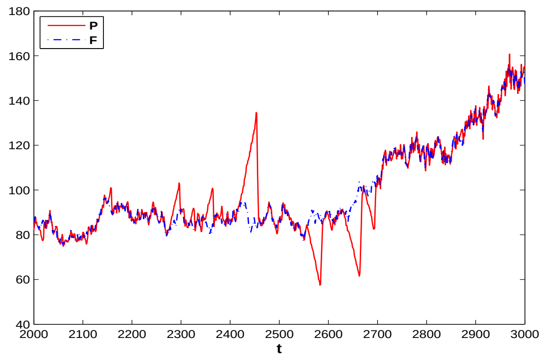


(g) Autocorrelation of the absolute return $ACF(|r|)$

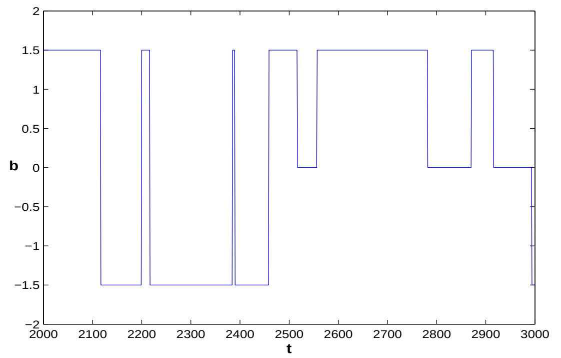


(h) Autocorrelation of the squared return $ACF(r^2)$

Figure 8: Simulation results.

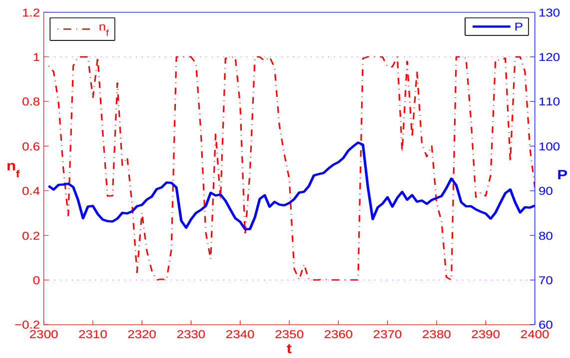


(a) Risky asset price and its fundamental price

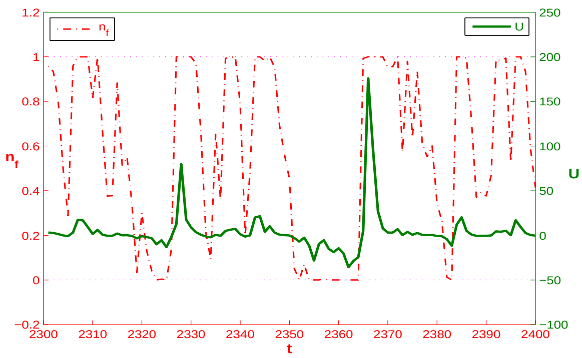


(b) B-coefficient

Figure 9: Enlargement of Figures 8(a), (b).



(a) Market fraction of the fundamentalists n_f and the risky asset price P



(b) Market fraction of the fundamentalists n_f and performance measure difference $U = U_f - U_c$

Figure 10: Herd behaviour

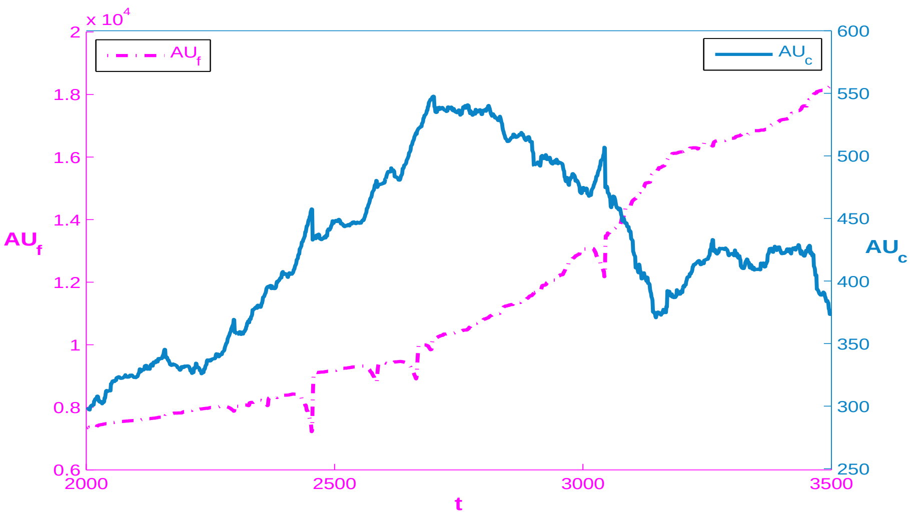


Figure 11: Accumulative performance measure of the fundamentalists AU_f and chartists AU_c, where $AU_{q,t} = AU_{q,t-1} + (P_t + y_t - RP_{t-1})D^q_{t-1} - C_q$ $(q = f, c)$.