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Suppression of Super-harmonic Resonance Response of a Forced Nonlinear System Using a Linear Vibration Absorber

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Abstract

Super-harmonic resonances may appear in a forced weakly nonlinear system of cubic nonlinearity, when the forcing frequency is approximately equal to one-third of the linearized natural frequency. In contrast with the corresponding linear oscillator, the free-oscillation term does not decay to zero despite of the presence of damping and the nonlinearity adjusts the frequency of the free-oscillation term to exactly three times the frequency of the excitation. Saddle-node bifurcations may appear in the frequency-response curve for the amplitude of the free-oscillation terms, which may lead to jump and hysteresis phenomenon.

A small linear vibration absorber is designed to suppress the super-harmonic resonance response of the forced oscillator of cubic nonlinearity. The absorber can be considered as a small mass-spring-damper oscillator in the sense that the mass and stiffness of the absorber are less than one-tenth of the values of the mass and linear stiffness of the forced nonlinear oscillator. It is shown that a small linear vibration absorber is effective in suppressing the super-harmonic resonance response of the system by transferring the vibrational energy from the main nonlinear oscillator to a small mass-spring-damper oscillator. Saddle-node bifurcations and jump phenomena can be easily eliminated by adding the small linear vibration absorber to the forced oscillator.

Key words: Nonlinear vibration, Super-harmonic resonances, Vibration suppression, Vibration absorber.

1. Introduction

In a forced single-degree-of-freedom weakly nonlinear system, nonlinear resonances may occur if the linearized natural frequency and the frequency of an external excitation satisfy a certain relationship. A small-amplitude excitation may produce a relatively large-amplitude response under primary resonance conditions, when the forcing frequency is in the neighbourhood of the linearized natural frequency. Super-harmonic resonance may also appear in the forced response of a weakly nonlinear system of cubic nonlinearity when the forcing frequency is approximately equal to one-third of the natural frequency (i.e. $\Omega \approx \omega_0 / 3$). The free-oscillation term does not decay to zero in spite of the presence of damping and in contrast with the linear case. As in the case of primary resonances, the frequency-response curve of the nonlinear system may exhibit saddle-node bifurcations, jump and hysteresis phenomena [1]. These behaviours may be unwanted in many applications because they can result in discontinuous behaviour.

Over the past decade, active control methods have been developed to suppress the nonlinear resonance vibrations of weakly nonlinear systems with parametric or external periodic excitations. These methods include time-delayed feedback control [2-6], a linear-plus-nonlinear feedback control [7,8], and a nonlinear parametric feedback control [9], and an internal resonance control technique [10-12]. The use of active controllers for vibration attenuation is not feasible in many applications, for reasons including cost, or required independent energy supply. A passive control approach is an alternative under these circumstances [13]. On the other hand, a passive system may be required as a back-up to prevent complete disaster in the event of the failure of active control methods. In controlling the vibrations of linear system, one possible method of reducing vibration levels is to add an extra system on the existing structure. The extra system, also known as vibration absorber, may be a simple mass-spring-damper system being attached at a single point of structure. The dynamic vibration absorber is designed such that the natural frequencies of the resulting system are away from the excitation frequency [14,15].

The main purpose of the present paper is to suppress the super-harmonic resonance vibrations of a weakly nonlinear system with periodic excitations by using a mass-spring-damper absorber. The absorber refers to here as a mass that is relatively light in comparison with the mass of primary system and is attached to the primary system by a linear spring and a linear damping (also called coupling). The damping coefficient and the spring stiffness are much lower than their counterpart, so that the absorber can be considered as a small attachment to the main system. The addition of an absorber to a one-degree-of-freedom weakly nonlinear system (a primary system) results in a new two degree-of-freedom weakly nonlinear system. The characteristics of the primary system attached by an absorber change only slightly in terms of the values of its linearized natural frequency, damping coefficient and frequency interval for nonlinear resonance, because the absorber is a small attachment and does not contribute significantly to the changes of these parameters.

2. Mathematical Modelling

The equations of motion for a weakly nonlinear oscillator with periodic external excitation attached by a small linear vibration absorber can be written as

$$m_1\ddot{x}_1 = -k_1x_1 - k_2x_1^3 - c_1\dot{x}_1 + k_3(x_2 - x_1) + c_2(\dot{x}_2 - \dot{x}_1) + f_0\cos(\Omega t),$$

$$m_2\ddot{x}_2 = -k_3(x_2 - x_1) - c_2(\dot{x}_2 - \dot{x}_1), \qquad (1)$$

where m_1 denotes the mass of the primary system and m_2 the mass of the small attachment. k_1, k_2 and c_1 represent the linear, nonlinear stiffness and damping coefficient in relation to mass m_1, respectively. The coupling stiffness and damping coefficient are k_3 and c_2. The displacements of the primary nonlinear system and the small attachment, as shown in Figure 1, are denoted by x_1 and x_2. An overdot indicates the differentiation with respect to time *t*.

Equation (1) can be simplified by dividing m_1 on both sides of the first equation and dividing m_2 on both sides of the second equation as

$$\ddot{x}_1 + \mu_1\dot{x}_1 + \omega_1^2x_1 - m\mu_2\dot{x}_2 - m\omega_2^2x_2 + \alpha x_1^3 = f\cos(\Omega t),$$

$$\ddot{x}_2 + \mu_2(\dot{x}_2 - \dot{x}_1) + \omega_2^2(x_2 - x_1) = 0\,, \qquad (2a,b)$$

where $\mu_1 = (c_1 + c_2)/m_1$, $\omega_1^2 = (k_1 + k_3)/m_1$, $m = m_2/m_1$, $\mu_2 = c_2/m_2$, $\omega_2^2 = k_3/m_2$, $\alpha = k_2/m_1$, $f = f_0/m_1$.

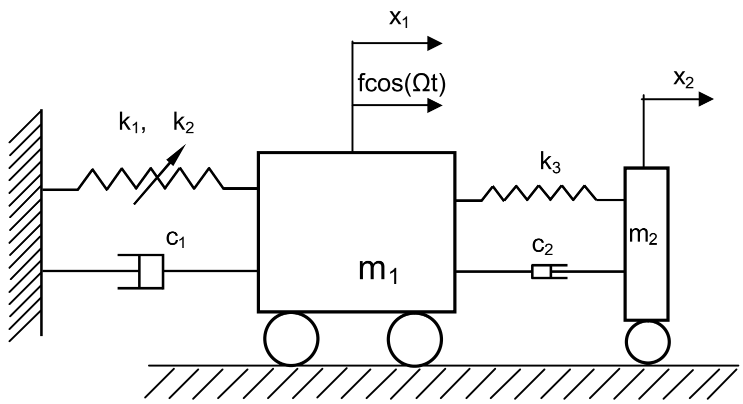

Figure 1: A main nonlinear system linearly coupled with a relatively light mass

It should be mentioned that the main purpose of the current research is to investigate the suppression of nonlinear vibrations of a forced nonlinear oscillator using a small attachment without adversely affecting the performance of the main oscillator. The small attached mass and the damping and spring stiffness of coupling can be considered as a perturbation to the primary oscillator, in a sense that the primary nonlinear system is weakly coupled with a small attachment. As a result, the stiffness and mass of the primary oscillator should be much larger than the stiffness of the linked spring and the mass of the small attachment. For the attached aborber, though its stiffness and mass are small in comparison with those of the primary system, the linear stiffness of the attachment is comparable with its mass and thus is assumed to be leading terms in equation (2b). In particular, all damping terms and nonlinear terms are assumed to be small and in the order of O(ε) in equation (2a) and the damping terms are considered to be in the order of O(ε) in equation (2b). From above discussions on the order of the coefficients, equation (2) can be rewritten as

$$\ddot{x}_1 + \varepsilon\mu_1\dot{x}_1 + \omega_1^2 x_1 - \varepsilon m\mu_2\dot{x}_2 - \varepsilon m\omega_2^2 x_2 + \varepsilon\alpha x_1^3 = f\cos(\Omega t)\,,$$

$$\ddot{x}_2 + \varepsilon\mu_2\dot{x}_2 + \omega_2^2 x_2 = \varepsilon\mu_2\dot{x}_1 + \omega_2^2 x_1 \qquad (3a,b)$$

where ε is a dimensionless parameter with $\varepsilon << 1$, the coefficients of the damping term and nonlinear term, μ_1, μ_2 and α in equation (2) have been re-scaled in terms of $\mu_1 = \varepsilon\,\overline{\mu}_1$, $\mu_2 = \varepsilon\,\overline{\mu}_2$, and $\alpha = \varepsilon\,\overline{\alpha}$, and the overbars in $\overline{\mu}_1$, $\overline{\mu}_2$ and $\overline{\alpha}$ have been removed for brevity. Equation (3) can be regarded as a weakly nonlinear main system being coupled to a linear system with an external excitation.

3. Perturbation Analysis

The method of multiple scales is employed to obtain a set of four averaged equations that determine the amplitudes and phases of the steady state solutions on a slow scale [1]. For the sake of simplicity, only the first-order approximate solutions will be sought in subsequent analysis. It is assumed that the solutions of equation (3) in the neighbourhood of the trivial equilibrium are represented by an expansion of the form

$$x_1(t;\varepsilon) = x_{10}(T_0, T_1) + \varepsilon x_{11}(T_0, T_1) + O(\varepsilon^2),$$

$$x_2(t;\varepsilon) = x_{20}(T_0, T_1) + \varepsilon x_{21}(T_0, T_1) + O(\varepsilon^2), \quad (4)$$

where ε is a non-dimensional small parameter, $T_0 = t$ is a fast scale associated with changes occurring at the frequencies ω_1 and Ω, and $T_1 = \varepsilon t$ is slow scale associated with modulations in the amplitude and phase caused by the non-linearity, damping and resonances. The derivatives of x_1 and x_2 with respect to t then become expansions in terms of partial derivatives with respect to T_0 and T_1.

Substituting the approximate solutions (4) into (3) and then balancing the like powers of ε, results in the following ordered perturbation equations:

$$\varepsilon^0 \quad D_0^2 x_{10} + \omega_1^2 x_{10} = f\cos(\Omega T_0), \quad D_0^2 x_{20} + \omega_2^2 x_{20} = \omega_2^2 x_{10}, \quad (5a,b)$$

$$\varepsilon \quad D_0^2 x_{11} + \omega_1^2 x_{11} = -2D_0 D_1 x_{10} - \mu_1 D_0 x_{10} + m\mu_2 D_0 x_{20} + m\omega_2^2 x_{20} - \alpha x_{10}^3,$$

$$D_0^2 x_{21} + \omega_2^2 x_{21} = -2D_0 D_1 x_{20} - \mu_2 D_0 x_{20} + \mu_2 D_0 x_{10} + \omega_2^2 x_{11}. \quad (6a,b)$$

The general solutions of equation (5) can be written as

$$x_{10} = A(T_1)\exp(i\omega_1 T_0) + F_0 \exp(i\Omega T_0) + cc, \quad (7a)$$

$$x_{20} = B(T_1)\exp(i\omega_2 T_0) + H_1 A(T_1)\exp(i\omega_1 T_0) + H_2 \exp(i\Omega T_0) + cc, \quad (7b)$$

where $A(T_1)$ and $B(T_1)$ are arbitrary functions of T_1 at this order of approximation, $F_0 = f/[2(\omega_1^2 - \Omega^2)]$, $H_1 = 1/[1-(\omega_1/\omega_2)^2]$, $H_2 = F_0/[1-(\Omega/\omega_2)^2]$, and cc stands for complex conjugate of the preceding terms.

Substituting the general solutions (7) into equation (6) yields

$$D_0^2 x_{11} + \omega_1^2 x_{11} = [m\omega_2^2 H_1 A - 6\alpha A F_0^2 - 3\alpha A^2 \bar{A} + i\omega_1(\mu_1 A + m\mu_2 H_1 A - 2A')$$

$$]\exp(i\omega_1 T_0) - \alpha F_0^3 \exp(3i\Omega T_0) + NST + cc, \quad (8)$$

$$D_0^2 x_{21} + \omega_2^2 x_{21} = -i\omega_2(\mu_2 B + 2B')\exp(i\omega_2 T_0) + \omega_2^2 x_{11}$$

$$+ i\omega_1(\mu_2 A - \mu_2 H_1 A - 2H_1 A')\exp(i\omega_1 T_0) + cc, \quad (9)$$

The particular solution x_{11} of equation (8) can be written as

$$x_{11} = PB\exp(i\omega_2 T_0) + \bar{P}\bar{B}\exp(-i\omega_2 T_0) + NST, \quad (10)$$

where $P = (m\omega_2^2 - im\mu_2\omega_2)/(\omega_1^2 - \omega_2^2)$, $\bar{P}$ is the conjugate of *P*, and *NST* stands for the terms that do not produce secular terms in seeking solution x_{21}.

For super-harmonic resonance response of the forced nonlinear oscillator, the nearness of Ω to $\frac{1}{3}\omega_1$ can be expressed by introducing the detuning parameter σ according to $3\Omega = \omega_1 + \varepsilon\sigma$. Eliminating the terms that lead to secular terms from equations (8) and (9) yields

$$i\omega_1(2A' + \mu_1 A - m\mu_2 H_1 A) - m\omega_2^2 H_1 A + 6\alpha F_0^2 A + 3\alpha A^2 \overline{A} + \alpha F_0^3 \exp(i\sigma_2 T_1) = 0$$

$$\omega_2^2 P - i\mu_2\omega_2 B - 2i\omega_2 B' = 0, \qquad (11)$$

where the prime indicates the differentiations with respect to the slow scale T_1.

The functions $A(T_1)$ and $B(T_1)$ can be expressed in the polar form:

$$A(T_1) = \tfrac{1}{2}a(T_1)\exp[i\beta(T_1)], \quad B(T_1) = \tfrac{1}{2}b(T_1)\exp[i\theta(T_1)], \qquad (12)$$

where $a(T_1)$, $b(T_1)$, $\beta(T_1)$ and $\theta(T_1)$ are real functions of time T_1.

Substituting equation (12) into equation (11) and then separating real and imaginary parts gives rise to

$$a' = \frac{g_1 - \mu_1}{2}a + K_1\sin(\gamma),$$

$$a\gamma' = (-\sigma + g_2 + h_{11})a + h_{21}a^3 + K_1\cos(\gamma),$$

$$b' = n_1 b, \qquad b\theta' = n_2 b, \qquad (13)$$

where $\gamma = \beta - \sigma T_1$, $g_1 = m\mu_2 H_1$, $K_1 = \alpha F_0^3/\omega_1$, $g_2 = -m\omega_2^2 H_1/(2\omega_1)$, $h_{11} = 3\alpha F_0^2/\omega_1$, $h_{21} = 3\alpha/(8\omega_1)$, $n_1 = -\mu_2/2 - \mu_2\omega_2^2/[2(\omega_1^2 - \omega_2^2)]$, $n_2 = -m\omega_2^3/[2(\omega_1^2 - \omega_2^2)]$.

Therefore for the first approximation

$$x_1 = a\cos(3\Omega + \gamma) + F_0\cos(\Omega t) + O(\varepsilon),$$

$$x_2 = aH_1\cos(3\Omega + \gamma) + H_2\cos(\Omega t) + O(\varepsilon). \qquad (14)$$

The frequency-response curve for super-harmonic resonances is determined by

$$\left[\frac{(g_1 - \mu_1)^2}{4} + \left(-\sigma + g_2 + h_{11} + h_{21}a^2\right)^2\right]a^2 = K_1^2. \qquad (15)$$

The stability of the steady-state motions depends on the eigenvalues of the coefficient matrix on the right-hand side of the perturbed averaged equations in terms of small perturbation. The three eigenvalues are found to be

$$\lambda_{1,2} = -\tfrac{1}{2}(\mu_1 - g_1) \pm \sqrt{4h_{21}^2 a^4 - (2h_{21}a^2 - \sigma + g_2 + h_{11})^2}, \quad \lambda_3 = -n_1.$$

4. Illustrative Examples

Numerical simulations have been performed under the following values of the system parameters: $m_1 = 10.0$ kg, $m_2 = 0.8$ kg, $c_1 = 0.1$ Ns/m, $c_2 = 0.08$ Ns/m,

$k_1 = 44.0$ N/m, $k_2 = 8.0$ N/m^3, $k_3 = 2.0$ N/m, unless otherwise specified. This combination of system parameters indicates that the mass ratio is $m = 0.08$ and the coupling stiffness is approximately 4.55% of the stiffness of the primary system. This set of system parameters confirms a small mass attachment to the primary system. The linearized natural frequencies of the primary system before and after being attached by the absorber are found to be approximately $\omega_{10} = 2.0976$, $\omega_1 = 2.1448$, and the natural frequency of the absorber be $\omega_2 = 1.5811$. The linearized natural frequencies of the primary system before and after the addition of absorber change slightly, only at approximately 2.19%. For this given set of system parameters, super-harmonic resonances may appear in the nonlinear oscillator (without being attached by vibration absorber) when the forcing frequency is approximately equal to 0.6992.

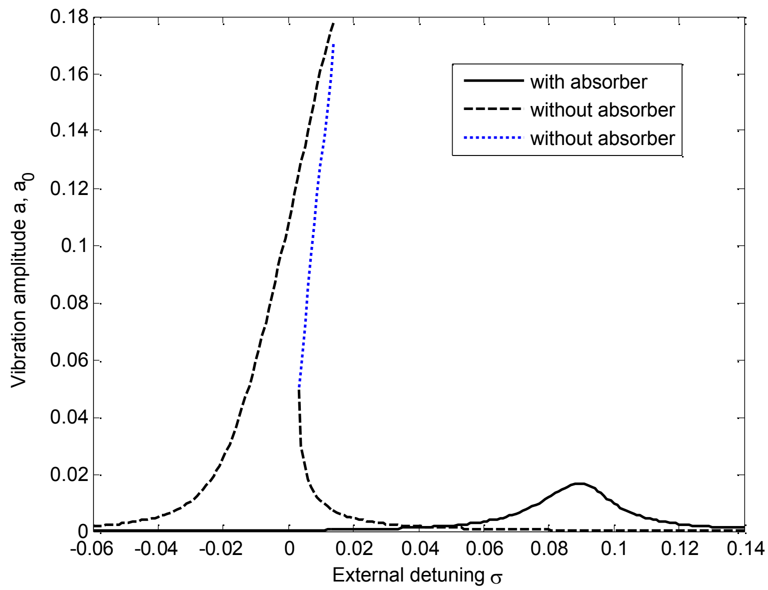

Figure 2: Frequency-response curves for Super-harmonic resonance response

The performance of vibration absorber on attenuation of super-harmonic resonance response of nonlinear oscillator can be clearly demonstrated with the help of frequency-response curves. Figure 2 shows the frequency-response curves of the primary system before and after the addition of the absorber for the amplitude of excitation $f_0 = 13.8$. Dashed lines and dotted line are used to represent the amplitudes of the stable and unstable solutions of the super-harmonic resonance response of the nonlinear oscillator alone, and solid lines are used to denote the amplitudes of stable solutions of the nonlinear oscillator attached with absorber, respectively. The horizontal axis represents an interval of external detuning $\sigma \in [-0.06, 0.14]$, which corresponds to a small interval of forcing frequency $\Omega \in [0.6792, 0.7459]$. Without adding the absorber, the peak amplitude of super-harmonic resonance response of the primary system is 0.1777 and saddle-node bifurcations occur in the frequency-response curve. In the interval $\sigma \in [0.0035, 0.014]$, two stable solutions coexist with a unstable solution in between. Jump-up phenomenon happens at $\sigma = 0.0035$ when decreasing forcing frequency from $\sigma = 0.14$, and jump-down phenomenon occurs at $\sigma = 0.014$ when increasing forcing frequency from $\sigma = -0.06$. After adding the absorber to the primary system, the peak amplitude of super-harmonic resonance response of the primary system has been greatly reduced to 0.0166. The interval of the multiple coexisting solutions disappears and the jump phenomena are eliminated. The super-harmonic resonance vibrations of the primary system have been significantly attenuated. As shown in Figure 2, the frequencies at which the amplitudes of super-harmonic resonance vibrations reach their maximum have shifted slightly from $\sigma = 0.014$ for the primary system alone to $\sigma = 0.089$ for the primary system with absorber. In terms of the frequency of excitation, the maximum amplitudes of super-harmonic resonance vibrations occur at $\Omega = 0.7039$ for the primary system without absorber and at $\Omega = 0.7289$ for the primary system with absorber.

5. Conclusion

The super-harmonic resonance response of a nonlinear oscillator can be suppressed by a linear vibration absorber which consists of a relatively light mass attached to a main nonlinear oscillator by a linear damper and a linear spring. The small attachment of the light mass can absorb vibrational energy without significantly modifying the primary system and adversely affecting its performance. The stiffness of the linked spring is much lower than the stiffness of the primary system itself. The linearized natural frequencies of the primary system before and after addition of vibration absorber change only slightly. It has been shown that a small linear vibration absorber is effective in suppressing the super-harmonic resonances of the nonlinear system. Saddle-node bifurcations and jump phenomena can be eliminated by adding a linear vibration absorber to the forced nonlinear oscillator.

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