A Cusp Model of Housing Price in Hong Kong

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Abstract

This paper presents a cusp catastrophe model for the Hong Kong housing market, to explain discontinuous change in housing prices. The paper begins by describing the significance of applying the catastrophe model to the analysis of discontinuous changes in housing prices. The study focuses on the housing price systems effects of equilibrium price level, interaction between housing supply and demand, and vacant units. Lagged supply makes housing price fall due to decreasing in demand for housing. The validity of applying a catastrophe model to housing price analysis is examined by testing yearly data from Hong Kong. The analysis result shows that the ratio of vacant and the equilibrium take-up units can be used for examining discontinuous change in housing prices.

Keywords: housing price, cusp model, discontinuous change, housing supply and demand, vacant units, cobweb theorem, stock flow model

1. Introduction

The Hong Kong residential market is unique in several aspects: restricted land supply, high price volatility, high appreciation rate, a small group of large developers, and a huge public housing sector (Lai & Wang, 1999). Housing prices in Hong Kong have experienced rapid changes over the last two decades. The highest price was in 1997, and then prices decreased rapidly (Figure 1). In particular, housing prices decreased over 50 percent, according to the statistics released by the Hong Kong Government (Hong Kong Monthly Digest of Statistics, various issues).

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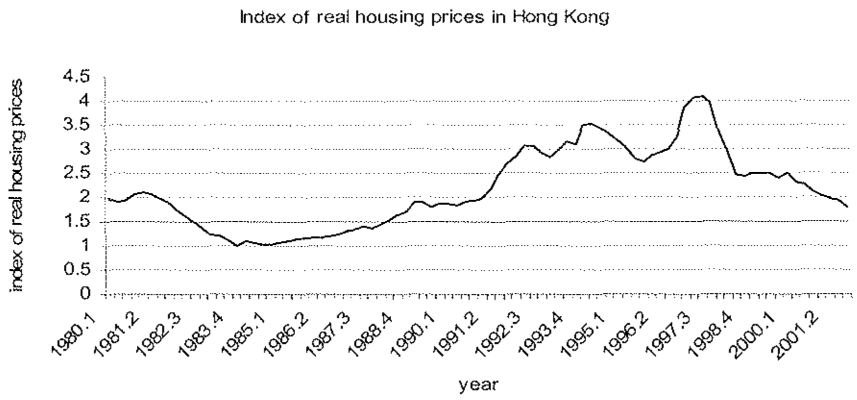

Figure 1 Index of Real Housing Price in Hong Kong
[Source: *Hong Kong Monthly Digest of Statistics*, various issues]

At the same time, new supply units of private domestic property[2] and vacant units increased and reached the highest level in 1999. Figure 2 depicts the relationship of housing supply, take-up, and vacant units over the last 20 years. Housing supply appears to fluctuate. These fluctuations may be affected by economic conditions such as interest rates in Hong Kong (Tse, 1998).

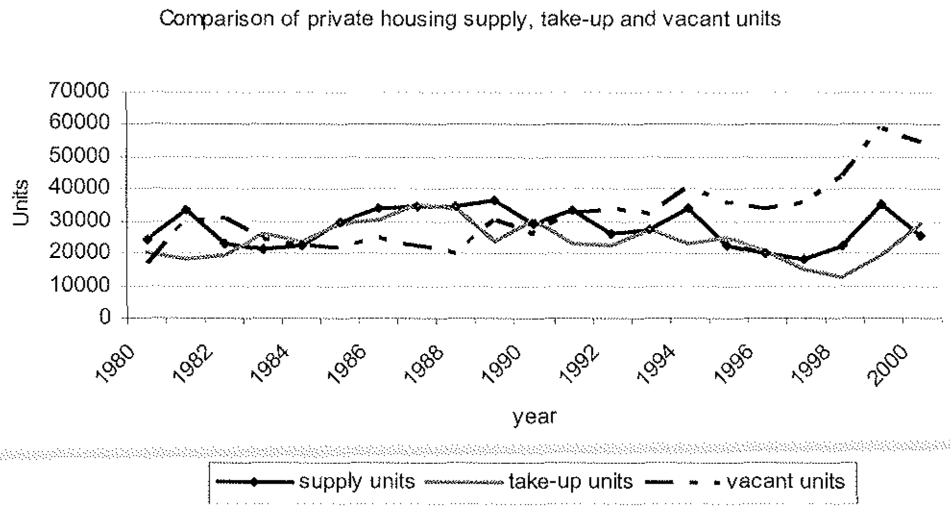

Figure 2 Relationship among Housing Supply, Take-up, and Vacant units from 1980 to 2000 in Hong Kong
[Source: *Hong Kong Property Review*, Rating and Valuation Department, Hong Kong]

[2] Private domestic units are defined as independent dwellings with separate cooking facilities and bathroom (and/or lavatory) (*Hong Kong Property Review*, Rating and Valuation Department, Hong Kong).

The number of vacant units has increased a great deal since 1992. The average vacancy rate[3] was 4.15 percent during the period 1980 to 2000, and the highest was 5.9 percent in 1999 (Figure 5.3).

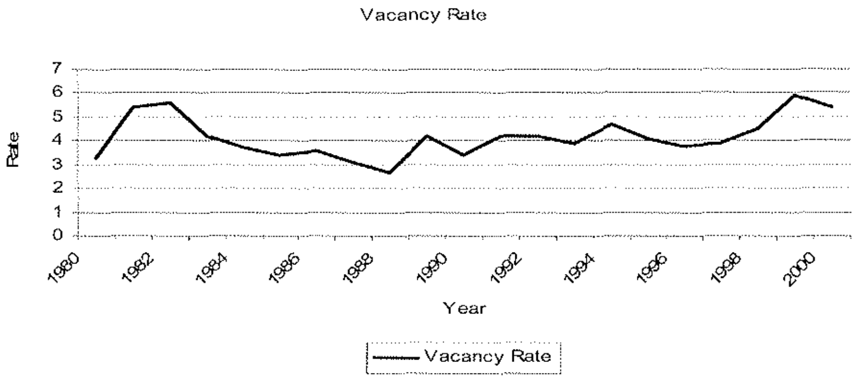

Figure 3 Housing Vacancy Rate from 1980 to 2000 in Hong Kong

[Source: *Hong Kong Property Review*, Rating and Valuation Department, Hong Kong]

Figure 4 shows the relationship between take-up units and supply of housing units during the period 1980 to 2001. By looking at 1996 to 1999, we can see that housing supply increased a great deal. Despite a contraction of the economy in Hong Kong during that period, housing supply increased considerably from 1997 to 1999. The effects of the large number of vacant units negatively affected the housing market in Hong Kong.

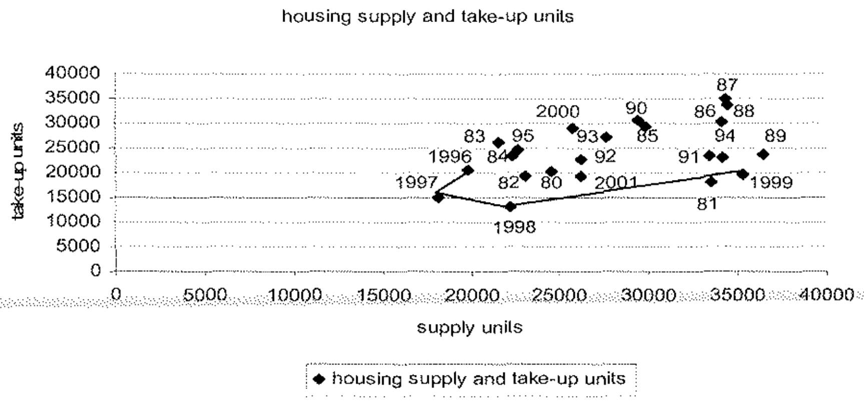

Figure 4 Housing Supply and Take-up Units over the Last Two Decades

[Source: *Hong Kong Monthly Digest of Statistics*, various issues]

[3] Vacancy rate refers to vacancies in respect to all private premises completed during the review year, and those completed earlier but not yet assessed for rating purposes, determined by inspection at the end of the year (Rating and Valuation Department, Hong Kong).

The literature states that the housing price has been studied by the stock-flow model (Fisher, 1992; DiPasquale & Wheaton, 1992, 1994, 1996). The authors suggested that the market clears quickly, and prices are adjusted to equate the demand for housing to the existing stock at any period of time. That is, the market price of housing will tend to change in a direction that will balance the quantity of desired houses with the quantity supplied by developers (Reichert, 1990; DiPasquale & Wheaton, 1994). The reason is based on the economic theory that a higher price will reduce the quantity of housing demanded by households, and increase the housing supply (Dubin, 1998; Ho & Ganesan, 1998). Housing prices are also gradually adjusted in the short term and achieve equilibrium of housing supply and demand in the long term (Kenny, 1999). Figure 5 illustrates a market that adjusts to a change in demand for housing. An increase in demand for housing will lead to an increase in housing price because of a highly inelastic supply curve in the short term. A new equilibrium position moves from *a* to *b*. In the long term, the quantity supplied is more responsive to a price change than in the short term. The more responsive housing supply will place downward pressure on housing price (from b to c) with the passage of time (Gwartney, 2000).

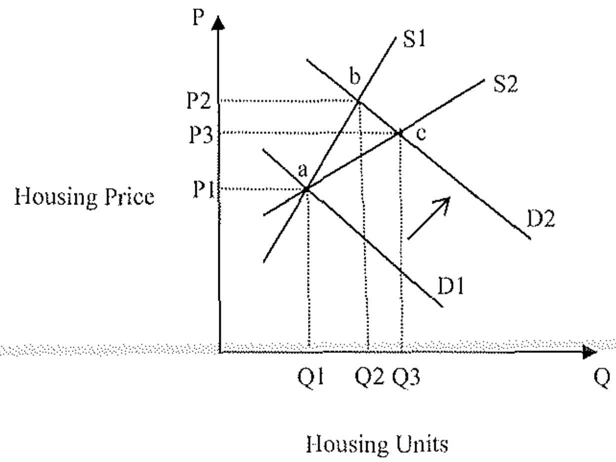

Figure 5 Time, Housing Supply, and Adjustment to an Increase in Demand

Figure 6 illustrates the housing market adjustment to a decrease in demand.

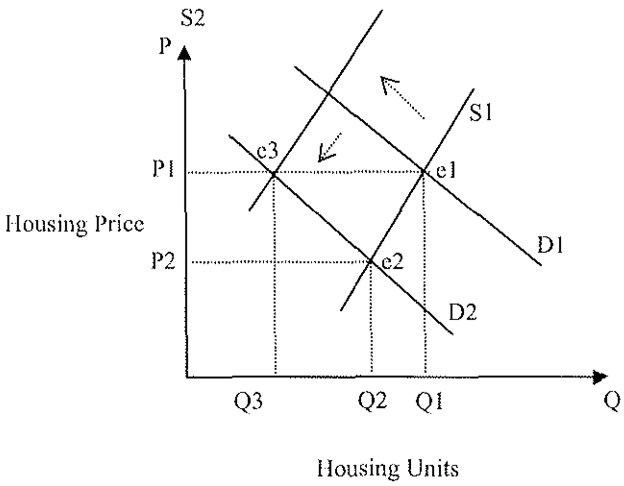

Figure 6 Market Adjustments to a Decrease in Demand

Initially, demand curve is D_1 and supply S_1 is at equilibrium e_1. When the decrease in demand shifts from D_1 to D_2, the housing price decreases from P_1 to P_2 at a new equilibrium position, e_2. A reduction in supply shifting the supply curve S_1 to S_2 causes an increase in the equilibrium price and a decrease in the equilibrium quantity at Q_3.

The situation will be different if there is an increase in housing supply from S_1 to S_2. A new equilibrium is achieved, and housing price decreases further to P_3 at equilibrium quantity Q_3 (Figure 7).

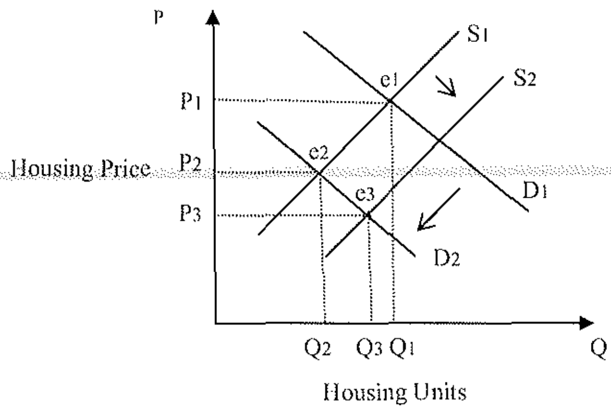

Figure 7 Market Adjustments to a Decrease in Demand in Hong Kong

The stock-flow model shows the effect of falling prices but does not consider construction lag-time for building and vacant units. The fluctuation of prices in markets with a production lag has been studied by the cobweb theorem. This theorem explains the path followed in moving towards an equilibrium price and quantity in the long term when there are time tags in the adjustment of either supply or demand to changes in price (Nellis & Parker, 2002). Figure 8 shows the cobweb market adjustment. At equilibrium price P_1, the quantity is Q_1, assuming that there is a change in demand for housing from D_1 to D_2. In the short term, the supply cannot adjust to the increase in demand because of the time it takes to finish construction. During this lag, the price is pushed to P_2, which is above the equilibrium price at point f, a new long-term equilibrium created by the intersection of D_2 and S_1. Assuming supply will reach Q_2, a temporarily fixed position, a surplus of supply results, since P_2 is above the long-term equilibrium. Due to the surplus, the price will fall to P_3, which is below the new long-term equilibrium at point f. Housing supply will adjust by a reduction in construction. This completes the trough cycle, bringing the new short-term supply Q_3 at point e. The cycle will then repeat itself, giving the supply points the appearance of a cobweb.

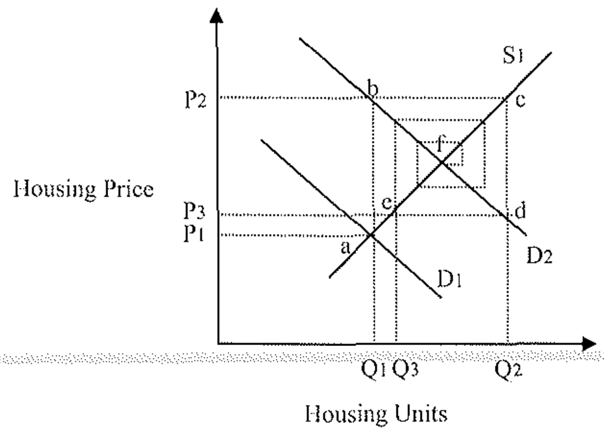

Figure 8 Cobweb Market Adjustment

The cobweb theorem indicates an effect of the change in current price and housing supply in the opposite direction (Hua & Chang, 1999). The main reason for the phenomenon may be lagged construction.

Neither model has focused on the housing price crash. A crash reflects a sharp decline or discontinuous change in price that is partially due to the market fundamentals (Fernandes, 1998). According to the cusp theory, discontinuous change in housing prices may be induced by the instability of the housing price system and the imbalance of housing stock and vacant units.

In the housing market, the main reasons for the housing price crash may be misinformed expectations and speculation (Kim & Kim, 1999). Housing is real property. People review the past and current behaviour of housing prices in the market and constantly adjust their expectations about the future, based on the available information (Cho, 1996; David, 1983). Some households purchase property for investment when they expect a rise in housing prices. The continuous rise in housing price provides an environment to earn profits for short term speculators. Price speculation involves buying or selling with the expectation of a future price change (Levin & Wright, 1997). Speculators are a result of self-reinforcement (Johansen *et al.* 2001). The speculative households have the same psychology (Johansen & Sornette, 1997) to purchase houses in the market. Johansen *et al.* (2001) developed a model for the stock market. Their evidence suggests that people imitate the behaviour of and expect to achieve the same return as other households that benefit from price increases, because they believe housing prices will continue to increase. The speculative activities slowly create bubbles in which the housing prices exceed the real value of property (Ho, 2000). Alternatively, if the housing price falls rapidly when the bubble breaks because of external factors, negative feedback occurs from households. Because of downturn expectations, many households withdraw from the housing market and monitor the housing price movement. Some speculators will leave the market by selling their property as quickly as possible at a marginal price, to avoid loss. This behaviour further increases the housing supply and puts more pressure on the housing price. Therefore, the speculators' behaviour in the housing market and their expectations of housing prices need to be considered in the modelling.

Housing demand increases faster because demographic factors like population growth, household formation rates, immigration, employment growth, income growth, expectations for future performance of the economy, and an upward price movement all push demand forward more rapidly (Douglas, 2002; Tutor2u, 2003). Housing supply grows more slowly in the short term because it is constrained by government ordinances, time requirements for land development and construction, and developers' confidence in selling housing units (Wong *et al.*, 1999; Tse, 1998). Housing prices rise as the demand for housing increases. To meet the increased demand, developers may speed up construction of housing units. An unfavourable event causes demand for housing to decrease rapidly. Lagged construction increases quantity supplied for housing in the current period, and housing prices fall rapidly. The effect of the special characteristics of the housing market results in an unstable system in housing prices.

A change in expectations that changes demand and the time lag in housing supply (Runeson, 2004) that accumulates considerable vacant units can be evaluated. In a private housing market, private developers supply housing. The production decision of firms is based on the expected profitability of building housing. Poterba (1984) suggested that the production decision of firms is based on the comparison of the production costs of housing and the current real price of housing. His estimation also indicates that the price of housing is a major determinant of new construction. There is a direct relationship between the price of housing and the number of units produced and offered for sale. Though the housing market will achieve a long-term market-clearing price, it is often characterised by significant deviations (Kenny, 1999). The quantity supplied will adjust the stock of dwellings according to quantity demanded and housing prices. When quantity demanded increases, housing prices rise, as housing supply is inelastic in the short term. As the price of housing increases, keeping other factors constant, developers pursue profits by increasing the number of units supplied on the market.

The housing supply is inelastic in the short term because of the time lag for the construction period, i.e., the period for commencing building and the buildings being completed (Tse, 1998). Lagged construction produces a greater supply of housing. A new supply to the market is measured by housing completions, as this is more accurate

(Coulson, 1999). A portion of new units may be taken up by house purchasers. The unsold units become vacant units available for the market. A new supply increases the housing market, even though there may be a decrease in demand for housing because of construction lags. In this circumstance, more vacant units accumulate in the market.

The sources of vacant units may come from an unsold new supply and from existing vacant or resold units (Chan, 2003). Adequate vacant units are necessary for mobility in the housing market. According to The UK Parliament (2002), vacancies are an essential part of the effective operation of the housing market, as unnecessary vacant homes are a wasted resource and allowances for the vacancy rate would be recommended four percent for the private sector. In Australia, a vacancy rate below three percent is considered as demand exceeding supply; whereas a rate above three percent indicates supply exceeding demand (Housing Accessibility, 2000). This may influence the price level. The adequate vacancy rate in Hong Kong has to be assessed.

It is important to study discontinuous change in housing prices to understand why they happen, when they happen, and how to avoid them. For this reason, a cusp catastrophic theory for analysing housing price change is used. In fact, studying the functional system of housing price stability is more important than studying only its effects. A sudden price fall is not only related to supply and demand changes, but also to its instability in the system. It requires understanding the special characteristics of the housing market for effective modelling.

2. The Cusp Model

Catastrophe illustrates those situations in which minor changes in one variable provoke an abrupt, 'catastrophic' change, or bifurcation, in another variable (Gilmore, 1981). It studies and classifies phenomenon characterised by sudden shifts in behaviour arising from small changes in circumstances (He & Zhao, 1989). Castrigiano & Hayes (1993) defined a catastrophe as a sudden transition resulting from a continuous parameter change. Catastrophe is a theory about singularities (Saunders, 1998). It is a mathematical model for describing discontinuous change: 'change in a course of events, change in an object's shape, change in a system's behaviour, and change in ideas themselves' (Woodcock & Davis, 1982). For housing prices, catastrophe can be defined as the phenomenon of an abrupt change in housing prices when other parameters change only slightly.

To illustrate discontinuous behaviour, Douglas (2002) gave the following example: a piece of wood bends and then suddenly breaks if it is bent slightly beyond the breaking point. The failure of the wood is a discontinuity in the distortion of the wood as it bends. The eruption of volcanoes and earthquakes are other examples of sudden discontinuous events. A discontinuous system behaves differently from a continuous system (Gilmore, 1981). A linear system exhibits continuous behaviour. Figure 9 is an example of the difference in the two systems. Assume the coordinate axis *x* is dependent or state variable, and the abscissa axis α is the independent or control variable. Figure 9(a) represents continuous behaviour. The solid line is the collection of (stable) equilibrium states of the system. To slowly increase α, the system will always move towards the nearest stable equilibrium state. Therefore, a small continuous increase or decrease in α will lead to a small continuous increase or decrease in *x*. In Figure 9(b), the solid line is the collection of stable equilibrium states. The dotted line is the collection of unstable equilibrium states. When $\alpha < 2$, because the nearest stable equilibrium state is always located at the lower solid line for slowly increasing α, no transitions will occur. At α=2, a stable and an unstable equilibrium state coalesce and disappear. A small continuous increase in α will lead to a large discontinuous change in *x*.

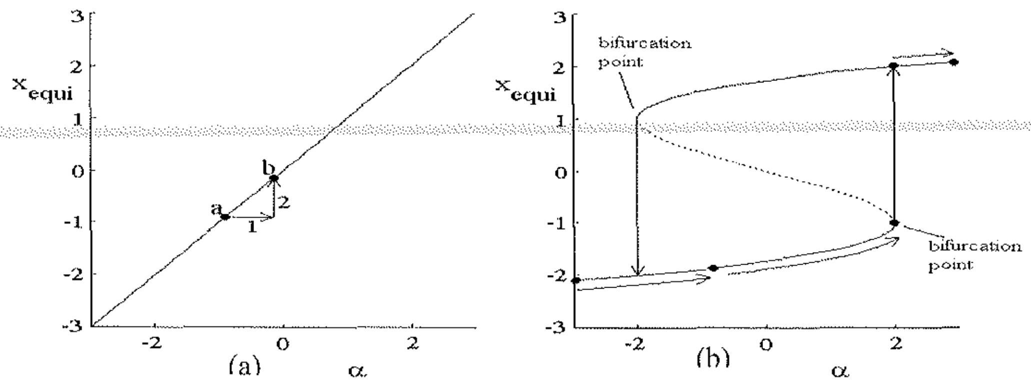

Figure 9 A Continuous and Discontinuous System [Source: Gilmore, 1981]

Catastrophe theory shows that the number of qualitatively different configurations of discontinuities that can occur depends not on the number of state (dependent) variables, which may be very large, but on the number of control (independent) variables, which is generally small (Tang *et al.*, 1993). There are only seven distinct types of catastrophe if the number of control variables is not greater than four (Castrigiano & Hayes, 1993). A cusp type catastrophe is commonly used (Hansen, 1993).

A cusp type catastrophe model has two control variables and one state variable. It illustrates how a change in control variables can give rise to discontinuous behaviour of the state variable, through the disappearance of stable steady states (Saunders, 1998). It is derived from the quadratic function as follows:

$$V(q,p,x) = \frac{1}{4}x^4 + \frac{p}{2}x^2 + qx \tag{1}$$

where x is the behaviour (state) variable and p, q are the control variables. The phase space describing the behaviour of x in relation to p and q is thus three-dimensional. The equilibrium surface, M, is defined by the equation

$$\nabla_x V = 0 \tag{2}$$

where the subscript x indicates the gradient ∇ is taken with respect to the behaviour variable x only. This surface is made up of all the critical points of V, i.e., all the equilibria (stable or otherwise) of the system, using M to denote a manifold, (a well-behaved, smooth surface). According to Tang *et al.* (1993), the equilibrium surface M for the cusp catastrophe is given by:

$$\nabla_x V = \frac{\partial V}{\partial x} = 0 \Rightarrow x^3 + px + q = 0 \tag{3}$$

The singularity set, S, is defined by the fold curve which is a subset of M consisting of all the degenerate critical points of V. They are the points at which $\nabla_x V = 0$ and

$$\nabla_x^2 V = 0 \tag{4}$$

where the matrix of second order partial derivatives is defined as follows:

$$\begin{bmatrix} \frac{\partial^2 V}{\partial x_1^2} & \frac{\partial^2 V}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 V}{\partial x_1 \partial x_n} \\ \frac{\partial^2 V}{\partial x_2 \partial x_1} & \frac{\partial^2 V}{\partial x_2^2} & \cdots & \frac{\partial^2 V}{\partial x_2 \partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 V}{\partial x_n \partial x_1} & \frac{\partial^2 V}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 V}{\partial x_n^2} \end{bmatrix}$$

x_i are the state variables given, since one has only one state variable x here,

$$\nabla_x^2 V = \frac{\partial^2 v}{\partial x^2} = 0 \Rightarrow 3x^2 + p = 0 \qquad (5)$$

By eliminating the behaviour variables x from equations 3 and 5, one can project the three-dimensional fold lines S down into the control space C, the pq plane, to obtain the two-dimensional curves defining the bifurcation set, B, which is the set of all points on C at which V is singular. Because equation 3 is a cubic equation, it has either one or three real roots. The number of real roots is determined by the discriminant (Zeeman, 1981):

$$\Delta = 4p^3 + 27q^2 = 0 \qquad (6)$$

If $\Delta \leq 0$ there are three real roots; otherwise, there is only one. The roots are distinct unless $\Delta = 0$, in which case either two coincide (if q and p are non-zero), or all three coincide.

Finally, the form of V is determined at every point in C. Changes can occur only when a control point crosses B, and it is sufficient to consider only one point within each of the regions into which B divides C.

A cusp catastrophe of a surface in a three-dimensional space is illustrated in Figure 10. The surface is a sheet that is folded to create an upper level, a lower level, and a cusp or fold. An event is defined as a transition from either the lower state to the upper state, or vice versa. If the transition occurs on the back edge of the surface following line A, the event is normal and continuous. If, however, the transition occurs along the front edge following line B, then the event is a sudden and abrupt change, which constitutes a catastrophe.

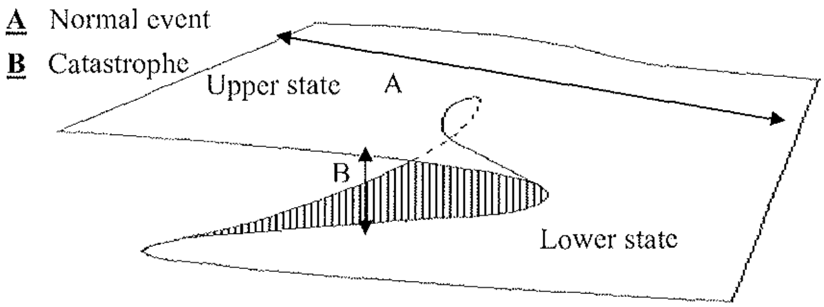

Figure 10 Catastrophe Model [Source: Zeeman, 1995]

Figure 11 illustrates what will occur for different paths in the neighbourhood of *B* if the surface is sketched. The set of equilibrium values is the surface defined by *x*, *p*, and *q*. If the state of the system is represented by a point *T* in the three-dimensional phase space with *x*, *p* and *q* as coordinates, the phase point *T* must always lie on the surface. In fact, it must always lie on either the top or the bottom sheet, because the middle sheet corresponds to an unstable equilibrium (Tang *et al.*, 1993) that is not physically realisable.

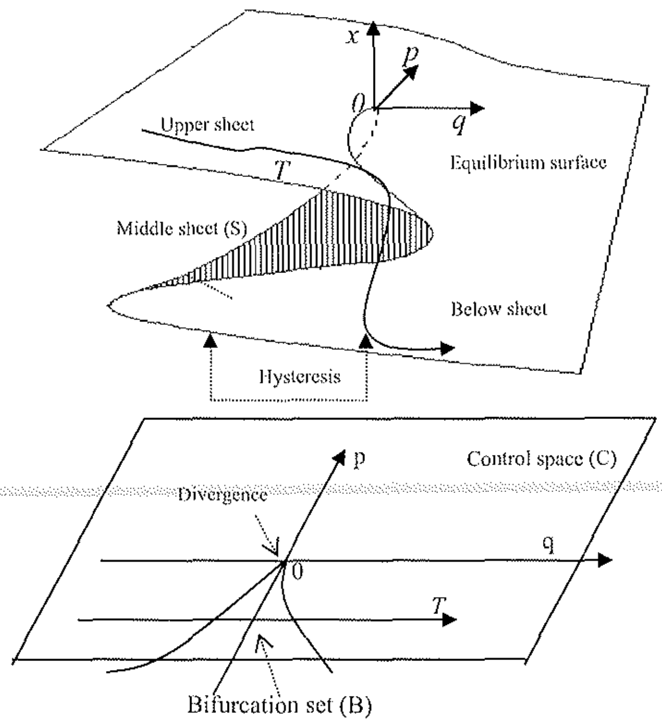

Figure 11 The Equilibrium Surface and Bifurcation Set of the Cusp Catastrophe [Source: Tang *et al.*, 1993]

The diagram can be interpreted as follows. T is a point on the p-q plane, which is called the control space. As the control variables p and q are altered, this control point (in two dimensions) traces out a path that is called the control trajectory. At the same time, the phase point (in three dimensions) moves along a trajectory in the equilibrium surface, directly above the control trajectory. Smooth variations in p and q almost always produce smooth variations in x. The only exceptions occur when the control trajectory crosses the bifurcation set B (described by equations 5 and 6), which is the projection onto the p-q plane of the fold lines S of the equilibrium surface. If the phase point happens to be on the surface that ends at a point on the fold lines, then it must jump to the other sheet. This brings about a sudden change in x (Tang *et al.*, 1993).

The five main features of the cusp catastrophe model are described by Zeeman, (1995) and Quantitative Minor Program (2002): 1) bimodality—the behaviour being predicted has an either/or tendency, 2) sudden transitions—sudden changes between one mode of behaviour and another, 3) hysteresis—transitions between behavioural models do not occur at the same place, 4) inaccessibility—certain behaviours are extremely unlikely given the input parameters, and 5) divergence—a small change in the input parameters can dramatically affect system stability.

The cusp catastrophe theory has been successfully applied in the animal behavioural sciences (Woodcock & Davis, 1982; Zeeman, 1976); physics, chemistry, and biology (Tang, 1993; Yin & Zheng, 1988; Thompson, 1975); sociology and economics (Fernandes, 1998; Xu & Tang, 2002); politics and public opinion (Tse, 2002); engineering (Chen & Leung, 1998) and psychology. Recently, Sornette and Zhou (2002) developed a stock market model while studying how materials fail under stress. The predicted results from the catastrophe analysis agree with experimental results. However, there is no previous application in the analysis of housing prices to date.

3. Development of Analytic Framework of Housing Prices

In order to analyse when and how price catastrophe may occur, it is necessary to study the effect of a change of expectations, demand, and the time lag in supply. This section develops an analytical framework for housing prices, by modifying the stock-flow model and the cobweb theorem.

3.1 The general assumptions

The model is developed using the following assumptions:

1) There is an open system of economic environment with sufficient information. The housing market is competitive, and housing products are assumed homogeneous (Muth, 1960). Households and developers go freely into the housing market to purchase and sell. Householders seek to maximise their utility derived from expenditures for both housing and non-housing related goods and services (Maclennan, 1982). Developers pursue profit maximisation. The existing housing stock plus additions to that housing stock are determined by developer behaviour in search of profits (DiPasquale, 1999) left after consideration of development costs (Megbolugbe & Cho, 1993). The behaviour of households and developers is rational and constant.
2) The housing market is always in equilibrium in the long term, but adjusts in the short term (Maclennan, 1882). Housing prices are determined only by quantity supplied and demand variables. A stable steady state of housing prices is achieved from the interaction between quantity supplied and demanded. A new stable state of housing prices is arrived at after the adjustment of supply and demand.
3) Housing demand includes consumption demand and investment demand (Reichert, 1990). Scarcity of land makes housing, property, and value appreciate (Tse, 1998). Some households may speculate if they expect future increases in housing prices (Ho, 2000).
4) Total housing supply or housing stock consists of a new supply of housing units and existing housing units (DiPasquale & Wheaton, 1994). The existing housing units include hoarding units, unsold new units, vacant units, and second-hand units for sale (Chan, 2003). This is a dynamic process of new supply. The decision of housing unit construction for new supply is adjusted, depending on the quantity demanded for housing units (DiPasquale, 1999).
5) There is a sound financial credits system that allows households to get into the

housing market (Chan, 1999).

6) In a stable system, vacant units adjust to the changes in supply and demand at an equilibrium point, i.e., a positive relationship exists between them.
7) The cusp catastrophe model is suitable for analysis of the housing market.

3.2 A model of instability of housing price system

The model aims to analyse changes in vacant units and the condition of instability of a housing market system, in order to diagnose the discontinuous housing prices.

According to the stock-flow model, housing prices are derived from the analysis of a series of supply and demand equilibrium. It is possible to study the dynamic effects on housing prices when housing supply adjusts at a given slope corresponding to housing demand changes, in line with literature reviewed. That is, a dynamic movement of housing price is examined, i.e. housing price from one equilibrium point move to another equilibrium point when housing supply and demand interacts.

Based on the cobweb theorem, as time lags, equilibrium price and quantity are achieved through the adjustment of either supply or demand to changes in price (Nellis & Parker, 2002). At an equilibrium position, vacant units are a proportion of the total housing stock in the market.

The model starts to build on a relationship between an equilibrium price and vacant units in a stable housing price system, indicated in Figure 12. The ordinate indicates the equilibrium state of housing price. The abscissa indicates quantity of housing units, which includes equilibrium quantity demanded and supply, and vacant units. It is defined as follows:

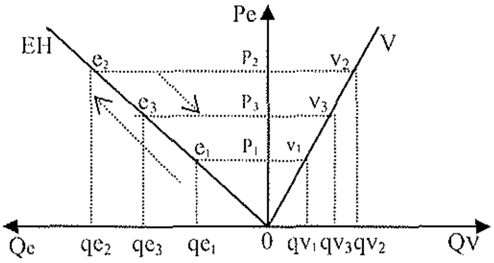

Figure 12 Relationship between an Equilibrium Price and Vacant Units in a Stable Price System

EH = the curve of housing supply and demand is balanced at an equilibrium price level, called the 'equilibrium curve'.

v = the curve of vacant units.

Q_e = quantity changes of housing units at each equilibrium level, i.e., $qe_{1,2,3.}$

Q_v = quantity changes of vacant units at each equilibrium level, i.e., $qv_{1,2,3}$.

P_e = the equilibrium level of housing price, i.e., $P_{1,2,3}$.

The housing market has special characteristics, such as durability (Maclennan, 1982; Megbolugbe *et al.*, 1991; Glaeser & Gyourko, 2003), heterogeneity (Megbolugbe & Cho, 1993, Ding, 2002), and a long construction period (Smith *et al.*, 1988; Tse, 1998), which are distinguished from other commodities. In a stable price system, it is assumed that when the housing market is at market equilibrium, it allows existent vacant units for mobility. Vacant units should be a proportion of the total housing supply in a stable system. Referring to Figure 12, the original is at an equilibrium point of e_1 with equilibrium price of P_1, and vacant units are at qv_1. A new equilibrium price is achieved at P_2 when housing supply adjusts to an increase in housing demand to an equilibrium position at e_2. The vacant units adjust to qv_2. Similarly, if there is a decrease in housing demand, housing supply reaches a new equilibrium at point e_3 and price P_3, and the vacant units change back to qv_3.

There will be a time lag between the decision to start work and the date of building completion for housing (Tse *et al.*, 1999). This means that the lag supply may not meet the increase in demand for housing. This leads to a higher price of housing in the short term, and the market may absorb vacant units. Alternatively if there is a sharp decreased in demand, housing prices fall, and the lag supply increases because of expectations of supply. The unsold new units and second-hand units for sale belong to vacant units in the housing market. Vacant units may be increased suddenly when there is a continuous increase in the supply of housing units. This causes pressure on the housing price, and the stable system may become instable. The change in price expectation may depend on previous price levels and recent forecasting (Cho, 1996; David, 1983; Pashigian, 2004). Households expect the price to fall further due to a pessimistic economy, so more units will be available for sale in the market. A discontinuous change in housing price cannot be avoided in this situation. The description of the unstable phenomenon of housing price, Figure 12, may be modified to Figure 13.

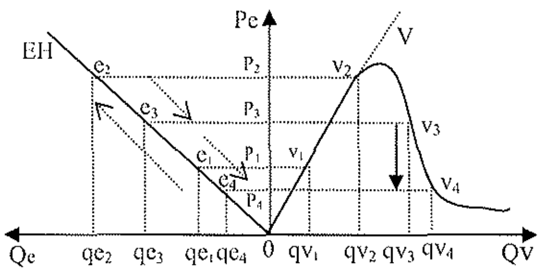

Figure 13 The Unstable System of Housing Prices

Referring to Figure 13, the housing market is at an equilibrium position e_2 with equilibrium price at P_2. An economic shock stimulates a decrease in demand for housing to an equilibrium position e_3. The lag supply adds vacant units to qv_3, in proportion to the total housing supply increase. Low expectation causes a decrease in demand further to the market equilibrium point, e_4; the lag supply and selling from existing stock adds more to vacant units to qv_4. The price crashes from P_3 to P_4 accordingly. The instability of the price system is established when the vacant units curve becomes non-linear, i.e., a rapid increase of vacant units. This figure thus explains the phenomenon during the period from 1996 to 1999, a vulnerable housing price system as shown in Figure 14 and Table 1. Take-up units have decreased since 1996 and increased in 1999. The housing supply increased considerably from 1997 to 1999. The vacant units increased rapidly from 1996 to 1999.

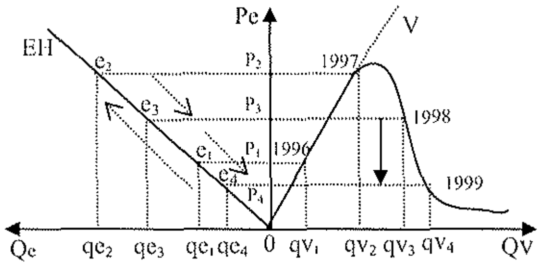

Figure 14 An Unstable Housing Price System in Hong Kong from 1996 to 1999

Table 1 Take-up, New Supply, and Vacant Units from 1996 to 2000 in Hong Kong

Year	Take-up Units	New Supply Units	Vacant Units
1996	20480	19870	34050
1997	15090	18200	35980
1998	13050	22280	43820
1999	19560	35320	59140
2000	29180	25790	54950

[Source: *Hong Kong Property Review*, various issues]

In order to understand the effect of housing price instability, it is necessary to test what makes prices fall catastrophically and under what circumstances. This is done in the following section.

4. Developing a Cusp Catastrophe Model of Housing Prices

This section applies catastrophe theory in conjunction with the stock-flow and cobweb theorem, described in the previous section, to reveal the cause of a stable and unstable equilibrium of housing prices. The purpose of developing a housing price catastrophe model is to explore the reasons for sudden changes of housing prices. Mathematical inference is used for testing the model. The model takes the methodology from Tang (1993), who developed a catastrophe model for the analysis of catastrophe in rock stability failure.

4.1 A cusp model of housing prices

The cusp model is a basic scientific research methodology to describe natural phenomenon by using mathematic models (Tang, 1993). Its standard form of potential function *V(x)* is shown in equation 1, and the bifurcation set is displayed in equations 2 to 6. In Figure 10, the state of a cusp system is described by a three-dimensional phase space. The phase point must always be on the surface, either the top or the bottom sheet, because the middle sheet corresponds to an unstable condition (He & Zhao, 1989) that is not realisable. A crossing of the bifurcation set (equation 6) brings a sudden change in the state. The basic characteristics of the cusp model show a simple foundation for modelling housing prices, especially finding the inflection point of prices and the precursory signals.

The idea of the model is to analyse the relationship of change in vacant units and the market equilibrium quantity. A housing price system is assumed stable and equal from one steady state to another. The stable condition becomes fragile at catastrophe. Any small external disturbance means a vulnerable condition from one stable condition to the new stable condition. By comparing two different stable conditions of parameter changes, such as the change of vacant units and the change of capital investments, unstable catastrophe housing prices can be analysed.

Table 2 The Definition of Symbols for Catastrophe Model of Housing Prices

Symbols	Definition	Symbols	Definition
P_e	Equilibrium point housing prices	Q_T	Total housing stock
Q_e	Equilibrium take-up quantity	Q_v	Quantity of vacant units
λ_e	Slope of equilibrium curve	λ_v	Slope of vacant units curve
Q_v^*	State variable, inflection point of vacant units	Q_{vm}	Vacant units corresponding to the maximum level of capital investment
λ_v^*	Slope of vacant units at the inflection point	ξ	A dimensionless parameter related to Q_n
K	Ratio of λ_e / λ_v	K^*	=1/K
V_{Qe}	The capital investment in equilibrium housing market	V_{Qv}	The capital investment of vacant units

The potential function is crucial for the establishment of a cusp model for housing prices. To develop a cusp model, the equilibrium quantity, vacant units, and the price level as variables are considered for modelling. The definition of symbols is displayed in Table 2.

It is assumed that the total housing stock (Q_T) can be denoted as the equilibrium quantity plus the vacant units:

$$Q_T = Q_e + Q_v \tag{7}$$

Q_e and Q_v are the equilibrium take-up quantity and vacant units at one period. The total housing stock is denoted as Q_T.

To simplify the model, it is assumed that equilibrium take-up quantity is in a linear form. According to Figure 15, the capital investment in the equilibrium housing market can be assumed to be a triangular shape (e.g., e_2-qe_2-0), described by the following equation:

$$V_{Q_e} = \int_0^{Q_e} f(Q_e)dQ_e = \frac{1}{2}Q_eP_e = \frac{1}{2}\lambda_e Q_e^2 \tag{8}$$

where $f(Q_e)$ or λ_e is the slope of the equilibrium quantity curve, and Q_e is the change of equilibrium take-up quantity of houses.

$$P_e = \lambda_e Q_e \tag{9}$$

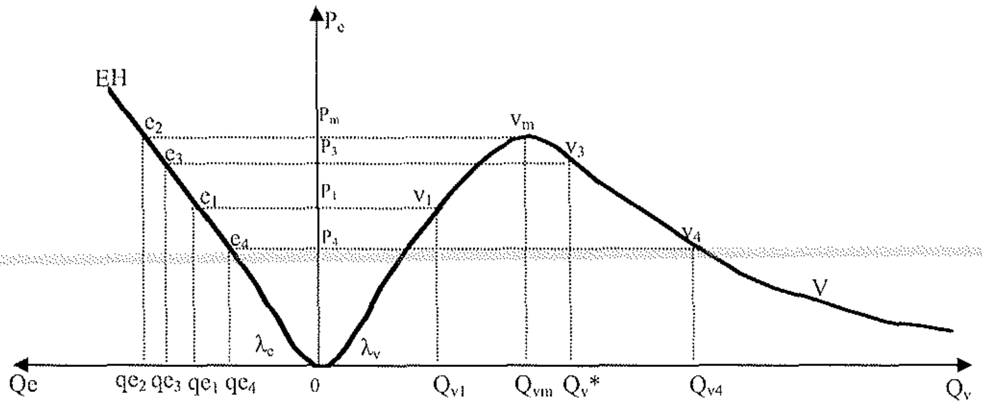

Figure 15 A Catastrophe Model of Housing Price

The vacant units curve depends on the change in the number of vacant units. In an unstable system, its curve is assumed to be non-linear, as described in Appendix 1, in the shape of nearly normal distribution.

According to Appendix 2, the function of the capital investment of vacant units is derived by:

$$V_{Q_v} = \int_{Q_v}^{Q_{vm}} f(Q_v)dQ_v = \lambda_v Q_{vm}\left[Q_{vm} - (Q_{vm} + Q_v)e^{-\frac{Q_v}{Q_{vm}}} \right] \tag{10}$$

$f(Q_v)$ is the slope of vacant units (λ_v). Take the vacant units Q_v as the state variable of the system. The potential function of total capital investment in the housing market can be derived as follows:

$$\begin{aligned} V_{Q_T} &= V_{Q_e} + V_{Q_v} \\ &= \int_0^{Q_e} \lambda_e(Q_e)dQ_e + \int_{Q_v}^{Q_{vm}} \lambda_v(Q_v)dQ_v \\ &= \frac{1}{2}\lambda_e Q_e^2 + \lambda_v Q_{vm}\left[Q_{vm} - (Q_{vm} + Q_v)e^{-\frac{Q_v}{Q_{vm}}} \right] \end{aligned} \tag{11}$$

The derivation of the potential function is shown in Appendix 3. According to equation 7, $Q_e = Q_T - Q_v$, equation 11 can be rewritten as follows:

$$V_{Q_T} = \frac{1}{2}\lambda_e (Q_T - Q_v)^2 + \lambda_v Q_{vm}\left[Q_{vm} - (Q_{vm} + Q_v)e^{-\frac{Q_v}{Q_{vm}}} \right] \tag{12}$$

The potential function of the total capital investment in housing property is derived. It is possible to discuss the behaviour of the system using the cusp catastrophe theory mentioned previously.

The equilibrium surface *M* may be given by:

$$\nabla_{Q_v} V_{Q_T} = \frac{\partial V_{Q_T}}{\partial Q_v} = \lambda_v Q_v e^{-\frac{Q_v}{Q_{vm}}} - \lambda_e (Q_T - Q_v) = 0 \tag{13}$$

The singularity set is as follows:

$$\nabla_{Q_v}^2 V_{Q_T} = \frac{\partial^2 V_{Q_T}}{\partial^2 Q_v} = \lambda_v (1 - \frac{Q_v}{Q_{vm}})e^{-\frac{Q_v}{Q_{vm}}} + \lambda_e = 0 \tag{14}$$

It is found that the equation is relevant to the slope of vacant units (λ_v), the equilibrium curve (λ_e), the quantity of vacant units at the highest equilibrium price lever (Q_{vm}) in Figure 15 and the state variable Q_v. The singularity at vanishing gradient demonstrates that catastrophe may occur and is determined only by the characteristics of the housing price system.

The cusp point can be obtained as follows:

$$\nabla^3_{Q_v} V_{Q_T} = \frac{\partial^3 V_{Q_T}}{\partial^3 Q_v} = \frac{\lambda_v}{Q_{vm}}\left(\frac{Q_v}{Q_{vm}} - 2\right)e^{-\frac{Q_v}{Q_{vm}}} = 0 \quad (15)$$

The solution to this equation at an inflection point is:

$$Q_v = Q_v^* = 2Q_{vm} \quad (16)$$

where $\frac{Q_v}{Q_{vm}} - 2 = 0$ (17)

In order to obtain a standard format of the cusp model, the Taylor series (Hans, 1986) expansion (Appendix 4) is used for equation 14 in the inflection point $Q_v = Q_v^*$ up to the third degree:

$$\begin{aligned}
&\lambda_v^* Q_v^* e^{-\frac{Q_v^*}{Q_{vm}}} - \lambda_e (Q_T - Q_v^*) + \\
&\left[\lambda_v^* \left(1 - \frac{Q_v^*}{Q_{vm}}\right)e^{-\frac{Q_v^*}{Q_{vm}}} + \lambda_e\right](Q_v - Q_v^*) + \\
&\left[\frac{\lambda_v^*}{2Q_{vm}}\left(\frac{Q_v^*}{Q_{vm}} - 2\right)e^{-\frac{Q_v^*}{Q_{vm}}}\right](Q_v - Q_v^*)^2 + \\
&\left[\frac{\lambda_v^*}{6Q_{vm}^2}\left(3 - \frac{Q_v^*}{Q_{vm}}\right)e^{-\frac{Q_v^*}{Q_{vm}}}\right](Q_v - Q_v^*)^3 = 0
\end{aligned} \quad (18)$$

Because $Q_v^* = 2Q_{vm}$, equation 18, can be simplified to:

$$\frac{3}{2}\left[1 - \frac{\lambda_e}{\lambda_v^* e^{-2}}\left(\frac{Q_T - Q_v^*}{Q_v^*}\right)\right] + \frac{3}{2}\left(\frac{\lambda_e}{\lambda_v^* e^{-2}} - 1\right)\left(\frac{Q_v - Q_v^*}{Q_v^*}\right) + \left(\frac{Q_v - Q_v^*}{Q_v^*}\right)^3 = 0 \quad (19)$$

or $$x^3 + px + q = 0 \quad (20)$$

The process of simplification is shown in Appendix 5. This comes to the standard format of the cusp catastrophe model of the housing price system when x is the dimensionless state variable, relates to vacant units. It is called inflection ratio.

$$x = \frac{Q_v - Q_v^*}{Q_v^*} \qquad (21)$$

and

$$p = \frac{3}{2}(K - 1) \qquad (22)$$

$$q = \frac{3}{2}(1 - K\xi) \qquad (23)$$

$$K = \frac{\lambda_e}{\lambda_v^* e^{-2}} = \frac{\lambda_e}{\lambda_v^*} \qquad K^* = \frac{1}{K} = \frac{\lambda_v^*}{\lambda_e} \qquad (24)$$

$$\xi = \frac{Q_T - Q_v^*}{Q_v^*} \qquad (25)$$

Equation 22 indicates that control variable p is determined by K^* which is the parameter that indicates the slope ratio of vacant units at the inflection point to the equilibrium curve. Equation 23 shows that control variable q is influenced by K & ξ, in which ξ is the ratio of equilibrium quantity to the vacant units at the inflection point; we can call it the take-up ratio.

The relationship between state variable x and control variables p and q is depicted by equation 20. The bifurcation set is determined by equation 6, which indicates the cusp at point *(0,0)*. The control space is divided into two parts by the bifurcation set. In the small area are three equilibrium points, of which two are stable and one is unstable. In the large area, only one stable equilibrium point exists. In the bifurcation set, there are two equilibrium states; one is stable and the other unstable. The system maintains stability if the control point (p, q) varies smoothly in the control space. However, equilibrium breaks if it crosses the bifurcation boundary and brings about a sudden change in state variable x.

4.2 Criteria for instability

The purpose of developing a discontinuous change in housing price model is to estimate the instability of the system and to detect when discontinuous change in price occurs. A housing price catastrophe occurs when there is a sudden disrupting change in housing prices. Equation 20 demonstrates the catastrophe model of housing prices. The necessary condition for the system to reach its catastrophe state is $p \leq 0$, that is, equation 22:

$$p = \frac{3}{2}(K-1) \leq 0 \qquad (26)$$

$$K - 1 \leq 0 \qquad (27)$$

From equation 24, $K^* = \frac{\lambda_v^*}{\lambda_e}$, one gets

$$\lambda_v^* - \lambda_e \leq 0 \qquad (28)$$

λ_e indicates the responsiveness of equilibrium quantity and price at the equilibrium position; λ_v^* is the change in the number of vacant units at the inflection point. Thus, equation 28 demonstrates that the necessary unstable condition is completely determined by the internal characteristics of the housing price system itself. If the slope of vacant units curve is equal to or less than the equilibrium curve, a disrupting change in housing prices occurs. By estimating the relationship of λ_e and λ_v^*, the stability of the price system and a discontinuous price can be studied.

The strength of change in housing price can vary, depending on the shape of the vacant units curve. The discontinuous price change occurs when a change of equilibrium take-up quantity is less than a change of vacant units. In this situation, price has to cross the bifurcation area and starts to jump. A moderate and stable supply of housing units will avoid the occurrence of discontinuous change in housing prices.

Equation 23 shows that there is a negative relationship between the control variable *q* and ξ, the change of vacant units at the inflection point from total stock of houses; that is, an increase in the parameter, ξ, corresponding to the reduction of control variable, q. To test the statement by inserting equations 22 and 23 into equation 6, the parameter, ξ, becomes

$$\xi = \frac{1}{K}\left[1 \pm \frac{\sqrt{2}}{3}(1-K)^{\frac{3}{2}}\right] \qquad \text{(Appendix 6)} \qquad (29)$$

For each ratio of K, there are two values of ξ. One is when $q>0$, in which q is located at the small value, corresponds to the right side of the bifurcation set, in which state the catastrophe only causes the sudden change of the mathematical structure (the number of equilibrium states and the stability) but does not cause any sudden change of the state variable, Q_v^*. Another situation is $q<0$, and q is crossing the bifurcation set from the left sheet of the model. The state of the system is unstable, and the sudden change of the state variable Q_v^* occurs (Figure 16).

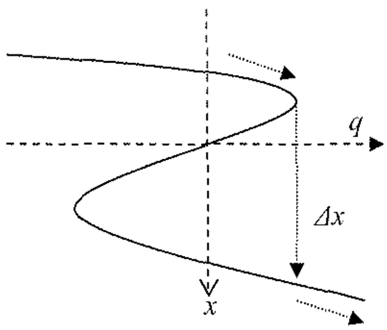

Figure 16 The State Jump when Crossing the Bifurcation Set

The main factors affecting instability of the housing system can be concluded from the mathematical inference above:

1) The behaviours of housing market performance are determined by the intrinsic properties of a system.
2) Lag supply is crucial to housing price changes. The lag supply increases the vacant units if there is a large decrease in demand. The rapid increase in vacancy rate may induce sudden changes in housing prices.
3) The critical and inflection points of catastrophe changes in housing prices are relevant to the ratio of slopes ($K=\lambda_v/\lambda_e$) for the vacant units curve and the

equilibrium curve. The instability of the housing market system can thus be analysed.

4) The discontinuous change in housing price occurs when the control variables ($p<0$, $q<0$) are less than zero in the cusp catastrophe model.
5) Housing stock is an element causing housing price change. However, it is not the factor contributing to a sudden change in housing prices in which the disrupted changes are determined by the market system itself.

5. Empirical Study for Validation of Cusp Housing Price Model

In this section, statistical data from the Hong Kong property market are applied in the developed model to validate the model. The procedures of the test are 1) obtaining data and pre-processing, 2) applications and results of validation.

Table 3 The Slope of Take-up and Vacant Units in Hong Kong

Year	Slope		Ratio K
	Vacant units	Take-up units	
80/81	89.42	-86.38	-1.04
81/82	72.83	70.37	1.03
82/83	-86.50	86.55	-1.00
83/84	-88.08	-88.59	0.99
84/85	-78.13	89.44	-0.87
85/86	88.35	84.56	1.04
86/87	-86.24	88.14	-0.98
87/88	-84.10	-79.72	1.05
88/89	88.38	-88.39	-1.00
89/90	-89.81	89.91	-1.00
90/91	86.32	-86.57	-1.00
91/92	58.46	-45.56	-1.28
92/93	-88.73	89.53	-0.99
93/94	87.28	-84.33	-1.03
94/95	-83.59	70.94	-1.18
95/96	-87.68	-88.84	0.99
96/97	63.12	-79.74	-0.79
97/98	81.28	-59.38	-1.37
98/99	88.88	87.31	1.02
99/00	-87.55	88.96	-0.98
00/01	-89.67	-87.95	1.02

5.1 The data and pre-process

The yearly data of overall private domestic premises are obtained from the *Monthly Digest of Statistics (*Hong Kong Government, various issues). The analysis covers the period from 1980 through 2000. The take-up units (in thousands) are used as proxy of market equilibrium quantity demanded. The take-up unit is the net increase in the number of units occupied in the year under review. The vacant units (in thousands) are used, and the data must be pre-processed before application. Real housing price index (1989=100) is calculated by dividing the consumer price index (Group C, October 1999—September 2000=100) for each period. First, we calculate the difference of time series data between each two periods. The slopes of take-up units and vacant units can be derived as Appendix 7. Table 3 summarises the results of data from the pre-processing.

5.2 Application of mathematical inferences

The applications of mathematical equations and their results are discussed, based on the derived equations.

- **Estimation for instability of the system**

Instability of the housing price system can be analysed in a number of ways. The cusp housing price model suggests that when 0≥K>-1, it is a precursory signal of housing prices. When $K \rightarrow -1$, the catastrophe phase of housing prices is represented by $K=\lambda_n/\lambda_d$. Figure 17 shows the ratio of slope for vacant units and the equilibrium curve for the last two decades. The period of precursory signals was the years 82/83, 84/85, 86/87, and 99/00. The housing price catastrophe was the years 89/90, 91/92, 94/95, and 97/98.

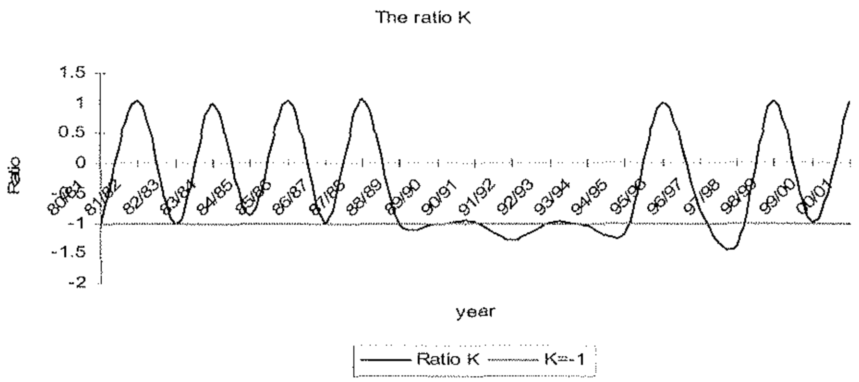

Figure 17 Ratio of Slope for Vacant Units and Equilibrium Curve

In equation 28, $\lambda_v^* - \lambda_e \leq 0$ is a criterion of unstable of housing prices. Figure 18 shows the housing price catastrophe for the period. During the years 84/85 and 96/97, results show a great deviation below zero. The years 86/87, 90/91, 93/94, and 99/00 indicate a minor deviation below zero.

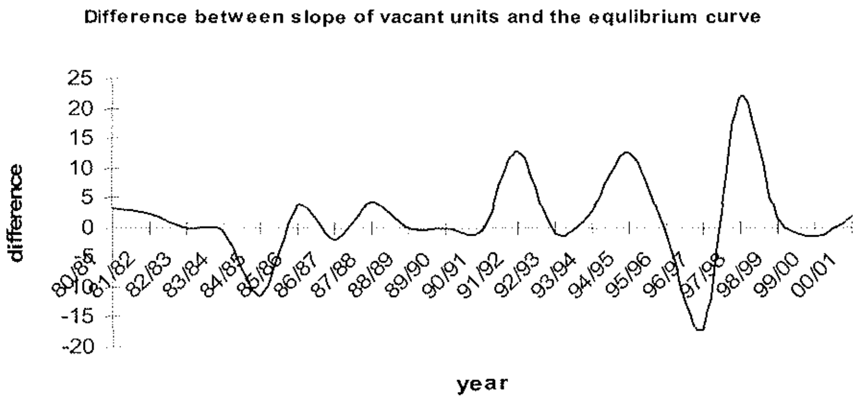

Figure 18 Difference between Slope of Vacant Units and Equilibrium curve ($\lambda_v^* - \lambda_e$)

Again, housing prices were inflected at the bar chart (in yellow) when the slope of take-up units (in blue) is greater than the slope of vacant units (in red) (Figure 19). Housing prices start to fall when the slope of the equilibrium curve appears much greater than the vacant units, such as in the years 81/82, 88/89, 91/92, 94/95, and 97/98. It is therefore demonstrated that *K* is a criterion for the instability of a housing system.

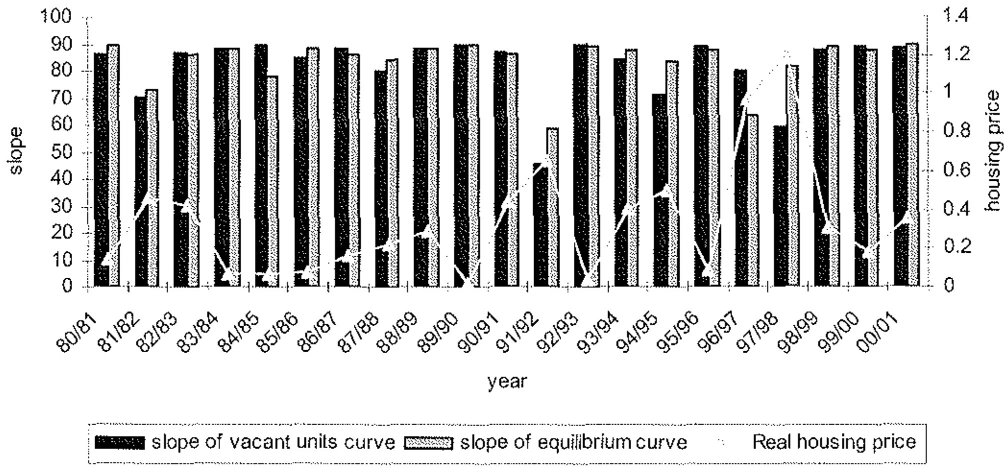

Figure 5.19 Relationship between Housing Prices and Slopes of Equilibrium Curve and Vacant Units

Empirical results from Figures 17 to 19 demonstrate consistent outcomes with the housing price booms and busts in Hong Kong (Ge & Lam, 2002) as Figure 1 or Table 1. For instance, housing prices have dropped substantially since 1981, 1989, 1994, 1997, and 2000. The main reasons for price reduction were demand decrease and excess of lag supply of housing units.

The research discovered that a discontinuous housing price occurred in a non-linear system. A dramatic fall in housing prices from the third quarter of 1997, for instance, occurred in an unstable non-linear price system. From 1994 to 1999, the take-up units decreased, but in the same period, the number of units supplied increased. The lagged supply added to the vacant units and supplied more housing on the market to meet the increased demand. The Asian financial crisis reduced the bubble and suppressed housing demand. In this situation, the price system changes from linear to non-linear, because of mass lagged supply accompanied by a decrease in housing demand. In the non-linear system, a discontinuous housing price occurs easily.

5.3 Validation

The cusp catastrophe housing price models are justified empirically, developed within economic theory and perfectly consistent with economic theory. Both descriptive and mathematical illustrations have validated the developed cusp housing price model that the results derived in the application of empirical studies are consistent with actual housing price behaviour in the market. The outcome of the catastrophe model demonstrates that the discontinuous housing price can be analysed by calculating the ratio of two slopes from a given ratio of vacant units and taken-up units. Because availability of total units in the market for the next period is known, whether or not a catastrophe will occur can be detected by estimating the short-term trend of taken-up units and the economic environment. A discontinuous housing price may occur when the trend of taken-up units decreases and the trend of a new supply and vacant units increases. Accordingly, given new units completions, we may project that the housing price will continuously drop as lack of demanded for housing, but the system is stable in the years 2000/2001 to 2002/2003 in Hong Kong, though the slope of the equilibrium curve is slightly higher than the curve of vacant units.

6. Conclusions

This paper develops a catastrophe housing price model to study the housing system and its instability in Hong Kong. By using techniques and methodology from the study of catastrophe in unstable rock failure (Tang, 1993), a theorem of discontinuous housing price is explored by applying the cusp model. The study demonstrates that the discontinuous housing price can be identified and explained by applying the derived catastrophe criteria.

The main findings from the paper are:

- Discontinuous housing price occurs in an unstable system as a non-linear curve of vacant units.
- The slopes of vacant units and the equilibrium curve determine the stability of the housing price system. An unstable state happens when the slope of the equilibrium curve is equal to the slope of vacant units in the same system with the opposite sign. Thus, discontinuous housing price can be detected.
- The trend or changes of housing price can be estimated by given changes of take-up units and vacant units, because units of supply are known by lagged construction and price levels.

- Catastrophe prices can be detected when there is an excess of vacant units for the equilibrium quantity demanded.
- Housing price behaviours can be clearly illustrated by the descriptive catastrophe model, which provides qualitative study for the system.

This study provides a qualitative study of housing price. The investigation has achieved the purposes of answering the 'why, when and how' of catastrophe housing price occurrence. The conclusion is that the catastrophe occurs because of instability in the housing system. Instability is caused by a rapid change of housing demand and lagged supply. Catastrophe occurs when the slope ratio for vacant units to the equilibrium curve is equal to or smaller than -1. Preventive methods may be any measures that can induce a stable demand and suitable adjustment of supply to meet the demand requirements. Maintaining an adequate vacancy rate may be crucial.

Appendix 1: The Assumption of Distribution for Vacant Units

To test a nonlinear curve of vacant units, it is necessary to demonstrate the vacant units are a random or stochastic distribution. Assume V is parameter that indicates the ratio of change of vacant units. When V=0 indicates there is no change in vacant units which does not happen normally; when V=1 indicates maximum changes of vacant units which does not appear frequently. Assume the initial quantity is at Q_0 and change to Q_1 at period one.

$$V = \frac{Q_1 - Q_0}{Q_0} \quad (1\text{-}1)$$

or

$$1 - V = \frac{Q_1}{Q_0} \quad (1\text{-}2)$$

The equation 1-1 shows the rate of vacant unit changes, and $(Q_1\text{-}Q_0)$ shows the difference between two periods. Thus, (1-V) denotes the ratio of current vacant units to the level of the last period. It is assumed that vacant units curve follows the normal distribution. This is because the number of vacant unit should be a random variable depending on the equilibrium market activity. The random variable of vacant units therefore is said to be distributed as a standard normal variable or $N(0,1)$. The normal distribution equation is given as follows:

$$\phi(P_v) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(P_{vx}-P_{v0})^2}{2\sigma^2}} \qquad N(P_{v0},\sigma^2),\ (\sigma>0)\ (P_{vx} \geq 0) \tag{1-3}$$

Where P_{v0} and σ are equilibrium price of housing supply and standard deviation respectively.

There is relationship between the equilibrium price and vacant units. The probability of change in vacant units (V) at the change of equilibrium housing price can be denoted as:

$$\frac{dV}{dP_v} = \phi(P_v) \tag{1-4}$$

Where $\phi(P_v)$ measures the equilibrium price for at a period. Assume that at the initial vacant unit equal zero, when V=*0*, it may be

$$V = \frac{1}{\sigma\sqrt{2\pi}} \int_0^{P_v} e^{-\frac{(P_{vx}-P_{v0})^2}{2\sigma^2}} dP_v \qquad (P_v \geq 0) \tag{1-5}$$

The formula above denotes the impact of vacant units expressed by the probability of equilibrium price. To satisfy all points of the curve in theory, the equation 1-6 includes negative price, which does not occur in reality. Thus the price is assumed to be approximately zero.

$$\int_{-\infty}^{0} \phi(P_v) \qquad dP_v \approx 0 \tag{1-6}$$

To combine the Equation 1-5 and 1-6 together results the Equation 1-7 as follows.

$$V = \phi\left(\frac{P_{vx} - P_{v0}}{\sigma}\right) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{P_v} \phi(P_v) dP_v \qquad (P_v \geq 0) \tag{1-7}$$

The impact on vacant unit is tested as a normal distribution from the equation listed above. The descriptions above demonstrate that vacant unit is a function of equilibrium housing prices and is nonlinear in nature.

Appendix 2: Derived a Function of Vacant Units Curve

The variable Q_v is said to be distributed which can be denoted by:

$$f(Q_v) = \frac{1}{\sqrt{2\pi}} e^{-Q_v^2/2} \qquad (-\infty < Q_v < \infty) \tag{2-1}$$

(Mustoe & Barry, 1998). However, $f(Q_v)$ cannot be integrated analytically, so the area under the curve is tabulated in statistical tables only. It is therefore not convenience for practical application. Thus, it assumes the distribution of vacant units in term of Weibull's distribution density function. That is

$$\phi(P_e) = \frac{m}{P_{v0}}(P_v^{m-1})e^{-(\frac{P_v^m}{P_{v0}})} \quad (2\text{-}2)$$

Where m is a shape parameter and P_{v0} is a measure of equilibrium housing price, a constant number.

According to equation 1-4, $\frac{dV}{dP_v} = \phi(P_v)$, the expression for vacant unit parameter is

$$V = \int_0^{P_v} \phi(P_v)dP_v$$

$$= \frac{m}{P_{v0}} \int_0^{P_v} P_v^{m-1} e^{-(\frac{P_v^m}{P_{v0}})} dP_v \quad (2\text{-}3)$$

$$= 1 - e^{-(\frac{P_v^m}{P_{v0}})}$$

When $m=1$, the equation becomes

$$V = 1 - e^{-\frac{P_v}{P_{v0}}} \quad (2\text{-}4)$$

Assume that for a given period, the vacant units and the highest the equilibrium price are given, their relationship can be expressed as follows:

$$P_v = \lambda_v Q_v e^{-\frac{Q_v}{Q_{vm}}} \quad (2\text{-}5)$$

λ_v is slope of vacant units at initial level and Q_{vm} is change of vacant units corresponding to the maximum level of price at a given period. In equation 2-5, there exists an inflection in the curve at the point $Q_v^* = 2Q_{vm}$, the absolute value of its slope is $\lambda_v^* = \lambda_v e^{-2}$. Q_v^* *and* λ_v^* are the inflection point and the corresponding slope respectively.

Appendix 3:The Deriving Process of the Potential Function of Housing Prices

Refer to Figure 15,

$$P_e = \lambda_e Q_e \tag{3-1}$$

$$V_{Q_e} = \int_0^{Q_e} f(Q_e)dQ_e = \frac{1}{2}P_e Q_e = \frac{1}{2}\lambda_e Q_e^2 \tag{3-2}$$

$$P_v = \lambda_v Q_v e^{-\frac{Q_v}{Q_{vm}}} \qquad \text{(Appendix 2)} \tag{3-3}$$

$$V_{Q_v} = \int_{Q_v}^{Q_{vm}} f(Q_v)dQ_v \tag{3-4}$$

$$V_{Q_v} = \int_{Q_v}^{Q_{vm}} \lambda_v Q_v e^{-\frac{Q_v}{Q_{vm}}} dQ_v \tag{3-5}$$

$$V_{Q_v} = \lambda_v \int_{Q_v}^{Q_{vm}} Q_v e^{-\frac{Q_v}{Q_{vm}}} dQ_v \tag{3-6}$$

$$V_{Q_v} = -\lambda_v Q_{vm} \int_{Q_v}^{Q_{vm}} Q_v de^{-\frac{Q_v}{Q_{vm}}} \tag{3-7}$$

$$V_{Q_v} = -\lambda_v Q_{vm}\left[Q_v e^{-\frac{Q_v}{Q_{vm}}} \Big|_{Q_v}^{Q_{vm}} - \int_{Q_v}^{Q_{vm}} e^{-\frac{Q_v}{Q_{vm}}} dQ_v \right] \tag{3-8}$$

$$V_{Q_v} = -\lambda_v Q_{vm}\left[Q_{vm}e^{-1} - Q_v e^{-\frac{Q_v}{Q_{vm}}} + Q_{vm}\int_{Q_v}^{Q_{vm}} de^{-\frac{Q_v}{Q_{vm}}} \right] \tag{3-9}$$

$$V_{Q_v} = -\lambda_v Q_{vm}\left[Q_{vm}e^{-1} - Q_v e^{-\frac{Q_v}{Q_{vm}}} + Q_{vm} e^{-\frac{Q_v}{Q_{vm}}} \Big|_{Q_v}^{Q_{vm}} \right] \tag{3-10}$$

$$V_{Q_v} = -\lambda_v Q_{vm}\left[Q_{vm}e^{-1} - Q_v e^{-\frac{Q_v}{Q_{vm}}} + Q_{vm}e^{-1} - Q_{vm} e^{-\frac{Q_v}{Q_{vm}}} \right] \tag{3-11}$$

$$V_{Q_v} = -\lambda_v Q_{vm}\left[\frac{2Q_{vm}}{e} - (Q_v + Q_{vm})e^{-\frac{Q_v}{Q_{vm}}} \right] \tag{3-12}$$

$$\approx -\lambda_v Q_{vm}$$

$$\because e \approx 2.718281828459$$

Also, assume that equilibrium curve is positive, so that the impact on vacant units should be negative. Therefore, the equation 3-12 becomes

$$V_{Q_v} = \lambda_v Q_{vm}\left[Q_{vm} - (Q_v + Q_{vm})e^{-\frac{Q_v}{Q_{vm}}} \right] \tag{3-13}$$

$$
\begin{aligned}
V_{Q_T} &= V_{Q_e} + V_{Q_v} \\
&= \int_0^{Q_e} f(Q_e)dQ_e + \int_{Q_v}^{Q_{vm}} f(Q_v)dQ_v \\
&= \frac{1}{2}\lambda_e Q_e^2 + \lambda_v Q_{vm}\left[Q_{vm} - (Q_v + Q_{vm})e^{-\frac{Q_v}{Q_{vm}}}\right]
\end{aligned}
\tag{3-14}
$$

Appendix 4: The Taylor Series Expansion

Given the function *f(x)*, the series expansion at the point *x=a* is

$$
\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} a_n (x-a)^n \\
&= f(a_0) + f'(a_0)(x-a_0) + \frac{f''(a_0)}{2!}(x-a_0)^2 \\
&+ \frac{f^{(n)}(a_0)}{n!}(x-a_0)^n + R_a(x)
\end{aligned}
$$

Where $a_n = \frac{1}{n!} f^{(n)}(a)$, *n=0, 1, 2, …* and

$0! = 1$, $f^{(0)}(a) = f(a)$.

Appendix 5: The Deriving of Catastrophe Model

The Equation 18 gives

$$
\begin{aligned}
&\lambda_v^* Q_v^* e^{-\frac{Q_v^*}{Q_{vm}}} - \lambda_e (Q_T - Q_v^*) + \\
&\left[\lambda_v^* (1 - \frac{Q_v^*}{Q_{vm}}) e^{-\frac{Q_v^*}{Q_{vm}}} + \lambda_e\right](Q_v - Q_v^*) + \\
&\left[\frac{\lambda_v^*}{2Q_{vm}} (\frac{Q_v^*}{Q_{vm}} - 2) e^{-\frac{Q_v^*}{Q_{vm}}}\right](Q_v - Q_v^*)^2 + \\
&\left[\frac{\lambda_v^*}{6Q_{vm}^2} (3 - \frac{Q_v^*}{Q_{vm}}) e^{-\frac{Q_v^*}{Q_{vm}}}\right](Q_v - Q_v^*)^3 = 0
\end{aligned}
\tag{5-1}
$$

Assume

$$
A = \frac{\lambda_v^*}{6Q_{vm}^2} (3 - \frac{Q_v^*}{Q_{vm}}) e^{-\frac{Q_v^*}{Q_{vm}}}
\tag{5-2}
$$

$$B = \frac{\lambda_v^*}{2Q_{vm}}\left(\frac{Q_v^*}{Q_{vm}} - 2\right)e^{-\frac{Q_v^*}{Q_{vm}}} \quad (5\text{-}3)$$

$$C = \lambda_v^*\left(1 - \frac{Q_v^*}{Q_{vm}}\right)e^{-\frac{Q_v^*}{Q_{vm}}} + \lambda_e \quad (5\text{-}4)$$

$$D = \lambda_v^* Q_v^* e^{-\frac{Q_v^*}{Q_{vm}}} - \lambda_e(Q_T - Q_v^*) \quad (5\text{-}5)$$

The Equation (5-1) is

$$A(Q_v - Q_v^*)^3 + B(Q_v - Q_v^*)^2 + C(Q_v - Q_v^*) + D = 0 \quad (5\text{-}6)$$

Because $Q_v^* = 2Q_{vm}$, thus $Q_{vm} = \frac{Q_v^*}{2}$ (5-7)

Insert Equation (5-6) into A, B, C and D that can be changed to:

$$A = \frac{\lambda_v^*}{6\left(\frac{Q_v^*}{2}\right)^2}\left(3 - \frac{Q_v^*}{\frac{Q_v^*}{2}}\right)e^{-\frac{2Q_v^*}{Q_v^*}} = \frac{2\lambda_v^*}{3(Q_v^*)^2}e^{-2} \quad (5\text{-}8)$$

$$B = \frac{\lambda_v^*}{2\left(\frac{Q_v^*}{2}\right)}\left(\frac{Q_v^*}{\frac{Q_v^*}{2}} - 2\right)e^{-\frac{2Q_v^*}{Q_v^*}} = 0 \quad (5\text{-}9)$$

$$C = \lambda_v^*\left(1 - \frac{2Q_v^*}{Q_v^*}\right)e^{-\frac{2Q_v^*}{Q_v^*}} + \lambda_e = \lambda_e - \lambda_v^* e^{-2} \quad (5\text{-}10)$$

$$D = \lambda_v^* Q_v^* e^{-2} - \lambda_e(Q_T - Q_v^*) \quad (5\text{-}11)$$

Thus, the Equation (5-6) becomes to

$$\left[\lambda_v^* Q_v^* e^{-2} - \lambda_e(Q_T - Q_v^*)\right] + \left(\lambda_e - \lambda_v^* e^{-2}\right)(Q_v - Q_v^*) + 0 + \left(\frac{2\lambda_v^*}{3(Q_v^*)^2}e^{-2}\right)(Q_v - Q_v^*)^3 = 0 \quad (5\text{-}12)$$

Rearrange each section

$$\left[\lambda_v^* Q_v^* e^{-2} - \lambda_e (Q_T - Q_v^*)\right] + \left[Q_v^* (\lambda_e - \lambda_v^* e^{-2})(\frac{Q_v - Q_v^*}{Q_v^*})\right] + \left[(Q_v^*)(\frac{2\lambda_v^* e^{-2}}{3})(\frac{Q_v - Q_v^*}{Q_v^*})^3\right] = 0 \qquad (5\text{-}13)$$

Assume $\frac{Q_v - Q_v^*}{Q_v^*} = x$

The Equation (5-13) becomes

$$Ax^3 + Cx + D = 0 \qquad (5\text{-}14)$$

Dividing A in the each side of Equation (5-14):

$$x^3 + \frac{C}{A}x + \frac{D}{A} = 0 \qquad (5\text{-}15)$$

Assume *C/A=p* and *D/A=q*, thus

$$x^3 + px + q = 0 \qquad (5\text{-}16)$$

Assume $e^{-2} \approx 1$

$$\frac{C}{A} = p = \frac{Q_v^*(\lambda_e - \lambda_v^* e^{-2})}{\frac{2}{3}Q_v^* \lambda_v^* e^{-2}} = \frac{3}{2}(\frac{\lambda_e}{\lambda_v} - 1) \qquad (5\text{-}17)$$

$$\frac{D}{A} = q = \frac{[\lambda_v Q_v^* e^{-2} - \lambda_e(Q_T - Q_v^*)]}{\frac{2}{3}Q_v^* \lambda_v^* e^{-2}} = \frac{3}{2}[1 - \frac{\lambda_e}{\lambda_v^*}(\frac{Q_T - Q_v^*}{Q_v^*})] \qquad (5\text{-}18)$$

Assume $K^* = \frac{\lambda_e}{\lambda_v^*}$, $\therefore p = \frac{3}{2}(K^* - 1)$ (5-19)

Assume $\xi = \frac{Q_T - Q_v^*}{Q_v^*}$ $\therefore q = \frac{3}{2}(1 - K^*\xi)$ (5-20)

Appendix 6: Deriving of ξ

Given $p = \frac{3}{2}(K - 1)$ (6-1)

$$q = \frac{3}{2}(1 - K\xi) \qquad (6\text{-}2)$$

$$4p^3 + 27q^2 = 0 \qquad (6\text{-}3)$$

Equation (6-1) and (6-2) insert to (6-4):

$$4\left[\frac{3}{2}(K-1)\right]^3+27\left[\frac{3}{2}(1-K\xi)^2\right]=0$$

$$\frac{27}{2}(K-1)^3+\frac{3^5}{2^2}(1-K\xi)^2=0$$

$$(1-K\xi)^2=\frac{2}{9}(1-K)^3 \qquad (6\text{-}4)$$

$$1-K\xi=\frac{\sqrt{2}}{3}(1-K)^{\frac{3}{2}}$$

$$\xi=\frac{1}{K}\left[1-\frac{\sqrt{2}}{3}(1-K)^{\frac{3}{2}}\right]$$

Appendix 7: Empirical Data Preprocessing

Housing price: Yearly private housing price index collected from the Monthly Digest Statistics in Hong Kong government publications for the period from 1980 to 2000. Real price (1989=100) are calculated by dividing consumer price index (group C, October 1999 – September 2000 = 100) for each period. The change of price is taken difference, *i.e.*, $HP_d = HP_t - HP_{t-1}$.

Take-up Units (Q_e) and Vacant units (Q_v): They are used as proxy of equilibrium quantity and vacant units. The year data are preprocessed as follows:

1) Differencing (Q_d): For example, $Q_d = Q_e - Q_{e-1}$;
2) Calculating slope: $S=Q_d/HP_d$
3) Taking slope: T=ATAN(S)
4) Calculating λ_e = T/3.14*180

To simplify, similar procedures apply for vacant units.

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