Experiments in the Dynamics of Phase Coupled Oscillators When Applied to Graph Colouring

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Abstract

This paper examines the capacity of networks of phase coupled oscillators to coordinate activity in a parallel, distributed fashion. To benchmark these networks of oscillators, we present empirical results from a study of the capacity of such networks to colour graphs. We generalise the update equation of Aihara et al. (2006) to an equation that can be applied to graphs requiring multiple colours. We find that our simple multi-phase model can colour some types of graphs, especially complete graphs and complete k-partite graphs with equal or a near equal number of vertices in each partition. A surprising empirical result is that the effectiveness of the approach appears to be more dependent upon the topology of the graph than the size of the graph.

Keywords: graph colouring, phase coupled oscillators.

1 Introduction

Observations of the phenomena of coupled oscillators date back to the early seventeenth century, when Christiaan Huygens noticed that the pendula of two of his clocks, suspended side-by-side, always settled into swinging in opposite directions, even after he disturbed the position of the pendula (Bennett *et al*., 2002; Strogatz, 2003). In 1680, Engelbert Kaempfer reported another form of phased coupled oscillation, in the synchronous flashing of hundreds of fireflies on trees along the Chao Phraya River in Thailand (Buck & Buck, 1976). Many similar instances of naturally occurring synchronization have since been discovered, such as in heart pacemaker cells and in neural networks (Camazine *et al*., 2001).

Fireflies generate light from the lantern in the abdomen; it usually takes about 800 milliseconds to recharge the lantern and 200 milliseconds to produce a spark; the process may then repeat. Formal models of this behaviour describe a single firefly as an oscillator with a phase $0 \leq \theta \leq 2\pi$ and period $\acute{\omega}$. For a large proportion of each cycle, the oscillator is recharging and therefore discharging is impossible. For the remaining portion of the cycle, the firefly/oscillator is ready to discharge or “fire”. If the firefly/oscillator is operating in isolation from other firefly/oscillators, then it fires at $\theta = 2\pi$. If a firefly/oscillator is not operating in isolation, has completed recharging, and sees sufficient light (stimulus) from neighbouring fireflies, the firefly/oscillator can adjust its phase slightly so as to bring itself closer to synchronization with the other firefly/oscillators (Camazine et al., 2001). Mirollo & Strogatz (1990) demonstrated, by mathematical proof and computer simulation, the conditions under which a fully connected network of oscillators will synchronise.

Networks of oscillators have properties that make them an interesting approach to coordinating activity in large networks of simple computational elements. First, the synchronization mechanism of the oscillators is parallel and distributed – no global coordination is required. Second, the oscillators can be implemented in hardware with very simple circuitry, making it a promising approach for massive networks of tiny processing elements. In fact, the approach has already received some attention for synchronization in ad-hoc sensor networks (Hong & Scaglione, 2003; Lucarelli & Wang, 2004; Werner-Allen et al., 2005), and the coordination of multi agent systems (Bettstetter, 2006; Spong, 2006).

1.1 Anti-phase Synchronisation with Two Oscillators

Phase coupling need not be confined to phase synchronisation (i.e. where the phase difference of oscillators is 0). In some applications, the desired effect may be to have the computational elements differentiate into two or more groups. One of the simplest models of anti-phase synchronisation was studied by Aihara et al. (2006). They studied the mating calls of rain frogs, which they modelled as a network of exactly two oscillators, where the oscillators were intended to interact in such a way that they would settle into having a phase difference of π.

In the Aihara *et al*. model, the two frogs/oscillators are denoted a and b. The phases of the frogs/oscillators are denoted θ_a, θ_b with the respective frogs calling when their phase is zero, and the frequency of the oscillators are denoted ω_a, ω_b. The dynamic of oscillator a in isolation from oscillator b is described by:

$$\frac{d(\theta_a)}{dt} = \omega_a \qquad \ldots (1)$$

By extending Ermentrout & Rinzel (1984)'s model of firefly entrainment, Aihara *et al*. specified the interaction between the two oscillators as follows:

$$\begin{aligned} \frac{d(\theta_a)}{dt} &= \omega_a - g(\theta_b - \theta_a - \alpha) \\ \frac{d(\theta_b)}{dt} &= \omega_b - g(\theta_a - \theta_b - \beta) \end{aligned} \qquad \ldots (2)$$

Where g is a 2π periodic function, and the constants α and β are frustration parameters (we will assume $\alpha = \beta$). Typically g is $K\sin(\theta_a - \theta_b - \beta)$ where the constant K is the coupling strength

Aihara *et al*. were able to show that a stable equilibrium phase difference $\varphi = \theta_a - \theta_b$ of π exists between the two oscillators provided $|\omega_a - \omega_b| << K$.

We performed a computer simulation of the Aihara *et al.* model. The phases of the two oscillators were randomly initialised and we used a coupling strength K of *0.1.* The waveforms in Figures 1 and 2 show the phases of the two oscillators, where they are initially out of phase by approximately 0.4 of a radian, finally reaching a stable anti-phase difference of ~3.14 from the 14th cycle onwards.

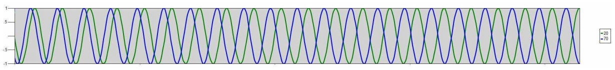

Figure 1: Evolution of the phases of two oscillators, which are eventually out of phase by π.

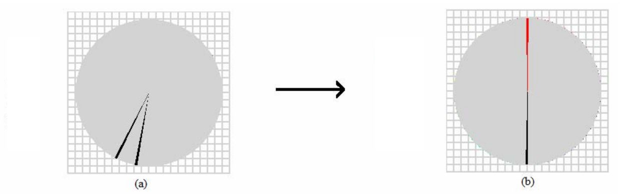

Figure 2: An alternative visualisation of the simulation from Figure 1. The wheel on the left shows the initial phase of each oscillator, which are similar. The wheel on the right shows the simulation at a later stage, when the phases of the two oscillators are separated by π.

1.2 Graph Colouring

In this paper, we further investigate the computational power of networks of oscillators. Like Hopfield & Tank (1985, p. 142), we believe that the computational power of such networks is best characterized by studying the behaviour of such networks when applied to difficult but well understood combinatorial optimization problems. Consequently, we have chosen to study the dynamics of networks of oscillators when applied to graph colouring.

The task of colouring a graph involves an assignment of colours to vertices in the graph such that no two vertices that share an edge have the same colour (Garey & Johnson, 1979). In our models, each node of the graph is an oscillator. Two oscillators are coupled if the respective nodes in the graph are connected. The colour of a node is represented by the phase of the oscillator. To visualise the graph colourings, we use a colour scheme that maps the phase of the oscillator in a 2π periodic system to a colour in the RGB (Red, Green, Blue) domain.

Wu (2002) conducted some simple experiments using oscillators to perform graph colouring. He conducted computer simulations for 300 graphs with the number of vertices ranging from 4 to 16, and where all graphs were known to be 2- or 3-colourable. His system coloured all 2-colourable graphs correctly, with a single exception, and also coloured approximately 80% of the 3-colourable graphs correctly. However, there are several limitations in Wu's study:

- Wu's approach to graph colouring was hybrid, where an initial colouring from the oscillators was subsequently "cleaned up" by an algorithm. Since our interest is in using oscillators to coordinate real networks, a hybrid approach is not practical: we need a purely parallel, distributed algorithm.
- Wu did not consider problems where more than three colours are required.
- Wu did not consider the effect of the graph topology on the effectiveness of the network of oscillators.
- Wu only considered the final state of his system, not the dynamics leading to the final state.

In this paper, we address these limitations in Wu's study. Furthermore, we generalise the Aihara *et al.* model so that it can be applied to more general graph colouring problems.

2 Two-Colouring in a Plane: The Ising Model

As a preliminary experiment, we chose to apply the Aihara *et al.* model to a simple and very well understood 2-colouring problem, the two-dimensional Ising Spin Problem (Kindermann & Snell, 1980). In this problem, which is illustrated in Figure 3, the nodes of the graph can be thought of as squares in a plane. Two nodes of the graph are adjacent if the corresponding squares share a common edge; therefore, each node is connected to four other nodes. It is obvious that such a graph can be 2-coloured, as shown in Figure 3.

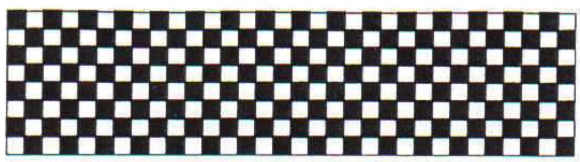

Figure 3: An example of the Ising Spin problem.

The Ising Spin problem is an interesting benchmark for two reasons. First, a planar mesh of computational elements is a realistic model of how a network of oscillators may be organised. Second, while the optimal solution is obvious, simple distributed algorithms – where each computational element can only "see" its four neighbours – do not reliably produce the optimal 2-colouring. For example, Lister (1992) showed that, for an 8×32 problem, like that shown in Figure 3, a simple iterative improvement algorithm only produces the optimal 2-colouring 15% of the time.

In performing our benchmark of networks of oscillators, we implemented the Aihara *et al*. anti-phase model, applying the equations from (2) above to each pair of connected oscillators.

Figure 4 illustrates six "snap shots" from a typical simulation of the system (with K = 0.1). Snap shot (a) in the figure shows the initial state of the 32x8 configuration where the oscillators were randomly initialised to a phase. The sequence of the states as indicated in part (b), (c), (d), (e) and (f) are the states of the oscillators after 150, 300, 450, 600 and 800 oscillator cycles. The final state of the system, as shown in (f) is the optimal solution, with all oscillators in anti-phase with their neighbours.

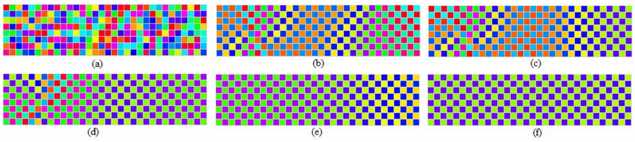

Figure 4: Stages in the convergence of the Ising model

Detailed examinations of the progress of the 32x8 oscillators demonstrate that the oscillators congregate into a number of groups and these groups slowly merge. For example, there are about 6 distinct groups in figure (b) as identifiable by the colour. The six groups begin to consolidate and increase in size in figure (c). In figure (d), two nearly synchronised groups start to dominate the right half of the network and in figure (e) the synchronised group converts the remaining oscillators on the left.

We ran the above simulation 100 times. Seventy-seven of the runs reached a global synchronised state after 1300 cycles. Investigation on the remaining twenty-three runs show that the oscillators in those runs form limit cycles. That is, the oscillators change their phases in a way that eventually brings them back to an identical set of phases; these changes then repeat. Figure 5 shows such a sequence of phases. Configuration (f) in the figure is identical to configuration (a).

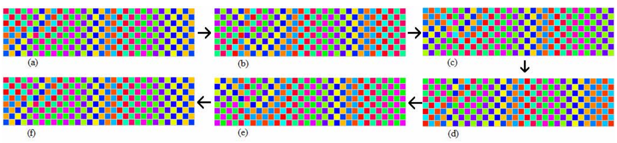

Figure 5: A limit cycle in a suboptimal run.

The occurrence of limit cycles is observable during simulations, as waves or rotating spiral-like patterns as shown in the following snapshots Figure 6. The simulation in this figure consists of 1024 vertices. The patterns can be seen from early in a simulation.

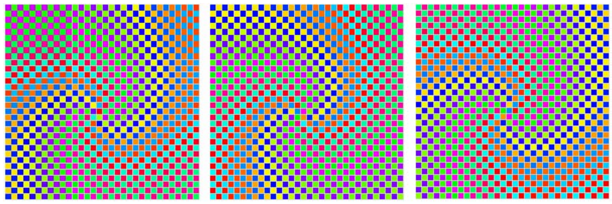

Figure 6: Rotating spirals in a simulation with 1024 vertices

3 Generalisation to a Multi-Phase Model

In order to perform an arbitrary *k*-colouring where *k* > 2, the phase coupled oscillators need to achieve a stable phase configurations with oscillators grouping into k phase-clusters. For example, for 3-fully connected oscillators, the phase difference between the oscillators should be near $2\pi/3$ (120 degrees). To admit such phase configurations, we generalised the Aihara *et al*. equations from section 1.1, as described in this section.

For a general case, where there are *n* fully connected interacting oscillators, we assume a mean field model (Kuramoto, 1984) to derive our generalisation of the Aihara *et al*. model:

$$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{N}\sum_{j=0}^{N}\sin(\theta_j - \theta_i - \beta), i = 1,..., N \qquad \cdots \ (3)$$

The frustration parameter β and the frequency ω are the same for all the oscillators in the system.

An attractive feature of this model is that there are no parameters that need to be tuned depending upon the number of colours required by a graph.

Below, we describe an empirical study that shows this model meets our initial requirement that the angle separation should be a multiple of 1/*n* for *n* fully connected oscillators.

3.1 Testing Multi-Phase Synchrony for n>=3

Fully connected networks are a realistic scenario to explore, as nodes connected by wireless could easily implement such a completely connected network topology, with the only necessary communication among the oscillators being a broadcast of their firings.

Figure 7 shows examples of colourings for small complete graphs. The leftmost portion of the diagram shows a graph, with 3 vertices. Next to it is a colour wheel showing the phases of the three oscillators, which are spread evenly, indicating a correct colouring. Beside that is another graph, with 4 vertices. Its associated colour wheel also shows that the phases of the oscillators evenly spread.

We have tested the update equation in (3) on complete graphs, up to *n*=100 vertices (larger graphs are not practical with our simulation software). We used a coupling strength *K*/*N*=0.1. We have found that our simulations reliably converge to good solutions for a wide variety of values of β, provided $\beta \geq 2\pi/n$.

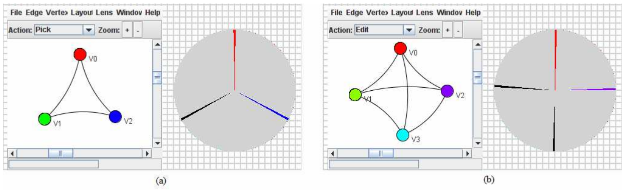

Figure 7: Angle separation in a 3 & 4 fully connected oscillators

The 3-colour problem in Figure 7 typically requires 20 oscillator cycles to reach a stable state, with a worst case of 30 oscillator cycles. The 4-colour problem in Figure 7 typically requires 50 oscillator cycles to give a reasonable phase difference between the oscillators, but requires an approximately 150 cycles to achieve a near perfect multiple $\pi/2$ phase difference.

Phase separation over a number of cycles is illustrated in Figure 8, for a complete graph of 8 nodes. The horizontal axis represents time. The vertical axis shows the phase of each of the eight oscillators when oscillator 0 fires. The phases of the oscillators are randomly initialised. Soon after the system starts, several oscillators are already in a near stable synchronisation. As the clock continues, the remaining pairs of oscillators (0 and 5) and (3 and 6) begin to separate evenly after 80 cycles. The oscillators ultimately synchronise at the 300th cycle with a near even phase difference of $\pi/4$.

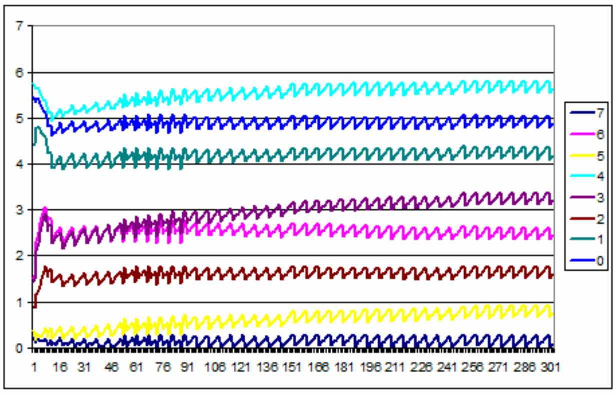

Figure 8: The phase of 8-fully connected oscillators over 300 cycles

4 Multi-phase Oscillators in Complete k-Partite Graphs

A complete *k-partite* graph has its vertices split into *k* partitions where (1) vertices in the same partitions are not connected, but (2) all nodes in each partition are connected to all nodes in the other partitions. Figure 9 illustrates k-partite graphs for k = 2, 3 and 7**.** Such graphs are an interesting case study to explore, as the optimal solutions are obvious (each partition requires one colour), and the results offer insight into the limitations of our generalisation of the Aihara *et al*. model.

4.1 Equal Complete k-Partite Graphs

The results from our experiment indicate that multi-phase coupled oscillators can reliably find a minimal graph colouring of complete k-partite graphs provided the number of vertices in each partition is equal. Figure 9 demonstrate colourings we have found using our update equation (3) for k-partite graphs with k=2, 3 and 7 using coupled oscillators. Colours are typically found in a small number of cycles and the system synchronises rapidly.

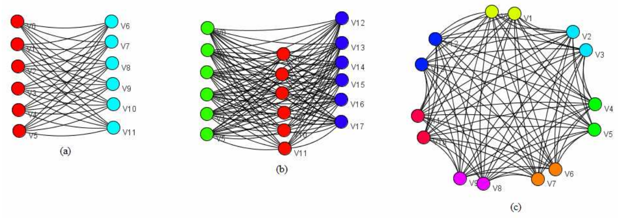

Figure 9: Complete 2, 3 and 7-partite graphs

4.2 Unequal Complete k-Partite Graphs

We performed tests where the number of vertices differs in the partitions of the complete k-partite graphs. We found that the quality of the results varies according to the size difference between partitions. Figure 10 demonstrates the colouring of k-partite graphs with unequal number of vertices in the partitions. Part (a) illustrates that good colourings can still occur if the number of oscillators in each partition is approximately equal, but part (b) demonstrates what happens as the size difference in the partitions grows.

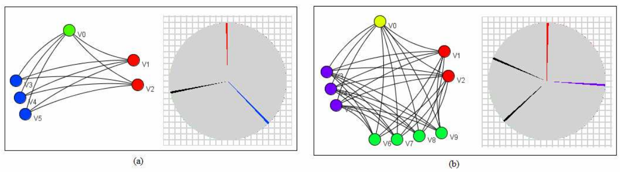

Figure 10: K-partite graphs with unequal number of vertices in each partition

4.3 A Further Illustration

A fundamental problem with colouring k-partite graphs with unequal partition sizes is more obviously illustrated on a simpler case that is not a k-partite graph. This case is illustrated in Figure 11. The graph shown can be thought of as containing two overlapping complete subgraphs of different sizes: vertices v_0-v_3 form one complete subgraph, (S1) and vertices v_3-v_5 form the other subgraph (S2). The four oscillators forming S1 tend to separate into equal phase differences corresponding to four colours (as illustrated by the leftmost colour wheel in the figure), while the three oscillators forming S2 tend to separate into equal phase differences corresponding to three colours (as illustrated by the middle colour wheel). The combined effect (as illustrated by the rightmost colour wheel) is a suboptimal solution.

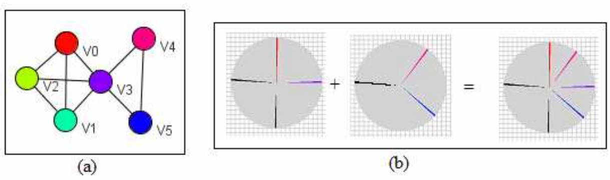

Figure 11: A simple illustration on two overlapping complete sub graphs

5 Multi-Phase Oscillators in a Plane: Experiments with Three Colours

Earlier, we examined the standard Ising Spin problem, where 2 colours are sufficient. This problem is easily generalised to forms that require 3 and 4 colours, by (for example) replacing the squares in the plane with tessellations of hexagons (3 colours) or adding extra connections to the squares so that the squares also connect diagonally (four colours).

We performed tests on four hexagonal topologies as shown in Figure 12. The top left graph in Figure 12 illustrates the simplest case for a hexagonal arrangement that can be 3-coloured. As the associated colour wheel illustrates, the oscillators forming this simple graph always synchronise with a minimum number of colours. For larger graphs, as illustrated in part (b), (c) and (d), there is an observable clustering of the oscillators phases into three groups (less obvious in part (d)), but the phases of the oscillators within those clusters remain separated.

The reason why oscillator phases remain separated is related to the reason why oscillators do not converge in k-partite graphs with unequal number of vertices in each partition. An inspection of the graphs in Figure 12 reveals that nodes in these graphs have unequal numbers of neighbours. Nodes at the periphery of the graphs can have as few as three neighbours, whereas nodes inside the graphs have as many as six neighbours. The dynamics of the update equation (3) has internal nodes and peripheral nodes having asymmetric effects on each other.

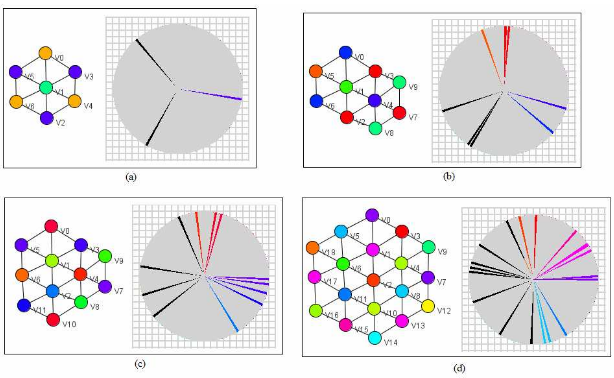

Figure 12: Colouring of hexagons

The graph colouring problem illustrated in Figure 12 scales to arbitrarily large numbers of nodes. As the size of such a network grows the ratio of peripheral elements to internal nodes decreases. Thus, the problem illustrated in Figure 12 may be less evident in large networks of computational elements.

6 Multi-phase Oscillators in Regular Graphs

The results described in the previous section, for tessellations of hexagons on a plane, show that the model does not colour the graph optimally. This suboptimal behaviour is at least partly due the unequal degree of vertices in the tessellation. For example, in Figure 12, the vertices on the fringe of the graphs typically have degree 3 or 4 while the inner vertices have degree 6.

To test whether the unequal degree of vertices completely explains such suboptimal behaviour, we performed tests on graphs where the vertices within each of the graphs have the same degree — we used graphs based on the Platonic solids and also the Ising Spin Problem on a torus.

6.1 Platonic Solids

Our experiments indicate that colourings of the first three Platonic solids, the tetrahedron, hexahedron and octahedron (all 3 and 4-regular graphs) are always optimal and are achieved in a small number of cycles. Figure 13 shows these results.

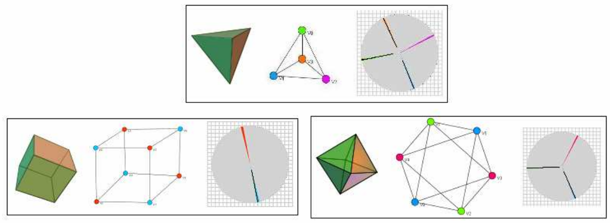

Figure 13: Colouring of simple solids

However, colourings are sub optimal for the dodecahedron (3-regular) and icosahedron (5-regular) as illustrated on Figure 14. Typically, the oscillators settle into 6 colours instead of the minimum of 3 and 4 respectively.

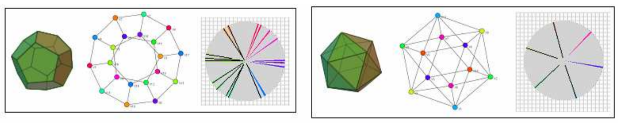

Figure 14: Colouring of dodecahedron & icosahedron

6.2 The Ising Model on a Torus

The torus formation of the Ising Spin Problem is achieved by taking a standard Euclidean Ising Spin Problem (as described in section 2), then connecting the upper and lower ends, and also the left and right ends. Consequently, all vertices have a degree of 4. This is illustrated in Figure 15. The results of 100 runs of such an 8×32 problem resulted in network convergence that was 50% faster than that of the 8×32 Euclidean Ising Spin Problem. However, only 64% of the runs attain an optimal synchronisation. Figure 15 illustrates a typical suboptimal solution, where subsets of the oscillators are optimal within their respective subsets, but the relationships between subsets is suboptimal.

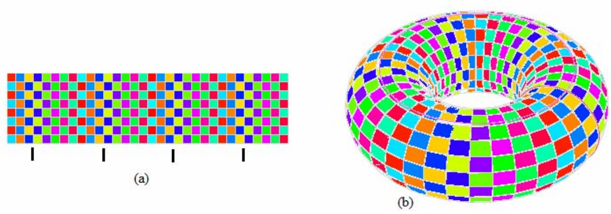

Figure 15: Sub optimal colouring on torus Ising

To illustrate this suboptimal behaviour further, we constructed a simple 1x7 Ising Spin Problem on a torus, which is a 2-regular ring graph. Ideally, the network should converge to 3 colours. Figure 16 shows a run of such a network at 200, 400, 600 and 800 oscillator cycles. At 200 cycles, the oscillators tend to form 2 clusters at π apart. As the run continues to 800 cycles, the oscillators spread out evenly. By way of contrast, in similar experiments with rings containing an even number of oscillators, the networks always converged to an optimal 2 colouring.

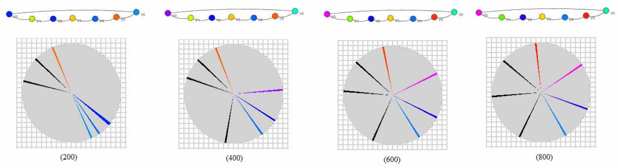

Figure 16: A simulation on a 1x7 ring

7 Characteristics of the Aihara Model

From our experiments, and as a direct consequence of the model in Equation (3), we observe the following general characteristics of networks of oscillators using our generalised Aihara model:

Observation 1: Convergence to a stable point in phase space does not imply a minimum colouring.

Observation 2: Convergence to a stable point in phase space does not imply an even phase separation among the oscillators.

Observation 3: The dynamic of the generalised Aihara equation is such that an oscillator connected to two other oscillators will move to a phase that is equidistant from the phase of the other two oscillators.

Observation 3': An oscillator O connected to two disjoint sets of oscillators, S1 and S2, where the oscillators within each set have the same phase, will move to a phase such that ratios of the phase separations from O to S1 and O to S2 will be proportional to the ratio of the sizes of S1 and S2.

Observation 4: The effectiveness of colouring graphs using networks of oscillators appears to be less dependent upon the size of the graph and more dependent on the graph topology — the degree of the vertices and the existence of odd or even cycles in the graph.

8 Conclusion

The purpose of carrying out this study was certainly not to find an algorithm guaranteed to minimally colour graphs – complexity theory suggests that such an algorithm does not exist. Instead, the purpose was to use graph colouring to benchmark the capacity of phase coupled oscillators to coordinate activity, in a parallel distributed fashion, within a network of simple computational elements. The results of our experiments clearly indicate that the basic oscillator phase coupling approach can effectively coordinate activity, in a parallel distributed fashion, in some types of graphs. Surprisingly, the size of graphs to be coloured is not the major determinant of effectiveness, but instead it is the topology of the graphs that most determines the effectiveness of this approach to graph colouring.

In this paper, our goal was to explore models close to the original biological source of the idea. Having identified some limitations of the pure biological approach, our future work will focus on overcoming these limitations using techniques that can be implemented in simple computational elements, without undermining the fundamental parallel, distributed nature of phase coupled oscillators. The remainder of the conclusion indicates some solutions to the problems identified in this paper.

The problem of suboptimal limit cycles, which was identified in the experiments on the standard Ising Spin problem, might be addressed by injecting a small amount of noise into the system (i.e. randomly perturbing the phase of oscillators).

A core issue to solve is the problem highlighted in the studies of k-partite graphs with unequal vertices in the partitions. One approach might be to decrease the effect of an oscillator as its phase approaches the phase of other oscillators. By a suitable formulation of the update equation, a group of closely synchronized oscillators could have the same effect on other oscillators as a single non-synchronized oscillator has on those other oscillators.

A solution to the problem highlighted in Figure 11 may only require (1) an initial global broadcast that communicates the number of colours (C) to be used in the colouring of a graph, and (2) a global agreement, via either a regular broadcasted synchronization signal, or via clocks aboard each processing element, that the final phase values of all oscillators will only differ by multiples of $2\pi/C$.

The characteristics of the Aihara model, and observation 3 in particular, indicate that a mechanism may be required whereby an oscillator can escape from having its phase trapped between the phases of two other oscillators. This might be achieved by introducing an annealing component into the way an oscillator alters its phase.

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