Homogenous and Heterogeneous Logical Proportions

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1 Introduction

Commonsense reasoning often relies on the perception of similarity as well as dissimilarity between objects or situations. Such a perception may be expressed and summarized by means of analogical proportions, i.e., statements of the form "A is to B as C is to D". Analogy is not a mere question of similarity between two objects (or situations), but rather a matter of proportion or relation between objects. This view dates back to Aristotle and was enforced by Scholastic philosophy. Indeed, an analogical proportion equates a relation between two objects with the relation between two other objects. As such, the analogical proportion "A is to B as C is to D" poses an analogy of proportionality by (implicitly) stating that the way the two objects A and B, otherwise similar, differ is the same way as the two objects C and D, which are similar in some respects, differ.

A propositional logic modeling of analogical proportions, viewed as a quaternary connective between the Boolean values of some feature pertaining to A, B, C, and D, has been recently proposed in [14]. This logical modeling amounts to precisely state that the difference between A and B is the same as the one between C and D, and that the difference between B and A is the same as the one between D and C. This view can then be proved to be equivalent to state that each time a Boolean feature is true for A and D (resp. A or D) it is also true for B and C (resp. B or C), and conversely. This latter point shows that a counterpart of a characteristic behavior of numerical geometrical proportions ($\frac{a}{b} = \frac{c}{d}$), or of numerical arithmetic proportions ($a - b = c - d$), namely that the product (resp. sum, in the second case) of the extremes is equal to the product (resp. the sum) of the means, is still observed in the logical setting.

However, analogical proportions are not the only type of quaternary statements relying on the ideas of similarity and dissimilarity that can be imagined. They turn out to be a special case of so-called *logical proportions* [17]. Roughly speaking, a logical proportion between four terms A, B, C, D equates similarity or dissimilarity evaluations about the pair (A, B) with similarity or dissimilarity evaluations about the pair (C, D). A set of 120 distinct logical proportions, whose formal expressions share the same structure as well as some remarkable properties, has been identified. Among them, 8 logical proportions stand out as being the only ones that enjoy a code independency property. Namely, their truth status remains unchanged when the truth values 0 and 1 are exchanged. These 8 proportions split into two groups, namely, 4 *homogeneous* ones (which include the analogical proportion) [22], and 4 *heterogeneous* logical proportions, which are dual in some sense of the former ones. The pairs (A, B) and (C, D) play symmetrical roles for homogeneous proportions, while it is not the case for the heterogeneous ones. However, both enjoy noticeable permutation properties.

Similarity and dissimilarity are naturally a matter of degrees. Thus, the extension of homogeneous and heterogeneous logical proportions when features are graded make sense in a multiple-valued logic setting. This makes these logical proportions closer to a symbolic counterpart of numerical proportions where the equality between ratios or differences of quantities may be approximate.

Besides, knowing three values, the statement of the equality of numerical ratios, or of numerical differences, involving a fourth unknown value, and expressing a proportionality relation, is useful for extrapolating this latter value. Similarly, the solving of logical proportion equations may be the basis of reasoning procedures. In particular, when an analogical proportion holds for a large number of features between four situations described by means of n binary features, one may make the plausible inference that the same type of proportion should also hold for a $(n+1)$th feature. If the truth value of this latter feature is known for three of the situations, and unknown for the fourth one, this value can thus be obtained as the solution of an analogical proportion equation.

The paper is organized as follows. In Section 2, the notion of logical proportions is introduced and formally defined. Then, a structural typology of the different families of logical proportions, as well as some noticeable properties, are presented. Section 3 is devoted to a more detailed study of homogeneous proportions. Section 4 deals with extensions of homogeneous proportions for handling non Boolean or unknown features. This is the case if the features are gradual, or if they are binary but may not apply. It may also happen that for some situations it is not known if a feature holds or not. The section investigates these three types of cases (gradual features, features non applicable, and missing information about a feature), where different multiple-valued logical calculi are involved. Section 5 focuses on heterogeneous proportions, studies their properties, and their extension to gradual properties. Section 6 discusses applications of homogeneous and heterogeneous proportions. Homogeneous logical proportions, especially analogical proportions, seem of interest for completing missing values in tables, a problem sometimes termed “matrix abduction” [1]. It amounts in the logical proportion setting to completing a series A, B, C with X such as (A, B, C, X) makes a proportion of a given type. Heterogeneous logical proportions are shown to be instrumental for picking out the item that does not fit in a list. Thus, the setting of logical proportions appears to be rich enough for coping with two different types of reasoning problems where the ideas of similarity and dissimilarity play a key role in both cases. Psychological quizzes or tests are used for illustrating this ability to exploit comparisons in reasoning.

This paper provides a synthesis of results that have appeared mostly in a series of papers by the authors [19, 18, 22].

2 Logical proportions

Before introducing the formal definitions, let us briefly clarify the notations used.

- When dealing with Boolean logic, $a, b, \ldots$ denote propositional variables (having 0 or 1 as truth value), and we use the standard symbols $\wedge, \vee$ to build up formulas (with parentheses when needed). For the negation operator, instead of using the standard $\neg$ symbol, we will use $\overline{a}$ to denote $\neg a$. This is done for saving space when writing long formulas. As usual $\top$ (resp. $\bot$) denotes the always true (resp. false) proposition.

- 0 and 1 denote the Boolean truth values, and a valuation v is just a function from the set of propositional variables to the set of truth values, i.e., $\{0, 1\}$ in the Boolean case, or $[0, 1]$ in the graded case.

- When we propose a new definition, we will use the symbol $\triangleq$ meaning definitional equality. The right hand side of the equation is the definition of the left hand side.

- When we consider syntactic identity, we use $=_{Id}$: for instance $a \wedge b =_{Id} a \wedge b$ but we do not have $a \wedge b =_{Id} b \wedge a$.

- Finally, the symbol $\equiv$ is reserved for the equivalence, i.e.,

$$a \to b \triangleq \overline{a} \vee b \qquad a \equiv b \triangleq (a \to b) \wedge (b \to a)$$

Logical proportions are Boolean formulas built upon what we called indicators. We introduce this concept in the next subsection and we investigate some fundamental properties.

2.1 Similarity and dissimilarity indicators

Generally speaking, the comparison of two items A and B relies on the representation of these items. For instance, the items may be represented as a set of features $\mathcal{A}$ and $\mathcal{B}$. Then, one may define a *similarity measure*. This is the aim of the well-known work of Amos Tversky [26], taking into account the common features, the specificities of A w.r.t. B, and the specificities of B w.r.t. A, respectively modeled by $\mathcal{A} \cap \mathcal{B}$, $\mathcal{A} \setminus \mathcal{B}$, and $\mathcal{B} \setminus \mathcal{A}$. Here, we are not looking for any global measure of similarity, we are rather interested in keeping track in what respect items are similar and in what respect they are dissimilar using Boolean indicators. This is why we adopt a logical setting: features are viewed as Boolean properties. Let P be such a property, which can be seen as a predicate: $P(A)$ may be true (in that case $\neg P(A)$ is false), or false.

When comparing two items A and B w.r.t. such a property P, it makes sense to consider A and B similar (w.r.t. property P):
- when $P(A) \wedge P(B)$ is true or
- when $\neg P(A) \wedge \neg P(B)$ is true.

In the remaining cases:
- when $\neg P(A) \wedge P(B)$ is true or
- when $P(A) \wedge \neg P(B)$ is true,

we can consider A and B as dissimilar w.r.t. property P.

Since $P(A)$ and $P(B)$ are ground formulas, they can simply be considered as Boolean variables, and denoted a and b by abstracting w.r.t. P. If the conjunction $a \wedge b$ is true, the property is satisfied by both items A and B, while the property is satisfied by neither A nor B if $\overline{a} \wedge \overline{b}$ is true. The property is true for A only (resp. B only) if $a \wedge \overline{b}$ (resp. $\overline{a} \wedge b$) is true. This is why we call such a conjunction of Boolean literals an *indicator*, and for a given pair of Boolean variables (a, b), we have exactly 4 distinct indicators:

- $a \wedge b$ and $\overline{a} \wedge \overline{b}$ that we call *similarity indicators*,
- $a \wedge \overline{b}$ and $\overline{a} \wedge b$ that we call *dissimilarity indicators*.

Let us observe that negating anyone of the two terms of a dissimilarity indicator turns it into a similarity indicator, and conversely. Hence, negating the two terms of an indicator yields an indicator of the same type.

2.2 Building logical proportions with indicators

When describing two elementary situations encoded by two Boolean variables a and b, one may use one of the four above indicators. Putting such a description in relation with what takes place with two other Boolean variables c and d in terms of some indicator, leads to state an equivalence between one indicator pertaining to the pair (a, b) and one indicator pertaining to the pair (c, d). However, one may consider that using *two* indicators to describe the status of 2 variables a and b may be more satisfactory from some symmetrization point of view than using only one indicator. For instance, using $\overline{a} \wedge b$ together with $a \wedge \overline{b}$ establishes the symmetry between a and b, or using $a \wedge \overline{b}$ together with $a \wedge b$ considers counter-examples as well as examples in context a, or using $\overline{a} \wedge \overline{b}$ together with $a \wedge b$ provides the same role to negative or positive features. Note that such symmetrizations occur for free with numerical proportions where for instance one can exchange a and b on the one hand, c and d on the other hand, still writing a unique equality. It is why we more particularly focus on proportions defined as the conjunction of *two distinct* equivalences between an indicator for the pair (a, b) and an indicator for the pair (c, d).

One may wonder about the simultaneous use of three indicators for comparing two Boolean variables. This would lead to three equivalences instead of two, which appears conceptually more complicated, and maybe farther from the idea of proportion inherited from the numerical setting. Then, for the sake of simplicity, we stick to the conjunctions of two equivalences between indicators in the following. This defines a so-called *logical proportion* [17, 19]. More formally, let us denote $I_{(a,b)}$ and $I'_{(a,b)}$[1] (resp. $I_{(c,d)}$ and $I'_{(c,d)}$) 2 indicators for (a, b) (resp. (c, d)). Then

Definition 1. *A logical proportion $T(a, b, c, d)$ is the conjunction of 2 distinct equivalences between indicators of the form*

$$I_{(a,b)} \equiv I_{(c,d)} \wedge I'_{(a,b)} \equiv I'_{(c,d)}$$

An example of such proportion is $((\overline{a} \wedge \overline{b}) \equiv (c \wedge \overline{d})) \wedge ((\overline{a} \wedge b) \equiv (\overline{c} \wedge d))$ where

- $I_{(a,b)} \triangleq \overline{a} \wedge \overline{b}, \quad I_{(c,d)} \triangleq c \wedge \overline{d},$
- $I'_{(a,b)} \triangleq \overline{a} \wedge b, \quad I'_{(c,d)} \triangleq \overline{c} \wedge d.$

Obviously, this formal definition goes beyond what may be expected from the informal idea of "logical proportion", since equivalences may be put between things that are not homogeneous (i.e., mixing similarity and dissimilarity indicators in various ways).

Let us first determine the number of logical proportions. To build an equivalence between indicators, we have to choose one indicator among four for the pair (a, b)

[1]Note that $I_{(a,b)}$ (or $I'_{(a,b)}$) refers to one element in the set $\{a \wedge b, \overline{a} \wedge b, a \wedge \overline{b}, \overline{a} \wedge \overline{b}\}$, and should not be considered as a functional symbol. Still, we use this notation for the sake of readability.

and similarly for the pair (c, d), we get $4 \times 4 = 16$ distinct equivalences. To build up a logical proportion, we first choose one equivalence among 16, and then the second equivalence has to be chosen among the 15 remaining ones, leading to $16 \times 15 = 240$ pairs of equivalences. Taking into account the commutativity of the Boolean conjunction, we finally get $240/2 = 120$ potentially distinct logical proportions . We shall see in subsection 2.4 that they are indeed distinct. We first provide a syntactic typology of the logical proportions.

2.3 Typology of logical proportions

Logical proportions can be classified according to the ways they are built up. At this stage, it makes sense to distinguish between two types of indicators: similarity indicators that are denoted by S, and dissimilarity indicators that are denoted by D: e.g., $D_{(a,b)} \in \{a \wedge \overline{b}, \overline{a} \wedge b\}$.

Depending on the way the indicators are chosen, one may mix the similarity and the dissimilarity indicators differently in the definition of a proportion.

This leads us to distinguish a specific subfamily of proportions, the so-called *degenerated proportions*: those ones involving only 3 distinct indicators in their definition. For instance

$$(a \wedge b \equiv \overline{c} \wedge d) \wedge (\overline{a} \wedge b \equiv \overline{c} \wedge d)$$

is such a proportion where $I_{(c,d)} =_{Id} I'_{(c,d)}$.

For the remaining proportions, it is required that all the indicators appearing in the definition of the proportion are distinct. At this stage, among the non-degenerated proportions, we can identify 4 subfamilies that we describe below:

- **The 4 homogeneous proportions**

  For these proportions, we do not mix different types of indicators in the 2 equivalences. The homogeneous proportions are of the form

  $$S_{(a,b)} \equiv S_{(c,d)} \wedge S'_{(a,b)} \equiv S'_{(c,d)}$$

  or

  $$D_{(a,b)} \equiv D_{(c,d)} \wedge D'_{(a,b)} \equiv D'_{(c,d)}$$

  Thus, it appears that only 4 proportions among 120 are homogeneous. They are (with their name):

  - *analogy* : $A(a, b, c, d)$, defined by

    $$((a \wedge \overline{b}) \equiv (c \wedge \overline{d})) \wedge ((\overline{a} \wedge b) \equiv (\overline{c} \wedge d))$$

- *reverse analogy*: $R(a,b,c,d)$, defined by

$$((a \wedge \overline{b}) \equiv (\overline{c} \wedge d)) \wedge ((\overline{a} \wedge b) \equiv (c \wedge \overline{d}))$$

- *paralogy* : $P(a,b,c,d)$, defined by

$$((a \wedge b) \equiv (c \wedge d)) \wedge ((\overline{a} \wedge \overline{b}) \equiv (\overline{c} \wedge \overline{d}))$$

- *inverse paralogy*: $I(a,b,c,d)$, defined by

$$((a \wedge b) \equiv (\overline{c} \wedge \overline{d})) \wedge ((\overline{a} \wedge \overline{b}) \equiv (c \wedge d))$$

Analogy already appeared under this form in [14]; paralogy and reverse analogy were first introduced in [16], and inverse paralogy in [19]. While the analogical proportion (analogy, for short) reads "a is to b as c is to d" and expresses that "a differs from b as c differs from d, and conversely b differs from a as d differs from c", reverse analogy expresses that "a differs from b as d differs from c, and conversely b differs from a as c differs from d", paralogy expresses that "what a and b have in common, c and d have it also" (positively and negatively). Paralogy is a given name. Finally, *inverse paralogy* expresses that "what a and b have in common, c and d miss it, and conversely". As can be seen, inverse paralogy expresses a form of antinomy between pairs (a,b) and (c,d). Note that we use two different words, "inverse" and "reverse", since the changes between analogy and reverse analogy on the one hand, and paralogy and inverse paralogy on the other hand, are not of the same nature. From now on, we denote analogy with A, reverse analogy with R, paralogy with P, inverse analogy with I. When we need to denote any unspecified proportion, we will use the letter T.

- **The 16 conditional proportions**

  Their expression is made of the conjunction of an equivalence between similarity indicators and of an equivalence between dissimilarity indicators. Thus, they are of the form

$$S_{(a,b)} \equiv S_{(c,d)} \wedge D_{(a,b)} \equiv D_{(c,d)}$$

  There are 16 conditional proportions (2×2 choices *per* equivalence). An example is

$$((a \wedge b) \equiv (c \wedge d)) \wedge ((a \wedge \overline{b}) \equiv (c \wedge \overline{d}))$$

Let us explain the term "conditional". It comes from the fact that these proportions express "equivalences" between conditional statements. Indeed, it has been advocated in [5] that a rule "if a then b" can be seen as a three valued entity that is called 'conditional object' and denoted $b|a$ [4]. This entity is:

- true if $a \wedge b$ is true. The elements making it true are the examples of the rule "if a then b",
- false if $a \wedge \overline{b}$ is true. The elements making it true are the counter-examples of the rule "if a then b",
- undefined if $\overline{a}$ is true. The rule "if a then b" is then not applicable.

Thus, the above proportion $((a \wedge b) \equiv (c \wedge d)) \wedge ((a \wedge \overline{b}) \equiv (c \wedge \overline{d}))$ may be denoted $b|a :: d|c$ combining the two conditional objects in the spirit of the usual notation for analogical proportion. Indeed, it expresses a semantical equivalence between the 2 rules "if a then b" and "if c then d" by stating that they have the same examples, i.e. $(a \wedge b) \equiv (c \wedge d))$ and the same counter-examples $(a \wedge \overline{b}) \equiv (c \wedge \overline{d})$.

It is worth noticing that such proportions have equivalent forms, e.g.:

$$(b|a :: d|c) \equiv (\overline{b}|a :: \overline{d}|c)$$

which agrees with the above semantics and more generally with the idea of conditioning. Indeed the examples "if a then b" are the counter-examples of "if a then $\overline{b}$", and vice-versa. Due to this remark, it is enough to consider the equivalences between one of the 4 conditional objects $a|b$, $b|a$, $a|\overline{b}$, $b|\overline{a}$, and the 4 other conditional objects built with (c, d), yielding 4×4 proportions as expected. Besides, 8 conditional proportions have been first considered in [19], but not the 8 remaining ones, since they do not satisfy the "full identity" property, discussed in the next section.

- **The 20 hybrid proportions**

  They are characterized by equivalences between similarity and dissimilarity indicators in their definitions. They are of the form.

$$S_{(a,b)} \equiv D_{(c,d)} \wedge S'_{(a,b)} \equiv D'_{(c,d)}$$

or

$$D_{(a,b)} \equiv S_{(c,d)} \wedge D'_{(a,b)} \equiv S'_{(c,d)}$$

or

$$S_{(a,b)} \equiv D_{(c,d)} \wedge D_{(a,b)} \equiv S_{(c,d)}.$$

There are 20 hybrid proportions: 2 of the first type, 2 of the second type, 16 of the third type since we have here 4 choices for an equivalence $S_{(a,b)} \equiv D_{(c,d)}$, and 4 choices for $D_{(a,b)} \equiv S_{(c,d)}$.

If we remember that negating anyone of the two terms of a dissimilarity indicator turns it into a similarity indicator, and conversely, we understand that changing a into $\overline{a}$ (and $\overline{a}$ into a), or applying a similar transformation with respect to b, c, or d, turns

- an hybrid proportion into an homogeneous or a conditional proportion;

- an homogeneous or a conditional proportion into an hybrid proportion.

This indicates the close relationship of hybrid proportions with homogeneous and conditional proportions. More precisely,

- on the one hand there are 4 hybrid proportions such that replacing a with $\overline{a}$ leads to the 4 homogeneous proportions A, R, P, I. They are obtained by the two first kinds of patterns for building hybrid proportions. Moreover, we shall see in the next section that they constitute with the 4 homogeneous proportions the 8 proportions that are the only ones satisfying "code independency" property.

- on the other hand, there are 16 remaining hybrid proportions, obtained by the third kind of pattern for building them. They can be written as the equivalence of 2 conditional objects, although they do not obey the conditional proportion pattern. For instance, $((\overline{a} \wedge b) \equiv (c \wedge d)) \wedge ((a \wedge b) \equiv (\overline{c} \wedge d))$ can be written as $\overline{a}|b :: c|d$. This proportion is indeed obtained from the conditional proportion $a|b :: c|d$ by changing a into $\overline{a}$. Thus, these 16 new equivalences between conditional objects are not of the form $a|b :: c|d$ (or equivalently $\overline{a}|b :: \overline{c}|d$) produced by the pattern of conditional proportions, but of a "mixed" form having an odd number of negated terms.

- **The 32 semi-hybrid proportions**

One half of their expressions involve indicators of the same type, while the other half requires equivalence between indicators of opposite types. They are of the form

$$S_{(a,b)} \equiv S_{(c,d)} \wedge S'_{(a,b)} \equiv D_{(c,d)}$$

or

$$S_{(a,b)} \equiv S_{(c,d)} \wedge D_{(a,b)} \equiv S'_{(c,d)}$$

or

$$D_{(a,b)} \equiv D_{(c,d)} \wedge S_{(a,b)} \equiv D'_{(c,d)}$$

or

$$D_{(a,b)} \equiv D_{(c,d)} \wedge D'_{(a,b)} \equiv S_{(c,d)}$$

There are 32 semi-hybrid proportions (8 of each kind: 4 choices for the first equivalence, times 2 choices for the element that is not of the same type as the three others (D or S) in the second equivalence). An example of semi-hybrid proportion is $((a \wedge b) \equiv (c \wedge d)) \wedge ((\overline{a} \wedge \overline{b} \equiv (\overline{c} \wedge d))$.

Applying a change from a to $\overline{a}$ (and $\overline{a}$ to a), or applying a similar transformation with respect to b, c, or d, turns a semi-hybrid proportion into a semi-hybrid proportion (since as already said, negating anyone of the two terms of a dissimilarity indicator turns it into a similarity indicator, and conversely). This contrasts with the hybrid proportion class which is not closed under such a transformation.

- **The 48 degenerated proportions**

  In all the above categories, the 4 indicators related by equivalence symbols should be all distinct. In degenerated proportions, there are only 3 different indicators and it is simpler to come back to our initial notation. With this notation, these proportions are of the form

$$I_{(a,b)} \equiv I_{(c,d)} \wedge I_{(a,b)} \equiv I'_{(c,d)}$$

or

$$I_{(a,b)} \equiv I_{(c,d)} \wedge I'_{(a,b)} \equiv I_{(c,d)}$$

Their number is easy to compute: we have to choose $I_{(a,b)}$ among 4 indicators and then to choose 2 distinct indicators among 4 pertaining to (c, d): we then get $4 * 6 = 24$ proportions of the first form. The same reasoning with the second kind of expression leads to a total of 48 degenerated proportions. Note that the change from a to $\overline{a}$ (and $\overline{a}$ to a), or a similar transformation with respect to b, c, or d, turns a degenerated proportion into a degenerated proportion.

It can be seen that degenerated proportions always involve a mutual exclusiveness condition between 2 positive or negative literals pertaining to either the pair (a, b) or the pair (c, d). Indeed, if we consider the first form, we get $I_{(a,b)} \equiv I_{(c,d)}$ on the one hand, and $I_{(c,d)} \equiv I'_{(c,d)}$ on the other hand, i.e. an equivalence between two syntactically distinct indicators pertaining to the same pair (c, d). There are 6 cases only:

- $(\overline{c} \wedge d) \equiv (c \wedge \overline{d})$ iff $c \equiv d$
- $(c \wedge d) \equiv (\overline{c} \wedge \overline{d})$ iff $c \equiv \overline{d}$
- $(c \wedge d) \equiv (c \wedge \overline{d})$ iff $c \equiv \bot$
- $(c \wedge d) \equiv (\overline{c} \wedge d)$ iff $d \equiv \bot$
- $(\overline{c} \wedge d) \equiv (\overline{c} \wedge \overline{d})$ iff $\overline{c} \equiv \bot$
- $(c \wedge \overline{d}) \equiv (\overline{c} \wedge \overline{d})$ iff $\overline{d} \equiv \bot$

Thus, we also have $I_{(a,b)} \equiv \bot$ (since we have $I_{(c,d)} \equiv \bot$ and $I'_{(c,d)} \equiv \bot$), which expresses a mutual exclusiveness condition. Since we have 4 possible choices for $I_{(a,b)}$, it yields $4 \times 6 = 24$ distinct proportions, and exchanging (a, b) with (c, d) gives the 24 other degenerated proportions. Generally speaking, degenerated proportions correspond to a mutual exclusiveness condition between component(s) or negation of component(s) of one of the pairs (a, b) or (c, d), together with

- either an identity condition pertaining to the other pair,

- or a tautology condition on one of the literals of the other pair without any constraint on the other literal.

2.4 Basic properties of logical proportions

In this subsection, we first establish a remarkable property that single out the logical proportion s among the whole set of quaternary Boolean formulas. In order to do that we need a lemma.

Lemma 1. *An equivalence between indicators has exactly 10 valid valuations.*

Proof: Such an equivalence $eq \triangleq I_{a,b} \equiv I_{c,d}$ is satisfied only when it matches one of the 2 patterns $1 = 1$ or $0 = 0$: due to the fact that 0 is an absorbing value for $\wedge$, these patterns correspond to the 10 valuations shown in Table 1 for the literals involved in the indicators (with obvious notation). Any other valuation[2] does not match anyone of the 2 previous patterns and will lead to the truth value 0 for the equivalence eq. $\square$

literal 1	*literal 2*	*literal 3*	*literal 4*	*pattern*
1	1	1	1	$1 = 1$
0	1	0	1	$0 = 0$
0	1	1	0	$0 = 0$
0	1	0	0	$0 = 0$
1	0	0	1	$0 = 0$
1	0	1	0	$0 = 0$
1	0	0	0	$0 = 0$
0	0	0	1	$0 = 0$
0	0	1	0	$0 = 0$
0	0	0	0	$0 = 0$

Table 1: 10 valid valuations for an equivalence between indicators

Proposition 1. *The truth table of a logical proportion has 6 and only 6 valuations with truth value 1.*

Proof: Since a logical proportion T is the conjunction $eq_1 \wedge eq_2$ of 2 equalities between indicators, with $eq_1 \neq eq_2$, it appears from Lemma 1 that T has a maximum of 10 valid valuations and a minimum of 4 valid valuations. Let us start from $eq1$, having 10 valid valuations which are candidate to validate T. Obviously, adding eq_2 to eq_1 will reduce the number of valid valuations for T. Let us assume eq_2 differs from eq_1 with only one literal (or negation operator). This is then a degenerated proportion. Without loss of generality, we can consider that the difference between eq_1 and eq_2 occurs on the first literal meaning eq_1 is $a \wedge l_2 \equiv l_3 \wedge l_4$ and eq_2 is $\overline{a} \wedge l_2 \equiv l_3 \wedge l_4$ or vice versa. It is then quite clear that the first valuation 1111 valid for eq_1 is not valid any more for T. It remains 9 candidates valuations. Finally any valuation starting with 01 is not valid any more and we have 3 such valuations. All the 6 remaining valuations are still valid for T. Which ends the proof when the 2 equalities differ from one negation (i.e. one literal). Now when they differ from 2 literals, two cases have to be considered:

[2]The only valuations considered in this paper pertain to 4-tuples of variables. In practice, a Boolean valuation v will be denoted by the values $v(a)v(b)v(c)v(d)$ without any blank space, e.g., 0100 is short for $v(a) = 0, v(b) = 1, v(c) = 0, v(d) = 0$.

- either the 2 literals where eq_1 differs from eq_2 are on the same side of an equivalence i.e. eq_2 is $l'_1 \wedge l'_2 \equiv l_3 \wedge l_4$ (degenerated proportion)
- or they are on different side i.e. eq_2 is $l'_1 \wedge l_2 \equiv l'_3 \wedge l_4$.

In the first case, the valuations $1111, 0010, 0001$ and 0000 are not valid any more, but all other ones remain valid. In the second case, the valuations $0100, 0110, 1001$ and 0001 are not valid anymore, but all the other ones remain valid. We are done for the case of 2 differences. When they differ from 3 literals, let us suppose l_4 appears in both equivalence, the valuations $1001, 0101, 0010$ and 0000 are not valid anymore and we stick with the 6 remaining ones. In the case where all the literals are different, obviously the 4 valuations containing only one occurrence of 1 are not valid anymore because they lead to an invalid pattern 0=1 or 1=0 for eq_2. And we have exactly 4 such valuations. It remains 6 valid valuations. □

Note that the negation of a logical proportion is not a logical proportion since such a negation has 10 valuations leading to true in its table. Besides, the 120 logical proportions are all distinct as shown below with the help of the following lemma.

Lemma 2. *Two equivalences between indicators have the same truth table iff they are identical.*

Proof: It is sufficient to show that if 2 equalities eq_1 and eq_2 have the same truth table, then they are syntactically identical. In other terms, we have to prove that $eq_1 \equiv eq_2$ implies $eq_1 =_{Id} eq_2$. Without loss of generality, let us assume that eq_1 contains a but eq_2 contains $\overline{a}$. Considering the unique valuation v such that $v(eq_1) = 1$ with the pattern $1 = 1$, v is such that $v(a) = 1$. By hypothesis, $v(eq_2) = 1$ but in that case with the pattern $0 = 0$ since $v(\overline{a}) = 0$. Let us now modify v into v' such that $v'(a) = v(a) = 0, v'(c) = v(c), v'(d) = v(d)$ and $v'(b) = v(b)$. Obviously v' does not validate eq_1 but validates eq_2 which contradicts the hypothesis. □

Proposition 2. *The truth tables of the 120 proportions are all distinct.*

Proof: We are going to show that, when 2 proportions $T \triangleq eq_1 \wedge eq_2$ and $T' \triangleq eq'_1 \wedge eq'_2$ have the same truth table, they are syntactically identical (up to a permutation of the 2 equalities). In other words, $T \equiv T'$ implies $T =_{Id} T'$. Starting from $T \equiv T'$, it amounts to show that if eq_1 is syntactically different from eq'_1, eq_1 is syntactically equal to eq'_2. This will complete the proof as a similar reasoning will show that eq_2 is, in the same context, syntactically equal to eq'_1.

In fact, if eq_1 is syntactically different from eq'_1, we can assume for instance without loss of generality that eq_1 contains a but eq'_1 contains $\overline{a}$. Let us consider the unique valuation σ, validating T and T', such that $\sigma(eq_1) = 1$ with the pattern $1 = 1$. Necessarily, this valuation σ is such that $\sigma(a) = 1$. By hypothesis, $\sigma(eq'_1) = 1$ but in that case with the pattern $0 = 0$ since $\sigma(\overline{a}) = 0$. Let us now modify σ into σ' such that $\sigma'(a) = \sigma(a) = 0, \sigma'(c) = \sigma(c), \sigma'(d) = \sigma(d)$ and $\sigma'(b) = \sigma(b)$. Obviously $\sigma'(T) = \sigma'(eq_1) = 0$ but $\sigma'(eq_1) = 1$ still following the pattern $0 = 0$. The only option for having $\sigma(T) = \sigma(T') = 0$ is thus to have $\sigma'(eq'_2) = 0$ which means a belongs to eq'_2. Continuing the same reasoning, we show that $eq_1 =_{Id} eq'_2$ and we infer that if $eq_1 \neq eq'_1$, necessarily $eq_1 =_{Id} eq_2$. $\square$

Combined with the fact that there are $C^6_{16} = 8008$ truth tables with 16 lines, this result makes logical proportions quite rare in the world of quaternary Boolean formulas.

An exhaustive investigation of the whole set of logical proportions with respect to various other properties has been done in [19, 22, 21]. In the next subsection, we focus on one of these properties which allows us to characterize a small subset of remarkable proportions.

2.5 Code independency

Just as a numerical proportion holds independently of the base used for encoding numbers, or of the system of units representing the quantities at hand, it seems desirable that a logical proportion should be independent of the way we encode items in terms of the truth or the falsity of features. It means that the formula defining a proportion T should be valid when we switch 0 to 1 and 1 to 0. The formal expression of this property, that we call *code independency*, writes:

$$T(a, b, c, d) \rightarrow T(\overline{a}, \overline{b}, \overline{c}, \overline{d})$$

Surprisingly, this property highlights the fact once more that a single equivalence would not lead to a satisfactory definition for a logical proportion. Indeed, a unique equivalence between indicators, denoted $l_1 \wedge l_2 \equiv l_3 \wedge l_4$, where the l_i's are literals does not satisfy *code independency*, as explained now. If we consider a valuation v such that $v(l_1) = v(l_2) = v(l_3) = 0$ and $v(l_4) = 1$, obviously v makes the equivalence valid since $v(l_1 \wedge l_2) = v(l_3 \wedge l_4) = 0$. But when we switch 0 to 1 and 1 to 0, it appears that the new valuation v' such that $v'(l_1) = v'(l_2) = v'(l_3) = 1$ and $v'(l_4) = 0$ does not validate the equivalence anymore. This shows that one equivalence is not enough if we are interested in "code independency". We have to consider at least 2 equivalences to capture this behavior. For instance, $(a \wedge b \equiv c \wedge d) \wedge (\overline{a} \wedge \overline{b} \equiv \overline{c} \wedge \overline{d})$ clearly satisfies code independency.

Unfortunately, being built as the conjunction of two equivalences is not a sufficient condition for code independency, and many logical proportions do not satisfy it. We have the following result:

Proposition 3. *There are exactly 8 proportions satisfying the code independency property: the 4 homogeneous proportions A, R, P, I, and 4 hybrid proportions (shown in Table 2).*

Proof: In fact, the code independency property implies a complete equivalence:

$$T(a, b, c, d) \leftrightarrow T(\overline{a}, \overline{b}, \overline{c}, \overline{d})$$

Since both $T(a, b, c, d)$ and $T(\overline{a}, \overline{b}, \overline{c}, \overline{d})$ are logical proportions, Proposition 2 tells us that the 2 proportions should be identical up to a permutation of the 2 equalities. This exactly means that the second equivalence is obtained from the first one by negating all the variables. Since we have 4×4 equalities between indicators, we can build exactly $16/2 = 8$ proportions satisfying code independency property: each time we choose an equivalence, we use it and its negated form to build up a suitable proportion. Since A, R, P, I are built this way, they satisfy code independency. □

$\mathbf{H_a}$	$\mathbf{H_b}$
$(\overline{a} \wedge b \equiv \overline{c} \wedge \overline{d}) \wedge (a \wedge \overline{b} \equiv c \wedge d)$	$(\overline{a} \wedge b \equiv c \wedge d) \wedge (a \wedge \overline{b} \equiv \overline{c} \wedge \overline{d})$
$\mathbf{H_c}$	$\mathbf{H_d}$
$(\overline{a} \wedge \overline{b} \equiv \overline{c} \wedge d) \wedge (a \wedge b \equiv c \wedge \overline{d})$	$(\overline{a} \wedge \overline{b} \equiv c \wedge \overline{d}) \wedge (a \wedge b \equiv \overline{c} \wedge d)$

Table 2: The 4 hybrid proportions satisfying code independency

As a consequence of this result, this set of 8 proportions stand out of the whole set of 120 proportions. This set of proportions is clearly divided in 2 subsets: the 4 homogeneous proportions on one hand, and the 4 remaining ones, that we call *heterogeneous* proportions, on the other hand. In the next two sections, we first investigate the 4 homogeneous proportions through the angle of a list of meaningful properties, as well as their interrelationships, and their extensions to multiple-valued settings. After which, we shall move to the study of the 4 heterogeneous proportions in Section 5.

3 The 4 homogeneous proportions

We investigate now the 4 homogeneous proportions A, R, P, I from a semantical point of view. When considered as Boolean formulas, their semantics is given via their truth tables (which have $2^4 = 16$ lines since these proportions involve 4 variables).

3.1 Boolean truth tables

Starting from their syntactic expressions, it is an easy game to build up the truth tables of proportions A, R, P, I: they are exhibited in Table 3, where only the valuations leading to the truth value 1, are shown. This means that all the other ones lead to the truth value 0. As expected, only 6 valuations among 16 in the tables lead to a truth value 1. We also observe that there are only 8 distinct valuations that appear in Table 3. This emphasizes their collective coherence as the whole class of homogeneous proportions. Moreover, they go by pairs where 0 and 1 are exchanged, thus pointing out their "code independency".

A				**R**				**P**				**I**			
0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0
1	1	1	1	1	1	1	1	1	1	1	1	0	0	1	1
0	0	1	1	0	0	1	1	1	0	0	1	1	0	0	1
1	1	0	0	1	1	0	0	0	1	1	0	0	1	1	0
0	1	0	1	0	1	1	0	0	1	0	1	0	1	0	1
1	0	1	0	1	0	0	1	1	0	1	0	1	0	1	0

Table 3: Analogy, Reverse analogy, Paralogy, Inverse paralogy truth tables

It is interesting to take a closer look at the truth tables of the four homogeneous proportions. First, one can observe in Table 3, that 8 possible valuations for (a, b, c, d) never appear among the patterns that make A, R, P, or I true: these 8 valuations are of the form $xxxy$, $xxyx$, $xyxx$, or $yxxx$ with $x \neq y$ and $(x, y) \in \{0, 1\}^2$. As can be seen, it corresponds to situations where $a = b$ and $c \neq d$, or $a \neq b$ and $c = d$, i.e., similarity holds between the components of one of the pairs, and dissimilarity holds in the other pair. Moreover, the truth table of each of the four homogeneous proportions, is built in the same manner:

1. 2 lines of the table correspond to the characteristic pattern of the proportion; namely the two lines where one of the two equivalences in its definition holds true under the form $1 \equiv 1$ (rather than $0 \equiv 0$). Thus,
   - A is characterized by the pattern $xyxy$ (corresponding to valuations 1010 and 0101), i.e. we have the same difference between a and b as between c and d;
   - R is characterized by the pattern $xyyx$ (corresponding to valuations 1001 and 0110), i.e., the differences between a and b and between c and d are

in opposite directions;

- P is characterized by the pattern $xxxx$ (corresponding to valuations 1111 and 0000), i.e., what a and b have in common, c and d have it also;
- I is characterized by the pattern $xxyy$ (corresponding to valuations 1100 and 0011), i.e. what a and b have in common, c and d do not have it, and conversely.

2. the 4 other lines of the truth table of an homogeneous proportion T are generated by the characteristic patterns of the two other proportions that are not opposed to T (in the sense that A and R are opposed, as well as P and I). For these four lines, the proportion holds true since its expression reduces to $(0 \equiv 0) \wedge (0 \equiv 0)$.

Thus, the six lines of the truth table of A that makes it true are induced by the characteristic patterns of A, P, and I[3], the six valuations that makes P true are induced by the characteristic patterns of P, A, and R, and so on for R and I.

3.2 Relevant properties

Before going deeper in the investigation, remember that the Boolean analogical proportion is supposed to be, in a Boolean setting, the counterpart of the classical numerical proportions. Then, it is interesting to consider Boolean counterparts of the properties satisfied by the numerical proportions, other than *code independency*. We list these properties below (with T denoting a logical proportion).

- *Full identity*: A numerical proportion holds when all the numbers are equal, i.e., $a = b = c = d$, which logically translates into

$$T(a, a, a, a)$$

- *Reflexivity*: A numerical proportion holds between (a, b) and (a, b) which logically translates into

$$T(a, b, a, b)$$

  Obviously, *reflexivity* entails *full identity*.

[3]The measure of analogical dissimilarity introduced in [13] is 0 for the valuations corresponding to the characteristic patterns of A, P, and I, maximal for the valuations corresponding to the characteristic patterns of R, and takes the same intermediary value for the 8 valuations characterized by one of the patterns $xxxy$, $xxyx$, $xyxx$, or $yxxx$.

- *Sameness*: A numerical proportion holds between (a, a) and (b, b), which logically translates into
$$T(a, a, b, b)$$
Still, *sameness* entails *full identity*.

- *symmetry* : We can exchange the pair (a, b) with the pair (c, d) in the numerical proportion, which logically translates into
$$T(a, b, c, d) \to T(c, d, a, b)$$

- *Central (and extreme) permutation* : This is a well known property of numerical proportions, which logically translates into
$$T(a, b, c, d) \to T(a, c, b, d) \text{ (central permutation)}$$
and
$$T(a, b, c, d) \to T(d, b, c, a) \text{ (extreme permutation)}$$

- *Transitivity*: This property that holds for numerical proportions is logically stated as follows
$$T(a, b, c, d) \wedge T(c, d, e, f) \to T(a, b, e, f)$$

- *Exchange-mirroring*: The negation operator can play for Boolean values the role of an inverse operator for numbers. A numerical proportion holds between a pair (a, b) and the pair(b^{-1}, a^{-1}), which logically translates into
$$T(a, b, \overline{b}, \overline{a})$$

- *Semi-mirroring*: Similarly it is worth to consider
$$T(a, b, \overline{a}, \overline{b})$$
This property is not satisfied by numerical proportions.

- *Negation-compatibility*: Similarly it is worth to consider
$$T(a, \overline{a}, b, \overline{b})$$
This property is also not satisfied by numerical proportions.

Property name	Formal definition	Proportion
full identity	$T(a, a, a, a)$	A,R,P
reflexivity	$T(a, b, a, b)$	A,P
reverse reflexivity	$T(a, b, b, a)$	R,P
sameness	$T(a, a, b, b)$	A,R
symmetry	$T(a, b, c, d) \to T(c, d, a, b)$	A,R,P,I
permutation of means	$T(a, b, c, d) \to T(a, c, b, d)$	A,I
permutation of extremes	$T(a, b, c, d) \to T(d, b, c, a)$	A,I
all permutations	$\forall i, j, T(a, b, c, d) \to T(p_{i,j}(a, b, c, d))$	I
transitivity	$T(a, b, c, d) \wedge T(c, d, e, f) \to T(a, b, e, f)$	A,P
semi-mirroring	$T(a, b, \overline{a}, \overline{b})$	R,I
exchange mirroring	$T(a, b, \overline{b}, \overline{a})$	A,I
negation compatib.	$T(a, \overline{a}, b, \overline{b})$	P,I

Table 4: Boolean properties of A, R, P, I

Investigating the homogeneous proportions with regard to the properties listed above can simply be done with an examination of the truth table of the target proportion. We summarize in Table 4 all the properties satisfied by A, R, P, I: the third column enumerates the homogeneous proportions satisfying the property, respectively named and described in the 1st and 2nd columns.
Note that the 4 homogeneous proportions satisfy symmetry: $T(a, b, c, d) = T(c, d, a, b)$, as well as many other properties. In particular, analogical proportion A enjoys properties that parallel properties of numerical proportions: full identity, reflexivity, symmetry, central and extreme permutations, and transitivity.

One can also establish properties linking the homogeneous proportions, which are easily deducible from their definitions in terms of indicators.

Proposition 4.
$A(a, b, c, d) \equiv R(a, b, d, c)$; $A(a, b, c, d) \equiv P(a, d, c, b)$; $A(a, b, c, d) \equiv I(\overline{a}, d, \overline{c}, b)$

As can be seen, homogeneous proportions are strongly linked together. Especially A, R, P are exchanged through simple permutation s; in that respect, I stands apart. Besides, A, R, P, I are mutually exclusive, as a simple examination of their truth tables reveals that their intersection is empty.

Proposition 5. $A(a, b, c, d) \wedge R(a, b, c, d) \wedge P(a, b, c, d) \wedge I(a, b, c, d) = \bot$

Lastly, having a closer look on the homogeneous proportions, we can easily build Table 5 which gives what $T(a, b, c, d) \wedge T(c, d, e, f)$ entails for the 4 homogeneous proportions.

chaining	result	transitivity
$A \wedge A$	A	yes
$R \wedge R$	A	no
$P \wedge P$	P	yes
$I \wedge I$	P	no
$A \wedge R$	R	
$P \wedge I$	I	

Table 5: Chaining properties for A, R, P, I

All these common properties explain why the homogeneous proportions stand out from the whole set of 120 logical proportions. It makes homogeneous proportions a worth considering Boolean counterpart of numerical proportions.

3.3 Characterization of homogeneous proportions by properties

Some subsets of the properties listed above are sufficient for characterizing one or more homogeneous proportions as unique among the 120 logical proportions. Let us start with the following result:

Proposition 6.

- *A, R, P are the unique proportions to satisfy full identity and code independency.*
- *A is the only proportion to satisfy sameness ($T(a, a, b, b)$) and reflexivity ($T(a, b, a, b)$).*
- *R is the only proportion to satisfy sameness and reverse reflexivity $T(a, b, b, a)$.*
- *P is the only proportion to satisfy reflexivity and reverse reflexivity.*
- *There is no proportion simultaneously satisfying sameness, reflexivity, and reverse reflexivity.*

Proof: The first statement comes from Proposition 3 giving the 8 proportions satisfying code independency, along with an immediate checking of the proportions syntactic form. For instance, H_a defined as $(a \wedge b \equiv c \wedge \overline{d}) \wedge (\overline{a} \wedge \overline{b} \equiv \overline{c} \wedge d)$ is definitely not valid for valuation 0000. The same reasoning applies to all the proportions other than A, R, P.
This is an easy proof for the first 3 following statements since each property generates a set of 4 valid valuations (and two of them yield 6 valid valuations). For instance,

sameness ($T(a, a, b, b)$ implies that valuations $1111, 0000, 0011, 1100$ should be valid and *reflexivity* ($T(a, b, a, b)$) implies that valuations $1111, 0000, 0101, 1010$, which is the truth table of A.
Let us consider the last statement, having the simultaneous satisfaction of the 3 properties leads to a truth table where the 8 valuations 0000, 1111, 1010, 0101, 0110, 1001, 0011, 1100 are valid: then this cannot be the truth table of a logical proportion. □

It is well known that a valid numerical proportion still holds when we exchange the extreme elements or the mean elements. And we have seen that A and I satisfy both of these permutations. In fact, there are 6 pairwise permutations of the 4 variables appearing in a proportion. So, the behavior of logical proportions w.r.t. these permutations is worth investigating. We denote the permutation of element i and j by $p_{i,j}$: for instance $p_{2,3}$ is the mean permutation while $p_{1,4}$ is the extreme permutation. We can establish the following result:

Proposition 7.
- *A is the only proportion to satisfy reflexivity and to be stable for $p_{1,4}$ (or $p_{2,3}$).*
- *A is the only proportion to satisfy sameness and to be stable for $p_{1,4}$ (or $p_{2,3}$).*
- *R is the only proportion to satisfy sameness and to be stable for $p_{1,3}$ (or $p_{2,4}$).*
- *R is the only proportion to satisfy reverse reflexivity and to be stable for $p_{1,3}$ (or $p_{2,4}$).*
- *P is the only proportion to satisfy reflexivity and to be stable for $p_{1,2}$ (or $p_{3,4}$).*
- *P is the only proportion to satisfy reverse reflexivity and to be stable for $p_{1,2}$ (or $p_{3,4}$).*
- *A and I are the only proportions to satisfy symmetry and to be stable for $p_{1,4}$ (or $p_{2,3}$).*
- *P and I are the only proportions to satisfy symmetry and to be stable for $p_{1,2}$ (or $p_{3,4}$).*
- *I is the* unique *logical proportion to satisfy the 6 permutations.*

Proof: The proofs are quite similar for the 8 first statements. Let us give an example for the first statement. *reflexivity* means that valuations $0000, 1111, 0011, 1100$ have to be valid. Adding stability for $p_{2,3}$ leads to add 0101 and 1010 as valid valuations. This is the truth table of A.

Let us consider the last statement which is a bit more tricky. It is easy to check that these permutations induce a partition of the set of valuations into 5 classes, each of them being closed for these 6 permutations:

- the class $\{0000\}$ and the class $\{1111\}$
- the class $\{0111, 1011, 1101, 1110\}$
- the class $\{1000, 0100, 0010, 0001\}$
- the class $\{0101, 1100, 0011, 1010, 1001, 0110\}$

Taking into account that a logical proportion is true for only 6 valuations (Proposition 1), we only have 3 options:
- a proportion valid for $\{0000\}$, $\{1111\}$ and $\{0111, 1011, 1101, 1110\}$,
- or for $\{0000\}$, $\{1111\}$ and $\{1000, 0100, 0010, 0001\}$,
- or for $\{0101, 1100, 0011, 1010, 1001, 0110\}$.

It appears that the latter class is just the truth table of inverse paralogy. Lemma 3 that we shall prove below allows us to complete the proof. □

Lemma 3. *A logical proportion cannot satisfies the class of valuation*

$$\{0111, 1011, 1101, 1110\} \text{ or the class } \{1000, 0100, 0010, 0001\}.$$

Proof: It is enough to show that this is the case for an equivalence between indicators. So let us consider such an equivalence $l_1 \wedge l_2 \equiv l_3 \wedge l_4$. If this equivalence is valid for $\{0111, 1011\}$, it means that its truth value does not change when we switch the truth value of the 2 first literals from 0 to 1: there are only 2 indicators for a and b satisfying this requirement: $a \wedge b$ and $\overline{a} \wedge \overline{b}$. On top of that, if this equivalence is still valid for $\{1101, 1110\}$, it means that its truth value does not change when we switch the truth value of the 2 last literals from 0 to 1: there are only 2 indicators for c and d satisfying this requirement: $c \wedge d$ and $\overline{c} \wedge \overline{d}$. Then the equivalence $l_1 \wedge l_2 \equiv l_3 \wedge l_4$ is just $a \wedge b \equiv c \wedge d$, $a \wedge b \equiv \overline{c} \wedge \overline{d}$, $a \wedge b \equiv \overline{c} \wedge \overline{d}$ or $\overline{a} \wedge \overline{b} \equiv \overline{c} \wedge \overline{d}$. None of these equivalences satisfies the whole class $\{0111, 1011, 1101, 1110\}$. The same reasoning applies for the other class. □

We summarize the results of this subsection by a pair of properties characterizing a subset of homogeneous proportions, in Table 6 and Table 7. An empty cell means that the corresponding properties do not characterize any subset of homogeneous proportion. For instance, the diagonal cells are all empty because an homogeneous proportion cannot be characterized with only one property.

	full identity	code indep.	symmetry	sameness	reflexivity	rev. reflexivity
full identity		A, R, P				
code indep.	A, R, P		A, R, P, I	A, R	A, P	R, P
symmetry		A, R, P, I		A, R		R, P
sameness		A, R	A, R		A	R
reflexivity		A, P		A		P
rev. reflexivity		R, P	R, P	R	P	

Table 6: Characteristic properties of A, R, P, I

	$p_{1,2}$	$p_{1,3}$	$p_{1,4}$	$p_{2,3}$	$p_{2,4}$	$p_{3,4}$
sameness		R	A	A	R	
reflexivity	P		A	A		P
rev. reflexivity	P	R			R	P
symmetry	P, I		A, I	A, I		P, I

Table 7: Characteristic properties of A, R, P, I w.r.t. permutations

To conclude this section, we establish a result which shows how singular I is among the set of homogeneous proportions.

Proposition 8.

- *A logical proportion satisfying 2 properties among semi-mirroring, negation-compatibility and exchange-mirroring satisfies the remaining one, and is unique. This is the inverse paralogy I.*
- *A logical proportion stable for 4 permutations is stable for the 2 remaining ones and is unique. This is the inverse paralogy I.*

Proof: Considering the first statement, let us choose for instance *semi-mirroring* and *negation-compatibility*. First of all, we can observe that, for a proportion T to satisfy *semi-mirroring*, means the 4 valuations 1010, 1001, 0110, 0101 are valid. For *negation-compatibility* to be satisfied, the 4 valuations 1100, 0011, 1001, 0110 should be valid. Then the truth table of a proportion satisfying both properties should contains all these valuations i.e. 1010, 1001, 0110, 0101, 1100, 0011. Thanks to Proposition 1, this is the truth table of inverse paralogy I. A similar reasoning applies for the other cases. Regarding the second statement, let us consider a proportion stable for 4 pairwise permutations: since such pairwise permutations generate the full group of permutations of 4 elements, it means this proportion is stable for any permutations. We can consider 2 cases:

- either such a proportion is valid for a valuation having an even number of 0 and other than 0000 and 1111. We can consider this is 0110 for instance. The stability leads to have $0011, 0110, 0101, 1001, 1010$ valid as well: this is the truth table for i.

- or such a proportion does not have a valid valuation with an even number of 0 other than 0000 and 1111. It means there is a valid valuation with an odd number of 0 like 1000. In that case, the stability w.r.t. the permutations leads to have $1000, 0100, 0010, 0001$ as valid valuations, which is not possible thanks to Lemma 3. □

3.4 Equation solving

The idea of proportion is closely related to the idea of extrapolation, i.e. to guess/-compute a new value on the ground of existing values. In the case of geometrical proportions, this leads to the well known "rule of three" where, knowing that $\frac{a}{b} = \frac{c}{x}$ holds, allows us to compute the value of x from a, b, c. In the Boolean setting, if for some reason it is believed or known that a logical proportion holds between 4 binary items, 3 of them being known, then one may try to infer the value of the 4th one, at least when this extrapolation leads to a unique value. For a proportion T, there are exactly 6 valuations v such that:

$$v(T(a, b, c, d)) = 1$$

In our context, the problem can be stated as follows. Given a logical proportion T and a valuation v such that $v(a), v(b), v(c)$ are known, does it exist a Boolean value x such that $v(T(a, b, c, d)) = 1$ when $v(d) = x$, and in that case, is this value unique?

We will refer to this problem as "the equation solving problem", and for the sake of simplicity, a propositional variable a is denoted as its truth value $v(a)$, and we use the equational notation $T(a, b, c, x) = 1$, where $x \in \{0, 1\}$ is unknown. First of all, it is easy to see that there are always cases where the equation has no solution. Indeed, the triple a, b, c may take $2^3 = 8$ values, while any proportion T is true only for 6 distinct valuations, leaving at least 2 cases with no solution. For instance, when we deal with analogy A, the equations $A(1, 0, 0, x)$ and $A(0, 1, 1, x)$ have no solution. We have the following results:

Proposition 9.
The analogical equation $A(a, b, c, x)$ is solvable iff $(a \equiv b) \vee (a \equiv c)$ holds. In that case, the unique solution is $x = a \equiv (b \equiv c)$.
The reverse analogical equation $R(a, b, c, x)$ is solvable iff $(b \equiv a) \vee (b \equiv c)$ holds. In that case, the unique solution is $x = b \equiv (a \equiv c)$.
The paralogical equation $P(a, b, c, x)$ is solvable iff $(c \equiv b) \vee (c \equiv a)$ holds.
In each of the three above cases, when it exists*, the unique solution is given by $x = c \equiv (a \equiv b)$, i.e. $x = a \equiv b \equiv c$.*
The inverse paralogical equation $I(a, b, c, x)$ is solvable iff $(a \not\equiv b) \vee (b \not\equiv c)$ holds. In that case, the unique solution is $x = c \not\equiv (a \not\equiv b)$.

Proof: By immediate investigation of the truth tables. □

The anthropologist, linguist and computer scientist Sheldon Klein [9, 10] was the first to propose to solve analogical equations of the form $A(a, b, c, x) = 1$, where x is unknown, as $x = c \equiv (a \equiv b)$, without however providing an explicit definition for $A(a, b, c, d)$, nor distinguishing between A, R, and P. As we can see, the first 3 homogeneous proportions A, R, P behave similarly. Still, their conditions of equation solvability differ. Moreover, it can be checked that at least 2 of these proportions are always simultaneously solvable. Besides, when they are solvable, there is a common expression that yields the solution.

3.5 Alternative writings for homogeneous proportions

When sticking to the Boolean setting, we can use standard equivalences to get alternative writings for A, R, P, I. First of all, using the De Morgan's laws and the fact that $p \equiv q$ is equivalent to $\overline{p} \equiv \overline{q}$, we get definitions where the internal $\wedge$ are replaced with $\vee$ as shown in Table 8. It means that, in a Boolean setting, indicators involving $\vee$ are a perfect replacement for indicators using $\wedge$.

A	R
$(a \vee \overline{b} \equiv c \vee \overline{d}) \wedge (\overline{a} \vee b \equiv \overline{c} \vee d)$	$(a \vee \overline{b} \equiv \overline{c} \vee d) \wedge (a \vee \overline{b} \equiv c \vee \overline{d})$
P	I
$(a \vee b \equiv c \vee d) \wedge (\overline{a} \vee \overline{b} \equiv \overline{c} \vee \overline{d})$	$(a \vee b \equiv \overline{c} \vee \overline{d}) \wedge (\overline{a} \vee \overline{b} \equiv c \vee d)$

Table 8: A, R, P, I definitions with $\vee$ operator

A more interesting option is to start from the definition of P with indicators

$$(a \wedge b \equiv c \wedge d) \wedge (\overline{a} \wedge \overline{b} \equiv \overline{c} \wedge \overline{d}) \ (P)$$

and to use again De Morgan's laws to rewrite the second equivalence. This leads to a definition of P without any negation that we denote P^*:

$$(a \wedge b \equiv c \wedge d) \wedge (a \vee b \equiv c \vee d) \ (P^*)$$

Then, considering the link between A and P established in Proposition 4, namely $A(a, b, c, d) \equiv P(a, d, c, b)$, it comes another definition for A, without any negation operator:

$$(a \wedge d \equiv b \wedge c) \wedge (a \vee d \equiv b \vee c) \ (A^*)$$

It is noticeable that this latter new definition exactly corresponds to what the psychologist Jean Piaget [15], called *logical proportion*! However, strangely enough, he has not developed their study nor pointed out their link with analogy.

Thus, since a and d are the extreme variables, b and c the mean variables, the analogical proportion $A(a, b, c, d)$ can be read as "the conjunction (resp. disjunction) of the extremes is equivalent to the conjunction (resp. disjunction) of the means".

Considering the link between A, R, P, I coming from Proposition 4, we can finally get alternative writing denoted A^*, R^*, P^* and I^* that are shown in Table 9.

A^*	R^*
$(a \wedge d \equiv b \wedge c) \wedge (a \vee d \equiv b \vee c)$	$(a \wedge c \equiv b \wedge d) \wedge (a \vee c \equiv b \vee d)$
P^*	I^*
$(a \wedge b \equiv c \wedge d) \wedge (a \vee b \equiv c \vee d)$	$(a \wedge b \equiv \overline{c} \wedge \overline{d}) \wedge (a \vee b \equiv \overline{c} \vee \overline{d})$

Table 9: A^*, R^*, P^*, I^* definitions

Since, in the Boolean setting, the equivalence $T(a,b,c,d) \equiv T^*(a,b,c,d)$ holds (where T denotes any homogeneous proportion among A, R, P, I), one could consider T^* as an alternative writing for T. It is interesting to note that this approach leads to rewrite A, R, P without any negation. We have to be aware that these equivalences, leading to alternative writings, are not necessarily valid outside the Boolean framework.

4 Homogeneous proportions: multiple-valued semantics

Ultimately, logical proportions, and in particular the homogeneous ones, could be used for practical applications where we have to deal with missing information or features whose satisfaction is a matter of degree. To cover such situations, extensions of the Boolean interpretation to multiple-valued logics (3-valued at least) is necessary. A formal way to cope with these situations is to extend the Boolean framework to a multiple-valued one by introducing truth values belonging to $[0,1]$. We should carefully distinguish between three cases:

- when feature satisfaction is a matter of degree instead of being binary, i.e., the truth value of a given feature may be an *intermediate* value between 0 and 1.
- when a feature does not make sense for a given item, i.e., the feature is *non applicable* to it.
- when *information* about some features *is missing*, i.e., we have no clue about the truth value of some features for some items, and the corresponding truth value is not known, i.e., *unknown*.

At this stage, two questions arise:

1. in a given model, what are the valuations that correspond to a "perfect" proportion of a given type (i.e., having 1 as truth value)? For instance, does $T(a,a,a,a)$ postulate still have to be satisfied by A, R, P, or can we consider models where $A(u,u,u,u) = u$, u being a truth value distinct from 0 and 1?
2. are there valuations that could be regarded as "imperfect" proportions of a given type (i.e., with a truth value distinct from 0 and 1) and in that case, what is their truth value?

We investigate these issues in the following subsections keeping in mind an essential principle: whatever the way we define the truth values, the Boolean model should be the limit case of our models when restricted to Boolean valuations.

4.1 Semantics for gradual features

When the satisfaction of features may be a matter of degree, we have to consider that the truth values belong to a linearly ordered scale $\mathcal{L}$. The simplest case is when $\mathcal{L} = \{0, \alpha, 1\}$, with the ordering $0 < \alpha < 1$, which can be generalized into a finite chain $\mathcal{L} = \{\alpha_0 = 0, \alpha_1, \cdots, \alpha_n = 1\}$ of ordered grades $0 < \alpha_1 < \cdots < 1$, or to an infinite chain using the real interval $[0, 1]$. A proposal for extending A in such cases has been advocated in [18]. It takes its source in the initial definition

$$A(a, b, c, d) = (a \wedge \bar{b} \equiv c \wedge \bar{d}) \wedge (\bar{a} \wedge b \equiv \bar{c} \wedge d),$$

where now

- i) the central $\wedge$ is taken as equal to min;
- ii) $s \equiv t$ is taken as $\min(s \to_L t, t \to_L s)$ where $\to_L$ is Łukasiewicz implication, defined by $s \to_L t = \min(1, 1 - s + t)$, for $\mathcal{L} = [0, 1]$ (in the discrete cases, we take $\alpha = 1/2$ and $\alpha_i = i/n$), and thus $s \equiv t = 1 - |s - t|$; note that $s \equiv t = (1 - s) \equiv (1 - t)$;
- iii) $s \wedge \bar{t} = \max(0, s - t) = 1 - (s \to_L t)$, i.e., $\wedge^{-}$is understood as expressing a bounded difference. Note that this choice preserves $A(a, b, c, d) = A(\bar{a}, \bar{b}, \bar{c}, \bar{d})$ for the involutive negation $\bar{x} = 1 - x$.

The resulting expression for $A(a, b, c, d)$ is given in Table 10. Then, we understand the truth value of $A(a, b, c, d)$ as the extent to which the truth values a, b, c, d makes an analogical proportion. For instance, in such a graded model, the truth value of $A(0.9, 1, 1, 1) = 0.9$, which fits the intuition. It can be checked that the semantics of $A(a, b, c, d)$ thus defined in the graded case, reduces to the previous definition when restricted to the Boolean case.

It is interesting to study in what cases $A(a, b, c, d) = 1$ (and in what cases $A(a, b, c, d) = 0$). Then it is clear that $A(a, b, c, d) = 1$ when $a - b = c - d$. When $a, b, c, d \in \{0, \alpha = 1/2, 1\}$, it yields the 19 following patterns 1111; 0000; $\alpha\alpha\alpha\alpha$; 1010; 0101; $1\alpha1\alpha$; $\alpha1\alpha1$; $0\alpha0\alpha$; $\alpha0\alpha0$; 1100; 0011; $11\alpha\alpha$; $\alpha\alpha11$; $\alpha\alpha00$; $00\alpha\alpha$; $1\alpha\alpha0$; $0\alpha\alpha1$; $\alpha10\alpha$; $\alpha01\alpha$.

This means that $A(a, b, c, d) = 1$ when the change from a to b has the same direction and the same intensity as the change from c to d. However, the last 4 patterns show that there is no need to have $a = b$ and $a = c$ while these conditions hold for the 15 first patterns, which are all of the form $xyxy$, $xxyy$, or $xxxx$. In contrast, note that the last 4 patterns exhibit 3 distinct values.

$A(a, b, c, d) = 0$ when $a - b = 1$ and $c \leq d$, or $b - a = 1$ and $d \leq c$, or $a \leq b$ and $c - d = 1$, or $b \leq a$ and $d - c = 1$. It means the 22 following patterns in the 3-valued

$A(a,b,c,d) =$ $1-\mid (a-b)-(c-d)\mid$ if $a \geq b$ and $c \geq d$, or $a \leq b$ and $c \leq d$ $1-max(\mid a-b\mid, \mid c-d\mid)$ if $a \leq b$ and $c \geq d$, or $a \geq b$ and $c \leq d$
$R(a,b,c,d) = A(a,b,d,c)$
$P^*(a,b,c,d) =$ $min(1-\lvert max(a,b)-max(c,d)\rvert, 1-\lvert min(a,b)-min(c,d)\rvert)$

Table 10: Graded definitions for A, R, P^*

case: 1110; 1101; 1011; 0111; 0001; 0010; 0100; 1000; 1001; 0110; $10\alpha\alpha$; $01\alpha\alpha$; $\alpha\alpha10$; $\alpha\alpha01$; 100α; 011α; $10\alpha1$; $\alpha001$; $0\alpha10$; $1\alpha01$; $01\alpha0$; $\alpha110$. Thus, $A(a,b,c,d) = 0$ when one change inside the pairs (a,b) and (c,d) is maximal, while the other pair shows no change or a change in the opposite direction.

Using $\mathcal{L} = \{0, \alpha, 1\}$, $A(a,b,c,d) = \alpha$ for 81 - 19 - 22 = 40 distinct patterns.

In [18], the graded extension of $R(a,b,c,d)$ is defined by permuting c and d in the definition of A, according to Proposition 4. But the extension of the paralogy is no longer obtained by permuting b and d in the definition of A (as Proposition 4 would suggest). In fact, the paralogical proportion is defined directly from P^* (thus changing $\bar{a} \wedge \bar{b} \equiv \bar{c} \wedge \bar{d}$ into $a \vee b \equiv c \vee d$), and taking $\wedge = \min$, $\vee = \max$, and $s \equiv t = 1 - |s-t|$, we obtain the definition in Table 10. If we now exchange b and d (using Proposition 4 again) in this definition, we get the graded version of A^* (which is no longer equivalent to A), namely

$$A^*(a,b,c,d) = min(1-|max(a,d)-max(b,c)|, 1-|min(a,d)-min(b,c)|)$$

This is the direct counterpart of the definition without negation of the analogical proportion in the Boolean case. This alternative extension still preserves $A^*(a,b,c,d) = A^*(\bar{a},\bar{b},\bar{c},\bar{d})$ for the involutive negation $\bar{x} = 1-x$. It can be checked that $A^*(a,b,c,d) = 1$ only for the 15 patterns with at most two distinct values (for which $A(a,b,c,d) = 1$), while $A^*(a,b,c,d) = \alpha$ for the 4 other patterns for which $A(a,b,c,d) = 1$, namely for $1\alpha\alpha0$; $0\alpha\alpha1$; $\alpha10\alpha$; $\alpha01\alpha$. Besides, $A^*(a,b,c,d) = 0$ for only 18 among the 22 patterns that make $A(a,b,c,d) = 0$. The 4 patterns for which $A^*(a,b,c,d) = \alpha$ (instead of 0) are $10\alpha\alpha$; $01\alpha\alpha$; $\alpha\alpha10$; $\alpha\alpha01$.

Using $\mathcal{L} = \{0, \alpha, 1\}$, $A^*(a,b,c,d) = \alpha$ for 81 - 15 - 18 = 48 distinct patterns.

Thus, it appears that $A^*(a, b, c, d)$ does not acknowledge as perfect the analogical proportion patterns where the amount of change between a and b is the same as between c and d and has the same direction, but where this change applies in different areas of the truth scale. Still, $A^*(a, b, c, d)$ remains half-true in these cases, for $\mathcal{L} = \{0, \alpha, 1\}$. When $\mathcal{L} = [0, 1]$, it can be checked that $A^*(a, b, c, d) \geq 1/2$ when $a - b = c - d$; in particular, $\forall a, b, A^*(a, b, a, b) = 1$, which corresponds to the case where $a = c$ and $b = d$. In the same spirit, if $\mathcal{L} = \{0, \alpha, 1\}$ as well as for $\mathcal{L} = [0, 1]$, $A^*(a, b, c, d) = 0$ when a change inside the pairs (a,b) and (c,d) is maximal, while the other pair shows a change in the opposite direction starting from 0 or 1. However, $A^*(1, 0, c, c) = min(c, 1-c)$ and A^* takes the same value for the 7 other permutations of $(1, 0, c, c)$ obtained by applying symmetry and/or central permutation.

As can be seen in Table 11, A^* and A also coincide on some patterns having intermediary truth values, but diverge on others. Generally speaking, A^* is smoother than A in the sense that more patterns have intermediary truth values with A^* than with A.

A	A^*
$A(1, 1, u, v) = 1 - \|u - v\|$	$A^*(1, 1, u, v) = 1 - \|u - v\|$
$A(1, 0, u, v) = u - v$ if $u \geq v$	$A^*(1, 0, u, v) = \min(u, 1 - v)$
$= 0$ if $u \leq v$	
$A(0, 1, u, v) = v - u$ if $u \leq v$	$A^*(0, 1, u, v) = min(v, 1 - u)$
$= 0$ if $u \geq v$	
$A(0, 0, u, v) = A(1, 1, u, v)$	$A^*(0, 0, u, v) = A^*(1, 1, u, v)$

Table 11: The two graded definitions of the analogical proportion in [0, 1]

Both A and A^* continue to satisfy the *symmetry property* (as P, R, and P^*, R^* with $R^*(a, b, c, d) = A^*(a, b, d, c) = P^*(a, c, d, b)$). However, only A^* still enjoys the *means permutation* and the *extremes permutation* properties. *This is no longer the case with* A, as shown by the following counter-example.

$A(0.8, 0.6, 1, 0.3) = 1- \mid (0.8 - 0.6) - (1 - 0.3) \mid = 1- \mid 0.2 - 0.7 \mid = 0.5$ since $0.8 \geq 0.6$ and $1 \geq 0.3$, and $A(0.8, 1, 0.6, 0.3) = 1 - \max(\mid 0.8 - 1 \mid, \mid 0.6 - 0.3 \mid) = 1 - \max(0.2, 0.3) = 0.7$ since $0.8 \leq 1$ and $0.6 \geq 0.3$.

But, as already mentioned, *both* A *and* A^* continue to satisfy the *code independency* property with respect to $\overline{a} = 1 - a$. Some more Boolean properties that remain valid in the multiple-valued case are summarized in Table 12.

Property name	Formal definition	Proportion
full identity	$T(a,a,a,a)$	A^*, A, R, P
reflexivity	$T(a,b,a,b)$	A^*, A, P
reverse reflexivity	$T(a,b,b,a)$	R,P
sameness	$T(a,a,b,b)$	A^*, A, R
symmetry	$T(a,b,c,d) \to T(c,d,a,b)$	A^*, A, R, P
permutation of means	$T(a,b,c,d) \to T(a,c,b,d)$	A^*
permutation of extremes	$T(a,b,c,d) \to T(d,b,c,a)$	A^*
all permutations	$\forall i,j, T(a,b,c,d) \to T(p_{i,j}(a,b,c,d))$	none
semi-mirroring	$T(a,b,\overline{a},\overline{b})$	R
exchange mirroring	$T(a,b,\overline{b},\overline{a})$	A
negation compatib.	$T(a,\overline{a},b,\overline{b})$	none

Table 12: Graded properties of A, A^*, R, P

4.2 Dealing with non-applicable features

The abbreviation 'n/a' (for *non applicable*) is currently used in data tables when an attribute does not apply, when a feature does not make sense for a particular item. However, the extensive use of 'n/a' may be often ambiguous when it also appears in the same tables when information is *non available* for some attribute values of some items. Indeed one has to carefully distinguish the case where the feature does apply to the item, but it is not known if the feature is true or is false for the item, from the case where the feature is neither true nor false for the item since the feature does not apply to it. The case of unknown truth values is discussed in the next section, while we now address the problem of dealing with genuinely non applicable features.

The idea of introducing a third truth value for 'non applicable' (na for short in the following) in the context of analogy can be already found in the pioneering work of Sheldon Klein [9, 10], which we already mentioned in the equation solving subsection 3.4. However, his handling of na is based on $(na \equiv na) = na$, which suggests that the evaluation of an analogical proportion where na appears may receive the truth value na, which is more in the spirit of understanding na as 'not available', or 'unknown'.

Indeed, although a property may be 'true', 'false', or 'non applicable' for an item, it seems natural to expect that $A(a,b,c,d)$ can only be 'true' or 'false', since $1na1na$ looks intuitively satisfactory as an analogical proportion, while $1na00$ is not. More precisely, in the context of non applicable properties, we have only 3 valuation patterns that should make an analogical proportion true: $xxxx$, $xyxy$, and $xxyy$, where $x, y \in \{0, 1, na\}$. Any other option should make it false, since $\{0, 1, na\}$ play the same role. This leads to acknowledge as true the 15 following valuations:

- 1111; 0000; $nananana$ corresponding to $xxxx$;

- 1010; 0101; $1na1na$; $na1na1$; $0na0na$; $na0na0$ corresponding to $xyxy$ with $x \neq y$;

- 1100; 0011; $11nana$; $nana11$; $nana00$; $00nana$ corresponding to $xxyy$ with $x \neq y$.

All the remaining valuations lead to false.

In other words, we are in a situation somewhat similar to the one encountered in the previous section in the case of a unique intermediary truth-value α between true and false, meaning 'half-true' (or equivalently 'half-false'), when we refuse the four valuations $1\alpha\alpha0$, $0\alpha\alpha1$, $\alpha01\alpha$ and $\alpha10\alpha$ as being true, *except that* now no valuation leads to the third truth value. It is possible to find logical definitions of the analogical proportion having the expected behavior for the truth values $\{0, 1, na\}$. A solution to get the exact truth table is:
- to order $\{0, 1, na\}$ as the chain $1 > na > 0$,
- to use Kleene conjunction and disjunction, see, e.g., [2], respectively defined by the minimum and the maximum according to the above ordering,
- to use the strong Kleene equivalence $\equiv$, where $x \equiv y = 1$ if and only if $x = y$, and $x \equiv y = 0$ otherwise,
- to define analogical proportion with A^* instead of A, namely

$$A^*(a, b, c, d) = (a \wedge d \equiv b \wedge c) \wedge (a \vee d \equiv b \vee c).$$

A counterpart to $A(a, b, c, d) = (a \setminus b \equiv c \setminus d) \wedge (b \setminus a \equiv d \setminus c)$ where $\setminus$ here denotes the Boolean logical connective corresponding to set difference, can also be found. However, since we do not want to have $1nana0$ true, the difference between 1 and na and the difference between na and 0 should not be the same, neither the same as between 1 and 0, nor between 1 and 1 for sure. Thus we need 4 distinct values for the difference. This is impossible with 3 truth values! This contrasts with the Boolean case where there are only two possible difference values needed. The solution is then to use *2* connectives for differences:

$x \setminus_1 y = 1$ if $x = 1$ and $y = 0$; $x \setminus_1 y = na$ if $x = 1$ and $y = na$; $x \setminus_1 y = 0$ otherwise;

$x \setminus_2 y = 1$ if $x = 1$ and $y = 0$; $x \setminus_2 y = na$ if $x = na$ and $y = 0$; $x \setminus_2 y = 0$ otherwise.

Then the definition of $A(a, b, c, d)$ becomes

$$(a \setminus_1 b \equiv c \setminus_1 d) \wedge (b \setminus_2 a \equiv d \setminus_2 c) \wedge (a \setminus_2 b \equiv c \setminus_2 d) \wedge (b \setminus_1 a \equiv d \setminus_1 c)$$

where $x \equiv y = 1$ iff $x = y$; $x \equiv y = 0$ otherwise; and $\wedge$ is any conjunction connective that coincides with classical conjunction on $\{0, 1\}$. This definition yields 1 for the 15 expected patterns and is 0 otherwise for the 81 - 15 = 66 remaining patterns.

It is even possible to find an expression for $A(a,b,c,d)$ where $\backslash_1$ and $\backslash_2$ are expressed in terms of a conjunction (denoted $\wedge^*$) and two distinct negation operators $\overline{(\cdot)}^1$ and $\overline{(\cdot)}^2$, i.e., where $x\backslash_1$ y is replaced by $x \wedge^* \bar{y}^1$ and $x \backslash_2 y$ is replaced by $x \wedge^* \bar{y}^2$. We obtain a definition for $A(a,b,c,d)$ under the form

$$(a \wedge^* \bar{b}^1 \equiv c \wedge^* \bar{d}^1) \wedge^* (b \wedge^* \bar{a}^2 \equiv d \wedge^* \bar{c}^2) \wedge^* (a \wedge^* \bar{b}^2 \equiv c \wedge^* \bar{d}^2) \wedge^* (b \wedge^* \bar{a}^1 \equiv d \wedge^* \bar{c}^1)$$

where the two negations are Post-like negations defined through a circular ordering of the three truth-values, where the negation of a value is the successor value in the ordering, namely $\overline{0}^1 = na; \overline{na}^1 = 1; \overline{1}^1 = 0$ and $\overline{0}^2 = 1; \overline{na}^2 = 0; \overline{1}^2 = na$. This acknowledges the fact that in some sense these three truth-values play similar roles. The non-standard three-valued conjunction $\wedge^*$, which is defined by

$x \wedge^* y = 1$ if $x = 1$ and $y = 1$
$x \wedge^* y = u$ if $x = na$ and $y = na$
$x \wedge^* y = 0$ otherwise

also agrees with this view, while coinciding with classical conjunction in the binary case.

As in the previous section, we summarize in Table 13 the properties of the Boolean case that remain valid in this 3-valued model where na, standing for non applicable, is the third truth value.

Property name	Formal definition	Proportion
full identity	$T(a,a,a,a)$	A,R,P
reflexivity	$T(a,b,a,b)$	A,P
reverse reflexivity	$T(a,b,b,a)$	R,P
sameness	$T(a,a,b,b)$	A,R
symmetry	$T(a,b,c,d) \to T(c,d,a,b)$	A,R,P
permutation of means	$T(a,b,c,d) \to T(a,c,b,d)$	A
permutation of extremes	$T(a,b,c,d) \to T(d,b,c,a)$	A
all permutations	$\forall i,j, T(a,b,c,d) \to T(p_{i,j}(a,b,c,d))$	none

Table 13: Properties of A, R, P with truth value na (as non applicable)

4.3 Dealing with unknown features

In this section, we briefly consider a situation that is quite different from the ones studied in the two previous sections. We assume now that the features used for describing situations are all binary (i.e., they can be only true or false), but their truth value may be unknown.

Thus, the possible states of information regarding a Boolean variable x pertaining to a given feature may be represented by one of the 3 truth value subsets $\{0\}, \{1\}$ or $\{0, 1\}$, corresponding respectively to the case where the truth value of x is false, true or unknown. We denote this state of information by $\tilde{x}$, which is a subset of $\{0, 1\}$. The evaluation of a logical proportion $T(a, b, c, d)$ then amounts to compute the state of information denoted $\mathcal{T}(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})$ about its truth value, knowing $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$. It is given by the standard set extension:

$$\mathcal{T}(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}) = \{v(T(a, b, c, d)) \mid v(a) \in \tilde{a}, v(b) \in \tilde{b}, v(c) \in \tilde{c}, v(d) \in \tilde{d}\}$$

where v denotes a Boolean valuation.

From now on, we focus on analogical proportion A only, but R, P and I could be handled in a similar manner. For instance, let us take the example $A(a, b, c, d)$ where $\tilde{a} = \{1\}, \tilde{b} = \{0\}, \tilde{c} = \tilde{d} = \{0, 1\}$. Applying the previous formula leads to

$$\mathcal{A}(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}) = \{0, 1\}$$

since the truth value of $A(a, b, c, d)$ may be 0 for the valuations 1001, 1000, 1011, and 1 for 1010.

Let us now consider the following expression $A(a, b, a, b)$ when $\tilde{a} = \tilde{b} = \{0, 1\}$. A similar computation leads to

$$\mathcal{A}(\tilde{a}, \tilde{b}, \tilde{a}, \tilde{b}) = \{1\}$$

since the truth value of $A(a, b, a, b)$ is 1 for any of the valuations 1010, 1111, 0101, or 0000. Similarly, the truth value of $A(a, a, a, a)$ is 1, even when $\tilde{a} = \{0, 1\}$.

But, the set of possible truth values for $A(a, b, c, d)$ is $\{0, 1\}$ when $\tilde{a} = \{0, 1\}, \tilde{b} = \{0, 1\}, \tilde{c} = \{0, 1\}, \tilde{d} = \{0, 1\}$. It should be clear that this does not mean that the Boolean variables a, b, c, d are equal; we just have the same state of information for all of them. This expresses that the full identity property does not hold any longer at the information level for analogical proportion. And this illustrates the fact that the logic of uncertainty is no longer truth functional, since the state of information about the truth value of $A(a, b, c, d)$ does not only depend on the state of information about the truth values of a, b, c, and d, but is also constrained by the existence of possible logical dependencies between these variables.

Nevertheless, some key properties of homogeneous proportions remain valid at the information level such as symmetry, or central and extreme permutations. Indeed it can be checked that, for instance, for symmetry:

$$\mathcal{A}(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}) = \mathcal{A}(\tilde{c}, \tilde{d}, \tilde{a}, \tilde{b})$$

Using the set extension evaluation of logical proportions in presence of incomplete information, we can compute the set of possible truth values of the analogical proportion for the different 4-tuples of states of information. We now denote by u the state $\{0, 1\}$, and respectively by 0 and 1, the states of information $\{0\}$ and $\{1\}$. A 4-tuple of states of information will be called *information pattern*, or pattern for short, and denoted by a 4-tuple of elements of $\{0, 1, u\}$ without blank space. For instance, $01u1$ is such a pattern and should be understood as the 4-tuple of states of information $(\{0\}, \{1\}, \{0, 1\}, \{1\})$.

Then, the 6 patterns $0000, 1111, 0011, 1100, 1010, 0101$ that makes A true in the Boolean case, and where u does not appear, are the only ones that are still true with the above view (for which we get the singleton $\{1\}$ as information state for $A(a, b, c, d)$). As soon as at least one state of information is u in the pattern, the state of information for $A(a, b, c, d)$ is u or 0. Indeed, for instance, $01u0$ leads to 0 since whatever the truth value of the 3rd variable, the analogical proportion does not hold. Thus, despite the lack of knowledge regarding the 3rd variable, we know the exact truth value of the proportion in this case, namely it is false. It appears that there are 18 patterns that lead to 0. They are the 10 patterns of the Boolean case and the 8 following ones: $01u0$, $0u10$, $u001$, $100u$, $10u1$, $1u01$, $u110$, $011u$. Thus, in the $81 - 6 - 18 = 57$ remaining cases, the state of information for $A(a, b, c, d)$ is u.

It can be checked that these results can be retrieved both with the initial definition of A or with A^* where complete ignorance u is handled with $\bar{\ }$, $\wedge, \vee$ as the strong Kleene connectives (see [2]) and $\equiv$ as Bochvar connective, where u is an absorbing element. The corresponding truth tables are recalled in Table 14. This

	$\bar{\ }$	$\wedge$	0	1	u	$\vee$	0	1	u	$\equiv$	0	1	u
0	1	0	0	0	0	0	0	1	u	0	1	0	u
1	0	1	0	1	u	1	1	1	1	1	0	1	u
u	u	u	0	u	u	u	u	1	u	u	u	u	u

Table 14: Truth tables for u as lack of knowledge

provides a way to extend the definition of the analogical proportion in case of lack of knowledge when no dependencies between the variables exist. As in the Boolean case, the definitions A (resp. R, P, I) and A^* (resp. R^*, P^*, I^*) are equivalent.

Nevertheless, this truth-functional calculus provides only a description of the evaluation of the patterns at the information level. Namely, it enables us to retrieve the tri-partition of the patterns in respectively 6, 18 and 57 patterns leading respectively to 1, 0 and u, but it does not account for the full calculus of the extended definition of logical proportions in presence of incomplete information, when dependencies take place between variables, for instance it can be checked that $A(a,b,a,b)$ and $A^*(a,b,a,b)$ when a and b are unknown does not yield 1 as expected, but u (this is just due to the fact that constraints $a=c$ and $b=d$ are ignored).

5 Heterogeneous proportions

As highlighted in the introduction, there are 4 other proportions that satisfy code independency, and as such stand out of the 120 logical proportions, namely the heterogeneous proportions, whose truth tables are given in Table 15.

$\mathbf{H_a}$					$\mathbf{H_b}$					$\mathbf{H_c}$					$\mathbf{H_d}$			
1	1	1	0		1	1	1	0		1	1	1	0		1	1	0	1
0	0	0	1		0	0	0	1		0	0	0	1		0	0	1	0
1	1	0	1		1	1	0	1		1	0	1	1		1	0	1	1
0	0	1	0		0	0	1	0		0	1	0	0		0	1	0	0
1	0	1	1		0	1	1	1		0	1	1	1		0	1	1	1
0	1	0	0		1	0	0	0		1	0	0	0		1	0	0	0

Table 15: H_a, H_b, H_c, H_d - Boolean truth tables

It is stunning to note that these truth tables exactly involve the 8 missing tuples of the homogeneous tables, i.e., those ones having an odd number of 0 and 1. It should not come as a surprise that they satisfy the same association properties as the homogeneous ones: for instance, any combination of 2 or 3 heterogeneous proportions is satisfiable, but the conjunction $H_a(a,b,c,d) \wedge H_b(a,b,c,d) \wedge H_c(a,b,c,d) \wedge H_d(a,b,c,d)$ is not satisfiable. This fact contributes to make the heterogeneous proportions the perfect dual of the homogeneous ones.

5.1 Properties

The formal definitions given in Table 2 lead to immediate Boolean equivalences between heterogeneous and homogeneous proportions that we summarize in Table 16.

Obviously, the heterogeneous proportions are strongly linked together: for instance, using the symmetry of I,

$$H_a(a,b,c,d) \equiv I(\overline{a},b,c,d) \equiv I(c,d,\overline{a},b) \equiv H_c(c,d,a,b).$$

We may consider two different ways for generating these proportions:

$\mathbf{H_a}$	$\mathbf{H_b}$
$H_a(a,b,c,d) \equiv I(\overline{a},b,c,d)$	$H_b(a,b,c,d) \equiv I(a,\overline{b},c,d)$
$H_a(a,b,c,d) \equiv P(\overline{a},b,\overline{c},\overline{d})$	$H_b(a,b,c,d) \equiv P(a,\overline{b},\overline{c},\overline{d})$
$H_a(a,b,c,d) \equiv P(a,\overline{b},c,d)$	$H_b(a,b,c,d) \equiv P(\overline{a},b,c,d)$
$\mathbf{H_c}$	$\mathbf{H_d}$
$H_c(a,b,c,d) \equiv I(a,b,\overline{c},d)$	$H_d(a,b,c,d) \equiv I(a,b,c,\overline{d})$
$H_c(a,b,c,d) \equiv P(\overline{a},\overline{b},\overline{c},d)$	$H_d(a,b,c,d) \equiv P(\overline{a},\overline{b},c,\overline{d})$
$H_c(a,b,c,d) \equiv P(a,b,c,\overline{d})$	$H_d(a,b,c,d) \equiv P(a,b,\overline{c},d)$

Table 16: Equivalences between heterogeneous and homogeneous proportions

- A *semantic viewpoint*: The *full identity* postulate $T(a,a,a,a)$ asserts that proportion T holds between identical values. Negating one variable position only generates an intruder, as in $T(\overline{a},a,a,a)$, $T(a,\overline{a},a,a)$, $T(a,a,\overline{a},a)$ and $T(a,a,a,\overline{a})$, and leads to new postulates respectively denoted T_a, T_b, T_c and T_d. We call the negated position the *intruder position*: for instance, T_a expresses the fact that the first position is an intruder. For a proportion, to satisfy the property T_a means that *the first variable may be an intruder*. Since each postulate T_a, T_b, T_c and T_d is validated by 2 distinct valuations, it is clear that 3 of them are enough to define a logical proportion having exactly 6 valid tuples. There is no proportion satisfying all these postulates since it leads to 8 valid tuples, which excludes any logical proportion. It can be easily checked that H_a satisfies T_b, T_c, T_d and does not satisfy T_a: then H_a is uniquely characterized by the conjunction of properties $T_b \wedge T_c \wedge T_d$. We can interpret $H_a(a,b,c,d)$ as the following assertion: *the first position is not an intruder and there is an intruder among the remaining positions.* As a consequence, $H_a(a,b,c,d)$ does not hold when there is no intruder (i.e., when there is an even number of 0), or when a is the intruder. The same reasoning applies to H_b, H_c, H_d.

- A *syntactic viewpoint*: Here we start from the definition of the inverse paralogy I: $(a \wedge b \equiv \overline{c} \wedge \overline{d}) \wedge (\overline{a} \wedge \overline{b} \equiv c \wedge d)$. To get the definition of an heterogeneous proportion satisfying postulates where the intruder is in position 4, 2 or 1 for instance, we add a negation on the 3rd variable in both equivalences defining I. Here we get H_c as:

$$(a \wedge b \equiv c \wedge \overline{d}) \wedge (\overline{a} \wedge \overline{b} \equiv \overline{c} \wedge d)$$

This process, allowing us to generate the 4 heterogeneous proportions, shows that, in some sense, they are "atomic perturbations" of I: for this reason and since they are heterogeneous proportions, they have been respectively denoted H_a, H_b, H_c and H_d where the subscript corresponds to:

- the postulate *which is not satisfied* by the corresponding proportion or, equivalently,

- the negated variable in the equivalence with I.

For instance $H_a(a,b,c,d) \equiv I(\overline{a},b,c,d)$, H_a satisfies T_b, T_c, T_d and does not satisfy T_a. This leads to another way to interpret $H_a(a,b,c,d)$. Since $H_a(a,b,c,d) \equiv I(\overline{a},b,c,d)$, when $H_a(a,b,c,d) = 1$, a is not the intruder, i.e., $\overline{a}$ is the value of the intruder. The different possible cases are as follows:

- $\overline{a}bcd=$ 1100 or 0011 and the intruder is b,

- or $\overline{a}bcd =$ 0101, 0110, 1010 or 1001 and the intruder is c or d.

In other words, there is an intruder in (a,b,c,d), which is not a, iff the properties common to $\overline{a}$ and b (positively or negatively) are not those common to c and d, and conversely.

As in the case of homogeneous proportions, the semantic properties of heterogeneous proportions are easily derived from their truth tables, which we summarize in Table 17. It is clear on their truth tables, that none of the heterogeneous propor-

Property name	Formal definition	Proportion
full identity	$T(a,a,a,a)$	none
reflexivity	$T(a,b,a,b)$	none
reverse reflexivity	$T(a,b,b,a)$	none
sameness	$T(a,a,b,b)$	none
symmetry	$T(a,b,c,d) \to T(c,d,a,b)$	none
means permut.	$T(a,b,c,d) \to T(a,c,b,d)$	H_a, H_d
extremes permut.	$T(a,b,c,d) \to T(d,b,c,a)$	H_c, H_b
all permutations	$\forall i,j, T(a,b,c,d) \to T(p_{i,j}(a,b,c,d))$	none
transitivity	$T(a,b,c,d) \wedge T(c,d,e,f) \to T(a,b,e,f)$	none
Ta	$T(\overline{a},a,a,a)$	H_b, H_c, H_d
Tb	$T(a,\overline{a},a,a)$	H_a, H_c, H_d
Tc	$T(a,a,\overline{a},a)$	H_a, H_b, H_d
Td	$T(a,a,a,\overline{a})$	H_a, H_b, H_c

Table 17: Properties of heterogeneous proportions

tions satisfy neither symmetry nor transitivity. From a practical viewpoint, these proportions are closely related with the idea of spotting the odd one out (the intruder), or if we prefer of picking the one that doesn't fit among 4 items. This will be further discussed in Section 6, but we first consider the extension of heterogeneous proportions to the case of graded properties with intermediate truth values.

5.2 Multiple-valued semantics

We extend here what has been done for homogeneous proportions and their multiple-valued semantics in Section 4.1. Roughly speaking, in the case of H_a, the graded truth value of $H_a(a, b, c, d)$ estimates how *far* we are from having a as an intruder.

Obviously the same questions as for homogeneous proportions arise but with a different interpretation:

1) what are the valuations that correspond to a "perfect" proportion of a given type (i.e., having 1 as truth degree)? For instance, we want the truth value of $H_a(0, u, 0, u)$ to be equal to 1 (as well as the truth value of $H_c(0, u, 0, u)$) because in that context, it is true that $a = 0$ (resp. c) cannot be the intruder, whatever the value of u.

2) are there valuations that could be regarded as *approximate* proportions of a given type (with an intermediate truth degree) and in that case, what is their truth value? For instance, in the valuation $(0.7, 1, 1, 0.9)$, it is likely that a is the intruder just because the other candidate, d, has a value very close to 1, and the closer d is to 1, the more likely a is the intruder: then the truth value of $H_a(0.7, 1, 1, 0.9)$ should be small and related to $1 - d = 0.1$ (since H_a excludes a as intruder).

The most rigorous way to proceed is to start from the definition of multiple-valued paralogy given in Section 4.1. This definition is based on P^*: it leads, for a three valued scale, to 15 valuations fully true, and 18 fully false. The 48 remaining patterns get intermediate truth value given by the following general formula

$$P^*(a, b, c, d) = min(1 - |max(a, b) - max(c, d)|, 1 - |min(a, b) - min(c, d)|)$$

which, thanks to the symmetry of P^* and stability w.r.t. the permutation of its two first variables, has the following behavior:

general case	case $u = v$
$P^*(1, 1, u, v) = \min(u, v)$	$P^*(1, 1, u, u) = u$
$P^*(1, 0, u, v) = \min(\max(u, v), 1-\min(u, v))$	$P^*(1, 0, u, u) = \min(u, 1-u)$
$P^*(0, 0, u, v) = 1 - \max(u, v)$	$P^*(0, 0, u, u) = 1 - u$

Starting from the equivalences given in Table 16, we get the multi-valued definition for H_a (and similar definitions for H_b, H_c, H_d), still leading to 15 true valuations, 18 false valuations and 48 with intermediate values in case of a three valued scale:

$$H_a(a,b,c,d) = \min(1-|\max(a,1-b)-\max(c,d)|, 1-|\min(a,1-b)-\min(c,d)|)$$

Let us note that $H_a(0,0,u,v) = H_a(1,1,u,v)$ due to the equality $H_a(0,0,u,v) = P(0,1,u,v) = P(1,0,u,v)$. We have:

general case	case $u = v$
$H_a(1,1,u,v) = \min(\max(u,v), 1-\min(u,v))$	$H_a(1,1,u,u) = \min(u,1-u)$
$H_a(1,0,u,v) = \min(u,v)$	$H_a(1,0,u,u) = u$
$H_a(0,1,u,v) = 1-\max(u,v)$	$H_a(1,0,u,u) = 1-u$

Let us analyze two examples to highlight the fact that the above definition really fits with the intuition.

- Considering the valuation $100u$, its truth value is:

  - u for P: if u is close to 1, we are close to the fully true paralogical proportion and the truth value is high. In the opposite case, u is close to 0 and we are close to a fully false paralogical proportion 1000.

  - 1-u for H_b, H_c, H_d: if u is close to 1, we are close to the valuation 1001 which is definitely not a valid valuation for H_b, H_c, H_d: so 1-u is a low truth value. But if u is close to 0, we are close to the valuation 1000 which is valid for H_b, H_c, H_d and 1-u is a high truth value.

  - finally 0 for H_a: whatever the value of u, 100u means "an intruder is in first position", when the semantics of H_a is just the opposite.

- Back to the graded valuation valuation 0.7 1 1 0.9 considered above:

  - regarding P, the truth value as given by the formula is 0.8, i.e., the valuation is close to be a true paralogy.

  - regarding the heterogeneous proportions, we understand that we have 2 candidate intruders namely $a = 0.7$ and $d = 0.9$. But they are not equivalent in terms of intrusion and it is more likely to be a than d. This is consistent with the fact that the truth value of $H_a(0.7, 1, 1, 0.9)$ is 0.1 (very low), but the truth value of $H_d(0.7, 1, 1, 0.9)$ is 0.3 (a bit higher).

  - in fact, 0.7 1 1 0.9 does not give a genuine impression that there is an intruder, which is in agreement with the fact that $H_b(0.7, 1, 1, 0.9) = H_c(0.7, 1, 1, 0.9) = 0.4$.

6 Applications

In this section, we provide an overview of the use of logical proportions for various reasoning purposes. Since we have distinguished two remarkable groups of proportions, differing both from a syntactic and a semantic viewpoint, it is not surprising that they can be used for two different styles of applications. On the one hand, the homogeneous proportions allow us to build up a missing item in a given sequence. On the other hand, the heterogeneous proportions are suitable for a dual task which is to pick up the odd one out in a set. Let us start by discussing the use of the homogeneous logical proportions, which is the most developed.

6.1 Using homogeneous proportion for finding missing values

From a general viewpoint, a homogeneous proportion between 4 items a, b, c, d expresses that the elements of the pair (a, b) are similar (or dissimilar) in a way that can be related to the way the elements of the pair (c, d) are similar (or dissimilar). The equation-solving process described above enables us to compute d from the knowledge of a, b, c, when possible. Obviously, in practical cases, the items to be considered cannot be simply described by a single Boolean (or multiple-valued) variable, and a straightforward extension, allowing to cope with more sophisticated representations, can be given for Boolean vectors in $\mathbb{B}^n$, as follows (where T denotes any logical proportion):

$$T(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}, \overrightarrow{d}) \text{ iff } \forall i \in [1, n],\ T(a_i, b_i, c_i, d_i).$$

The solving process of the equation $T(\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}, \overrightarrow{x})$ is still effective: instead of getting one Boolean value, we get a Boolean vector, by solving equations componentwise, computing d_i from a_i, b_i, and c_i (provided that the solution exists). This can be illustrated on a sequence of 3 pictures to be completed (see Figure 1, as it is often the case in IQ tests . Indeed, a noticeable part of the IQ tests are based on providing incomplete analogical proportions (see, e.g., [6]). Usually, the 3 first items A, B, C are given and the 4th item X has to be chosen among several plausible candidates. In this case, the homogeneous logical proportion method applies straightforwardly. The items A, B, C in the example of Figure 1 can be described respectively by vectors $(1, 0, 1, 0, 1)$, $(1, 0, 0, 1, 1)$, $(0, 1, 1, 0, 1)$, where the vector components refer respectively to the presence (or not) of a square, of a triangle, of a star, of a circle, and of a black point. Assuming that an analogical proportion should hold, by solving componentwise the analogical proportion equations expressing that $A(a_i, b_i, c_i, x_i)$ holds true for $i = 1, 5$, we easily get $X = (0, 1, 0, 1, 1)$, which corresponds to the result exhibited in Figure 1. Note that X is directly computed with

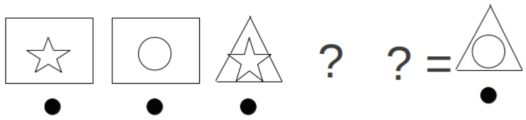

Figure 1: IQ test: Graphical analogy

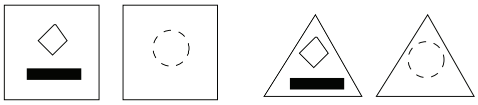

Figure 2: Analogy with a graded feature

this method, rather than chosen among a set of more or less “distant” potential solutions that would be given. In case the analogical equation has no solution for some component, on may try if another homogeneous proportion would fit for all the features. It would not be difficult to build examples of sequences of 4 pictures, where the display of squares, triangles, stars, circles and black dots is different from Figure 1, and where the fourth picture would be obtained via one of the three other homogeneous proportions R, P, or I, rather than via A as in Figure 1.

Moreover, Figure 2 illustrates the idea of having graded features, where here the presence of a circle is a matter of degree (the more densely dotted the circle, the higher the degree α of presence of a circle (in Figure 2 the analogical proportion $A(0, \alpha, 0, \alpha)$ clearly holds for the ‘circle’ variable).

In the above example, the problem is handled at a rather high conceptual level that requires that triangles, circles and so on be identified in the pictures. However, it has been pointed out [20] that the analogical proportion-based technique can still be applied at the pixel level. Then a black and white picture is represented by the Boolean vector made of its *bitmap* description that acknowledges (or not) the black color of each pixel. This supposes that all the geometric shapes (squares, triangles, stars, circles) use exactly the same pixels in all cases. Then, the proportion-based procedure automatically builds the associated geometric figure (when it exists), without introducing any knowledge about triangle, circle, etc.

Lastly, let us mention that it may be convenient to have extensions of the proportions allowing for the explicit handling of functional symbols, as in, e.g., the analogical proportion $A(x, f(x), y, f(y))$, for handling more sophisticated sequences of pictures (where for instance, elements are reversed from one picture to another), or analogical proportions quizzes like "*abc* is to *abd* as *ijk* is to ?" (where we have to encode that d is the successor of c); see [3].

6.2 Classification and matrix abduction

We now consider variants of the process described in the previous subsection, when it is first checked that an homogeneous proportion holds on a series of n features between 4 items, and on this basis, one extrapolates that the same logical proportion still holds for a $(n+1)$th feature of interest, which is known only for the first 3 items. Solving the logical proportion equation corresponding to this latter feature then enables us to compute a plausible value of this feature for the 4th item. Classification problems are an important instance of this situation where the $(n+1)$th feature refers to the class of the item while the n other features pertain to its description. See [13] for the case of binary features, where very good results are reported on classification benchmarks. The graded version A has been used for handling numerical features in classification problems (also with promising experimental results [23]), while A^* has not been experienced yet. It is still unclear if A^* may be more suitable for classification purposes.

The problem of completing a matrix where some values are missing is quite close to the classification problem, and thus different methods may be thought of in order to deal with this issue. Whatever the technique, the main question is to know if the extra knowledge that we may have about the problem and the available data carry sufficient information for an accurate reconstruction of the missing cells. This is not always the case. We focus here on a particular case, called "matrix abduction problem", using [1]'s terminology. It consists in guessing plausible values for cells having empty information in a matrix where each line corresponds to a situation described according to different binary features (each column corresponds to a particular feature).

Let us consider the screen example used by [1], where computer screens are described by 6 characteristic features: P is for price over £450, C for self collection, I for screen bigger than 24 inch, R for reaction time below 4ms, D for dot size less than 0.275, and S for stereophonic; 1 means "yes" and 0 means "no". We have 3 screens (screen 1, screen 2 and screen 4) whose characteristics are known and screen 3 where the truth value of the attribute S is missing (see Table 18).

To tackle such a common sense problem, a general idea (which may be also found in classification) is that replacing an unknown value by either 1 or 0 should result in the least possible *perturbation* of the matrix. This idea may be implemented in diverse ways. In [1] the idea is to choose the value that least perturbs the pre-

	P	C	I	R	D	S
$screen1$	0	1	0	1	0	1
$screen2$	0	0	1	1	0	1
$screen3$	0	0	0	0	1	?
$screen4$	1	1	0	0	1	1

Table 18: The screen example

existing partial ordering between the column vectors of the matrix. In [25], the idea is rather to respect betweenness and parallelism relations that hold in conceptual spaces. We suggest here to enforce an homogeneous proportion T that already holds for completely known features.

Assume we have a Boolean vector incompletely describing a situation with respect to a set of $n+1$ considered features, say $v = (v_1, ..., v_n, x_{n+1})$, where for simplicity we assume that only x_{n+1} is unknown. For trying to make a plausible guess of the value of x_{n+1}, we have a collection (which may be rather small) of completely informed examples $e^i = (e^i_1, ..., e^i_n, e^i_{n+1})$ for $i = 1, n$. Then one may have at least three strategies:

i) comparing v to each e^i separately, and using a k-nearest neighbors approach, extending the idea that $T(e, e, e, v)$ should hold true and has $v = e$ as solution.

ii) looking for pairs e^i, e^j such that $T(e^i_h, v_h, v_h, e^j_h)$ makes a continuous homogeneous proportion T for a maximal number of features h, implementing the idea of having v_h between e^i_h and e^j_h ; observe however, that in the Boolean case, this would force to have the trivial situations $T(1, 1, 1, 1)$ or $T(0, 0, 0, 0)$ on a maximal number of features, and tolerate some "approximate" patterns $T(1, 1, 1, 0)$, $T(0, 1, 1, 1)$, $T(0, 0, 0, 1)$, or $T(1, 0, 0, 0)$, while rejecting patterns $T(0, 1, 1, 0)$ and $T(1, 0, 0, 1)$.

iii) looking for triples e^i, e^j, e^k such that $T(e^i_h, e^j_h, e^k_h, v_h)$ makes an homogeneous proportion T for a maximal number of features h.

In cases ii) or iii), the principle amounts to say that if an homogeneous proportion holds for a number of features as great as possible among features h such that $1 \leq h \leq n$, it should still hold for feature $n+1$, which provides an equation for finding x_{n+1} if solvable. If there are several triples that are equally good in terms of numbers of features for which the proportion holds, but lead to different predictions, one may then consider that there is no acceptable plausible value for x_{n+1}.

The application of the first strategy on the above example yields 1 considering that screen 3 is already identical to screen 4 on 3 features. Using the second strategy, we observe that screen 3 is only in "between" screen 2 and screen 4 in the sense described above, leading again to 1 as a solution.

Using the third strategy that should involve 4 distinct items, we can observe that the analogical proportion $A(screen\ 1, screen\ 2, screen\ 4, screen\ 3)$ holds componentwise for features C, R, and D (while it fails with proportions P and I). Again we get 1 as a solution for ensuring an analogical proportion (namely $A(1,1,1,1)$) on S. Observe also that whatever the order in which the screens are considered, an homogeneous proportion holds for features C, R, D, and S. Considering other triples (if available) may lead to other equations having 0 as a solution. A prediction based on the triple making an homogeneous proportion with the incompletely described item on a maximal number of features, should be preferred. In case of ties on this maximal number of features between concurrent triples leading to opposite predictions, no prediction can be given. It is worth noticing that in [1], the use of 0 and 1 in the Boolean coding in their matrices is not just a matter of convention and we cannot exchange the 2 values since it will change the ordering. This is not the case with our approach since A, R, P, I satisfy code independency. The screen example is clearly a toy example but, in [1], similar examples are discussed which could also be handled using homogeneous proportions.

6.3 Analogical proportions in Raven's tests

Among the picture-based IQ tests (the use of pictures avoids the bias of a cultural background), the so-called Raven's Progressive Matrices [24] are considered as a reference for estimating the reasoning component of "the general intelligence". Recently a computational model for solving Raven's Progressive Matrices has been investigated in [11]. This model combines qualitative spatial representations with analogical comparison via structure-mapping [7]. In the following, we suggest with an example that the Boolean proportion approach can be also used for solving such a test (see [20, 3] for other examples).

Each Raven test is constituted with a 3x3 matrix $pic[i, j]$ of pictures where the last picture $pic[3, 3]$ is missing and has to be chosen among a panel of 8 candidate pictures. An example is given in Figure 3 and its solution in Figure 4. We assume that the Raven matrices can be understood in the following way, with respect to rows and columns:

$$\forall i \in [1, 2], \exists f \text{ such that } pic[i, 3] = f(pic[i, 1], pic[i, 2])$$

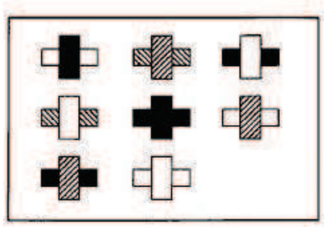

Figure 3: An example of Raven matrix

Figure 4: Raven matrix: the solution

$$\forall j \in [1,2], \exists g \text{ such that } pic[3,j] = g(pic[1,j], pic[2,j])$$

The two complete rows (resp. columns) are supposed to help to discover f (resp. g) and to predict the missing picture $pic([3,3])$ as $f(pic[3,1], pic[3,2])$ (resp. $g(pic[1,3], pic[2,3])$).

Obviously, these tests do not fit the standard equation solving scheme, but they follow an extended one telling us that $A((a,b), f(a,b), (c,d), f(c,d))$ holds for lines and $A((a,b), g(a,b), (c,d), g(c,d))$ for columns, i.e.

$$A((pic[1,1], pic[1,2]), pic[1,3], (pic[2,1], pic[2,2]), pic[2,3])$$

$$A((pic[1,1], pic[2,1]), pic[3,1], (pic[1,2], pic[2,2]), pic[3,2])$$

Thus, in that case, we have to consider a pair of cells $(pic[i,1], pic[i,2])$ as the first element of an analogical proportion, and then the pair $((pic[i,1], pic[i,2]), pic[i,3])$ provides the 2 first element a and b of the analogical proportion we are considering. In terms of coding, in the example of Figure 3, we may consider the pictures as represented by a pair (or vector) (hr, vr) with one horizontal rectangle hr and a vertical one vr, each of these rectangles having one color among $Black, White, Grey$, we have then the following obvious encoding of the matrix in Table 19.

It leads to the following analogical patterns (using the traditional notation for analogical proportion $a : b :: c : d$ instead of $A(a,b,c,d)$):

	1	2	3
1	WB	GG	BW
2	GW	BB	WG
3	BG	WW	?i?ii

Table 19: Raven matrix: a coding

(WB,GG) : BW :: (GW, BB) : WG (1st and 2nd rows)
(WB,GG) : BW :: (BG,WW) : ?i?ii (1st and 3rd rows)
where BW = f(WB,GG) and WG = f(GW,BB).

(WB,GW) : BG :: (GG, BB) : WW (1st and 2nd columns)
(WB,GW) : BG :: (BW,WG) : ?i?ii (1st and 3rd columns)
where BG = f(WB,GW) and WW = g(GG,BB),

or if we prefer, since analogical proportions holds componentwise, we have the following valid proportions
- for the horizontal bars:
(W,G) : B :: (G, B) : W (horizontal analysis)
(W,G) : B :: (B,W) : ?i (horizontal analysis)
(W,G) : B :: (G, B) : W (vertical analysis)
(W,G) : B :: (B,W) : ?i (vertical analysis)
- for the vertical bars:
(B,G) : W :: (W, B) : G (horizontal analysis)
(B,G) : W :: (G,W) : ?ii (horizontal analysis)
(B,W) : G :: (G, B) : W (vertical analysis)
(B,W) : G :: (W,G) : ?ii (vertical analysis)

One can observe that the item (B, W) appears only in the analogical proportions with question marks for horizontal bars, while the items (G, W) and (W, G) appear only in the analogical proportions with question marks for vertical bars. Analogical proportions coming from both horizontal or vertical analysis are insufficient for concluding here. However, we can consider the Raven matrix provides a set of analogical associations without any distinction between those ones coming from the horizontal bars and those ones coming from vertical bars. In other words, we now relax the componentwise reading by considering that what applies to horizontal bars, may apply to vertical bars, and vice-versa. With this viewpoint, it appears that the pair (B, W) and the pair (W, G) are respectively associated to G (vertical association for vertical bar) and B (horizontal association for horizontal bar), which encodes the expected solution GB (as pictured in Figure 4). Note also that (G, W) cannot help predicting ?ii.

6.4 Using heterogeneous proportions "to pick up the one which does not fit"

As it is the case for homogeneous proportions, heterogeneous proportions can also be related to the solving of some type of quiz problem. As we have seen, the truth tables of the heterogeneous proportions highlight a Boolean value (0 or 1) which is different from the 3 remaining ones. It is then natural to think in terms of exception or intruder in a sequence of 4 items: the heterogeneous proportions play a dual role with regard to homogeneous proportions. Given a sequence of objects, they allow to distinguish the object which does not follow the "logic" of the sequence. As a consequence, heterogeneous proportions are suitable for the "Finding the odd one out" problem where a complete sequence of items being given, we have to find the item that does not fit with the other ones and which is, in some sense, an intruder or an anomaly. On this basis, a complete battery of IQ tests has been recently proposed in [8]. Solving 'Find the odd one out' tests (which are visual tests) has been recently tackled in [12] by using analogical pairing between fractal representation of the pictures. It is worth noticing that the approach of these authors takes its root in the idea of analogical proportion. However, this method relies on the use of similarity/dissimilarity measures rather than referring to a formal logical view of analogical proportion. In the following, we show that an opposite type (in some sense) of proportions, namely heterogeneous proportions, provides a convenient way to code and to tackle this problem.

Let us first consider the case of 4 items: obviously, if these items are completely different in many respects, there is no notion of intruder. The intruder comes as soon as there is a kind of unique dissimilarity among an obvious set of similarities or identities. Let us start with a simple case where each item a can be represented as a Boolean vector $a_1, \ldots, a_n$ where n is the number of attributes and $a_i \in \{0, 1\}$. Let us consider the simple example $(lorry, bus, bicycle, car)$ (where the obvious intruder is $bicycle$) shown in Figure 5 where $n = 5$ with a straightforward coding.

When considering the item componentwise, we see that:

- for $i = 1, 3, 5$, $H_a(a_i, b_i, c_i, d_i) = H_b(a_i, b_i, c_i, d_i) = H_c(a_i, b_i, c_i, d_i) = H_d(a_i, b_i, c_i, d_i) = 0$.
- for $i = 2, 4$, $H_a(a_i, b_i, c_i, d_i) = H_b(a_i, b_i, c_i, d_i) = H_d(a_i, b_i, c_i, d_i) = 1$.

	canMove	hasEngine	onRoad	has4Wheel	canFly
lorry	1	1	1	1	0
bus	1	1	1	1	0
bicycle	1	0	1	0	0
car	1	1	1	1	0

Figure 5: A simple quiz and its Boolean coding

- for $i = 2, 4$, $H_c(a_i, b_i, c_i, d_i) = 0$

The indexes $1, 3$ and 5 are not useful to pick up the intruder because all the proportions have the same truth value. This is not the case for the indexes 2 and 4: H_a for instance, being equal to 1, insures that there is an intruder (which is not the first element). The intruder is then given by the proportion having the value 0: for instance, $H_c(a_i, b_i, c_i, d_i) = 0$ means that the fact that c is not an intruder is false, which exactly means that c is the intruder for component j. In our example, H_c is 0 on both components 2 and 4: this exactly leads to consider the third element *bicycle*, intruder for the components 2 and 4, as the global intruder. It may be the case that, we do not get the same intruder depending on the component: in that case, a majority vote may be applied and we choose as intruder the one which is intruder for the maximum number of components.

Thanks to the multiple-valued extension, this method can be generalized to the non Boolean case where each item a is represented as a real vector $a_1, \ldots, a_n$ and $a_i \in [0, 1]$. Then, the truth values of H_a, H_b, H_c and H_d on some features may be close to 0, which means that there is no clear intruder according to these features. Let us focus on the other features that are not identical. For each such index j, we can compute the 4 values $H_a(a_j, b_j, c_j, d_j)$, $H_b(a_j, b_j, c_j, d_j)$, $H_c(a_j, b_j, c_j, d_j)$ and $H_d(a_j, b_j, c_j, d_j)$. We know that they cannot be all equal (or close to) 1 since their conjunction is not satisfiable: in fact, exactly one proportion has to be close to 0, thus spotting out the intruder for that component. Applying again a majority vote, we shall consider as global intruder the one which is intruder for the maximum number of components.

In the case where we have to 'Find the odd one out' among more than 4 items, diverse options are available. We may consider all the subsets of 4 items. For each such subset, we apply the previous method to exhibit an intruder (if any). Then the global intruder will be the one which is intruder for the maximum number of subsets.

7 Conclusion

The Boolean modeling of logical proportions which relate 4 items in terms of similarity and dissimilarity, and which may be viewed as a counterpart to numerical proportions, has led to identify a set of 120 distinct proportions. All these logical proportions have the same type of truth table, namely they are true for exactly 6 valuations (and thus false for the 10 remaining valuations). Among this set, only 8 proportions satisfy a so-called code-independency property which makes sure that the evaluation of the proportion remains unchanged when the truth values of the 4 components are changed into their complement (1 is changed into 0, and 0 into 1). This property is important since it ensures that the evaluation of logical proportions will not depend on the positive or negative encoding of the features of the considered items. This set of 8 remarkable proportions divides into 4 homogeneous proportions, and the 4 heterogeneous proportions. These two subsets can be strongly contrasted and appear to be complementary. The 6 valuation patterns that make true homogeneous proportions have all an even number of 1 (and consequently of 0), while for heterogeneous proportions the numbers are odd. Homogeneous proportions are symmetrical, while heterogeneous ones are not. Both types of proportions satisfy remarkable permutation properties. Interestingly enough, these two subsets of logical proportions can be related to two types of IQ tests or quizzes respectively of the type “Find the missing item” and of the type “Find the odd one out”. Thus, both from a formal viewpoint and from an applicative viewpoint, heterogeneous proportions appear as a perfect dual of the homogeneous ones. Ultimately, logical proportions provide an elegant unique framework for dealing with IQ tests, from Raven progressive matrices to Find the odd one out quizzes, in a uniform way. Generally speaking, beyond these illustrations, logical proportions still constitute an intriguing set of quaternary connectives, including diverse subsets with remarkable properties, that look interesting for different reasoning purposes where the ideas of similarity and dissimilarity play a role.

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