"This is the peer reviewed version of the following article: [International Journal of Robust and Nonlinear Control, 2017***], which has been published in final form at [***http://dx.doi.org/10.1002/rnc.3852***]. This article may be used for non-commercial purposes in accordance with Wiley Terms and Conditions for Self-Archiving."***

$\mathscr{H}_2$-based optimal sparse sliding mode control for networked control systems

Ahmadreza Argha, Li Li, and Steven W. Su
Email: steven.su@uts.edu.au
Faculty of Engineering and Information Technology
University of Technology Sydney
PO Box 123, Broadway, NSW 2007, Australia

August 29, 2017

Abstract

This paper is devoted to the problem of designing a sparsely distributed sliding mode control for networked systems. Indeed, this note employs a distributed sliding mode control framework by exploiting (some of) other subsystems' information to improve the performance of each local controller so that it can widen the applicability region of the given scheme. To do so, different from the traditional schemes in the literature, a novel approach is proposed to design the sliding surface, in which the level of required control effort is taken into account during the sliding surface design based on the $\mathscr{H}_2$ control. We then use this novel scheme to provide an innovative less-complex procedure that explores sparse control networks to satisfy the underlying control objective. Besides, the proposed scheme to design the sliding surface makes it possible to avoid unbounded growth of control effort during the sparsification of the control network structure. Illustrative examples are presented to show the effectiveness of the proposed approach.

keywords Networked control systems, $\mathscr{H}_2$-based optimal sparse sliding mode control, distributed control systems, linear matrix inequality.

1 Introduction

Control systems that utilize spatially distributed components have been studied for a while. In early control systems, the information of the distributed sensors was transmitted to a central station (controller) through direct hard-wired links and then the generated control commands were sent to the spatially distributed actuators. Thanks to the recent advances in communication technology, efficient communication networks have been used in control systems, which has opened a new research area to consider the influences of communication networks on control systems [21].

Spatially distributed control systems with communication networks in their loop have been also regarded as networked control systems (NCSs). Utilizing a centralized control scheme in NCSs requires the central controller to have access to the states of all subsystems' plants, which is not practical as it needs a larger and more costly control network. On the other hand, decentralized or distributed control architectures have been proposed and used in the literature [36, 30, 1, 11]. The general idea behind the decentralized control scheme is to use only the local state information in order to control the subsystems and thus there is no control network. This can be effective only when the interconnections between the subsystems are not strong [35, 25]. In other words, when the interconnections are strong, utilizing distributed control frameworks has been considered. In this strategy, each subsystem can exploit local state as well as the state of some other subsystems. Hence, compared to the decentralized control scheme, distributed control scheme can ensure the stability of the overall system in the presence of stronger subsystem interconnections [31]. Meantime, it also has less complexity and improved computational aspects compared to the centralized control scheme.

An outstanding research implemented on the sliding mode control (SMC) has been the decentralized SMC for large-scale interconnected systems; see [32, 33, 20, 17, 34] and the references therein. However, in the literature the distributed SMC, for the cases that the interconnections between different subsystems are not weak, has received less attention and hence requires more investigations. To achieve better closed-loop control performance, this paper, as the first step, provides a methodology to design a distributed SMC for a network with an arbitrarily given control topology, by establishing some level of communication between the different sub-controllers. It is also assumed that communication networks do not have any data packet loss, bandwidth limitation, or network delays. We will show that the proposed distributed

SMC has the ability to cover all the cases such as decentralized, distributed, and sparsely distributed topologies. Then, we will consider the problem of finding a sparse control network structure that can satisfy the control objective. This issue is basically vital in the design of massively distributed control networks, such as smart grid systems [21].

Although the SMC is now a well-known strategy, from the standpoint of constraining the available control action, all the traditional methods considered in the literature have shortcomings [5]. This drawback basically comes from the nature of the SMC design process which contains two separate stages. During the synthesizing the sliding function, there is no sense of the control action level that is required to induce and retain sliding. This issue is more crucial in this paper when it comes to sparsify the control network structure, as with no limits on the available control actions, it may result in the high level of control efforts that each subsystem's controller requires to apply, which is not a practical case. To deal with this problem, [19] proposes a scheme to design a sliding surface which minimizes a cost functional of the system state and control input. However, the method given in [19] has several limitations. As the method in [19] needs to ensure that at least one eigenvalue of the closed-loop system (for single input systems) is a real value, not necessarily any arbitrary weighting matrices in the objective function can result in a sliding mode control. Hence, this reference either reselects the weighting matrices or approximates the closed-loop system eigenvalues so that a set of eigenvalues are generated which can be split between the range-space and null-space dynamics. However, no precise scheme is given on how to reselect the weighting matrices. Further, the approximation of eigenvalues may lead to a loss in optimality and possibly robustness. In order to resolve the limitations of [19], [26] proposes a framework in which a weighting matrix is computed that is tried to be the closest to the desired one and also results in the desired eigenvalues. The SMC then can be designed according to the obtained eigenvalues and weighting matrix. However, both methods in [19, 26] are only applicable to single input systems. Alternatively, [5] considers this problem and proposes two new frameworks. However, the proposed methods in [5] rely on a special system coordinate transformation which bounds the possible adoption of these methods to our control structure sparsification problem. This paper alternatively develops a different approach by which we can deal with an $\mathscr{H}_2$ based optimal structured SMC problem.

Recently, the issue of designing a control network with minimum communication links has been studied in the literature [21, 23, 24, 22, 14, 15, 16]. As an illustration, [14] proposes a non-convex condition which is solved numerically by exploiting a convex reweighted ℓ_1 norm approximation. Furthermore, [21] considers the problem of finding the sparsest control/observer network that satisfies the obtained stability condition. Roughly speaking, the sparsity is formulated in terms of cardinality (ℓ_0 quasi-norm) of the feedback gain in these references, which is then relaxed by the (weighted) ℓ_1-norm (see [3]). In this paper in order to address the problem of designing a sparse SMC controller, a specific form of fictitious system, whose matrices contain the control network structure, is derived. This makes the well-developed weighted ℓ_1 algorithm infeasible to our problem. Alternatively, this paper proposes a heuristic scheme to obtain the sparse sliding mode controller.

The rest of this paper is as follows. Section 2 describes the problem formulation and preliminaries. Section 3 presents the structured $\mathscr{H}_2$ based sliding mode control. Section 4 demonstrates our heuristic method to solve the problem of finding a favourable sparse structure for the control network. Effectiveness of the proposed sparsely distributed SMC is studied by numerical examples in Section 5. Finally, Section 6 concludes this paper.

Notation: $\left[\Sigma_{ij}\right]_{q\times q}$ is a block matrix with block entries Σ_{ij}, $i=1, \cdots, q$, $j=1, \cdots, q$. $\mathrm{diag}\,[\Sigma_i]_{i=1}^{q}$ is a block-diagonal matrix with block entries Σ_i, $i=1, \cdots, q$. Moreover, $\mathrm{col}(v_i)_{i=1}^{q}$ denotes a block-vector with block entries v_i, $i=1, \cdots, q$. $\{\circ\}$ denotes an operator for $\Xi=[\xi_{ij}]_{h\times h}$ in which $\xi_{ij}\in\mathbb{R}$ and $W=[W_{ij}]_{h\times h}$ in which $W_{ij}\in\mathbb{R}^{r_i\times s_j}$ such that $\Xi\circ W=[\xi_{ij}W_{ij}]_{h\times h}$. $\mathbb{1}_{p\times q}$ denotes a $p\times q$ matrix with all entries equal to 1.

2 Problem Formulation and Preliminaries

Consider a large scale networked system consisting of h subsystems,

$$\dot{x}_i(t)=A_ix_i(t)+\sum_{j=1,\ j\neq i}^{h}A_{ij}x_j(t)+B_i[u_i(t)+f_i(x_i)], \tag{1}$$

where $x_i\in\mathbb{R}^{n_i}$ and $u_i\in\mathbb{R}^{m_i}$ are the state vector and control input vector of the i-th subsystem, respectively. The matrices in (1) are constant and of appropriate dimensions. Besides, $A_{ij}\neq 0$ if the sub-system j influences directly the sub-system i. Without loss of generality, it is also assumed that $\mathrm{rank}(B_i)=m_i$. $f_i(x_i)\in\mathbb{R}^{m_i}$ is the matched uncertainty.

To design the sliding surface in this paper it is assumed that the system in (1) is in a special coordinate (see e.g. [5]) which is more stringent than what is considered in the well-known regular form coordinate. Thus, it is assumed that the input distribution matrix in (1) has the following form

$$B_i = \begin{bmatrix} 0 \\ I_{m_i} \end{bmatrix}, \tag{2}$$

Define

$$\begin{aligned} x(t) &:= \mathrm{col}(x_i(t))_{i=1}^h, \quad u(t) := \mathrm{col}(u_i(t))_{i=1}^h, \\ f(x) &:= \mathrm{col}(f_i(x_i))_{i=1}^h, \end{aligned} \tag{3}$$

and

$$A := \mathrm{diag}[A_i]_{i=1}^h + \left[A_{ij}\right]_{h\times h}, \; B := \mathrm{diag}[B_i]_{i=1}^h, \tag{4}$$

in which $A_{ii} = 0$. Using (1), (3) and (4), the overall system can be written as

$$\dot{x}(t) = Ax(t) + B[u(t) + f(x)]. \tag{5}$$

It is assumed in this note that some additional states from other subsystems are utilized to improve the performance of the control loop. This idea is different from the decentralized controller and will lead to a distributed control structure. Note that the control network may differ from the system network. Our objective here is to design an $\mathscr{H}_2$-based optimal distributed SMC, exploiting feedback from (some of) other subsystems, to stabilize the overall system in (5) through a sparse control network.

Definition 1. A matrix is said to be a structure matrix if its elements are either 0 or 1. The structure matrix of a block matrix $Y = [Y_{ij}]_{h\times h}$ with $Y_{ij} \in \mathbb{R}^{r_i \times s_j}$ is $\mathscr{S}(Y) \triangleq [s_{ij}]_{h\times h}$ with

$$s_{ij} = \begin{cases} 0 & \text{if} \quad Y_{ij} = 0, \quad i \neq j \\ 1 & \text{otherwise.} \end{cases}$$

Notice that the structure matrix defined above is similar to the well-known adjacency matrix in graph theory; see [10]. However, unlike the adjacency matrix, the diagonal entries in the structure matrix will be assumed to be 1.

Definition 2. Two matrices Y_1 and Y_2 are said to have the same structure if $\mathscr{S}(Y_1) = \mathscr{S}(Y_2)$.

Definition 3. The structure matrix $\mathscr{S}_1 \triangleq [s^1_{ij}]_{h\times h}$ is said to be a sub-structure matrix of $\mathscr{S}_2 \triangleq [s^2_{ij}]_{h\times h}$ if $s^2_{ij} - s^1_{ij} \geq 0$. We denote this as $\mathscr{S}_1 \subseteq \mathscr{S}_2$.

Now, consider the following linear sliding function

$$\sigma(t) = Sx(t), \tag{6}$$

where $\sigma(t) := \mathrm{col}(\sigma_i(t))_{i=1}^h$ and the block diagonal matrix $S := \mathrm{diag}[S_i]_{i=1}^h$ will be designed later such that SB is nonsingular.

During the ideal sliding motion the sliding function satisfies:

$$\sigma(t) = 0, \quad \forall t > t_s, \tag{7}$$

where $t_s > 0$ denotes the time that sliding motion starts. Due to the special system coordinate explained before, the overall switching function S matrix may be parameterized as

$$S = \mathrm{diag}[\bar{S}_i]_{i=1}^h \cdot \mathrm{diag}\left[\begin{bmatrix} M_i & I_{m_i} \end{bmatrix}\right]_{i=1}^h, \tag{8}$$

where $M_i \in \mathbb{R}^{m_i\times(n_i-m_i)}$ and $\bar{S}_i$ are nonsingular matrices having no influence on the overall reduced-order sliding motion. Now, the controller is assumed to be of the following structure:

$$u_i(t) = -(S_iB_i)^{-1}\left\{(S_iA_i - \Phi_iS_i)x_i(t) + S_i\sum_{j=1}^h \gamma_{ij}A_{ij}x_j(t)\right\} + \vartheta_i(t), \tag{9}$$

where $\Phi_i \in \mathbb{R}^{m_i\times m_i}$ is a stable matrix, γ_{ij} denotes the ij-th element of the structure matrix Γ of the control network, that is, $\gamma_{ij} = 1$ if $i = j$, or the ij-th link exists in the control network and $\gamma_{ij} = 0$ otherwise, and $\vartheta_i(t) \in \mathbb{R}^{m_i}$ denotes the nonlinear part of the controller.

Assumption 1. There exist known continuous functions $\rho_i(\cdot)$ and $\mu_i(\cdot)$ such that for $i = 1,\cdots,h$:

1) $\|f_i(x_i)\| \leq \rho_i(x_i)$,

2) $\left\|\sum_{j=1,j\neq i}^h (1-\gamma_{ij})A_{ij}x_j\right\| \leq \mu_i(\bar{\gamma}_i^T \circ x)$,

where $\bar{\gamma}_i$ implies the i-th row of $(\mathbb{1}_{h\times h} - \Gamma)$.

Then the nonlinear part of the controller has the following form

$$\vartheta_i(t) = -(S_iB_i)^{-1}\left\{\|S_iB_i\|\,\rho_i(x_i) + \kappa_i(x_i)\right\}\frac{\sigma_i(t)}{\|\sigma_i(t)\|} \quad \text{if } \sigma_i(t) \neq 0, \tag{10}$$

in which $\kappa_i(x_i)$ is a gain to be designed later in this section.

Besides, we need to design the sliding function matrix so that the resulting reduced $(n_i - m_i)$ order sliding mode dynamics are stable. Thus, our next problem is to design sliding matrices S_i ensuring overall stability and an additional $\mathscr{H}_2$ performance specification. Notice that the role of the term $(S_iB_i)^{-1}\Phi_iS_ix_i(t)$ in the controller (9) is to govern the convergence rate to the sliding manifold in association with the nonlinear part $\vartheta_i(t)$. Here, similar to [5], it is assumed that $\Phi_i = \lambda_i I_{m_i}$, where $\lambda_i < 0$ is a given constant value. Note that unlike in [5], λ_i can also belong to the spectrum of A_i. Owing to the special form of Φ_i, it can commute with S_i and then the control law $u_i(t)$ in (9) can be written as

$$u_i(t) = (S_iB_i)^{-1}S_i\left\{A_{\lambda,i}x_i(t) - \sum_{j=1}^{h}\gamma_{ij}A_{ij}x_j(t)\right\} + \vartheta_i(t), \tag{11}$$

where $A_{\lambda,i} = \lambda_i I_{n_i} - A_i$. Then the compact control law is

$$u(t) = (SB)^{-1}S(\Gamma \circ A_\lambda)x(t) + \vartheta(t), \tag{12}$$

where $A_\lambda = \mathrm{diag}[\lambda_i I_{n_i}]_{i=1}^{h} - A$, $\Gamma = [\gamma_{ij}]_{h\times h}$ and $\vartheta(t) = \mathrm{col}(\vartheta_i(t))_{i=1}^{h}$.

We now aim to show that the controller (11), (10) drives the system state to the composite sliding surface (6). Further in what follows, we assume the known sliding surface matrix $S := \mathrm{diag}[S_i]_{i=1}^{h}$ and its design will be derived in the next section.

Theorem 1. Consider the NCSs in (1). Under Assumption 1, the sparse controller (11), (10) drives the state of the system (1) to the composite sliding surface (6) and maintains a sliding motion if $\kappa_i(x_i)$ satisfies

$$\sum_{i=1}^{h}\kappa_i(x_i) > \sum_{i=1}^{h}\|S_i\|\,\mu_i(\bar{\gamma}_i \circ x) \tag{13}$$

where S_i are given sliding function matrices and $\mu_i(\cdot)$ are determined by Assumption 1.

Proof. The dynamics of σ_i of subsystem i can be derived by taking the time derivative of (6), substituting in the state equation (1), and using the controller (11), (10), i.e.,

$$\begin{aligned}\dot{\sigma}_i(t) =&\lambda_i\sigma_i(t) + S_i\sum_{j=1,j\neq i}^{h}(1-\gamma_{ij})A_{ij}x_j(t) - [\kappa_i(x_i) + \|S_iB_i\|\,\rho_i(x_i)]\,\frac{\sigma_i(t)}{\|\sigma_i(t)\|}\\ &+ S_iB_if_i(x_i).\end{aligned} \tag{14}$$

Now we will prove that the following composite reachability condition is satisfied [12]:

$$\sum_{i=1}^{h}\frac{\sigma_i^T\dot{\sigma}_i}{\|\sigma_i\|} < 0. \tag{15}$$

It follows from (14) and Assumption 1 that

$$\begin{aligned}\frac{\sigma_i^T\dot{\sigma}_i}{\|\sigma_i\|} \leq&\lambda_i\,\|\sigma_i\| + \|S_i\|\,\mu_i(\bar{\gamma}_i\circ x) - \kappa_i(x_i) + \|S_iB_if_i(x_i)\| - \|S_iB_i\|\,\rho_i(x_i)\\ \leq&\|S_i\|\,\mu_i(\bar{\gamma}_i\circ x) - \kappa_i(x_i).\end{aligned} \tag{16}$$

Finally, if $\kappa_i(x_i)$ satisfies (13), the composite reachability condition (15) holds. $\square$

Remark 1. An obvious choice for $\mu_i(\bar{\gamma}_i^T\circ x)$ is $\sum_{j=1,j\neq i}^{h}\left\|(1-\gamma_{ij})A_{ij}\right\|\left\|x_j\right\|$, and as a consequence, $\kappa_i(x_i) = \sum_{j=1,j\neq i}^{h}\left\|S_j\right\|\left\|(1-\gamma_{ji})A_{ji}\right\|\|x_i\| + \varepsilon_i$, with $\varepsilon_i > 0$ a small given scalar, satisfies the condition (13).

Note that thanks to the special structure of S and B, the controller can be written as

$$u(t) = \mathrm{diag}\left[\begin{bmatrix}M_i & I_{m_i}\end{bmatrix}\right]_{i=1}^{h}(\Gamma\circ A_\lambda)x(t) + \vartheta(t), \tag{17}$$

With different structure matrix Γ, the controller (12) can explain various topologies. The decentralized control strategy can be obtained by $\gamma_{ij} = \begin{cases}1 & \text{if} \quad i=j\\ 0 & \text{otherwise}\end{cases}$ which means that the local controllers use only local state information to control the given subsystem. In this case, there is no control network in the system. It is worth noting that most of the proposed decentralized SMC in the literature can be covered by the controller (12) with $\Gamma = I_h$. When $\Gamma = \mathbb{1}_{h\times h}$, or more accurately $\Gamma = \mathscr{S}(A)$, a fully distributed control system is achieved, i.e. each subsystem uses its own state as well as the states of all other physically coupled subsystems. In other words, the control network is structurally same as the system network. As the third alternative, the structure matrix Γ can generate a middle-of-the-road solution, between fully distributed control approaches and decentralized ones, regarded as sparsely distributed control systems. This could be beneficial when some constraints on communication requirements among local controllers exist, and hence, the control network could not have the same structure as the plant network.

It should also be mentioned that while decentralized controllers (i.e. $\Gamma = I_h$) are preferable in terms of minimum communication costs, they have shortcomings in terms of stabilizing the overall system if the interactions between subsystems are not weak. On the other hand, a fully distributed control scheme can significantly improve the stabilization potential of interconnected systems, but at the expense of a maximal communication overhead. Motivated by these issues, this paper considers a third alternative in which structures that offer maximal improvement in system performance at the minimal cost in information exchange are explored and exploited.

To this end, we design an $\mathscr{H}_2$-based SMC for interconnected systems with imposing a priori constraints on communication requirements (known Γ) among subsystems in the next section. Section 4 will establish an optimization framework to obtain a trade-off between the $\mathscr{H}_2$ performance and the sparsity of the control network structure.

Remark 2. The control network should always be a subset of the dynamics network, that is, $\Gamma \subseteq \mathscr{S}(A)$. In other words, if $A_{ij} = 0$ (subsystem j does not influence i-th subsystem), then $\gamma_{ij} = 0$.

3 Optimal Structured SMC Design Problem

This section aims to design sliding matrices S_i while ensuring overall stability and penalizing the level of required control effort to maintain sliding as well as the stability of the reduced order interconnected systems. In order to cope with the above problem we may resort to select the switching function matrices S_i, with given λ_i, while ensuring overall stability and the stability of the reduced order interconnected systems, so that the linear control part of (17) minimizes the cost functional

$$J := \int_0^\infty \left\{ x^T(\tau) Q x(\tau) + u^T(\tau) R u(\tau) \right\} d\tau, \tag{18}$$

where $Q \in \mathbb{R}^{n\times n}$, with $n = \sum_{i=1}^{h} n_i$, is a given positive semi-definite matrix, and $R := \mathrm{diag}[R_i]_{i=1}^{h} \in \mathbb{R}^{m\times m}$, with $m = \sum_{i=1}^{h} m_i$, is a given block diagonal s.p.d matrix ($0 < R_i \in \mathbb{R}^{m_i\times m_i}$).

3.1 $\mathscr{H}_2$ based optimal structured static output feedback

Consider the controller in (17) contains only the linear part, hence

$$\begin{aligned} \dot{x}(t) =& Ax(t) + w(t) + Bu(t) \\ z(t) =& \tilde{C}_z x(t) + \tilde{D}_z u(t) \\ u(t) =& \mathrm{diag}\left[\begin{bmatrix} M_i & I_{m_i} \end{bmatrix}\right]_{i=1}^{h} (\Gamma \circ A_\lambda) x(t), \end{aligned} \tag{19}$$

where $w(t) := \mathrm{col}(w_i(t))_{i=1}^{h}$ is a fictitious exogenous disturbance, $z(t) := \mathrm{col}(z_i(t))_{i=1}^{h}$, and $z_i(t) \in \mathbb{R}^{q_i}$ is the performance output vector of the i-th subsystem and

$$\tilde{C}_z := \begin{bmatrix} Q^{\frac{1}{2}} \\ 0 \end{bmatrix}, \; \tilde{D}_z := \begin{bmatrix} 0 \\ R^{\frac{1}{2}} \end{bmatrix}. \tag{20}$$

In order to cope with the optimal SMC problem explained previously, this paper then will endeavour to choose block diagonal matrix S so that the obtained closed-loop system by applying the linear control law in (19) minimizes

$$J := \|T_{wz}\|_2^2, \tag{21}$$

where $\|T_{wz}\|_2$ denotes the $\mathscr{H}_2$-norm of the closed loop transfer function from $w(t)$ to $z(t)$.

Remark 3. It should be noted that designing the sliding surface with only the linear part of the controller is a standard scheme in the existing literature of SMC; see e.g. [28].

The linear controller in (19) can be rewritten as $u(t) = Fx(t)$ in which

$$
\begin{aligned}
F =& \operatorname{diag}\left[\begin{bmatrix} M_i & I_{m_i}\end{bmatrix}\right]_{i=1}^{h} (\Gamma \circ A_\lambda) \qquad (22)\\
=& \left(\operatorname{diag}[M_i]_{i=1}^{h} \operatorname{diag}\left[\begin{bmatrix} I_{n_i-m_i} & 0_{(n_i-m_i)\times m_i}\end{bmatrix}\right]_{i=1}^{h} + \operatorname{diag}\left[\begin{bmatrix} 0 & I_{m_i}\end{bmatrix}\right]_{i=1}^{h} \right)(\Gamma \circ A_\lambda).
\end{aligned}
$$

As seen M_i are the design freedoms in this new framework. Let us obtain the closed-loop system as

$$
\begin{aligned}
A + BF =& A + \operatorname{diag}\left[\begin{bmatrix} 0 & 0 \\ 0 & I_{m_i}\end{bmatrix}\right]_{i=1}^{h} (\Gamma \circ A_\lambda)\\
&+ B\operatorname{diag}[M_i]_{i=1}^{h} \operatorname{diag}\left[\begin{bmatrix} I_{n_i-m_i} & 0_{(n_i-m_i)\times m_i}\end{bmatrix}\right]_{i=1}^{h} (\Gamma \circ A_\lambda)\\
\triangleq& A_c + BMC, \qquad (23)
\end{aligned}
$$

where $M = \operatorname{diag}[M_i]_{i=1}^{h}$ and

$$
\begin{aligned}
A_c &= A + \operatorname{diag}\left[\begin{bmatrix} 0 & 0 \\ 0 & I_{m_i}\end{bmatrix}\right]_{i=1}^{h} (\Gamma \circ A_\lambda), \qquad (24)\\
C &= \operatorname{diag}\left[\begin{bmatrix} I_{n_i-m_i} & 0_{(n_i-m_i)\times m_i}\end{bmatrix}\right]_{i=1}^{h} (\Gamma \circ A_\lambda).
\end{aligned}
$$

Now consider the fictitious system

$$
\begin{aligned}
\dot{x}(t) &= A_c x(t) + w(t) + B\bar{u}(t)\\
z(t) &= C_z x(t) + D_z \bar{u}(t)\\
y(t) &= Cx(t), \qquad (25)
\end{aligned}
$$

where $\bar{u}(t) = My(t)$ and

$$
C_z = \begin{bmatrix} Q^{\frac{1}{2}} \\ R^{\frac{1}{2}} \operatorname{diag}\left[\begin{bmatrix} 0 & I_{m_i}\end{bmatrix}\right]_{i=1}^{h} (\Gamma \circ A_\lambda) \end{bmatrix}, \; D_z = \begin{bmatrix} 0 \\ R^{\frac{1}{2}} \end{bmatrix}. \qquad (26)
$$

From this new viewpoint, the problem of designing $\mathscr{H}_2$ state feedback SMC (19)-(21) can be regarded as a static output feedback LQ problem for the fictitious system (A_c, B, C), given in (25). Specifically, minimizing the $\mathscr{H}_2$-norm of the T_{wz} (see (21)) subject to (19) is equivalent to minimizing the $\mathscr{H}_2$-norm of (25) with respect to the static output feedback gain M.
Different methods have been proposed in the literature to deal with the static

output feedback LQ problem in (25) most of which utilize iterative processes [27, 9, 6, 4, 13], and thus, their solutions and convergence depend on the initial conditions. Among the aforementioned schemes, for no particular reason, we adopt the iterative LMI method proposed in [13], referred to as the scaled min-max algorithm, to find $\boldsymbol{M}$ and thus $\boldsymbol{S}$; see Algorithm 1 in the Appendix section.

Remark 4. Notice that the value of the $\mathscr{H}_2$ cost obtained from Algorithm 1 is not the true one, due to the conservatism introduced by assuming the block-diagonal structure for $\boldsymbol{M}$. Nevertheless, the true value can be computed by solving the following Lyapunov equation

$$P_{true}(A+BF)+(A+BF)^T P_{true}+Q+F^T RF=0. \tag{27}$$

Then one can find the $\mathscr{H}_2$ cost as $\sqrt{\mathrm{trace}(P_{true})}$.

3.2 Stability analysis of sliding mode dynamics

It should be noted that the $\mathscr{H}_2$ based method presented in the previous subsection may not necessarily stabilize the sliding mode dynamics. This subsection aims to impose an additional reduced order stability constraint on the previously proposed optimization problem. Let us rewrite the system in (1) as

$$\begin{bmatrix}\dot{x}_{i1}(t)\\ \dot{x}_{i2}(t)\end{bmatrix}=\begin{bmatrix}A_{i11} & A_{i12}\\ A_{i21} & A_{i22}\end{bmatrix}\begin{bmatrix}x_{i1}(t)\\ x_{i2}(t)\end{bmatrix}+\sum_{j=1,\ j\neq i}^{h}\begin{bmatrix}A_{ij11} & A_{ij12}\\ A_{ij21} & A_{ij22}\end{bmatrix}\begin{bmatrix}x_{j1}(t)\\ x_{j2}(t)\end{bmatrix}+B_i[u_i(t)+f_i(x_i)]. \tag{28}$$

Now by applying the equivalent control:

$$u_{eq,i}=\begin{bmatrix}M_i & I_{m_i}\end{bmatrix}\left\{A_{\lambda,i}x_i(t)-\sum_{j=1}^{h}A_{ij}x_j(t)\right\}-f_i(x_i), \tag{29}$$

and using the nonsingular coordinate transformations $\boldsymbol{T}=\mathrm{diag}[T_i]_{i=1}^{h}$ with $T_i=\left[\begin{smallmatrix}I & 0\\ M_i & I\end{smallmatrix}\right]$, in the new coordinates, i.e. $\bar{x}=\boldsymbol{T}x$, we can write

$$\begin{bmatrix}\dot{x}_{i1}(t)\\ \dot{\sigma}_i(t)\end{bmatrix}=\begin{bmatrix}\bar{A}_{i11} & A_{i12}\\ 0 & \lambda_i I_{m_i}\end{bmatrix}\begin{bmatrix}x_{i1}(t)\\ \sigma_i(t)\end{bmatrix}+\sum_{j=1,\ j\neq i}^{h}\begin{bmatrix}\bar{A}_{ij11} & A_{ij12}\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_{j1}(t)\\ \sigma_j(t)\end{bmatrix}, \tag{30}$$

where $\bar{A}_{i11} = A_{i11} - A_{i12}M_i$ and $\bar{A}_{ij11} = A_{ij11} - A_{ij12}M_j$. Obviously, (30) includes a reduced order interconnected system composed of h subsystems with dimension $n_i - m_i$. Note that this reduced order system is same as the reduced-order system resulted by the SMC (17) as they both have the same sliding surface. Therefore the stability of the system (30) will infer the stabilty of the reduced order system in Section 2, thus guaranteeing the stabilizing of the proposed SMC. Now, a stability analysis is considered for the system (30). Let the overall closed-loop system, obtained by the overall equivalent control, be

$$\dot{\bar{x}}(t) = \bar{A}^r_{cl}\bar{x}(t). \tag{31}$$

It can readily be shown that the stability of $\bar{A}^r_{cl}$ is equivalent to the stability of $A^r_{cl} = A^r_c + BMC^r$, where A^r_c and C^r are obtained from (24) by letting $\Gamma = \mathbb{1}_{h\times h}$, that is no structure imposed. Now in order to ensure the stability of the sliding mode dynamics, we augment the $\mathscr{H}_2$ problem in (41) (see Appendix A) by including (42) with an s.p.d decision variable $\bar{P} > 0$. It is not hard to show that the obtained $M = \mathrm{diag}[M_i]_{i=1}^h$ ensures the stability of the following composite reduced order dynamics:

$$\dot{x}_r(t) = \begin{bmatrix} \bar{A}_{111} & \cdots & \bar{A}_{1h11} \\ \vdots & \ddots & \vdots \\ \bar{A}_{h111} & \cdots & \bar{A}_{h11} \end{bmatrix} x_r(t) \tag{32}$$

where $x_r = \mathrm{col}(x_{i1})_{i=1}^h$. Note that the obtained switching function matrices S_i are completely determined by choice of M_i.

We finally summarize the proposed structured $\mathscr{H}_2$ based SMC in the following theorem.

Theorem 2. Assume that Algorithm 1, with a given structure matrix Γ, has a solution $M = \mathrm{diag}[M_i]_{i=1}^h$ for some $\delta > 0$. Then the $\mathscr{H}_2$ performance constraint $\|T_{wz}\|_2^2 < \delta$ on the system (19) is ensured. After the reaching time t_s, the resulting reduced $n_i - m_i$ $(i = 1, \cdots, h)$ order sliding mode dynamics, obtained by applying the control law in (11) and (10) to the system (1), is asymptotically stable.

Proof. The proof of this theorem is already presented in the previously given method to select the sliding function matrix. □

Remark 5.

- As can be seen, in this paper, the sliding function as well as the sliding matrix $\boldsymbol{S}$ are obtained through a novel unified framework which can cover different topologies of control network such as fully decentralized, fully distributed and sparsely distributed topologies. To the best of authors' knowledge, this is the first paper dedicated to the design of sparsely distributed SMC for interconnected systems.

- More importantly, as stated in the introduction, broadly speaking, all the traditional methods proposed for the design of SMC have shortcomings in terms of taking into account the control action that is necessary to induce and maintain sliding. This drawback basically comes from the nature of the SMC design process which contains two separate stages. During synthesizing the sliding function, there is no sense of the control action level that is required to induce and maintain sliding. For example, the optimal quadratic method proposed in [29] that resembles the traditional LQR methods, deliberately assigns zero penalty to the control effort and, in other words, adopts the principle of cheap control. This issue is more crucial in this paper when it comes to sparsify the control network structure, as with no limits on the available control actions, it may result in the high level of control efforts that each subsystem's controller is required to apply, which is not a practical case. Indeed, we select the switching function matrix $\boldsymbol{S}$ while ensuring overall stability as well as the stability of the reduced order interconnected systems, so that the linear control part of (17) minimizes the cost functional (18). This objective ends up a novel optimal framework for the design of $\boldsymbol{S}$ in Section 3. As explained, the problem of designing $\mathscr{H}_2$ state feedback SMC is reformulated as a static output feedback LQ problem for the fictitious system $(\boldsymbol{A}_c, \boldsymbol{B}, \boldsymbol{C})$, given in (25). This is definitely a novelty in the field of SMC design for interconnected systems.

4 Sparsification of Control Network

Previous sections have studied the problem of designing $\mathscr{H}_2$-based SMC for NCSs with imposing a priori constraints on communication requirements among subsystems. In other words, the structure matrix Γ in (17) is assumed to be known a priori. The objective in this section is to establish an optimization framework which indeed aims to obtain a trade-off between the $\mathscr{H}_2$ performance and the sparsity of the control network structure. Indeed, one can say that the main objective here is to minimize the cost of the control network utilized to control (stabilize) the system. Here, we assume that the general costs, including the construction and data transferring costs etc, are identical for all the links. Hence, the minimization of the control network costs can intuitively be considered as the minimization of the number of links in the control network structure or equivalently finding the sparsest control network structure that can satisfy a global control objective. On the other hand, minimizing the control network structure for the SMC without taking into account the control costs, may not result in applicable results. Here we propose a way to minimize the control performance and the communication costs simultaneously. We formulate this problem as

$$\begin{aligned} \min \quad & J(\Gamma, M) + \eta \,\mathrm{card}(\Gamma) \\ \text{subject to } & \Gamma \subseteq \mathscr{S}(A),\ \mathscr{S}(M) = I, \text{ and (25)}, \end{aligned} \tag{33}$$

where J is the square of the $\mathscr{H}_2$ norm of the closed-loop transfer function from $w(t)$ to $z(t)$ in (25), $\Gamma = [\gamma_{ij}]_{h\times h}$ and $\mathrm{card}(\cdot)$ denotes the cardinality function (the number of nonzero elements of a matrix). Besides, $\eta \geq 0$ is a given constant which captures a trade-off between the $\mathscr{H}_2$ performance and the sparsity of the controller structure. For example a larger η will lead to a sparser Γ and $\eta = 0$, which means $\Gamma = \mathscr{S}(A)$, converts the problem to a distributed SMC with the objective function (21). The optimization problem in (33) is a mixed-binary problem which, broadly speaking, requires an intractable combinatorial search to achieve the solution.

Notice that the cardinality function, in optimization problems such as (33), is usually approximated by the ℓ_1 norm of the optimization variable [2] or the so-called weighted ℓ_1 norm [3]. Since the weighted ℓ_1 norm is not implementable (the required weights should be calculated based on the unknown feedback gain), a reweighted algorithm is proposed in [3], and further used by [14] to design sparse feedback gains. This algorithm solves weighted optimization problems iteratively in which the weights are updated inversely proportional to the strength of individual (block) entries of feedback gain in the previous iteration. However, the existing reweighted algorithms are not applicable to the optimization problem in (33), as the system matrices A_c and C in the fictitious system (25), involve the structure matrix Γ. Instead, in this note, we will consider a heuristic scheme by relaxing the constraint on the variables γ_{ij}, $i \neq j$ from the binary variables, 0 or 1, to the constraint of $0 \leq \gamma^r_{ij} \leq 1$, $i \neq j$, where

$$\gamma^r_{ij} = \frac{\|F_{ij}\|_F}{\max\limits_{\gamma_{ij}=1} \|F_{ij}\|_F} \qquad i \neq j, \tag{34}$$

in which F_{ij} denotes the ij-th entry of the control feedback gain F in (22) and $\|\cdot\|_F$ is the Frobenius norm. Indeed γ^r_{ij} can be considered as the normalized strength of the coupling feedback gain F_{ij}. This scheme works by first finding the normalized strengths of all the coupling feedbacks and then removing the links corresponding to the weaker feedback gains one-by-one until the stability of the overall closed-loop system is violated. Indeed, by assigning a normalized weight to each link according to the contribution of its corresponding feedback gain in the control objective, this process will reduce the probability of loosing the stability by removing a link. This also can lead to a more computationally efficient method compared to an exhaustive search without taking into account the strength of the coupled feedback gains.

Procedure 1.

1) Initialize $\Gamma = \mathscr{S}(A)$ and $l = 1$, in which l denotes the iteration number.

2) Solve Algorithm 1 (refer to Appendix) to find P and M. If the LMIs in (41) and (42) are feasible, $\Gamma^l \leftarrow \Gamma$ and $J_s(l) = J(\Gamma^l) + \eta \mathbf{Card}(\Gamma^l)$, otherwise terminate the search and the problem has no solution.

3) Find γ^r_{ij} as in (34) for all $\gamma_{ij} = 1$, $i \neq j$. Sort the set $\{\gamma^r_{ij}\}$ in ascending order.

4) Set γ_{ij} corresponding to the l-th entry of $\{\gamma^r_{ij}\}$ to zero and $l = l + 1$.

5) Solve Algorithm 1. If the LMIs in (41) and (42) are feasible, $\Gamma^l \leftarrow \Gamma$, then compute the objective function in (33) to find $J_s(l) = J(\Gamma^l) + \eta \mathbf{Card}(\Gamma^l)$, if $l \leq \mathbf{Card}(\mathscr{S}(A))$, return to Step 4, otherwise go to Step 6.

6) Find $l^\star = \arg\min\limits_{l} J_s(l)$ and return its corresponding $\Gamma^{l^\star}$.

Remark 6. It should be noted that random truncation of the distributed controller $(\Gamma = \mathscr{S}(A))$ may lead to a feedback that cannot stabilize the overall system. In contrast, the proposed method here is a systematic way to reduce the number of links in the control network structure while preserving the stability of the overall closed-loop system.

Notice that in order to obtain the result from the above procedure, at most $\sum_{l=1}^{\mathscr{A}} \mathscr{E}_l$ convex problems need to be solved, where $\mathscr{A} = \mathbf{Card}(\mathscr{S}(A))$ and $\mathscr{E}_l$ denotes the number of iterations that is required for Algorithm 1 at the l-th iteration of Procedure 1. This is in contrast to $\sum_{i=1}^{2^{\mathscr{A}}} \mathscr{E}_i$ in the case of carrying an exhaustive search on the binary variables. Besides, roughly speaking, all the methods in the literature to solve an $\mathscr{H}_2$ static output feedback problem utilize iterative processes, and their solutions and more importantly their convergence depend quite significantly on the initial conditions. It is difficult to ensure Procedure 1 to achieve the global minimum or even a local one. However, our extensive computational experiments show that this algorithm can provide an effective means to achieve an acceptable trade-off between the control performance and the sparsity of the control network structure. Compared to the exhaustive search, Procedure 1 proposes a simple suboptimal relaxation scheme, which is much more computationally attractive.

Remark 7. The reference [14] uses the alternating direction of multiplier method (ADMM) to address the sparsity-promoting optimal control problem. Specifically, ADMM is utilized to identify the control structure which strikes a balance between $\mathscr{H}_2$ performance of the closed-loop system and the sparsity of the controller. This reference also exploits weighted ℓ_1-norm as a convex relaxation of the cardinality function. Although the convergence of the weighted ℓ_1-optimization to a local minimum is demonstrated in [8], convergence of the proposed algorithm to the global minimum may not be ensured. While the LMI methods are not the best approaches for the problems with large dimensions, they provide a simple and tractable solution, especially with more and more powerful computing facilities nowadays. Therefore, we preferred to employ LMI approaches in this paper to address the problem of designing sparsely distributed SMC for NCSs and leave using more scalable schemes such as ADMM for the future work. Firstly, the problem formulation in this manuscript is substantially different from the ones used for sparsely distributed state feedback (SF) and static output feedback in the literature (e.g. [21, 23, 24, 22, 14, 15, 16]), in that we originally aimed at using SMC strategy and as a result the problem formulation is very different (please see the system matrices in (24)). Note also that the sparse SF obtained using the method in [14] may not necessarily be a feasible solution to the SMC problem [5]. Moreover, as seen the system matrices of the fictitious system in (25) include the topology matrix. Hence, we are not able to exploit the conventional ℓ_1 algorithms proposed in the associated literature for sparsity pattern recognition. Additionally, the particular formulation here makes it hard to use ADMM in order to solve the sparsity-promoting optimal SMC problem or more specifically identify the favorable controller patterns. However, we will consider the possibility of exploiting the ADMM approach in the future work, albeit by exploiting a different fictitious formulation.

5 Numerical Examples

5.1 Example 1

Consider an interconnected system which includes three inverted pendulums mounted on coupled carts [21, 18]. The linearized system equations are also given in [21]. Define $x_i = [x_{i,1}, x_{i,2}, x_{i,3}, x_{i,4}]^T = [\theta_i, \dot{\theta}_i, \mathbf{x}_i, \dot{\mathbf{x}}_i]^T$,

$$A_i = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{M_i+m}{M_i\ell}g & 0 & \frac{k_i}{M_i\ell} & \frac{c_i+b_i}{M_i\ell} \\ 0 & 0 & 0 & 1 \\ \frac{-m}{M_i}g & 0 & \frac{-k_i}{M_i} & \frac{-c_i-b_i}{M_i\ell} \end{bmatrix}, \; A_{ij} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-k_{ij}}{M_i\ell} & \frac{-b_{ij}}{M_i\ell} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{k_{ij}}{M_i} & \frac{b_{ij}}{M_i} \end{bmatrix}$$

$$B_i = \begin{bmatrix} 0 & \frac{-1}{M_i\ell} & 0 & \frac{1}{M_i} \end{bmatrix}^T,$$

for $(i,j) \in \{(1,2),(2,1),(2,3),(3,2)\}$, in which $k_i = \sum_{j\in \mathsf{J}_i} k_{ij}$ and $b_i = \sum_{j\in \mathsf{J}_i} b_{ij}$, where $\mathsf{J}_i := \{j \mid [\mathscr{S}(A)]_{ij} = 1, j \neq i\}$. Besides, c_i, $b_{ij} = b_{ji}$, $k_{ij} = k_{ji}$ and ℓ are friction, damper, spring coefficients and pendulum length respectively. It is also assumed that the moment of inertia of each pendulum is zero. Besides, the system parameters are assumed as $M_1 = 4,\ M_2 = 3,\ M_3 = 5,\ m = 0.2,\ g = 10,\ \ell = 4,\ k_{12} = k_{21} = 1,\ k_{23} = k_{32} = 1,\ b_{12} = b_{21} = 1,\ b_{23} = b_{32} = 0.2,\ c_1 = 0.4,\ c_2 = 0.2$ and $c_3 = 0.1$. Notice that here it is assumed that entire system states are available. Transformation matrices T_i are utilized to transform the subsystems to the form given in (2). The performance weights are set as $Q = I_n$ and $R = 0.1I_m$. We have also chosen $\lambda = -1$. Procedure 1, with three different parameters $\eta = 0.01$, $\eta = 0.05$ and $\eta = 0.1$, is solved and the corresponding results are given in Table 1. Furthermore, the initial conditions

Table 1: Results of Example 1

η	$\alpha^\star_{12}$	$\alpha^\star_{13}$	$\alpha^\star_{21}$	$\alpha^\star_{23}$	$\alpha^\star_{31}$	$\alpha^\star_{32}$
0.001	1	0	1	0	0	0
0.002	1	0	1	0	0	0
0.01	0	0	0	0	0	0

for Algorithm 1, which is required to be solved for addressing the suboptimal LQ static output feedback problem in Section 3, are set to $\mu_{sol} = 1$, $Y_{sol} = I_n$ and $\bar{Y}_{sol} = I_n$ and the parameter $\delta = 180$. As seen, the larger parameter η results in a more sparse control network.

5.2 Example 2

Consider the system (1) with the following parameters:

$$A = \left[\begin{array}{ccc|cc|cc} 0 & 0 & -3.6 & 0 & 0 & 0 & 0 \\ -0.2 & 7.2 & -0.4 & -0.1 & 0 & 0.3 & 0 \\ 0.3 & 0 & 0 & 3.0 & 0.2 & 0 & 0 \\ \hline 0 & 0.3 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.2 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1.0 \\ 0 & 0.3 & 0 & -0.5 & 0.1 & 0 & 0 \end{array}\right],$$

$$B = \left[\begin{array}{cc|c|c} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

Note that all three open-loop local subsystems are unstable. The performance weights Q and R are set to identity matrices and we choose $\lambda = -4$. We firstly use Procedure 1 with three different parameters $\eta = 0.001$, $\eta = 0.01$ and $\eta = 0.1$. The corresponding results are given in Table 2. The initial conditions for Algorithm 1, which solves the suboptimal LQ static output feedback problem, are $\mu_{sol} = 0.1$, $Y_{sol} = I_n$ and $\bar{Y}_{sol} = I_n$ and the parameter

Table 2: Results of Example 2

η	$\alpha_{12}^{\star}$	$\alpha_{13}^{\star}$	$\alpha_{21}^{\star}$	$\alpha_{23}^{\star}$	$\alpha_{31}^{\star}$	$\alpha_{32}^{\star}$
0.001	1	1	1	1	1	1
0.01	1	0	0	0	0	0
0.1	0	0	0	0	0	0

$\delta = 190$. One can see that as the regularization parameter η increases, the control network becomes more sparse.

5.2.1 Comparison 1

For comparison let us consider a standard sparse state-feedback LQR controller with the given choices of Q and R. In doing so, assume that there exists a stabilizing $\Gamma \circ F_{lqr}$ with $\Gamma = [\gamma_{ij}]_{h\times h}$ and $F_{lqr} \in \mathbb{R}^{m\times n}$ minimizing the following cost functional,

$$J = \text{trace}(\bar{X}_{lqr}), \tag{35}$$

where $\bar{X}_{lqr} = \text{diag}[\bar{X}_i]_{i=1}^{h} > 0$ is obtained from solving the Lyapunov inequality,

$$\begin{aligned}[A+B(\Gamma\circ F_{lqr})]^T\bar{X}_{lqr}+&\bar{X}_{lqr}[A+B(\Gamma\circ F_{lqr})]\\ &+Q+(\Gamma\circ F_{lqr})^T R(\Gamma\circ F_{lqr}) < 0.\end{aligned} \tag{36}$$

The above inequality is not convex with respect to $\bar{X}_{lqr}$ and $\Gamma\circ F_{lqr}$. However, it can be convexified through variable changing. Letting $X_{lqr} = \bar{X}_{lqr}^{-1}$ and pre and post multiplying X_{lqr} to (36), we have

$$\begin{aligned}X_{lqr}[A+B(\Gamma\circ F_{lqr})]^T + [A+B(\Gamma\circ F_{lqr})]X_{lqr} + X_{lqr}QX_{lqr}&\\ +X_{lqr}(\Gamma\circ F_{lqr})^T R(\Gamma\circ F_{lqr})X_{lqr} < 0.&\end{aligned} \tag{37}$$

Having the convex constraint in (37), the minimization problem explained in (35) can be cast as an optimization problem utilizing LMI approach,

$$\text{minimize} \quad \text{trace}\ (Z_s) \quad \text{subject to}$$

$$\begin{bmatrix} AX_{lqr}+X_{lqr}A^T+BY_{lqr}+Y_{lqr}^TB^T & \star & \star \\ Q^{\frac{1}{2}}X_{lqr} & -I & \star \\ R^{\frac{1}{2}}Y_{lqr} & 0 & -I \end{bmatrix} < 0, \tag{38}$$

$$\begin{bmatrix} -Z_s & \star \\ I & -X_{lqr} \end{bmatrix} < 0, \tag{39}$$

where $Y_{lqr} = (\Gamma \circ F_{lqr})X_{lqr} = \Gamma \circ (F_{lqr}X_{lqr}) = \Gamma \circ \bar{Y}_{lqr}$ with $\bar{Y}_{lqr} \in \mathbb{R}^{m\times n}$, and Z_s is a slack variable. Thus, the structural state-feedback can be obtained as $\Gamma \circ F_{lqr} = Y_{lqr}X_{lqr}^{-1}$.

Remark 8. It is easy to realize that

$$\mathscr{S}(X_{lqr}^{-1}) = \mathscr{S}(X_{lqr}) = I,$$

and since $\mathscr{S}(Y_{lqr}) \subseteq \Gamma$, thus

$$\mathscr{S}(Y_{lqr}X_{lqr}^{-1}) \subseteq \Gamma.$$

This means that the structural state feedback gain $\Gamma \circ F_{lqr}$ obtained from $Y_{lqr}X_{lqr}^{-1}$ has the desired structure Γ.

Solving the minimization problem in (38) and (39) gives a bound of 6.3837 on the $\mathscr{H}_2$ cost for decentralized structure. However, again it should be noted that due to the conservatism introduced by enforcing a block-diagonal structure on X_{lqr}, this is not the true value of $\mathscr{H}_2$ cost and it can be obtained as 5.8276 from solving the Lyapunov equation in (27) with the resulting F_{lqr}. Notice also that the true value of $\mathscr{H}_2$ cost achieved from Algorithm 1 for decentralized structure is 6.3080.

5.2.2 Comparison 2

We now consider a reweighted ℓ_1 algorithm for finding an optimal sparse state feedback gains; e.g. see [14]. This problem can be cast as

$$\begin{aligned}
&\text{Minimize } \operatorname{trace}(Z_s) + \bar{\eta} \left\| W \circ V_{lqr} \right\|_{\ell_1} \qquad (40)\\
&\text{subject to}\\
&\begin{bmatrix} AX_{lqr} + X_{lqr}A^T + BV_{lqr} + V_{lqr}^T B^T & \star & \star \\ Q^{\frac{1}{2}} X_{lqr} & -I & \star \\ R^{\frac{1}{2}} V_{lqr} & 0 & -I \end{bmatrix} < 0,\\
&\begin{bmatrix} -Z_s & \star \\ I & -X_{lqr} \end{bmatrix} < 0,
\end{aligned}$$

where $0 < X_{lqr} \in \mathbb{R}^{n\times n}$, $V_{lqr} \in \mathbb{R}^{m\times n}$, Z_s is a slack variable and W is a given weighting matrix with the same dimension of $\mathscr{S}(F_{lqr})$. We then exploit Algorithm 2 with $\bar{\eta} = 0.01$ to find the sparse state feedback matrix. This algorithm suggests the decentralized structure for the control network with a true value $\mathscr{H}_2$ cost of 5.8276. Also, notice that the existing reweighted ℓ_1 algorithm, for minimizing the network structure, is not applicable to our problem which was indeed the design of a sparse distributed $\mathscr{H}_2$-based SMC and rearranged as an LQ static output feedback problem.

6 Conclusions

This paper has developed a distributed sliding mode control framework by using (some of) other subsystems' states. Indeed this issue has been considered to widen the applicability region of the decentralized SMC in which each subsystem's controller uses only local information. Furthermore, an approach is proposed for the sliding surface design in which the level of required control effort is taken into account. Then this novel scheme has been utilized to present a heuristic algorithm that provides an effective means of selecting an overall sliding manifold through a trade-off between the performance and the sparsity of the controller. Indeed, the novel scheme proposed here to design the sliding surface helps avoid excessively large control effort. Illustrative examples have been used to demonstrate the effectiveness of the proposed approach.

A Algorithm for solving the LQ static output feedback problem

Consider the system in (25). As mentioned the objective is to design $M = \text{diag}\,[M_i]_{i=1}^{h}$ so that the $\mathscr{H}_2$ norm from $w(t)$ to $z(t)$ is less than a given constant δ while the stability of the composite sliding mode dynamics is ensured. According to e.g. [13], this problem can be cast as finding two symmetric $P > 0$ and $\bar{P}$ such that

$$\begin{aligned} &PA_{cl} + A_{cl}^T P + C_{cl}^T C_{cl} < 0 \\ &\text{trace}(P) < \delta, \end{aligned} \tag{41}$$

$$\bar{P}A_{cl}^r + \left(A_{cl}^r\right)^T \bar{P} < 0, \tag{42}$$

in which $A_{cl} = A_c + BMC$, $A_{cl}^r = A_c^r + BMC^r$ and $C_{cl} = C_z + D_z MC$, and A_c^r and C^r are defined in Section 3.2. To deal with this problem, [13] proposes the so-called iterative scaled min-max method. To explain this method, we need to introduce four scalar variables ν, β, ψ, ϖ and four symmetric matrices $0 < X \in \mathbb{R}^{n\times n}$, $0 < Y \in \mathbb{R}^{n\times n}$, $0 < \bar{X} \in \mathbb{R}^{n\times n}$ and $0 < \bar{Y} \in \mathbb{R}^{n\times n}$. Now the scaled min-max algorithm can be summarized as follows.

Algorithm 1. 1) Initialize Y_{sol}, $\bar{Y}_{sol}$, $\beta_{sol} > 0$ and set $\varepsilon > 0$ (termination scalar), $l = 1$ (iteration number).

2) Solve

$$\begin{aligned} &\min_{X,\bar{X},\varpi} \quad \psi_l \\ &I \le Y_{sol}^{\frac{1}{2}} X Y_{sol}^{\frac{1}{2}} \le \psi_l I \\ &I \le \bar{Y}_{sol}^{\frac{1}{2}} \bar{X} \bar{Y}_{sol}^{\frac{1}{2}} \le \psi_l I \\ &1 \le \varpi \beta_{sol} \le \psi_l \\ &\begin{bmatrix} B \\ D_z \end{bmatrix}^{\perp} \begin{bmatrix} A_c X + X A_c^T & X C_z^T \\ C_z X & -\varpi I \end{bmatrix} \begin{bmatrix} B \\ D_z \end{bmatrix}^{\perp T} \le -I \\ &B^{\perp}\left(A_c^r \bar{X} + \bar{X}\left(A_c^r\right)^T\right) B^{\perp T} \le -I, \end{aligned}$$

to find X_{sol}, $\bar{X}_{sol}$ and ϖ_{sol}.

3) With given X_{sol}, $\bar{X}_{sol}$ and ϖ_{sol} solve

$$
\begin{aligned}
&\max_{Y,\bar{Y},\beta} \quad \nu_l \\
&\nu_l I \leq X_{sol}^{\frac{1}{2}} Y X_{sol}^{\frac{1}{2}} \leq I \\
&\nu_l I \leq \bar{X}_{sol}^{\frac{1}{2}} \bar{Y} \bar{X}_{sol}^{\frac{1}{2}} \leq I \\
&\nu_l \leq \varpi_{sol} \beta \leq 1 \\
&\begin{bmatrix} C^{T\perp}(YA_c + A_c^T Y + \beta C_z^T C_z) C^{T\perp T} & \beta I \\ \beta I & -I \end{bmatrix} \leq 0 \\
&\begin{bmatrix} \mathrm{trace}(Y) - \delta\beta & \beta \\ \beta & -I \end{bmatrix} \leq 0 \\
&C^{T\perp}\left(\bar{Y} A_c^r + (A_c^r)^T \bar{Y}\right) C^{T\perp T} \leq 0,
\end{aligned}
$$

to find Y_{sol} and β_{sol}.

4) If $\lambda_{\min}(Y_{sol}) < \varepsilon$ or $\lambda_{\min}(\bar{Y}_{sol}) < \varepsilon$ or $\beta_{sol} < \varepsilon$ then stop, the algorithm does not converge.

5) If $\psi_l - \nu_l < \varepsilon$, go to Step 6, otherwise $l = l+1$ and return to Step 2.

6) Return $P = \beta^{-1} Y$ and $\bar{P} = \beta^{-1} \bar{Y}$.

If the algorithm converges to the solution, then $\psi_l \to 1$, $\nu_l \to 1$, $X \to Y^{-1}$, $\bar{X} \to \bar{Y}^{-1}$ and $\beta \to \varpi^{-1}$. The required M then can be obtained by solving (41) and (42) with given P and $\bar{P}$.

B Reweighted ℓ_1 minimization method

Using the reweighted ℓ_1 norm for promoting sparsity has been considered in e.g. [7, 2]. Define the matrix $N = \mathbb{1}_{m \times n}$. It can be shown that (e.g. see [7, 2]) the optimization problem in (40) is equivalent to

$$\begin{aligned}
&\text{Minimize } \mathrm{trace}(Z_s) + \bar{\eta}\,\mathrm{trace}(N^T G) \qquad (43)\\
&\text{subject to}\\
&\begin{bmatrix} AX_{lqr} + X_{lqr}A^T + BV_{lqr} + V_{lqr}^T B^T & \star & \star \\ Q^{\frac{1}{2}} X_{lqr} & -I & \star \\ R^{\frac{1}{2}} V_{lqr} & 0 & -I \end{bmatrix} < 0,\\
&\begin{bmatrix} -Z_s & \star \\ I & -X_{lqr} \end{bmatrix} < 0,\\
&-G \leq W \circ V_{lqr} \leq G,
\end{aligned}$$

where $0 < X_{lqr} \in \mathbb{R}^{n\times n}$, $V_{lqr} \in \mathbb{R}^{m\times n}$, Z_s is a slack variable, W denotes the weighting matrix and the last inequality is element-wise with $G \in \mathbb{R}^{m\times n}$ whose entries are nonnegative. Then the algorithm to solve the above optimization problem is as the following,

Algorithm 2.

1) With given $\varepsilon > 0$, $\alpha > 0$ and $\bar{\eta} > 0$, initialize $W = \mathbb{1}_{h\times h}$, $l = 1$ and $V^l = 0$.

2) Solve the minimization problem (43) to obtain $F_{lqr}^\star = V_{lqr}^\star X_{lqr}^{\star^{-1}}$.

3) Update $W_{ij} = \frac{1}{\left\| \left(V_{lqr}^\star\right)_{ij} \right\|_F + \varepsilon}$ and form $W = [W_{ij}]_{h\times h}$.

5) If $\left\| V_{lqr}^\star - V^l \right\| \leq \alpha$ go to Step 6, else $V^l = V_{lqr}^\star$, $l = l+1$ and return to Step 2.

6) Return $F_{lqr}^\star$.

Solving Algorithm 2 gives the most effective sparse structure of F_{lqr}. Then, by ignoring the unnecessary entries in F_{lqr} we find the structure matrix Γ. Eventually, the optimal structured feedback matrix is obtained by solving the problem in (38) and (39). This procedure is considered in e.g. [7].

References

[1] J. Baillieul and P. Antsaklis. Control and communication challenges in networked real-time systems. Proceedings of the IEEE, 95(1):9–28, 2007.

[2] S. Boyd and L. Vandenberghe. Convex optimization. Cambridge University Press, 2004.

[3] E. J. Candes, M. B. Wakin, and S. P. Boyd. Enhancing sparsity by reweighted ℓ_1 minimization. Journal of Fourier analysis and applications, 14(5-6):877–905, 2008.

[4] MC De Oliveira and JC Geromel. Numerical comparison of output feedback design methods. In American Control Conference, 1997. Proceedings of the 1997, volume 1, pages 72–76. IEEE, 1997.

[5] C. Edwards. A practical method for the design of sliding mode controllers using linear matrix inequalities. Automatica, 40:1761–1769, 2004.

[6] Laurent El Ghaoui, Francois Oustry, and Mustapha AitRami. A cone complementarity linearization algorithm for static output-feedback and related problems. IEEE transactions on automatic control, 42(8):1171–1176, 1997.

[7] M. Fardad, F. Lin, and M. R. Jovanovic. Sparsity-promoting optimal control for a class of distributed systems. In Proc. the American Control Conference, pages 2050–2055, San Francisco, CA, USA, 2011.

[8] M. Fazel, Haitham Hindi, and Stephen P. Boyd. Log-det heuristic for matrix rank minimization with applications to Hankel and Euclidean distance matrices. In Proc. the American Control Conference, pages 2156–2162, Denver, Colorado, USA, June 4-6, 2003.

[9] Jose C Geromel, CC De Souza, and RE Skelton. Static output feedback controllers: stability and convexity. IEEE Transactions on Automatic Control, 43(1):120–125, 1998.

[10] C. Godsil and G. Royle. Algebraic Graph Theory. Springer, 2001.

[11] J. Hespanha, P. Naghshtabrizi, and Y. Xu. A survey of recent results in networked control systems. Proceedings of the IEEE, 95(1):138–162, 2007.

[12] Kou-Cheng Hsu. Decentralized variable-structure control design for uncertain large-scale systems with series nonlinearities. International Journal of Control, 68(6):1231–1240, 1997.

[13] T. Iwasaki and F. Skelton. Linear quadratic suboptimal control with static output feedback. Systems and Control Letters, 23:421—430, 1994.

[14] F. Lin, M. Fardad, and M. Jovanovic. Augmented lagrangian approach to design of structured optimal state feedback gains. IEEE Trans. Autom. Control, 56(12):2923–2929, 2011.

[15] Fu Lin, Makan Fardad, and Mihailo R Jovanović. Sparse feedback synthesis via the alternating direction method of multipliers. In American Control Conference (ACC), 2012, pages 4765–4770. IEEE, 2012.

[16] Fu Lin, Makan Fardad, and Mihailo R Jovanović. Design of optimal sparse feedback gains via the alternating direction method of multipliers. IEEE Transactions on Automatic Control, 58(9):2426–2431, 2013.

[17] M. S. Mahmoud and A. Qureshi. Decentralized sliding-mode output-feedback control of interconnected discrete-delay systems. Automatica, 48(5):808–814, 2012.

[18] K. Ogata. Modern control engineering. Prentice-Hall Inc., 1997.

[19] Y. Pan and K. Furuta. VSS controller design for discrete-time systems. Control-Theory Adv. Technol, 10(4):669—687, 1994.

[20] A. Qureshi and M. A. Abido. Decentralized discrete-time quasi-sliding mode control of uncertain linear interconnected systems. International Journal of Control, Automation and Systems, 12(2):349–357, 2014.

[21] Mohammad Razeghi-Jahromi and Alireza Seyedi. Stabilization of networked control systems with sparse observer-controller networks. Automatic Control, IEEE Transactions on, 60(6):1686–1691, 2015.

[22] Simone Schuler, Ping Li, James Lam, and Frank Allgöwer. Design of structured dynamic output-feedback controllers for interconnected systems. International Journal of Control, 84(12):2081–2091, 2011.

[23] Simone Schuler, Ulrich Münz, and Frank Allgöwer. Decentralized state feedback control for interconnected process systems. In IFAC Symposium on Advanced Control of Chemical Processes, Furama Riverfront, Singapore, pages 1–10, 2012.

[24] Simone Schuler, Ulrich Münz, and Frank Allgöwer. Decentralized state feedback control for interconnected systems with application to power systems. Journal of Process Control, 24(2):379–388, 2014.

[25] D. Siljak. Decentralized control of complex systems. Dover Publications, 2012.

[26] C. Y. Tang and E. A. Misawab. Sliding surface design for discrete VSS using LQR technique with a preset real eigenvalue. Systems and Control Letters, 45(1):1—7, Jan, 2002.

[27] A Trofino and Vladimir Kucera. Stabilization via static output feedback. IEEE Transactions on Automatic Control, 38(5):764–765, 1993.

[28] V. Utkin. Sliding modes in control optimization. Springer, London, 1992.

[29] VI Utkin. Methods for constructing discontinuity planes in multidimensional variable structure systems. Automation and remote Control, 39:1466–1470, 1978.

[30] F. Wang and D. Liu. Networked control systems: theory and applications. Springer, 2008.

[31] X. Wang and M. Lemmon. Event-triggering in distributed networked control systems. IEEE Transactions on Automatic Control, 56(3):586–601, 2011.

[32] X.-G. Yan, C. Edwards, and S. K. Spurgeon. Decentralized robust sliding mode control for a class of nonlinear interconnected systems by static output feedback. Automatica, 40(4):613—620, Apr. 2004.

[33] X.-G. Yan, S. K. Spurgeon, and C. Edwards. Decentralised sliding mode control for nonminimum phase interconnected systems based on a reduced-order compensator. Automatica, 42(10):1821–1828, Oct. 2006.

[34] Xing-Gang Yan, Sarah K Spurgeon, and Christopher Edwards. Decentralized output feedback sliding mode control of nonlinear large-scale systems with uncertainties. Journal of optimization theory and applications, 119(3):597–614, 2003.

[35] A. Zecevic and D. Siljak. Global low-rank enhancement of decentralized control for large-scale systems. IEEE Transactions on Automatic Control, 50(5):740–744, 2005.

[36] W. Zhang, M. Branicky, and S. Phillips. Stability of networked control systems. IEEE Control Systems, 21(1):84–99, 2001.