Single frequency sound propagation in flat waveguides with locally reactive impedance boundaries

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A coherent image source method is presented for evaluating single frequency sound propagation from a point source in a flat waveguide with two infinite and parallel locally reactive boundaries. The method starts from formulating reflections of the spherical sound radiation into integrals of plane wave expansion, and the analytical evaluation of the integrals is simplified by introducing a physically plausible assumption that wave front shapes remain the same before and after each reflection on a reflective boundary. The proposed model can determine coherently the sound fields at arbitrary receiver locations in a flat waveguide, even when one boundary is highly sound absorptive. Being compared with the classical wave theory and the existing coherent ray-based methods, it is shown that the proposed method provides considerable accuracy and advantages to predict sound propagation in flat waveguides with a sound absorptive ceiling and a reflective floor over a broad frequency range, particularly at large distances from the source where the existing methods are problematic. © 2011 Acoustical Society of America. [DOI: 10.1121/1.3608124]

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I. INTRODUCTION

Speech privacy and noise control problems are often encountered in open-plan offices and appear serious when these rooms are large. In such rooms, usually, lateral dimensions are much larger than the height, the ceiling is lined with sound absorptive material and the floor is reflective. A simplified model for such rooms can be a flat waveguide with two infinite and parallel boundaries (ceiling and floor) with locally reactive material.

The sound field inside a flat waveguide can be calculated by many methods. Kuttruff proposed analytical formulas to predict the sound energy distribution in flat waveguides based on the incoherent image source method from the 1980s. Many other researchers have studied sound propagation in similar spaces, such as large fitted factories, workshops and dining rooms. These studies are either based on the incoherent image source methods or the ray-tracing techniques, which neglect the interference effect among the direct sound and the multiple reflections. However, it has been shown that the accuracy of prediction can be improved if a coherent model is used. Further, in some applications, such as that for predicting sound fields of speech or narrow band noise, interference from different sound waves must be taken into account to provide meaningful prediction.

There are some studies on the coherent ray-based prediction model for sound fields in bounded spaces. Dance et al. have developed an interference model to predict the sound pressure in industrial enclosures by considering the sound propagating phase shift and coherent summation of different reflected waves. However, in their model, the inherent phase shift of sound waves before and after reflections on impedance boundaries are not taken into account and the incidence angle of each reflection was not properly considered. It is also found that the method cannot provide accurate results for receivers far away from the reflecting surfaces.

A few years later, Wang and Bradley employed the image source method to calculate coherently the total field from a point source in open-plan offices using the plane wave reflection coefficients for the successive reflections of spherical sound radiation of the point source. Unfortunately, little theoretical analysis was provided in their paper.

Brekhovskikh described the sound field in a flat-layered homogeneous media as the sum of the direct sound and the multiple reflections from image sources, where each reflection is formulated as a plane wave expansion integral. Although the solution can be used in flat waveguides directly, large errors are observed in the circumstances with a sound absorptive ceiling and a reflective floor, especially at receivers far away from the reflecting surfaces. It is also found that the method cannot provide accurate results for receivers far away from the reflecting surfaces.

In light of Brekhovskikh’s work and based on the solution of the spherical wave reflection on an infinite plane, Gensane and Santon proposed a generalized solution to effectively model successive reflections of spherical sound radiation from a point source in bounded spaces with several boundaries. The plane wave reflection form, which is just the first-order approximate solution of the spherical wave reflection on a plane, was adopted in their study to develop a complex image source method. The method was shown to be able to predict coherently the sound field in bounded spaces with sufficiently hard boundaries at high frequencies.

According to the concepts of Gensane and Santon, Lemire and Nicolas replaced the reflection coefficient used in the method of Gensane and Santon with a more accurate solution of the spherical wave reflection on an infinite plane to numerically investigate the sound field in flat waveguides. The method proposed by Lemire and Nicolas has
been the basis of coherent prediction models recently developed by Iu and Li, Lam and Li, and Law et al. for sound field study in narrow street canyons, long spaces, and tunnels, respectively. Good accuracy from these models was observed in spaces with reflective boundaries. Unfortunately in the method of Lemire and Nicolas, it is implicitly assumed that the wave front shapes remain spherical after each successive reflection of the initial spherical wave incidence, and this requires that the boundaries can be approximated as infinite and hard ones. It would be interesting to investigate the accuracy degree of this method for spaces with absorptive boundaries, particularly when some of the boundaries are highly sound absorptive.

Westwood has proposed a ray-based method for sound fields in flat waveguides with a penetrable bottom boundary and an idealized sound-soft top boundary to model the shallow water ocean environment. In his method, the multiple reflections of a spherical wave incidence on the interfaces were also formulated as integrals from plane wave expansion and then the integrals were evaluated numerically or with the first-order approximation similar to the method of Gensane and Santon. Unfortunately, no further analytical solutions were given.

In review of these previous studies, there are needs of research toward a simple but accurate coherent model to predict the sound propagation in flat waveguides with a sound absorptive ceiling and a reflective floor, particularly at large distances from the source compared to the space height. This is the objective of this paper.

II. THEORETICAL METHOD
A. Formulation of the problem

Figure 1(a) shows a vertical section of a flat waveguide, where the infinite ceiling and floor planes are assumed to be locally reactive with uniform normalized specific admittance of $\beta_c$ and $\beta_f$, respectively. The height between the ceiling and floor is $h$, a point source is located at $(0, z_s)$ and a receiver is at $(r, z)$ with $r$ being the horizontal distance from the source to receiver. The time-dependent factor $e^{-j\omega t}$ is suppressed for simplicity throughout this paper.

Based on the plane wave expansion of a spherical wave, the Green function in free field can be expressed as

$$\frac{e^{jkR_0}}{4\pi R_0} = \frac{jk}{8\pi^2} \int_0^{2\pi} \int_0^{\pi/2-j\infty} e^{jkR_0 \sin \theta \, d\theta \, d\varphi},$$  

where $k = k(\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$, $k$ is the wave number, and $R_0$ is the distance vector from the source to the receiver with a magnitude of $R_0$. The parameters $\theta$ and $\varphi$ represent the azimuth angles characterizing the direction of each plane wave propagation in vertical and horizontal planes, respectively.

In the absence of the ceiling, the first reflection of the spherical wave incidence can be obtained by superposition of all the elementary plane wave reflections on the floor plane as

$$P_{-1} = \frac{jk}{8\pi^2} \int_0^{2\pi} \int_0^{\pi/2-j\infty} e^{kR_{-1} \sin \theta \, d\theta \, d\varphi},$$

where $P_{-1}$ denotes the field of the first reflection on the floor and $R_{-1} = (R_{-1} \sin \varphi_{-1} \cos \varphi_{-1}, R_{-1} \sin \varphi_{-1} \sin \varphi_{-1}, R_{-1} \cos \varphi_{-1})$ is the corresponding distance vector from the image source $S_{-1}$ to the receiver, as shown in Fig. 1(b). $V_f(\theta)$ represents the plane wave reflection coefficient of the floor at the incidence angle $\theta$ and is given by

$$V_f(\theta) = \frac{\cos \theta - \beta_f}{\cos \theta + \beta_f}.$$  

When the ceiling presents, the total field can be constructed with the rays from an infinite number of image sources, in which $n = 0, \pm 1, \pm 2, \ldots$, and $P_n$ represents the ray contribution from the $n$th image source, where a positive $n$ is for...
an image source located in the source series above the ceiling and a negative \( n \) is for that located below the floor as shown in Fig. 1(b). Particularly \( P_0 \) denotes the direct sound from the real source. Similar to Eq. (2), the ray contribution from the path with \( n_p \) floor reflections and \( n_c \) ceiling reflections can be expressed with an integral as

\[
P_a = \frac{jk}{8\pi^2} \int_0^{2\pi} \int_0^{\pi/2-j\infty} e^{jkR_n} [V_f(\theta)]^{n_p} [V_c(\theta)]^{n_c} \sin \theta \vartheta \vartheta \rho. \tag{5}
\]

where \( R_n = (R_{n1} \sin \theta_n \cos \varphi_n, R_{n2} \sin \theta_n \sin \varphi_n, R_n \cos \theta_n) \) is the distance vector from the \( n \)th image source \( S_n \) to the receiver, \( \theta_n \) and \( \varphi_n \) denote the azimuth angles to characterize the direction of \( R_n \) in vertical and horizontal planes, respectively. \( V_c(\theta) \) is the plane wave reflection coefficient on the ceiling boundary,

\[
V_c(\theta) = \frac{\cos \theta - \beta_c}{\cos \theta + \beta_c}. \tag{6}
\]

The boundary reflection order \( n_p \) and \( n_c \) can be determined from the image source order \( n \) with a rule of

\[
n_p = \frac{|n|}{2} - \frac{1}{2} \text{sgn}(n) \text{rem}(|n|, 2), \tag{7a}
\]

and

\[
n_c = \frac{|n|}{2} + \frac{1}{2} \text{sgn}(n) \text{rem}(|n|, 2), \tag{7b}
\]

where \( \text{sgn}(\cdot) \) denotes the signum function that is +1 for the positive input and equals −1 for the negative input, and \( \text{rem}(|n|, 2) \) represents the remainder of \( |n| \) after division by 2.

Defining an overall coefficient \( V(\theta) \) to replace the term \( [V_f(\theta)]^{n_p} [V_c(\theta)]^{n_c} \) and using the geometrical identity of \( r = R_{n1} \sin \theta_n, P_n \) in Eq. (5) is rewritten as

\[
P_n = \frac{jk}{8\pi^2} \int_0^{2\pi} \int_0^{\pi/2-j\infty} \exp \left\{ \frac{jkR_n}{R_c} \cos \varphi \cos \varphi_n + \sin \varphi \sin \varphi_n + R_n \cos \theta_n \right\} V(\theta) \sin \theta \vartheta \vartheta \rho. \tag{8}
\]

and then can be further transformed into

\[
P_n = \frac{jk}{8\pi^2} \int_0^{2\pi} \int_0^{\pi/2-j\infty} \exp \left\{ \frac{jkR_n}{R_c} \cos \varphi - \varphi_n \right\} d\varphi \times \exp (R_n \cos \theta \cos \theta_n) V(\theta) \sin \theta \vartheta \vartheta \rho. \tag{9}
\]

As the function \( V(\theta) \) is independent of integration variable \( \varphi \), by using \( 14, 24 \)

\[
\int_0^{2\pi} \exp (jkR_n \cos \theta - \varphi) d\varphi = 2\pi I_0(k R_n \sin \theta), \tag{10}
\]

where \( I_0(\cdot) \) represents the Bessel function of the zeroth order, Eq. (9) can be simplified as

\[
P_n = \frac{jk}{4\pi} \int_0^{\pi/2-j\infty} \exp (jkR_n \cos \theta - \varphi) V(\theta) I_0(k R_n \sin \theta) \times \sin \theta \vartheta \vartheta \rho. \tag{11}
\]

Using the identities for a complex number \( u \) that \( J_0(u) = [H_0^0(u) + H_0^0(u)]/2 \) and \( H_0^0(\varphi - u) = -H_0^0(u) \), Eq. (11) can be written as

\[
P_a = \frac{jk}{8\pi^2} \int_0^{2\pi} \int_0^{\pi/2-j\infty} V(\theta) \sin \theta \vartheta \vartheta \rho. \tag{12}
\]

where \( H_0^0(\cdot) \) and \( H_0^0(\cdot) \) are the Hankel functions of the first and second kind with the zeroth order. Now the total field from a point source in a flat waveguide is explicitly formulated by Eq. (4) accompanied by the integral equation (12).

B. The coherent image source method

As shown in Fig. 1(b), \( P_0 \) is the direct sound \( e^{ikR_0/4\pi R_0} \), and \( P_1 \) and \( P_{-1} \) are the fields of single reflections whose integral expressions with Eq. (12) can be evaluated with the exact (integral) solution provided by Nobile and Hayek, \( 25 \)

\[
P_1 = \frac{e^{jkR_1}}{4\pi R_1} \int_1 \left\{ 1 - \frac{4jk\beta BR_1}{\beta_c + \cos \theta_1} I(k R_1, \beta_c, \theta_1) \right\}, \tag{13a}
\]

\[
P_{-1} = \frac{e^{jkR_{-1}}}{4\pi R_{-1}} \int_1 \left\{ 1 - \frac{4jk\beta BR_1}{\beta_f + \cos \theta_{-1}} I(k R_{-1}, \beta_f, \theta_{-1}) \right\}, \tag{13b}
\]

where \( I(\cdot) \) is an integral function given by

\[
I(k R_n, \beta, \theta_n) = \int_0^\infty \frac{e^{-jkR_n}(\varphi + 2\beta)}{\sqrt{1 - \varphi^2/H - 2\beta t/H}} dt, \tag{14}
\]

in which \( \beta \) is the normal specific admittance of the plane that the reflection takes place on

\[
B = -j\sqrt{1 + \beta \cos \theta_n - (1 - \beta^2)^{1/2} \sin \theta_n}, \tag{15a}
\]

and

\[
H = 1 + \beta \cos \theta_n + (1 - \beta^2)^{1/2} \sin \theta_n. \tag{15b}
\]

For the ray with multiple reflections \( |n| \geq 2 \) on boundaries, although the exact solution of integral expression of \( P_n \) is generally not possible, it is feasible to approximate such integral analytically with the method of steepest descents for larger \( kr \). \( 14, 16 \) A second-order approximate solution provided by Brekhovskikh \( 14 \) in term of asymptotic series with accuracy to terms of \( 1/(kr R_n)^2 \) is

\[
P_n = \frac{e^{jkR_n}}{4\pi R_n} V(\theta_n) - \frac{e^{jkR_n}}{4\pi R_n} \frac{j}{2} V''(\theta_n) \cos \theta_n + V''''(\theta_n) \tag{16}
\]

where \( V'(\theta_n) \) and \( V''(\theta_n) \) represent the first and second derivative of \( V(\theta) \) at \( \theta_n \), respectively, and \( V(\theta) = [V_f(\theta)]^{n_p} \)


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Although Eq. (16) can be used in flat waveguides directly, its convergence needs to be analyzed before applications into the circumstance concerned in this paper. After substituting the expressions of $V_{r}(\theta)$ and $V_{s}(\theta)$ in Eqs. (3) and (6) into Eq. (16), a common factor of $[\cos \theta_{n} + \beta_{r}]^{n_{v}}$ can be found in all series terms in Eq. (16).

In the current circumstance, the floor is acoustical reflective, which leads $|\beta_{r}| \rightarrow 0$. If a receiver is far from the source compared to the waveguide height, which is a situation often encountered in large open-plan offices, the incidence angle $\theta_{n}$ of rays at the first few orders whose contributions are significant in the total field, will approach $\pi/2$ as shown in Fig. 1(b). This causes $(\cos \theta_{n} + \beta_{r}) \rightarrow 0$, and then limits the convergence radius of the asymptotic series in Eq. (16). Hence the second-order approximation in Eq. (16) might not be sufficiently accurate for the current problem theoretically.

Using explicit higher orders of the asymptotic series in Eq. (16) to improve the approximation is not successful as the exponential complexity is encountered as the order increases\textsuperscript{14,15} and such an extension appears not sensible either. This indicates that the coefficient $Q_{sr}$ can be assumed to be uniform for different spatial parts of the wave front in the reflection on a hard boundary, no matter what shape the incident wave fronts are. Thus, according to Eq. (17), each reflection of a ray on a reflective boundary that is denoted by RB, regardless of the reflection being the first or the successive one in the ray propagation from the spherical radiation, can be heuristically approximated by

$$P_{r} \approx P_{\text{ref}}Q_{\text{ref}}(\text{IS}, R[\text{RB}]),$$

where $P_{r}$ denotes the ray field at receiver after a reflection on RB, and $P_{\text{ref}}$ represents the expected ray field at the receiver if the boundary RB is rigid, which is just the perfectly reflected field on this boundary. $Q_{\text{ref}}(\text{IS}, R[\text{RB}])$, determined by Eq. (18), is used to replace the single reflection coefficient $Q_{sr}$ for reflections on the reflective boundary RB and corresponds to the ray from the image source IS to receiver R. Equation (19) shows that each reflection of a ray on a reflective boundary can be evaluated by multiplying the corresponding perfectly reflected field with the spherical wave reflection coefficient on this boundary.

In the current circumstances, the floor boundary is reflective. Thus, before and after each reflection on it, the wave front shape can be assumed to remain the same. The ray field alterations after each reflection on it can be quantified by a weighting factor $Q_{\text{ref}}(\text{IS}, R[\text{FB}])$ that depends on the floor boundary (FB) and the ray geometry according to Eq. (19). Therefore in the propagation of a ray with reflection order $n_{f}$ on the floor and $n_{c}$ on the ceiling shown in Fig. 1(b), after each “transmission” through the floor or its images, the propagation field of the ray can be once weighted by the reflection coefficient $Q_{\text{ref}}(S_{n}, R[\text{FB}])$. After the ray field being weighted for $n_{f}$ times due to transmission through the floor and its images in propagation, the field contribution of the ray, $P_{n}$, can be approximated as

$$P_{n} = \left[Q_{\text{ref}}(S_{n}, R[\text{FB}])\right]^{n_{f}} e^{jkR_{n}\cos \theta_{n}} \sin \theta_{n} d\theta,$$

where $Q_{\text{ref}}(S_{n}, R[\text{FB}])$ can be determined from Eq. (18). Being compared to Eq. (12), the integral of $P_{n}$ in Eq. (20) has been simplified, and now the integrand involves only the reflection coefficient on the ceiling.

The integral in Eq. (20) describes the field of the ray after $n_{f}$ times reflections on the absorbent ceiling. This can be evaluated by the second-order approximation in Eq. (16) with ensured accuracy because $|\beta_{r}|$ does not approach zero and for the asymptotic series there is no singularity similar to that from $(\cos \theta_{n} + \beta_{r}) \rightarrow 0$ when $\theta_{n} \rightarrow \pi/2$. So $P_{n}$ can be further approximated for larger $kr$ by
Two numerical cases are investigated to access the prediction performance with the proposed method and the existing coherent ray-based methods.
The predictions of the sound pressure level (SPL) and Santon15 provides accurate results only when the horizontal distance between the source and receiver is smaller than two times of the waveguide height, and the modified method of Lemire and Nicolas11 for better accuracy and applicability.11,25 The ceiling of the flat waveguide is assumed to be highly sound absorptive in the first case and to be moderately absorptive in the second, whereas the floor keeps reflective and the waveguide height is always 2 m unless otherwise stated.

In the first case, parameters of the porous material in the suspended ceiling are \( \sigma = 30 \text{ cgs}, \ \Omega = 1, \ T = 1, \ S_p = 0.25, \ d = 0.1 \text{ m} \), and \( L = 0.2 \text{ m} \), whereas the moderately absorbent one is defined with the same parameters except that \( \sigma = 150 \text{ cgs} \). The floor is assumed to be a reflective wooden one with parameters of \( \sigma = 10 \ 000 \text{ cgs}, \ \Omega = 0.2, \ T = 1.4, \ S_p = 0.5, \) and \( d = 0.01 \text{ m} \).

by replacing the approximate spherical wave reflection coefficient employed in it, the term \( R_p + (1 - R_p)F(\omega) \), in Eq. (7) of Ref. 11, with the exact integral one in Eq. (18) proposed by Nobile and Hayek25 for better accuracy and applicability.11,25 The ceiling of the flat waveguide is assumed to be highly sound absorptive in the first case and to be moderately absorptive in the second, whereas the floor keeps reflective and the waveguide height is always 2 m unless otherwise stated.

In the first case, parameters of the porous material in the suspended ceiling are \( \sigma = 30 \text{ cgs} \) (where 1 cgs = 1 kPa \( s \) \( m^{-2} \)), \( \Omega = 1, \ T = 1, \ S_p = 0.25, \) and \( d = 0.1 \text{ m} \) to represent a highly sound absorptive wool layer31 and the backing cavity depth \( L \) is 0.2 m. The selected parameters of the floor wood layer are \( \sigma = 10 \ 000 \text{ cgs}, \ \Omega = 0.2, \ T = 1.4, \ S_p = 0.5, \) and \( d = 0.01 \text{ m} \) to represent a high density wooden flooring.31,32 The predictions of the sound pressure level (SPL) are first investigated versus the horizontal distance between the source and receiver at 1 000 Hz, where heights of the source and receivers are chosen to be 0.25 m and the normal incident absorption coefficients of the ceiling and floor are 0.84 and 0.02 respectively.

The corresponding results with the proposed method, the wave theory, and the exiting ray-based methods are compared in Fig. 3. It can be seen that the results from the proposed method and those from the wave theory almost overlap over the range of \( r/h \) from 1 to 50. The method of Gensane and Santon15 provides accurate results only when the horizontal distance between the source and receiver is smaller than two times of the waveguide height, and the modified method of Lemire and Nicolas11 is accurate with \( r/h \) below 3. Although the method of Brekhovskikh,14 which is the second-order approximate solution in Eq. (16) for the flat waveguide, predicts much better than the latter two existing image source methods, large discrepancies are found between the results from it and those from the wave theory after \( r \) becomes larger than 8 times that of \( h \). This indicates that when receivers are far away from the source compared to the height of flat waveguides, the existing coherent ray-based methods are hard to predict the sound propagation accurately. On the other hand, the excellent prediction agreement between the existing ray-based methods and the wave theory at receivers rather close to the source provides cross-validation for the numerical calculations with the wave theory at the receivers far from the source, because the eigenvalues

FIG. 2. Spectra of normal incident absorption coefficient of three types of boundaries used in the numerical simulations, where the highly absorbent ceiling is defined by \( \sigma = 30 \text{ cgs}, \ \Omega = 1, \ T = 1, \ S_p = 0.25, \ d = 0.1 \text{ m} \), and \( L = 0.2 \text{ m} \), whereas the moderately absorbent one is defined with the same parameters except that \( \sigma = 150 \text{ cgs} \). The floor is assumed to be a reflective wooden one with parameters of \( \sigma = 10 \ 000 \text{ cgs}, \ \Omega = 0.2, \ T = 1.4, \ S_p = 0.5, \) and \( d = 0.01 \text{ m} \).

FIG. 3. (Color online) Predictions of the sound pressure levels (SPLs) at frequency 1000 Hz vs the horizontal distance (\( r \)) between the source and receiver in a flat waveguide with height (\( h \)) of 2 m, where all the source and receivers are located at 0.25 m above the floor. The ceiling is the highly sound absorbent type and the floor is reflective. The solid line represents the results with the proposed method, the circles are those in the wave theory, the dash-dotted line with cross markers is with the ray method from Brekhovskikh (Ref. 14) (Brekhovskikh’s method), the dash-dotted line is from the modified method of Lemire and Nicolas (Ref. 11) (Lemire’s method), and the dotted line is with the method of Gensane and Santon (Ref. 15) (Gensane’s method).

FIG. 4. Spectra of sound pressure levels (SPLs) in a flat waveguide with height (\( h \)) of 2 m, where the source is located at (0, 0.25 m) and receivers are at (\( r \), 0.25 m) and pairs of results corresponding to different source/receiver distances are shifted to fit on a single figure. The ceiling is the highly sound absorbent type and the floor is reflective. The solid lines represent the results with the proposed method, and the dotted lines with cross markers are those in the wave theory.
from the wave theory solution at a given frequency are exactly the same for the sound field at different receivers.

Computations have been carried out in this case to further validate the proposed method in the frequency range from 100 to 2 000 Hz, where receivers are chosen with a set of horizontal distances from the source of 5, 10, and 20 times of the waveguide height. Figures 4 and 5 show the results corresponding to the situations where the source and receivers are close to the sound reflective floor ($z_s = z = 0.25$ m) or the absorbent ceiling ($z_s = z = 1.8$ m), respectively, so as to simulate the propagation of noise from the floor-standing electrical appliances or the ceiling-mounted ventilation devices in usual open-plan offices. The situation for sound propagations at the height of a seated person ($z_s = z = 1.2$ m) is also investigated and the corresponding results are presented in Fig. 6.

As shown from Figs. 4–6, the prediction of SPLs with the proposed method always agree well with the wave theory over a broad frequency range, except small differences observed at some lower audio frequencies, which may be explained with two reasons. First the accuracy of Eq. (16) is ensured in theory for large $kr$. Second at these low frequencies, the magnitude of $\beta_i$ becomes small from the evaluation by the surface admittance model in Eqs. (B1), (B3a), and (B3b). This leads to large $(\cos \theta_i + \beta_i)^{-1}$ in the common factor of the series in Eq. (24), as well as large $(\cos \theta_i + \beta_f)^{-1}$ for the series in Eq. (16) as $\theta_i \rightarrow \pi/2$, which can reduce the degree of approximation for $Q_{abs}(S_n, R|CB, n_c)$ significantly.

In the second numerical case, the ceiling is defined with the same parameters used in the first case except that the flow resistivity $\sigma$ of the porous material changes to be 150 cgs. The parameters of the floor are the same with those in the first case. The corresponding SPL vs $r/h$ curves predicted with the proposed method, the wave theory, and the existing ray-based methods are compared in Fig. 7, where $z_S = z = 0.25$ m and the frequency is 1 000 Hz still for examination, whereas the normal incident absorption coefficient of the ceiling becomes 0.55.

In Fig. 7, the agreement between the proposed method and the wave theory is remarkably good with $r/h$ ranging from 1 to 50. The method of Brekhovskikh\textsuperscript{14} is shown with errors for an $r/h$ ratio larger than 7, whereas the performance from the methods of Lemire and Nicolas\textsuperscript{11} and Gensane and Santon\textsuperscript{15} seems better by comparison with that in Fig. 3 of the first case, which might be explained by less wave front shape distortion after each reflection on the ceiling as it is less absorbent. In this case, the spectra of SPL in the frequency range from 100 to 2 000 Hz, predicted with the proposed method and the wave theory, are also compared with the same considerations on source/receiver geometries in the first case. The results corresponding to the situation where $z_s = z = 1.2$ m are presented in Fig. 8, which shows that predictions from the proposed method are in good agreement with the reference method over a broad frequency range again, except small differences at some low frequencies.
More similar results were obtained by selecting other source/receiver geometries and other parameters of the ceiling and floor extensively, but they are not presented here for reasons of brevity. These results support that the proposed method is valid for predicting the sound propagation in flat waveguides with a sound absorptive ceiling and a reflective floor over a broad frequency range, even at large distances from the source compared to the space height while the ceiling can be highly sound absorptive.

The computational time with the proposed method is also investigated versus the frequency and the space height, by comparison to that from the numerical evaluation of the wave theory. The boundary parameters in the first numerical case are used for these investigations and the waveguide height is assumed to be 2, 3, 4, and 5 m in sequence. The location of source is (0, 0.25 m) and those of receivers are (5h, 0.25 m). The corresponding computational time with these two methods in the frequency range from 300 to 2000 Hz with different space heights are presented in Fig. 9. It is shown that the computational time with the wave theory increases remarkably as the actuating frequency increasing, whereas the general trend of that with the proposed method is observed to be a little decrease at higher frequencies, despite of fluctuations that may be due to field coherence between different reflected waves during truncating image source series at single frequencies.

Numerical results presented in Fig. 10 may explain the frequency dependence of computational time with the wave theory, which are carried out to validate the convergence of the total field computations in Eq. (A1). It is indicated from Fig. 10 that more eigenvalue roots in Eq. (A6) are needed to be found to include more high-order modes into total field computation at higher frequencies for convergence, which increases the computational time with the wave theory. However for the proposed image source method, generally less image source orders are needed for convergence at higher frequencies as the boundaries lined with porous materials become more acoustically absorbent at higher frequencies as shown in Fig. 2.

From Fig. 9, it is also shown that with larger space heights, more computational time is needed for the evaluation of the wave theory, but a little less time is needed for the calculation of the proposed method. This is because larger space

![FIG. 8. Spectra of sound pressure levels (SPLs) in a flat waveguide with height (h) of 2 m, where the source is located at (0, 1.2 m) and receivers are at (r, 1.2 m) and pairs of results corresponding to different source/receiver distances are shifted to fit on a single figure. The ceiling is the moderately sound absorbent type and the floor is reflective. The solid lines represent the results with the proposed method, and the dotted lines with cross markers are those in the wave theory.](image1)

![FIG. 9. (Color online) Spectra of the computational time with the proposed method and the numerical evaluation of the wave theory to predict the sound fields in flat waveguides with different heights (h) from 2 to 5 m, where the source is located at (0, 0.25 m) and receivers are at (5h, 0.25 m). The ceiling is the highly sound absorbent type and the floor is reflective. The lines represent the computational time results with the proposed method, and the lines with markers are those with the wave theory.](image2)

![FIG. 10. The convergence of sound pressure level (SPL) in the total field evaluated with the wave theory vs the maximum eigenvalue amplitude ($|k_m|$) in truncating the infinite mode series in total field computations in Eq. (A1) at four different actuating frequencies ($f$), where $k$ is the actuating wave number. The flat waveguide is 2 m high, the source is located at (0, 0.25 m) and the receiver is at (10 m, 0.25 m). The ceiling is the highly sound absorbent type and the floor is reflective.](image3)
dimensions lead to higher distribution density of the eigenvalue roots in a given interval with the wave theory, but they cause faster energy fading of the image sources in the proposed method due to geometric attenuation. It should be noted that the evaluation of the wave theory may be speeded up with some other numerical methods or other configurations. Nevertheless, it is indicated from Fig. 9 that for predicting the sound fields in the flat waveguides with absorptive boundaries, the proposed method is preferable to the wave theory at higher frequencies and for spaces with larger dimensions.

From the above-mentioned results for sound fields in flat waveguides with a sound absorptive ceiling and a reflective floor, it is clear that the proposed coherent image source method is valid and can provide more accurate predictions than the existing coherent ray-based methods for various source/receiver geometries. Although it is found from the numerical results that the proposed method can have reasonable accuracy at certain low frequencies, it must be remarked that the method is based on two assumptions: (i) $kr$ being large and (ii) wave front shape remaining the same before and after each reflection on a reflective boundary. Moreover, numerical results validate the idea proposed in Eq. (23) of separating the reflections on the reflective boundaries from those on absorbent ones in applying reflection coefficients in the image source method. This idea is expected to be useful for developing an accurate and versatile coherent image source method for sound fields in practical enclosures with absorbent boundaries.

**IV. CONCLUSION**

An accurate coherent image source method has been proposed in this paper to predict single frequency sound propagation in flat waveguides with a sound absorptive ceiling and a reflective floor. By expending a spherical wave into plane wave integrals, the method avoids the intractability in analytically evaluating the integrals of reflection waves. This is achieved by introducing a physically plausible assumption that wave front shapes remain the same before and after each reflection on a reflective boundary. The proposed coherent model can be used to predict sound propagation from a point source to arbitrary receiver locations in flat waveguides with highly absorptive ceiling. The method has been numerically validated with the wave theory for various source/receiver geometries over a broad frequency range. It is shown that the proposed method can accurately predict the single frequency sound propagation, even if at large distances from the source compared to the waveguide height whereas the existing coherent ray-based method are inaccurate. It is also shown that the proposed method is preferable to the wave theory at higher frequencies and with larger space dimensions. The proposed method should be useful for sound field prediction in open-plan offices.

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**APPENDIX A: THE WAVE THEORY SOLUTION FOR THE SOUND FIELD FROM A POINT SOURCE IN FLAT WAVEGUIDES**

Based on the wave theory in Refs. 27 and 28, the sound field from a point source in flat waveguides can be solved as a summation of normal modes along directions $z$ and $r$ in Fig. 1(a). It is expressed as

$$P_{\text{tot}} = \sum_{m=0}^{\infty} \frac{jA}{4} \int_{-\infty}^{\infty} \left| \psi_m(z) \right|^2 e^{-\beta m z} H_0^1(K r) dz,$$

where $A$ is the amplitude of the point source and assumed to be unit, and $z$ and $z_S$ are heights of the source and receiver, respectively, and $r$ denotes the source/receiver horizontal distance. The term $\psi_m(z)$ is the wave number. The eigenvalue $\psi_m(\cdot)$:

$$\left| \psi_m(z) \right|^2 = \int_0^h \left| \psi_m(z) \right|^2 dz.$$  \hspace{1cm} (A2)

The eigenfunction $\psi_m(\cdot)$ can be expressed as

$$\psi_m(z) = \omega_m + \gamma_m e^{-j\omega_m},$$  \hspace{1cm} (A3)

where $\omega_m$ denotes the eigenvalue in the $z$ direction and $\gamma_m$ is a complex constant related to $k_m$. Correspondingly the wave number in the $r$ direction in Eq. (A1), $K_r$, is determined from $k_m$ by

$$K_r = \sqrt{k^2 - k_m^2},$$  \hspace{1cm} (A4)

where the complex square root is taken to yield a result with a positive real part and $k$ is the wave number. The eigenvalue $k_m$ and constant $\gamma_m$ are obtained from the boundary condition equations along the pair of planes in the posed problem, which are

$$\begin{cases}
\frac{\partial \psi_m(0)}{\partial z} + j k \beta_f \psi_m(0) = 0 \\
\frac{\partial \psi_m(h)}{\partial z} - j k \beta_c \psi_m(h) = 0
\end{cases}.$$

Hence the eigenvalue equations can be obtained as

$$\frac{k_m + k \beta_f}{k_m - k \beta_c} = e^{2j\omega_m},$$  \hspace{1cm} (A6)

and

$$\gamma_m = \frac{k_m + k \beta_f}{k_m - k \beta_f}.$$  \hspace{1cm} (A7)
It should be noted that the analytical solution of Eq. (A6) is available only if the boundaries are rigid. For boundaries with complex admittance, transcendental equation (A6) becomes complex and may be solved by approximations or classical numerical approaches, such as Newton’s iteration if the boundaries are nearly rigid with sufficiently low admittance amplitude. When boundaries are absorptive with large admittance amplitude, the number of eigenvalue roots in Eq. (A6) and their possible locations in a given interval become hard to know, to which the classical numerical methods in solving nonlinear equations become inapplicable because they need one “good” single initial guess to yield a root. To ensure high accuracy of evaluation with the wave theory as a validation benchmark, a general numerical scheme can be used to solve all the eigenvalue roots in Eq. (A6) in a given interval with robustness, which avoids selecting the initial guesses in eigenvalue solution.

The solution in Eq. (A1) is expressed as the superposition of an infinite mode series, which however can be truncated at a finite number of modes in practice. To validate the truncation criterion considered in this paper for convergence of the total field computations, sound pressure levels in the total field at frequencies of 800, 1000, 1400, and 2000 Hz are examined versus the maximum eigenvalue amplitude \( |k_{mn}| \) considered for truncating the infinite mode series. The corresponding results based on parameters in the first numerical case in Sec. III are presented in Fig. 10, which show that the total field becomes convergent with \( |k_{mn}| \) larger than about 0.6 times of the actuating wave number \( k \) at different frequencies. Therefore \( |k_{mn}| \) is chosen to be 1.5 times \( k \) in this paper to ensure high evaluation accuracy of the wave theory as a reference method.

**APPENDIX B: EVALUATION OF BOUNDARY SURFACE ADMITTANCE FOR THE CEILING AND FLOOR**

The surface admittance of the suspended ceiling, \( \beta_c \), and that of the wooden floor, \( \beta_f \), can be evaluated, respectively, according to classic electrical transmission line analogy,

\[
\beta_c = \frac{Z_a + jZ_d \tan(k_a d)}{Z_a Z_d + jZ_a \tan(k_a d)} \quad \text{(B1)}
\]

and

\[
\beta_f = \frac{1}{Z_a \coth(jk_a d)} \quad \text{(B2)}
\]

where \( Z_a \) in Eq. (B1) represents the specific characteristic impedance of the backing cavity in the ceiling and can be \( -j/\tan(2\pi f L/c) \) in this paper, in which \( f \) is the frequency of interest, \( L \) denotes the cavity depth, and \( c \) is the sound velocity in air. Parameter \( d \) represents the thickness of the ceiling porous material layer in Eq. (B1) or the floor wood layer in Eq. (B2). \( Z_a \) and \( k_a \) denote, respectively, the specific characteristic impedance and the propagation constant inside the ceiling porous material in Eq. (B1) or inside the floor wood layer in Eq. (B2), which can be evaluated by Attenborough’s “three-parameter” approximation

\[
Z_a = \frac{(4T/3) - 4j\Omega \sigma_e}{2\pi f \rho_0} \left( \frac{2\pi f}{c k_a} \right), \quad \text{(B3a)}
\]

\[
k_a = \frac{2\pi f}{c} \sqrt{-\left(\frac{\gamma - 1}{\gamma}N P_r \right)}, \quad \text{(B3b)}
\]

where \( \Omega \) and \( T \) are the porosity and tortuosity inside the media, respectively, \( \rho_0 \) and \( \gamma \) denote the density and the ratio of specific heats for air separately, where \( \rho_0 = 1.293 \) kg m\(^{-3}\) and \( \gamma = 1.4 \) in simulations at the room temperature. Parameter \( a \) inside Eq. (B3b) is defined by \( a = 4/3 - [(\gamma - 1)/\gamma]N P_r \), where \( N P_r \) is the Prandtl number for air and can be 0.71 at the room temperature. In Eqs. (B3a) and (B3b), \( \sigma_e \) denotes an effective flow resistivity and is defined by \( \sigma_e = 4S_p^2 \sigma/\Omega \), where \( S_p \) is the pore shape factor and \( \sigma \) is the flow resistivity of the media with a unit of cgs (1 cgs = 1 kPa s m\(^{-2}\)).