Pricing and Hedging of Long-Dated Commodity Derivatives

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by

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Certificate of Authorship and Originality

I certify that the work in this thesis has not previously been submitted for a degree nor has it been submitted as part of requirements for a degree except as fully acknowledged within the text.

I also certify that the thesis has been written by me. Any help that I have received in my research work and the preparation of the thesis itself has been acknowledged. In addition, I certify that all information sources and literature used are indicated in the thesis.

Signed .....

Date .....

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Contents

Abstract

Commodity markets have grown substantially over the last decade and significantly contribute to all major financial sectors such as hedge funds, investment funds and insurance. Crude oil derivatives, in particular, are the most actively traded commodity derivative in which the market for long-dated contracts have tripled over the last 10 years. Given the rapid development and increasing importance of long-dated commodity derivatives contracts, models that can accurately evaluate and hedge this type of contracts become of critical importance.

Early commodity pricing models proposed in the literature are spot price models with convenience yields either modelled as a function of the spot price or as a correlated stochastic process. These models may have desired features of commodity prices such as mean-reversion and seasonality. However, futures prices from this type of models are endogenously derived. Consequently, futures prices of different maturities are highly correlated. Multi-factor spot price models may remedy this issue. Aiming to model the entire term structure of commodity futures price curve, several authors have proposed commodity pricing models within the Heath, Jarrow & Morton (1992) (hereafter HJM) framework, with different levels of generality. These models, albeit having captured empirically observed features of commodity derivatives, such as unspanned stochastic volatility and hump volatility structures, may not be suitable to price and/or hedge long-dated commodity derivatives as they assume deterministic interest rates. Models featuring stochastic volatility and stochastic interest rates have been studied for equity and FX markets, known as hybrid models, and yet the research in commodity derivatives markets is limited.

The main contributions of this thesis include:

- ▷ *Pricing of long-dated commodity derivatives with stochastic volatility and stochastic interest rates – Chapter 2*. This chapter develops a class of forward price

models within the HJM framework for commodity derivatives that incorporates stochastic volatility and stochastic interest rates and allows a correlation structure between the underlying processes. The functional form of the futures price volatility is specified, so that the model admits finite dimensional realisations and retains affine representations; henceforth, quasi-analytical European futures option pricing formulae can be obtained. A sensitivity analysis of the model parameters on pricing long-dated contracts is conducted, and the results are discussed.

▷ *Empirical pricing performance on long-dated crude oil derivatives – Chapter 3*. This chapter conducts an empirical study on the pricing performance of stochastic volatility/stochastic interest-rate models on long-dated crude oil derivatives. Forward price stochastic volatility models for commodity derivatives with deterministic and stochastic interest-rate specifications are considered that allow for a full correlation structure. By using historical crude oil futures and option prices, the proposed models are estimated, and the associated computational issues and results are discussed.

▷ *Hedging of futures options with stochastic interest rates – Chapter 4*. This chapter studies hedging of long-dated futures options with spot price models incorporating stochastic interest rates, a modified version of the Rabinovitch (1989) model. Several hedging schemes are considered including delta hedging and interest-rate hedging. The impact of the model parameters, such as the volatility of the interest rates, the long-term level of the interest rates, and the correlation on the hedging performance is investigated. Hedging long-dated futures options with shorter maturity derivatives is also considered.

▷ *Empirical hedging performance on long-dated crude oil derivatives – Chapter 5*. This chapter conducts an empirical study on hedging long-dated crude oil derivatives with the stochastic volatility/stochastic interest-rate models developed in Chapter 2. Delta hedging, gamma hedging, vega hedging and interest-rate hedging are considered, and the corresponding hedge ratios are computed by using factor hedging. The hedging performance of long-dated crude oil options is assessed with a variety of hedging instruments, such as futures and options with shorter maturities.

CHAPTER 1

Introduction

1.1. Literature Review and Motivation

The role of commodity markets in the financial sector has substantially increased over the last decade. A record high of $277 billion invested in commodity exchange-traded products was observed in 2009[1] (which was 50 times higher than the decade earlier) with the crude oil market being the most active commodity market. A variety of new products become available including exchange-traded products, managed futures funds, and hedge funds that boost activity in both short-term trading and long-term investment strategies [see Morana (2013)]. With the disappointing performance of the equity index market, commodity markets along with real estate become the new promising alternative investment vehicles with the commodity index persistently outperforming the S&P 500 index (see Figure 1.1) over the last decade. The crude oil futures and options are the world's most actively traded commodity derivatives forming a major part of these activities. The average daily open interest in crude oil futures contracts of all maturities has increased from 503,662 contracts in 2003 to 1,677,627 in 2013. Even though the most active contracts are short-dated, the market for long-dated contracts has also substantially increased. The maturities of crude oil futures contracts and options on futures listed on CME Group have extended to 9 years in recent years. The average daily open interest in crude oil futures contracts with maturities of two years or more were at 41,601 in December 2003, and have reached a record high of 202,964 contracts in 2008. Motivated by the increasing importance of long-dated crude oil exchange traded contracts for hedging, speculation

[1]Source: www.barchart.com/articles/etf/commodityindex.

and mostly investment purposes in the financial sector, we examine the pricing performance of long-dated crude oil derivatives.

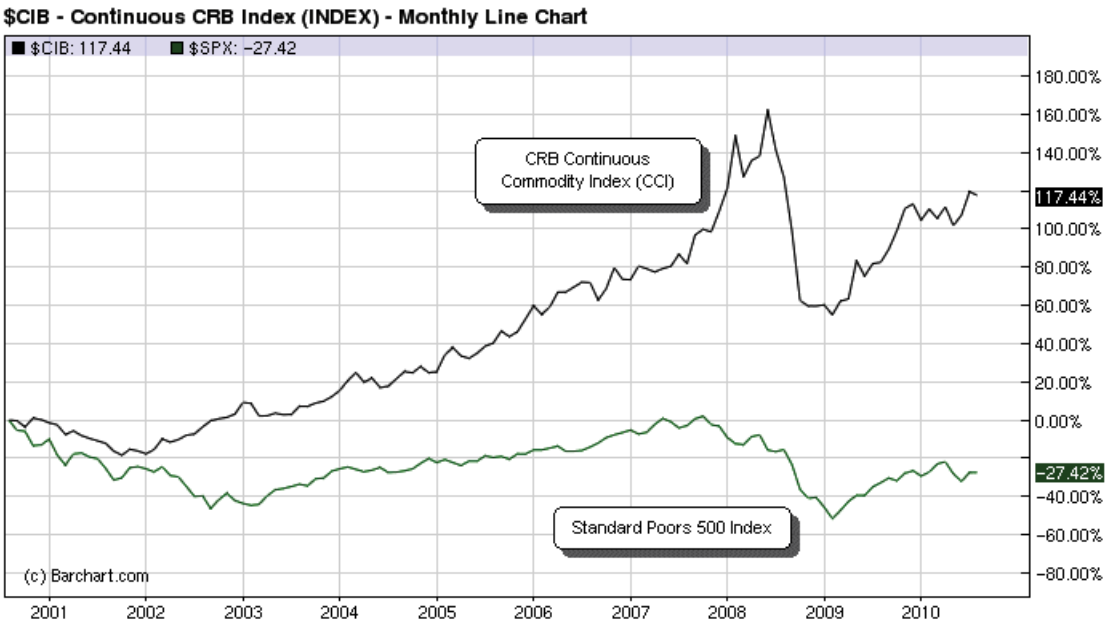

FIGURE 1.1. **CRB Continuous Commodity Index versus the S&P 500 Index.** Commodity index persistently outperformed the S&P 500 index. Source: www.barchart.com/articles/etf/commodityindex.

1.1.1. Option pricing models. The main drawbacks of the option pricing model introduced in the seminal paper by Black & Scholes (1973) comes from the assumptions of constant volatility and constant interest rates. While these assumptions admit a simple closed-form valuation formulae for European vanilla options, the constant volatility assumption usually overprices options near at-the-money and underprices deep-in-the-money or out-of-the-money options, and this discrepancy is more pronounced as the maturity increases.

1.1.1.1. *Stochastic volatility.* The pioneering work of Hull & White (1987), Stein & Stein (1991) and Heston (1993) allow the volatility in the Black & Scholes (1973) model

to be stochastic. By generalising the volatility of the spot asset price, this type of stochastic volatility models has the flexibility to capture the implied volatility surface. As a consequence, it produces option prices much closer to the market prices. The model proposed by Hull & White (1987) assumes that the variance process follows a geometric Brownian motion. This model has the advantage that the variance process is always positive and allows a non-zero correlation between the variance process and the spot price process. However, closed-form solutions for option valuation are not available, and the expectation of the variance process increases without bounds as time-to-maturity increases. The model introduced by Stein & Stein (1991) features a mean-reverting stochastic volatility process, but assumes zero correlation between the volatility process and the spot price process. The model admits closed-form solutions and the expectation of the volatility process converges to its long-term mean as time-to-maturity increases, however, there is a non-zero probability that the volatility process becomes negative. Heston (1993) models the dynamics of the spot price process by a geometric Brownian motion with a correlated stochastic volatility process following the Cox, Ingersoll & Ross (1985) process. Heston demonstrates that the explicit solution of the characteristic function of the logarithm of the spot price is available, and can be obtained by an application of the Fourier inversion technique pricing formulae for European call and put options. Schöbel & Zhu (1999) generalise the model proposed by Stein & Stein (1991) by allowing a non-zero correlation between the volatility process and the spot price process, and yet admits closed-form solutions for option valuation.

1.1.1.2. *Stochastic interest rates.* Another limitation of the Black & Scholes (1973) is the assumption of constant interest rates. Merton (1973) relaxes the assumption of constant interest rates in Black & Scholes (1973) model by assuming that the dynamic of the bond prices follow a stochastic differential equation, which depends on the interest rates. However, Merton (1973) only points out that if the interest rates are deterministic, then his general formula simplifies to the Black-Scholes formula. Rabinovitch (1989) proposes a tractable two-factor stochastic interest-rate model to value call options on stocks

and bonds. Turnbull & Milne (1991) and Amin & Jarrow (1992) both price contingent claims under the presence of stochastic interest rates. By using the asymptotic expansion approach, Kim & Kunitomo (1999) are able to derive an explicit solution to approximate option prices under a more general interest-rate specification. Scott (1997) proposes a spot price jump-diffusion model featuring stochastic volatility and stochastic interest rates with closed form solutions for prices on European stock options. By calibrating the model, he conducts an empirical analysis and results reveal that stock returns are negatively correlated with volatility and interest rates. He concludes that stochastic volatility and stochastic interest rates have a significant impact on option prices, particularly on the prices of long-dated options.

Bakshi, Cao & Chen (1997) and Bakshi, Cao & Chen (2000) have conducted an empirical analysis to investigate to what extent a more general stochastic model may bring to pricing and hedging performance on equity options. Bakshi et al. (1997) conclude that incorporating stochastic interest rates does not enhance pricing and hedging performance. This result can be attributed to the maturity of the index options being less than one year. Bakshi et al. (2000) conduct a similar empirical research on the pricing and hedging of options but with an emphasis on long-term options. By using long-term equity anticipation securities with maturities up to three years, Bakshi et al. (2000) empirically investigate the pricing and hedging performances on four option pricing models, namely the Black & Scholes (1973) model, the stochastic volatility model, the stochastic volatility/stochastic interest-rate model and the stochastic volatility jump model. Their results on pricing performance show that stochastic volatility jump model performs the best for short-term puts and the stochastic volatility model performs the best for long-term puts. They do not find evidence that stochastic volatility/stochastic interest-rate models lead to consistent improvement in pricing performance. However, in their hedging exercise, they find that the stochastic volatility stochastic interest-rate model helps improve empirical performance, when it devises hedges of long-term options. Fergusson & Platen (2015) develop a model to price long-dated equity index options with stochastic interest rates

under the benchmark approach [see Platen (2006)].

1.1.1.3. *Hybrid pricing models.* In recent years, markets for long-dated derivatives have become far more liquid and have attracted the attention of major market participants such as hedge funds and insurance companies.[2] Motivated by the new market for long-dated derivatives and the empirical findings, a new type of so-called hybrid pricing models has been proposed in the literature. This type of models typically features a geometric Brownian motion for the spot price process with mean-reverting stochastic volatility and stochastic interest-rate processes. Depending on the type of mean-reverting process and the correlation structure, the model may require some numerical approximations before leading to closed-form option pricing formulae. A hybrid spot price model was introduced by van Haastrecht, Lord, Pelsser & Schrager (2009) and Grzelak, Oosterlee & van Weeren (2012) by combining the stochastic volatility model by Schöbel & Zhu (1999) as the spot price process and the Hull & White (1990) model as the stochastic interest-rate process, while allowing full correlations between the spot price process, its stochastic volatility process and the interest-rate process. This model is dubbed as the Schöbel-Zhu Hull-White (hereafter SZHW) model. The model was applied by van Haastrecht et al. (2009) to the valuation of insurance options with long-term equity or foreign exchange (FX) exposure. The key advantage of this model is that it admits a closed-form pricing formula for European-style vanilla options; however, there is a positive probability that the interest-rate process or the stochastic volatility process becomes negative. Grzelak & Oosterlee (2011) propose two hybrid spot price models that specifically target the shortcomings of the SZHW model by replacing the mean-reverting Hull & White (1990) processes with the square-root processes [see Cox et al. (1985)]. While these models ensure that the stochastic volatility process is positive (Heston-Hull-White) or both the

[2]For example, Long-term Equity Anticipation Securities (LEAPS) are long-dated (more than 1 year) put and call options on common stocks, equity indexes or American depositary receipts (ADRs), Power reverse dual-currency notes are long-dated FX hybrid products or option contracts on crude oil listed on New York Mercantile Exchange (NYMEX) that extends to 9 years.

stochastic volatility process and stochastic interest-rate processes are positive (Heston-Cox-Ingersoll-Ross); the downside is that a closed-form option pricing formula cannot be obtained. However, the authors propose approximations to obtain analytical characteristic functions. Grzelak & Oosterlee (2012) apply the (Heston-Hull-White) model to value FX options where both domestic and foreign interest-rate processes are modelled by the Hull & White (1990) process. Even though equity and FX markets have been extensively studied with hybrid pricing models, yet, there is limited literature on commodity markets. This thesis aims to contribute to closing this gap.

1.1.2. Commodity Pricing Models. The Black (1976) model is the earliest commodity pricing model. It is based on the Black & Scholes (1973) model, and it is very popular among practitioners, although it lacks many features of interest. This model assumes that the cost-of-carry formula holds and that net convenience yields are constant. Another drawback is that describing the futures price dynamics only by a geometric Brownian motion does not capture commodity price properties, for instance, changes in the shape of the futures curves and mean-reversion. In the spirit of Black & Scholes (1973), the early commodity pricing models specify exogenously a stochastic process for the spot price dynamics, and then futures contracts are set to be equal to the expected future spot price under the risk-neutral measure. Brennan & Schwartz (1985) consider a model where the spot price of the commodity follows a geometric Brownian motion and a convenience yield that is a deterministic function of the spot price. This model does not capture the mean-reversion behaviour of market observable commodity prices. Gibson & Schwartz (1990) develop a two-factor joint diffusion process where the spot price follows a geometric Brownian motion with constant drift and volatility, and the net convenience yield follows an Ornstein-Uhlenbeck process, thereby the mean-reversion property of commodity prices is induced by the convenience yield process. However, this model does not feature stochastic volatility nor stochastic interest rates, and examines only the

pricing performance of relatively short-dated futures contracts.

Schwartz (1997) proposes and investigates the pricing performance of three commodity spot pricing models. The one-factor model features a mean-reversion process and models the logarithm of spot prices. The two-factor and three-factor models follow the Gibson & Schwartz (1990) model with an additional correlated interest-rate process following the Ornstein-Uhlenbeck process in the three-factor model. By employing Kalman filter and fitting to futures contracts only, the pricing performance of the three commodity derivative pricing models was empirically investigated. The one-factor model could not adequately capture the market prices of futures contracts, and the two-factor model greatly improves the ability to describe the empirically observed price behaviour of copper, crude oil and gold. Using crude oil futures contracts with a maturity of up to 17 months, the term structure of futures prices implied by the two- and three-factor models are indistinguishable. However, using proprietary crude oil forward prices with maturity of up to 9 years, the two- and three-factor models can imply quite different term structure of futures prices. The three-factor model with stochastic interest rates (6% infinite maturity discount bond yield is assumed) performs the best, and he confirms the importance of the stochastic interest-rate process for longer maturity contracts.

Hilliard & Reis (1998) propose a commodity pricing model with stochastic convenience yields, stochastic interest rates and jumps in spot prices, and investigate how these features impact futures, forwards and futures options. Schwartz & Smith (2000) assume that the price changes from long-dated futures contracts represent fundamental modifications which are expected to persist, whereas changes from the short-dated futures contracts represent ephemeral price shocks that are not expected to persist. Thus, they propose a two-factor model where the first factor represents short-run deviations and the second process represents the equilibrium level. Their model provides a better fitting to medium-term futures prices rather than to short-term and long-term futures prices. None of these models though feature stochastic volatility, an important feature for pricing long-dated

contracts [see Bakshi et al. (2000) and Cortazar, Gutierrez & Ortega (2016)]. Furthermore, the above-mentioned models are spot price models. A representative among the more recent literature of pricing commodity contingent claims with spot price models includes Cortazar & Schwartz (2003), Casassus & Collin-Dufresne (2005), Geman (2005), Geman & Nguyen (2005), Cortazar & Naranjo (2006), Dempster, Medova & Tang (2008) and Fusai, Marena & Roncoroni (2008). [3]

The main drawback of commodity spot price models is that the whole observed term structure of futures prices is implied. All spot price models, regardless of the level of generality, derive the futures or forward prices endogenously, by no-arbitrary arguments. This approach imposes restriction on the term structure of the futures prices. As a consequence, it is difficult for a spot commodity model to capture certain features of the term structure of futures prices. Furthermore, commodity spot price models require the modelling of the unobservable convenience yield, leading to more complicated models with additional parameters that make model estimation more demanding and less intuitive.

1.1.2.1. *Commodity forward price models.* In their seminal paper, Heath et al. (1992) (hereafter HJM) propose a general framework to model the evolution of instantaneous forward rates that, by construction, is consistent with the currently observed yield curves, and depends entirely on the forward rate volatility. Along these lines, Reisman (1991) proposes a commodity derivative pricing model under the HJM framework, and derives the spot price and the cost of carry processes from the dynamic of the futures prices. Cortazar & Schwartz (1994) apply the HJM framework with deterministic volatility specifications and theoretical results developed in Reisman (1991) to analyse the daily prices for all copper futures over a sample period of twelve years. Miltersen & Schwartz (1998) develop a commodity forward price option pricing model, where both interest rates and

[3]Even though the spot-price Fusai et al. (2008) model is able to perfectly fit the initial forward curve, unlike a forward-price model, it does not have any state variables to capture the change of the forward curve. It is not able to adapt to tomorrow's forward curve and as a consequence, tomorrow's forward curve is implied from the model.

convenience yields are stochastic. They demonstrate that by imposing suitable specifications on the interest-rate process and convenience yield process, closed-form solutions generalising the Black-Scholes formulae can be obtained.

Miltersen (2003) develops a commodity pricing model of spot prices and spot convenience yields that follow the Hull & White (1990) model such that the model matches the term structure of forward and futures prices and the term structure of forward and futures price volatilities. Crosby (2008) develops a jump-diffusion commodity model where the dynamics of futures price follows the HJM framework, and allows long-dated futures contracts to jump by smaller magnitudes than short-dated futures contracts. One implication is that the implied volatilities that this model produces are more skewed for short-dated contracts than long-dated contracts. However, all above-mentioned models assume deterministic volatility.

Trolle & Schwartz (2009*b*) introduce a commodity derivative pricing model within the HJM framework featuring unspanned stochastic volatility. By using crude oil futures contracts with maturities of up to 5 years and futures options with maturities of up to about 1 year, they empirically demonstrate the existence of unspanned stochastic volatility components. Thus, crude oil options are not redundant contracts, in the sense that they cannot be fully replicated by portfolios consisting only the underlying futures contracts. By fitting their model to a longer dataset of crude oil futures and options, Chiarella, Kang, Nikitopoulos & Tô (2013) consider a commodity pricing model within the Heath et al. (1992) framework, and demonstrate that a hump-shaped crude oil futures volatility structure provides a better fit to futures and option prices, and improves hedging performance. Pilz & Schlögl (2013) model commodity forward prices with stochastic interest rates driven by a multi-currency London Interbank Offered Rate (hereafter, LIBOR) Market Model and achieve a consistent cross-sectional calibration (i.e., single day) of the model to at-the-money market data for interest-rate options, commodity options and historically

estimated correlations.[4] Cortazar et al. (2016) investigate the pricing performance of different models on commodity prices, namely crude oil, gold and copper. The constant volatility model fits futures prices better, but the fitting to options improves significantly when a stochastic volatility model is considered on the expense of additional computational effort. Chiarella, Kang, Nikitopoulos & Tô (2016) present an alternative approach to study the return-volatility relationship in commodity futures markets, and analyse this relation in the crude oil futures markets and the gold futures markets. However, most of these studies assume deterministic interest rates. Thus, they may not be suitable for evaluation of long-dated derivative contracts.

1.1.3. Hedging of long-dated commodity derivatives. The collapse of the 14th largest German company – Metallgesellschaft A.G. (hereafter, MG) – at the end of 1993 has sparked research on understanding the risks associated with long-dated crude oil commitments and their method of hedging with short-dated futures contracts. In late 1993, MG had accumulated a total of over 150 million barrels of crude oil of commitments that had to deliver at fixed prices for every month spreading over the next 10 years. To 'lock-in' a profit, MG committed a one-to-one hedge which essentially held a long position in short-dated futures contracts and short-dated swap contracts with the size of the long position equivalent to the total number of barrels of crude oil in its forward commitments. Tragically, during most of the year of 1993 the prices of spot and nearby futures contracts declined significantly. Since futures contracts are traded in exchanges, and they are marked-to-market, they result in heavy losses from margin calls. MG's long-dated short positions in its forward contracts have made some gains, however most of the gains would not be realised until their corresponding forward contracts matured. Consequently, this so-called 'stack-and-roll' hedging strategy suffered a loss of 1.3 billion dollars, and

[4]Cross-sectional calibration to market data for interest-rate and commodity option volatility surfaces is treated in a stochastic volatility extension of the LIBOR Market Model in Karlsson, Pilz & Schlogl (2016).

nearly sent the German company to bankruptcy.[5]

One of the earlier works on hedging using futures contracts is Ederington (1979). This paper evaluates the hedging performance of using bonds, treasury bills and agricultural futures as instruments to hedge their corresponding spot prices. The hedge ratios are defined as the amount of futures positions proportional to the spot position and are evaluated as the amount of futures positions to minimise the portfolio variance. By arguing that one's expectations of the near future will be affected more by unexpected changes in the cash price than one's expectation of the more distant futures, the author hypothesises that the hedging performance would decline as one uses more distant contracts as hedging instruments, which is demonstrated in his empirical results.

Motivated by the debacle of Metallgesellschaft A.G., several papers specifically investigated the flaw of the stack-and-roll hedging strategy, and concluded that while MG was hedging the market risk, it was taking a huge risk on the basis. Edwards & Canter (1995) conduct a comprehensive investigation on the collapse of MG and suggest that MG was exposed to basis risk, funding risk and credit risk. They suggest that MG's barrel for barrel stack-and-roll strategy by rolling over short-dated futures when they mature is the culprit for the collapse of the company. They also propose a minimum-variance hedge strategy which could minimise the funding risk. Schwartz (1997), by using the Kalman filter methodology to estimate the parameters of his three proposed models, analyses the implications on the hedging ratios of short-term futures contracts to hedge long-term crude oil commitments. Bond contracts as hedging instruments are required in addition to the two futures contracts in his three-factor stochastic interest-rate model. Schwartz (1997) gives a brief explanation of the hedge ratios derived by the estimated model parameters, but it does not investigate the hedging performance of the models.

[5] 150 German and international banks rescued the company with a massive $1.9 billion dollar operation.

Brennan & Crew (1997) construct several hedging schemes for long-dated commodity commitments using short-dated futures contracts, and compare their hedging performances. The hedging schemes are the simple stack and roll hedge and the minimum variance hedges as well as more advanced hedges devised from two stochastic convenience yield models, namely Brennan (1991) and Gibson & Schwartz (1990). Their empirical results show that the hedging performance from the two stochastic convenience yield models perform significantly better than the two simpler methods. In all schemes, the hedging performance in terms of both mean hedging errors and standard deviation of hedging errors worsen as the maturities of the commitments increase. The authors point out that the longest maturity of the available futures contracts is only 24 months, which is much shorter than the forward commitments (10 years) in the case of MG. Another drawback is that the interest rates are assumed to be deterministic in these two stochastic convenience yield models.

Neuberger (1999), unlike similar papers in the literature that concentrate on using one futures contract as hedging instrument, analyses the problem in hedging a long-dated commodity commitment using multiple short-dated commodity futures contracts. The model that he proposes is very different from the dynamic models in the literature, in which, at the beginning of the month, a new longest maturity futures contract is listed, and its price is assumed to be a linear function of the prices of those traded contracts plus a noise term. Empirical results show that 85% of the risk can be expected to be removed of a six-year oil commitment by hedging with medium maturity futures contracts. However, the maturities of the shorter maturity futures contracts and the number of futures contracts are chosen rather arbitrarily, and the high number of futures contracts required in this strategy makes the application impractical.

Other papers in the literature that consider using multiple futures contracts to hedge long-dated commodity commitments include Veld-Merkoulova & De Roon (2003), Bühler, Korn & Schöbel (2004) and Shiraya & Takahashi (2012). Veld-Merkoulova & De Roon

(2003), by proposing a term structure model of futures convenience yields, develop a strategy that minimises both spot price risks and basis risks by using two futures contracts with different maturities. Empirical results show that this two-futures strategy outperforms the simple stack-and-roll hedge substantially in hedging performance. Bühler et al. (2004) empirically demonstrate that the oil market is characterised by two pricing regimes – when the spot oil price is high (low), the sensitivity of the futures price is low (high). They then propose a continuous-time, partial equilibrium, two-regime model and show that their model implies a relatively high (low) hedge ratios when oil prices are low (high). Shiraya & Takahashi (2012) propose a mean reversion Gaussian model of commodity spot prices and futures prices are derived endogenously. In their hedging analysis, using 3 futures contracts with different maturities to hedge the long-dated forward contract, they calculate the hedging positions by matching their sensitivities with different uncertainty. They empirically demonstrate that their Gaussian model outperforms the stack and roll model used by MG.

The literature on hedging long-dated forward commodity commitments is well developed, yet hedging of long-dated commodity derivative positions, in particular options, is relatively limited. Trolle & Schwartz (2009*b*) propose a stochastic volatility commodity derivative model that features unspanned volatility. Their empirical results demonstrate that adding options to the set of hedging instruments improves the hedging performance of volatility trades such as straddles, compared to using only futures as hedging instruments. This confirms that unspanned components exist in the volatility in the crude oil derivatives markets. Hence, crude oil options are not redundant contracts. Chiarella et al. (2013) propose a commodity forward price model featuring stochastic volatility that allows a hump-shaped volatility structure. Using factor hedging to derive hedging ratios to hedge a straddle over a period of six months (around August, 2008), they demonstrate that delta-gamma and delta-vega hedge are more effective than delta hedge, and that the hump-shaped volatility specification improves hedging performance compared to the exponentially decaying volatility specification typically used in the literature. Both of these

models assume deterministic interest rates, and consider hedging of short-dated option positions. This thesis aims to extend this research by incorporating stochastic interest rates and examining the hedging of long-dated commodity option positions.

1.2. Thesis Structure

There is a fast growing market for long-dated commodity derivative contracts, and the development of suitable models to accurately evaluate and hedge these contracts is of crucial importance. Empirical evidence in equity markets and FX markets suggests that stochastic volatility and stochastic interest rates are essential features when dealing with long-dated derivative contracts [see Bakshi et al. (2000) and Cortazar et al. (2016)]. This class of models incorporating stochastic volatility and stochastic interest rates is well-known as hybrid pricing models. Consequently, equity and FX markets have been extensively studied with hybrid pricing models [see, for instance, Amin & Jarrow (1992), Amin & Ng (1993), van Haastrecht et al. (2009), van Haastrecht & Pelsser (2011) and Grzelak et al. (2012)]. Yet, there is limited literature on commodity markets. Aiming to develop models suited for pricing and hedging long-dated contracts, this thesis studies commodity derivative pricing models featuring stochastic volatility and stochastic interest rates. Furthermore, two empirical studies are conducted to assess pricing and hedging performance of these models in the most liquid commodity derivatives market, the crude oil market.

There are four parts in this thesis. The first part, covered in Chapter 2, develops a class of forward price models for pricing commodity derivatives that incorporates stochastic volatility and stochastic interest rates. The second part, presented in Chapter 3, empirically investigates the pricing performance of the stochastic volatility/stochastic interest-rate models developed in Chapter 2 on crude oil long-dated derivatives. The third part, discussed in Chapter 4, examines hedging of futures options with spot price models that incorporate stochastic interest rates. The fourth part, discussed in Chapter 5, empirically

investigates the hedging of long-dated crude oil derivatives.

1.2.1. Commodity derivative models with stochastic volatility and stochastic interest rates. Chapter 2 develops a class of forward price models with stochastic volatility and stochastic interest rates for pricing commodity derivatives. The model is within the Heath et al. (1992) framework with the stochastic volatility modelled by an Ornstein-Uhlenbeck process and the interest-rate process modelled by a Hull & White (1990) process. Thus the proposed model can be considered as an extension of the Schöbel & Zhu (1999) to a Schöbel-Zhu-Hull-White type of model suitable for pricing commodity derivatives. The proposed models have the following advantages: firstly, as forward price models, they fit the entire initial forward curve by construction. Secondly, commodity futures prices are modelled directly rather than generating their dynamics endogenously from the spot price and the convenience yield. Traditionally, commodity spot price models require the modelling of the unobserved convenience yield and the spot price process to derive the dynamics of the futures prices. Thirdly, the proposed models allow a full correlation structure between the underlying futures price process, the stochastic volatility process, and the stochastic interest-rate process. Fourthly, the models can accommodate empirically observed volatility features such as unspanned stochastic volatility components, exponentially decaying – hump-shaped volatility structures [see Trolle & Schwartz (2009*b*) and Chiarella et al. (2013)]. In general though, models under the HJM framework have infinite dimensional state-space.

The main contributions of Chapter 2 are twofold: firstly, by imposing suitable structures on the volatility of futures price processes, the proposed models admit finite dimensional realisations and retain affine representations. Following the method presented in Duffie, Pan & Singleton (2000), quasi-analytical pricing formulae for European vanilla futures option prices can be obtained via an application of Fourier inversion technique. Therefore, this class of models is well suited for estimation and calibration applications. Since

these models can be fitted to both futures and option prices, they can potentially depict well the forward curves and the volatility smiles. Secondly, a sensitivity analysis is conducted to gauge the impact of the parameters of the interest-rate process to commodity derivatives prices. The analysis reveals that the correlation between stochastic interest rates and futures prices plays an important role on pricing long-dated commodity derivatives. In addition, commodity derivative prices of long-dated contracts are more sensitive to the interest-rate volatility rather than the long-term level of interest rates.

1.2.2. Empirical pricing performance to long-dated crude oil derivatives. Numerous empirical studies in equity markets and FX markets for pricing long-dated contracts have been conducted, some with stochastic interest rates, such as Rindell (1995), some with stochastic volatility, and, some with both [see Bakshi et al. (1997), Grzelak et al. (2012) and Bakshi et al. (2000)]. In general, stochastic volatility and stochastic interest rates matter for pricing long-dated contracts. In commodity markets, the literature is far more limited, with some models assuming stochastic interest rates, such as Schwartz (1997), some stochastic volatility such as Trolle & Schwartz (2009*a*), Chiarella et al. (2013) and Cortazar et al. (2016), but not both.

Chapter 3 assesses the impact of including stochastic interest rates beyond stochastic volatility to pricing long-dated crude oil derivative contracts, which are the most liquid long-dated commodity derivative contracts.[6] The key contributions of Chapter 3 are: Firstly, it considers two stochastic volatility forward price models, one model with deterministic interest-rate specifications and one model with stochastic interest rates as proposed in Chapter 2. Both models lead to affine term structures for futures prices and quasi-analytical European vanilla futures option pricing equations. Secondly, by using

[6]Crude oil is the most liquid commodity derivatives at CME Group.
References:
http://www.infinitytrading.com/futures/energy-futures/crude-oil-futures.
https://tradingsim.com/blog/top-10-liquid-futures-contracts/.
http://www.investopedia.com/articles/active-trading/090215/analyzing-5-most-liquid-commodity-futures.asp.

extended Kalman filter maximum log-likelihood methodology, the models parameters are estimated from historical time series of both crude oil futures prices and crude oil futures option prices. Thirdly, in-sample and out-of-sample pricing performance of the proposed models is carried out under different market conditions. One of the periods under consideration is $2005 - 2007$ that is characterised by relatively volatile interest rates and the other period, $2011 - 2012$, in which the interest-rate market was very low and stable.

Several important findings emerged from these investigations. Stochastic interest rates matter when pricing long-dated crude oil derivatives, especially when the volatility of the interest rates is high. In addition, increasing the dimension of the model (from two dimensions to three dimensions) does improve fitting to data, but it does not improve the pricing performance [see also Schwartz & Smith (2000) and Cortazar et al. (2016)]. Furthermore, there is empirical evidence for a negative correlation between crude oil futures prices and interest rates that is aligned with the empirical findings of Arora & Tanner (2013) in the crude oil spot market. Finally, the correlation between the stochastic volatility process and the stochastic interest-rate process has negligible impact on prices of long-dated options.

1.2.3. Hedging of futures options with stochastic interest rates. Interest-rate risk becomes more pronounced to derivatives with longer maturities, and models with stochastic interest rates tend to improve pricing and hedging performance on long-dated contracts. Hence, many stock option pricing models with stochastic interest rates, such as Rabinovitch (1989), Amin & Jarrow (1992) and Kim & Kunitomo (1999), have been developed and provide pricing formulae for long-dated contracts. However, hedging of long-dated contracts has not been sufficiently studied.

Chapter 4 investigates the hedging of long-dated futures options by using the Rabinovitch (1989) model for pricing options on futures, where the spot asset price process follows

a geometric Brownian motion, and the stochastic interest-rate process is modelled by an Ornstein-Uhlenbeck process. This chapter makes several contributions. Firstly, by using a Monte-Carlo simulation approach, the hedging performance of long-dated futures options for a variety of hedging schemes such as delta hedging and interest-rate hedging is assessed. The impact of the model parameters such as the interest rates volatility, the long-term level of the interest rates and the correlation to the hedging performance is thoroughly investigated. Secondly, to gauge the contribution of the stochastic interest-rate specifications to hedging long-dated option positions, hedge ratios from the deterministic interest-rate Black (1976) model and the stochastic interest-rate two-factor Rabinovitch (1989) model are considered and compared. Thirdly, the contribution of the discretisation error in the proposed hedging schemes is evaluated. Fourthly, hedging long-dated option contracts with a range of short-dated hedging instruments such as futures and forwards is examined and the impact of maturity mismatch between the target option to be hedged and hedging instruments is investigated. Fifthly, the factor hedging approach, a hedging approach suitable for multi-dimensional models [see Clewlow & Strickland (2000) and Chiarella et al. (2013)] is introduced, and the numerical efficiency of the approach is validated.

This chapter demonstrates both theoretically and numerically, that forward and futures contracts with the same maturity as the futures option can replicate the forward price of the option. However, when using short-dated contracts to hedge options with longer maturities, forward or futures contracts alone can no longer hedge the interest-rate risk. Adding bond contracts to the hedging portfolio is necessary in order to hedge away interest-rate risk.

1.2.4. Empirical hedging performance to long-dated crude oil derivatives. In 1993, the German company Metallgesellschaft suffered massive losses in its short-term futures contracts which could have sent the company to bankruptcy. This debacle has ignited a

number of research papers to investigate the methods and risks involving in hedging long-dated commodity derivatives, including Edwards & Canter (1995) and Brennan & Crew (1997), Neuberger (1999), Veld-Merkoulova & De Roon (2003), Bühler et al. (2004) and Shiraya & Takahashi (2012). Recent empirical studies on hedging crude oil derivatives positions, such as Trolle & Schwartz (2009*b*) and Chiarella et al. (2013), use models with unspanned stochastic volatility. However, this class of models is restricted to deterministic interest-rate specifications.

Chapter 5 empirically investigates the hedging performance of the stochastic volatility stochastic interest-rate model proposed in Chapter 2. In particular, this chapter aims to hedge a long-dated futures option on crude oil for a number of years using futures and option contracts traded in NYMEX. The factor hedging is applied to devise hedge ratios under several different hedging schemes – delta, delta-IR, delta-vega, delta-gamma, delta-vega-IR and delta-gamma-IR. For comparison purposes, both stochastic and deterministic interest-rate models are considered to compute the required hedge ratios. In addition, hedging of long-dated options contracts with short-dated futures contracts is examined, and the impact of increasing the maturity of the hedging instruments is investigated.

Several conclusions can be drawn from this empirical analysis. Firstly, interest-rate hedging overall improves hedging performance of long-dated contracts with the hedge being more effective during periods of high interest-rate volatility, for instance, around the Global Financial Crisis (GFC), compared to recent years where interest-rate volatility is very low. Secondly, using hedging instruments with longer maturities (compared to shorter maturities) reduces the hedging error, potentially due to the reduced basis risk associated to the infrequent rolling–the–hedge forward application of longer maturity hedging instruments. Thirdly, during GFC, stochastic interest-rate model demonstrates noticeably better hedging performance than the deterministic counterpart but only marginal improvement during pre-crisis and no noticeable improvement after 2010.

CHAPTER 2

Pricing of long-dated commodity derivatives with stochastic volatility and stochastic interest rates

This chapter presents a class of forward price models for the term structure of commodity futures prices that incorporates stochastic volatility and stochastic interest rates. Correlations between the futures price process, futures volatility process, and interest-rate process are allowed to be non-zero. The functional form for the futures price volatility is specified so that the model admits finite dimensional realisations and retains affine representations; henceforth, quasi-analytical European vanilla futures option pricing formulae can be obtained. The impact of the parameters of the stochastic interest-rate process to commodity option prices is also numerically investigated. This chapter is based on the working paper of Cheng, Nikitopoulos & Schlögl (2015).

2.1. Introduction

Derivative pricing models with stochastic volatility and stochastic interest rates, the so-called hybrid pricing models have emerged for spot markets such as equities or foreign exchanges. Some of the early models do not include correlations, or sufficient number of factors, and many do not derive closed form derivative pricing formulae [see, for instance, Amin & Jarrow (1992), Amin & Ng (1993), Bakshi et al. (1997), Grzelak & Oosterlee (2011)]. In particular, van Haastrecht et al. (2009), van Haastrecht & Pelsser (2011) and Grzelak et al. (2012) discuss numerical solutions of models combining the stochastic volatility Schöbel & Zhu (1999) model as the spot price process and the Hull & White (1990) model as the stochastic interest-rate process while allowing full correlations between the spot-price process, its stochastic volatility process and the interest-rate process. Equity and FX markets have been extensively studied with hybrid pricing models,

yet there is limited literature on commodity markets, which is a fast growing market for long-dated derivative contracts.

In this chapter, aiming to develop a model suited for pricing long-dated contracts, a commodity derivative pricing model featuring stochastic volatility and stochastic interest rates is considered. The stochastic interest-rate process is modelled by a Hull & White (1990) process, and the volatility is modelled by an Ornstein-Uhlenbeck process. The model is within the Heath et al. (1992) framework; thus, it fits the entire initial forward curve by construction rather than generating it endogenously from the spot price process, as in the spot pricing models. It is a continuous time multi-factor stochastic volatility model that allows for multiple volatility factors with flexible volatility structures. Empirical evidence in the crude oil market demonstrates that exponential decaying or hump-shaped function forms are typical structures of its volatility factors [see Chiarella et al. (2013)]. In addition, the proposed model allows a full correlation structure between the underlying futures price process, the stochastic volatility process and the stochastic interest-rate process. This feature is very important as empirical evidence has revealed that volatility is unspanned in commodity markets [see Trolle & Schwartz (2009*b*)].

One of the issues of using HJM models is that in general they have infinite dimensional state-space. By selecting suitable volatility structures, [1] the forward price model can be reduced to a finite dimensional state-space. Furthermore, these volatility specifications are flexible enough to generate a wide range of shapes for the futures price volatility surface including exponentially decaying and hump-shaped structures. This class of models, however, by itself does not conform to the general structure of affine term structure models. By introducing latent stochastic variables, this class of models has an affirm term

[1]The HJM model in general is only Markovian in the entire forward curve. Thus, it requires an infinite number of state variables to determine its current state. However, several papers [see, for example, Chiarella & Kwon (2001), Björk, Landén & Svensson (2004) and Björk, Blix & Landén (2006)] have proposed appropriate conditions on the volatility structure so that HJM models admit finite dimensional Markovian representations. This greatly improves its tractability to formulate closed-form option pricing formulae and it is important for its suitability in numerical evaluation techniques such as Monte Carlo simulations, finite difference, tree methods or Kalman filter.

structure representation, hence, quasi-analytical European vanilla futures option pricing formulae can be obtained. Therefore, this class of models is well suited for estimation and calibration applications. Consequently, these models can be fitted to both futures and option prices so that they have the potential to capture well the forward curves and the volatility smiles. Other mean-reverting processes, for example, the square-root process [see Cox et al. (1985)] can be used to model stochastic interest rates or stochastic volatility but some approximations are required in order to obtain closed-form solutions.

To better understand the impact of stochastic interest rates to the prices of futures options, a sensitivity analysis is performed to gauge the impact of the parameters of the interest-rate process to futures option prices. Results show that the impact of the correlation between the stochastic interest-rate process and the stochastic volatility process to option prices is negligible even when the time-to-maturity is twenty years. However, the impact of the correlation between stochastic interest rates and futures prices to the option prices is noticeable for longer maturity options. A certain long-term level of interest rates applies the same discounting factor to the future payoff of the options with the same maturity, and it is independent to the correlations between the stochastic interest-rate process and other processes. The interest-rate volatility impacts the option prices more severely than the long-term level of interest rates. When the volatility of interest rates is high, there is a nonlinear (convex) relationship between prices of long-dated options and the correlation coefficient between futures prices and interest rates. Furthermore, when the correlation between futures prices and interest rates is negative, option prices are more sensitive to interest-rate volatility.

The remaining of the chapter is structured as follows. Section 2.2 presents the proposed forward price model with stochastic volatility and stochastic interest rates for pricing commodity futures prices and demonstrates that for certain volatility specifications, the model can be reduced to a finite dimensional state-space. Section 2.3 shows that the model can be within the class of affine term structure models by introducing a latent stochastic variable;

it, thus, derives the formula for pricing European vanilla options on futures. Numerical investigations are conducted and the results are discussed in Section 2.4. Conclusions are drawn in Section 2.5.

2.2. Commodity Futures Prices

We consider a filtered probability space $(\Omega, \mathcal{F}_T, \mathbb{F}, \mathbb{P}), T \in [0, \infty)$ satisfying the usual conditions.[2] Here Ω is the state-space, $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ is a set of σ-algebras representing measurable events, and $\mathbb{P}$ is the historical (real-world) probability measure. We introduce $\sigma = \{\sigma_t; t \in [0, T]\}$ an n-dimensional generic stochastic volatility process modelling the uncertainty in the commodity market. We let $F(t, T, \sigma_t)$ be the futures price of the commodity at time $t \geq 0$, for delivery at time $T \in [t, \infty)$ and the current state of the stochastic volatility process $\sigma_t \in \Re^n$. The spot price at time t of the underlying commodity, namely, $S(t, \sigma_t)$, is obtained by taking the limit of the futures price as $T \to t$, i.e. $S(t, \sigma_t) = \lim_{T\to t} F(t, T, \sigma_t), t \in [0, T]$. We denote $r = \{r(t); t \in [0, T]\}$ the possibly multi-dimensional stochastic instantaneous short-rate process. Duffie (2001) by using no-arbitrage arguments demonstrates that the futures price process $F(t, T, \sigma_t)$ is a martingale under the equivalent risk-neutral probability measure with respect to the continuously compounded spot interest rate, denoted by $\mathbb{Q}$, namely

$$F(t, T, \sigma_t) = \mathbb{E}^{\mathbb{Q}}\big[S(T, \sigma_T)|\mathcal{F}_t\big].$$

Thus, the commodity futures price process must follow a stochastic differential equation with zero drift. Aiming to model the futures prices directly, we follow the HJM framework and assume that the commodity futures price process follows a driftless stochastic differential equation under the risk-neutral measure of the form:

$$\frac{dF(t, T, \sigma_t)}{F(t, T, \sigma_t)} = \sum_{i=1}^{n} \sigma_i^F(t, T, \sigma_t) dW_i^x(t), \tag{2.2.1}$$

[2]The usual conditions satisfied by a filtered complete probability space are: (a) $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets of $\mathcal{F}$ and (b) the filtration is right continuous.

where $\sigma_i^F(t,T,\sigma_t)$ are the $\mathbb{F}$-adapted futures price volatility processes for all $T > t$ and $W^x(t) = \{W_1^x(t),\ldots,W_n^x(t)\}$ is an n-dimensional Wiener process under the risk-neutral probability measure $\mathbb{Q}$ driving the commodity futures prices. The volatility process $\sigma_t = \{\sigma_i(t),\ldots,\sigma_n(t)\}$ is an n-dimensional well-defined Markovian process following the dynamics:

$$d\sigma_i(t) = \mu_i^\sigma(t,\sigma_t)dt + \sigma_i^\sigma(t,\sigma_t)dW_i^\sigma(t), \tag{2.2.2}$$

for $i \in \{1,\ldots,n\}$, where $\mu_i^\sigma(t,\sigma_t)$ and $\sigma_i^\sigma(t,\sigma_t)$ are integrable and square-integrable real-valued functions, respectively. We further specify the functional form of the drift and the volatility of the stochastic volatility process, and we consider an extended version of the n-dimensional Schöbel-Zhu-Hull-White (ESZHW hereafter) model by incorporating a multi-factor mean-reverting stochastic interest-rate process to the SZHW model [3] as follows:

$$\frac{dF(t,T,\sigma_t)}{F(t,T,\sigma_t)} = \sum_{i=1}^{n} \sigma_i^F(t,T,\sigma_t)dW_i^x(t), \tag{2.2.3}$$

where, for $i = 1,2,\ldots,n$,

$$d\sigma_i(t) = \kappa_i(\overline{\sigma}_i - \sigma_i(t))dt + \gamma_i dW_i^\sigma(t), \tag{2.2.4}$$

and

$$r(t) = \overline{r}(t) + \sum_{j=1}^{N} r_j(t), \tag{2.2.5}$$

$$dr_j(t) = -\lambda_j(t)r_j(t)dt + \theta_j dW_j^r(t), \quad \text{for } j = 1,2,\ldots,N.$$

Note that $W^\sigma(t) = \{W_1^\sigma(t),\ldots,W_n^\sigma(t)\}$ is an n-dimensional Wiener process under the risk-neutral probability measure driving the stochastic volatility process $\sigma_t = \{\sigma_1(t),\ldots,\sigma_n(t)\}$, $W^r(t) = \{W_1^r(t),\ldots,W_N^r(t)\}$ is an N-dimensional Wiener process under the risk-neutral probability measure driving the instantaneous short-rate process $r(t)$, for all $t \in [0,T]$, $\kappa_1,\ldots,\kappa_n$, $\overline{\sigma}_1,\ldots,\overline{\sigma}_n$ and $\theta_1,\ldots,\theta_N$ are constants, and $\{\lambda_i\}_{i=1,\ldots,N}$ and $\overline{r}$ are deterministic functions of time t. We further make the following assumptions on the correlation

[3] see van Haastrecht et al. (2009).

structure of the Wiener processes:

$$
\begin{aligned}
dW_i^x(t)dW_j^x(t) &= \begin{cases} dt, \text{ if } i = j, \\ 0, \text{ otherwise} \end{cases} \\
dW_i^\sigma(t)dW_j^\sigma(t) &= \begin{cases} dt, \text{ if } i = j, \\ 0, \text{ otherwise} \end{cases} \\
dW_{\hat{i}}^r(t)dW_{\hat{j}}^r(t) &= \begin{cases} dt, \text{ if } \hat{i} = \hat{j}, \\ 0, \text{ otherwise} \end{cases} \\
dW_i^x(t)dW_j^\sigma(t) &= \begin{cases} \rho_i^{x\sigma} dt, \text{ if } i = j, \\ 0, \text{ otherwise} \end{cases} \\
dW_i^x(t)dW_{\hat{j}}^r(t) &= \begin{cases} \rho_i^{xr} dt, \text{ if } \hat{j} = 1, \\ 0, \text{ otherwise} \end{cases} \\
dW_i^\sigma(t)dW_{\hat{j}}^r(t) &= \begin{cases} \rho_i^{r\sigma} dt, \text{ if } \hat{j} = 1, \\ 0, \text{ otherwise} \end{cases}
\end{aligned}
\tag{2.2.6}
$$

for $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, n\}$, $\hat{i} \in \{1, \ldots, N\}$ and $\hat{j} \in \{1, \ldots, N\}$. The above-mentioned specifications entail the feature of unspanned stochastic volatility in the model. More specifically, when the Wiener processes $W_i^x(t)$ and $W_i^\sigma(t)$ are correlated, futures contracts can be used to partially hedge the volatility risk of the derivatives, while when the Wiener processes $W_i^x(t)$ and $W_i^\sigma(t)$ are uncorrelated, the volatility risk of the derivatives is unhedgeable by futures contracts. Note that for modelling convenience, we assume that only the first Wiener process of the interest-rate process $W_1^r(t)$ can be correlated with the futures price process and the futures volatility process. If the Wiener processes of the interest-rate process are uncorrelated, they can be disentangled from the expectation

of the option's payoff function.

Let $X(t,T) = \log F(t,T,\sigma_t)$ be the natural logarithm of the futures price process then by an application of Itô's lemma, it follows that:

$$dX(t,T) = -\frac{1}{2}\sum_{i=1}^{n}\left(\sigma_i^F(t,T,\sigma_t)\right)^2 dt + \sum_{i=1}^{n}\sigma_i^F(t,T,\sigma_t)dW_i^x(t). \tag{2.2.7}$$

In Appendix 2.1 we show that the spot price follows the SDE:

$$\frac{dS(t,\sigma_t)}{S(t,\sigma_t)} = \zeta(t,\sigma_t)dt + \sum_{i=1}^{n}\sigma_i^F(t,t,\sigma_t)dW_i^x(t), \tag{2.2.8}$$

with the instantaneous spot cost of carry $\zeta(t,\sigma_t)$ satisfying the relationship

$$\begin{aligned}\zeta(t,\sigma_t) =& \frac{\partial}{\partial t}\log F(0,t,\sigma_t) - \sum_{i=1}^{n}\int_0^t \sigma_i^F(u,t,\sigma(u))\frac{\partial}{\partial t}\sigma_i^F(u,t,\sigma(u))\,du \\ &+ \sum_{i=1}^{n}\int_0^t \frac{\partial}{\partial t}\sigma_i^F(u,t,\sigma(u))\,dW_i^x(u).\end{aligned}$$

2.2.1. Finite Dimensional Realisations. The commodity HJM model (2.2.3) is Markovian in an infinite dimensional state-space due to the fact that the futures price curve is an infinite dimensional object. For the system to admit finite dimensional realisations, it is necessary to impose certain functional forms to the volatility terms $\sigma_i^F(t,T,\sigma_t)$ [see Chiarella & Kwon (2003)]. Employing methods of Lie algebra, Björk et al. (2004) and Björk et al. (2006) show that the volatility terms $\sigma_i^F(t,T,\sigma_t)$ admit finite dimensional realisations, if and only if $\sigma_i^F(t,T,\sigma_t)$ can be expressed in the following form:

$$\sigma_i^F(t,T,\sigma_t) = \varpi_i(T-t)\alpha_i(t,\sigma_t),$$

where $\alpha_i : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ are square-integrable real-valued functions and $\varpi_i : \mathbb{R} \to \mathbb{R}$ are quasi-exponential functions. A quasi-exponential function $f : \mathbb{R} \to \mathbb{R}$ has the general form

$$f(x) = \sum_i e^{m_i x} + \sum_j e^{n_j x}[p_j(x)\cos(k_j x) + q_j(x)\sin(k_j x)],$$

where m_i, n_i and k_i are real numbers and p_j and q_j are real polynomials. To reduce the system (2.2.3) to a finite dimensional state-space, we assume a simplified version of this form where the commodity futures price volatility process is assumed to be:

$$\begin{aligned}\sigma_i^F(t,T,\sigma_t) &= \varpi_i(T-t)\alpha_i(t,\sigma_t)\\ &= (\xi_{0i}+\xi_i(T-t))e^{-\eta_i(T-t)}\sigma_i(t),\end{aligned} \tag{2.2.9}$$

with ξ_{0i}, ξ_i and $\eta_i \in \mathbb{R}$ for all $i \in \{1, \cdots, n\}$. This volatility specification allows for a variety of volatility structures such as exponentially decaying and hump-shaped structures which are typical volatility structures in the commodity market, see for example Chiarella et al. (2013) or Trolle & Schwartz (2009*a*).

PROPOSITION 2.1. *Using volatility specifications of* (2.2.9)*, the dynamics* (2.2.7) *can derive the logarithm of the instantaneous futures prices at time* t *with maturity* T *in terms of* $6n$ *state variables, namely* $x_i(t), y_i(t), z_i(t), \phi_i(t), \psi_i(t)$ *and* $\sigma_i(t)$*:*

$$\begin{aligned}\log F(t,T,\sigma_t) &= \log F(0,T,\sigma_0)\\ &\quad -\frac{1}{2}\sum_{i=1}^{n}\Big(\gamma_{1i}(T-t)x_i(t)+\gamma_{2i}(T-t)y_i(t)+\gamma_{3i}(T-t)z_i(t)\Big)\\ &\quad +\sum_{i=1}^{n}\Big(\beta_{1i}(T-t)\phi_i(t)+\beta_{2i}(T-t)\psi_i(t)\Big),\end{aligned} \tag{2.2.10}$$

where for $i = 1, \ldots, n$ *the deterministic functions are defined as:*

$$\begin{aligned}\beta_{1i}(T-t) &= \varpi_i(T-t) = \big(\xi_{i0}+\xi_i(T-t)\big)e^{-\eta_i(T-t)},\\ \beta_{2i}(T-t) &= \xi_i e^{-\eta_i(T-t)},\\ \gamma_{1i}(T-t) &= \beta_{1i}(T-t)^2,\\ \gamma_{2i}(T-t) &= 2\beta_{1i}(T-t)\beta_{2i}(T-t),\\ \gamma_{3i}(T-t) &= \beta_{2i}(T-t)^2,\end{aligned}$$

and the state variables $x_i, y_i, z_i, \phi_i, \psi_i$ *satisfy the following SDE:*

$$\begin{aligned} dx_i(t) &= \big(-2\eta_i x_i(t) + \sigma_i^2(t)\big)\, dt, \\ dy_i(t) &= \big(-2\eta_i y_i(t) + x_i(t)\big)\, dt, \\ dz_i(t) &= \big(-2\eta_i z_i(t) + 2y_i(t)\big)\, dt, \\ d\phi_i(t) &= -\eta_i \phi_i(t) dt + \sigma_i(t) dW_i^x(t), \\ d\psi_i(t) &= \big(-\eta_i \psi_i(t) + \phi_i(t)\big) dt, \end{aligned} \tag{2.2.11}$$

subject to the initial condition $x_i(0) = y_i(0) = z_i(0) = \phi_i(0) = \psi_i(0) = 0$. *The above-mentioned* $5n$ *state variables are associated with the* n *stochastic volatility processes* $\sigma_i(t)$, $i \in \{1, \ldots, n\}$ *see equations* (2.2.4) *so that the whole system includes* $6n$ *state variables.*

PROOF. see Appendix 2.2 □

The stochastic interest-rate process $r(t)$ does not affect the futures prices process per se, but it matters when we consider pricing options on futures contracts.

2.3. Affine Class Transformation

The initially infinite-dimensional Markovian model is now reduced to a model with $6n + N$ state variables. The additional N state variables are from the stochastic interest-rate process specified by the equation (2.2.5). Note that this system is not affine. When the volatility specifications (2.2.9) are applied to the dynamics (2.2.7), the $\sigma_i^2(t)$ term is not an affine transformation of $\sigma_i(t)$, which would be required in order for the system to admit a closed-form characteristic function of $X(t,T) = \log F(t,T,\sigma_t)$, see Duffie et al. (2000) section 2.2 for the details. By introducing a latent stochastic variable $\nu_i(t) \triangleq \sigma_i^2(t)$ with $\nu_t = \{\nu_1(t), \ldots, \nu_n(t)\}$, then this system can be transformed to the following affine

system:

$$dX(t,T) = -\frac{1}{2}\sum_{i=1}^{n}\beta_{1i}^2(T-t)\nu_i(t)dt + \sum_{i=1}^{n}\beta_{1i}(T-t)\sqrt{\nu_i(t)}dW_i^x(t),$$

where, for $i = 1, 2, \ldots, n$,

$$\begin{aligned} d\sigma_i(t) &= \kappa_i(\overline{\sigma}_i - \sigma_i(t))dt + \gamma_i dW_i^\sigma(t), \\ d\nu_i(t) &= 2\kappa_i\big(\overline{\sigma}_i\sigma_i(t) + \frac{\gamma_i^2}{2\kappa_i} - \nu_i(t)\big)dt + 2\gamma_i\sqrt{\nu_i(t)}dW_i^\sigma(t), \end{aligned}$$

and

$$\begin{aligned} r(t) &= \overline{r}(t) + \sum_{j=1}^{N} r_j(t), \qquad (2.3.1) \\ dr_j(t) &= -\lambda_j(t)r_j(t)dt + \theta_j dW_j^r(t), \quad \text{for } j = 1, 2, \ldots, N, \end{aligned}$$

with the correlations of the Wiener processes are the same as those in (2.2.6). For $t \leq T_o \leq T$ and $\upsilon \in \mathbb{C}$, the $r_1(t)$-discounted characteristic functions of the logarithm of the futures prices $\phi(t) \triangleq \phi(t, X(t,T), r_1(t), \nu_t, \sigma_t; \upsilon, T_o, T)$:

$$\begin{aligned} \phi(t;\upsilon,T_o,T) &\triangleq \mathbb{E}_t^{\mathbb{Q}}\big[e^{-\int_t^{T_o} r_1(u)\,du}\exp\big\{\upsilon\log F(T_o,T,\sigma_{T_o})\big\}\big] \\ &= \mathbb{E}_t^{\mathbb{Q}}\big[e^{-\int_t^{T_o} r_1(u)\,du}\exp\big\{\upsilon X(T_o,T)\big\}\big] \end{aligned} \qquad (2.3.2)$$

can be expressed as:

$$\begin{aligned} \phi(t;\upsilon,T_o,T) = \exp\big\{&A(t;\upsilon,T_o) + B(t;\upsilon,T_o)X(t,T) + C(t;\upsilon,T_o)r_1(t) \\ &+ \sum_{i=1}^{n} D_i(t;\upsilon,T_o)\nu_i(t) + \sum_{i=1}^{n} E_i(t;\upsilon,T_o)\sigma_i(t)\}. \end{aligned} \qquad (2.3.3)$$

LEMMA 2.1. *The functions* $A(t;\upsilon,T_o)$, $B(t;\upsilon,T_o)$, $C(t;\upsilon,T_o)$, $D_i(t;\upsilon,T_o)$ *and* $E_i(t;\upsilon,T_o)$ *in equation* (2.3.3) *satisfy the following complex-valued Ricatti ordinary differential equations:*

$$\begin{aligned}
\frac{\partial B}{\partial t} =& 0,\\
\frac{\partial C}{\partial t} =& \lambda_1 C + 1,\\
\frac{\partial D_i}{\partial t} =& -\frac{1}{2}\beta_{1i}^2(T-t)(B-1)B - 2(\rho_i^{x\sigma}\beta_{1i}(T-t)\gamma_i B - \kappa_i)D_i - 2\gamma_i^2 D_i^2,\\
\frac{\partial E_i}{\partial t} =& -2\bar{\sigma}_i\kappa_i D_i - \rho_i^{xr}\theta_1\beta_{1i}(T-t)BC - 2\rho_i^{r\sigma}\theta_1\gamma_i CD\\
&\quad - (2\gamma_i^2 D_i - \kappa_i + \rho_i^{x\sigma}\beta_{1i}(T-t)\gamma_i B)E_i,\\
\frac{\partial A}{\partial t} =& -\frac{1}{2}\theta_1^2 C^2 - \sum_{i=1}^{n}\gamma_i^2 D_i - \sum_{i=1}^{n}(\kappa_i\bar{\sigma}_i + \frac{1}{2}\gamma_i^2 E_i + \rho_i^{r\sigma}\theta_1\gamma_i C)E_i,
\end{aligned} \tag{2.3.4}$$

where $i \in \{1,\ldots,n\}$, *subject to the terminal condition* $\phi(T_o) = e^{\upsilon X(T_o,T)}$.

PROOF. see Appendix 2.3 □

In the next section we present the quasi-analytical pricing formulae for European vanilla options on futures that this affine transformation leads to.

Pricing of European Option on Futures. We denote with Call$(t, F(t,T,\sigma_t);T_o)$ and Put$(t, F(t,T,\sigma_t);T_o)$ the price of the European style call and put options, respectively, with maturity T_o and strike K on the futures price $F(t,T,\sigma_t)$ maturing at time T. The price of a call option can be expressed as the discounted expected payoff under the risk-neutral measure:

$$\text{Call}(t, F(t,T,\sigma_t);T_o) = \mathbb{E}_t^{\mathbb{Q}}\Big(e^{-\int_t^{T_o} r(s)\,ds}\big(e^{X(T_o,T)} - K\big)^+\Big). \tag{2.3.5}$$

By using the Fourier inversion technique Duffie et al. (2000) provide a semi-analytical formula for the price of European-style vanilla options under the class of affine term structure. With a slight modification of the pricing equation in Duffie et al. (2000), equation

(2.3.5) can be expressed as:

$$\begin{aligned}\text{Call}(t, F(t,T,\sigma_t); T_o) = e^{-\int_t^{T_o} \overline{r}(s)\,ds} \prod_{i=2}^{N} \mathbb{E}_t^{\mathbb{Q}}\big[e^{-\int_t^{T_o} r_i(s)\,ds}\big] \times \\ \big[G_{1,-1}(-\log K) - KG_{0,-1}(-\log K)\big],\end{aligned} \tag{2.3.6}$$

where

$$G_{a,b}(y) = \frac{\phi(t; a, T_o, T)}{2} - \frac{1}{\pi}\int_0^{\infty} \frac{\Im[\phi(t; a + \mathbf{i}bu, T_o, T)e^{-\mathbf{i}uy}]}{u}\,du. \tag{2.3.7}$$

Note that $\mathbf{i}^2 = -1$ and $\Im(x + \mathbf{i}y) = y$. Note that, the product starts at $i = 2$ because the equation (2.3.6) is conditional to the specifications of the correlation structure (2.2.6) that allows only the first factor of the interest-rate process to be correlated with the futures price process. For European put options, the discounted expected payoff is:

$$\begin{aligned}\text{Put}(t, F(t,T,\sigma_t); T_o) &= \mathbb{E}_t^{\mathbb{Q}}\Big(e^{-\int_t^{T_o} r(s)\,ds}\big(K - e^{X(T_o,T)}\big)^+\Big) \\ &= e^{-\int_t^{T_o} \overline{r}(s)\,ds} \prod_{i=2}^{N} \mathbb{E}_t^{\mathbb{Q}}\big[e^{-\int_t^{T_o} r_i(s)\,ds}\big] \times \\ &\quad \big[KG_{0,1}(\log K) - G_{1,1}(\log K)\big].\end{aligned} \tag{2.3.8}$$

With finite dimensional realisations and affine class transformation the commodity pricing model we propose in equation (2.2.3) admits a closed-form option pricing formula. Quasi-analytical option pricing formulae greatly facilitate model estimation and calibration, as we shall see in a subsequent chapter where the model is estimated by using historical data from the most active commodity derivatives market, namely the crude oil market.

Note that the proposed model is not limited to pricing only commodity futures options. The model can be easily adjusted to price any type of options on futures, for instance, options on index futures. The model characteristics are examined next by performing a sensitivity analysis.

2.4. Numerical investigations

In this section, we numerically investigate the implications of allowing the interest-rate processes to be stochastic and possibly correlated to both of the future price processes, and the future stochastic volatility processes. To be specific, we want to assess the effect of the following parameters of the interest-rate model to commodity futures options:

- ρ^{xr} — the correlation between the future price process and the interest-rate process.
- $\rho^{r\sigma}$ — the correlation between the future price volatility process and the interest-rate process.
- $\overline{r}$ — the long-term level of interest rates.
- θ — the volatility of the interest-rate process.

The correlation $\rho^{x\sigma}$ between the futures price process and the volatility process has been well studied [see, for example, Schöbel & Zhu (1999)], so we omit the discussion here. In this numerical exercise, aiming to amplify and concentrate solely on the impact of the stochastic interest rates, we set $\rho^{x\sigma} = 0$. For simplicity, we choose the one-dimensional version of the ESZHW model [that is equations (2.2.3) and (2.2.4) with $n = 1$ and equation (2.2.5) with $N = 1$] with $\overline{\sigma}(t) = \overline{\sigma}$. Furthermore, we simplify the volatility structure of the futures price process by assuming $\xi_{01} = 1, \xi_1 = 0$ and $\eta_1 = 0$ [see equation (2.2.9)], thus, reducing it to a constant volatility, namely $\sigma^F(t, T, \sigma_t) = \sigma_t$. This allows us to target our investigations specifically on the impact of the parameters of the stochastic interest rates to option prices. The one-dimensional ESZHW model to be used in this analysis is summarised below:

$$\frac{dF(t, T, \sigma_t)}{F(t, T, \sigma_t)} = \sigma_t \, dW^x(t), \tag{2.4.1}$$

$$d\sigma_t = \kappa(\overline{\sigma} - \sigma_t)dt + \gamma dW^\sigma(t), \tag{2.4.2}$$

$$dr(t) = \lambda(\overline{r} - r(t))dt + \theta dW^r(t), \tag{2.4.3}$$

with,

$$dW^x(t)dW^r(t) = \rho^{xr}dt,$$

$$dW^x(t)dW^\sigma(t) = 0,$$

$$dW^r(t)dW^\sigma(t) = \rho^{r\sigma}dt.$$

Table 2.1 displays the parameter values used in our numerical investigations, where F_0, r_0, σ_0 is the initial futures price, the initial interest rate and the initial stochastic volatility respectively.

TABLE 2.1. **Parameter Values**

$\rho^{x\sigma}$	λ	$\overline{r}$	θ	κ	$\overline{\sigma}$	γ	F_0	K	r_0	σ_0
0	4	0.05	0.05	4	0.5	0.6	100	100	0.05	0.05

TABLE 2.2. The table displays the parameter values use in the sensitivity analysis.

We next investigate the following questions. What is the impact of the correlation coefficients ρ^{xr} and $\rho^{r\sigma}$ to the model option prices? How does the impact of these correlation coefficients to option prices charge as the maturity of the options increases? What is the impact of the stochastic interest-rate parameters, long-term level $\overline{r}$ and volatility θ?

2.4.1. The correlation coefficients ρ^{xr} and $\rho^{r\sigma}$. To gauge the impacts of ρ^{xr} and $\rho^{r\sigma}$ to the option prices, we vary the correlations from -0.75 to 0.75 with 0.25 as the incremental value, and compare these prices with the option prices of the restricted models with $\rho^{xr} = \rho^{r\sigma} = 0$. We are thus considering the following ratio:

$$\text{Ratio}(\rho^{xr}, \rho^{r\sigma}; T) = \frac{\text{OptionPrice}(\rho^{xr}, \rho^{r\sigma}; T)}{\text{OptionPrice}(0, 0; T)}. \tag{2.4.4}$$

This ratio as a function of the two correlations is plotted in Figure 2.1 for four call options maturities, namely $0.5, 3, 10$ and 20 years. From the surface plots of the ratio as a function of ρ^{xr} and $\rho^{r\sigma}$ two observations can be made. The first is that the ratios are virtually

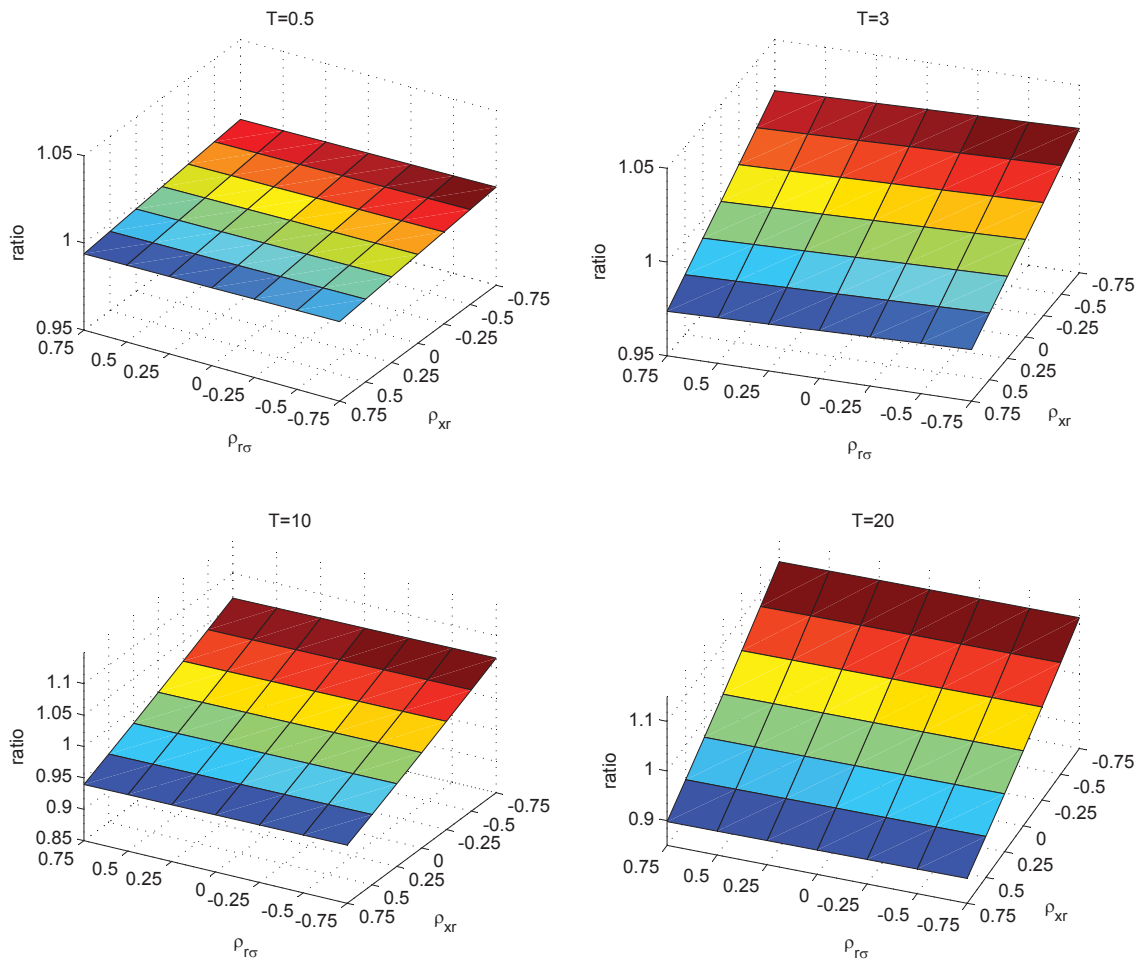

FIGURE 2.1. **Impact of the correlation coefficients ρ^{xr} and $\rho^{r\sigma}$ on call option prices.** The plots display the ratio of option prices (between prices from a model with full correlation structure, and prices from a restricted model with zero correlation) for a range of correlation coefficient parameter values and for four maturities; 0.5, 3, 10 and 20 years.

unchanged as $\rho^{r\sigma}$ varies from -0.75 to 0.75 in all four plots. This implies that the correlation between the interest-rate process and the future price volatility process has minimal impact on call option prices. The second observation is that the correlation between the interest-rate process and the future price process has minimal impact for short maturities (T = 0.5 years), but the impact gradually becomes more pronounced as the maturity increases. At $T = 20$, the price difference between taking correlation (ρ^{xr}) into account

and assuming zero correlation ($\rho^{xr} = 0$) can be more than 10%.

Figure 2.2 depicts the cross-section of Figure 2.1 with $\rho^{r\sigma} = 0$. From this plot, we see that when the futures price process and the interest-rate process are very negatively (positively) correlated, the call option price is more expensive (less expensive) than the option prices computed when ignoring this correlation. The effect is more pronounced for options with longer maturities. We can intuitively understand the impact of the correlation coefficient ρ^{xr} to the option prices by considering paths of the futures price process and the interest-rate process. Consider paths that lead to increasing futures prices such that a call option at maturity is in-the-money. In these paths, a negative correlation between futures prices and interest rates implies decreasing interest rates, hence the discounted payoffs at maturity of these particular paths are relatively expensive. Conversely, when the correlation is positive, the discounted payoffs at maturity are relatively less expensive. To sum up, a negative (positive) correlation means that we are discounting the payoffs of the in-the-money paths less (more) severely; hence the price of the option is more (less) expensive.

2.4.2. The long-term level of interest rates $\overline{r}$. We now turn our attention to the parameter $\overline{r}$ — the long-term level of interest rates. Figure 2.3 shows the option prices and their ratios when the long-term level of interest rates $\overline{r}$ varies from 1% to 5% to 10% for the two maturities 0.5 and 20 years. We set $\rho^{r\sigma} = 0$, thus, consider the ratio of option prices given $\rho^{xr}, T, \overline{r}$ as:

$$\text{Ratio}(\rho^{xr}; T, \overline{r}) = \frac{\text{OptionPrice}(\rho^{xr}; T, \overline{r})}{\text{OptionPrice}(0; T, \overline{r})}. \tag{2.4.5}$$

One interesting observation from Figure 2.3 is that the level of $\overline{r}$ has no impact on the ratio of the option prices. This can be easily seen from the term $e^{-\int_t^{T_o} \overline{r}(s)\, ds}$ in (2.3.6). In this analysis we set $\overline{r}(t) = \overline{r}$ so that the integral reduces to $e^{-\overline{r}(T_o - t)}$, and it cancels out in (2.4.5). The implication on option pricing is that by assuming that the interest-rate volatility θ does not vary, the correlation ρ^{xr} impacts the option price equally during

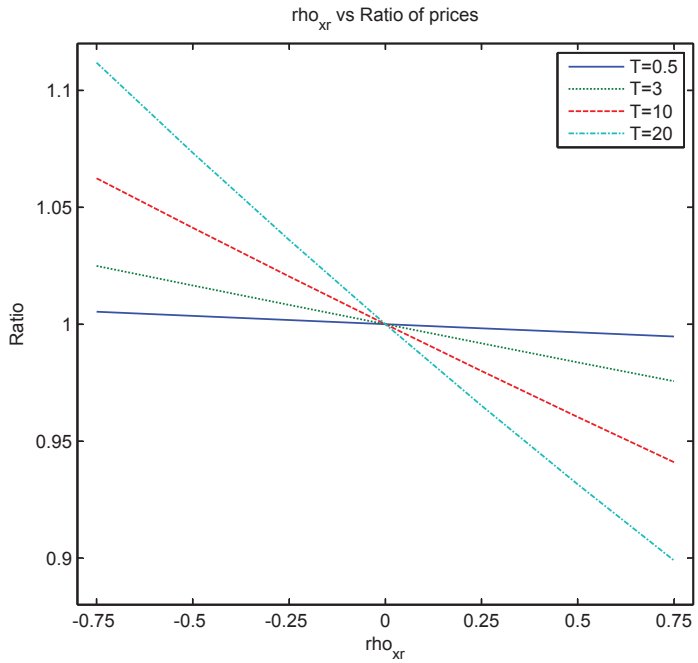

FIGURE 2.2. **Impact of the correlation between futures prices and interest rates on call option prices.** The plot shows the ratio of option prices (between prices from a model with full correlation structure and prices from a restricted model with $\rho^{r\sigma} = 0$) for a range of correlation coefficient parameter values and for four different maturities; 0.5, 3, 10 and 20 years.

a period of higher interest rates and a period of lower interest rates. Higher long-term interest rates merely discount the future payout more severely. However, the interest-rate volatility θ has a significant impact on option prices variance and this is outlined in the next subsection.

2.4.3. The interest-rate volatility θ. Figure 2.4 plots call option prices, and the option ratios for varying ρ^{xr} with four parameter values of interest-rate volatility θ and two maturities are $T = 0.5$ and $T = 20$. Comparing Figure 2.4 to Figure 2.3, we conclude that the interest-rate volatility has a substantial impact to long-dated futures option prices. For long-dated options, and when θ is very high, there is a nonlinear (convex) relationship between ρ^{xr} and the option prices. More specifically, the higher the volatility of interest rates, the higher (lower) the options prices when interest rates are negatively (positively) correlated with the futures prices. In addition, option prices from models with interest

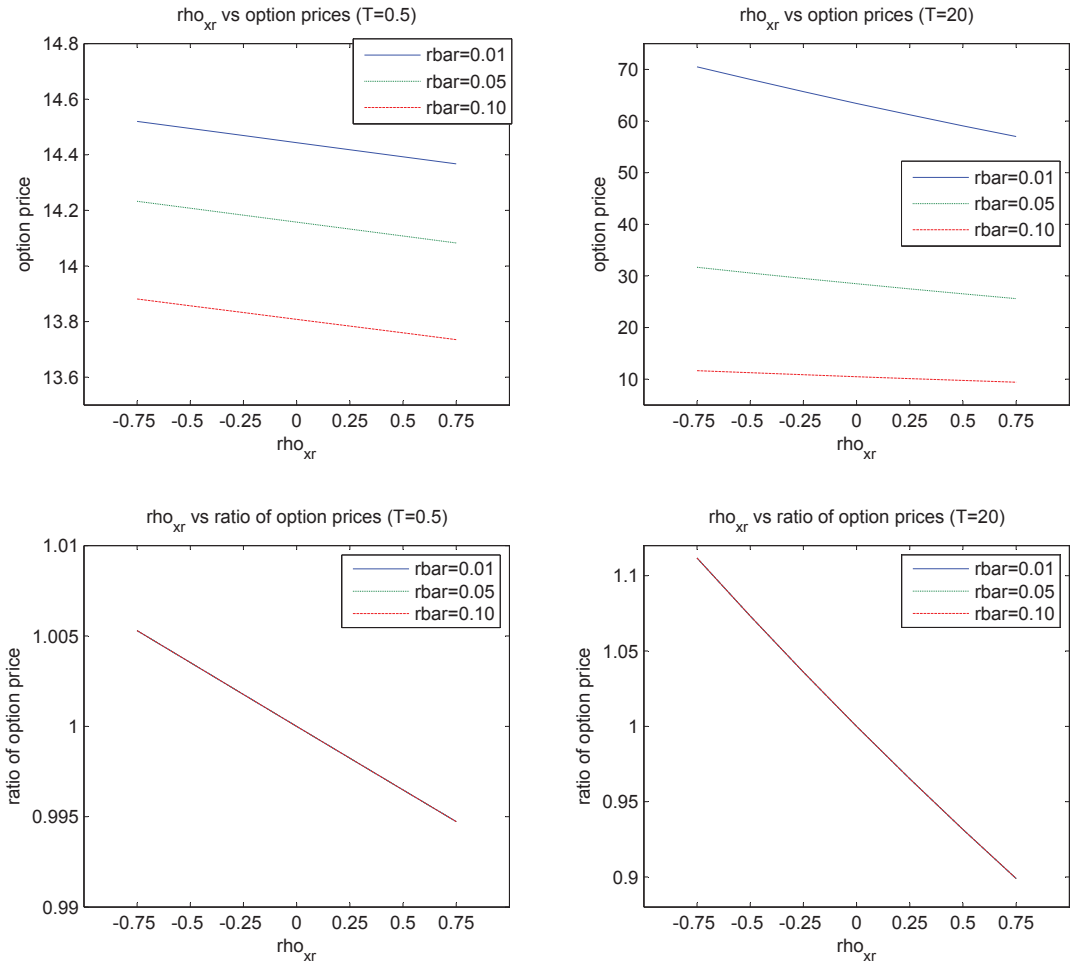

FIGURE 2.3. **Impact of the long term level of interest rates on call option prices.** This plot displays option prices (top two graphs) and the ratios of option prices (bottom two graphs) for two different maturities, $T = 0.5$ and $T = 20$, and for $\{\overline{r}, r_0\} = \{0.01, 0.01\}, \{0.05, 0.05\}, \{0.10, 0.10\}$.

rates being negatively correlated to the futures prices are more sensitive to the volatility of interest rates.

2.5. Conclusion

In this chapter, we propose a commodity pricing model featuring stochastic volatility and stochastic interest rates. The functional form of the volatility function is chosen such that the model admits finite dimensional realisations and leads to affine representations, hence, quasi-analytical European vanilla option pricing formulae can be obtained. Numerical investigations to gauge the impact of parameters of the interest-rate model to option prices

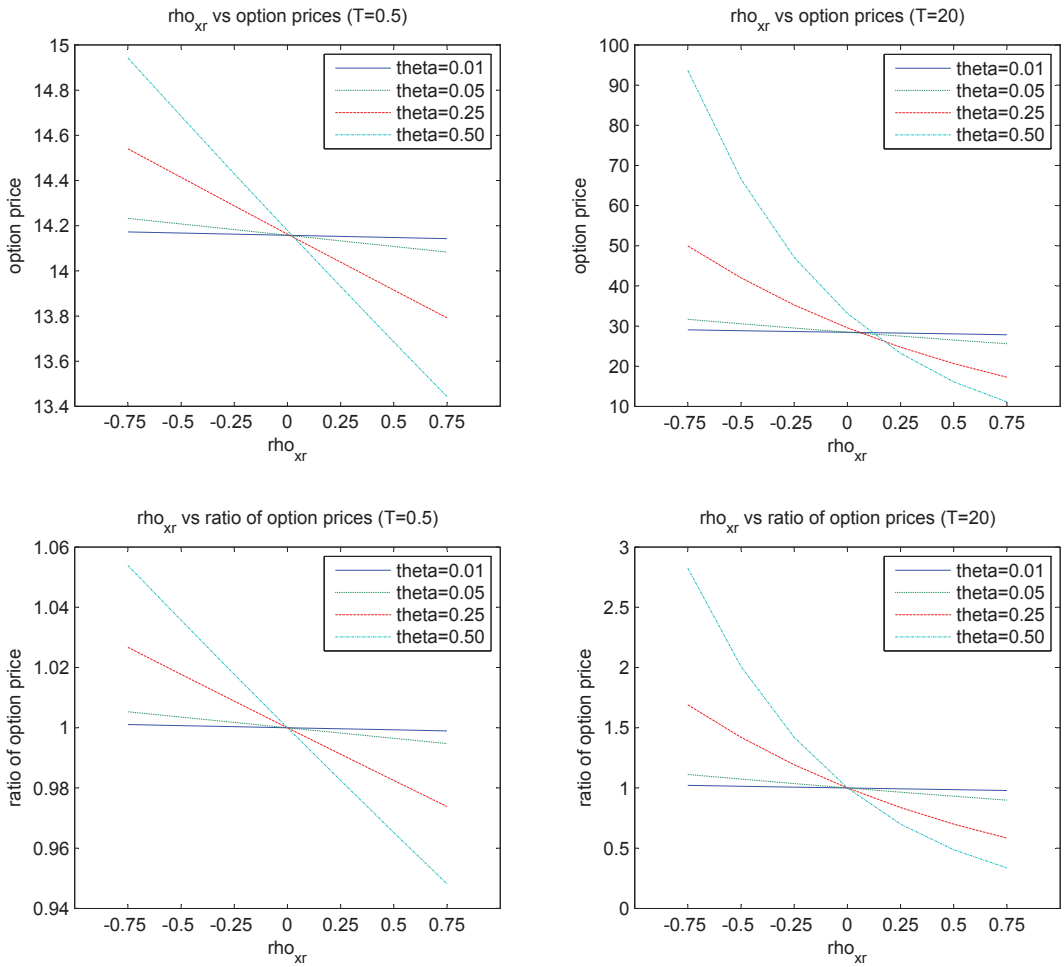

FIGURE 2.4. **Impact of the volatility of interest rates on call option prices.** The plot displays option prices (top two graphs) and ratios of option prices (bottom two graphs) for two different maturities $T = 0.5$ and $T = 20$ and for $\theta = 0.01, 0.05, 0.25, 0.50$.

are performed.

Several conclusions can be drawn from these investigations. Firstly, the correlation between the stochastic interest rates and the stochastic futures price volatility process has negligible impact on the prices even for very long-dated options. Secondly, the correlation between the stochastic interest rates and the stochastic futures price process has noticeable impact on prices of long-dated options, but remains negligible for short-dated options. Thirdly, the impact of the long-term level of interest rates to option prices does not depend on the correlation between stochastic interest rates and the stochastic futures

price process. Lastly, as the volatility of interest rates increases, the value of the option increases with the impact be more pronounced for longer-maturity options and when the correlation between futures prices and interest rates is negative.

The implications of the results from our sensitivity analysis may be relevant to practitioners who trade derivative contracts with long maturities, especially, at a time when volatility of interest rates is high, and there exists a strong negative correlation between the interest rates and futures prices. A company who underwrites options in this situation using models with deterministic interest rates could be selling them at prices lower than from models that incorporate stochastic interest rates; therefore, exposing the company to interest-rate risk. However, at a time when the volatility of interest rates is low or when there is low correlation between the interest rates and futures prices, a model with deterministic interest rates may be sufficient. In the next chapter, we present an empirical study on the proposed model that using the most liquid commodity market, namely the crude oil market.

Appendix 2.1 Instantaneous Spot Cost of Carry

Starting with the SDE of the logarithm of the futures price process $X(t,T) = \log F(t,T,\sigma_t)$, an application of the Itô's lemma derives

$$dX(t,T) = -\frac{1}{2}\sum_{i=1}^{n}\left(\sigma_i^F(t,T,\sigma_t)\right)^2 dt + \sum_{i=1}^{n}\sigma_i^F(t,T,\sigma_t)dW_i^x(t). \tag{A.1.1}$$

By integrating (A.1.1) we get:

$$F(t,T,\sigma_t) = \\ F(0,T,\sigma_0)\exp\Big(-\frac{1}{2}\sum_{i=1}^{n}\int_0^t\left(\sigma_i^F(u,T,\sigma_u)\right)^2 du + \sum_{i=1}^{n}\int_0^t \sigma_i^F(u,T,\sigma_u)\,dW_i^x(u)\Big). \tag{A.1.2}$$

By letting the maturity T approaches time t, (A.1.2) derives the dynamics of the commodity spot price as:

$$S(t,\sigma_t) = \lim_{T\to t} F(t,T,\sigma_t) \\ = F(0,t,\sigma_0)\exp\Big(-\frac{1}{2}\sum_{i=1}^{n}\int_0^t\left(\sigma_i^F(u,t,\sigma_u)\right)^2 du + \sum_{i=1}^{n}\int_0^t \sigma_i^F(u,t,\sigma_u)\,dW_i^x(u)\Big), \tag{A.1.3}$$

equivalently,

$$\log S(t,\sigma_t) = \log F(0,t,\sigma_0) \\ -\frac{1}{2}\sum_{i=1}^{n}\int_0^t\left(\sigma_i^F(u,t,\sigma_u)\right)^2 du + \sum_{i=1}^{n}\int_0^t \sigma_i^F(u,t,\sigma_u)\,dW_i^x(u)\Big). \tag{A.1.4}$$

By applying the stochastic Leibniz differential rule, we obtain (2.2.8).

Appendix 2.2 Finite Dimensional Realisation for the Futures Process

Starting from (A.1.2) we need to calculate two integrals:

$$I = \int_0^t \sigma_i^F(u, T, \sigma_u)\, dW_x(u) \tag{A.2.1}$$

$$II = \int_0^t \left(\sigma_i^F(u, T, \sigma_u)\right)^2 du \tag{A.2.2}$$

We substitute the volatility specifications (2.2.9) to get:

$$\begin{aligned}
I =& \int_0^t \big(\xi_{0i} + \xi_i(T-u)\big) e^{-\eta_i(T-u)} \sigma_i(u)\, dW_i^x(u) \\
=& \int_0^t \big(\xi_{0i} + \xi_i(T-t) + \xi_i(t-u)\big) e^{-\eta_i(T-t)} e^{-\eta_i(t-u)} \sigma_i(u)\, dW_i^x(u) \\
=& \big(\xi_{0i} + \xi_i(T-t)\big) e^{-\eta_i(T-t)} \int_0^t e^{-\eta_i(t-u)} \sigma_i(u)\, dW_i^x(u) \\
&+ \xi_i e^{-\eta_i(T-t)} \int_0^t (t-u) e^{-\eta_i(t-u)} \sigma_i(u)\, dW_i^x(u) \\
=& \beta_{1i}(T-t)\phi_i(t) + \beta_{2i}(T-t)\psi_i(t),
\end{aligned}$$

where the deterministic functions are defined as:

$$\begin{aligned}
\beta_{1i}(T-t) &= \big(\xi_{0i} + \xi_i(T-t)\big) e^{-\eta_i(T-t)}, \\
\beta_{2i}(T-t) &= \xi_i e^{-\eta_i(T-t)},
\end{aligned}$$

and the state variables are defined by:

$$\begin{aligned}
\phi_i(t) &= \int_0^t e^{-\eta_i(t-u)} \sigma_i(u)\, dW_i^x(u), \\
\psi_i(t) &= \int_0^t (t-u) e^{-\eta_i(t-u)} \sigma_i(u)\, dW_i^x(u).
\end{aligned}$$

Next

$$
\begin{aligned}
II &= \int_0^t \left(\sigma_i^F(u,T,\sigma_u)\right)^2 du \\
&= \int_0^t \left(\xi_{0i} + \xi_i(T-u)\right)^2 e^{-2\eta_i(T-u)}\sigma_i(u)^2\, du \\
&= \int_0^t \Big(\beta_{1i}(T-t) + \beta_{2i}(T-t)(t-u)\Big)^2 e^{-2\eta_i(t-u)}\sigma_i(u)^2\, du \\
&= \gamma_{1i}(T-t)x_i(t) + \gamma_{2i}(T-t)y_i(t) + \gamma_{3i}(T-t)z_i(t),
\end{aligned}
$$

where the deterministic functions are defined as:

$$
\begin{aligned}
\gamma_{1i}(T-t) &= \beta_{1i}(T-t)^2, \\
\gamma_{2i}(T-t) &= 2\beta_{1i}(T-t)\beta_{2i}(T-t), \\
\gamma_{3i}(T-t) &= \beta_{2i}(T-t)^2,
\end{aligned}
$$

and the state variables are defined as:

$$
\begin{aligned}
x_i(t) &= \int_0^t e^{-2\eta_i(t-u)}\sigma_i(u)^2\, du \\
y_i(t) &= \int_0^t (t-u)e^{-2\eta_i(t-u)}\sigma_i(u)^2\, du \\
z_i(t) &= \int_0^t (t-u)^2 e^{-2\eta_i(t-u)}\sigma_i(u)^2\, du.
\end{aligned}
$$

Alternatively, the $5n$ state variables $\phi_i(t), \psi_i(t), x_i(t), y_i(t)$ and $z_i(t)$ can be expressed in differential form as (2.2.11).

Appendix 2.3 Derivation of the Riccati ODE

Starting from (2.3.2), first observation is that this equation it not a martingale, but multiplying it by a discount factor $e^{-\int_0^t y_1(u)\,du}$, it can be turned into a martingale. Defining

$\phi_t = \phi(t, x(t), r_1(t), \nu_t, \sigma_t)$ and by applying Itô's lemma we have

$$d\big(e^{-\int_0^t r_1(u)\,du}\phi(t)\big) = e^{-\int_0^t r_1(u)\,du}\big(-r_1(t)\phi(t)dt + d\phi(t)\big)$$

Since $e^{-\int_0^t r_1(u)\,du}\phi(t)$ is a martingale, its drift term must be zero and as a result we have:

$$-r_1(t)\phi(t) + \text{drift term of } d\phi(t) = 0. \tag{A.3.1}$$

Using the correlation structure in (2.2.6) we have:

$$
\begin{aligned}
d\phi(t) &= \frac{\partial \phi}{\partial t}dt + \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial r_1}dr_1 + \sum_{i=1}^{n}\frac{\partial \phi}{\partial \nu_i}d\nu_i + \sum_{i=1}^{n}\frac{\partial \phi}{\partial \sigma_i}d\sigma_i + \frac{1}{2}\frac{\partial^2 \phi}{\partial x^2}dx^2 \\
&+ \frac{1}{2}\frac{\partial^2 \phi}{\partial r_1^2}dr_1^2 + \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2 \phi}{\partial \nu_i^2}d\nu^2 + \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2 \phi}{\partial \sigma_i^2}d\sigma^2 + \frac{\partial^2 \phi}{\partial x \partial r_1}dxdr_1 + \sum_{i=1}^{n}\frac{\partial^2 \phi}{\partial x \partial \nu_i}dxd\nu_i \\
&+ \sum_{i=1}^{n}\frac{\partial^2 \phi}{\partial x \partial \sigma_i}dxd\sigma_i + \sum_{i=1}^{n}\frac{\partial^2 \phi}{\partial r_1 \partial \nu_i}dr_1 d\nu_i + \sum_{i=1}^{n}\frac{\partial^2 \phi}{\partial r_1 \partial \sigma_i}dr_1 d\sigma_i \\
&+ \underbrace{\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 \phi}{\partial \nu_i \partial \sigma_j}d\nu_j d\sigma_j}_{=0 \text{ if } i\neq j} + \underbrace{\sum_{i=1}^{n}\sum_{j>i}^{n}\frac{\partial^2 \phi}{\partial \nu_i \partial \nu_j}d\nu_j d\nu_j}_{=0} + \underbrace{\sum_{i=1}^{n}\sum_{j>i}^{n}\frac{\partial^2 \phi}{\partial \sigma_i \partial \sigma_j}d\sigma_j d\sigma_j}_{=0} \\
&= \phi(t)\Bigg(\frac{\partial A}{\partial t} + x\frac{\partial B}{\partial t} + r_1\frac{\partial C}{\partial t} + \sum_{i=1}^{n}\Big(\nu_i\frac{\partial D_i}{\partial t} + \sigma_i\frac{\partial E_i}{\partial t}\Big)\Bigg)dt \\
&+ \phi(t)\Bigg(B\Big(-\frac{1}{2}\sum_{i=1}^{n}\beta_{1i}^2(T-t)\nu_i(t)dt + \sum_{i=1}^{n}\beta_{1i}(T-t)\sqrt{\nu_i(t)}dW_i^x(t)\Big)\Bigg) \\
&+ \phi(t)\Bigg(C\Big(-\lambda r_1(t))dt + \theta dW_r(t)\Big)\Bigg) \\
&+ \phi(t)\Bigg(\sum_{i=1}^{n}D_i\Big((-2\nu_i(t)\kappa_i + 2\kappa_i\overline{\sigma}_i\sigma_i(t) + \gamma_i^2)dt + 2\gamma_i\sqrt{\nu_i(t)}dW_i^\sigma(t)\Big)\Bigg) \\
&+ \phi(t)\Bigg(\sum_{i=1}^{n}E_i\Big(\kappa_i(\overline{\sigma}_i - \sigma_i(t))dt + \gamma_i dW_i^\sigma(t)\Big)\Bigg) \\
&+ \frac{1}{2}\phi(t)B^2\Big(\sum_{i=1}^{n}\beta_{1i}^2(T-t)\nu_i(t)dt\Big) + \frac{1}{2}\phi(t)C^2\theta^2 dt \\
&+ \frac{1}{2}\phi(t)\Bigg(4\sum_{i=1}^{n}D_i^2\gamma_i^2\nu_i dt\Bigg) + \frac{1}{2}\phi(t)\sum_{i=1}^{n}E_i^2\gamma_i^2 dt \\
&+ \phi(t)BC\sum_{i=1}^{n}\theta\beta_{1i}(T-t)\sigma_i(t)\rho_i^{xr}dt + 2\phi(t)B\sum_{i=1}^{n}D_i\gamma_i\rho_i^{x\sigma}\nu_i(t)\beta_{1i}(T-t)dt \\
&+ B\phi(t)\sum_{i=1}^{n}E_i\rho_i^{x\sigma}\gamma_i\beta_{1i}(T-t)\sigma_j dt + 2\phi(t)C\sum_{i=1}^{n}D_i\theta\gamma_i\rho_i^{\sigma r}\sigma_i(t)dt \\
&+ C\phi(t)\sum_{i=1}^{n}E_i\rho_i^{\sigma r}\theta\gamma_i dt + 2\phi(t)\sum_{i=1}^{n}D_jE_i\gamma_i^2\sigma_i(t)dt.
\end{aligned}
$$

Combining this result with (A.3.1) we get:

$$
\begin{aligned}
0 = {} & \frac{\partial A}{\partial t} + x(t)\frac{\partial B}{\partial t} + r_1(t)\frac{\partial C}{\partial t} + \sum_{i=1}^{n} \nu_i(t)\frac{\partial D_i}{\partial t} + \sum_{i=1}^{n} \sigma_i(t)\frac{\partial E_i}{\partial t} - \frac{1}{2}B\sum_{i=1}^{n} \beta_{1i}^2(T-t)\nu_i(t) \\
& - C\lambda r_1(t) + \sum_{i=1}^{n} D_i(-2\nu_i(t)\kappa_i + 2\kappa_i\overline{\sigma}_i\sigma_i(t) + \gamma_i^2) \\
& + \sum_{i=1}^{n} E_i\kappa_i(\overline{\sigma}_i - \sigma_i(t)) + \frac{1}{2}B^2\sum_{i=1}^{n} \beta_{1i}^2(T-t)\nu_i(t) + \frac{1}{2}C^2\theta^2 \\
& + 2\sum_{i=1}^{n} D_i^2\gamma_i^2\nu_i(t) + \frac{1}{2}\sum_{i=1}^{n} E_i^2\gamma_i^2 + BC\sum_{i=1}^{n} \theta\beta_{1i}(T-t)\sigma_i(t)\rho_i^{xr} \\
& + 2B\sum_{i=1}^{n} D_i\gamma_i\rho_i^{x\sigma}\nu_i(t)\beta_{1i}(T-t) + B\sum_{i=1}^{n} E_i\rho_i^{x\sigma}\gamma_i\beta_{1i}(T-t)\sigma_i(t) \\
& + 2C\sum_{i=1}^{n} D_i\theta\gamma_i\rho_i^{\sigma r}\sigma_i(t) + C\sum_{i=1}^{n} E_i\rho_i^{\sigma r}\theta\gamma_i + 2\sum_{i=1}^{n} D_iE_i\gamma_i^2\sigma_i(t) - r_1(t),
\end{aligned}
$$

rearranging, we get:

$$
\begin{aligned}
0 &= x(t)\frac{\partial B}{\partial t}, \\
0 &= r_1(t)\frac{\partial C}{\partial t} - \lambda r_1(t)C - r_1(t), \\
0 &= \nu_i(t)\frac{\partial D_i}{\partial t} - \frac{1}{2}\beta_{1i}^2(T-t)\nu_i(t)B - 2\kappa_i\nu_i(t)D_i + \frac{1}{2}\beta_{1i}^2(T-t)\nu_i(t)B^2, \\
&\quad + 2\nu_i(t)\gamma_i^2D_i^2 + 2\rho_i^{x\sigma}\beta_{1i}(T-t)\gamma_i\nu_i(t)BD_i, \\
0 &= \sigma_i(t)\frac{\partial E_i}{\partial t} + 2\bar{\sigma}_i\kappa_i\sigma_i(t)D_i - \kappa_i\sigma_i(t)E_i + \rho_i^{xr}\theta\beta_{1i}(T-t)\sigma_i(t)BC, \\
&\quad + \rho_i^{x\sigma}\beta_{1i}(T-t)\gamma_i\sigma_i(t)BE_i + 2\rho_i^{r\sigma}\theta\gamma_i\sigma_i(t)CD_i + 2\gamma_i^2\sigma_i(t)D_jE_i, \\
0 &= \frac{\partial A}{\partial t} + \frac{1}{2}\theta^2C^2 + \sum_{i=1}^{n} \gamma_i^2D_i + \sum_{i=1}^{n} \kappa_i\bar{\sigma}_iE_i + \frac{1}{2}\sum_{i=1}^{n} \gamma_i^2E_i^2 \\
&\quad + \sum_{i=1}^{n} \rho_i^{r\sigma}\theta\gamma_iCE_i,
\end{aligned}
$$

where $i \in \{1, \ldots, n\}$, subject to the terminal condition $\phi(T_0) = e^{\upsilon X(T_o, T_o)}$. This implies that $A(T_0; \upsilon, T_0) = C(T_0; \upsilon, T_0) = D_i(T_0; \upsilon, T_0) = E_i(T_0; \upsilon, T_0) = 0, \forall i \in \{1, \ldots, n\}$ and $B(T_0; \upsilon, T_0) = \upsilon$.

CHAPTER 3

Empirical pricing performance on long-dated crude oil derivatives

This chapter empirically investigates to what extent stochastic interest rates beyond stochastic volatility may improve the pricing performance of long-dated crude oil derivatives. Forward price stochastic volatility models for commodity derivatives with deterministic and stochastic interest-rate specifications are considered that allow for a full correlation structure between the underlying model factors. A description of the crude oil derivative dataset and its summary statistics are provided. The proposed models can be recast into a state-space discrete evolution form, hence Kalman filter log-likelihood methodology is employed to estimate the model parameters from historical crude oil futures and option prices. The related computational issues, methods to overcome them, the estimated results and its implications are discussed. This chapter is based on the working paper of Cheng, Nikitopoulos & Schlögl (2016*a*).

3.1. Introduction

Crude oil futures and options are the world's most actively traded commodity derivatives playing an important role in financial activities such as short-term trading and long-term investment strategies. The liquidity of commodity futures contracts has increased, for instance, the average daily open interest of crude oil futures contracts of all maturities has increased from 503,662 contracts in 2003 to 1,677,627 contracts in 2013. Even though the most active contracts are short-dated, the market for long-dated contracts has also substantially increased. The maturities of the crude oil futures contracts and the options on futures contracts have extended from 18 months in 1990, to over 9 years in recent years. The average daily open interest in crude oil futures contracts with maturities of two years or more was 41,601 contracts in December 2003 and reached a record high of 202,964

contracts in 2008. Figure 3.1 displays the average daily open interest of crude oil futures contracts for each year from 2003 to 2013, and reveals a considerable trading activity for long-dated crude oil futures contracts over recent years. Motivated by the increasing importance of long-dated commodity derivative contracts to the financial markets, we make theoretical and empirical contributions on the pricing of long-dated commodity derivatives.

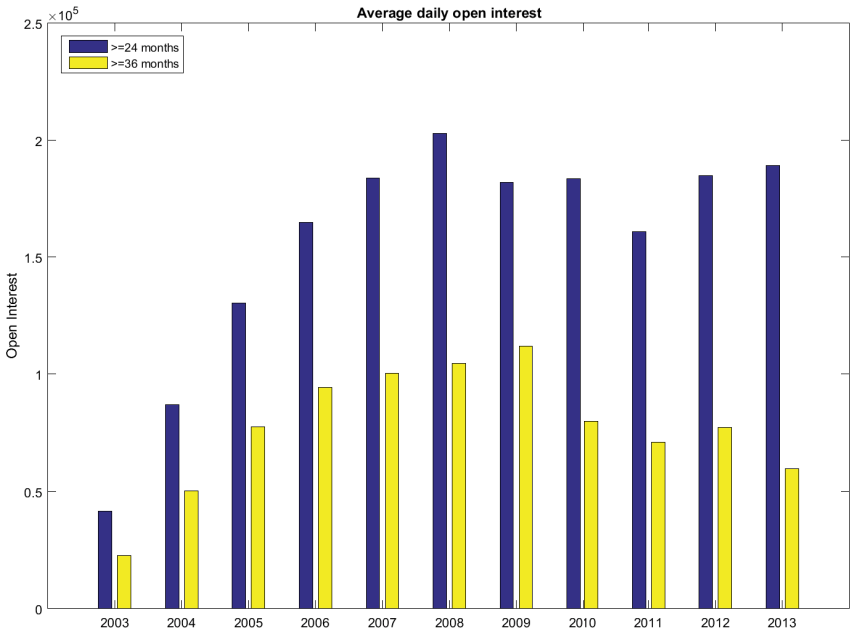

FIGURE 3.1. **Average daily open interest.** The average daily open interest is calculated by summing all the open interest at the end of each trading day with maturities more than or equal to 24 months (for ≥ 24) or 36 months (for ≥ 36) over a given year and then dividing this sum by the number of trading days in that year.

Studies dealing with pricing of long-dated derivatives typically use spot price models with stochastic interest rates and/or stochastic volatility, and are applied mostly in equity or Foreign exchange (FX) markets. Using European stock index options with maturities of up to two years from the Swedish option market, Rindell (1995) demonstrates empirically that the stochastic interest-rate option pricing model by Amin & Jarrow (1992)

outperforms the original Black & Scholes (1973) model that assumes constant interest rates. Schwartz (1997) confirms the importance of the stochastic interest rates for pricing longer maturity crude oil futures contracts, yet with deterministic volatility specifications. Bakshi et al. (1997) and Bakshi et al. (2000) perform empirical investigations on a variety of equity option pricing models, and demonstrate that stochastic volatility specifications tend to improve pricing and hedging performance of long-dated contracts, while stochastic interest rates tend to improve hedging performance. In response, a class of hybrid pricing models emerged, predominantly with applications in equity, insurance and foreign exchange markets, for instance Ballotta & Haberman (2003), Schrager & Pelsser (2004), van Haastrecht et al. (2009), van Haastrecht & Pelsser (2011) and Grzelak et al. (2012). Studies in the empirical literature in commodity markets, and, in particular, crude oil markets with stochastic volatility models include Trolle & Schwartz (2009*a*), Chiarella et al. (2013) and Cortazar et al. (2016). However, all models developed in these studies assume deterministic interest rates. As a result, it remains an open question to what extent these models are able to price long-dated derivative contracts, especially during a period where the interest rates are more volatile.

Motivated by the increasing importance of long-dated commodity derivative contracts to the financial market as well as the limited literature dedicated to empirical research on pricing long-dated commodity derivatives, we consider an empirical investigation of the impact of including stochastic interest rates beyond stochastic volatility to pricing long-dated crude oil derivative contracts. We employ two multi-dimensional stochastic volatility Heath et al. (1992) type models, one model with deterministic interest-rate specifications and one model with stochastic interest rates as proposed in Chapter 2. The models feature a full correlation structure and fit the entire initial forward curve by construction rather than generating it endogenously from the spot price as required in the spot pricing models. This class of models has an affine term structure representation that leads to quasi-analytical European vanilla futures option pricing equations. Furthermore, this

class of forward price models allows us to model directly multiple futures prices simultaneously.[1] Spot price models require the modelling of the spot price and the interest rates, as well as making assumptions on the convenience yield to be able to specify the futures prices. The proposed approach directly models the full term structure of futures prices and provides tractable prices for futures options (something that would be far more complicated to obtain with spot price models). Thus, our model can be estimated by fitting to both futures prices and option prices. One should also note that any seasonality in the spot price will be reflected in the term structure of futures prices, which the model fits by construction.

From an empirical point of view, our main contributions are threefold. Firstly, we estimate the parameters of the proposed hybrid type of models, from historical time series of both crude oil futures prices and crude oil futures option prices (since, is the most active commodity derivatives market), by using extended Kalman filter maximum log-likelihood methodology. Due to the large number of parameters required to be empirically estimated, the estimation process is treated in three stages. In stage one, we estimate the parameters of the interest-rate models in isolation by fitting the implied yields to US Treasury yields. In stage two, we run a sensitivity analysis on the correlations to determine their impacts on the pricing of long-dated crude oil options. In stage three, we estimate the remaining model parameters. We find evidence for a negative correlation between the crude oil futures price process and the interest-rate process, especially, over periods of high interest-rate volatility that is aligned with the empirical findings of Arora & Tanner (2013) from the crude oil spot market. We also find an insignificant impact of the correlation between stochastic volatility and interest rates to option prices.

Secondly, we evaluate the in-sample and out-of-sample ability of stochastic interest-rate models to improve pricing performance on long-dated crude oil derivatives compared to

[1]Note that it is the presence of convenience yields which permits us to specify directly the prices of futures to several maturities, simultaneously on the same underlying, without introducing inconsistency to the model.

models with deterministic interest-rate specifications. To assess the impact of stochastic interest rates under different market conditions, two subsamples are considered: the period $2005 - 2007$ that is characterised by relatively volatile interest rates, and the period $2011 - 2012$ that exhibits very low interest rates and a very stable interest-rate market. According to the numerical analysis conducted in Chapter 2, the volatility of the interest rates itself, rather than the level of the interest rates, contributes the most to the pricing of long-dated commodity derivatives. By comparing the pricing errors of the two models (deterministic interest-rate model against stochastic interest-rate model), we find that the stochastic interest-rate counterpart performs better when compared to the deterministic interest-rate counterpart, an effect that augments as the maturity of the crude oil futures options increases. These results are more pronounced during the period of relatively high volatility of the interest rates, consistent with the numerical findings in Chapter 2. During the period of very low interest rates and interest-rate volatility, there are no noticeable differences between the stochastic and deterministic interest-rate models.

Thirdly, we investigate the dimensionality of the model required to provide better pricing performance on long-dated crude oil derivatives. We find that three-dimensional models provide better fit to futures prices of all maturities compared to two-dimensional models. However, for long-dated crude oil option prices, increasing the model dimensionality does not tend to improve the pricing performance [also see Schwartz & Smith (2000) and Cortazar et al. (2016) for similar conclusions].

The remainder of this chapter is structured as follows. Section 3.2 presents a term structure model for pricing commodity derivatives incorporating stochastic volatility and stochastic interest rates. The crude oil derivative data used in our empirical analysis is also presented. Section 3.3 provides the details of the estimation methodology. Section 3.4 presents the estimation results and discusses the empirical findings of pricing long-dated crude oil derivatives. Section 3.5 concludes.

3.2. Model and data description

This section presents a brief description of the HJM term structure models used in the empirical analysis. The forward price models for commodity derivatives used in this chapter incorporate stochastic volatility and stochastic interest rates, and they can be easily adjusted to allow for deterministic interest-rate specifications. The details of the proposed models and the derivation of their European option pricing formulae can be found in Chapter 2. Using these models, we assess the contribution of stochastic interest rates beyond stochastic volatility when pricing long-dated commodity derivatives. We first present the general stochastic volatilitystochastic interest-rate models, and then we specify a time-deterministic interest-rate model.

3.2.1. The stochastic interest-rate model. We consider $\sigma = \{\sigma_t; t \in [0, T]\}$ an n-dimensional stochastic volatility process, and we denote as $F(t, T, \sigma_t)$ the futures price of the commodity at time $t \geq 0$, for delivery at time $T \in [t, \infty)$ and as $r = \{r(t); t \in [0, T]\}$ the N-dimensional instantaneous short-rate process. We further consider the extended version of the N-dimensional Schöbel-Zhu-Hull-White model, under the risk-neutral measure, as shown below:

$$\frac{dF(t, T, \sigma_t)}{F(t, T, \sigma_t)} = \sum_{i=1}^{n} \sigma_i^F(t, T, \sigma_t) dW_i^x(t), \tag{3.2.1}$$

where, for $i = 1, 2, \ldots, n$,

$$d\sigma_i(t) = \kappa_i(\overline{\sigma}_i - \sigma_i(t))dt + \gamma_i dW_i^\sigma(t), , \tag{3.2.2}$$

$$r(t) = \overline{r}(t) + \sum_{j=1}^{N} r_j(t), \tag{3.2.3}$$

and for $j = 1, 2, \ldots, N$,

$$dr_j(t) = -\lambda_j(t) r_j(t) dt + \theta_j dW_j^r(t).$$

The correlation structure is given by equation (2.2.6). The functional form of the volatility term structure $\sigma_i^F(t, T, \sigma_t)$ is specified as follows:

$$\sigma_i^F(t, T, \sigma_t) = (\xi_{0i} + \xi_i(T - t))e^{-\eta_i(T-t)}\sigma_i(t) \tag{3.2.4}$$

with ξ_{0i}, ξ_i, and $\eta_i \in \mathbb{R}$ for all $i \in \{1, \ldots, n\}$. This volatility specification allows for a variety of volatility structures such as exponentially decaying and hump-shaped which are typical volatility structures in commodity market and admits finite dimensional realisations. A detailed description of the model and the associated formulae can be found in Chapter 2 in this thesis. We denote with Call$(t, F(t, T, \sigma_t); T_o)$ the price of the European style call option with maturity T_o and strike K on the futures price $F(t, T, \sigma_t)$ maturing at time T. The price of a call option can be expressed as the discounted expected payoff under the risk-neutral measure of the form:

$$\text{Call}(t, F(t, T, \sigma_t); T_o) = \mathbb{E}_t^{\mathbb{Q}}\Big(e^{-\int_t^{T_o} r(s)\,ds}\big(e^{X(T_o,T)} - K\big)^+\Big) \tag{3.2.5}$$

where $X(t, T) = \log F(t, T, \sigma_t)$. By using Fourier inversion technique, Duffie et al. (2000) provide a semi-analytical formula for the price of European-style vanilla options under the class of affine term structure. With a slight modification of the pricing equation in Duffie et al. (2000), the equation (3.2.5) can be expressed as:[2]

$$\begin{aligned}\text{Call}(t, F(t, T, \sigma_t); T_o) = e^{-\int_t^{T_o} \overline{r}(s)\,ds} \prod_{i=2}^{N} \mathbb{E}_t^{\mathbb{Q}}\big[e^{-\int_t^{T_o} r_i(s)\,ds}\big] \times \\ \big[G_{1,-1}(-\log K) - KG_{0,-1}(-\log K)\big],\end{aligned} \tag{3.2.6}$$

where

$$G_{a,b}(y) = \frac{\phi(t; a, T_o, T)}{2} - \frac{1}{\pi}\int_0^\infty \frac{\Im[\phi(t; a + \mathbf{i}bu, T_o, T)e^{-\mathbf{i}uy}]}{u}\,du. \tag{3.2.7}$$

[2]The product starts at $i = 2$ because the state variables $r_2, \ldots, r_N$ are uncorrelated to other state variables. However r_1 is correlated.

The characteristic function ϕ can be expressed as:

$$\begin{aligned}\phi(t;\upsilon,T_o,T) = \exp\big\{&A(t;\upsilon,T_o) + B(t;\upsilon,T_o)X(t,T) + C(t;\upsilon,T_o)r_1(t)\\ &+\sum_{i=1}^{n} D_i(t;\upsilon,T_o)\nu_i(t) + \sum_{i=1}^{n} E_i(t;\upsilon,T_o)\sigma_i(t)\}.\end{aligned} \tag{3.2.8}$$

where the functions $A(t;\upsilon,T_o), B(t;\upsilon,T_o), C(t;\upsilon,T_o), D_i(t;\upsilon,T_o)$ and $E_i(t;\upsilon,T_o)$ in equation (3.2.8) satisfy the complex-valued Ricatti ordinary differential equations. Note that $\mathbf{i}^2 = -1$ and $\Im(x+\mathbf{i}y) = y$. The logarithm of the instantaneous futures prices at time t with maturity T can be expressed in terms of $6n$ state variables, namely $x_i(t)$, $y_i(t)$, $z_i(t)$, $\phi_i(t)$, $\psi_i(t)$ and $\sigma_i(t)$ (see Proposition 2.1 for details):

$$\begin{aligned}\log F(t,T,\sigma_t) = \log F(0,T,\sigma_0) - \frac{1}{2}\sum_{i=1}^{n}\Big(&\gamma_{1i}(T-t)x_i(t) + \gamma_{2i}(T-t)y_i(t) + \gamma_{3i}(T-t)z_i(t)\Big)\\ &+\sum_{i=1}^{n}\Big(\beta_{1i}(T-t)\phi_i(t) + \beta_{2i}(T-t)\psi_i(t)\Big),\end{aligned} \tag{3.2.9}$$

where for $i = 1,\ldots,n$ the deterministic functions are defined as:

$$\beta_{1i}(T-t) = \big(\xi_{i0} + \xi_i(T-t)\big)e^{-\eta_i(T-t)}, \quad \beta_{2i}(T-t) = \xi_i e^{-\eta_i(T-t)},$$

$$\gamma_{1i}(T-t) = \beta_{1i}(T-t)^2, \quad \gamma_{2i}(T-t) = 2\beta_{1i}(T-t)\beta_{2i}(T-t), \quad \gamma_{3i}(T-t) = \beta_{2i}(T-t)^2,$$

and the state variables $x_i, y_i, z_i, \phi_i, \psi_i$ satisfy the following SDE:

$$\begin{aligned}dx_i(t) &= \big(-2\eta_i x_i(t) + \sigma_i^2(t)\big)\,dt,\\ dy_i(t) &= \big(-2\eta_i y_i(t) + x_i(t)\big)\,dt,\\ dz_i(t) &= \big(-2\eta_i z_i(t) + 2y_i(t)\big)\,dt,\\ d\phi_i(t) &= -\eta_i\phi_i(t)dt + \sigma_i(t)dW_i^x(t),\\ d\psi_i(t) &= \big(-\eta_i\psi_i(t) + \phi_i(t)\big)dt,\end{aligned} \tag{3.2.10}$$

subject to the initial condition $x_i(0) = y_i(0) = z_i(0) = \phi_i(0) = \psi_i(0) = 0$.

Due to these quasi-analytical option pricing formulae, the model can be estimated by fitting to historical data from the crude oil derivatives market. For estimation applications,

we need to account for the market price of risk and the market price of volatility risk, namely Λ_i and Λ_i^σ. By introducing $\mathbb{W}_i(t)$ and $\mathbb{W}_i^\sigma(t)$ standard Wiener processes under the physical measure $\mathbb{P}$, we have the following dynamics:

$$d\mathbb{W}_i(t) = dW_i(t) - \Lambda_i \sqrt{\nu_i(t)}\, dt,$$
$$d\mathbb{W}_i^\sigma(t) = dW_i^\sigma(t) - \Lambda_i^\sigma \sqrt{\nu_i(t)}\, dt.$$

For comparison purposes, we also consider a model with time-deterministic interest rates, which is described next.

3.2.2. The deterministic interest-rate model. The dynamics of the futures price process and the stochastic volatility process remain the same as specified in equations (3.2.1) and (3.2.2), but the stochastic interest-rate process is replaced by a deterministic discount function. This discount function $P_{NS}(t,T)$ is obtained by fitting a Nelson & Siegel (1987) curve each trading day to US Treasure yields with maturities of $1, 2, 3$ and 5 years similar to Chiarella et al. (2013). Nelson and Siegel propose a functional form of the time-t instantaneous forward rate as follows:

$$f_{NS}(t,T) = b_0 + b_1 e^{-a(T-t)} + b_2 a(T-t) e^{-a(T-t)}$$

where a, b_0, b_1 and b_2 are parameters to be determined. The yield to maturity on a US Treasure bill is:

$$\begin{aligned} y_{NS}(t,T) &= \frac{\int_t^T f_{NS}(t,u)\, du}{T-t} \\ &= b_0 + \frac{(b_1+b_2)(1-e^{-a(T-t)})}{a(T-t)} - b_2 e^{-a(T-t)}. \end{aligned} \tag{3.2.11}$$

In each trading day, we determine the parameters by minimising the square of the errors between the observable treasury yields and the implied yields $y_{NS}(t,T)$. The discount

function for the option with maturity T is simply $P_{NS}(t,T) = e^{-y_{NS}(t,T)(T-t)}$.

We next conduct an empirical study using crude oil derivatives. The aim of the study is to gauge the impact of stochastic interest rates beyond stochastic volatility on pricing long-dated crude oil derivatives.

3.2.3. Data description. The sensitivity analysis conducted in Section 2.4 reveals that the correlation between the stochastic interest rates and the stochastic futures price process has a noticeable impact on prices of long-dated options but remains negligible for short-dated options [see Figure 2.2]. Furthermore, Figure 2.4 reveals that as the volatility of interest rates increases, the value of the option increases with the impact being more pronounced for longer-maturity options, while the long-term level of the interest rates does not impact option prices in a similar manner. Accordingly, we select two two-year periods to estimate the model parameters and assess the contribution of stochastic interest rates on long-dated crude oil derivative prices. The first period is chosen before the financial crisis from 1st August 2005 to 31st July 2007 as it represents a period of relatively high interest rates (over 4.6% , see Table 3.1) and high interest-rate volatility. The second period is chosen after the financial crisis from 1st January 2011 to 31st December 2012, and it represents a period of extremely low interest rates, also featuring low volatilities (below 0.5% for maturities under three years, see Table 3.1). The rationale here is that higher interest-rate volatility has more impact on option prices with long maturities.[3]

3.2.3.1. *Interest-rate data.* We use the US Treasury yields[4] as the proxy to estimate the parameters in the interest-rate process in our model as well as to convert the prices of American options to European options, as required when we consider the crude oil options. The reason for this choice over other rates such as LIBOR is that the options

[3]Chapter 2 provides a detailed analysis of the impact of different interest-rate parameters have on the option price.

[4]Data were obtained from www.treasury.gov.

in the dataset are exchange-traded options, hence, there is no counterparty credit risk involved. There are over ten different maturities in the dataset, and we choose only four sets of yields (the 1-year, 2-year, 3-year and 5-year yields) that best match the maturities of our options. Figure 3.2 displays the evolution of the US Treasury yields from 2005 to 2015, including the sample periods used in our empirical study. Summary statistics of the yields are presented in Table 3.1. Note that the standard deviation of the prices of 5-year bond yields estimated in the period between January 2011 and December 2012 (0.539%) is almost double that (0.288%) of the corresponding yield between the period August 2005 and July 2007. The reason for this seemingly higher volatility is due to the fact that the 5-year yield at the beginning of 2011 was around 2% and it increased to 2.4% in early February 2011, and then it quickly plummeted to less than 1% in August 2011. To better quantify the volatility of the interest rates during those two periods, we also present the linearly as well as nonlinearly detrended standard deviation of the interest rates. The linearity and nonlinearity are removed by subtracting a least-squares polynomial fit of degree 1 and degree 2 respectively. We observe that the nonlinearly detrended standard deviation of the 5-year yield is lower during 2011 period.

3.2.3.2. *Crude oil derivatives data.* We use Light Sweet Crude Oil (WTI) futures and options traded on the NYMEX[5] which is one of the richest dataset available on commodity derivatives. The number of available futures contracts per day has increased from 24 on the 3rd January 2005 and reached over 40 in 2013. The futures dataset has $145,805$ lines of data and the option dataset has close to 5 million lines of data. Due to the enormous amount of data, for estimation purposes we make a selection of contracts based on their liquidity and we use the open interest of the futures as the proxy of its liquidity. Liquidity is generally very low for contracts with less than 14 days to expiration while for contracts with more than 14 days to expiration liquidity increases significantly. Consequently we use only contracts with more than 14 days to expiration. Figure 3.3 shows the open interest of the futures contracts by the time-to-maturity for the first nine months and then

[5]The database has been provided by CME.

Period: August 2005 - July 2007				
Maturity in years	1	2	3	5
Mean	4.773%	4.682%	4.648%	4.640%
Standard Deviation	0.389%	0.309%	0.297%	0.288%
Detrended S.D.	0.262%	0.250%	0.251%	0.247%
Nonlinear Detrended S.D.	0.115%	0.177%	0.198%	0.210%
Kurtosis	0.411	0.268	0.012	-0.252
Skewness	-1.225	-0.797	-0.558	-0.334
Period: January 2011 - December 2012				
Maturity in years	1	2	3	5
Mean	0.178%	0.363%	0.565%	1.140%
Standard Deviation	0.052%	0.173%	0.310%	0.539%
Detrended S.D.	0.049%	0.122%	0.194%	0.264%
Nonlinear Detrended S.D.	0.036%	0.086%	0.131%	0.172%
Kurtosis	-0.026	0.783	0.343	-0.520
Skewness	0.450	1.415	1.324	0.991

TABLE 3.1. **Descriptive Statistics of US Treasury yields.** The table displays the descriptive statistics for the 1-, 2-, 3- and 5-year US Treasury yields from August 2, 2005 to July 31, 2007 and from January 1, 2011 to December 31, 2012. The first period represents a period with high volatility of interest rates and the second represents a period with low levels and volatility of interest rates.

the open interest of the futures contracts with maturities more than one year by calendar month. It is clear that liquidity is mainly concentrated on short-maturity contracts and on the December contracts. Thus, we select the first seven monthly contracts, then the next three contracts with maturity either on March, June, September and December, and then all contracts with maturity in December. We also filter out abnormalities such as futures price of zero and zero open interest. Thus, on a daily basis, we use around $10-15$ futures contracts for the period 2005 to 2007 and $15-17$ futures contracts for the period 2011 to 2012, extending our dataset of futures maturities to eight years.

For the crude oil futures option dataset, we select the options with the underlying futures contracts being the one used in the crude oil futures dataset. That is, we take options on the first seven monthly futures contracts and the next three contracts with maturity in either March, June, September, and all December contracts with maturities of up to

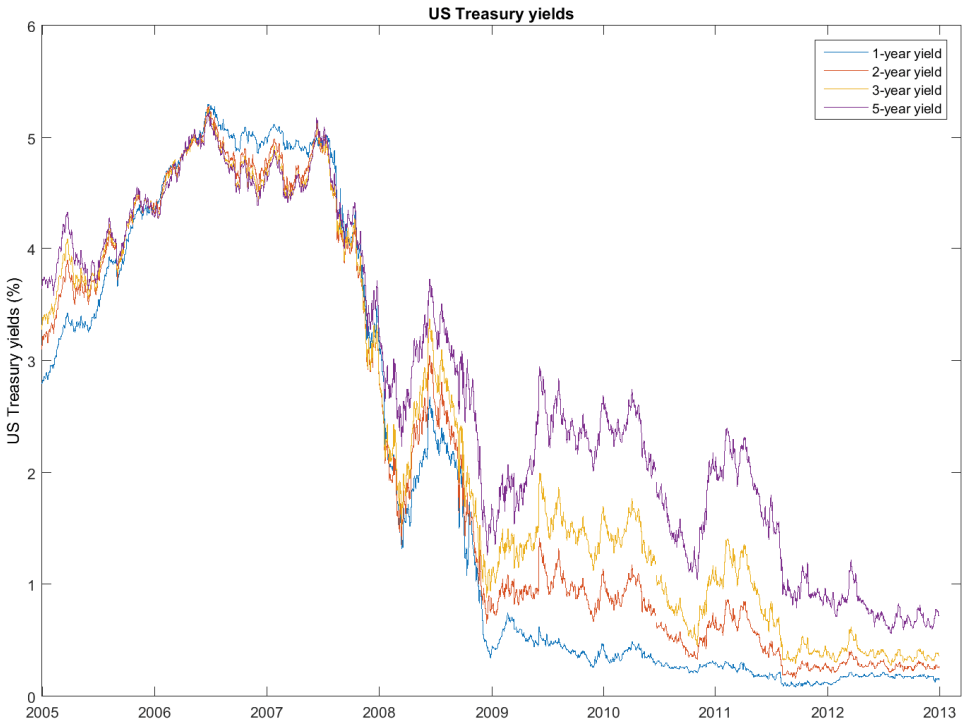

FIGURE 3.2. US Treasury yields.

5 years. For each option maturity, we consider six moneyness intervals, 0.86–0.905, 0.905–0.955, 0.955–1.005, 1.005–1.055, 1.055–1.105 and 1.105–1.15. We define moneyness as the option strike divided by the price of the underlying futures contract. In each of the moneyness interval, we choose either call or put options, in order to use only the out-of-the-money (OTM) and at-the-money (ATM) options that are closest to the interval mean. In order to reduce the computational overhead, we select the OTM and ATM options as they are generally more liquid. Beside that, the OTM options have lower early exercise approximation errors. On a daily basis we use around $50 - 77$ options for the period $2005 - 2007$ and $77 - 95$ options for the period $2011 - 2012$. After sorting the data, we are left with $74,073$ futures contracts and $297,878$ option contracts over a period of $5,103$ trading days.

3.2.3.3. *American to European options conversion.* The prices of the options in the dataset are American-style options whereas our pricing model is for European-style options. For the conversion of American prices for European prices, we first back out the

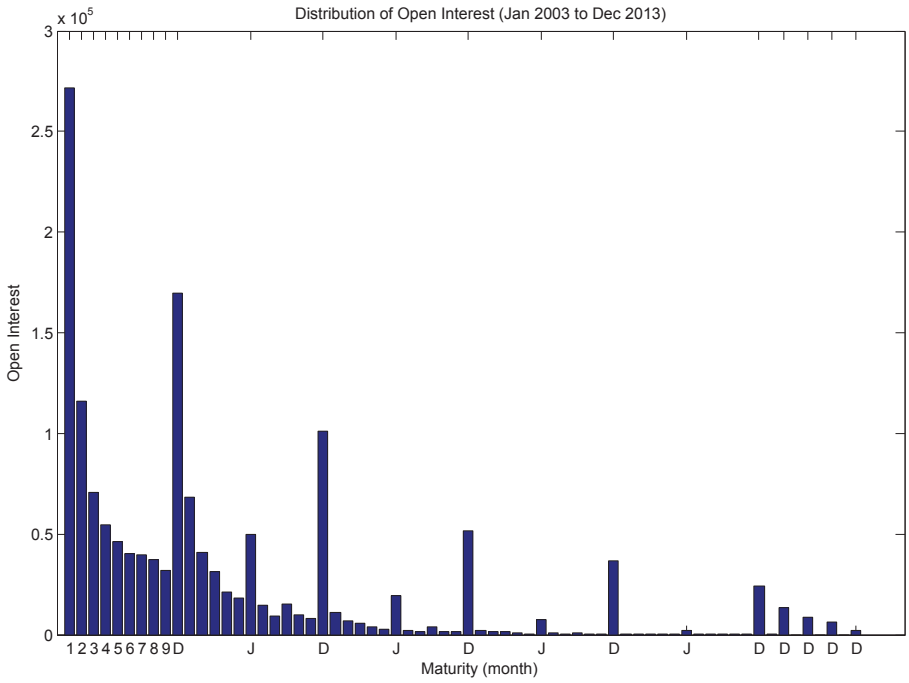

FIGURE 3.3. **Liquidity of crude oil futures contracts by calendar month.** The plot shows the liquidity of the first nine months from the trading day as well as December and June contracts in the following years. It shows that futures contracts with next-month delivery date are the most liquid, and liquidity gradually decreases over the following months. Futures contracts with maturities in December are very liquid even after a few years and June contracts are moderately liquid. Contracts with maturities of more than two years are very illiquid in other months.

implied volatility from the prices of American options using the approximation method provided by Barone-Adesi and Whaley, and then we use the Black (1976) formula together with this implied volatility to compute the prices of the corresponding European options [6]. The only parameter that is not immediately available is the constant interest-rate yield for the corresponding maturity of the options. To find out the constant interest-rate yield corresponding to the maturity of the options to be converted, we use the yield of the one-month treasury bill as the instantaneous short-rate together with the interest-rate parameters that we estimate in Section 3.4.1 to calculate the no arbitrage zero coupon bond

[6]An alternative solution is to apply the least-square approach as outlined in Longstaff & Schwartz (2001) which uses Monte Carlo simulation to calculate American option prices.

price corresponding to its maturity. Then we can obtain the constant interest yield from the bond price.

3.3. Estimation method

Several methodologies have been proposed in the literature to estimate the parameters of stochastic models, such as efficient method of moments, see Gallant, Hsieh & Tauchen (1997), maximum likelihood estimation, see Chen & Scott (1993) and the Kalman filter method. Duffee & Stanton (2012) perform an extensive analysis and comparison on these methods and conclude that the Kalman filter is the best method among these three. In this paper, we adopt the Kalman filter methodology to estimate the parameters of the model. For the purpose of parameter estimation, we let $\overline{r}(t), \lambda_i(t)$ and $\theta_i(t)$ to be constants for all i, i.e. $\overline{r}(t) = \overline{r}, \lambda_i(t) = \lambda_i$ and $\theta_i(t) = \theta_i, \forall i$. With these time-varying functions taken as constants the option pricing formula (3.2.6) is reduced to:

$$\text{Call}(t, F(t,T,\sigma_t); T_o) = e^{-\overline{r}(T_o - t)} \prod_{i=2}^{N} \exp(-A_i(T_o - t) r_i(t) + D_i(T_o - t)) \times \\ \big[G_{1,-1}(-\log K) - K G_{0,-1}(-\log K)\big], \tag{3.3.1}$$

where

$$A_i(\tau) = \frac{1 - e^{-\lambda_i \tau}}{\lambda_i},$$
$$D_i(\tau) = \Big(-\frac{\theta_i^2}{2\lambda_i^2} \Big) \Big(A_i(\tau) - \tau \Big) - \frac{\theta_i^2 A_i(\tau)^2}{4\lambda_i}.$$

There are $34 + 2N$ parameters[7] that need to be estimated in our three-dimensional model and $24 + 2N$ for the two-dimensional model. Due to this large number of parameters, it is computationally expensive to estimate all the parameters at once. Thus, we subdivide the task into three steps as follow.

[7]Three of each $\xi_0, \xi, \eta, \kappa, \gamma, \rho_{xr}, \rho_{x\sigma}, \rho_{r\sigma}, \Lambda, \Lambda^\sigma$, N of each λ_i, θ_i and one of each $\overline{r}, f_0, \sigma_f, \sigma_o$. $\Lambda, \Lambda^\sigma, f_0, \sigma_f, \sigma_o$ are the market price of risk, market price of volatility risk, initial time-homogenous futures curve, measurement noise of the futures and options respectively. See Appendix 3.2 and 3.3 for details.

(1) We firstly estimate the stochastic interest-rate process in isolation by using US Treasury yields, with an assumption that the parameters of the interest-rate process are not affected by commodity futures prices.

(2) We perform a sensitivity analysis over the correlations of the model to determine their impact on futures and option prices, and see if there are any correlations with negligible impact on the prices.

(3) We estimate the remaining parameters of the model.

We estimate the parameters of the interest-rate dynamics by using Kalman filter and maximum likelihood estimates (see Appendix 3.1 for details). The estimation method for the futures price volatility processes σ_i^F and the volatility processes $\sigma_i(t)$ involves using the extended Kalman filter and maximum likelihood (see Appendix 3.3 for details), where the model is re-expressed in a state-space form which consists of the system equations and the observation equation, and then the maximum likelihood method is employed to estimate the state variables.

The system equations describe the evolution of the underlying state variables. In this model, the state vector is[8] $X_t = \{X_t^i, i = 1, 2, \ldots, n\}$ where X_t^i consists seven state variables $x_i(t), y_i(t), z_i(t), \phi_i(t), \psi_i(t), \sigma_t^i$ and ν_t^i, see (3.2.10). This system can be put in a state-space discrete evolution form as follows:

$$\mathbf{X}_{t+1} = \mathbf{\Phi}_0 + \mathbf{\Phi}_X \mathbf{X}_t + \omega_{t+1}. \tag{3.3.2}$$

The ω_t with $t = 0, 1, 2, \ldots$ are independent of each other, with zero mean and with the covariance matrices conditional on time t a deterministic function of the state variable $\sigma(t)$. The observation equation is:

$$\mathbf{Z}_t = h(\mathbf{X}_t, \mathbf{u}_t),\ \mathbf{u}_t \sim \text{i.i.d. } N(\mathbf{1}, \mathbf{\Omega}), \tag{3.3.3}$$

[8]Note that we use $X(t, T)$ as the logarithm of the futures prices in (3.2.5). In this section on the Kalman filter, we re-define X_t to be a vector of state variables.

where $\mathbf{u}_t$ is a vector of i.i.d. multiplicative Gaussian measurement errors with covariance matrix Ω. The observation equation can be constructed by (3.2.9) which relates the logarithm of the futures prices linearly to the state variables $x_i(t), y_i(t), z_i(t), \phi_i(t)$ and $\psi_i(t)$. However, equation (3.2.6) relates option prices to the state variables through non-linear expressions. The application of the extended Kalman filter for parameter estimation involves linearising the observation function h and making the assumption that the disturbance terms $\{\omega_t\}_{t=0,1,\ldots}$ in equation (3.3.2) follow multivariate normal distributions. From the Kalman filter recursions, we can compute the likelihood function.

When interest rates are assumed to be stochastic, the interest rates needed for discounting at each future date are calculated from the estimated values of the state variables of the interest-rate model. In particular, we have the estimated parameters of the interest-rate model $\bar{r}, \lambda_j, \theta_j$, where $j = 1, 2, 3$ (for the three-dimensional model specifications). On each date, the Kalman filter updates $r_j(t)$. With known parameters and state variables, we use equation (3.2.6) to calculate the theoretical option prices. Note that r_2 and r_3 are independent, so they do not appear in the main option pricing function $G_{a,b}(y)$. When interest rates are assumed to be deterministic, the rates used for discounting at each future date are specified by equation (3.2.11).

3.3.1. Computational details. The program is written in Matlab. The log-likelihood function is maximised using Matlab's "fminsearch" routine to search for the minimal point of the negative of the log-likelihood function. "fminsearch" is an unconstrained nonlinear optimisation routine and it is derivatives-free. The Ricatti ordinary differential equations of the characteristic function ϕ in equation (3.2.8) is solved by Matlab's "ode23" which is an automatic step-size Runge-Kutta-Fehlberg integration method. This method uses lower order formulae compared to other ODE-methods which can be less accurate; but the advantage is that this method is fast compared to other methods. The

integral in (3.2.6) is computed by the Gauss-Legendre quadrature formula with 19 integration points and truncating the integral at 33 and we find that these numbers of integration points and truncation of the integral provide a good trade-off between computational time and accuracy.

3.3.1.1. *Methods to reduce computational time.* One of the biggest challenges is the formidable amount of computational time required for the estimation of parameters. Although this model admits quasi-analytic solutions for option pricing, complex-valued numerical ODE approximation together with complex-valued numerical integration are needed for each option price. Furthermore, for each day of the crude oil data, a numerical Jacobian needs to be calculated for the linearisation of option prices in the Kalman filter update. To complete one day of the data, which typically involves the calculation of around 70 options and its Jacobian for the Kalman filter update may take 10 to 15 minutes on my desktop running a second generation quad-core i7 processer. So that is about $5,000$ to $7,500$ minutes for the program to process two years (about 500 trading days) of data in order to calculate one log-likelihood. Matlab's "fminsearch" routine may take several hundreds of iterations for it to converge to a local maximum. The key observation to massively reduce the computational time is that, given a set of parameters of the model, the characteristic function $\phi(t; a + \mathbf{i}bu, T_o, T)$ [see (3.2.7)] is a function of a, b, u, T_o and T. T_o and T are set to be the same because all the crude oil options traded in CME expire only a few days before the underlying futures contracts. So for each iteration we precalculate six tables for the characteristic functions. These are $\phi(t; 0, T, T)$, $\phi(t; 1, T, T)$, $\phi(t; 1-\mathbf{i}u, T, T)$, $\phi(t; -\mathbf{i}u, T, T)$, $\phi(t; 1+\mathbf{i}u, T, T)$ and $\phi(t; \mathbf{i}u, T, T)$ where values of the variable u are determined by the 19 integration points on the interval from 0 to 33 calculated using Gauss-Legendre quadrature, and the values of the maturity T are $14, 15, 16, \ldots, 1850$ because the shortest maturity is only 14 days, and the longest maturity is five years. Observing the fact that each calculation of the characteristic function is independent of the others, we also take advantage of the parallel toolbox available in Matlab by using Matlab's "parfor" loop. This reduces the time required to process one

iteration (2 years of data) from $5,000$ to $7,500$ minutes to around 2 minutes. The total time required for the parameters to converge to a local maximum can still take a few days.

3.4. Estimation results

As part of the estimation applications, we conduct several investigations into the pricing performance of the models developed in Section 3.2. Firstly, we discuss the statistical significance and the economic significance of the estimated parameters over two distinct time periods, characterised by different market conditions, i.e., low against high interest-rate volatility environments. Secondly, we evaluate the ability of stochastic interest-rate models to improve pricing performance on long-dated crude oil derivatives compared to models with deterministic interest-rate specifications. Thirdly, we assess both in-sample and out-of-sample pricing performance on long-dated crude oil options with maturities up to five years. Fourthly, given the multi-dimensional nature of our models, we investigate the sufficient number of dimensions required for the models to provide satisfactory levels of pricing performance, and we discuss the trade-off between computational effort and numerical accuracy.

3.4.1. Interest-rate process. The estimation results of the multi-dimensional affine term structure models for the interest-rate process are summarised in Table 3.2. We estimated the interest-rate process for $N = 1, 2, 3$, to determine the number of model dimensions required to provide a satisfactory fit to the interest-rate data. The results reveal that the three-dimensional affine term structure model provides the best fit for all maturities. This is consistent with results by Litterman & Scheinkman (1991) and Fan, Gupta & Ritchken (2007) in the swaption market. Therefore, we consider the three-dimensional version of interest-rate models in the subsequent analysis.

The estimated long-term mean level $\overline{r}$ in the first period is 4.96%, which is much higher than the long-term mean level of -0.20% estimated in the second period. The same also holds for the volatility parameters θ_i of the interest-rate process. Thus, the parameter

estimates are consistent with the statistical properties of the interest rates over these two periods, [see Table 3.1]. The August 2005–July 2007 period is characterised by high levels of interest rates, with high volatility, while the January 2011–December 2012 period exhibits much lower interest rates and corresponding volatility.

Period 1: Aug 05 – Jul 07

	1 dimension	2 dimensions		3 dimensions		
	$i = 1$	$i = 1$	$i = 2$	$i = 1$	$i = 2$	$i = 3$
λ_{0i}	0.0492	-0.0401	3.9081	0.0752	0.2766	1.5860
θ_i	0.0058	0.0058	0.0164	0.0152	0.0080	0.0130
$\overline{r}$	4.1592%	3.9112%			4.9611%	
$\log$ L	11255	12803			13240	
rmse 1yr	4.1033%	1.0425%			0.4885%	
rmse 2yr	1.1798%	0.4110%			0.2823%	
rmse 3yr	0.6045%	0.3392%			0.1559%	
rmse 5yr	0.5367%	0.1346%			0.0476%	

Period 2: Jan 11 – Dec 12

	1 dimension	2 dimensions		3 dimensions		
	$i = 1$	$i = 1$	$i = 2$	$i = 1$	$i = 2$	$i = 3$
λ_{0i}	0.2161	-0.3913	0.5144	0.1223	-0.4987	-1.2297
θ_i	0.0017	0.0027	0.0025	0.0021	0.0019	3.98E-05
$\overline{r}$	0.0376%	-0.2828%			-0.2018%	
$\log$ L	12174	13057			13595	
rmse 1yr	47.6749%	22.5815%			15.3528%	
rmse 2yr	9.8461%	5.6910%			3.8036%	
rmse 3yr	4.7325%	2.8091%			1.2320%	
rmse 5yr	0.9182%	0.3006%			0.0661%	

TABLE 3.2. **Parameter estimates of the interest-rate process.** The table displays the parameter estimates and root mean-square errors (RMSE) of multi-dimensional affine term structure models for $N = 1, 2, 3$ over two periods. The first period represents a period with high volatility of interest rates and the second represents a period with low levels and volatility of interest rates. Three-dimensional models provide the best fit for all maturities.

3.4.2. Sensitivity analysis of the correlations. A sensitivity analysis of the correlation between the stochastic interest-rate process and the futures price process ρ^{xr} as well as the correlation between the stochastic interest-rate process and the stochastic volatility process $\rho^{r\sigma}$ is performed to determine their impact on the futures and option prices. We firstly estimate the models developed in Section 3.2 assuming zero correlations between the futures price process and the interest-rate process (i.e., $\rho_i^{xr} = 0$) for both periods

(2005 − 2007 and 2011 − 2012). The estimation results of fitting the two-dimensional model and the three-dimensional model to crude oil futures and options are shown on Table 3.3 and Table 3.4, respectively. Then we use these parameters estimated in these two models to price call options with a different correlation structure. In our analysis, we select an in-the-money (ITM) option with a maturity of 4,000 days and a strike of $100. These numbers are chosen for illustration purposes only and the results are presented in Table 3.5.

	Period 1: Aug 2005 – Jul 2007		Period 2: Jan 2011 – Dec 2012	
	$i = 1$	$i = 2$	$i = 1$	$i = 2$
ξ_{0i}	0.0564	1.2693	0.9683	0.1988
ξ_i	0.1034	0.0297	0.0015	0.0046
η_i	0.0872	0.1892	0.0049	0.2704
κ_i	0.0859	0.0193	0.0053	0.1222
γ_i	0.0181	0.0488	0.0205	0.2114
ρ_i^{xr}	0	0	0	0
$\rho_i^{x\sigma}$	-0.4126	-0.4416	-0.4376	0.0127
Λ_i	0.1145	0.4785	-0.0130	0.2238
Λ_i^{σ}	-0.0090	-0.0917	-2.1178	0.7654
f_0	7.305		3.675	
σ_f	2.00%		2.00%	
σ_O	6.87%		6.95%	
$\log \mathrm{L}$	-60139		-62160	
RMSE Futures	2.8635%		1.6565%	
RMSE Imp.Vol. 4mth	1.2075%		1.7917%	
RMSE Imp.Vol. 12mth	1.4787%		1.7302%	
RMSE Imp.Vol. 2yr	1.9997%		1.4997%	
RMSE Imp.Vol. 3yr	2.3210%		0.9885%	
RMSE Imp.Vol. 4yr	2.4634%		1.0235%	
RMSE Imp.Vol. 5yr	2.6908%		1.3064%	

TABLE 3.3. **Parameter estimates of a two-dimensional model for crude oil futures and options with $\rho^{xr} = 0$.** The table displays the maximum-likelihood estimates for the two-dimensional model specifications over the two two-year periods, namely, August, 2005 to July, 2007 and January 2011 to December 2012. The model assumes zero correlations between the futures price process and the interest-rate process. Note that f_0 is the time-homogenous futures price at time 0, namely $F(0,t) = f_0, \forall t$. The quantities σ_f and σ_o are the standard deviations of the log futures prices measurements errors and the option price measurement errors, respectively. We normalised the long run mean of the volatility process, $\overline{\sigma}_i$, to one to achieve identification.

	Period 1: Aug 2005 – Jul 2007			Period 2: Jan 2011 – Dec 2012		
	i=1	i=2	i=3	i=1	i=2	i=3
ξ_{0i}	0.0295	0.3164	2.0226	0.6734	0.3183	0.0143
ξ_i	0.2710	0.0334	0.0068	0.0037	0.0104	0.0519
η_i	0.1748	0.3526	0.0209	0.0053	0.2972	0.0597
κ_i	0.0287	0.0670	-0.0319	0.0105	0.1606	0.0025
γ_i	-0.0311	1.0054	0.0064	-0.0516	-0.2799	0.0750
ρ_i^{xr}	0	0	0	0	0	0
$\rho_i^{x\sigma}$	-0.0894	-0.0962	-0.1767	0.4008	0.0015	0.0159
Λ_i	-0.3211	1.6172	-16.4285	0.2504	0.0037	-0.0586
Λ_i^{σ}	1.1087	0.1652	-1.8788	-0.2761	0.8139	-1.0503
f_0		9.0851			5.5110	
σ_F		1.27%			1.66%	
σ_O		3.15%			11.34%	
$\log \mathrm{L}$		-38670			-31096	
RMSE Futures		1.2995%			1.2535%	
RMSE Imp.Vol. 4mth		1.1827%			1.7927%	
RMSE Imp.Vol. 12mth		1.6173%			1.7064%	
RMSE Imp.Vol. 2yr		1.9285%			1.5372%	
RMSE Imp.Vol. 3yr		2.1934%			1.0613%	
RMSE Imp.Vol. 4yr		2.5390%			0.8927%	
RMSE Imp.Vol. 5yr		2.9145%			0.9096%	

TABLE 3.4. **Parameter estimates of a three-dimensional model for crude oil futures and options with $\rho^{xr} = 0$.** The table displays the maximum-likelihood estimates for the three-dimensional model specifications over the two two-year periods, August, 2005 to July, 2007 and January 2011 to December 2012. The model assumes zero correlations between the futures price process and the interest-rate process. Note that f_0 is the time-homogenous futures price at time 0, namely $F(0,t) = f_0, \forall t$. The quantities σ_f and σ_o are the standard deviations of the log futures prices measurements errors and the option price measurement errors, respectively. We normalised the long run mean of the volatility process, $\overline{\sigma}_i$, to one to achieve identification.

From this analysis, we make several conclusions. Firstly, the impact of the correlation between the stochastic interest-rate process and the stochastic volatility process ($\rho_{r\sigma}$) on the option prices is insignificant even for maturity as long as $4,000$ days (see Table 3.5). For $\rho^{xr} = 0$,[9] we see that the percentage difference of the option price (comparing $\rho^{r\sigma} = 0.50$ and $\rho^{r\sigma} = 0$) is $(10.51 - 10.416)/10.51 = 0.894\%$ for the period $2005 - 2007$ and the percentage difference of the option price (comparing $\rho^{r\sigma} = 0.50$ and $\rho^{r\sigma} = 0$) is $(28.201 - 28.187)/28.201 = 0.05\%$ for the period $2011 - 2012$. Secondly, the correlation between the stochastic interest-rate process and the underlying futures price process (ρ^{xr})

[9]Throughout this section, I use the shorthanded notation $\rho^{xr} = 0$ to refer to $\rho_1^{xr} = \rho_2^{xr} = \rho_3^{xr} = 0$, and the same interpretation for $\rho^{r\sigma}$.

Period 2005-2007

		$\rho^{r\sigma}$						
		-0.50	-0.30	-0.15	0	0.15	0.30	0.50
ρ^{xr}	-0.50	10.967	10.922	10.889	10.857	10.825	10.794	10.753
	-0.30	10.823	10.78	10.748	10.717	10.686	10.656	10.617
	-0.15	10.717	10.675	10.644	10.613	10.583	10.554	10.516
	0	10.611	10.57	10.54	10.51	10.481	10.453	10.416
	0.15	10.506	10.466	10.437	10.408	10.38	10.353	10.317
	0.30	10.403	10.364	10.335	10.307	10.28	10.253	10.218
	0.50	10.266	10.228	10.201	10.174	10.148	10.122	10.089

Period 2011-2012

		$\rho^{r\sigma}$						
		-0.50	-0.30	-0.15	0	0.15	0.30	0.50
ρ^{xr}	-0.50	28.605	28.6	28.595	28.591	28.587	28.583	28.577
	-0.30	28.449	28.443	28.439	28.435	28.43	28.426	28.421
	-0.15	28.332	28.326	28.322	28.318	28.314	28.309	28.304
	0	28.215	28.209	28.205	28.201	28.197	28.193	28.187
	0.15	28.098	28.093	28.089	28.085	28.081	28.077	28.071
	0.30	27.982	27.977	27.973	27.969	27.965	27.961	27.955
	0.50	27.828	27.823	27.819	27.815	27.811	27.807	27.802

TABLE 3.5. **Call Option prices for varying correlation coefficients.** Futures=100, strike=100, maturity=4000 days. We denote for example $\rho^{xr} = -0.50$ to mean $\rho_1^{xr} = \rho_2^{xr} = \rho_3^{xr} = -0.50$ and similarly for $\rho^{r\sigma}$.

has an impact on the option prices. More specifically, the impact to the option prices in the period $2005 - 2007$ is more than twice compared to the impact in the period from $2011-2012$. For instance, the percentage difference of the option price (comparing $\rho^{xr} = 0$ and $\rho^{xr} = -0.50$) is $(10.51 - 10.857)/10.51 = -3.30\%$ for the period $2005 - 2007$ and the percentage difference of the option price (comparing $\rho^{xr} = 0$ and $\rho^{xr} = -0.50$) is $(28.201 - 28.591)/28.201 = -1.38\%$ for the period $2011 - 2012$. Consequently, as the impact of the correlation between the stochastic interest-rate process and the stochastic volatility process on the option price is negligible, we set $\rho_1^{r\sigma} = \rho_2^{r\sigma} = \rho_3^{r\sigma} = 0$ (see Chapter 2 for similar conclusions) and we only estimate ρ_i^{xr}. One can view this technical evidence as follows. The correlation between the underlying futures prices and interest rates is important because when the path of futures prices is increasing, the value of the pay-off at maturity would increase for a call option. If this correlation is negative, interest rates would have a higher tendency to be lower as the underlying futures prices increase.

Therefore, the pay-off at maturity would be discounted at a lower rate, hence higher call option price if there is a negative correlation between futures price and interest rates. By a similar argument, positive correlation would increase put option prices. Given a negative correlation between the volatility and the interest rates, lower paths of interest rates would tend to associate with higher volatility. But higher volatility means higher expected payoff at maturity given it is above the strike for call options or below the strike for put options, rather than higher pay-off at maturity. Consequently, the correlation between the volatility and the interest rates affects the conditional expected payoff, rather than the payoff itself, the impact of this correlation is much less when compared to the correlation between the futures prices and the interest rates.

3.4.3. The correlation ρ_i^{xr}. Table 3.6 and Table 3.7 present the estimates of a two-dimensional and a three-dimensional model, respectively, when the correlation coefficient ρ_i^{xr} between the futures price process and interest-rate process is non-zero. We find that the absolute values of the correlation coefficients between the stochastic interest-rate process and futures price process are quite high, ranging from -0.64 to 0.59, underscoring the important (though not invariant) relationship between interest rates and futures prices. We observe that, especially, over the high interest-rate volatility period $2005-2007$, these correlations are always negative. Studies such as Akram (2009), Arora & Tanner (2013) and Frankel (2014) provide empirical evidence for a negative relationship between oil prices and interest rates. Akram (2009) conducts an empirical analysis based on structural VAR models estimated on quarterly data over the period 1990 – 2007. One of his results suggests that there is a negative relationship between the real oil prices and real interest rates. Furthermore, Arora & Tanner (2013) suggest that oil price consistently falls with unexpected rises in short-term real interest rates through the whole sample period from 1975 to 2012. Another result their paper suggests is that oil prices have become more responsive to long-term real interest rates over time. Frankel (2014) presents and estimates a “carry trade” model of crude oil prices and other storable commodities. Their empirical results support the hypothesis that low interest rates contribute to the upward pressure

on real commodity prices via a high demand for inventories. Even though our empirical analysis does not refer to the correlations between the actual financial observables as the above studies do, it reveals a negative correlation between innovations of the crude oil futures prices and innovations in the interest rates process. This implies that crude oil futures prices and crude oil spot prices have similar response to changes in the interest rates.

Furthermore, we compare the results presented in Table 3.6 and Table 3.7 with the results from models that ignore the correlation coefficient between the futures price process and interest-rate process; thus, assume $\rho_i^{xr} = 0$, see Table 3.3 and Table 3.4. In the low interest-rate period $2011 - 2012$ there is no improvement in the fit of both futures prices and implied volatility by incorporating the correlation ρ_i^{xr}. This is mainly because during that period, interest rates and their volatility are very low and the interest-rate process has very little impact in the option prices. However, we observe some improvement in the RMSE of the implied volatility in the period $2005 - 2007$ when incorporating the correlation ρ^{xr}. For instance, the RMSE of the implied volatility of options with five years to maturity improves from 2.6908% to 2.4606%. If interest rates were more volatile, we would expect an even more substantial improvement as it has been shown in Chapter 2.

3.4.4. Pricing performance of long-dated derivatives. Table 3.6 and Table 3.7 demonstrate that the three-dimensional model outperforms the two-dimensional model, producing lower RMSE when fitting to futures prices for the period $2005 - 2007$ and the period $2011 - 2012$. However, when fitting to the implied volatility, the improvement in the RMSEs between the 2-factor model and the 3-factor model is minimal: in both periods, and especially, for the shorter maturity options. For the period $2005 - 2007$, the two-dimensional model seems to perform slightly better when the very long maturities of the 4-year and 5-year implied volatility is considered. As it has been shown also in Schwartz & Smith (2000) and Cortazar et al. (2016), increasing the dimensionality of

	Period 1: Aug 2005 – Jul 2007		Period 2: Jan 2011 – Dec 2012	
	$i = 1$	$i = 2$	$i = 1$	$i = 2$
ξ_{0i}	0.0564	1.2693	0.9683	0.1988
ξ_i	0.1034	0.0297	0.0015	0.0046
η_i	0.0872	0.1892	0.0049	0.2704
κ_i	0.0859	0.0193	0.0053	0.1222
γ_i	0.0181	0.0488	0.0205	0.2114
ρ_i^{xr}	-0.6166	-0.3014	0.5420	-0.6409
$\rho_i^{x\sigma}$	-0.4126	-0.4416	-0.4376	0.0127
Λ_i	0.1145	0.4785	-0.0130	0.2238
Λ_i^{σ}	-0.0090	-0.0917	-2.1178	0.7654
f_0	7.305		3.675	
σ_f	2.00%		2.00%	
σ_O	6.87%		6.95%	
$\log \mathrm{L}$	-59977		-61299	
RMSE Futures	2.8408%		1.6563%	
RMSE Imp.Vol. 4mth	1.2174%		1.7916%	
RMSE Imp.Vol. 12mth	1.4736%		1.7233%	
RMSE Imp.Vol. 2yr	1.9605%		1.4987%	
RMSE Imp.Vol. 3yr	2.2144%		0.9987%	
RMSE Imp.Vol. 4yr	2.2947%		1.0082%	
RMSE Imp.Vol. 5yr	2.4606%		1.2727%	

TABLE 3.6. **Parameter estimates of a two-dimensional model for crude oil futures and options.** The table displays the maximum-likelihood estimates for the two-dimensional model specifications for the two two-year periods, namely, August, 2005 to July, 2007 and January 2011 to December 2012. Here f_0 is the time-homogenous futures price at time 0, namely $F(0,t) = f_0, \forall t$. The quantities σ_f and σ_o are the standard deviations of the log futures prices measurements errors and the option price measurement errors, respectively. We normalise the long run mean of the volatility process, $\overline{\sigma}_i$, to one to achieve identification.

multi-dimensional models does not necessarily improve pricing on long-dated commodity contracts. Overall, the two-dimensional forward price models would provide a satisfactory fit to implied volatilities for longer maturities, while a three-dimensional forward price models would be required for a satisfactory fit to the term structure of longer maturities futures contracts.

In addition, we examine the impact of including stochastic interest rates when pricing long-dated crude oil derivatives. It is well documented that stochastic volatility alone improves pricing performance on long-dated equity derivatives, see Bakshi et al. (2000), as well as long-dated commodity derivatives, see Cortazar et al. (2016). Yet, stochastic

	Period 1: Aug 2005 – Jul 2007			Period 2: Jan 2011 – Dec 2012		
	i=1	i=2	i=3	i=1	i=2	i=3
ξ_{0i}	0.0295	0.3164	2.0226	0.6734	0.3183	0.0143
ξ_i	0.2710	0.0334	0.0068	0.0037	0.0104	0.0519
η_i	0.1748	0.3526	0.0209	0.0053	0.2972	0.0597
κ_i	0.0287	0.0670	-0.0319	0.0105	0.1606	0.0025
γ_i	-0.0311	1.0054	0.0064	-0.0516	-0.2799	0.0750
ρ_i^{xr}	-0.6222	-0.3822	-0.5444	-0.3167	0.59167	-0.4583
$\rho_i^{x\sigma}$	-0.0894	-0.0962	-0.1767	0.4008	0.0015	0.0159
Λ_i	-0.3211	1.6172	-16.4285	0.2504	0.0037	-0.0586
Λ_i^σ	1.1087	0.1652	-1.8788	-0.2761	0.8139	-1.0503
f_0		9.0851			5.5110	
σ_F		1.27%			1.66%	
σ_O		3.15%			11.34%	
$\log \mathrm{L}$		-35893			-31372	
RMSE Futures		1.3067%			1.2452%	
RMSE Imp.Vol. 4mth		1.1792%			1.7925%	
RMSE Imp.Vol. 12mth		1.6091%			1.7078%	
RMSE Imp.Vol. 2yr		1.8815%			1.5345%	
RMSE Imp.Vol. 3yr		2.0380%			1.0497%	
RMSE Imp.Vol. 4yr		2.3487%			0.8897%	
RMSE Imp.Vol. 5yr		2.6539%			0.9139%	

TABLE 3.7. **Parameter estimates of a three-dimensional model for crude oil futures and options.** The table displays the maximum-likelihood estimates for the three-dimensional model specifications for the two two-year periods, August, 2005 to July, 2007 and January 2011 to December 2012. Here f_0 is the time-homogenous futures price at time 0, namely $F(0,t) = f_0, \forall t$. The quantities σ_f and σ_o are the standard deviations of the log futures prices measurements errors and the option price measurement errors, respectively. We normalise the long run mean of the volatility process, $\overline{\sigma}_i$, to one to achieve identification.

interest rates are important when considering long-dated commodity commitments, see Hilliard & Reis (1998) and Grzelak & Oosterlee (2011). Thus, we compare the pricing performance of our proposed three-dimensional models that include stochastic volatility and stochastic interest rates with equivalent models (same parameter estimates as the proposed models) but with deterministic interest rates (i.e., discounted by the corresponding US Treasury yields). We also use the model parameters estimated in the previous section to assess out-of-sample performance by re-running the scenarios with extended data. The data extends from previously 31st July 2007 to 31st December 2007 for the first period and from 31st December 2012 to 31st December 2013 for the second period. Figure 3.5 displays the average of the RMSE across all maturities between the volatility implied by

the market option prices on that day and the implied volatility from the estimated model (as described in Section 3.2.3.2) for the two sample periods used in our analysis. Figure 3.5 and Figure 3.6 display the average of the daily RMSE of the implied volatility for the maturities of 2, 3, 4 and 5 years, respectively. Since the liquidity of the crude oil long-dated contracts is concentrated in December contracts, we assess the model fit to December contracts which are in the second, third, fourth and fifth year to maturity. The results of the in-sample and out-of-sample analysis are also included and summarised in Table 3.8.

Firstly, we observe that models that incorporate stochastic interest rates consistently improve pricing performance, yet the improvement becomes more evident as the maturity of the derivative contracts increases. For maturities of up to two-years, the improvement is minimal or not discernible for both the in-sample and the out-of-sample analysis. However, Table 3.8 reveals that the improvement on implied volatilities reaches 50 basis points on average, for maturities up to 5 years. Table 3.9 displays the maximum absolute difference between the volatility implied by the market crude oil option prices and the model implied volatility (of the deterministic interest-rate model and the stochastic interest-rate model) observed in the in-sample period and the out-of-sample period. In general, the stochastic interest model provides lower absolute differences compared to the deterministic interest-rate model, which increase as the options maturity increases.[10] Taking into consideration that the current bid-ask spreads for 0.5-year, 1.5-year and 2.5-year options, expressed in terms of implied volatility differences, are around 1.06%, 1.41% and 2.67%[11], respectively, the improvement in the maximum absolute differences in the out-of-sample analysis are comparable to the implied volatility differences of the bid-ask spread prices,

[10]The stochastic interest-rate model provides slightly higher absolute differences compared to the deterministic one, in the in-sample period between January 2011 and December 2012, where the interest-rate market experienced low volatility. Thus, the contribution of stochastic interest rates is negligible.

[11]These numbers represent the average bid-ask spreads (expressed in terms of implied volatilities differences) of the corresponding maturities of Dec-2016, Dec-2017 and Dec-2018 option contracts observed in June 2016. For example, for the implied volatility of the Dec-2017 contracts, we use the Dec-2017 futures prices, 1.5-year Treasury yields and a minimisation routine to match the implied volatility to the Dec-2017 futures option prices.

especially in the period of high interest-rate volatility during August 2005 and July 2007. This observation underscores the importance of stochastic interest-rate models for pricing long-dated commodity derivatives. In particular, given that when hedging long-dated commodity derivative contracts, this effect would accumulate over the repeated rebalancing of a dynamic hedging strategy, one would expect the inclusion of stochastic interest rates in the model to improve its hedging performance.

Secondly, note that stochastic interest rates become relevant and important when interest-rate volatility is relatively high. During the period January 2011 – December 2012, i.e., in the period where interest rates were low, marginal improvement in pricing performance is observed for all maturities. Thus, when interest rates are not volatile, stochastic interest rates will not improve in-sample pricing performance, but they do contribute on improving out-of-sample pricing performance to some extent.

	Period 1: Aug 05 – Jul 07				Period 2: Jan 11 – Dec 12			
	IS		OS		IS		OS	
	Sto	Det	Sto	Det	Sto	Det	Sto	Det
4mth	1.18%	1.18%	1.30%	1.34%	1.75%	1.75%	1.88%	1.88%
12mth	1.61%	1.63%	2.43%	2.59%	1.72%	1.72%	1.64%	1.67%
2year	1.90%	1.98%	2.38%	2.58%	1.53%	1.51%	1.59%	1.65%
3year	2.39%	2.68%	2.50%	3.15%	1.27%	1.18%	1.43%	1.60%
4year	2.75%	3.13%	2.87%	3.77%	1.02%	0.89%	1.54%	1.81%
5year	3.11%	3.58%	3.23%	4.06%	1.04%	0.89%	1.91%	2.26%

TABLE 3.8. **RMSE of implied volatility.** The table displays the RMSE between the observed implied volatility and the implied volatility from the estimated model for different maturities. In-sample and out-of-sample analysis is included.

3.5. Conclusion

In this paper, we propose commodity derivative pricing models featuring stochastic volatility and stochastic interest rates which allow non-zero correlations between the underlying futures price process, the stochastic volatility process and the stochastic interest-rate process. These models are within the HJM framework. Thus, they can generate the forward

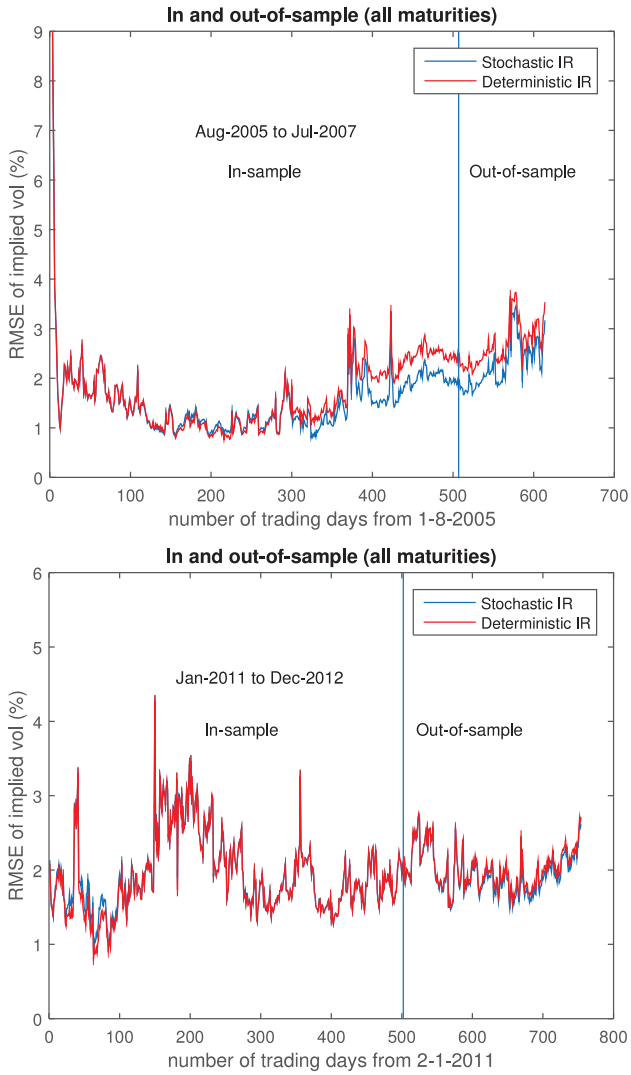

FIGURE 3.4. **RMSE of implied volatility for all maturities.** The graph displays the average of the daily RMSE of the implied volatility over all maturities. The top panel displays the fitting starting from August 2005 and the bottom panel displays the fitting starting from January 2011. The RMSE is defined to be between the volatility implied by observed market prices of the options and the implied volatility from the estimated model.

curve exogenously rather than endogenously from the spot prices as in the spot pricing models. This forward price model is then augmented by Hull and White type of correlated interest-rate process and a Vasicek type volatility process. For a generalised hump-shaped volatility structure, the model admits a finite dimensional affine state-space and has closed-form characteristic function hence quasi-analytical option prices can be obtained. Estimation of the parameters of the model is performed by using the extended Kalman filter and maximum likelihood method on two two-year periods, one between

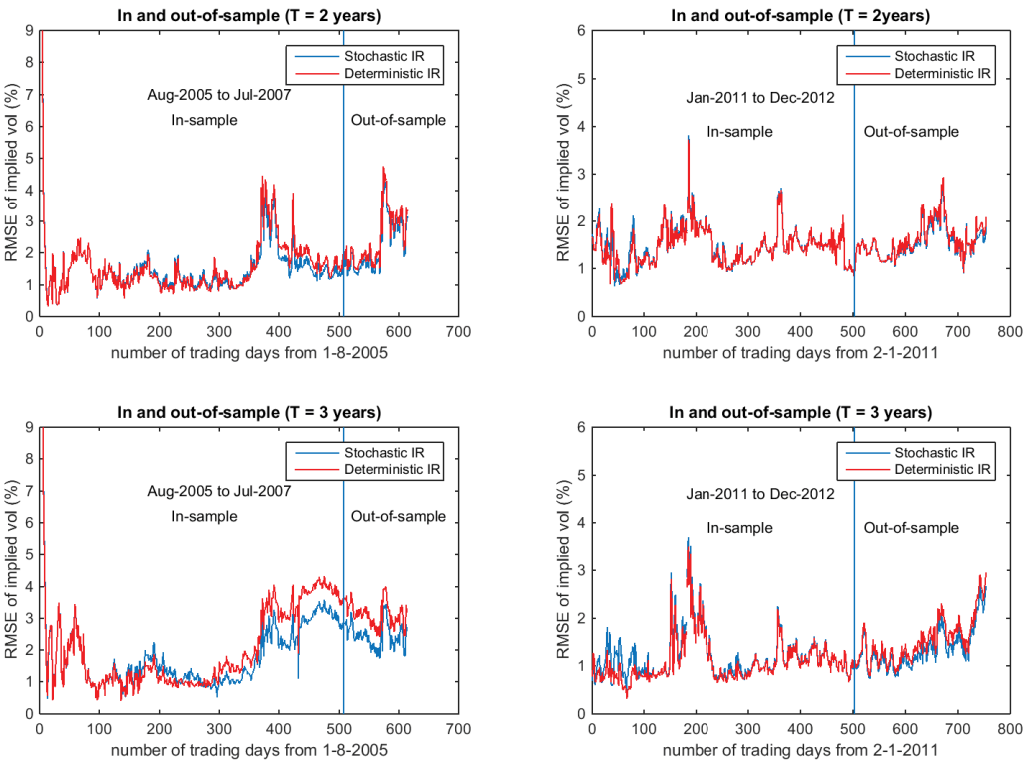

FIGURE 3.5. **RMSE of implied volatility with 2-year and 3-year maturities.** The four graphs show the daily RMSE of the implied volatility (the RMSE between the observed implied volatility and the implied volatility from the estimated model. The two graphs on the left show the RMSE of options fitted to December contracts, which are in the second and third year starting from August 2005. The two graphs on the right have a starting day in January 2011, during the period when the interest-rate volatility was low.

	Period 1: Aug 05 – Jul 07				Period 1: Jan 11 – Dec 12			
	IS		OS		IS		OS	
	Sto	Det	Sto	Det	Sto	Det	Sto	Det
1year	4.24%	4.35%	4.98%	5.21%	5.49%	5.55%	4.81%	4.80%
2year	4.65%	5.04%	4.98%	5.20%	6.55%	6.45%	4.67%	4.81%
3year	4.98%	4.99%	4.19%	4.75%	4.78%	4.58%	3.64%	3.92%
4year	5.63%	5.96%	4.11%	5.07%	3.49%	2.95%	3.55%	3.89%
5year	6.81%	6.79%	4.93%	6.48%	4.07%	3.14%	4.67%	5.08%

TABLE 3.9. **Maximum absolute difference in implied volatilities.** The table displays the maximum absolute difference between the volatility implied by the market crude oil option prices and the model implied volatility (from the deterministic interest rates model and the stochastic interest rates model) observed in the in-sample (IS) period and the out-of-sample (OS) period, over a range of maturities.

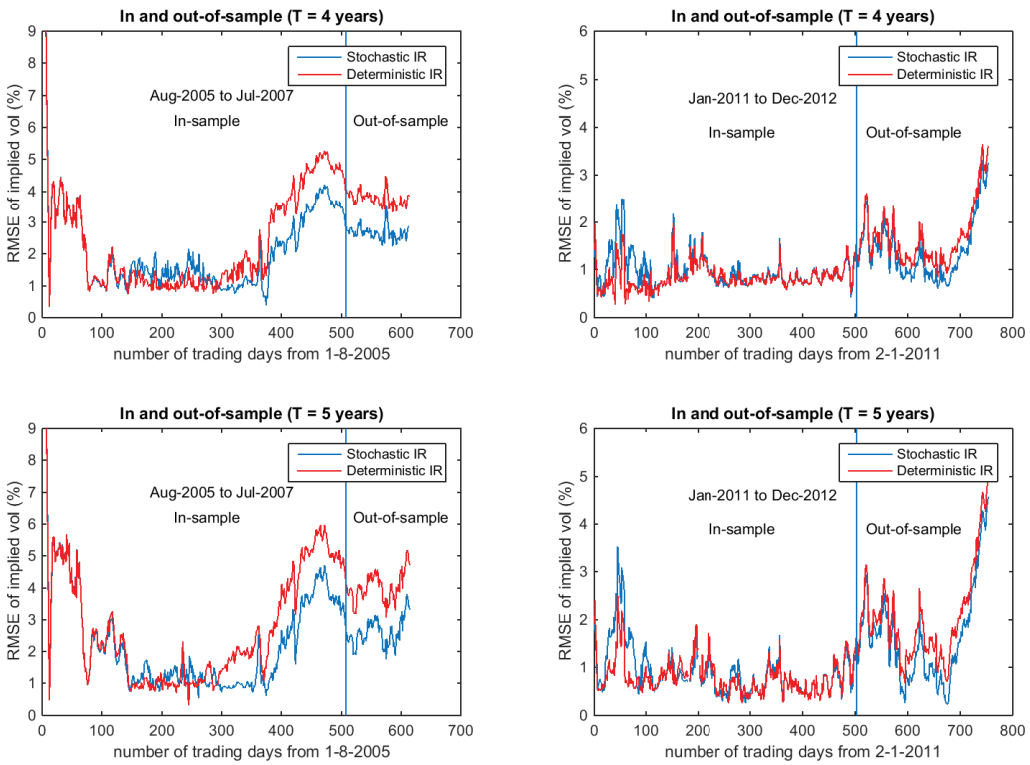

FIGURE 3.6. **RMSE of implied volatility with 4-year and 5-year maturities.** The four graphs show the daily RMSE of the implied volatility (the RMSE between the observed implied volatility and the implied volatility from the estimated model). The two graphs on the left show the RMSE of options fitted to December contracts which are in the fourth and fifth year starting from August 2005. The two graphs on the right have a starting day in January 2011, during the period when the interest-rate volatility was low.

August 2005 and July 2007 and one between January 2011 and December 2012. The first period represents a period with high interest rates whereas the second period represents a period with low interest rates. Empirical results suggest that the correlation of the futures process and the stochastic volatility process is negatively correlated which shows that the futures price volatility is partially unspanned and this result is consistent with the literature see Gruber & Vigfusson (2012). By taking into account the correlation between stochastic interest rates and the futures process, we find that it improves the pricing performance of longer maturity option contracts especially when the interest rates are high. In addition, the interest rates are negatively correlated to the commodity futures prices. Further, by sensitivity analysis we show that the correlation between the interest-rate process and the stochastic volatility process has negligible impact on option pricing. However, a

noticeable difference in the option pricing occurs when the correlation between interest rates and futures prices is relatively high. This difference is more pronounced in a period of high interest rates than in a period of low interest rates.

The pricing performance of modelling stochastic interest rates in addition to stochastic volatility is empirically assessed on long-dated crude oil derivatives. The stochastic volatility/stochastic interest rates, forward price model developed in Chapter 2 is considered as well as the nested version of deterministic interest-rate specifications. The model parameters were estimated from historical time series of crude oil futures prices and option prices over two periods characterised by different interest-rate market conditions; one period with high interest-rate volatility and one period with very low interest-rate volatility.

The empirical analysis leads to the following conclusions. Firstly, stochastic volatility forward price models that incorporate stochastic interest rates generally improve pricing performance on long-dated crude oil derivatives, especially for maturities of two years or more. Secondly, this improvement to the pricing performance is more pronounced when the interest-rate volatility is high. Thirdly, three-dimensional models provide better fit to futures prices of all maturities compared to two-dimensional models. However, for long-dated crude oil option prices, increasing the model dimensionality does not tend to improve the pricing performance. Fourth, there is empirical evidence for a negative correlation between the futures price process and the interest-rate process especially over periods of high interest-rate volatility. This enhances the impact of the inclusion of stochastic interest rates on model performance. Fifth, the correlation between the futures price volatility process and the interest-rate process has negligible impact on the pricing of long-dated crude oil contracts.

These empirical results presented in this chapter may well provide useful insights to practitioners. Increasing the dimension of multi-dimensional forward-price commodity models may improve the model fit to historical market data, but it comes with additional computational effort. Our results show that for long-dated crude oil commodities adding more factors does not improve the pricing performance but a minimum two-dimensional models are required. Another point that would be of interest to practitioners is that stochastic interest rates matter for crude oil derivatives with maturities of two years and above and only if the volatility of interest rates is high. Otherwise using deterministic interest rates will be sufficient. From a historical perspective, interest rates experience high volatility from time to time, for instance, our empirical estimates of the model (3), presented in Table 3.2, reveal a short-interest-rate volatility of 215 basis points between August 2005 and July 2007.[12] While it is quite rare to encounter cases with such a high interest-rate volatility, especially, in current financial conditions, modelers should be mindful of the impact of stochastic interest rates when pricing long-dated commodity derivative contracts.

[12]Under the three-dimensional model assumptions (3), the aggregate volatility across all factors of the short-rate process can be computed by the square root of the sum of the squares of the θ_i, for $i = 1, 2, 3$.

Appendix 3.1 Estimation of N-dimensional affine term structure models

We use Kalman filter to estimate the parameters for the N-dimensional affine term structure models. Then we estimate and compare their corresponding log-likelihoods and RMSEs. The N-dimensional affine term structure models have the form:

$$r(t) = \overline{r} + \sum_{i=1}^{N} y_i(t),$$

$$dy_i(t) = -\lambda_i y_i(t)dt + \theta_i \, dW_i(t), \; i \in 1, 2, \ldots, N$$

$$dW_i \, dW_j = \begin{cases} dt \text{ if } i = j, \\ 0 \text{ otherwise.} \end{cases}$$

Solving the SDE of $y_i(t)$ we get:

$$y_i(t) = y_i(s)e^{-\lambda_i(t-s)} + \theta_i \int_s^t e^{-\lambda_i(t-u)} \, dW_i(u).$$

Redefining the notations $y_t^i \triangleq y_i(t)$ and $y_{t-1}^i \triangleq y_i(t - \Delta t)$ where $\Delta t = t - s$, it can be re-expressed into a state-space form as follow:

$$y_t^i = \Phi_i y_{t-1}^i + \omega_{it}, \; \omega_{it} \sim \text{iid}\, N(0, Q_i),$$

where

$$\Phi_i = e^{-\lambda_i \Delta t},$$

$$Q_i = \frac{\theta_i^2}{2\lambda_i}\big(1 - e^{-2\lambda_i \Delta t}\big) = \frac{\theta_i^2}{2\lambda_i}\big(1 - \Phi_i^2\big).$$

In the real world, the instantaneous risk-free rates do not exist. However, the government bonds with time-to-maturities $\tau = \{\tau_1, \tau_2, \ldots, \tau_n\}$ are observable and are traded frequently in exchanges. They can be used as proxies to estimate the parameters in our

model. We choose the bonds with maturities of one year, two years, three years and five years to be our proxies. Now, let the time-to-maturity of a j-year bond be τ_j then the arbitrage-free bond pricing formula can be expressed as:

$$
\begin{aligned}
B(r_t, T_j) &= \mathbb{E}^{\mathbb{Q}}\big[e^{-\int_t^{t+\tau_j} r(u)\,du}|\mathcal{F}_t\big], \text{with } T_j = t + \tau_j \\
&= e^{-\overline{r}\tau_j} \prod_{i=1}^{N} \mathbb{E}^{\mathbb{Q}}\big[e^{-\int_t^{t+\tau_j} y_i(u)\,du}|\mathcal{F}_t\big], \\
&= e^{-\overline{r}\tau_j} \prod_{i=1}^{N} \exp(-A_i(\tau_j)y_t^i + D_i(\tau_j)), \qquad \text{(A.1.1)} \\
A_i(\tau_j) &= \frac{1 - e^{-\lambda_i \tau_j}}{\lambda_i}, \\
D_i(\tau_j) &= \Big(-\frac{\theta_i^2}{2\lambda_i^2}\Big)\Big(A_i(\tau_j) - \tau_j\Big) - \frac{\theta_i^2 A_i(\tau_j)^2}{4\lambda_i}.
\end{aligned}
$$

The objective now is that given the observable bond prices, Kalman filter is used together with maximum likelihood method to estimate the parameters $\overline{r}$, λ_i and θ_i.

Let the observable bond price at time t with time-to-maturities τ_j be $z(t, T_j)$ where $T_j = t + \tau_j$ and define the logarithm of this bond price to be $Z(t, T_j) = \log z(t, T_j)$. So that the logarithm of bond price is a linear function of the hidden state variable r_t. We further let

$Y(t,T_j) = Z(t,T_j) + \overline{r}\tau_j - \sum_{i=1}^{N} D_i(\tau_j)$. The Kalman filter equation becomes:

$$\begin{bmatrix} y_t^1 \\ y_t^2 \\ \vdots \\ y_t^N \end{bmatrix} = \begin{bmatrix} \Phi_1 & 0 & \dots & 0 \\ 0 & \Phi_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \Phi_N \end{bmatrix} \begin{bmatrix} y_{t-1}^1 \\ y_{t-1}^2 \\ \vdots \\ y_{t-1}^N \end{bmatrix} + \omega_t,\ \omega_t \sim \text{iid}\, N(0,Q),$$

$$\begin{bmatrix} Y(\tau_1) \\ Y(\tau_2) \\ Y(\tau_3) \\ Y(\tau_5) \end{bmatrix} = - \begin{bmatrix} A_1(\tau_1) & A_2(\tau_1) & \dots & A_N(\tau_1) \\ A_1(\tau_2) & A_2(\tau_2) & \dots & A_N(\tau_2) \\ A_1(\tau_3) & A_2(\tau_3) & \dots & A_N(\tau_3) \\ A_1(\tau_5) & A_2(\tau_5) & \dots & A_N(\tau_5) \end{bmatrix} \begin{bmatrix} y_t^1 \\ y_t^2 \\ \vdots \\ y_t^N \end{bmatrix} + u_t,$$

$$u_t \sim \text{iid}\, N(0,\Omega),$$

where

$$Q = \begin{bmatrix} Q_1 & 0 & \dots & 0 \\ 0 & Q_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & Q_N \end{bmatrix}$$

$$\Omega = \begin{bmatrix} \omega^2 & 0 & \dots & 0 \\ 0 & \omega^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \omega^2 \end{bmatrix}.$$

The above Kalman filter equation can be re-expressed in matrix notation as:

$$y_t = \Phi y_{t-1} + \omega_t,\ \omega_t \sim \text{iid}\, N(0,Q),$$

$$Y(\tau) = -Ay_t + u_t, u_t \sim \text{iid}\, N(0,\Omega).$$

To apply the maximum likelihood method, we would need to find a set of parameters which maximises the log-likelihood function. Let the log-likelihood function be a function of $\xi \triangleq \{\overline{r}, \lambda_1, \ldots, \lambda_N, \theta_1, \ldots, \theta_N, \Omega\}$, ie:

$$\log L = \log L(\xi).$$

Now, given a set of parameters ξ we calculate its corresponding likelihood $\log L$ as follows:

Initialisation:

$$\hat{y}_{0|0} = \mathbb{E}(y_0) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$P_{0|0} = \mathrm{var}(y_0) = \mathbf{0}_{N\times N},$$

The Kalman filter yields,

$$\hat{y}_{t|t-1} = \Phi \hat{y}_{t-1|t-1},$$

$$P_{t|t-1} = \Phi P_{t-1|t-1} \Phi' + Q,$$

and

$$\hat{y}_{t|t} = \hat{y}_{t|t-1} + P_{t|t-1}(-A)' F_t^{-1} \varepsilon_t,$$

$$P_{t|t} = P_{t|t-1} - P_{t|t-1}(-A)' F_t^{-1}(-A)\, P_{t|t-1},$$

where

$$\varepsilon_t = Y(t,T) + A(\tau) y_t$$

$$F_t = -A(\tau)\, P_{t|t-1}(-A(\tau))' + \Omega.$$

The log-likelihood function $\log L(\xi)$ is constructed as:

$$\log L = -\frac{1}{2}N\log(2\pi) - \frac{1}{2}\sum_{t=1}^{T}\log|F_t| - \frac{1}{2}\sum_{t=1}^{T}\epsilon_t' F_t^{-1}\epsilon_t.$$

Because the $-\frac{1}{2}N\log(2\pi)$ part is a constant for any choice of ξ, it, therefore, has no contribution to the maximisation problem, and can be dropped for estimation proposes.

Appendix 3.2 The system equation

From equations (3.2.9) we get the dynamics of the model as,

$$\begin{aligned}\log F(t,T,\sigma_t) &= \log F(0,T)\\ &\quad - \frac{1}{2}\sum_{i=1}^{n}\Big(\gamma_{1i}(T-t)x_i(t) + \gamma_{2i}(T-t)y_i(t) + \gamma_{3i}(T-t)z_i(t)\Big)\\ &\qquad + \sum_{i=1}^{n}\Big(\beta_{1i}(T-t)\phi_i(t) + \beta_{2i}(T-t)\psi_i(t)\Big),\end{aligned}$$

where the deterministic functions are defined as:

$$\begin{aligned}\beta_{1i}(T-t) &= \big(\xi_{i0} + \xi_i(T-t)\big)e^{-\eta_i(T-t)},\\ \beta_{2i}(T-t) &= \xi_i e^{-\eta_i(T-t)},\\ \gamma_{1i}(T-t) &= \beta_{1i}(T-t)^2,\\ \gamma_{2i}(T-t) &= 2\beta_{1i}(T-t)\beta_{2i}(T-t),\\ \gamma_{3i}(T-t) &= \beta_{2i}(T-t)^2,\end{aligned}$$

and the state variables satisfy the following SDE under the risk-neutral measure:

$$
\begin{aligned}
dx_i(t) &= \big(-2\eta_i x_i(t) + \nu_i(t)\big)\, dt, \\
dy_i(t) &= \big(-2\eta_i y_i(t) + x_i(t)\big)\, dt, \\
dz_i(t) &= \big(-2\eta_i z_i(t) + 2y_i(t)\big)\, dt, \\
d\phi_i(t) &= -\eta_i \phi_i(t) dt + \sqrt{\nu_i(t)} dW_i^x(t), \\
d\psi_i(t) &= \big(-\eta_i \psi_i(t) + \phi_i(t)\big) dt, \\
d\sigma_i(t) &= \kappa_i(\overline{\sigma}_i - \sigma_i(t)) dt + \gamma_i dW_i^\sigma(t), \\
d\nu_i(t) &= (-2\nu_i(t)\kappa_i + 2\kappa_i \overline{\sigma}_i \sigma_i(t) + \gamma_i^2) dt + 2\gamma_i \sqrt{\nu_i(t)} dW_i^\sigma(t).
\end{aligned}
$$

However, in the physical world we need to account for the market price of risk and the market price of volatility risk namely Λ_i and Λ_i^σ. They are specified as:

$$
\begin{aligned}
d\mathbb{W}_i(t) &= dW_i(t) - \Lambda_i \sqrt{\nu_i(t)}\, dt, \\
d\mathbb{W}_i^\sigma(t) &= dW_i^\sigma(t) - \Lambda_i^\sigma \sqrt{\nu_i(t)}\, dt,
\end{aligned}
$$

where, $\mathbb{W}_i(t)$ and $\mathbb{W}_i^\sigma(t)$ are Brownian motions under the physical measure. The dynamics of the state variables under the physical measure is:

$$
\begin{aligned}
dx_i(t) &= \big(-2\eta_i x_i(t) + \nu_i(t)\big)\,dt,\\
dy_i(t) &= \big(-2\eta_i y_i(t) + x_i(t)\big)\,dt,\\
dz_i(t) &= \big(-2\eta_i z_i(t) + 2y_i(t)\big)\,dt,\\
d\phi_i(t) &= \big(-\eta_i\phi_i(t) + \Lambda_i\nu_i(t)\big)dt + \sqrt{\nu_i(t)}d\mathbb{W}_i(t),\\
d\psi_i(t) &= \big(-\eta_i\psi_i(t) + \phi_i(t)\big)dt,\\
d\sigma_i(t) &= \big(\kappa_i\overline{\sigma}_i - (\kappa_i - \gamma_i\Lambda_i^\sigma)\sigma_i(t)\big)dt + \gamma_i\, d\mathbb{W}_i^\sigma(t),\\
d\nu_i(t) &= \big((2\gamma_i\Lambda_i^\sigma - 2\kappa_i)\nu_i(t) + 2\kappa_i\overline{\sigma}_i\sigma_i(t) + \gamma_i^2\big)dt + 2\gamma_i\sqrt{\nu_i(t)}d\mathbb{W}_i^\sigma(t).
\end{aligned}
$$

This equation can be more succinctly written in matrix form as:

$$
dX_t^i = (\Psi_i - \mathcal{K}_i X_t^i)dt + \Sigma_i d\overline{\mathbb{W}}_i(t)
$$

where $X_t^i = (x_i(t), y_i(t), z_i(t), \phi_i(t), \psi_i(t), \sigma_i(t), \nu_i(t))'$, $\overline{\mathbb{W}}_i(t) = (\mathbb{W}_i(t), \mathbb{W}_i^\sigma(t))'$ and

$$
\Psi_i = \begin{pmatrix} 0\\ 0\\ 0\\ 0\\ 0\\ \kappa_i\overline{\sigma}_i\\ \gamma_i^2 \end{pmatrix}
\quad
\mathcal{K}_i = \begin{pmatrix}
2\eta_i & 0 & 0 & 0 & 0 & 0 & -1\\
-1 & 2\eta_i & 0 & 0 & 0 & 0 & 0\\
0 & -2 & 2\eta_i & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \eta_i & 0 & 0 & -\Lambda_i\\
0 & 0 & 0 & -1 & \eta_i & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \kappa_i - \gamma_i\Lambda_i^\sigma & 0\\
0 & 0 & 0 & 0 & 0 & -2\kappa_i\overline{\sigma}_i & 2\kappa_i - 2\gamma_i\Lambda_i^\sigma
\end{pmatrix},
$$

$$\Sigma_i = (\Sigma_i^1 + \sqrt{\nu_t^i}\Sigma_i^2) \cdot R_i^{\frac{1}{2}},\ \Sigma_i^1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \gamma_i \\ 0 & 0 \end{pmatrix},\ \Sigma_i^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 2\gamma_i \end{pmatrix}$$

where $R_i^{\frac{1}{2}}$ is the 2×2 Cholesky decomposition of the correlation matrix for the Wiener Processese:

$$R_i^{\frac{1}{2}} = \begin{pmatrix} 1 & 0 \\ \rho_i^{x\sigma} & \sqrt{1-(\rho_i^{x\sigma})^2} \end{pmatrix}.$$

Note that $R_i^{\frac{1}{2}}$ is just a notation and it means:

$$R_i \triangleq R_i^{\frac{1}{2}}(R_i^{\frac{1}{2}})' = \begin{pmatrix} 1 & \rho_i^{x\sigma} \\ \rho_i^{x\sigma} & 1 \end{pmatrix}.$$

Applying the stochastic chain rule to $e^{\mathcal{K}_i t}X_t^i$, we have

$$\begin{aligned} d\big(e^{\mathcal{K}_i t}X_t^i\big) &= e^{\mathcal{K}_i t}\mathcal{K}_i X_t^i dt + e^{\mathcal{K}_i t} dX_t^i \\ &= e^{\mathcal{K}_i t}\Psi_i dt + e^{\mathcal{K}_i t}\Sigma_i d\overline{\mathbb{W}}_i(t). \end{aligned}$$

It follows that $X_s^i, s > t$ is given by

$$X_s^i = e^{-\mathcal{K}_i(s-t)}X_t^i + \int_t^s e^{-\mathcal{K}_i(s-u)}\Psi_i\, du + \int_t^s e^{-\mathcal{K}_i(s-u)}\Sigma_i\, d\overline{\mathbb{W}}_i(u). \tag{A.2.1}$$

When we discretise equation (A.2.1) by letting $t \in \{0, \Delta t, 2\Delta t, \ldots\}$ we get:

$$X_{t+1}^i = \underbrace{e^{-\mathcal{K}_i \Delta t}}_{=\Phi_X^i} X_t^i + \underbrace{\int_t^{t+\Delta t} e^{-\mathcal{K}_i(t+\Delta t-u)}\Psi_i\, du}_{=\Phi_0^i} + \underbrace{\int_t^{t+\Delta t} e^{-\mathcal{K}_i(t+\Delta t-u)}\Sigma_i\, d\overline{\mathbb{W}}_i(u)}_{\omega_{t+1}^i}.$$

The mean of X_s^i conditional on the information at time t is given by:

$$\mathbb{E}_t\big[X_s^i\big] = e^{-\mathcal{K}_i(s-t)}X_t^i + \int_t^s e^{-\mathcal{K}_i(s-u)}\Psi_i\,du,$$

and covariance matrix of X_s^i conditional on the information at time t is given by:

$$\begin{aligned}
\mathrm{cov}_t\left(X_s^i,(X_s^i)'\right) =& \mathbb{E}_t\Big[\Big(\int_t^s e^{-\mathcal{K}_i(s-u)}\Sigma_i\,d\overline{\mathbb{W}}_i(u)\Big)\Big(\int_t^s e^{-\mathcal{K}_i(s-u)}\Sigma_i\,d\overline{\mathbb{W}}_i(u)\Big)'\Big]\\
=& \int_t^s e^{-\mathcal{K}_i(s-u)}\Sigma_i^1 R_i(\Sigma_i^1)'(e^{-\mathcal{K}_i(s-u)})'du\\
&+\int_t^s \mathbb{E}_t(\sigma^i(u))e^{-\mathcal{K}_i(s-u)}\Sigma_i^1 R_i(\Sigma_i^2)'(e^{-\mathcal{K}_i(s-u)})'du\\
&+\int_t^s \mathbb{E}_t(\sigma^i(u))e^{-\mathcal{K}_i(s-u)}\Sigma_i^2 R_i(\Sigma_i^1)'(e^{-\mathcal{K}_i(s-u)})'du\\
&+\int_t^s \mathbb{E}_t(\nu^i(u))e^{-\mathcal{K}_i(s-u)}\Sigma_i^2 R_i(\Sigma_i^2)'(e^{-\mathcal{K}_i(s-u)})'du.
\end{aligned}$$

We know that the conditional distribution of $\sigma_i(u)$ given information at time t is normally distributed with mean and variance as follow:

$$\begin{aligned}
\mathbb{E}_t\big(\sigma_i(u)\big) &= \sigma_i(t)e^{-\kappa_i(u-t)} + \overline{\sigma}(1-e^{-\kappa_i(u-t)}),\\
\mathrm{var}_t\left(\sigma_i(u)\right) &= \frac{\gamma_i^2}{2\kappa_i}(1-e^{-2\kappa_i(u-t)}).
\end{aligned}$$

So,

$$\begin{aligned}
\mathbb{E}_t\big(\nu^i(u)\big) &= \mathbb{E}_t\big(\sigma_i^2(u)\big)\\
&= \mathrm{var}_t\left(\sigma_i(u)\right) + \Big(\mathbb{E}_t\big(\sigma_i(u)\big)\Big)^2.
\end{aligned}$$

Putting the three dimensions together, we obtain:

$$X_t = \begin{pmatrix} X_t^1\\ X_t^2\\ X_t^3\end{pmatrix}, \overline{\mathbb{W}}(t) = \begin{pmatrix} \overline{\mathbb{W}}_1(t)\\ \overline{\mathbb{W}}_2(t)\\ \overline{\mathbb{W}}_3(t)\end{pmatrix},$$

$$\Psi = \begin{pmatrix} \Psi^1 \\ \Psi^2 \\ \Psi^3 \end{pmatrix}, \mathcal{K} = \begin{pmatrix} \mathcal{K}_1 & 0 & 0 \\ 0 & \mathcal{K}_2 & 0 \\ 0 & 0 & \mathcal{K}_3 \end{pmatrix}, \Sigma = \begin{pmatrix} \Sigma_1 & 0 & 0 \\ 0 & \Sigma_2 & 0 \\ 0 & 0 & \Sigma_3 \end{pmatrix},$$

$$\mathrm{cov}_t(X_s, (X_s)') = \begin{pmatrix} \mathrm{cov}_t(X_s^1, (X_s^1)') & 0 & 0 \\ 0 & \mathrm{cov}_t(X_s^2, (X_s^2)') & 0 \\ 0 & 0 & \mathrm{cov}_t(X_s^3, (X_s^3)') \end{pmatrix}.$$

The system equation, therefore can be written in discrete form as:

$$X_t = \Phi_0 + \Phi_X X_{t-1} + \omega_t,\ \omega_t \sim\ N(0, Q_t), \tag{A.2.2}$$

$$(\omega_t \text{ and } \omega_\tau \text{ are independent for } t \neq \tau) \tag{A.2.3}$$

where

$$\Phi_0 = \int_t^{t+\Delta t} e^{-\mathcal{K}(t+\Delta t-u)} \Psi\, du,$$
$$\Phi_X = e^{-\mathcal{K}\Delta t}.$$

Appendix 3.3 The extended Kalman filter and the maximum likelihood method

There are two sets of equations in this model. One is the system equation as shown in equation (A.2.2). This equation describes the evolution of the unobservable states

$$X_t = (X_t^1, X_t^2, \ldots)',$$

where $X_t^i = (x_i(t), y_i(t), z_i(t), \phi_i(t), \psi_i(t), \sigma_i(t), \nu_i(t))'$. The other equation is the observation equation that links the unobservable state variables with the market-observable variables (i.e. futures and option prices) and is of the form:

$$Z_t = h(X_t, u_t),\ u_t \sim \text{i.i.d.}\, N(\mathbf{1}, \Omega), \tag{A.3.1}$$

here we assume that u_t and u_τ are independent for $u \neq \tau$. We model the noise u_t to be multiplicative with the cross-correlation of the noise to be zero (i.e. Ω is a diagonal matrix). For estimation purposes, we assume that $\Omega = \begin{bmatrix} \Omega_f & 0 \\ 0 & \Omega_O \end{bmatrix}$ where Ω_f is a diagonal matrix with constant noise σ_f and Ω_O with σ_O. σ_f and σ_O are the measurement noise and they are to be estimated from Kalman filter. Note that for options, the function h is not a linear function, see equation (3.2.6). It is however a linear function for futures, see equation (3.2.9).

Let the number of futures observed in the market at time t be m, i.e. $F_{t,1}, \ldots, F_{t,m}$. Let the number of options on futures observed in the market at time t be n, i.e. $O_{t,1}, \ldots, O_{t,n}$ then we have:

$$Z_t = (\log F_{t,1}, \ldots, \log F_{t,m}, O_{t,1}, \ldots, O_{t,n})'.$$

Let $\hat{X}_t = \mathbb{E}_t[X_t]$ and $\hat{X}_{t|t-1} = \mathbb{E}_{t-1}[X_t]$ denote the expectation of X_t given information at time t and time $t-1$ respectively and let P_t and $P_{t|t-1}$ denote the corresponding estimation error covariance matrices. That is

$$\mathrm{var}(X_t - \hat{X}_{t|t-1}) = P_{t|t-1},$$

$$\mathrm{var}(X_t - \hat{X}_t) = P_t.$$

We linearise the h-function around $\hat{X}_{t|t-1}$ by Taylor series expansion and we ignore higher order terms:

$$h(X_t) = h(\hat{X}_{t|t-1}) + H(\hat{X}_{t|t-1})(X_t - \hat{X}_{t|t-1}) + \text{Higher Order Terms}.$$

The function H is the Jacobian of $h(\cdot)$, and it is given by

$$H \triangleq \begin{pmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \vdots & & \ddots & \vdots \\ \frac{\partial h_n}{\partial x_1} & \frac{\partial h_n}{\partial x_2} & \cdots & \frac{\partial h_n}{\partial x_n} \end{pmatrix},$$

where $h(X) = (h_1(X), h_2(X), \ldots, h_n(X))'$ and $X = (X_1, X_2, \ldots, X_n)'$. For simplicity, we shall let H_t to represent $H(\hat{X}_{t|t-1})$. The Kalman filter yields

$$\hat{X}_{t|t-1} = \Phi_0 + \Phi_X \hat{X}_{t-1},$$

$$P_{t|t-1} = \Phi_X P_{t-1} \Phi_X' + Q_t,$$

and

$$\hat{X}_t = \hat{X}_{t|t-1} + P_{t|t-1} H_t' F_t^{-1} \epsilon_t,$$

$$P_t = P_{t|t-1} - P_{t|t-1} H_t' F_t^{-1} H_t P_{t|t-1},$$

where

$$\epsilon_t = Z_t - h(\hat{X}_{t|t-1}),$$

$$F_t = H_t P_{t|t-1} H_t' + D_k \Omega D_k',$$

where

$$D_k \triangleq \frac{\partial h(X, u)}{\partial u}|_u.$$

The log-likelihood function is constructed as:

$$\log L = -\frac{1}{2}\log(2\pi)\sum_{t=1}^{T} N_t - \frac{1}{2}\sum_{t=1}^{T}\log|F_t| - \frac{1}{2}\sum_{t=1}^{T}\epsilon_t' F_t^{-1}\epsilon_t,$$

where N_t here represents the sum of the number of future and option contracts of day t.

The log-futures price dynamics that we develop earlier in equation (3.2.9) is time-inhomogeneous and fits the initial futures curve by construction. For the purpose of estimation, it is more convenient to work with a model that the initial futures curve is time-homogeneous. We achieve this by assuming that the initial futures curve $F(0, T) = f_0$ for all time T, where f_0 is a constant representing the long-term futures price (futures with infinite maturity) which is then estimated in the Kalman filter. In the estimation process, the long-run mean of the volatility process $\overline{\sigma}_i$ is normalised to one to achieve identification [see Dai & Singleton (2000)].

CHAPTER 4

Hedging futures options with stochastic interest rates

This chapter presents a simulation study of hedging long-dated futures options, in the Rabinovitch (1989) model which assumes correlated dynamics between spot asset prices and interest rates. To investigate the hedging performance of the proposed model on long-dated futures options, several hedging schemes are considered including delta hedging and interest-rate hedging. The impact of the model parameters, such as the interest-rate volatility, the long-term level of the interest rates and the correlation, to the hedging performance is also investigated. Finally, the effect of hedging long-dated futures options with shorter maturity derivatives is examined. This chapter is based on the working paper of Cheng, Nikitopoulos & Schlögl (2016*b*).

4.1. Introduction

Typically, the sensitivity of an option's price with respect to the interest rate, that is the partial derivatives of the option price with respect to the interest rate, is an increasing function with time-to-maturity. Thus, interest-rate risk should be more relevant to derivatives with longer maturities and models with stochastic interest rates tend to improve pricing and hedging performance on long-dated contracts [see Bakshi et al. (2000) for supporting evidence in equity markets]. A representation literature on spot option pricing models with stochastic interest rates includes Rabinovitch (1989), Amin & Jarrow (1992) and Kim & Kunitomo (1999). Scott (1997) demonstrates that stochastic volatility and stochastic interest rates have a significant impact on stock option prices, particularly on the prices of long-dated option contracts. Schwartz (1997), Brennan & Crew (1997), Bühler et al. (2004) and Shiraya & Takahashi (2012) have considered hedging with futures contracts of long-term commodity commitments in spot or forward markets, yet most of these

models do not assume stochastic interest rates. Very limited literature has addressed the hedging of long-dated futures options, under a stochastic interest-rate economy.

This chapter aims to gauge the impact of interest-rate risk in futures option positions and examine the hedging of this risk in a simulated environment. In order to isolate the impact of stochastic volatility and its correlation to pricing and hedging derivatives, we pursue a more tractable and simple model with only market and interest-rate risks. We employ the two-factor Rabinovitch (1989) model to price options on futures and compute their corresponding hedge ratios. Under this model, the spot asset price process follows a geometric Brownian motion, the stochastic interest-rate process is modelled by an Ornstein-Uhlenbeck process and their dynamics are correlated. The model is one of the most tractable spot price models accommodating stochastic volatility and leads to a modified Black & Scholes (1973) option pricing formula.

The hedging of futures option positions is discussed where both futures and forward contracts are considered as hedging instruments. Under the assumption of the Rabinovitch (1989) model, we proof mathematically that forward contracts can hedge both the underlying market risk and interest-rate risk simultaneously, as long as the maturities of the forward contracts coincide with the maturity of the futures option. Furthermore, with a suitable convexity adjustment, futures contracts with the same maturity as the option, can also hedge both the market risk and the interest-rate risk of the futures option positions. To gauge the contribution of the stochastic interest-rate specifications to hedging long-dated option positions, we consider several hedging strategies including the hedge ratios derived from the deterministic interest-rate Black (1976) model and the stochastic interest-rate two-factor Rabinovitch (1989) model. We also introduce the factor hedging approach [see Clewlow & Strickland (2000) and Chiarella et al. (2013)] which is well suited for hedging with more general multi-dimensional models, and we validate the numerical efficiency of the approach.

A Monte-Carlo simulation approach is employed to numerically investigate the hedging performance of long-dated futures options for a variety of hedging schemes such as delta hedging and interest-rate hedging. The stock price process and the interest-rate process are generated using the Euler scheme under the historical measure with the market price of risk and the market price of interest-rate risk introduced to the drift terms of the processes in the Rabinovitch (1989) model. The contribution of the discretisation error in the proposed hedging schemes is evaluated, as well as the impact of the model parameters such as the interest-rate volatility, the long-term mean of interest rates and the correlation between the spot price process and the interest process. We find that when interest-rate volatility is high, the models with stochastic interest rates significantly improve the hedging performance of futures options positions compared to the models with deterministic interest-rate specifications. This improvement becomes more pronounced for options with longer maturities.

We finally examine the effect of hedging long-dated options with isntruments of shorter maturities. Within the stochastic interest-rate model, we numerically validate using forward contracts as hedging instruments with the hedge ratio calculated according to the Black (1976) model (using the volatility of the forward), with a balance in the continuously compounded savings account. We find that this can replicate the forward price of the option in the limit. However, when using short-dated contracts to hedge options with longer maturities, forward or futures contracts alone can no longer hedge the interest-rate risk. Adding bond contracts to the hedging portfolio is necessary in order to mitigate the interest-rate risk.

The remainder of the chapter is structured as follows. Section 4.2 presents the Rabinovitch (1989) spot price model featuring stochastic interest rates and gives pricing equations for forwards, futures and futures options. Section 4.3 describes the hedging methodology including a variety of hedge ratios and hedge schemes such as delta hedge and interest-rate hedge. Numerical investigations to assess the contribution of stochastic interest rates

to hedging long-dated futures options as well as reflections on the results are presented in Section 4.4. Section 4.5 concludes.

4.2. Model description

We consider a filtered probability space $(\Omega, \mathcal{F}_T, \mathbb{F}, \mathbb{P}), T \in [0, \infty)$ satisfying the usual conditions.[1] Here Ω is the state space, $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ is a set of σ-algebras representing measurable events and $\mathbb{P}$ is the historical (real-world) probability measure. The Rabinovitch (1989) model with spot asset price process $S(t)$ and correlated stochastic interest-rate process $r(t)$ is specified by the following dynamics under the historical measure $\mathbb{P}$:

$$dS(t) = (r(t) + \varpi_1\sigma)S(t)dt + \sigma S(t)dW_1^{\mathbb{P}}(t), \tag{4.2.1}$$

$$dr(t) = \big(\lambda(\bar{r} - r(t)) + \varpi_2\theta\big)dt + \theta dW_2^{\mathbb{P}}(t), \tag{4.2.2}$$

$$\rho dt = dW_1^{\mathbb{P}}(t)dW_2^{\mathbb{P}}(t).$$

Each parameter of the set $\Psi = \{\sigma, \lambda, \bar{r}, \theta, \rho, \Lambda_1 \text{ and } \Lambda_2\}$ is a constant and the initial state variables are $S(0) = S_0$ and $r(0) = r_0$. The market price of spot asset price risk and interest-rate risk are ϖ_1 and ϖ_2, respectively. The long-term level of the interest-rate process and the rate of reversion to the long-term level of the interest-rate process are $\bar{r}$ and λ, respectively. The volatilities of the spot asset price process and the interest-rate process are σ and θ, respectively. Thus, under the spot risk-neutral measure $\mathbb{Q}$, the dynamics of the spot asset price are expressed as:

$$dS(t) = r(t)S(t)dt + \sigma S(t)dW_1(t), \tag{4.2.3}$$

$$dr(t) = \lambda\big(\bar{r} - r(t)\big)dt + \theta dW_2(t), \tag{4.2.4}$$

$$\rho dt = dW_1(t)dW_2(t).$$

[1]The usual conditions satisfied by a filtered complete probability space are: (a) $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets of $\mathcal{F}$ and (b) the filtration is right continuous.

Under these model specifications, derivatives on this underlying asset are priced next, including futures, forwards, options on the spot and options on futures.

4.2.1. Futures price. At time t, the term structure of futures prices $\{F(t,T)\}_{T\in[t,T^*]}$ (where T^* is the maximum maturity under consideration) is determined by the spot risk-neutral expectation of the future spot asset prices, [see Cox, Ingersoll & Ross (1981)]:

$$\begin{aligned} F(t,T) &= \mathbb{E}^{\mathbb{Q}}\big[S(T)|\mathcal{F}_t\big] && (4.2.5)\\ &= S(t)\exp\big(-\frac{1}{2}\sigma^2(T-t)\big)\exp\Big(M+\frac{1}{2}V^2\Big) \\ &= S(t)\exp\Big(M+\frac{1}{2}V_1^2+V_3^2\Big), && (4.2.6)\end{aligned}$$

where

$$M \triangleq (r(t)-\bar{r})A(t,T,\lambda)+\bar{r}(T-t), \tag{4.2.7}$$

$$V^2 \triangleq V_1^2+V_2^2+2V_3^2, \tag{4.2.8}$$

with

$$\begin{aligned} V_1^2 &= \frac{\theta^2}{\lambda^2}\Big((T-t)-2A(t,T,\lambda)+A(t,T,2\lambda)\Big), && (4.2.9)\\ V_2^2 &= \sigma^2(T-t), \\ V_3^2 &= \frac{\rho\theta\sigma}{\lambda}\big((T-t)-A(t,T,\lambda)\big), \\ A(t,T,\lambda) &= \frac{1}{\lambda}\big(1-e^{-\lambda(T-t)}\big). \end{aligned}$$

V_1^2 denotes the variance of the stochastic interest rates, accumulated from time t to T, which is equal to $\mathrm{var}^{\mathbb{Q}}\big[\int_t^T r(u)\,du|\mathcal{F}_t\big]$. V_2^2 denotes the variance of the logarithm of the spot asset price process contributed by just the market risk $W_1^{\mathbb{P}}(t)$, accumulated from time t to T, which is equal to $\mathrm{var}^{\mathbb{Q}}\big[\sigma W_1^{\mathbb{P}}(T)|\mathcal{F}_t\big]$. V_3^2 denotes the cross variance, accumulated from time t to T, which is equal to $\frac{\rho\theta\sigma}{\lambda}\big((T-t)-A(t,T,\lambda)\big)$. The details of the derivation of the expectation of the future spot asset price (4.2.5) given the dynamics (4.2.1) and (4.2.2) can be found in Appendix 4.1.

4.2.2. Forward price. The price at time t of a zero-coupon bond maturing at time T is denoted by $B(t, T, r(t))$. If the interest rate $r(t)$ follows the dynamics (4.2.4), then the bond price $B(t, T, r(t))$ is expressed as

$$B(t, T, r(t)) = \exp\big(- M + \frac{1}{2}V_1^2\big), \tag{4.2.10}$$

where M and V_1^2 are shown in equations (4.2.7) and (4.2.9). At time t, the term structure of forward prices $\{\text{For}(t, T)\}_{T\in[t,T^*]}$ is determined by the $\mathbb{T}$-forward expectation of the future spot asset prices:[2]

$$\text{For}(t, T) = \mathbb{E}^{\mathbb{T}}\big[S(T)|\mathcal{F}_t\big] \tag{4.2.11}$$

$$= \frac{S(t)}{B(t, T, r(t))} = S(t)\exp\big(M - \frac{1}{2}V_1^2\big). \tag{4.2.12}$$

Equations (4.2.6) and (4.2.12) reveal that the difference between the futures and forward prices is affected by the interest-rate volatility, and this is true even when the instantaneous correlation between the spot and the interest-rate process is zero. Evidently, futures prices and forward prices do not depend on the variance of the spot asset price process V_2^2. If the spot asset price process is uncorrelated to the interest-rate process (namely $\rho = 0$), there is still a convexity adjustment for futures prices which depends on the variance of the stochastic interest rates. Thus, forward prices are different to the futures prices even when this correlation is zero. From (4.2.6) and (4.2.12), we consequently obtain the associated convexity adjustment (which is critical to hedging applications) as

$$\frac{\text{For}(t, T)}{F(t, T)} = \exp\big(- V_1^2 - V_3^2\big), \tag{4.2.13}$$

that reduces to $\exp\big(- V_1^2\big)$ for zero correlation. However, under deterministic interest-rate specifications, futures prices and forward prices would be the same. Thus, under the assumption of stochastic interest rates, the forward price and futures price are different. For a rigorous proof, see Appendix 4.2.

[2]See Lemma 9.6.2 and equation (9.30) on page 341 in Musiela & Rutkowski (2006).

4.2.3. Option price. We consider next futures options with the underlying futures contract maturing at the same time T as the maturity of the futures option. At maturity, the futures price converges to the spot asset price, that is, at time T, $F(T,T) = S(T)$. If $C_f(F(t,T), r(t), T-t; K)$ and $C(S(t), r(t), T-t; K)$ denote the prices of call futures option and the call spot option, respectively, then the price of a European call futures option is the expectation of the discounted future payoff at maturity under the spot risk-neutral measure which reduces to the price of a European call spot option:

$$\begin{aligned} C_f(F(t,T), r(t), T-t; K) &= \mathbb{E}^{\mathbb{Q}}\Big[e^{-\int_t^T r(u)\,du}\big(F(T,T)-K\big)^+|\mathcal{F}_t\Big] \\ &= \mathbb{E}^{\mathbb{Q}}\Big[e^{-\int_t^T r(u)\,du}\big(S(T)-K\big)^+|\mathcal{F}_t\Big] \\ &= C(S(t), r(t), T-t; K), \end{aligned}$$

Similar arguments apply to the price of a put futures option $P_f(F(t,T), r(t), T-t; K)$ and the price of a put spot option $P(F(t,T), r(t), T-t; K)$:

$$\begin{aligned} P_f(F(t,T), r(t), T-t; K) &= \mathbb{E}^{\mathbb{Q}}\Big[e^{-\int_t^T r(u)\,du}\big(K-F(T,T)\big)^+|\mathcal{F}_t\Big] \\ &= \mathbb{E}^{\mathbb{Q}}\Big[e^{-\int_t^T r(u)\,du}\big(K-S(T)\big)^+|\mathcal{F}_t\Big] \\ &= P(S(t), r(t), T-t; K). \end{aligned}$$

Since both the dynamics of $e^{-\int_t^T r(u)\,du}|r(t)$ and $S(T)|S(t)$ are log-normally distributed and the product of two log-normally distributed random variables is also log-normally distributed, then Black-Scholes-like European option pricing formula can be applied with some modifications. Its derivation can be found in Rabinovitch (1989). When $S(t)$ follows the dynamics (4.2.3) and (4.2.4), then the price at time t of a European call option with payoff $\big(S(T)-K\big)^+$ at time T is:[3]

$$C(S(t), r(t), T-t; K) = S(t)N(d_1) - KB(t,T,r(t))N(d_2), \tag{4.2.14}$$

[3]Equation (5) of Rabinovitch (1989) should be $\delta(\tau) = -\nu B(\tau)$ which leads to $+2\rho\sigma(\tau - B)\nu/q$ in equation (8) of Rabinovitch (1989).

and the price of the corresponding put is

$$P(S(t), r(t), T-t; K) = KB(t, T, r(t))N(-d_2) - S(t)N(-d_1), \tag{4.2.15}$$

where

$$d_1 = \left(\log\left(\frac{S(t)}{KB(t,T,r(t))}\right) + \frac{V^2}{2}\right)\Big/ V\ , \tag{4.2.16}$$

$$d_2 = d_1 - V, \tag{4.2.17}$$

where V^2 is defined in (4.2.8) and $N(x)$ denotes the standard normal cumulative distribution function.

We discuss next the hedging of risks associated with positions in futures options under the proposed model by employing a variety of hedge ratios and hedge schemes.

4.3. Hedging delta and interest-rate risk

We aim to hedge the risk of a position in futures options arising from changes in the underlying asset, known as delta, as well as from changes in the interest rates. Typically, delta hedges require positions in the underlying asset (futures contracts or forward contracts) while hedging the interest-rate risk would require positions in interest-rate-sensitive assets, such as bonds. Under the model specification of Section 4.2, the following proposition demonstrates that a position in a forward contract can hedge simultaneously both delta and interest-rate risk of the forward price of an option. Note that the forward price of the option is computed by

$$\frac{C(S(t), r(t), T-t; K)}{B(t, T, r(t))}.$$

PROPOSITION 4.1. *Under the Rabinovitch (1989) model, a position in forward contracts with a balance of the hedge made up of the continuously compounded savings account replicates the forward price of an option. Thus, no separate hedge of the interest-rate risk is required.*

PROOF. Divide the call option formula (4.2.14) by the bond price $B(t, T, r(t))$ to obtain:

$$\frac{C(S(t), r(t), T-t; K)}{B(t, T, r(t))} = \text{For}(t, T)N(d_1) - KN(d_2). \tag{4.3.1}$$

Note that $\frac{C(S(t),r(t),T-t;K)}{B(t,T,r(t))}$ denotes the forward option price. By taking the partial derivatives with respect to the forward price For(t), we obtain:

$$\frac{\partial\Big(\frac{C(S(t),r(t),T-t;K)}{B(t,T,r(t))}\Big)}{\partial\text{For}(t, T)} = N(d_1). \tag{4.3.2}$$

By Itô's Lemma the diffusion part of the stochastic process

$$\frac{C(S(t), r(t), T-t; K)}{B(t, T, r(t))}$$

is exactly the same as the diffusion part of the stochastic process of a position of $N(d_1)$ in the forward For(t, T). Thus, by constructing a hedging portfolio consisting of $N(d_1)$ amount of the forward For(t, T), with the balance of the hedge made up of the continuously compounded savings account, hedges all the risk. There is no need to separately hedge the interest-rate risk. The hedging portfolio replicates the forward option price that collapses to $C(S(T), r(T), 0; K) = \big[S(T) - K\big]^+$ at maturity. □

We may also consider futures contracts (instead of forward contracts) as hedging instruments. Futures contracts are exchange-traded contracts with several desirable features over the forward contracts, which are over-the-counter contracts. First of all, there are no default risks because the exchange acts as an intermediary, guaranteeing delivery and payment by the use of a clearing house. Futures contracts are standardised and futures prices are mark-to-market, which provide a realised profit and loss daily, unlike forward

contracts where the profit and loss will not be realised until the maturity of the contract.

Thus, under the Rabinovitch (1989) model specifications, the forward option price (with its underlying futures contract maturing at the same time as the option) can be replicated by holding a position in the forward contract with the balance invested in a continuously compounded savings account. However, it is unknown to what extent holding a position in the futures contract with a balance invested in a savings account can replicate the forward option price in the presence of interest-rate risk. We will answer this question in Section 4.4 where we conduct a numerical analysis using Monte Carlo simulations.

4.3.1. Three ways to compute delta. We employ three different methods to calculate the number of hedging instruments required to hedge the futures option. Firstly, we hedge the futures option under the assumption of deterministic interest rates, i.e., the hedge ratios are the Black-Scholes deltas. Then we hedge the futures option assuming stochastic interest rates, i.e., the hedge ratios are calculated using the Rabinovitch (1989) model. In the third method, hedge ratios are derived from *factor hedging*, which can be easily generalised to hedge risk generated from multi-factor / multi-dimensional models. To calculate the number of bond contracts needed to hedge the interest-rate risk of the option position, when necessary, we use hedge ratios derived from factor hedging.

4.3.1.1. *Black-Scholes' delta.* Firstly, the number of the forward contracts are determined by the Black & Scholes (1973) delta, denoted by $\delta_t^{\text{BS,For}}$, that assumes constant interest rates, and it is given by $N\big(d_1^{\text{BS,For}}(t)\big)$ with

$$d_1^{\text{BS,For}}(t) = \frac{\log\left(\frac{S(t)}{K}\right) + \big(\bar{r} + \frac{1}{2}\sigma^2\big)(T-t)}{\sigma\sqrt{T-t}}, \tag{4.3.3}$$

where, under this model, the future interest rates until maturity are assumed to be fixed which equals to the long-term average $\bar{r}$. Unlike the forward price of options which have payoff at maturity, futures contracts are mark-to-market with profits or losses realised at the end of each trading day. Because of this, the number of the futures contracts to hold

must be adjusted and it is given by:

$$\delta_t^{\text{BS,Fut}} = \delta_t^{\text{BS,For}} e^{-\bar{r}(T-t)}. \tag{4.3.4}$$

4.3.1.2. *Rabinovitch's delta.* Under the stochastic interest-rate model of Rabinovitch (1989), the number of forward contracts required for hedging, denoted by $\delta_t^{\text{R,For}}$, is equal to $N\big(d_1^{\text{R,For}}(t)\big)$ with $d_1^{\text{R,For}}(t)$ specified by (4.2.16). To determine the number of futures contracts required for hedging, we need to additionally multiply $N\big(d_1^{\text{R,For}}(t)\big)$ by the convexity adjustment $e^{-V_1^2-V_3^2}$, as shown in equation (4.2.13).[4] As a result, we have $\delta_t^{\text{R,Fut}} = \delta_t^{\text{R,For}} e^{-V_1^2-V_3^2} B\big(t, T, r(t)\big)$.

4.3.1.3. *Delta hedge by factor hedging.* Hedge ratios derived from the Black-Scholes' and Rabinovitch's delta are suited for one-factor model specifications, similar to the ones treated in this chapter. However, for more general multi-dimensional models (that provide better fit to market data, see Section 2.2), we propose hedge ratios obtained from factor hedging. In this chapter, we only consider one-dimensional models, so we reduce the factor hedging specifications to one factor only. However, these investigations will allow us to compare the performance of hedge ratios from factor hedging with alternative hedge ratios. To delta hedge a forward option's position by the factor hedging method, we attempt to immunise the profits and losses (hereafter P/L) of a forward call option due to a small movement from the spot asset price by adding some position in forward contracts. On the trading day t_k, we determine $\delta_k^{\text{F,For}}$ units of the forward contracts required for delta hedging as follows. We determine two spot asset prices S^u and S^d for the next trading day t_{k+1} given a one-standard-deviation up and down shock[5] to the spot asset price dynamics

[4]We note that we do not adjust our Black-Scholes' delta for futures by this convexity adjustment because in the Black-Scholes interest rates are deterministic, hence the futures price equals the forward price.

[5]Chiarella et al. (2013) perform a simulation over a number of shocks instead of using a deterministic one-standard-deviation jumps. We have compared both methods and we do not find any noticeable difference between them in the present context, so we use one-standard-deviation up and down jumps for the sake of simplicity and computational efficiency.

(4.2.1), respectively. Thus:[6]

$$S^u \triangleq S_k \exp\Big(\big(r_k + \Lambda_1\sigma - \frac{1}{2}\sigma^2\big)\Delta t + \sigma\sqrt{\Delta t}\Big),$$
$$S^d \triangleq S_k \exp\Big(\big(r_k + \Lambda_1\sigma - \frac{1}{2}\sigma^2\big)\Delta t - \sigma\sqrt{\Delta t}\Big),$$

where $\Delta t = 1/252$ represents the time interval of one trading day. Then the delta hedge ratio δ_k^F is determined by:

$$\delta_k^{\text{F,For}} = \left(\frac{C(S^u, r_k)}{B(t_k, T, r_k)} - \frac{C(S^d, r_k)}{B(t_k, T, r_k)}\right) \Big/ \left(\frac{S^u}{B(t_k, T, r_k)} - \frac{S^d}{B(t_k, T, r_k)}\right)$$

$$= \frac{C(S^u, r_k, T - t_k; K) - C(S^d, r_k, T - t_k; K)}{S^u - S^d}. \tag{4.3.5}$$

Similarly to Black-Scholes' delta and Rabinovitch's delta, to determine the number of futures contracts required for hedging, we need to adjust the factor hedging delta in (4.3.5) by the convexity adjustment $e^{-V_1^2 - V_3^2}$:

$$\delta_k^{\text{F,Fut}} = \delta_k^{\text{F,For}} e^{-V_1^2 - V_3^2} B\big(t_k, T, r_k\big). \tag{4.3.6}$$

The replicating portfolio consisting of a position in the forward contracts (or futures contract), and money in the savings account would be able to replicate the forward option price at all times and provide a payoff that matches exactly the payoff of the spot option. Thus, the forward (or futures) contracts hedge both the delta risk and the interest-rate risk of the options positions. This holds under the assumption that the maturity of the hedging instruments is the same as the maturity of the option to be hedged.

4.3.2. Hedging with short-dated hedging instruments. In this section, we consider hedging of long-dated options with short-dated contracts. The short-dated forwards and futures hedge the delta risk of the longer-dated option positions, yet bonds are additionally

[6]We define the shorthanded notation of $S(t_k)$ by S_k and $r(t_k)$ by r_k.

required to hedge the interest-rate risk. Thus, we assume the following hedging instruments: shorter maturity forward and futures contracts, while the maturity of the bonds is assumed to be the same as the maturity of the futures option.[7] This also allows us to gauge the impact of the basis risk emerging due to rolling the hedge forward with shorter maturity forward or futures contracts.

4.3.2.1. *Delta hedging.* For delta hedging, we adjust the amount of hedging instruments by a factor equal to the bond price maturing at the same time as the hedging instrument, further divided by the bond price maturing at the same time as the futures option (see Appendix 4.3 for details). We specifically denote the maturity of the option as T, and the maturity of the hedging instrument as T_F. To reduce confusion in the notation, in this section, we explicitly show the dependency of the maturity and current time in V^2, V_1^2, V_2^2 and V_3^2 in equation (4.2.8) by $V^2(t,T), V_1^2(t,T), V_2^2(t,T)$ and $V_3^2(t,T)$. The Rabinovitch delta for forward contracts from Section 4.3.1.2 is generalised to:

$$\delta_t^{\text{R,For}} = N\big(d_1^{\text{R,For}}(t)\big)\frac{B\big(t,T_F,r(t)\big)}{B\big(t,T,r(t)\big)}$$

and the Rabinovitch delta for futures contracts is given by:

$$\begin{aligned}\delta_t^{\text{R,Fut}} &= \delta_k^{\text{R,For}} e^{-V_1^2(t,T_F)-V_3^2(t,T_F)} B\big(t,T,r(t)\big)\\ &= N\big(d_1^{\text{R,For}}(t)\big)e^{-V_1^2(t,T_F)-V_3^2(t,T_F)} B\big(t,T,r(t)\big)\frac{B\big(t,T_F,r(t)\big)}{B\big(t,T,r(t)\big)}\\ &= N\big(d_1^{\text{R,For}}(t)\big)e^{-V_1^2(t,T_F)-V_3^2(t,T_F)} B\big(t,T_F,r(t)\big).\end{aligned}$$

We note that $d_1^{\text{R,For}}$ is defined exactly as in equation (4.2.16).

[7] In practice, long-dated bond contracts are liquid. Thus, we may assume that there is no need to use short-dated bonds as hedging instruments.

The deltas for forward and futures contracts calculated using factor hedging method are generalised as

$$\begin{aligned}\delta_k^{\text{F,For}} &= \frac{B\big(t_k, T_F, r_k\big)}{B\big(t_k, T, r_k\big)} \left(\frac{C(S^u, r_k)}{B(t_k, T, r_k)} - \frac{C(S^d, r_k)}{B(t_k, T, r_k)}\right) \Bigg/ \left(\frac{S^u}{B(t_k, T, r_k)} - \frac{S^d}{B(t_k, T, r_k)}\right) \\ &= \left(\frac{C(S^u, r_k)}{B(t_k, T, r_k)} - \frac{C(S^d, r_k)}{B(t_k, T, r_k)}\right) \Bigg/ \left(\frac{S^u}{B(t_k, T_F, r_k)} - \frac{S^d}{B(t_k, T_F, r_k)}\right) \qquad (4.3.7)\end{aligned}$$

and

$$\delta_t^{\text{F,Fut}} = \delta_k^{\text{F,For}} e^{-V_1^2 - V_3^2} B\big(t_k, T, r_k\big),$$

respectively. Interest-rate hedging is discussed next by adding bonds to the hedging portfolios.

4.3.2.2. *Delta-IR hedging.* To hedge the delta and interest-rate risk by factor hedging,[8] we firstly immunise the P/L of a forward option due to a small movement in the market risk, e.g. delta hedge by factor hedging. The step to calculate the amount of forward or futures contracts required to hedge is exactly the same as in equation (4.3.7). We further immunise the residual risk due to a small movement in the interest-rate risk. We calculate the up and down movements of the interest-rate process r^u and r^d by:

$$\begin{aligned} r^u &= r_k + \lambda\big(\bar{r} - r_k + \Lambda_2\theta\big)\Delta t + \theta\sqrt{\Delta t}, \\ r^d &= r_k + \lambda\big(\bar{r} - r_k + \Lambda_2\theta\big)\Delta t - \theta\sqrt{\Delta t}. \end{aligned}$$

From these two movements, the up and down interest-rate shocked bond and forward prices (with maturity equal to T_F that can be different to the maturity of the futures option

[8]Aiming to generalise this hedging application to multi-dimensional models, we examine delta-IR (interest-rate) hedging by using the factor hedging since it can be easily extended to more general models.

T) are:

$$\begin{aligned} B^{ur} &= B(t_k, T_F, r^u), \\ B^{dr} &= B(t_k, T_F, r^d), \\ \text{For}^{ur} &= \frac{S_k}{B^{ur}}, \\ \text{For}^{dr} &= \frac{S_k}{B^{dr}}. \end{aligned}$$

We also let:

$$\begin{aligned} C^{ur} &= C(S_k, r^u, T - t_k; K), \\ C^{dr} &= C(S_k, r^d, T - t_k; K). \end{aligned}$$

The change of the time-T forward value of this hedging portfolio (long a call and short δ forward) solely due to the up and down interest-rate shocks in forward time T is:

$$\frac{C^{ur}}{B^{ur}} - \frac{C^{dr}}{B^{dr}} - \delta_k^{\text{F,For}}\big(\text{For}^{ur}(t_k, T_F) - \text{For}^{dr}(t_k, T_F)\big), \tag{4.3.8}$$

where $\delta_k^{\text{F,For}}$ is given in equation (4.3.7). This change of the time-T forward value of the hedging portfolio due to the interest-rate shocks is to be further immunised by holding Φ number of bonds maturing at time T. The change of the time-T forward value of Φ number of bonds (maturing at T) due to the up and down interest-rate shocks is:

$$\Phi_k \frac{B(t_k, T, r^u) - B(t_k, T, r^d)}{B\big(t_k, T, r_k\big)}. \tag{4.3.9}$$

The amount of bond contracts Φ required for hedging is computed by equating equations (4.3.8) and (4.3.9):

$$\Phi_k = \left(\frac{\frac{C^{ur}}{B^{ur}} - \frac{C^{dr}}{B^{dr}} - \delta_k^{\text{F,For}}\big(\text{For}^{ur}(t_k, T_F) - \text{For}^{dr}(t_k, T_F)\big)}{B(t_k, T, r^u) - B(t_k, T, r^d)} \right) B\big(t_k, T, r_k\big). \tag{4.3.10}$$

We note that the maturity of the bond is the same as the maturity of the futures option. Only the futures or forward contract as hedging instrument is allowed to have a shorter maturity. The formula (4.3.10) should be adjusted by the suitable convexity when futures

contracts are used for hedging.

4.4. Numerical investigations

In this section, we perform numerical investigations to gauge the impact of stochastic interest rates on hedging long-dated futures options. Firstly, we assess the contribution of the discretisation error in our hedging applications. Secondly, we compare the performance of different hedging schemes, including hedging ratios from deterministic interest-rate specifications, stochastic interest-rate specifications and factor hedging. For each hedging scheme, we consider different hedging instruments, for instance, forwards, futures and bonds with varying maturities. Lastly, the impact of the model parameters such as the long-term mean of interest rates ($\bar{r}$) and the interest-rate volatility (θ) is also evaluated.

Let the set of trading days be $\{t_k\}, k = 1, \ldots, N$ where N is the total number of trading days and the option, futures and forwards all have the same maturity, denoted by $T > T_N$. Given a hedge frequency h representing the frequency that the number of futures, forwards or bond contracts is allowed to change in a day, in each simulation, we can calculate the standard deviations of the profits and losses of the forward call option over a period of N trading days by:

$$\text{SD}_j = \sqrt{\sum_{k=2}^{N\times h} \left(\text{P/L}_{j,k}\right)^2}, \tag{4.4.1}$$

$$\text{P/L}_{j,k} \triangleq CB^j_{t_k} - CB^j_{t_{k-1}}, \tag{4.4.2}$$

where $CB^j_t = \frac{C^j_t}{B^j_t}$ denotes the forward option price and $j = 1, \ldots, M$ denotes the jth realisation of the Monte Carlo simulation. Further, we let $C^j_{t_k} = C(S^j(t_k), r^j(t_k), T - t_k; K)$ to denote the simulated option price at time t_k using the jth simulated spot asset price $S^j(t_k)$, and the jth simulated interest rates $r^j(t_k)$ under the historical measure as specified in (4.2.1) and (4.2.2), where both spot asset prices and interest rates evolve stochastically.

Similarly we let $B^j_{t_k} = B(t_k, T, r^j(t_k))$ to be the simulated bond price at time t_k using the simulated interest rates $r^j(t_k)$ under the historical measure.

The jth realised standard deviation of a hedging portfolio using Black-Scholes' delta is defined similarly to (4.4.1) with a time-T forward value[9] of P/L$_k$ defined as follows:

$$\begin{aligned}
\text{P/L}^{\text{BS,For}}_{j,k} &\triangleq C^j_{t_k} e^{\bar{r}(T-t_k)} - C^j_{t_{k-1}} e^{\bar{r}(T-t_{k-1})} - \delta^{\text{BS,For}}_{t_{k-1}} \Big(\text{For}(t_k, T) - \text{For}(t_{k-1}, T)\Big) \\
&\quad - C^j_{t_{k-1}} e^{\bar{r}(T-t_k)} \big(e^{\bar{r}(t_k - t_{k-1})} - 1\big) \\
&= C^j_{t_k} e^{\bar{r}(T-t_k)} - C^j_{t_{k-1}} e^{\bar{r}(T-t_{k-1})} - \delta^{\text{BS,For}}_{t_{k-1}} \Big(\text{For}(t_k, T) - \text{For}(t_{k-1}, T)\Big) \\
&\quad - C^j_{t_{k-1}} \big(e^{\bar{r}(T-t_{k-1})} - e^{\bar{r}(T-t_k)}\big) \qquad (4.4.3)
\end{aligned}$$

and

$$\begin{aligned}
\text{P/L}^{\text{BS,Fut}}_{j,k} &\triangleq C^j_{t_k} e^{\bar{r}(T-t_k)} - C^j_{t_{k-1}} e^{\bar{r}(T-t_{k-1})} \\
&\quad - \delta^{\text{BS,Fut}}_{t_{k-1}} \Big(F(t_k, T) - F(t_{k-1}, T)\Big) e^{\bar{r}(T-t_k)} - C^j_{t_{k-1}} \big(e^{\bar{r}(T-t_{k-1})} - e^{\bar{r}(T-t_k)}\big). \qquad (4.4.4)
\end{aligned}$$

The P/L at time t_k of a jth realised standard deviation of a hedging portfolio using Rabinovitch's delta is:

$$\begin{aligned}
\text{P/L}^{\text{R,For}}_{j,k} &\triangleq CB^j_{t_k} - CB^j_{t_{k-1}} - \delta^{\text{R,For}}_{t_{k-1}} \Big(\text{For}(t_k, T) - \text{For}(t_{k-1}, T)\Big) \\
&\qquad - \frac{C^j_{t_{k-1}}}{B\big(t_k, T, r_k\big)} \Big(1/B\big(t_{k-1}, t_k, r_{k-1}\big) - 1\Big). \qquad (4.4.5)
\end{aligned}$$

or for futures contracts as hedging instruments, we have:

$$\begin{aligned}
\text{P/L}^{\text{R,Fut}}_{j,k} &\triangleq CB^j_{t_k} - CB^j_{t_{k-1}} - \delta^{\text{R,Fut}}_{t_{k-1}} \Big(F(t_k, T) - F(t_{k-1}, T)\Big) \Big/ B\big(t_k, T, r_k\big) \\
&\qquad - \frac{C^j_{t_{k-1}}}{B\big(t_k, T, r_k\big)} \Big(1/B\big(t_{k-1}, t_k, r_{k-1}\big) - 1\Big). \qquad (4.4.6)
\end{aligned}$$

[9]To get the forward value under the Black-Scholes world, we assume the interest rate is fixed till maturity. So the forward value of \$1 is \$ $e^{r_k(T-t_k)}$.

The P/L at time t_k of a jth realised standard deviation of a hedging portfolio using delta calculated by the factor hedging method, $\text{P/L}_{j,k}^{\text{F,For}}$, is defined similarly to (4.4.5), but with $\delta_{t_{k-1}}^{\text{R,For}}$ replaced by $\delta_{t_{k-1}}^{\text{F,For}}$. $\text{P/L}_{j,k}^{\text{F,Fut}}$ is defined similarly to (4.4.6) using $\delta_{t_{k-1}}^{\text{F,Fut}}$.

The P/L at time t_k of a jth realised standard deviation of a hedging portfolio using both forwards and bonds, with the maturity of the short-dated forwards as $T_F < T$, is:

$$\begin{aligned}
\text{P/L}_{j,k}^{\text{F,For,IR}} \triangleq & CB_{t_k}^j - CB_{t_{k-1}}^j - \delta_{t_{k-1}}^{\text{R,For}}\Big(\text{For}(t_k, T_F) - \text{For}(t_{k-1}, T_F)\Big) \\
& - \Phi_{k-1}\Big(B\big(t_k, T, r_k\big) - B\big(t_{k-1}, T, r_{k-1}\big)\Big)\Big/B\big(t_k, T, r_k\big) \\
& - \frac{C_{t_{k-1}}^j - \Phi_{k-1}B(t_{k-1}, T, r_{k-1})}{B\big(t_k, T, r_k\big)}\Big(1/B\big(t_{k-1}, t_k, r_{k-1}\big) - 1\Big) \\
= & CB_{t_k}^j - CB_{t_{k-1}}^j - \delta_{t_{k-1}}^{\text{R,For}}\Big(\text{For}(t_k, T_F) - \text{For}(t_{k-1}, T_F)\Big) \\
& - \Phi_{k-1}\Big(1 - \frac{B\big(t_{k-1}, T, r_{k-1}\big)}{B\big(t_k, T, r_k\big)B\big(t_{k-1}, t_k, r_{k-1}\big)}\Big) \\
& - \frac{C_{t_{k-1}}^j}{B\big(t_k, T, r_k\big)}\Big(1/B\big(t_{k-1}, t_k, r_{k-1}\big) - 1\Big)
\end{aligned}$$

and $\text{P/L}_{j,k}^{\text{F,Fut,IR}}$ at time t_k of a hedging portfolio using both futures and bonds is defined similarly.

Proposition (4.1) demonstrates that a hedged position consisting of forward or futures contracts (with a maturity equals to the futures option's maturity), and the balance of the hedge in a continuously compounded savings account is enough to replicate the forward option price. One requirement for this hedging portfolio to exactly replicate the forward option price is that it has to be re-balanced continuously, which is impossible to execute in practice. If the portfolio cannot be continuously re-balanced, hedging error due to discretisation exists. We assess the contribution of discretisation error when the forward option price is replicated by using six different hedging schemes, and their performance is assessed as the hedging frequency increases. Furthermore, we assess hedging performance when the maturity of the hedging instruments match with the maturity of the option

to be hedged, and when their maturities do not match (e.g., use shorter maturity hedging instruments).

4.4.1. Hedging with matching maturity. In this section, we assume that the maturity of the option and the hedging instruments (forward and futures) is the same and is set to 1500 trading days for our numerical investigations. The number M of the simulations is 1000, and the option is hedged from day 1 to day 1000. For consistency and comparability, σ are chosen such that V^2 in equation (4.2.8) remains the same as we vary θ. The option price is struck at-the-money forward (that is, $K = \text{For}(t, T)$) so d_1 in equation (4.2.16) reduces to $\frac{V}{2}$ and with V remaining the same across different θ, the price of the initial options with different θ are the same. However, the forward price of the options will differ slightly because the bond prices change for different values of θ. Table 4.1 provides parameter values used in the numerical hedging analysis, and we consider the following six different hedging schemes:

(1) Black-Scholes delta with forward contracts as hedging instrument,
(2) Black-Scholes delta with futures contracts as hedging instrument,
(3) Rabinovitch delta using forwards contracts as hedging instrument,
(4) Rabinovitch delta with futures contracts as hedging instrument,
(5) Factor hedging method with forward contracts as hedging instrument,
(6) Factor hedging method with futures contracts as hedging instrument.

Tables 4.2 and 4.3 show the simulated hedging performance, measured by the average of the standard deviation of the forward price of the hedging portfolios over 1000 paths consisting of an option and a number of futures or forwards derived by using three different hedging schemes as described above and for increasing hedging frequency. Table 4.2 considers a long-term mean of interest rates of 5% (around the last 12-year empirical average), while Table 4.3 considers a long-term mean of interest rates around 1% of which is the currently observed level of interest rates.

TABLE 4.1. Parameters Values

Parameters	Values	Notes
M	1000	Number of simulations
N	1000	Number of trading days
T	1500	Option maturity
T_F	1500	Hedging Fut / For maturity
λ	0.4	mean-reversion rate
Λ_1	0	market price of risk
Λ_2	0	market price of interest-rate risk
σ	varying	volatility
$\bar{r}$	$\{5\%, 1\%\}$	long-term level of interest rates
θ	$\{8\%, 3\%, 1\%\}$	volatility of interest rates
ρ	0%	interest-rate correlation
$S(0)$	\$50	initial stock price
$r(0)$	$\bar{r}$	initial interest rate
K	at-the-money forward	strike
Type	Call	option type

$\bar{r} = 5\%, \theta = 8\%$							
Hedge Frequency	Call/Bond	BS For	BS Fut	Rab For	Rab Fut	Factor For	Factor Fut
1	18.9006	7.8440	7.0645	0.3866	0.3418	0.3866	0.3418
10	18.2189	7.3175	6.5420	0.1130	0.0977	0.1130	0.0977
100	18.8607	7.5770	6.7804	0.0364	0.0318	0.0364	0.0318
$\bar{r} = 5\%, \theta = 3\%$							
Hedge Frequency	Call/Bond	BS For	BS Fut	Rab For	Rab Fut	Factor For	Factor Fut
1	16.4546	2.0149	1.9708	0.2328	0.2223	0.2329	0.2223
10	15.9600	1.9582	1.9172	0.0723	0.0688	0.0723	0.0688
100	16.1199	1.9334	1.8933	0.0227	0.0216	0.0227	0.0216
$\bar{r} = 5\%, \theta = 1\%$							
Hedge Frequency	Call/Bond	BS For	BS Fut	Rab For	Rab Fut	Factor For	Factor Fut
1	16.0233	0.6620	0.6598	0.2105	0.2092	0.2105	0.2093
10	15.5760	0.6152	0.6136	0.0654	0.0650	0.0654	0.0650
100	15.8806	0.6237	0.6222	0.0210	0.0209	0.0210	0.0209

TABLE 4.2. **Discretisation hedging error with increasing hedging frequency when $\bar{r} = 5\%$ and volatility of interest rates are 8%, 3% and 1%, respectively.** The table displays the discretisation hedging error, when a 1500-day option is hedged by using forwards/futures with the same maturity. The option is hedged over 1000 days.

According to Proposition 4.1, the hedging portfolio consisted of a position in futures or forwards, calculated using the Rabinovitch delta or the factor hedging method and the balance in the continuously compounded savings account is enough to replicate the forward

$\bar{r} = 1\%, \theta = 8\%$							
Hedge Frequency	Call/Bond	BS For	BS Fut	Rab For	Rab Fut	Factor For	Factor Fut
1	14.2534	5.7584	5.1510	0.2445	0.2053	0.2445	0.2053
10	14.4185	5.7653	5.1542	0.0785	0.0659	0.0785	0.0659
100	14.1518	5.8260	5.2197	0.0246	0.0206	0.0246	0.0206
$\bar{r} = 1\%, \theta = 3\%$							
Hedge Frequency	Call/Bond	BS For	BS Fut	Rab For	Rab Fut	Factor For	Factor Fut
1	12.5001	1.5301	1.4959	0.1571	0.1478	0.1571	0.1478
10	12.5569	1.5211	1.4894	0.0492	0.0464	0.0492	0.0464
100	12.1457	1.4881	1.4564	0.0156	0.0147	0.0156	0.0147
$\bar{r} = 1\%, \theta = 1\%$							
Hedge Frequency	Call/Bond	BS For	BS Fut	Rab For	Rab Fut	Factor For	Factor Fut
1	12.3066	0.5083	0.5063	0.1464	0.1453	0.1464	0.1453
10	12.2661	0.4812	0.4799	0.0460	0.0457	0.0460	0.0457
100	11.6745	0.4680	0.4669	0.0144	0.0143	0.0144	0.0143

TABLE 4.3. **Discretisation hedging error with increasing hedging frequency when $\bar{r} = 1\%$ and volatility of interest rates are 8%, 3% and 1%, respectively.** The table displays the discretisation hedging error, when a 1500-day option is hedged by using forwards/futures with the same maturity. The option is hedged over 1000 days.

option price. Thus, the errors that we see in the tables above by using the Rabinovitch or the factor hedging delta are due to discretisation only, and as expected, the errors are reduced by a factor of $\sqrt{10}$ as we increase the hedging frequency from 1 to 10 and from 10 to 100. The variance obtained from the hedged position with the number of futures or forward contracts calculated by using the Black-Scholes delta does not perform well. Indeed, the errors do not decrease as the hedging frequency increases. This simulation exercise demonstrates that using futures or forward contracts alone, with their maturities matching the maturity of the option, can hedge both the underlying market risk, and interest risk and as a consequence, additional interest-rate-sensitive contracts such as bonds are not required.

Next, we compare the hedging performance of the Rabinovitch model that assumes stochastic interest rates to the Black model that assumes deterministic interest rates. In a very high interest-rate volatility environment the improvement provided by the stochastic

interest-rate model over the deterministic interest-rate model is significant. From Table 4.2, and when interest-rate volatility is at 8%, the hedging error of using forwards as hedging instruments reduces from 7.8440 in the Black model to 0.3866 in the Rabinovitch model; that is a reduction of over 95%. However, when interest-rate volatility is at 1%, the reduction is only 68% $(1 - 0.2105/0.6620)$. Similar conclusions can be drawn using results from Table 4.3 when the long-term level of the interest rates is low. We also observe that the long-term level of the interest rates $\bar{r}$ does not have any structural impact to the hedging performance.

Finally, both tables validate the robustness of the hedge ratios computed by the factor hedging, since, the errors from Rabinovitch hedging are identical to the errors from factor delta hedging.

The hedging exercise performed in this section sheds some light on making a decision between assuming a model with stochastic interest rates, which is more involved mathematically and computationally demanding, or assuming a deterministic interest-rate model, which is simpler and easier to be estimated empirically. We conclude that the interest-rate volatility is the single most decisive factor when one considers stochastic interest rates. In the next section, a numerical investigation is conducted when hedging instruments with maturities that do not match the maturity of the futures option.

4.4.2. Hedging with maturity mismatch. In practice, when hedging long-dated futures options, it is not always possible to hedge with the underlying futures contract. Long-dated options are hedged with shorter maturity futures or forward contracts and then the hedge is rolled over to another contract. Since only market risk and interest-rate risk are present in our model, the interest-rate risk is the risk that reflects the basis risk introduced when we roll the hedge forward. The following numerical investigations allow us to gauge the level of deterioration in the hedging performance as the frequency of

rolling the hedge forward is increasing.

Table 4.4 displays the parameter values, number of simulations, maturities of the option, futures and forward for the numerical hedging analysis conducted when the hedge is rolled forward. We assume that the call option in this simulation experiment matures in $T = 2000$ trading days, with hedging instruments futures or forwards maturing in $2000, 1800, 1200$ and 600 trading days denoted as T_{F1}, T_{F2}, T_{F3} and T_{F4}, respectively. We hedge the option for 500 trading days, so there is no rolling-over required in the futures or forward contracts. The six hedging schemes and their associated hedge ratios considered in these investigations are:

(1) Rabinovitch delta with forward contracts as hedging instrument,

(2) Rabinovitch delta with futures contracts as hedging instrument,

(3) Factor delta hedging with forward contracts as hedging instrument,

(4) Factor delta hedging with futures contracts as hedging instrument,

(5) Factor delta and interest-rate hedging with forward contracts and bonds as hedging instruments,

(6) Factor delta and interest-rate hedging with futures contracts and bonds as hedging instruments.

Table 4.5 and Table 4.6 present the hedging performance of these hedging schemes with increasing hedging frequency, maturity mismatches and zero correlation coefficient between the asset price process and the interest-rate process. In particular, Table 4.5 considers a relatively high interest-rate volatility of 8%, while Table 4.6 considers a low interest-rate volatility environment with interest-rate volatility of 1%. By comparing the results in these two tables, we conclude that adding a position in bond contracts provides a substantial improvement when interest rates are volatile compared to when interest rates are stable.[10] For instance, in Table 4.5 with maturities of hedging instrument equal to

[10]This does not hold, when the maturity of the hedging instruments match the options maturity. Using only forward or futures can hedge both the market risk and interest-rate risk so bonds are not required, see Section 4.4.1.

TABLE 4.4. Parameters Values

Parameters	Values	Notes
M	1000	Number of simulations
N	500	Number of trading days
T	2000	Option maturity
T_{F1}	2000	Hedging Fut / For maturity 1
T_{F2}	1800	Hedging Fut / For maturity 2
T_{F3}	1200	Hedging Fut / For maturity 3
T_{F4}	600	Hedging futures maturity 4
T_B	2000	Hedging bond maturity
λ	0.4	mean-reversion rate
Λ_1	0	market price of risk
Λ_2	0	market price of interest-rate risk
σ	varying	volatility
$\bar{r}$	5%	long-term level of interest rates
θ	$\{8\%, 1\%\}$	volatility of interest rates
ρ	$\{0\%, -50\%\}$	interest-rate correlation
$S(0)$	\$50	initial stock price
$r(0)$	$\bar{r}$	initial interest rate
K	at-the-money forward	strike
Type	Call	option type

1200 trading days (T_{F3}), and with hedging frequency equals to 1 day, the standard deviation of the portfolio with forwards alone is 2.3242 and 0.2550, when bond contracts are added. This reduction in the standard deviation is nearly by a factor of 10. However, in Table 4.6, the corresponding results are 0.3173 and 0.1502, with a reduction of factor by about 2. So, in practice, if we cannot re-balance the portfolio daily and if the volatility of the interest rates θ is lower than 1%, adding bond contracts as hedging instruments only provide only marginal improvement.

From the results of the scenarios where the maturities of futures or forwards match the option's maturity (T_{F1}), futures or forward contracts alone can hedge both market and interest-rate risks (because their standard deviation are reduced as we increase the hedging frequency). Furthermore, as expected, adding bond contracts in the hedging portfolio does not further reduce the standard deviation.

When the maturities of futures and forwards equal to 1800 trading days (T_{F2}), we observe that there is a reduction in the standard deviation as the hedging frequency increases, but the reduction factor is no longer $\sqrt{10}$ (for example, from the Rabinovitch forward column in Table 4.5, the standard deviation is reduced from 0.4226 to 0.3373 to 0.3295, and similarly to futures contracts). However, there is a significant reduction of the variance when the hedging portfolio consists of bond contracts and futures (or forward) contracts. As the hedging frequency increases, the hedging portfolio consisting of bond contracts and futures (or forward) contracts reduces the standard deviation by a factor of $\sqrt{10}$. Thus, we conclude that when futures or forwards contracts with shorter maturities are used as hedging instruments, the interest-rate risk cannot be hedged by futures or forward contracts alone. The results of the scenarios with hedging instruments maturing in 1200 trading days, and 600 trading days are also consistent with these findings. Comparing the hedging performance of using forward contracts that mature in day 1800 (T_{F2}) and in day 1200 (T_{F3}), over a high interest-rate volatility environment ($\theta = 8\%$), and with a hedge frequency of 1, the hedging performance deteriorates by more than five-fold, from 0.4226 to 2.3242. However, during a lower interest-rate volatility environment ($\theta = 1\%$) the hedging performance deteriorates only by over twofold, from 0.1555 to 0.3173.

Tables 4.7 and 4.8 display the results of similar investigations as in Tables 4.5 and 4.6 by allowing the correlation ρ to be -50%. Overall, the correlation has no impact to the conclusions drawn before. A maturity mismatch introduces an interest-rate risk that cannot be hedged by futures contracts only and as a result, bond contracts should be added to reduce this interest-rate risk.

4.5. Conclusion

In this paper, we analyse the impact of interest-rate risk on futures options positions and examine the hedging of this risk. Due to its tractability, the Rabinovitch (1989) model is considered with correlated dynamics between spot asset prices and interest rates. The underlying market risk and the interest-rate risk of a position in futures options is hedged by

$T_{F1} = 2000, \theta = 8\%, \rho = 0\%$							
Hedge Frequency	Call/Bond	Rab For	Rab Fut	Factor For	Factor Fut	For with B	Fut with B
1	15.1403	0.2520	0.2220	0.2521	0.2220	0.2521	0.2220
10	14.8047	0.0764	0.0654	0.0764	0.0654	0.0764	0.0654
100	14.8824	0.0237	0.0204	0.0237	0.0204	0.0237	0.0204
$T_{F2} = 1800, \theta = 8\%, \rho = 0\%$							
Hedge Frequency	Call/Bond	Rab For	Rab Fut	Factor For	Factor Fut	For with B	Fut with B
1	15.1403	0.4226	0.3996	0.4219	0.3987	0.2521	0.2070
10	14.8047	0.3373	0.3348	0.3372	0.3348	0.0765	0.0644
100	14.8824	0.3295	0.3293	0.3295	0.3293	0.0237	0.0198
$T_{F3} = 1200, \theta = 8\%, \rho = 0\%$							
Hedge Frequency	Call/Bond	Rab For	Rab Fut	Factor For	Factor Fut	For with B	Fut with B
1	15.1403	2.3242	2.3222	2.3234	2.3214	0.2550	0.2246
10	14.8047	2.2492	2.2490	2.2491	2.2489	0.0778	0.0695
100	14.8824	2.2522	2.2522	2.2522	2.2522	0.0241	0.0215
$T_{F4} = 600, \theta = 8\%, \rho = 0\%$							
Hedge Frequency	Call/Bond	Rab For	Rab Fut	Factor For	Factor Fut	For with B	Fut with B
1	15.1403	7.4311	7.4309	7.4308	7.4306	0.2760	0.2683
10	14.8047	7.2242	7.2242	7.2242	7.2242	0.0845	0.0823
100	14.8824	7.2375	7.2375	7.2375	7.2375	0.0263	0.0256

TABLE 4.5. **Hedging performance with increasing hedging frequency, maturity mismatch and $\theta = 8\%$.** The forward price of a long-dated option (Call/Bond) is hedged by using futures or forwards with various maturities. The four different maturities T_{F1}, T_{F2}, T_{F3} and T_{F4} represent the maturities of the hedging futures and forwards with 2000, 1800, 1200 and 600 trading days, respectively. In all cases, the maturity of the bond contracts is 2000 trading days.

using several hedge ratios and schemes such as factor hedging. In particular, we study the impact of the discretisation error to the hedging values (such as interest-rate volatility) to the hedging applications. We also examine hedging long-dated options with shorter maturity hedging instruments.

We show mathematically and numerically that, under the Rabinovitch (1989) model specifications, the forward option price can be replicated by forward contracts with maturities equal to the maturity of the option with the balance invested in the continuously compounded savings account. This implies that forward contracts can hedge both the interest-rate risk and the underlying market risk of options positions. As a consequence, there is

$T_{F1} = 2000, \theta = 1\%, \rho = 0\%$							
Hedge Frequency	Call/Bond	Rab For	Rab Fut	Factor For	Factor Fut	For with B	Fut with B
1	12.8962	0.1499	0.1486	0.1499	0.1486	0.1499	0.1486
10	12.7605	0.0471	0.0467	0.0471	0.0467	0.0471	0.0467
100	12.9293	0.0150	0.0149	0.0150	0.0149	0.0150	0.0149
$T_{F2} = 1800, \theta = 1\%, \rho = 0\%$							
Hedge Frequency	Call/Bond	Rab For	Rab Fut	Factor For	Factor Fut	For with B	Fut with B
1	12.8962	0.1555	0.1543	0.1555	0.1544	0.1499	0.1488
10	12.7605	0.0621	0.0618	0.0621	0.0618	0.0471	0.0467
100	12.9293	0.0434	0.0434	0.0434	0.0434	0.0150	0.0149
$T_{F3} = 1200, \theta = 1\%, \rho = 0\%$							
Hedge Frequency	Call/Bond	Rab For	Rab Fut	Factor For	Factor Fut	For with B	Fut with B
1	12.8962	0.3173	0.3169	0.3174	0.3170	0.1502	0.1494
10	12.7605	0.2781	0.2781	0.2781	0.2781	0.0472	0.0469
100	12.9293	0.2788	0.2788	0.2788	0.2788	0.0150	0.0150
$T_{F4} = 600, \theta = 1\%, \rho = 0\%$							
Hedge Frequency	Call/Bond	Rab For	Rab Fut	Factor For	Factor Fut	For with B	Fut with B
1	12.8962	0.9077	0.9077	0.9078	0.9077	0.1514	0.1512
10	12.7605	0.8816	0.8816	0.8816	0.8816	0.0476	0.0475
100	12.9293	0.8947	0.8947	0.8947	0.8947	0.0152	0.0151

TABLE 4.6. **Hedging performance with increasing hedging frequency, maturity mismatch and $\theta = 1\%$.** The forward price of a long-dated option (Call/Bond) is hedged by using futures or forwards with various maturities. The four different maturities T_{F1}, T_{F2}, T_{F3} and T_{F4} represent the maturities of the hedging futures and forwards with 2000, 1800, 1200 and 600 trading days, respectively. In all cases, the maturity of the bond contracts is 2000 trading days.

no need to separately hedge interest-rate risk. This also holds when we use futures contracts as hedging instruments, but convexity adjustment must be considered in calculating the hedging ratio of futures contracts.

One requirement is that the maturities of the hedging instruments must match the maturity of the option, in order for the forward (or futures) contracts to hedge both risks simultaneously. Numerical results demonstrate that when the maturities of forward (or futures) contracts and the maturity of the option differ slightly (maturity of 2000 trading days for options and 1800 for futures/forwards), the hedging performance deteriorates noticeably. To improve the hedging performance, the interest-rate risk must be hedged separately

$T_{F1} = 2000, \theta = 8\%, \rho = -50\%$							
Hedge Frequency	Call/Bond	Rab For	Rab Fut	Factor For	Factor Fut	For with B	Fut with B
1	12.8207	0.1893	0.2048	0.1894	0.2047	0.1894	0.2048
10	12.7255	0.0582	0.0629	0.0582	0.0629	0.0582	0.0629
100	12.6190	0.0183	0.0197	0.0183	0.0197	0.0183	0.0197
$T_{F2} = 1800, \theta = 8\%, \rho = -50\%$							
Hedge Frequency	Call/Bond	Rab For	Rab Fut	Factor For	Factor Fut	For with B	Fut with B
1	12.8207	0.3693	0.3592	0.3693	0.3591	0.1898	0.1692
10	12.7255	0.3181	0.3170	0.3181	0.3170	0.0584	0.0523
100	12.6190	0.3097	0.3096	0.3097	0.3096	0.0183	0.0165
$T_{F3} = 1200, \theta = 8\%, \rho = -50\%$							
Hedge Frequency	Call/Bond	Rab For	Rab Fut	Factor For	Factor Fut	For with B	Fut with B
1	12.8207	2.1702	2.1695	2.1702	2.1694	0.1960	0.1850
10	12.7255	2.1419	2.1418	2.1419	2.1418	0.0604	0.0571
100	12.6190	2.1184	2.1184	2.1184	2.1184	0.0190	0.0180
$T_{F4} = 600, \theta = 8\%, \rho = -50\%$							
Hedge Frequency	Call/Bond	Rab For	Rab Fut	Factor For	Factor Fut	For with B	Fut with B
1	12.8207	6.9531	6.9532	6.9532	6.9533	0.2385	0.2418
10	12.7255	6.8812	6.8812	6.8812	6.8812	0.0737	0.0747
100	12.6190	6.8075	6.8075	6.8075	6.8075	0.0231	0.0234

TABLE 4.7. **Hedging performance with increasing hedging frequency, maturity mismatch, $\theta = 8\%$ and $\rho = -50\%$.** In this table, the forward price of a long-dated option is hedged by using bonds and futures or forwards with various maturities. The four different maturities T_{F1}, T_{F2}, T_{F3} and T_{F4} represent the maturities of the hedging futures and forwards with 2000, 1800, 1200 and 600 trading days respectively. In all cases, the maturity of the bond contracts is 2000 trading days.

by holding positions in bond contracts. Finally, when the interest-rate volatility is high, adding bond contracts can improve the hedging performance substantially, and the improvement builds up as we increase the hedging frequency. We have also validated the numerical efficiency of factor hedging that is suited for multi-dimensional models, similar to the ones considered in the next chapter.

These numerical investigations provide useful insights on the sources and the nature of the interest-rate risk present in option futures positions, with the objective to better understand it and manage it. The impact of interest-rate risk becomes more influential with

$T_{F1} = 2000, \theta = 1\%, \rho = -50\%$							
Hedge Frequency	Call/Bond	Rab For	Rab Fut	Factor For	Factor Fut	For with B	Fut with B
1	12.5936	0.1425	0.1522	0.1425	0.1523	0.1425	0.1523
10	12.6937	0.0454	0.0485	0.0454	0.0485	0.0454	0.0485
100	12.6386	0.0142	0.0152	0.0142	0.0152	0.0142	0.0152
$T_{F2} = 1800, \theta = 1\%, \rho = -50\%$							
Hedge Frequency	Call/Bond	Rab For	Rab Fut	Factor For	Factor Fut	For with B	Fut with B
1	12.5936	0.1482	0.1521	0.1481	0.1520	0.1424	0.1465
10	12.6937	0.0611	0.0620	0.0611	0.0620	0.0453	0.0466
100	12.6386	0.0430	0.0431	0.0429	0.0431	0.0142	0.0146
$T_{F3} = 1200, \theta = 1\%, \rho = -50\%$							
Hedge Frequency	Call/Bond	Rab For	Rab Fut	Factor For	Factor Fut	For with B	Fut with B
1	12.5936	0.3114	0.3130	0.3110	0.3126	0.1418	0.1454
10	12.6937	0.2806	0.2808	0.2805	0.2807	0.0451	0.0463
100	12.6386	0.2772	0.2772	0.2772	0.2772	0.0142	0.0145
$T_{F4} = 600, \theta = 1\%, \rho = -50\%$							
Hedge Frequency	Call/Bond	Rab For	Rab Fut	Factor For	Factor Fut	For with B	Fut with B
1	12.5936	0.8977	0.8980	0.8973	0.8976	0.1408	0.1430
10	12.6937	0.8904	0.8904	0.8903	0.8904	0.0448	0.0455
100	12.6386	0.8897	0.8897	0.8897	0.8897	0.0141	0.0143

TABLE 4.8. **Hedging performance with increasing hedging frequency, maturity mismatch, $\theta = 1\%$ and $\rho = -50\%$.** In this table the forward price of a long-dated option is hedged by using bonds and futures or forwards with various maturities. The four different maturities T_{F1}, T_{F2}, T_{F3} and T_{F4} represent the maturities of the hedging futures and forwards with 2000, 1800, 1200 and 600 trading days respectively. In all cases, the maturity of the bond contracts is 2000 trading days.

increasing options maturity. Thus, the investigations are concentrated to long-dated contracts. Motivated by the significant increase in trading volumes of longer-dated commodity derivatives, especially crude oil options, the next chapter empirically analyses the hedging performance on long-dated crude oil futures option market data.

Appendix 4.1 Derivation of the futures prices

In this section, we derive an expression for the futures prices given the following dynamics of the spot asset price and the spot interest rate under the spot risk-neutral measure:

$$dS(t) = r(t)S(t)dt + \sigma S(t)dW_1(t), \tag{A.1.1}$$

$$dr(t) = \lambda\big(\bar{r} - r(t)\big)dt + \theta dW_2(t), \tag{A.1.2}$$

$$\rho dt = dW_1(t)dW_2(t).$$

The futures price with a maturity $T \geq t$ is the expectation at time t of the future spot asset price at time T under the risk neutral measure, i.e.,

$$\begin{aligned} F(t,T) &= \mathbb{E}^{\mathbb{Q}}\big[S(T)|\mathcal{F}_t\big] \\ &= S(t)\exp\big(-\frac{1}{2}\sigma^2(T-t)\big)\mathbb{E}^{\mathbb{Q}}\big[\exp\big(\int_t^T r(u)\,du + \sigma W^1(T-t)\big)|\mathcal{F}_t\big] \\ &= S(t)\exp\big(-\frac{1}{2}\sigma^2(T-t)\big)\mathbb{E}^{\mathbb{Q}}\big[\exp\big(Y\big)|\mathcal{F}_t\big] \end{aligned}$$

where

$$Y = \int_t^T r(u)\,du + \sigma W^1(T-t).$$

Since the random variable Y is a sum of an integral of Gaussian random variables and a Wiener process, which also has a Gaussian distribution, Y also has a Gaussian distribution. So we need to find the expectation and the variance of Y, and then use the moment generating function to get an expression of $F(t,T)$.

Now, we re-write the two correlated Wiener processes into two independent Wiener processes as follow,

$$\begin{aligned} dW_1(t) &= dB^1(t), \\ dW_2(t) &= \rho dB^1(t) + \sqrt{(1-\rho^2)}dB^2(t), \\ 0 &= dB^1(t)dB^2(t) \end{aligned}$$

so the dynamics of the interest-rate process becomes

$$dr(t) = \lambda\big(\bar{r} - r(t)\big)dt + \theta\rho dB^1(t) + \theta\sqrt{(1-\rho^2)}dB^2(t).$$

From A.1.1, we can derive the future stock price at time $T \geq t$ given the current stock price, at time t is:

$$S(T) = S(t)\exp\Big(-\frac{1}{2}\sigma^2(T-t)\big) + \int_t^T r(u)\,du + \sigma B^1(T-t)\Big).$$

We can further obtain an expression for $\int_t^T r(s)\,ds, s \geq t$ by firstly deriving $r(s)$:

$$r(s) = r(t)e^{-\lambda(s-t)} + \bar{r}\big(1 - e^{-\lambda(s-t)}\big) + \theta\rho\int_t^s e^{-\lambda(s-u)}\,dB^1(u) + \theta\sqrt{(1-\rho^2)}\int_t^s e^{-\lambda(s-u)}\,dB^2(u)$$

then by integrating $r(s)$ from t to T and by applying Fubini's theorem we obtain:

$$\int_t^T r(s)\,ds = \big(r(t) - \bar{r}\big)A(t,T,\lambda) + \bar{r}(T-t) + \theta\rho\int_t^T A(u,T,\lambda)\,dB^1(u) + \theta\sqrt{(1-\rho^2)}\int_t^T A(u,T,\lambda)\,dB^2(u)$$

where

$$A(t,T,\lambda) = \frac{1}{\lambda}\big(1 - e^{-\lambda(T-t)}\big).$$

The expectation of Y is:

$$\begin{aligned} M \triangleq \mathbb{E}^{\mathbb{Q}}\big[Y|\mathcal{F}_t\big] &= \mathbb{E}^{\mathbb{Q}}\big[\int_t^T r(u)\,du + \sigma W^1(T-t)|\mathcal{F}_t\big] \\ &= \mathbb{E}^{\mathbb{Q}}\big[\int_t^T r(u)\,du|\mathcal{F}_t\big] \\ &= (r(t) - \bar{r})A(t,T,\lambda) + \bar{r}(T-t). \end{aligned}$$

The variance of Y is:

$$\begin{aligned} V^2 \triangleq \operatorname{var}^{\mathbb{Q}}\left[Y|\mathcal{F}_t\right] =& \operatorname{var}^{\mathbb{Q}}\left[\int_t^T r(u)\,du|\mathcal{F}_t\right] + \operatorname{var}^{\mathbb{Q}}\left[\sigma W^1(T-t)|\mathcal{F}_t\right] \\ &+ 2\operatorname{cov}^{\mathbb{Q}}\Big(\int_t^T r(u)\,du, \sigma W^1(T-t)|\mathcal{F}_t\Big) \\ =& V_1^2 + V_2^2 + 2V_3^2 \end{aligned}$$

where,

$$\begin{aligned} V_1^2 &= \operatorname{var}^{\mathbb{Q}}\left[\int_t^T r(u)\,du|\mathcal{F}_t\right] \\ V_2^2 &= \operatorname{var}^{\mathbb{Q}}\left[\sigma W^1(T-t)|\mathcal{F}_t\right] \\ V_3^2 &= \operatorname{cov}^{\mathbb{Q}}\Big(\int_t^T r(u)\,du, \sigma W^1(T-t)|\mathcal{F}_t\Big) \end{aligned}$$

Now, we have:

$$\begin{aligned} V_1^2 &= \operatorname{var}^{\mathbb{Q}}\left[\theta\int_t^T A(u,T,\lambda)\,dW_2(u)|\mathcal{F}_t\right] \\ &= \theta^2\mathbb{E}^{\mathbb{Q}}\Big[\Big(\int_t^T A(u,T,\lambda)\,dW_2(u)\Big)^2|\mathcal{F}_t\Big] \\ &= \theta^2\int_t^T A^2(u,T)\,du \quad \text{(Itô isometry)} \\ &= \frac{\theta^2}{\lambda^2}\int_t^T \left(1-e^{-\lambda(T-u)}\right)^2 du \\ &= \frac{\theta^2}{\lambda^2}\Big((T-t) - 2A(t,T,\lambda) + A(t,T,2\lambda)\Big). \end{aligned}$$

and

$$V_2^2 = \sigma^2(T-t)$$

Finally:

$$\begin{aligned}
V_3^2 &= \operatorname{cov}^{\mathbb{Q}}\big(\theta\rho\int_t^T A(u,T,\lambda)\,dB^1(u), \sigma W^1(T-t)|\mathcal{F}_t\big)\\
&= \rho\theta\sigma\operatorname{cov}^{\mathbb{Q}}\big(\int_t^T A(u,T,\lambda)\,dB^1(u), W^1(T-t)|\mathcal{F}_t\big)\\
&= \rho\theta\sigma\mathbb{E}^{\mathbb{Q}}\big[\int_t^T A(u,T,\lambda)\,dB^1(u)\int_t^T dW_1(u)|\mathcal{F}_t\big]\\
&= \rho\theta\sigma\int_t^T A(u,T,\lambda)\,du\\
&= \frac{\rho\theta\sigma}{\lambda}\big((T-t)-A(t,T,\lambda)\big).
\end{aligned}$$

Putting all together we have:

$$\begin{aligned}
F(t,T) &= \mathbb{E}^{\mathbb{Q}}\big[S(T)|\mathcal{F}_t\big]\\
&= S(t)\exp\big(-\frac{1}{2}\sigma^2(T-t)\big)\exp\Big(M+\frac{1}{2}V^2\Big)\\
&= S(t)\exp\Big(M+\frac{1}{2}V_1^2+V_3^2\Big).
\end{aligned}$$

Appendix 4.2 Forward and futures prices under stochastic interest rates

From Musiela & Rutkowski (2006), we have that:[11]

$$F(t,T) = \mathbb{E}^{\mathbb{Q}}\big[S(T)|\mathcal{F}_t\big] \tag{A.2.1}$$

and[12],[13]

$$\begin{aligned}
\text{For}(t,T) &= \mathbb{E}^{\mathbb{T}}\big[S(T)|\mathcal{F}_t\big]\\
&= \frac{S(t)}{B(t,T,r(t))}.
\end{aligned} \tag{A.2.2}$$

[11]See Definition 11.5.1 on page 418, Musiela & Rutkowski (2006).
[12]See Lemma 9.6.2 on page 342, Musiela & Rutkowski (2006).
[13]See equation 9.30, on page 341, Musiela & Rutkowski (2006).

The futures price at time t maturing at time T is:

$$\begin{aligned} F(t,T) &= \mathbb{E}^{\mathbb{Q}}\big[S(T)|\mathcal{F}_t\big] \\ &= \mathbb{E}^{\mathbb{Q}}\big[S(t)\exp\Big(-\frac{1}{2}\sigma^2(T-t)+\int_t^T r(u)\,du+\sigma W_1(T-t)\Big)|\mathcal{F}_t\big] \\ &= S(t)\exp\Big(M+\frac{1}{2}V_1^2+V_3^2\Big), \end{aligned}$$

We also have:[14]

$$\begin{aligned} B(t,T,r(t)) &= \mathbb{E}^{\mathbb{Q}}\big[\exp(-\int_t^T r(u)\,du)|\mathcal{F}_t\big] \\ &= \exp\Big(-M+\frac{1}{2}V_1^2\Big). \end{aligned}$$

So $F(t,T) \neq \text{For}(t,T)$. The quotient of forward and futures is called the convexity adjustment that we would need to consider when we hedge an option using futures contracts and it is given by:

$$\frac{\text{For}(t,T)}{F(t,T)} = e^{-V_1^2-V_3^2}. \tag{A.2.3}$$

When $\rho = 0$, we have

$$\frac{\text{For}(t,T)}{F(t,T)} = e^{-V_1^2}. \tag{A.2.4}$$

Appendix 4.3 Using short-dated forward to hedge long-dated options

Let the maturity of the option be T and the maturity of the hedging forward contract be T_F. Let the forward price of the underlying forward contract of the option be $F(t,T)$ and the forward price of the hedging forward contract be $F(t,T_F)$. We assume that $T_F \leq T$.

$$F(t,T) = \frac{S(t)}{B(t,T)} \Leftrightarrow S(t) = B(t,T)F(t,T)$$

and

$$F(t,T_F) = \frac{S(t)}{B(t,T_F)} \Leftrightarrow S(t) = B(t,T_F)F(t,T_F).$$

[14]See equation (9.14) on page 333, Musiela & Rutkowski (2006).

We have:

$$F(t,T) = \frac{B(t,T_F)F(t,T_F)}{B(t,T)}$$

and

$$\frac{\partial F(t,T)}{\partial F(t,T_F)} = \frac{B(t,T_F)}{B(t,T)}.$$

If δ amount of forward contracts with maturity T is required to hedge the option, we would need to use $\delta\frac{B(t,T_F)}{B(t,T)}$ amount of forward contracts with maturity T_F to hedge.

CHAPTER 5

Empirical hedging performance on long-dated crude oil derivatives

This chapter conducts an empirical study on hedging long-dated crude oil futures options with forward price models developed in Chapter 2, incorporating stochastic interest rates and stochastic volatility. Several hedging schemes are considered including delta, gamma, vega and interest-rate hedging. The factor hedging method is applied to the proposed multi-dimensional models, and the corresponding hedge ratios are computed and their implications discussed. Hedging instruments with different maturities are also considered to gauge the impact of hedging long-dated crude oil options with futures of shorter maturities. This chapter is based on the working paper of Cheng, Nikitopoulos & Schlögl (2016*c*).

5.1. Introduction

Motivated by the debacle of the German company Metallgesellschaft (MG) at the end of 1993, several research papers have investigated the methods and risks in hedging long-dated over-the-counter forward commodity contracts by using short-dated contracts, typically short-dated futures. The investigation conducted in Edwards & Canter (1995) and Brennan & Crew (1997) conclude that MG's stack-and-roll hedging strategy was flawed, and it exposed the company to huge basis risks. While these earlier papers mainly concentrate on using a single short-dated futures contract to hedge long-dated forward commitments, literature on hedging forward commitments with several short-dated futures contracts includes Neuberger (1999), Veld-Merkoulova & De Roon (2003), Bühler et al. (2004) and Shiraya & Takahashi (2012).

Trolle & Schwartz (2009*b*) propose a multi-dimensional stochastic volatility model for commodity derivatives featuring unspanned stochastic volatility. They show that adding options to the set of hedging instruments significantly improves hedging of volatility trades, such as straddles, compared to using futures only as hedging instruments. Dempster et al. (2008) propose a four-factor model for two spot prices and their convenience yields and develop closed-form pricing and hedging formulae for options on spot and futures spreads of commodity. Chiarella et al. (2013) empirically demonstrate that hump-shaped volatility specifications reduce the hedging error of crude oil volatility trades (straddles) compared to the exponential volatility specification counterpart. However, all these papers assume deterministic interest rates, and it is unclear to what extent these models can provide adequate hedges for long-dated options positions. The research literature on hedging long-dated commodity option positions is rather limited and this chapter aims to make a contribution by empirically investigating the hedging of crude oil futures options with maturities up to six years.

In this hedging analysis, we use the Light Sweet Crude Oil (WTI) futures and option dataset from the NYMEX[1] spanning a 6-year period from January 2006 to December 2011. A call futures option maturing in December 2011 is hedged from 3rd July, 2006 to 31st October, 2011. During this 6-year period, a number of major events happened, such as the GFC and the Arab Spring and Libyan revolution that had significant impact on spot crude oil prices, crude oil futures, and its options, as well as interest rates. The gross domestic product growth rates of China and India have increased exponentially during the decade of the 2000s. To support the growth of the economy, the consumption of energy also increased. In particular, China's demand of crude oil grew at 7.2% annual logarithmic rate between 1991 and 2006. This phenomenal growth rate, among other factors, increased the demand in crude oil. On the supply side, Saudi Arabia, the biggest oil exporter in the world, had actually reduced its crude oil production in 2007. Consequently, crude oil prices increased sharply, sending the price to a high of $145 per barrel on 3rd

[1]The database was provided by CME.

July, 2008, which was immediately followed by a spectacular collapse in prices, and by the end of 2008, the spot crude oil price was below $40 per barrel [see Hamilton (2009) and Hamilton (2008)]. The US Treasury yields were above 4.5% before the July 2007 and had decreased steadily to nearly zero from the end of 2008, after the global financial crisis [see Figure 3.2].

The empirical analysis in this paper considers a position in a long-dated call option on futures with a maturity of December 2011, and this option is hedged over five years using several hedging schemes such as delta hedge, delta-vega hedge and delta-gamma hedge. For the purpose of comparison, two models are used to compute the suitable hedge ratios; one with stochastic interest rates as modelled in Chapter 2 and one with deterministic interest rates fitted to a Nelson & Siegel (1987) curve. These models are estimated from historical crude oil futures and option prices and Treasury yields by using the extended Kalman filter; see Section 3.3 for a similar estimation application. The hedge ratios for delta-interest-rate (delta-IR), delta-vega-interest-rate (delta-vega-IR) and delta-gamma-interest-rate (delta-gamma-IR) hedges are also computed, but they are limited to the stochastic interest-rate model. The hedge ratios are derived using the factor-hedging methodology which is presented in details in Section 5.2 and the effectiveness of which has been verified in Section 4.4.

From this empirical analysis, several observations have emerged. Firstly, when an interest-rate hedge is added to the delta, gamma and vega hedge, there is a consistent improvement to the hedging performance, especially, when shorter maturity contracts are used to roll the hedge forward (thus, more basis risk is present). Secondly, because of the high interest-rate volatility, interest-rate hedging was more important during the GFC than in recent years. Over periods of high interest-rate volatility (for instance, pre-GFC and during GFC), and when shorter maturity hedging contracts are used, the delta-IR hedge consistently improves hedging performance compared to delta hedge, while adding a gamma or vega hedge to the delta hedge worsens hedging performance. Thirdly, due to lower

basis risks, using hedging instruments with maturities closer to the maturity of the option to be hedged reduces the hedging error. Thirdly, the hedging performance from the stochastic interest-rate model is consistently better than the hedging performance from the deterministic interest-rate model, with the effect being more pronounced during the GFC. However, there is only marginal improvement over the deterministic interest-rate model during pre-crisis period, and no noticeable improvement after 2010.

The remainder of the chapter is structured as follows. Section 5.2 presents the application of the factor hedging methodology on the three-dimensional stochastic volatility/stochastic interest-rate forward price model developed in Chapter 2. It also derives the hedge ratios for a variety of hedging schemes including delta, delta-IR, delta-vega, delta-vega-IR, delta-gamma and delta-gamma-IR. Section 5.3 describes the methodology to assess hedging performance on long-dated crude oil futures options, including the details of the dataset used in this hedging analysis. Section 5.4 presents the empirical results and discusses their implications. Section 5.5 concludes.

5.2. Factor hedging for a stochastic volatility/stochastic interest-rate model

Factor hedging is a broad hedging method that allows one to hedge simultaneously multiple factors and multiple dimensions impacting the forward curve of commodities, the instantaneous volatility component and the interest-rate variation and subsequently the value of commodity derivatives portfolios. By considering the n-dimensional stochastic volatility and N-dimensional stochastic interest-rate model developed in Chapter 2, to hedge the n-dimensional forward rate risks [that is $W_i(t)$ for $i = 1, 2, \ldots, n$], it is necessary to use n number of hedging instruments such as futures contracts. To further hedge the n number of volatility risks [that is $W_i^\sigma(t)$ for $i = 1, 2, \ldots, n$], it is required to use an additional n number of volatility-sensitive hedging instruments such as futures options. Since the proposed model considers stochastic interest rates, the N-dimensional

interest-rate risks (that is $W_i^r(t)$ for $i = 1, 2, \ldots, N$) should be hedged by using N number of interest-rate-sensitive contracts such as bonds. For completeness, we next present the model used to compute the required hedge ratios.

5.2.1. Model Description. The model assumes that the time t-futures price $F(t, T, \sigma_t)$ of a commodity, for delivery at time T, evolves as follows:

$$\frac{dF(t, T, \sigma_t)}{F(t, T, \sigma_t)} = \sum_{i=1}^{n} \sigma_i^F(t, T, \sigma_t) dW_i^x(t),$$

where $\sigma_t = \{\sigma_1(t), \ldots, \sigma_n(t)\}$ and for $i = 1, 2, \ldots, n$,

$$d\sigma_i(t) = \kappa_i(\overline{\sigma}_i - \sigma_i(t))dt + \gamma_i dW_i^\sigma(t)$$

and

$$r(t) = \overline{r}(t) + \sum_{j=1}^{N} r_j(t)$$

where, for $j = 1, 2, \ldots, N$,

$$dr_j(t) = -\lambda_j(t) r_j(t) dt + \theta_j dW_j^r(t).$$

The functional form of the volatility term structure $\sigma_i^F(t, T, \sigma_t)$ is specified as follows:

$$\sigma_i^F(t, T, \sigma_t) = (\xi_{0i} + \xi_i(T - t))e^{-\eta_i(T-t)}\sigma_i(t)$$

with ξ_{0i}, ξ_i, and $\eta_i \in \mathbb{R}$ for all $i \in \{1, 2, 3\}$. For a comparison of the hedging performance between the stochastic and the deterministic interest-rate model, we also consider the deterministic interest-rate counterpart of this three-dimensional model, as described in Section 3.2.2.

5.2.2. Delta hedging. For an n-dimensional model, factor delta hedging requires n hedging instruments such as futures contracts with different maturities is denoted by $F(t, T_j, \sigma_t), j = 1, \ldots, n$. Let $\Upsilon(t, T_M, \sigma_t)$ be the value of the futures option to be hedged

and T_M be its maturity. Let $\Delta\Upsilon^{\delta}_{H,i}$ and $\Delta\Upsilon^{\delta}_{i}$ denote the change in price of the portfolio and the change in the price of the option on futures (which is the target to be hedged) by the ith shock of the uncertainty in the futures curve (that is $dW^{x}_{i}(t)$) respectively and δ_j be the position corresponding to the jth hedging instrument, we have:

$$\Delta\Upsilon^{\delta}_{H,i} \triangleq \Delta\Upsilon^{\delta}_{i} + \delta_1\Delta F_i(t,T_1,\sigma_t) + \delta_2\Delta F_i(t,T_2,\sigma_t) + \ldots + \delta_n\Delta F_i(t,T_n,\sigma_t). \quad (5.2.1)$$

For an n-dimensional model with n amount of hedging instruments, the set of equations in equation (5.2.1) forms a system of n linear equations which can be solved exactly by matrix inversion. However, from numerical results, some of the values of δ_i produced by using this method may be unnecessarily large, and consequently, lead to very large profit and loss of the hedging portfolio. We use an alternative and more general method by simply minimising the sum of the squared hedging errors, $\Delta\Upsilon^{\delta}_{H,i}$:

$$\underset{\delta_1,\ldots,\delta_n}{\text{minimise}}\Big\{(\Delta\Upsilon^{\delta}_{H,1})^2 + (\Delta\Upsilon^{\delta}_{H,2})^2 + \ldots + (\Delta\Upsilon^{\delta}_{H,n})^2\Big\} \quad (5.2.2)$$

with a constraint that $\delta_i \leq \ell$.[2] The changes in the value of the hedging instruments $\Delta F_i(t,T_j,\sigma_t)$ can be approximated by the discretisation of the stochastic differential equation, as follows:[3]

$$\begin{aligned}\Delta F_i(t,T_j,\sigma_t) &= \Big(F_i(t,T_j,\sigma_t)\sigma_i(t,T_j)\Delta W_i(t)\Big) - \Big(F(t,T_j,\sigma_t)\sigma_i(t,T_j)\big(-\Delta W_i(t)\big)\Big)\\ &= 2F(t,T_j,\sigma_t)\sigma_i(t,T_j)\Delta W_i(t).\end{aligned}$$

Lastly, we can calculate the change in the option on futures $\Delta\Upsilon^{\delta}_{i}$ by the ith shock as:

$$\Delta\Upsilon^{\delta}_{i}(t,T_j) = \Upsilon\Big(F_{i,U}(t,T_j,\sigma_t)\Big) - \Upsilon\Big(F_{i,D}(t,T_j,\sigma_t)\Big), \quad (5.2.3)$$

[2] $\ell = 1$ seems to be a good choice to give low hedging errors and the hedging results are stable by varying ℓ slightly.

[3] This first order approximation is sufficient for the study at hand, because any additional accuracy on the calculation of the hedge would be drowned out by the fact that the model is only an approximation of the empirical reality.

where $F_{i,U}$ and $F_{i,D}$ denote the 'up' and 'down' moves of the price of the underlying hedging instrument (e.g. futures price):

$$F_{i,U}(t, T_j, \sigma_t) = F(t, T_j, \sigma_t) + F(t, T_j, \sigma_t)\sigma_i(t, T_j)\Delta W_i(t) \tag{5.2.4}$$

$$F_{i,D}(t, T_j, \sigma_t) = F(t, T_j, \sigma_t) - F(t, T_j, \sigma_t)\sigma_i(t, T_j)\Delta W_i(t). \tag{5.2.5}$$

We note that the change in $W_i(t)$ is normally distributed, so an 'up' or 'down' move is just a notation. An 'up' move does not necessarily mean $\Delta W_i(t) > 0$.

5.2.3. Delta-Vega hedging. The condition in equation (5.2.2) only immunises small risks generated from the uncertainty that directly impacts the underlying asset, for instance, the futures curve. It cannot mitigate risks originating from an instantaneous volatility which may be stochastic. In order to account for the risks of a stochastic volatility process, an additional number of n futures options may be used as hedging instruments to simultaneously immunise both volatility and futures price risks.[4] Let v_i be the number of futures options that have values of $\Psi(t, T_j, \sigma_t)$ for $j = 1, \ldots, n$. The number of futures options $v_1, v_2, \ldots, v_n$ are determined similarly to equation (5.2.2) such that the overall changes in the value of the hedged portfolio due to volatility risk are minimised. Setting up the hedged portfolio, we have:

$$\begin{aligned}\Delta\Upsilon^{\sigma}_{H,i} \triangleq \Delta\Upsilon^{\sigma}_{i} + v_1\Delta\Psi^{\sigma}_{i}\big(F(t,T_1),t,T_1\big) + v_2\Delta\Psi^{\sigma}_{i}\big(F(t,T_2),t,T_2\big) + \\ \ldots + v_n\Delta\Psi^{\sigma}_{i}\big(F(t,T_n),t,T_n\big),\end{aligned} \tag{5.2.6}$$

where $\Delta\Psi^{\sigma}_{i}\big(F(t,T_j),t,T_j\big) = \Psi\big(F(t,T_j),t,T_j,\sigma^{iU}_t\big) - \Psi\big(F(t,T_j),t,T_j,\sigma^{iD}_t\big)$. σ^{iU}_t represents the vector of stochastic volatility processes where the ith element $\sigma_i(t)$ has been shocked by an 'up' movement in $W^{\sigma}_i(t)$, denoted by $\sigma_{i,U}$ and it can be obtained as follows:

$$\sigma_{i,U} = \sigma_i(t) + \kappa_i\big(\overline{\sigma}_i - \sigma_i(t)\big)\Delta t + \gamma_i\Delta W^{\sigma}_i(t).$$

[4]Note that we are not using a futures option as a hedging instrument that matches exactly both the maturity and strike of the target futures option that we are hedging because if we can use an exactly matching option to hedge then the hedged portfolio would have a value of zero at all times.

σ_t^{iD} is interpreted in a similar way:

$$\sigma_{i,D} = \sigma_i(t) + \kappa_i\big(\overline{\sigma}_i - \sigma_i(t)\big)\Delta t - \gamma_i \Delta W_i^{\sigma}(t).$$

The hedging portfolio consisting of one target option to be hedged and n additional options, with number of contracts v_i, which are determined by a similar minimisation procedure to the one outlined in equation (5.2.2). This portfolio, however, is not delta-neutral. To make this hedging portfolio delta-neutral also, an additional n number of futures contracts with number of positions δ_i are included in the vega-neutral portfolio. Then the number of positions δ_i are determined such that the portfolio is both delta- and vega-neutral. Thus, a portfolio that is simultaneously delta- and vega-neutral should satisfy:

$$\begin{aligned}\Upsilon_H^{\delta\sigma} \triangleq \Upsilon + \delta_1 F(t,T_1) + \delta_2 F(t,T_2) + \ldots + \delta_n F(t,T_n) + \\ v_1 \Psi\big(F(t,T_1),t,T_1\big) + v_2 \Psi\big(F(t,T_2),t,T_2\big) + \ldots + v_n \Psi\big(F(t,T_n),t,T_n\big).\end{aligned} \tag{5.2.7}$$

Since the 'up' and 'down' shocks of the stochastic volatility process σ_t have no impact on futures prices, we can determine $v_1, \ldots, v_n$ by applying n volatility shocks to equation (5.2.8) to calculate $\Delta\Upsilon_{H,i}^{\sigma}$ and minimise its squared hedging errors as shown in equation (5.2.2). The next step is to determine $\delta_i, \ldots, \delta_n$ by applying n shocks to equation (5.2.7) (with v_i determined previous and are held as constants), and then minimising its squared errors.

5.2.4. Delta-Gamma hedging. Delta hedging can provide sufficient protection against small price changes, but not against larger price changes. To hedge larger price changes, a second order hedging is required to take into account the curvature of option prices, in other words, gamma hedging. Gamma measures the rate of change of the option's delta with respect to the underlying, which is equivalent to the second derivative of the option price with respect to the underlying. If gamma is low, re-balancing of the portfolio infrequently may be sufficient, because the delta of the option does not vary much when the underlying moves. However, if gamma is high, it is necessary to re-balance the portfolio frequently because when the underlying moves, the delta of the option is not accurate

anymore. Hence, it cannot be used as an efficient hedge against market risk. The hedging portfolio is the same as the portfolio in equation (5.2.7), but we use a different method to calculate the positions $v_1, v_2, \ldots, v_n$. To determine the amount of positions, we construct the change of the hedging portfolio given by:

$$\Delta\Upsilon^{\gamma}_{H,i} \triangleq \Delta\Upsilon^{\gamma}_{i} + v_1\Delta\Psi^{\gamma}_{i}\big(F(t,T_1),t,T_1\big) + v_2\Delta\Psi^{\gamma}_{i}\big(F(t,T_2),t,T_2\big) + \\ \ldots + v_n\Delta\Psi^{\gamma}_{i}\big(F(t,T_n),t,T_n\big), \tag{5.2.8}$$

where

$$\Delta\Psi^{\gamma}_{i}\big(F(t,T_j),t,T_j\big) = \Psi\big(F_{i,U}(t,T_j),t,T_j,\sigma_t\big) - 2\Psi\big(F(t,T_j),t,T_j,\sigma_t\big) \\ + \Psi\big(F_{i,D}(t,T_j),t,T_j,\sigma_t\big).$$

$F_{i,U}$ and $F_{i,D}$ are the 'up' and 'down' moves of the futures price as defined in equation (5.2.4) and equation (5.2.5). The change of the target futures option $\Delta\Upsilon^{\gamma}_{i}$ is calculated similar to $\Delta\Psi^{\gamma}_{i}\big(F(t,T_1),t,T_1\big)$. The rest of the steps to determine $v_1, \ldots, v_n$ are exactly the same as in subsection 5.2.2.

5.2.5. Delta-IR, Delta-Vega-IR and Delta-Gamma-IR hedging. To immunise N risks from the interest-rate shocks, N additional bond contracts with different maturities are required. All together, to apply a delta-vega-IR factor hedge , we need n number of futures with different maturities, n number of options on futures with different maturities and N number of bond contracts with different maturities. Since the shocks from the stochastic volatility process have no impact on bonds and futures, the first step is to determine $v_1, \ldots, v_n$, which are exactly the same as in equation (5.2.8). The shocks from the interest-rate process have an impact on futures options as well as on bonds, so the next step is to determine the number of bond contracts by minimising the sum of the squared errors of the hedging portfolio consisting options and bonds. The construction of a delta-gamma-IR factor hedge and the method to determine the number of the bond contracts follow the same procedure as the delta-vega-IR factor hedge, and the detail is, therefore,

omitted. To construct a delta-IR hedge we need n number of futures with different maturities, and N number of bond contracts with different maturities. Options are not used as hedging instruments to construct this delta-IR hedge.

5.3. Hedging Futures Options

Crude oil futures options with maturities beyond five years are hedged by using the above mentioned hedging schemes. The forward price model developed in Chapter 2 that incorporates stochastic volatility, and stochastic interest rates is employed to compute the required hedging ratios. For a comparison of the hedging performance between the stochastic and the deterministic interest-rate model, we also consider the deterministic interest-rate counterpart of this three-dimensional model, as fitted to a Nelson & Siegel (1987) curve, see Section 3.2.2 for details. To capture the impact of stochastic interest rates in the hedging performance, hedging under deterministic interest-rate specifications is compared to stochastic interest-rate specifications. Furthermore, to assess the hedging performance of these schemes in terms of the choice of the hedging instruments, several scenarios are considered for varying maturities of the hedging instruments.

5.3.1. Methodology. A 6-year crude oil dataset of futures and options is used in the investigations starting from January 2006 till December 2011. Crude oil futures options that mature in December are considered as the target option to hedge because December contracts are the only contracts with maturities over five years in our dataset. June contracts are also liquid but they have a maximum maturity of only four years in the dataset. There are only a few futures options with a maturity of December 2011, and with non-zero open interest that persist throughout the whole 5-year period. Thus, the futures option with a strike of $\$62$ and a maturity of December 2011 is used as the target option to be hedged.

The 6-year crude oil dataset of futures and options is sub-divided into 12 half-year periods. The first sub-period is between January 2006 and June 2006, the second is between July 2006 and December 2006, the third sub-period is between January 2007 and June 2007 and so on. The last sub-period, which is the 12th, is between July 2011 and October 2011. Since a crude oil option on futures contract ceases trading around the 17th of a month prior to the maturity month of the futures option, a December contract would cease trading around the 17th of November,[5] and in the dataset, we exclude options with maturities under 14 calendar days, the hedging performance analysis is terminated at the end of October 2011. For each of the 12 sub-periods, a set of model parameters is estimated for the stochastic interest-rate model and for the deterministic interest-rate model following the Kalman filter methodology outlined in equation (3.3).

We use model parameters from previous sub-period together with the current state variables and equation (3.2.3) to estimate the hedge ratio for the current sub-period. For example, during the hedging analysis between July 2006 and December 2006, the model parameters estimated in the first sub-period are used together with the up-to-date state variables to derive the hedging ratios. This procedure is followed until October 2011. The idea of this is that out-of-sample model parameters are used when we derive the hedge ratios, i.e., the hedge ratios only use market information which is available at the time for which they are constructed.

To compare differences in the hedging performance of a long-dated option between using longer maturity hedging instruments and shorter maturity hedging instruments, four scenarios are investigated where the hedging instruments have varying maturities. For the three-dimensional model used, as illustrated in Section 5.2, to construct a delta-hedged portfolio, it is necessary to use three hedging instruments, thus, futures contracts with three different maturities. In theory any three different maturities would be fine. But in practice, the liquidity of futures contracts with a maturity of more than a year decreases

[5]See CME's Crude Oil Futures Contract Specs and Crude Oil Options Contract Specs.

significantly with an exception of futures maturing in June or December. Only futures contracts maturing in December are available with a maturity of more than 3 years. So in this empirical analysis, June and December futures contracts with maturities of less than 3 years are considered, and only December contracts with a maturity longer than 3 years.

Scenario 1	**Scenario 2**	**Scenario 3**	**Scenario 4**
Dec-2006	Dec-2007	Dec-2008	Dec-2009
Jun-2007	Jun-2008	Jun-2009	Dec-2010
Dec-2007	Dec-2008	Dec-2009	Dec-2011
Jun-2008	Jun-2009	Jun-2010	Jun-2012
Dec-2008	Dec-2009	Dec-2010	Dec-2012
Jun-2009	Jun-2010	Jun-2011	Jun-2013
Dec-2009	Dec-2010	Dec-2011	
Jun-2010	Jun-2011	Jun-2012	
Dec-2010	Dec-2011	Dec-2012	
Jun-2011	Jun-2012	Jun-2013	
Dec-2011	Dec-2012		
Jun-2012	Jun-2013		
Dec-2012			
Jun-2013			

TABLE 5.1. **Different maturities of contracts used in the four scenarios.** The table displays different maturities of contracts used in these four scenarios. For instance, in Scenario 4 and assuming delta-hedging, a call option on futures is hedged in 3rd July 2006 by using 3 futures contracts with maturities of Dec 2009, Dec 2010 and Dec 2011. June contracts are not included here because on 3rd July 2006, futures with a maturity of June 2010 are not available. As the call option approaches the end of October 2009, the futures with a maturity of Dec 2009 gets rolled over to Jun 2012, while the other two futures contracts with maturities of Dec 2010 and Dec 2011 remain.

In the first scenario, hedging instruments with the three most adjacent June or December contracts are used. For instance, if the trading date is 15th of July, 2006, then hedging instruments with maturities of December 2006, June 2007 and December 2007 are used. By the end of October 2006, the December 2006 contract gets rolled over to June 2008 (and keeping the other two contracts), and so on. The second scenario is similar to the first scenario except that in the beginning of the hedging period the first hedging contract matures in December 2007, followed by June 2008 and December 2008. As the trading

date gets to the end of October 2007, the December 2007 contract gets rolled over to June 2009, and so on. The third and fourth scenarios follow this idea. Table 5.1 shows the maturities of contracts used in these four scenarios. To construct a delta-vega or delta-gamma hedged portfolio, three additional options on futures are required. The maturities of these options are selected according to Table 5.1. Their strikes are selected based on a combination of liquidity and moneyness. We first filter out near-the-money options with strikes that are $\pm 15\%$ away from the at-the-money strike and then out of these strikes, we select the strike with the highest number of open interest.

5.3.2. Monte Carlo simulation. It is unclear what is the best approach to choose the size of the 'up' and 'down' movements ($\Delta W_i(t), \Delta W_i^{\sigma}(t)$ and $\Delta W_j^{r}(t)$) introduced in Section 5.2. So, we follow Chiarella et al. (2013) and determine the size of the movements by using a Monte Carlo simulation approach.

We first consider the Delta hedging scheme outlined in Section 5.2.2. Let k be the index of the Monte Carlo simulation with 1000 iterations (that is, $k = 1, 2, \ldots, 1000$). Let t_d be the trading day and $d = 1, 2, \ldots, 1333$ be the index of the trading day.[6] At t_1, we generate n number of independent ΔW_i by drawing n random samples from a normally distributed random number generator with mean of 0 and variance of $\frac{1}{252}$. Using these n samples and following the steps outlined in Section 5.2.2, we can construct a hedging portfolio consisting the option to be hedged and n number of futures with different maturities and with hedge ratios $\delta_1^k, \ldots, \delta_n^k$. We add the index k in these hedge ratios to explicitly show their dependencies on the n random samples realised in iteration k. The profit and loss (P/L) of the delta-hedged portfolio on trading day t_2 is defined as:

$$\text{P/L}_2^{\delta,k} \triangleq \Delta\Upsilon_2^{\delta} + \delta_1^k \Delta F(t_2, T_1) + \delta_2^k \Delta F(t_2, T_2) + \ldots + \delta_n^k \Delta F(t_2, T_n), \tag{5.3.1}$$

[6]Since we start the hedging scheme from July 2006, we have t_1 representing the trading day of the 3rd of July, 2006, which is a Monday. We note that the 4th of July, 2006 is a public holiday (Independence Day) in the United States, so t_2 represents the trading day of the 5th of July, 2006. t_{1333} represents the last trading day under consideration which is on the 31st of October, 2011.

where $\Delta F(t_2, T_1) = F(t_2, T_1) - \Delta F(t_1, T_1)$ represents the difference of market quoted futures prices between t_1 and t_2 [7] and $\Delta\Upsilon_2^\delta$ represents the difference of the market quoted option prices from t_1 to t_2. In each of the 1000 simulations, we have a total of 1332 P/Ls ($\text{P/L}_2^{\delta,k}$ to $\text{P/L}_{1333}^{\delta,k}$) and we define the root-mean-square errors (RMSEs) as follow:

$$\begin{aligned}
\text{RMSE}_{\text{total}}^{\delta,k} &= \sqrt{\frac{1}{1332}\sum_{d=2}^{1333}\left(\text{P/L}_d^{\delta,k}\right)^2} \\
\text{RMSE}_1^{\delta,k} &= \sqrt{\frac{1}{50}\sum_{d=2}^{51}\left(\text{P/L}_d^{\delta,k}\right)^2} \\
\text{RMSE}_2^{\delta,k} &= \sqrt{\frac{1}{50}\sum_{d=52}^{101}\left(\text{P/L}_d^{\delta,k}\right)^2} \\
&\vdots \\
\text{RMSE}_{27}^{\delta,k} &= \sqrt{\frac{1}{32}\sum_{d=1302}^{1333}\left(\text{P/L}_d^{\delta,k}\right)^2}.
\end{aligned}$$

To calculate the total average of the RMSEs, we simply take the average over 1000 simulations. That is:

$$\begin{aligned}
\text{RMSE}_{\text{total}}^{\delta} &= \frac{1}{1000}\sum_{k=1}^{1000}\text{RMSE}_{\text{total}}^{\delta,k} \\
\text{RMSE}_1^{\delta} &= \frac{1}{1000}\sum_{k=1}^{1000}\text{RMSE}_1^{\delta,k} \\
\text{RMSE}_2^{\delta} &= \frac{1}{1000}\sum_{k=1}^{1000}\text{RMSE}_2^{\delta,k} \\
&\vdots \\
\text{RMSE}_{27}^{\delta} &= \frac{1}{1000}\sum_{k=1}^{1000}\text{RMSE}_{27}^{\delta,k}.
\end{aligned}$$

[7] We drop the dependency of σ_t in these futures prices because these are quoted prices rather than model prices.

In each path, seven hedging schemes[8] are considered, namely: unhedged, delta, delta-IR, delta-vega, delta-vega-IR, delta-gamma and delta-gamma-IR. The results are the RMSEs of the profits and losses of the hedging portfolios. So in each of the 1000 simulations, seven RMSEs represent the variability of the hedging portfolios under different hedging schemes for the whole six-year period.[9] To illustrate the variability of the hedging portfolios at various times over the six-year period, the whole period from July 2006 to the end of 2011 is sub-divided into 27 sub-periods. Each of the sub-periods consists 50 trading days, except the last one. So now, given a scenario, the RMSE represents the RMSE of the profits and losses of the hedging portfolio per path per sub-period per hedging scheme. Then, the average of the RMSE over 1000 simulations for each hedging scheme is computed, for each of the scenarios. Thus, we have 27 by 7 RMSEs, as shown in Figure 5.4.

5.4. Empirical results

The results of our empirical hedging application are depicted in Figure 5.4 and Figure 5.5, where the RMSE of the unhedged portfolio are compared with the RMSE of the hedged portfolios for the six different hedging schemes,[10] under the proposed three-dimensional stochastic volatility/stochastic interest-rate model. Figure 5.4 presents the hedging performance of Scenario 1 and 2, where short-dated futures contracts are used as hedge instruments. Figure 5.5 presents the results for Scenario 3 and 4, where longer-dated contracts are used to hedge (see Table 5.1 for exact contract maturities). Thus, the results are separated into 27 groups of 7 bars. The first group, labelled 'Aug06', represents the RMSE over a 50-trading day period from 5th of July, 2006 to 13th of September, 2006, inclusive. The second group, without label, represents the RMSE over a 50-trading day period from 14th of September, 2006 to 24th of November, 2006 inclusive. We continue

[8]strictly speaking, one unhedged portfolio with just the option itself and six hedging portfolios.

[9]Note that the hedging schemes are all run on the same underlying empirical data set covering 1333 days of daily price history, and as explained above, the Monte Carlo simulations only serve to provide the dW "bumps" used to calculate the positions in each hedge instrument in the factor hedge.

[10]The six hedging schemes are: delta, delta-IR, delta-vega, delta-vega-IR, delta-gamma and delta-gamma-IR.

similarly, so the last group, labelled 'Oct11', represents the RMSE over a 32-trading day period from 15th of September, 2011 to 31st of October, 2011.

Next, we compare the overall hedging error between hedged positions and unhedged positions, and then discuss the performance of each hedging scheme. Finally, we evaluate the performance of hedges based on stochastic interest-rate models to those based on deterministic interest-rate models.

5.4.1. Unhedged vs. hedged positions. Table 5.2 reports the improvement in the overall hedging error for a range of hedging schemes compared to the unhedged position. Several conclusions can be drawn. Firstly, interest-rate hedge incrementally but consistently improves hedging performance when it is added to delta, delta-vega and delta-gamma hedging. When longer maturity contracts are used as hedging instruments (thus, lower basis risk exists), such as Scenario 4, then the improvement is about 1% while the improvement nearly doubles when shorter maturity contracts are used as hedging instruments. When short-dated contracts are used as hedging instruments, it is necessary to more frequently rollover the hedge, and as a result, a higher basis risk is present. Consequently, part of the basis risk is essentially managed by hedging the interest-rate risks. This is in line with what would be expected theoretically, since, the difference between futures prices for different maturities is in partly due to interest rates (though convenience yields would also play a role).

Secondly, longer maturity hedging instruments (Scenario 4) consistently provide a better hedging performance across all hedging schemes compared to the shorter maturity hedging instruments (e.g. Scenario 1). With shorter maturity hedging instruments, the improvement between the different schemes is marginal, ranging between 73.95% and 75.56%. With longer maturity hedging instruments (e.g. Scenario 4), there is a larger

improvement across all schemes, ranging between 82.87% and 85.77%, with the delta-gamma-IR hedge outperforming all other schemes (see Table 5.2). These results for the improvement hold over the entire period, from July 2006 to October 2011, as well as, over three sub-periods in this period characterised as the pre-GFC (July 2006 to December 2007), the GFC (January 2008 to June 2009) and the post-GFC (July 2009 to October 2011), see Table 5.2. As expected, over the more volatile periods, the hedges are less effective, yet the magnitude of improvement between Scenario 1 and 4 is far more substantial. For example, between January 2008 to June 2009 (GFC period), delta-vega-IR hedge improves RMSEs (over the unhedged) by 68.38% under Scenario 1, and by 81.13% under Scenario 4.

Three-dimensional stochastic volatility/stochastic interest-rate model						
Scenario	delta	delta-IR	delta-vega	delta-vega-IR	delta-gamma	delta-gamma-IR
	July 2006 – October 2011					
1	74.16%	75.08%	73.95%	74.40%	74.68%	75.56%
2	74.88%	75.66%	74.78%	75.68%	75.42%	76.42%
3	76.68%	77.33%	76.52%	77.46%	76.98%	77.64%
4	82.87%	83.02%	84.18%	84.67%	85.15%	85.77%
	July 2006 – December 2007					
1	44.04%	48.21%	44.56%	43.07%	45.72%	46.58%
2	55.23%	57.53%	57.11%	57.28%	57.93%	59.22%
3	71.48%	71.26%	72.86%	73.71%	72.11%	72.92%
4	78.15%	78.76%	84.63%	84.90%	81.37%	81.69%
	January 2008 – June 2009					
1	67.93%	69.72%	67.66%	68.38%	68.24%	69.54%
2	67.93%	69.72%	67.76%	69.08%	68.24%	69.71%
3	69.05%	70.62%	69.24%	70.63%	69.41%	70.27%
4	78.20%	78.89%	80.49%	81.13%	80.95%	81.88%
	July 2009 – October 2011					
1	90.21%	89.02%	90.41%	90.47%	91.92%	91.99%
2	90.21%	89.02%	90.41%	90.45%	91.92%	91.98%
3	90.21%	89.02%	90.41%	90.48%	91.92%	91.98%
4	90.36%	89.58%	91.08%	91.13%	92.26%	92.33%

TABLE 5.2. **Improvement of hedging schemes over unhedged positions.** This table shows the percentage improvement of the RMSE of the hedged position over the unhedged portfolio for a range of hedging schemes. The improvement is defined to be $1 - \frac{\text{RMSE}_{\text{total}}^{\text{hedged}}}{\text{RMSE}_{\text{total}}^{\text{unhedged}}}$.

5.4.2. Comparison of hedging schemes. Figure 5.4 and Figure 5.5 present the bar plots of RMSEs for Scenario 1 and 2, and Scenario 3 and 4. Figure 5.6 and Figure 5.7 reproduce the same results but with the results of unhedged portfolio omitted (thus, changing the scale of the graph and making it easier to discern differences between the errors for the various hedged positions). A thorough inspection of the hedging performance of the different hedging schemes reveals different patterns over the course of these five years.

Figures 5.1 depicts the annualised monthly standard deviations of the US Treasury yields. Figures 5.2 plots futures prices for four December contracts and Figures 5.3 plots the annualised standard deviation of the logarithm of futures prices, between 2005 and 2013. These figures reveal that there is a period of high volatility of interest rates, high volatility of crude oil futures prices till the beginning of 2008, then there is a period of extreme variation in yields and futures prices between 2008 and 2009 associated with the GFC, and then after that, a period of relatively low variation in interest rates, especially, the 1-year and 2-year yields. Thus, the pre-GFC period, the GFC period and the post-GFC period, as specified above, have been used to discuss the results.

As we have already discussed in Section 5.4.1, an interest-rate hedge typically provides an improvement in the hedging performance, when combined with the other hedging schemes including delta, delta-gamma and delta-vega hedging. This improvement is stronger when the futures option is hedged with shorter maturity contracts, and weakens as the maturity of the hedging instruments increases (thus, lowering the basis risk). During the GFC period, the improvement due to combining an IR hedge with the other hedging schemes such as delta, delta-vega and delta-gamma is substantial (reaching 4.17% for delta hedges, 1.40% for delta-vega hedges and 1.47% for delta-gamma hedges). In the post-GFC period, yields were far less volatile compared to the pre-GFC and GFC periods. Accordingly, the inclusion of IR hedge has a marginal improvement in the pre-GFC period, and no perceptible improvement in the post-GFC period. These observations are consistent with the numerical findings in Chapter 3, where it is demonstrated that during

periods of low interest-rate volatility, stochastic interest rates marginally improve hedging performance. However, during the high interest-rate volatility period of the GFC (see Figures 5.1), adding IR hedge over delta, delta-vega and delta-gamma hedging, considerably reduces the hedging errors.

Furthermore, Table 5.3 reveals that delta-IR hedge outperforms consistently and substantially all other hedging schemes, when the market experiences high interest-rate volatility and the hedge is subject to high basis risk (e.g. see Scenario 1, where shorter maturity hedging instruments are considered, see top panel of Figures 5.5). In particular, during pre-GFC and GFC period, adding IR hedge to delta hedging typically improves hedging performance, while adding gamma or vega to delta hedging sometimes worsens hedging performance. This effect is more evident, when more basis risk is present. Note also that, over periods of extremely high volatility such as during the GFC, the hedge is subject to model risk, since models would not be able to capture well these market conditions. One would expect the delta-vega or the delta-gamma hedging schemes to be more sensitive to this type of model misspecification, i.e., to a mismatch between the assumed stochastic dynamics and the true process generating empirical data, since three short-dated options and three short-dated futures are used to hedge the long-dated option. Consequently, it is a key observation from this analysis that for hedging applications that are subject to model risk or basis risk, an IR hedge tends to improve further delta hedging, something that other hedging schemes, such as gamma and vega, do not attain.[11]

On the other hand, under Scenario 4, that provides a hedge with the least basis risk compared to the other scenarios, the delta-vega-IR hedge tends to perform better during the pre-GFC, while the delta-gamma-IR hedge tends to perform marginally better during the GFC period. In addition, the delta-gamma-IR hedge typically performs the best in the

[11]Even though a stochastic volatility model is used, delta-vega hedge does not provide any noticeable improvement, especially when high basis risk is present such as in Scenario 1. A volatility trade such as a strangle or straddle would have probably been more sensitive to vega hedging compared to an outright options positions.

post-GFC period (see bottom panel of Figures 5.7). During the post-GFC period, the volatility of the futures contracts is low, which makes a volatility hedge less necessary (or even counterproductive). During post-GFC period, where the interest-rate volatilities and levels have been unusually low, the IR hedges do not typically provide further improvement compared to the other hedges. These latter results do not generally depend on the basis risk present in the hedging applications.

Finally, delta-gamma hedging tends to perform better compared to delta-vega hedging. Thus, when hedging long-dated option positions, a gamma hedge seems to be more efficient compared to a vega hedge, and both would improve a delta-only hedge.

5.4.3. Deterministic vs. stochastic interest rates. Table 5.3 displays the percentage improvement of hedging errors (over the unhedged positions) between models with stochastic interest rates and deterministic interest rates, for a range of hedging schemes. Delta, vega and gamma hedges are considered and the option position is hedged using hedge ratios from stochastic interest-rate and deterministic interest-rate model specifications. The objective is to access the effect of stochastic interest-rate model specifications compared to deterministic interest-rate model specifications, on the performance of the non-IR (delta, vega, gamma) hedge ratios, so IR hedges are not considered in this instance. Hedging with hedge ratios from the stochastic interest-rate model consistently improves hedging performance compared to the deterministic interest-rate model for all hedging schemes and scenarios. When considering the improvement over the entire period, from July 2006 to October 2011 (see top panel of Table 5.3), there are marginal differences between hedging with hedge ratios from stochastic and deterministic interest-rate models. However, when comparing the improvement over the three sub-periods, the pre-GFC (July 2006 to December 2007), the GFC (January 2008 to June 2009) and the post-GFC (July 2009 to October 2011), there are noticeable differences. The pre-GFC period and GFC period, where relatively high interest-rate volatility was present, hedging

with hedge ratios from stochastic interest-rate models (compared to deterministic) provides a substantial improvement. As expected, hedging with shorter maturity contracts (Scenarios 1, 2 and 3) yields less efficient hedges, predominantly due to the additional basis risk involved, but the improvement resulting from modelling stochastic interest rates is more marked. For example, in the pre-GFC period and the GFC period the improvement in the delta-vega hedges is 3.11% and 2.76%, respectively under Scenario 1, but only 0.64% and 0.69% respectively under Scenario 4. In the post-GFC period, the improvement is small for all hedging schemes. Thus, hedging with hedge ratios from stochastic interest-rate models in periods of stable market conditions (in particular low interest-rate volatility periods) does not provide any advantage.

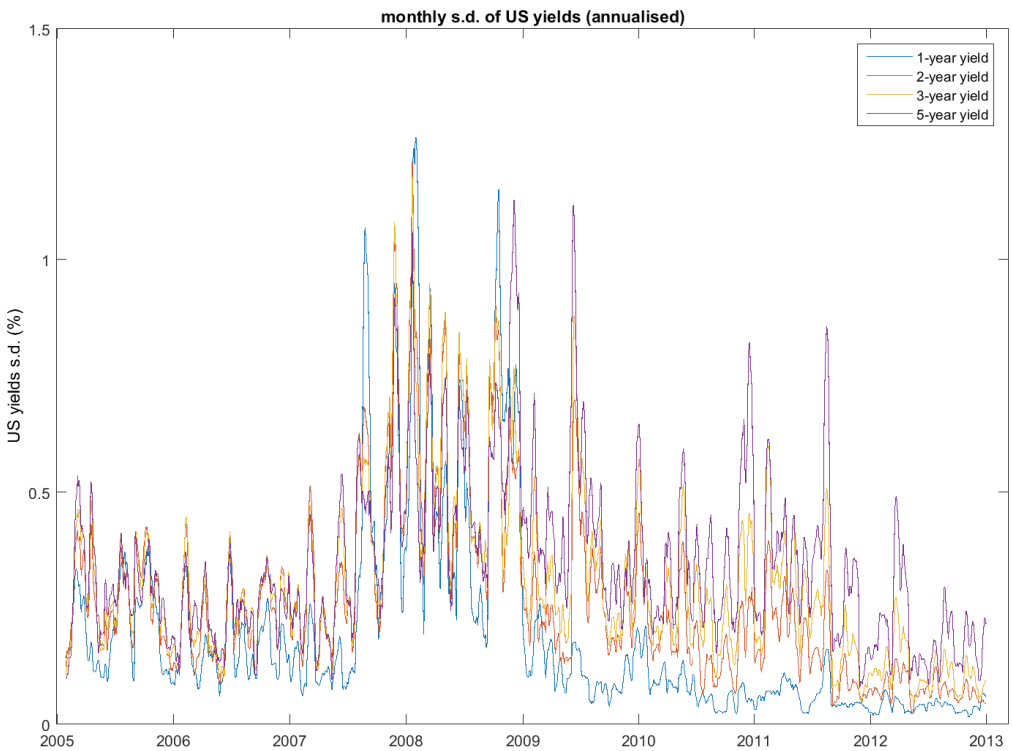

FIGURE 5.1. **Annualised monthly standard deviations of the US Treasury yields.** This plot presents annualised monthly standard deviations are calculated based on computing the standard deviations of the US Treasury yields over 20 trading days and then multiplied by $\sqrt{12}$.

Three-dimensional model with Stochastic IR (s) and Deterministic IR (d)						
Scenario	delta (s)	delta (d)	delta-vega (s)	delta-vega (d)	delta-gamma (s)	delta-gamma (d)
	July 2006 – October 2011					
1	74.16%	72.95%	73.95%	72.87%	74.68%	73.85%
2	74.88%	73.55%	74.78%	73.73%	75.42%	74.68%
3	76.68%	75.33%	76.52%	75.43%	76.98%	75.58%
4	82.87%	81.23%	84.18%	82.83%	85.15%	83.80%
	July 2006 – December 2007					
1	44.04%	42.17%	44.56%	41.45%	45.72%	43.26%
2	55.23%	55.05%	57.11%	53.98%	57.93%	54.46%
3	71.48%	69.67%	72.86%	70.99%	72.11%	72.02%
4	78.15%	77.34%	84.63%	84.00%	81.37%	80.48%
	January 2008 – June 2009					
1	67.93%	65.96%	67.66%	64.90%	68.24%	65.84%
2	67.93%	66.74%	67.76%	64.16%	68.24%	65.86%
3	69.05%	66.63%	69.24%	67.91%	69.41%	66.71%
4	78.20%	77.22%	80.49%	79.80%	80.95%	80.66%
	July 2009 – October 2011					
1	90.21%	88.61%	90.41%	90.15%	91.92%	91.55%
2	90.21%	88.61%	90.41%	90.15%	91.92%	91.55%
3	90.21%	88.61%	90.41%	90.15%	91.92%	91.55%
4	90.36%	89.18%	91.08%	90.92%	92.26%	92.02%

TABLE 5.3. **Comparison between stochastic interest-rate model and deterministic interest-rate model.** This table shows the percentage improvement of the RMSE over the unhedged portfolio between the stochastic interest-rate model and deterministic interest-rate model. We note that there is no interest-rate hedging (i.e., no bonds are used to hedge) in here. In both delta(s) and delta(d) columns, only futures are used as hedging instruments with hedge ratios derived from the stochastic interest-rate model and deterministic interest-rate model, respectively. The other columns follow the same idea.

5.5. Conclusion

This chapter aims to apply the stochastic volatility/stochastic interest-rate forward price model developed in Chapter 2 to hedge long-dated crude oil options over a number of years. The Light Sweet Crude Oil (WTI) futures and option dataset from the NYMEX spanning a 6-year period is used in the empirical analysis. Several hedging schemes are applied to hedge a option on futures from the beginning of 2006 to the end of 2011, including delta-, gamma-, vega- and interest-rate risk hedges. The factor hedging methodology is introduced and its ability to simultaneously hedge multi–dimensional risks impacting

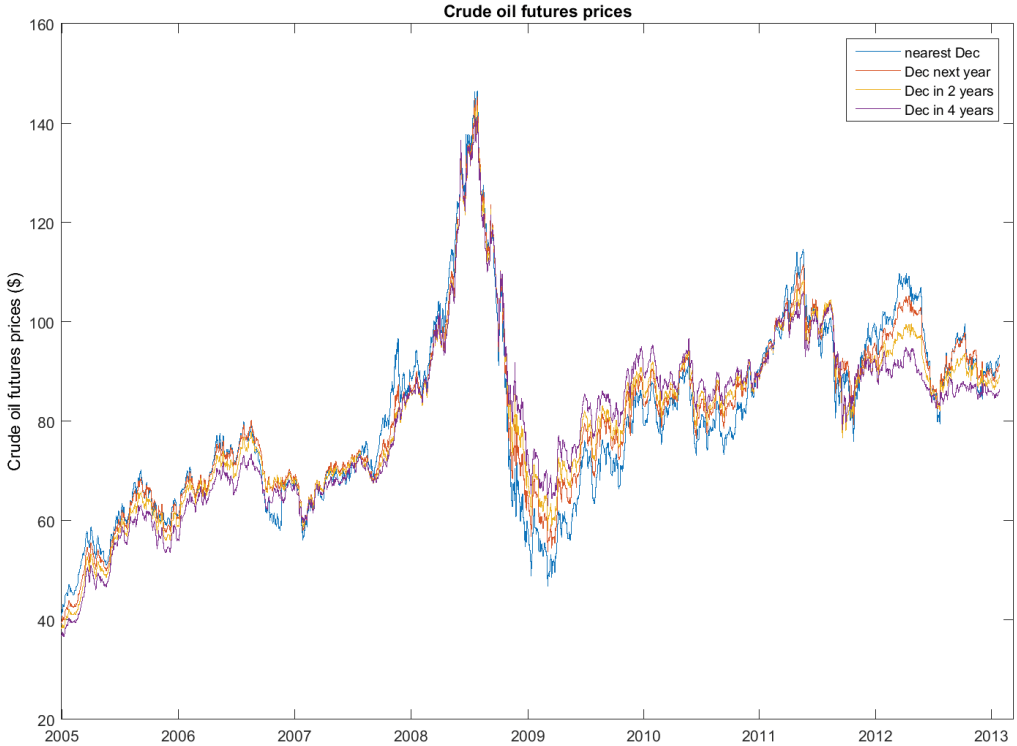

FIGURE 5.2. **Futures prices for four December contracts.** This plot presents the futures prices for four December contracts over eight years.

the forward curve, the stochastic volatility and stochastic interest rates, is demonstrated.

Several conclusions from the empirical analysis can be drawn. Firstly, using hedging instruments with maturities that are closer to the maturity of the option reduces the hedging error. This is predominantly due to the fact that hedging instruments with longer maturities require to roll forward the hedge less frequently; hence, leading to lower basis risks. Secondly, delta-gamma hedging is overall more effective than delta-vega hedging, especially, when the maturities of the hedging instruments are getting shorter. For hedging instruments with longer maturities, delta-vega performs better compared to delta-gamma in the pre-GFC period.

Thirdly, interest-rate hedging consistently improves hedging performance when it is added to delta, delta-vega and delta-gamma hedging, with the improvement being more evident when the shorter maturity contracts are used as hedging instruments (thus, when basis

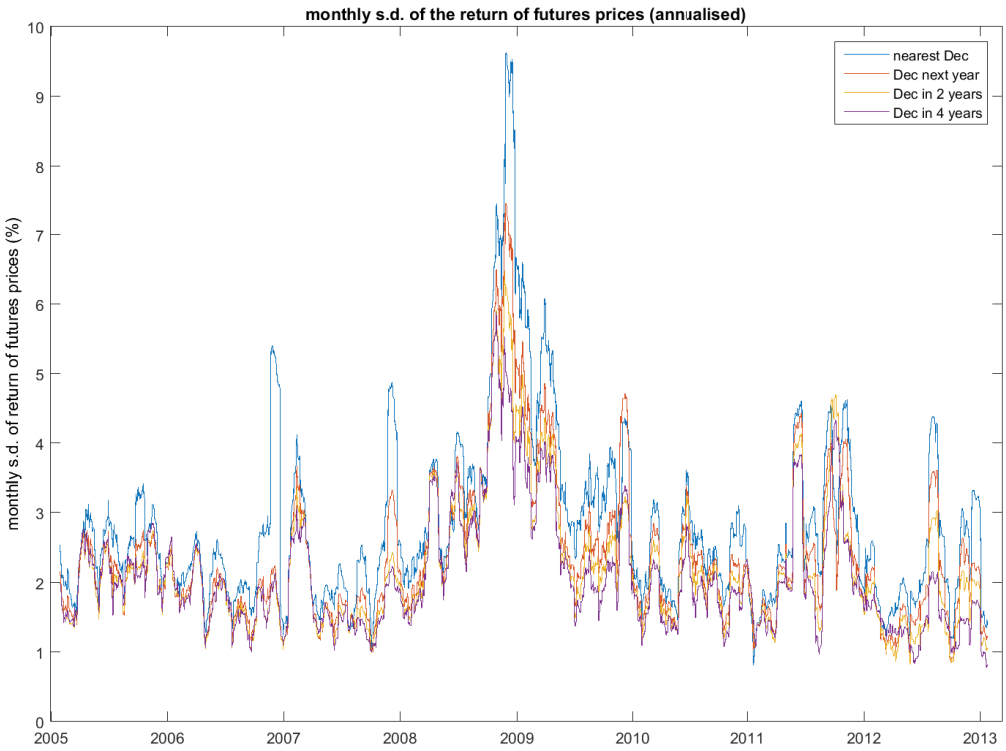

FIGURE 5.3. **Annualised standard deviations of the log-return of futures prices for four December contracts.** This plot presents the annualised standard deviations of the log-return of futures prices for four December contracts over eight years.

risk is higher). Consequently, part of the basis risk is essentially managed via the hedging of the interest-rate risk. Furthermore, interest-rate hedging was more effective during the GFC when interest-rate volatility was very high, while interest-rate hedging does not improve the performance of the hedge in the post-GFC period, where interest rates volatility was extremely low.

Fourthly, comparing the hedging performance between a stochastic interest-rate model and a deterministic interest-rate model, during the GFC, there is a significant improvement from the stochastic interest-rate model over the deterministic interest-rate model. However, there is only marginal hedging improvement from the stochastic interest-rate model during the pre-crisis period and no noticeable improvement at all after 2010.

Lastly, delta-IR hedge often outperforms delta-vega and delta-gamma hedges. When hedging is carried out with shorter maturity hedging instruments and over periods of high interest rates volatility, adding IR hedge to delta hedge improves hedging performance, while adding gamma or vega hedge to the delta hedge deteriorates the hedge. This is a key conclusion: when we have more exposure to model risk (due to turbulent market conditions) and basis risk (due to a mismatch between the maturity of the option to be hedged and the hedge instruments), IR hedge beyond delta hedge can consistently outperform all other hedging schemes.

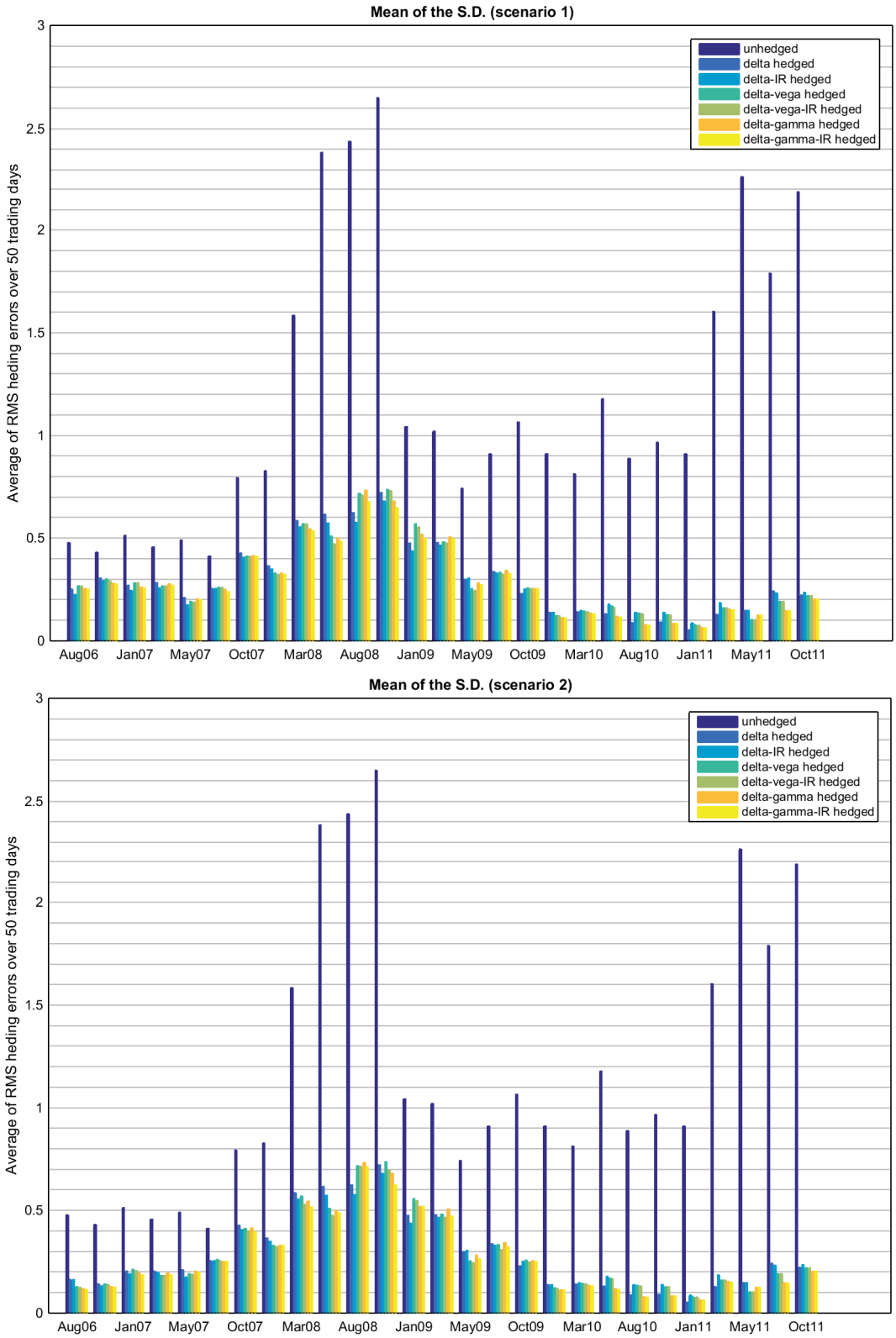

FIGURE 5.4. **RMSEs for Scenario 1 and 2.** These plots compare the average RMSEs over 1000 simulation paths of various hedging schemes under Scenario 1 (top) and Scenario 2 (bottom) with an unhedged position.

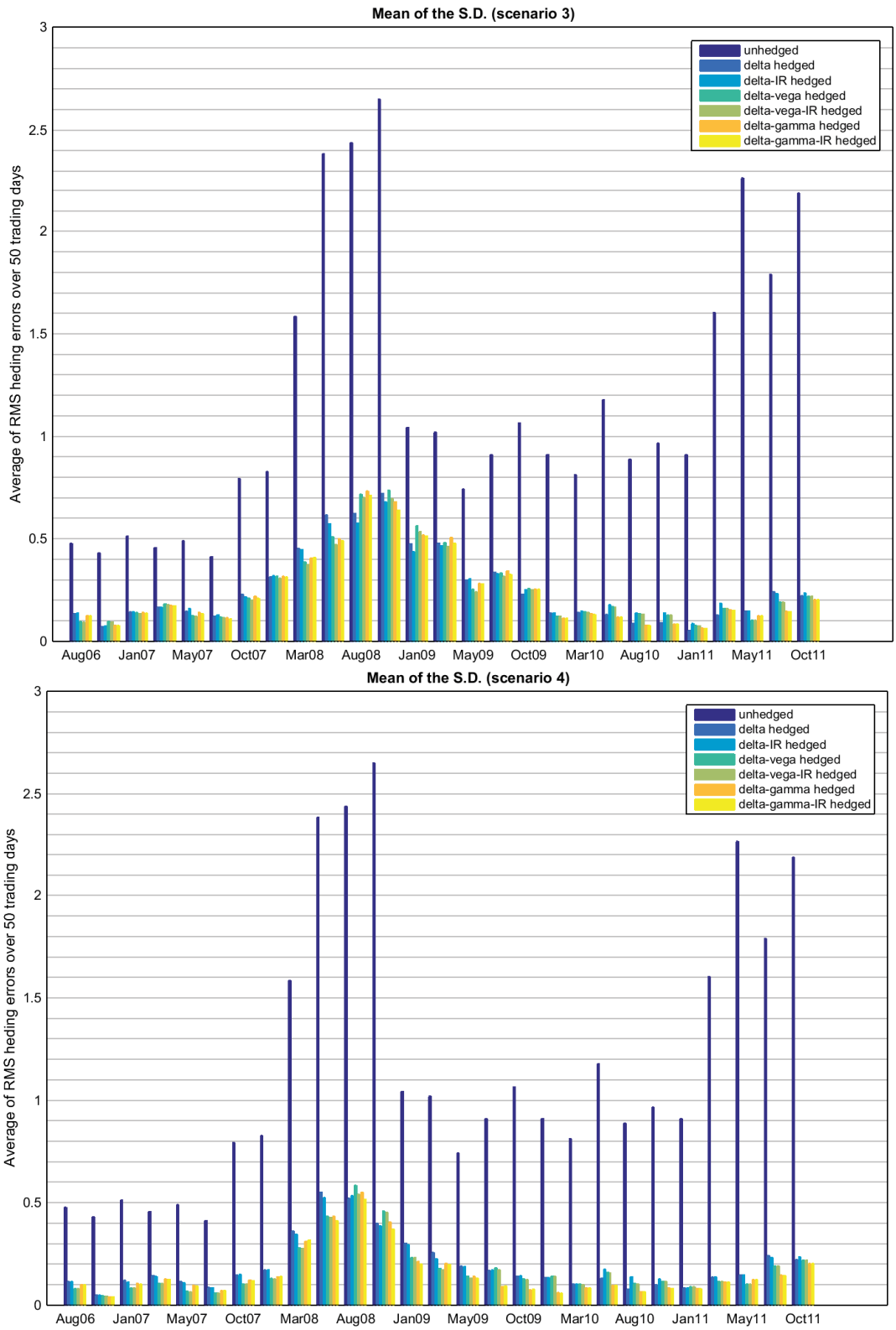

FIGURE 5.5. **RMSEs for Scenario 3 and 4.** These plots compare the average RMSEs over 1000 simulation paths of various hedging schemes under Scenario 3 (top) and Scenario 4 (bottom) with an unhedged position.

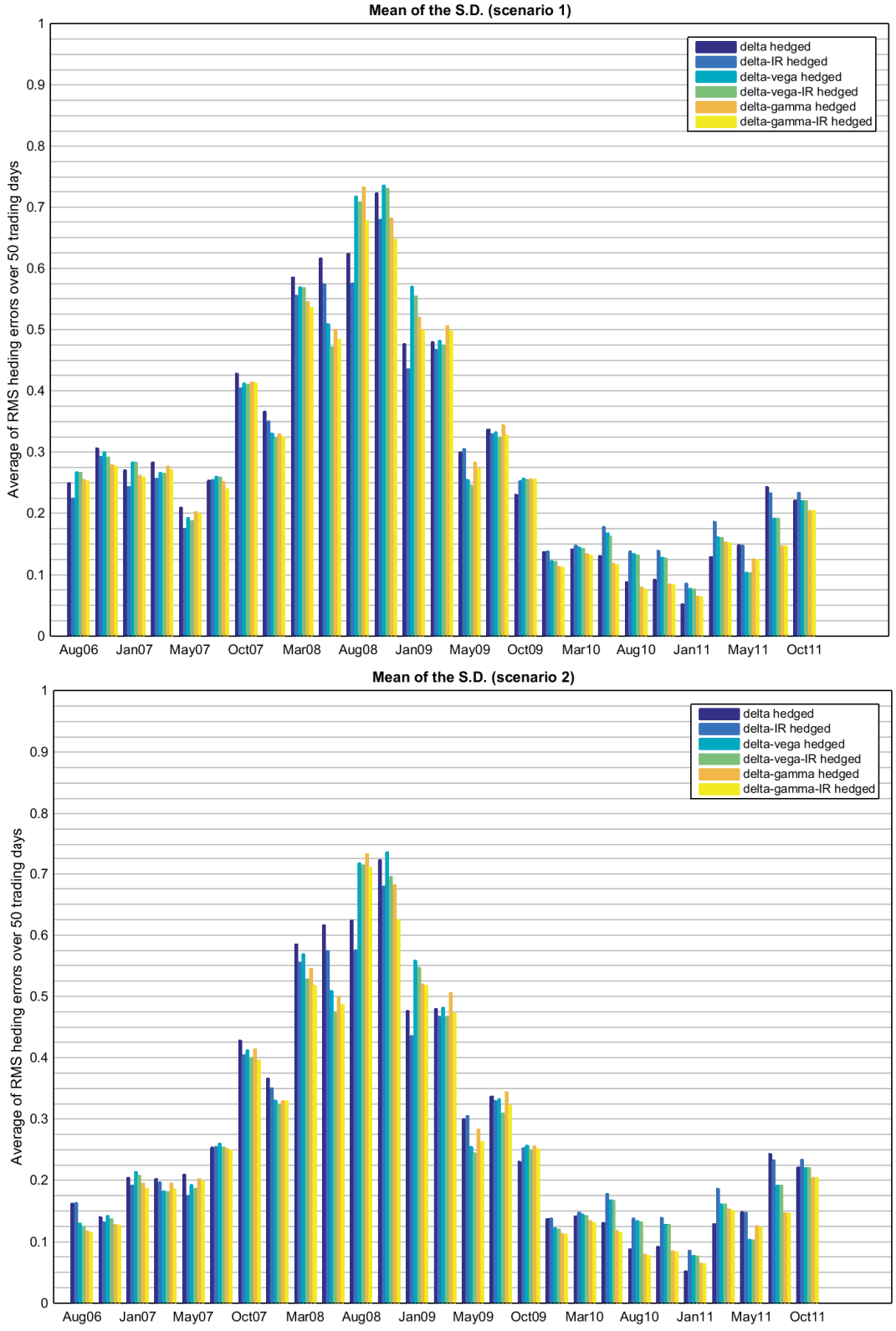

FIGURE 5.6. **RMSEs for Scenario 1 and 2.** These plots compare the average RMSEs over 1000 simulation paths of various hedging schemes under Scenario 1 (top) and Scenario 2 (bottom).

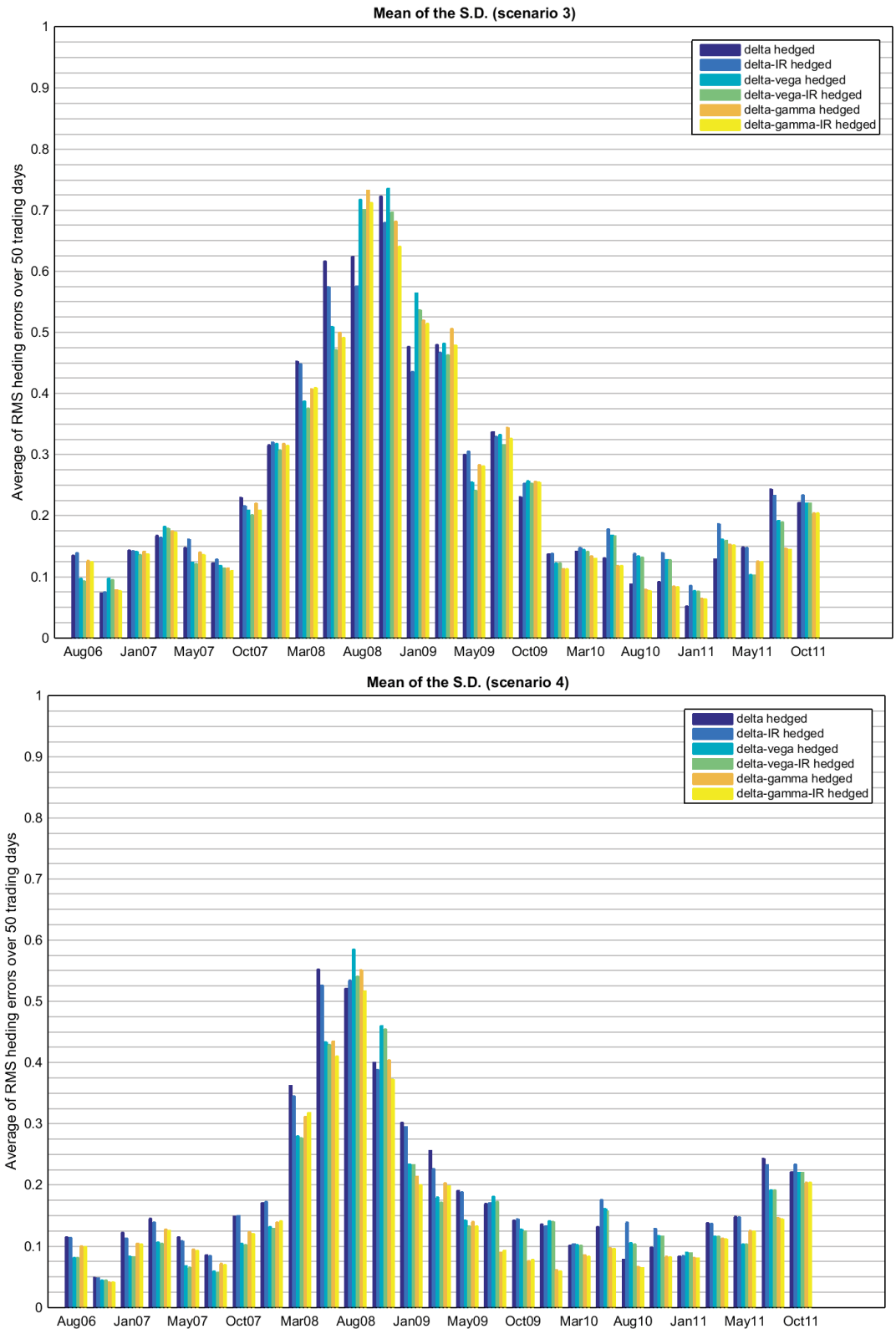

FIGURE 5.7. **RMSEs for Scenario 3 and 4.** These plots compare the average RMSEs over 1000 simulation paths of various hedging schemes under Scenario 3 (top) and Scenario 4 (bottom).

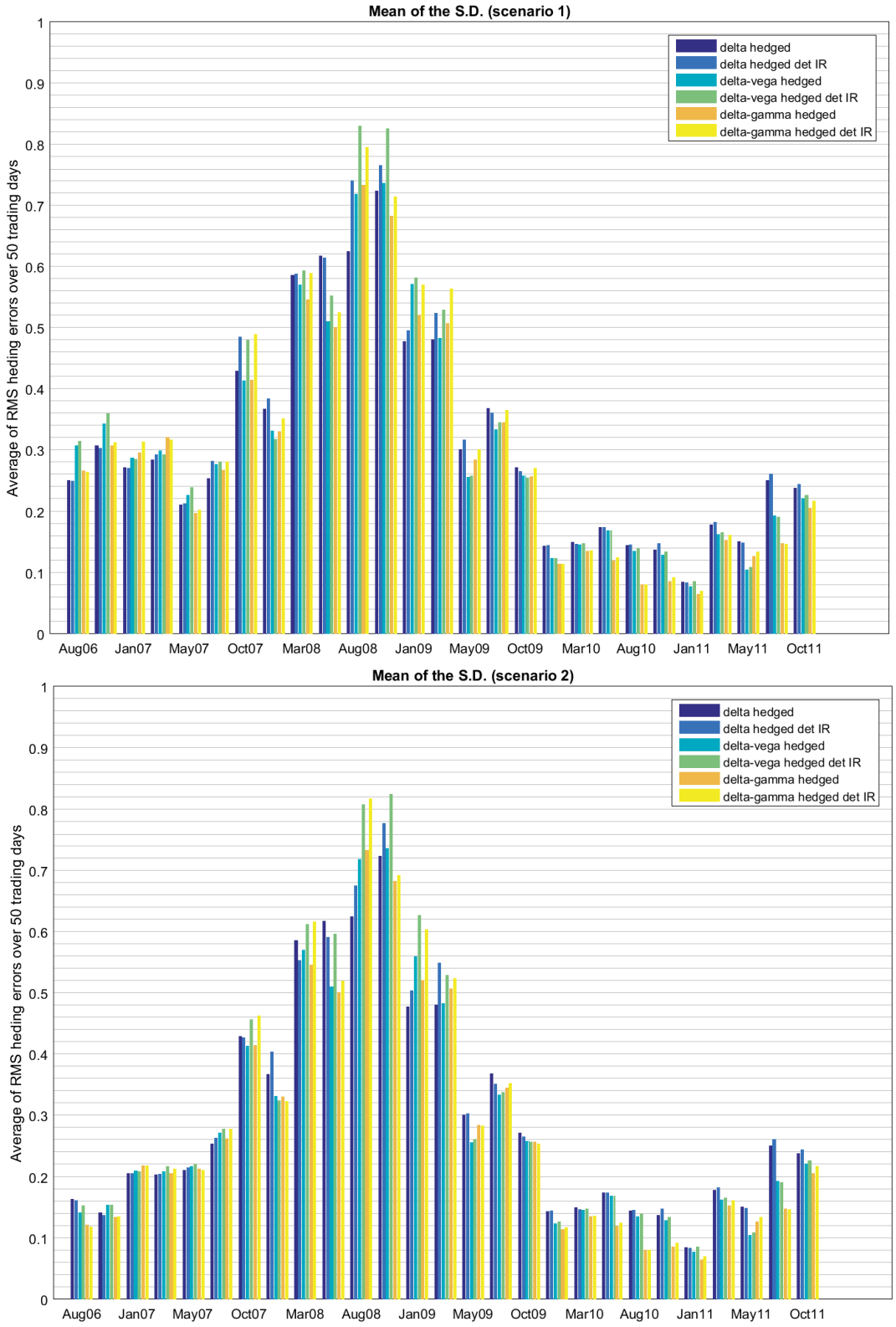

FIGURE 5.8. **RMSEs for Scenario 1 and 2.** These plots compare the average RMSEs of portfolios assuming stochastic and deterministic interest rates under Scenario 1 (top) and Scenario 2 (bottom).

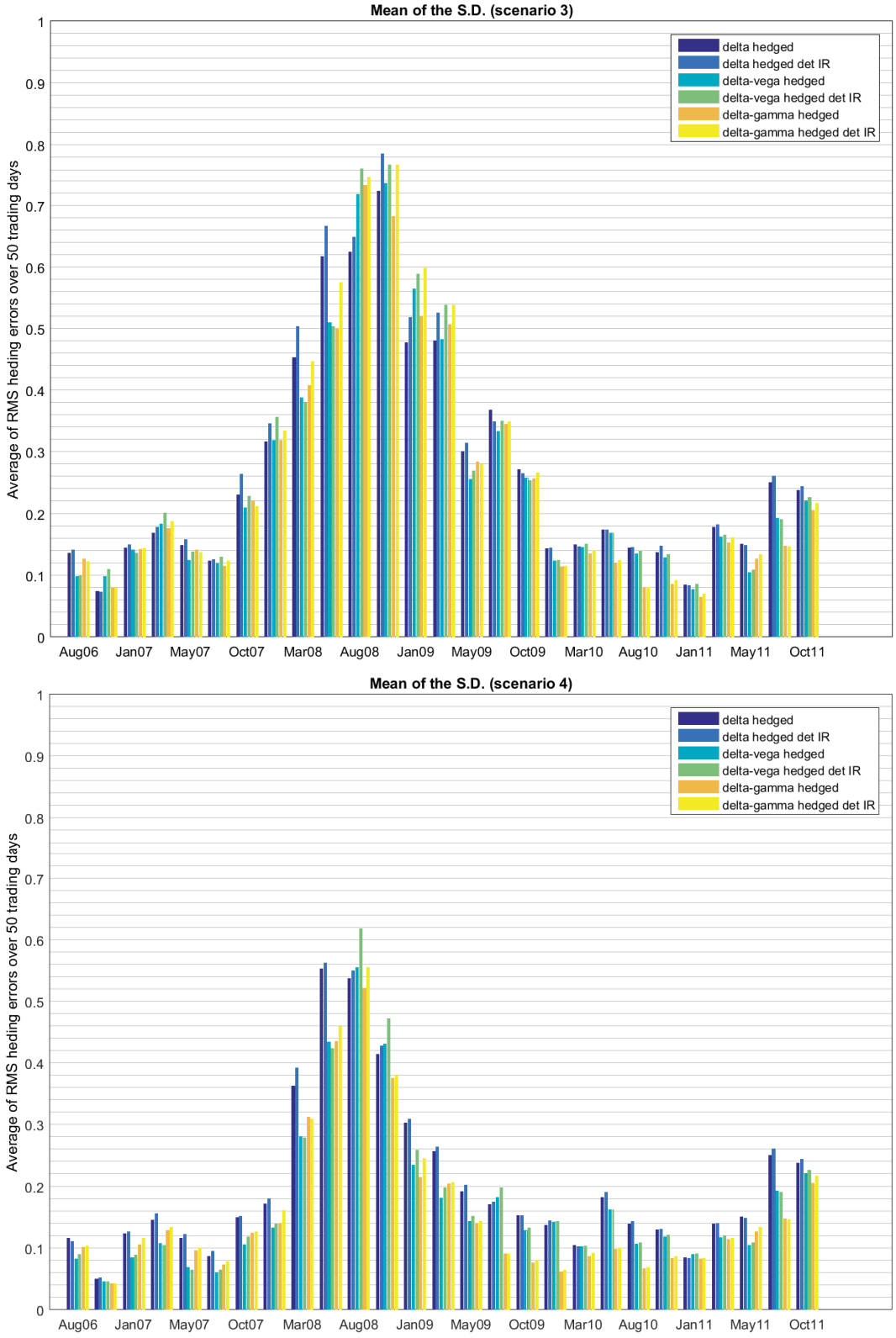

FIGURE 5.9. **RMSEs for Scenario 3 and 4.** These plots compare the average RMSEs of portfolios assuming stochastic and deterministic interest rates under Scenario 3 (top) and Scenario 4 (bottom).

CHAPTER 6

Conclusions

The market has seen a substantial growth in commodity contracts in the last decade with the average daily open interest in crude oil futures contracts of all maturities reaching 1,677,627 in 2013. This phenomenal growth of crude oil contracts is accompanied with an increase in the maturity of futures and option contracts to 9 years in recent years. Motivated by the development of long-dated crude oil contracts in hedging, speculation and investment purposes in the financial sector, this thesis develops a commodity pricing model under the HJM framework, and empirically examines the pricing and hedging performance of long-dated crude oil derivatives.

Since the publication of the seminal paper by Black & Scholes (1973) some research papers have pointed out the unrealistic assumption of constant volatility of the model by comparing to market quotes of option prices and by empirical observations that the distribution of stock prices is leptokurtic. Most prominently, Hull & White (1987), Stein & Stein (1991), Heston (1993), Schöbel & Zhu (1999) and other local volatility models have successfully extended the Black & Scholes (1973) model with various generalisations. The extension to consider stochastic interest rates has not received much attention in the earlier years due to the fact that for short- and medium-term options, the impact of interest-rate risk is of second order compared to volatility or market risks. Bakshi et al. (1997) and Bakshi et al. (2000) conduct a thorough empirical analysis on the merits and drawbacks of option pricing models with different generalisations. Their results show that only for long-term equity anticipation securities with maturities up to three years, models incorporating stochastic interest rates demonstrate some improvements over their deterministic counterparts. Rabinovitch (1989), Turnbull & Milne (1991), Amin & Jarrow

(1992), Kim & Kunitomo (1999), and recently, Fergusson & Platen (2015) have developed the theory and derived solutions for models incorporating stochastic interest rates, but they have not empirically investigated whether featuring stochastic interest rates may provide any benefits.

Inspired by both the importance of stochastic volatility and the growth of long-dated contracts, a new breed of models, namely the hybrid open pricing models, featuring stochastic volatility and stochastic interest rates have emerged. This type of models typically feature a geometric Brownian motion for the spot price process with mean-reverting stochastic volatility and stochastic interest-rate processes. Depending on the type of mean-reverting process and the correlation structure, the model may require some numerical approximations in order to emit closed-form option pricing formulae. Notable papers in this category are van Haastrecht et al. (2009), Grzelak et al. (2012), Grzelak & Oosterlee (2011) and Grzelak & Oosterlee (2012).

One of the earliest commodity pricing models is the Black (1976), and it is based on the Black & Scholes (1973) model. Although this model features some unrealistic assumptions such as the cost-of-carry formula holds, and that net convenience yields are constant, due to the similarity to the original Black-Scholes model, this model has been the most popular commodity pricing model among the industries. Because the convenience yields are assumed to be constant, the Black (1976) model cannot capture mean-revisions as observed in commodity prices. A representative literature of pricing commodity contingent with spot price models with various generalisation of the convenience yields includes Brennan & Schwartz (1985), Gibson & Schwartz (1990), Schwartz (1997), Hilliard & Reis (1998), and Schwartz & Smith (2000). A more recent representative literature includes Cortazar & Schwartz (2003), Casassus & Collin-Dufresne (2005), Geman (2005), Geman & Nguyen (2005), Cortazar & Naranjo (2006), Dempster et al. (2008) and Fusai et al. (2008). The main drawback of spot price commodity models is that the convenience

yields are not observable which make the model estimation more demanding and less intuitive.

Since the seminal paper of Heath et al. (1992), several authors have proposed commodity pricing models where the forward prices are taken as input to the models rather than implied by no-arbitrage arguments from the spot price models. Reisman (1991) proposes a commodity derivatives pricing model under the HJM framework and Cortazar & Schwartz (1994) apply the HJM model with deterministic volatility specifications to analyse the daily prices for all copper futures over a number of years. Trolle & Schwartz (2009*b*) introduce a commodity derivatives pricing model under the HJM framework featuring unspanned stochastic volatility, and Chiarella et al. (2013) consider a commodity forward price model and demonstrate that a hump-shaped crude oil futures volatility structure provides better fit to futures and option prices and improves hedging performance. Other representative literature includes Miltersen & Schwartz (1998), Miltersen (2003), Crosby (2008), Pilz & Schlögl (2013) and Cortazar et al. (2016).

The failure of the German company Metallgesellschaft A.G. sparks a number of research papers to investigate the methods and risks in hedging long-dated over-the-counter commodity contracts by rolling over shorter maturity futures contracts traded in exchanges. Edwards & Canter (1995) conclude that the stack-and-roll hedging strategy MG deployed is essentially flawed and exposed the company to huge basis risk. Other representative literature on hedging commodity commitments includes Brennan & Crew (1997), Neuberger (1999), Veld-Merkoulova & De Roon (2003), Bühler et al. (2004) and Shiraya & Takahashi (2012). The literature on hedging long-dated forward commodity commitments is well developed, however, hedging of long-dated commodity derivatives positions, in particular options, is rather limited. This thesis aims to extend this research by incorporating stochastic interest rates and examining the hedging of long-dated option positions.

6.1. Commodity derivative models with stochastic volatility and stochastic interest rates

Chapter 2 develops a class of forward price models for pricing commodity derivatives that incorporates stochastic volatility and stochastic interest rates, and allows non-zero correlations between the underlying processes. This class of models has the ability to fit the entire initial forward curve by construction, whereby bypassing the modelling and estimation of the unobservable convenience yields. In general, models under the Heath et al. (1992) framework have an infinite dimensional state-space. By suitable specification of the volatility term-structure, the infinite dimensional state-space can be collapsed to a finite dimensional state-space that provides quasi-analytical solutions for futures option prices.

A sensitivity analysis is performed to gauge the impact of the model parameters to commodity derivatives prices. The sensitivity analysis reveals that option prices are sensitive to the volatility of interest rates with the effect being more pronounced for longer-maturity options, especially when the correlation between futures prices and interest rates is negative. Furthermore, the correlation between the interest-rate process and the futures price process has noticeable impact on the prices of long-dated futures options, while the correlation between the interest-rate process and the futures price volatility process does not impact option prices.

6.2. Empirical pricing performance on long-dated crude oil derivatives

Chapter 3 conducts an extensive empirical study on the impact of featuring stochastic interest rates beyond stochastic volatility in the pricing of long-dated crude oil derivatives contracts. To gauge the impact of stochastic interest rates, two stochastic volatility forward price models are considered, as proposed in Chapter 2, with one featuring stochastic interest rates and one featuring deterministic interest rates. The extended Kalman

filter maximum log-likelihood methodology is employed to estimate the model parameters from historical time series of both crude oil futures prices and crude oil futures option prices. In-sample and out-of-sample analysis of the pricing performance of the proposed models are carried out under two periods, with one representing a period of relatively volatile interest rates, and the other one with a very stable and low level of interest rates.

Empirical results demonstrate that stochastic interest rates are important on the pricing of long-dated crude oil derivatives, especially when the volatility of the interest rates is high. Furthermore, there is empirical evidence of a negative correlation between crude oil futures prices and interest rates. Increasing the dimension of the model (from two to three dimensions) does improve fitting to data but it does not improve the pricing performance [see also Schwartz & Smith (2000) and Cortazar et al. (2016)].

6.3. Hedging futures options with stochastic interest rates

Using the Monte-Carlo simulation approach, Chapter 4 investigates the hedging performance of long-dated futures options under the Rabinovitch (1989) stochastic interest-rate model. This chapter firstly derives the futures, forward and option prices, and demonstrates both mathematically and numerically that using only the forward contracts as hedging instruments, both market and interest-rate risks can be eliminated. For comparison purposes, the hedging results from the Black (1976) model are compared with the results from the Rabinovitch (1989) model. The impact of the model parameters such as the interest-rate volatility, the long-term level of the interest rates and the correlation to the hedging performance, is thoroughly investigated. Further, hedging long-dated option contracts with a range of short-dated hedging instruments such as futures and forwards is examined and the impact of maturity mismatch between target hedge contract and hedging instruments is investigated.

Under this model, and when the maturity of the hedging instruments match the maturity of the option, forward contracts and futures contracts can hedge both the market risk and the

interest-rate risk of the options positions. When the hedge is rolled forward with shorter maturity hedging instruments, then bond contracts are additionally required to hedge the interest-rate risk. This requirement becomes more pronounced for contracts with a longer maturity, and amplifies as the interest-rate volatility increases. Factor hedging ratios are also considered, which are suited for multi-dimensional models, and their numerical efficiency is validated.

6.4. Empirical hedging performance on long-dated crude oil derivatives

Chapter 5 empirically investigates the hedging performance of the stochastic volatility / stochastic interest-rate model proposed in Chapter 2. The study considers the hedging of a long-dated option on crude oil futures over six years (from January 2006 to October 2011). The factor hedging methodology is applied to derive hedge ratios for a variety of schemes, including delta, delta-IR delta-vega, delta-gamma, delta-vega-IR and delta-gamma-IR. The corresponding hedge ratios are estimated by using historical crude oil futures prices, crude oil option prices and Treasury yields, under both stochastic and deterministic interest-rate model specifications. The hedging of long-dated crude oil futures options with hedging instruments of shorter maturities is also investigated.

Empirical study demonstrates that the hedge ratios from stochastic interest-rate models consistently improve hedging performance over hedge ratios from deterministic interest-rate models; an improvement that becomes more pronounced over periods with high interest-rate volatility such as during the GFC. In particular, during GFC, hedging with the stochastic interest-rate model outperforms the deterministic counterpart, but provides only marginal improvement during pre-crisis, and no noticeable improvement after 2010. Furthermore, an interest-rate hedge consistently improves hedging beyond delta, gamma and vega hedging, especially when shorter maturity contracts are used to roll the hedge forward. Finally, when the market experiences high interest-rate volatility, and the hedge is subject to high basis risk, adding interest-rate hedge to delta hedge provides an improvement, while adding gamma and/or vega to the delta hedge worsens performance.

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