On lower and upper bounds for Asian-type options: a unified approach

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Abstract. In the context of dealing with financial risk management problems it is desirable to have accurate bounds for option prices in situations when pricing formulae do not exist in the closed form. A unified approach for obtaining upper and lower bounds for Asian-type options, including options on VWAP, is proposed in this paper. The bounds obtained are applicable to the continuous and discrete-time frameworks for the case of time-dependent interest rates. Numerical examples are provided to illustrate the accuracy of the bounds.
Keywords: Asian options; Lower and upper bounds; Volume-weighted average price, Options on VWAP.

1. Introduction. We aim to obtain accurate bounds for option prices

$$C_T = Ee^{-R_T} F_T(S),$$

where $R_t = \int_0^t r_s ds$, r_s is an interest rate, $F_T(S)$ is an Asian-type payoff of the option written on the stock price $S = (S_t, 0 \le t \le T)$, T is the maturity time. (We assume that all random processes are defined on the *filtered probability space* $(\Omega, \{F_t\}_{t\ge 0}, P)$).

The typical payoff for Asian-type options is

$$F_T(S) = (\int_0^T (S_u - K) d\mu(u))^+, \tag{1}$$

where $x^+ = \max[x, 0] = (-x)^-$ for any x, K is a fixed strike, $\mu(u)$ is a distribution function on the interval $[0, T]$. Using the notation

$$\overline{h} = \int_0^T h_u \mu(du), h \in H,$$

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where H is the class of adapted random processes $h = (h_s, 0 \leq s \leq T)$ such that $\int_0^T |h_u|\mu(du) = \overline{|h|} < \infty$ *a.s.*, we can rewrite (1) as follows

$$F_T(S) = (\overline{S} - K)^+ = \overline{(S - K)}^+. \tag{2}$$

In relation to discretely monitored options (**DMO**) or continuously monitored options (**CMO**) the distribution function μ can be discrete or continuous respectively. This setup also includes the case of call options on the volume-weighted average price (VWAP), that is

$$A_T := \frac{\sum\limits_{t_j \leq T} S_{t_j} U_{t_j}}{\sum\limits_{t_j \leq T} U_{t_j}}, \ F_T(S) = (A_T - K)^+,$$

where U_{t_j} is a traded volume at the moment t_j. By setting

$$\mu(u) := \frac{\sum\limits_{t_j \leq u} U_{t_j}}{\sum\limits_{t_j \leq T} U_{t_j}} \ , 0 \leq u \leq T,$$

we obtain the representations (1) and (2) for options on VWAP.

Below we develop a unified approach to obtaining lower and upper bounds for Asian-type DMO and CMO including VWAP with a general term structure of interest rate.

The presentation of classical Asian payoffs in the form (1) was mentioned by Rogers and Shi [9] and Večeř ([14]) where they used the PDE approach for finding C_T for CMO under the geometric Brownian motion (**gBm**) model and constant interest rates. Thus, using the notation (1) we can consider an essentially wider class of options compare to the papers [9] and ([14]).

The paper [9] generated a flow of related results about lower and upper bounds under different settings. We would like to mention here the pioneering paper by Curran [3] and the unpublished paper by Thompson [12]; in fact, the latter contains some ideas which we are developing further here. One can find in literature many other similar modifications of lower and upper bounds, see e.g. ([1]), [16] and ([7]). We would like to mention the paper by Chen and Lyuu [2] containing intensive numerical results for CMO under the gBm model, and the paper by Lemmens et al [6] which discusses DMO based on bounds for geometric Levy processes. In [6] comparisons to other approaches were presented; in particular, among other methods, comparisons to the recursive integration method developed by Fusai and Meucci [4] and the method utilising comonotonic bounds (e.g. [13]) were given.

Note that all above cited papers based on the assumption that the interest rate process is constant. Below we illustrate numerically that for long-dated contacts the price of the Asian option can be essentially different if one takes into account a term structure of interest rate.

The case of floating strikes, that is options with the payoff $F_T(S) = (\overline{S} - S_T)^+$, can be reduced to the case (1) and is not discussed here.

2. Lower and Upper bounds.

Our main result, which we use for the derivation of lower and upper bounds below, is given in the following

Theorem 1. *Let z be a real number. Then*

$$C_T = \sup_{z,h\in H} Ee^{-R_T}(\overline{S} - K)I\{\overline{h} > z\} \tag{3}$$

$$= \inf_{h\in H} Ee^{-R_T}\overline{(S - K(1 + h - \overline{h}))^+} \tag{4}$$

where both supremum and infimum are attained by taking

$$h_u = S_u/K \tag{5}$$

and $z = 1$.

Proof. For any $h \in H$ and z

$$(\overline{S} - K)^+ I\{\overline{h} > z\} = (\overline{S} - K)I\{\overline{h} > z\} + (\overline{S} - K)^- I\{\overline{h} > z\}$$

$$\geq (\overline{S} - K)I\{\overline{h} > z\},$$

thus we obtain

$$C_T = Ee^{-R_T}(\overline{S} - K)^+ \geq \sup_{z,h\in H} Ee^{-R_T}(\overline{S} - K)I\{\overline{h} > z\}. \tag{6}$$

Since $(\overline{S} - K)^+ = (\overline{S} - K)I\{\overline{S}/K > 1\}$, the equalities in (6) and correspondingly in (3) *are attained* when $z = 1$ and $\overline{h} = \overline{S}/K$.

To prove (4) we note that for any $h \in H$

$$C_T = Ee^{-R_T}\overline{(S - K)}^+$$

$$= Ee^{-R_T}\overline{(S - K(1 + h - \overline{h}))}^+ \leq Ee^{-R_T}\overline{(S - K(1 + h - \overline{h}))^+}, \tag{7}$$

where the last inequality is due to convexity of x^+. This implies that the C_T is not greater than infimum of the RHS of (7) over $h \in H$. The equality in (3) *is attained* when $h_u = S_u/K$ since for the latter case

$$\overline{(S - K(1 + h - \overline{h}))^+} = \overline{(S - K(1 + S/K - \overline{S}/K))^+}$$
$$= \overline{(\overline{S} - K)}^+ = (\overline{S} - K)^+.$$

Remark 1. *The proof of this result exploits only the property of an indicator function for the part (3), Jensen inequality for the part (4) and, of course, these elements were used in many papers including the cited above. Our main observation consists in noting that (3)=(4) and both supremum and infimum are attained on the same function; beside we claim that this is true not only for DMO and CMO under the gBm model but also for options on stocks with general structure and this includes the case of VWAP as well.*

Further we use the notation

$$X_t := \log(S_t/S_0)$$

and assume that the discounted process $e^{-R_t}S_t = S_0 e^{X_t - R_t}$ is a martingale with respect to the filtration $\{F_t\}_{t\geq 0}$, as required by the non-arbitrage theory (see e.g. [5]).

Theorem 1 implies that for all $h \in H$ the following lower and upper bounds hold

$$C_T \geq LB0 := S_0 \sup_z E e^{-R_T} (\overline{e^X} - \frac{K}{S_0}) I\{\overline{h} > z\}, \tag{8}$$

$$C_T \leq UB0 := S_0 E e^{-R_T} \overline{(e^X - \frac{K}{S_0}(1 + h - \overline{h}))^+}. \tag{9}$$

To find a process h producing accurate bounds we need to take into account a complexity of calculations of the joint distribution of $(X, h, \overline{h})$. Obviously, the problem can be made computationally affordable when h_u is a linear function of X_u, that is under the choice

$$h_u = a(u)X_u + b(u)$$

with some nonrandom functions $a(u)$ and $b(u)$. Since both inequalities (8) and (9) are, in fact, equalities when (5) holds, one may try to match the first moments of h_u and S_u/K that is to set

$$Eh_u = E(S_u/K),\ Var(h_u) = Var(S_u/K).$$

In this paper we apply **another** simple choice with $a(u) = a = const$ and $b(u) = 0$ i.e.

$$h_u = aX_u \tag{10}$$

where the constant a needs to be chosen in the upper bound. For the latter case we have

$$C_T \geq LB1 := S_0 \sup_z E e^{-R_T} (\overline{e^X} - \frac{K}{S_0}) I\{\overline{X} > z\}, \tag{11}$$

$$C_T \leq UB1 := S_0 \inf_a E\overline{e^{-R_T}(e^X - \frac{K}{S_0}(1 + aX - a\overline{X}))^+}. \qquad (12)$$

Note that the calculation of the lower bound (11) does not depend on a choice of the constant a.

Remark 2. *The lower bound (11) was, in fact, used in [12] for the case of CMO; for the case DMO it was used in [2], both under the gBm model; see other similar bounds e.g. in [16]. The upper bound (12) seems to be new.*

Remark 3. *Assuming that $R = (R_t, 0 \leq t \leq T)$ and $X = (X_t, 0 \leq t \leq T)$ are independent processes, we can easily obtain another lower bound which appears originally in [3]:*

$$C_T \geq LB2 := S_0 E\overline{e^{-R_T}(E(e^X|\overline{X}) - \frac{K}{S_0})^+}. \qquad (13)$$

This bound holds due to the equality $C_T = S_0 E e^{-R_T}\{E(\overline{e^X - \frac{K}{S_0})}^+|\overline{h})\}$ and convexity of x^+.

Note that under the additional assumption

$$g(x) := \overline{E(e^X|\overline{X} = x)} \text{ is an increasing function of } x, \qquad (14)$$

we have

$$LB1 \geq LB2.$$

Indeed, one can see that

$$\begin{aligned} LB2 &= S_0 E\overline{e^{-R_T}(E(e^X|\overline{X}) - \frac{K}{S_0})}I\{E\overline{(e^X|\overline{X})} > \frac{K}{S_0}\} \\ &= S_0 E\overline{e^{-R_T}(E(e^X|\overline{X}) - \frac{K}{S_0})}I\{\overline{X} > g^{-1}(\frac{K}{S_0})\}, \end{aligned}$$

where g^{-1} is the inverse function. Now it is clear that $LB2$ does not exceed $LB1$ since one can use the obvious representation

$$LB1 = S_0 \sup_z E\overline{e^{-R_T}(E(e^X|\overline{X}) - \frac{K}{S_0})}I\{\overline{X} > z\}.$$

It is easy to check that the condition (14) holds in the classical model where X is a Brownian motion and r_t is a nonrandom function.

3. The case of Gaussian returns.

Here we suppose that the process $X = (X_u, 0 \leq u \leq T)$ is Gaussian. To simplify the exposition we also suppose that the process r_t is nonrandom. The case of stochastic interest rates which are independent of S_t, can be treated in a similar way.

The pair $(X_u, \overline{X})$, obviously, has a Gaussian distribution with

$$Cov(X_u, \overline{X}) = \int_0^T Cov(X_u, X_s) d\mu(s), \tag{15}$$

$$Var(\overline{X}) = \int_0^T \int_0^T Cov(X_u, X_s) d\mu(u) d\mu(s). \tag{16}$$

Below we consider a numerical example which corresponds to the gBm model with

$$X_u = R_u + \sigma W_u - \sigma^2/2\ u,$$

where W_u is a standard Bm.

1) Bounds for arithmetic Asian options.

For the case of DMO we assume that $\mu(u)$ is an uniform discrete distribution on (0,T] with jumps at points

$$u_i = \frac{i}{N} T,\ i = 1, ..., N,$$

where N is the number of time units (e.g. trading days).

From (15) we obtain

$$\kappa(u_i) := cov(W_{u_i}, \overline{W}) = \sum_{j=1}^{N} \min(u_i, s_j) T/N = u_i(T - \frac{u_i}{2} + \frac{T}{2N}),$$

$$V_N := Var(\overline{W}) = \frac{T}{3}(1 + \frac{3}{2N} + \frac{1}{2N^2}).$$

Note that letting $N \to \infty$ one can obtain the characteristics needed for the pricing of CMO as well.

For numerical illustrations and comparisons we consider the set of parameters $S_0 = K = 100,\ \sigma = 0.3,$ the interest rate

$$r_s = 0.09(1 + c/2 \sin(2\pi s)), \tag{17}$$

where the parameter $c = 0$ or $c = 1$.

One can speed up calculations of the bounds using the function $erfc(x)$. For example, using the Girsanov transformation we have obtained the following expression for the lower bound

$$LB1 =$$

$$\frac{e^{-R_T} S_0}{2T\ N} \max_z [\sum_i e^{R_{u_i}} erfc\{\sqrt{V_N/2}(z - \sigma\kappa(u_i))\} - \frac{K}{S_0} erfc\{\sqrt{V_N/2} z\}].$$

It takes less than a quarter of second with Mathematica for any σ to find this lower bound. Computing the upper bounds UB2 is also relatively fast (up to 7 seconds using Mathematica for fixed a) but essentially slower with use of the command *FindMinimum* in Mathematica. The optimal value of a for the upper bound (12) is usually found in the interval $(0.7, 1)$. In fact, we found that UB1 with the choice $a = 1$ produces a reasonable accuracy.

In Table 1 the numerical results for LB1 and UB1 obtained with Mathematica are reported with three decimal digits. We provide the calculated bounds for two cases $c = 0$ and $c = 1$ in (17); the results for $c = 1$ are formatted in bold and placed in brackets. As an estimate for the price we consider a midpoint of the interval $(LB1, UB1)$:

$$\hat{C}_T = \frac{LB1 + UB1}{2}.$$

The following bound is valid for the relative error of $\hat{C}_T$:

$$|\hat{C}_T/C_T - 1|100\% = (UB1/LB1 - 1)50\%.$$

Table 1.

T	N	$LB1$	$UB1$	$error$ % for $\hat{C}_T$
	10	12.162 (**12.135**)	12.259 (**12.239**)	0.4 (**0.42**)
1	50	11.782 (**11.785**)	11.829 (**11.807**)	0.1 (**0.11**)
	∞	11.718 (**11.741**)	11.731 (**11.769**)	0.03 (**0.11**)
	10	56.344 (**60.769**)	57.233 (**61.568**)	0.78 (**0.68**)
9	50	56.073 (**60.066**)	56.419 (**60.506**)	0.3 (**0.37**)
	∞	56.012 (**60.014**)	56.146 (**60.197**)	0.17 (**0.15**)

As it might be anticipated, the prices for options with longer maturities (here $T = 9$) depend essentially on a term structure of interest rate.

2) Bounds on DMO and CMO on VWAP options.

In [8] we applied the method of matching moments for finding approximations for options on VWAP under the assumption that S_t is a gBm and the volume process U_t is a squared Ornstein-Uhlenbeck process and assuming that S_t.and U_t are independent, $r_t = r = const$. The key point in the approach used in [8] was the development of technique for finding the function

$$g = (g_t := E\frac{U_t}{\overline{U}}, 0 \leq t \leq T).$$

Again with the choice of $h_t = aX_t$, Proposition 1 implies the following bounds

$$C_T \geq LB1 = S_0 e^{-rT} \sup_z E\ \overline{(e^X - \frac{K}{S_0})gI\{\overline{X} > z\}},$$

$$C_T \leq UB1 = S_0 e^{-rT} \inf_a E\ \overline{(e^X - \frac{K}{S_0}(1 + aX - a\overline{X}))^+ g},$$

where the averaging is supposed to be with respect to an uniform discrete or continuous distribution on $(0, T]$ for DMO or CMO cases respectively.

The method for calculation of the function g suggested in [8] is based on the formula

$$g_t = \int_0^\infty \frac{\partial}{\partial z} \mathbb{E}\left(e^{zU_t - qV_T}\right)\Big|_{z=0} dq,$$

which leads to an analytical representation for g for the case under consideration.

For numerical illustrations we consider the case of CMO with the following parameters (to match the related results from Stace (2007), (2007a) who used a different approach via PDE):

$$dS_t = 0.1\ S_t dt + \sigma S_t dW_t,\ \ S_0 = 110, T = 1,\ \ K = 100,$$

$$U_t = X_t^2, dX_t = 2(22 - X_t)dt + 5dW_t, X_0 = 22.$$

Table 2

σ	$LB1$	$MC\ (error)$	$UB1$
0.1	14.198	14.199 (0.0019)	14.204
0.5	19.612	19.6406 (0.0083)	19.650
0.8	25.591	25.642 (0.014)	25.784

For Monte Carlo we used 10 million trajectories and 500 discretisation points.

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