First-price Auction Implements Efficient Investments*

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Abstract

This note shows that the first-price auction fully implements efficient investments when agents make not only *ex ante* but also *ex post* investments. The essential assumptions of our model are that (i) each agent can invest before and after participating in the auction under the same cost function and (ii) the cost functions are common knowledge among agents. In any equilibrium of our model, the most efficient agent always wins and makes the efficient level of investment.

Keywords: first-price auction, investment efficiency, full implementation

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1 Introduction

In the literature of auction theory, a number of papers have examined the *ex ante* investment incentives in the auction mechanisms (Tan, 1992; Piccione and Tan, 1996; Arozamena and Cantillon, 2004). Our companion paper Tomoeda (2017) analyzes full implementability of efficient investments in the general framework of mechanism design. The main result of Tomoeda (2017) is that a novel concept *commitment-proofness* is sufficient and necessary for fully implementing efficient investments when *ex post* investment is possible and the social choice function is allocatively efficient. One limitation of this theorem, however, is that we cannot apply it to allocatively inefficient mechanisms, which include an important class of mechanisms: *asymmetric* first-price auctions (FPA).

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In this note, we show that the first-price auction implements efficient investments in any perfect Bayesian equilibrium when agents can invest *ex post* as well as *ex ante*. In particular, we show that only one agent whose investment is most efficient will actually invest in any equilibrium of our model. We consider a single-item and private-value auction. Investment is modeled as a choice of (a distribution of) private valuations of the item, which makes the first-price auction naturally asymmetric. The timeline of the game is as follows. First, each agent simultaneously chooses a distribution over the valuations of the item, and the valuation is drawn from the distribution. Then, each agent participates in the first-price auction with the knowledge of her own valuation and the distributions of all other agents' valuations. After the outcome of the auction is determined, agents can make further investments. Technically, we apply the equilibrium characterization results of Maskin and Riley (2000, 2003).

2 Model

Let $I \equiv \{1, 2, ..., n\}$ be a finite set of agents. The set of alternatives is defined by $\Omega \equiv \{\omega_i\}_{i \in I}$. For each agent $i \in I$, alternative ω_i means that i obtains the item. The valuation a_i of the item for each $i \in I$ is in an interval $[0, \alpha_i]$ where $\alpha_i \in \mathbb{R}_+$.

Agents can make investments to change their own private valuations. Let $\mathcal{F}_i$ be the set of cumulative distribution functions of valuations which can be chosen as an *ex ante* investment by agent i. Let $\mathcal{F} \equiv \times_{i \in I} \mathcal{F}_i$. We assume that $\mathcal{F}_i$ includes all cumulative distribution functions F_i of a_i that satisfy the following conditions: (i) the support $[\underline{a}_i^{F_i}, \overline{a}_i^{F_i}]$ of F_i is in $[0, \alpha_i]$, (ii) F_i is twice continuously differentiable and (iii) F_i's derivative is strictly positive on $[\underline{a}_i^{F_i}, \overline{a}_i^{F_i}]$, and (iv) $F^i(\underline{a}_i^{F_i}) > 0$.

The cost of investment is defined in the following way. For each agent $i \in I$, the cost of deterministic investment $a_i \in [0, \alpha_i]$ is given by a cost function $c_i : [0, \alpha_i] \to \mathbb{R}_+$. $c_i(\cdot)$ satisfies the following conditions: (i) $c_i(0) = 0$, (ii) $c_i(\cdot)$ is twice continuously differentiable, (iii) $c_i'(a_i) > 0$ for any $a_i \in [0, \alpha_i]$, (iv) $c_i''(a_i) > 0$ for any $a_i \in [0, \alpha_i]$, and (v) $c_i'(\alpha_i) = 1$.[1] Since *ex ante* investment can be a distribution over valuations, the cost of each uncertain investment $F_i \in \mathcal{F}_i$ is defined as the expected cost of valuations. We denote it by $\gamma_i^{c_i} : \mathcal{F}_i \to \mathbb{R}_+$ and it is defined as

$$\gamma_i^{c_i}(F_i) \equiv \int_0^{\alpha_i} c_i(a) dF_i(a).$$

[1] If we do not assume condition (v), we need some more specific conditions on $c_i''(\cdot)$ to ensure the existence of a Bayesian equilibrium in FPA following Maskin and Riley (2000, 2003). Moreover, under those conditions, we can show that no valuation a with $c_i'(a) > 1$ is chosen in equilibrium anyway. Therefore, we just assume condition (v) to focus on investment choices a with $c_i'(a) \leq 1$, and avoid other specific assumptions on $c_i''(\cdot)$.

If agent i increases her valuation from $a_i \in [0, \alpha_i]$ to $\tilde{a}_i \in [a_i, \alpha_i]$ after the auction stage, the cost of *ex post* investment that i incurs is $c_i(\tilde{a}_i) - c_i(a_i)$. Although investments may be uncertain, we assume that the cost functions $\{c_i(\cdot)\}_{i\in I}$ and $\{\alpha_i\}_{i\in I}$ are common knowledge among agents.

We assume that investment is irreversible, i.e., if the pre-auction valuation is a_i, then the agent can only choose a new valuation from $[a_i, \alpha_i]$ *ex post*. The timeline of the investment game and the informational assumption are summarized as follows:

1. Each agent $i \in I$ simultaneously chooses a distribution $F_i \in \mathcal{F}_i$. $(F_i)_{i\in I}$ is observable to all agents. The valuation $a_i \in [0, \alpha_i]$ of the item is drawn from F_i for each agent i.

2. Agents participate in the first-price auction. Each agent i knows her own valuation a_i, but does not know the realizations of other agents' valuations $(a_j)_{j\in I\setminus\{i\}}$.

3. After the auction is run, each agent may make an additional investment. That is, agent i can choose a valuation $\tilde{a}_i$ from $[a_i, \alpha_i]$.

3 Investments in the First-Price Auction

In the first-price auction, each agent submits a non-negative sealed bid $b_i \in \mathbb{R}_+$. The bidder with the highest bid wins and pays her own bid. If two or more bids tie, we use the *Vickrey tie-breaking rule*, in which each agent submits a non-negative sealed tie-breaker $t_i \in \mathbb{R}_+$. If more than one bidders tie with a bid b_i, the bidder i with the highest tie-breaker among them wins, and she pays $b_i + \max_{j\in I\setminus\{i\}}\{t_j | b_j = b_i\}$. If there is a tie for the highest tie-breaker, we randomize among those who make this bid with equal probability. The Vickrey tie-breaking rule plays a crucial role in ensuring the existence of a Bayesian equilibrium in FPA.[2] Let $w : \mathbb{R}^{2n}_+ \to 2^I \times \{0, 1\}$ represent the set of bidders whose bids are highest and whose tie-breakers are highest among them:

$$w(b,t) = \begin{cases} (\arg\max_{i\in I}\{b_i\}, 0) & \text{if } |\arg\max_{i\in I}\{b_i\}| = 1, \\ (\arg\max_{j\in I}\{t_j | j \in \arg\max_{i\in I}\{b_i\}\}, 1) & \text{if } |\arg\max_{i\in I}\{b_i\}| \geq 2, \end{cases}$$

where $b \equiv (b_1, ..., b_n)$ and $t \equiv (t_1, ..., t_n)$.

For each agent $i \in I$, a strategy is defined by $(F_i, (\beta_i^{\tilde{F}})_{\tilde{F}\in\mathcal{F}}, \tilde{a}_i)$. $F_i \in \mathcal{F}_i$ is the choice of *ex ante* investment. $\beta_i^{\tilde{F}} : [\underline{a}_i^{\tilde{F}_i}, \overline{a}_i^{\tilde{F}_i}] \to \Delta(\mathbb{R}^2_+)$ is the (mixed) bidding strategy (which also

[2]See the proof of Theorem 1 and footnote 4 for more details.

specifies the tie-breaker) for each profile of distributions $\tilde{F} \in \mathcal{F}$. $\tilde{a}_i : [0, \alpha_i] \times \Omega \to [0, \alpha_i]$ is the *ex post* investment strategy which satisfies $\tilde{a}_i(a_i, \omega) \in [a_i, \alpha_i]$ for any $a_i \in [0, \alpha_i]$ and $\omega \in \Omega$. For any profile of (on-path) strategies $(F, \beta^F, \tilde{a})$, the interim utility $u_i^{FPA}(F, a_i, \beta^F, \tilde{a})$ for each realized valuation a_i is defined as

$$u_i^{FPA}(F, a_i, \beta^F, \tilde{a}) \equiv \int_0^{\alpha_n} ... \int_0^{\alpha_{i+1}} \int_0^{\alpha_{i-1}} ... \int_0^{\alpha_1} \int_{\mathbb{R}^2_+} ... \int_{\mathbb{R}^2_+} \Big[\big(\tilde{a}_i(a_i, \omega_i) - b_i\big) \mathbb{1}_{\{\omega(b,t)=(\{i\},0)\}}$$

$$+ \big(\tilde{a}_i(a_i, \omega_i) - b_i - \max_{j \in I \setminus \{i\}} \{t_j | b_j = b_i\}\big) \sum_{S \in \{S' \in 2^I | i \in S'\}} \frac{1}{|S|} \mathbb{1}_{\{\omega(b,t)=(S,1)\}}$$

$$- \sum_{j \in I} c_i(\tilde{a}_i(a_i, \omega_j)) \Big(\mathbb{1}_{\{\omega(b,t)=(\{j\},0)\}} + \sum_{S \in \{S' \in 2^I | j \in S'\}} \frac{1}{|S|} \mathbb{1}_{\{\omega(b,t)=(S,1)\}} \Big) + c_i(a_i) \Big]$$

$$\big(\beta_1^F(a_1)\big)(b_1, t_1) d(b_1, t_1) ... \big(\beta_n^F(a_n)\big)(b_n, t_n) d(b_n, t_n) dF_1(a_1) ... dF_{i-1}(a_{i-1}) dF_{i+1}(a_{i+1}) ... dF_n(a_n).$$

The *ex ante* utility $U_i^{FPA}(F, \beta^F, \tilde{a})$ is defined by taking the expectation of the interim utility with respect to a_i and subtracting the cost of *ex ante* investment:

$$U_i^{FPA}(F, \beta^F, \tilde{a}) \equiv -\gamma_i^{c_i}(F_i) + \int_0^{\alpha_i} u_i^{FPA}(F, a_i, \beta^F, \tilde{a}) dF_i(a_i).$$

A perfect Bayesian equilibrium of this game is defined in the following way.

Definition 1. A *perfect Bayesian equilibrium* of the first-price auction with an investment game is a profile of strategies $(F, (\beta^{\tilde{F}})_{\tilde{F} \in \mathcal{F}}, \tilde{a})$ that satisfies the following conditions:

1. For each $i \in I$, the *ex post* investment strategy $\tilde{a}_i : [0, \alpha_i] \times \Omega \to [0, \alpha_i]$ satisfies

$$\tilde{a}_i(a_i, \omega_i) \in \arg\max_{a \in [a_i, \alpha_i]} \Big[a - c_i(a) \Big] \text{ and}$$

$$\tilde{a}_i(a_i, \omega_j) = a_i \text{ for any } j \neq i$$

   for each $a_i \in [0, \alpha_i]$.

2. For each $i \in I$ and $\tilde{F} \in \mathcal{F}$, the bidding strategy $\beta_i^{\tilde{F}} : [\underline{a}_i^{\tilde{F}_i}, \overline{a}_i^{\tilde{F}_i}] \to \Delta(\mathbb{R}^2_+)$ satisfies

$$\beta_i^{\tilde{F}} \in \arg\max_{\hat{\beta}_i^{\tilde{F}} : [\underline{a}_i^{\tilde{F}_i}, \overline{a}_i^{\tilde{F}_i}] \to \Delta(\mathbb{R}^2_+)} u_i^{FPA}(\tilde{F}, a_i, (\hat{\beta}_i^{\tilde{F}}, \beta_{-i}^{\tilde{F}}), \tilde{a})$$

   for each $a_i \in [\underline{a}_i^{\tilde{F}_i}, \overline{a}_i^{\tilde{F}_i}]$ given other agents' bidding strategies $\beta_{-i}^{\tilde{F}}$.

3. For each $i \in I$, the choice of cumulative distribution function $F_i \in \mathcal{F}_i$ satisfies

$$F_i \in \arg\max_{\hat{F}_i \in \mathcal{F}_i} U_i^{FPA}((\hat{F}_i, F_{-i}), \beta^{(\hat{F}_i, F_{-i})}, \tilde{a})$$

   given other agents' choices $F_{-i} \in \mathcal{F}_{-i}$.

As required by Maskin and Riley (2003), we assume that agents never bid strictly higher than their own reservation prices at the auction stage. That is, if agent i's realized valuation is a_i, her bid is at most $b^{c_i}(a_i) \equiv \max_{a \in [a_i, \alpha_i]} \{a - (c_i(a) - c_i(a_i))\}$.

We say that the first-price auction with an investment game achieves full efficiency in a perfect Bayesian equilibrium if the equilibrium achieves efficiency in both allocations and investments.

Definition 2. *The first-price auction with an investment game achieves full efficiency in a perfect Bayesian equilibrium* $(F, (\beta^{\tilde{F}})_{\tilde{F} \in \mathcal{F}}, \tilde{a})$ if the following conditions are satisfied:

1. Allocative Efficiency: For any $a \in \times_{i \in I} [\underline{a}_i^{F_i}, \overline{a}_i^{F_i}]$, if

$$\int_{\mathbb{R}_+^2} \cdots \int_{\mathbb{R}_+^2} \mathbb{1}_{\{i \in \arg\max_{j \in I} \{b_j\} \cap \arg\max_{j \in I} \{t_j | b_j = b_i\}\}} \big(\beta_1^F(a_1)\big)(b_1, t_1) d(b_1, t_1) ... \big(\beta_n^F(a_n)\big)(b_n, t_n) d(b_n, t_n) > 0,$$

then

$$i \in \arg\max_{j \in I} \Big\{ \tilde{a}_j(a_j, \omega_j) - \big[c_j(\tilde{a}_j(a_j, \omega_j)) - c_j(a_j)\big] \Big\}.$$

2. Investment Efficiency:

$$F \in \arg\max_{\hat{F} \in \mathcal{F}} \Big[-\sum_{i \in I} \gamma_i^{c_i}(\hat{F}_i) + \int_0^{\alpha_n} \cdots \int_0^{\alpha_1} \max_{i \in I} \big\{ \tilde{a}_i(a_i, \omega_i) - \big[c_i(\tilde{a}_i(a_i, \omega_i)) - c_i(a_i)\big] \big\} d\hat{F}_1(a_1) ... d\hat{F}_n(a_n) \Big].$$

Note that we do not need to require conditions on the *ex post* investment strategy $\tilde{a}$ because it is always socially optimal in equilibrium.

Our main result is that the first-price auction with an investment game always achieves full efficiency in any perfect Bayesian equilibrium. In particular, only one of the agents with lowest cost functions invests and no other agents make positive investments in any equilibrium.

Theorem 1. *The first-price auction with an investment game achieves full efficiency in any perfect Bayesian equilibrium.*

Proof. Let $b^{c_i}(a_i)$ be i's reservation price in the auction stage when a_i is drawn *ex ante*. Since the *ex ante* cost is sunk and the *ex post* investment strategy is always optimal, it is

$$b^{c_i}(a_i) \equiv \max_{a \in [a_i, \alpha_i]} \Big\{ a - \big(c_i(a) - c_i(a_i)\big) \Big\} = \alpha_i - c_i(\alpha_i) + c_i(a_i).$$

From our assumptions on $c_i(\cdot)$, $b^{c_i}(\cdot)$ is twice continuously differentiable and $(b^{c_i})'(a_i) > 0$ for any $a_i \in [0, \alpha_i]$.

Let $G_i^{F_i}$ be the c.d.f. of the reservation prices when F_i is chosen *ex ante*, i.e., $G_i^{F_i}(b) = F_i\big((b^{c_i})^{-1}(b)\big)$ for any $b \in [b^{c_i}(0), \alpha_i]$. By the properties of $b^{c_i}(\cdot)$ and the assumptions on F_i,

$G_i^{F_i}$ satisfies the following properties assumed by Maskin and Riley (2000, 2003): (i) $G_i^{F_i}$ is twice continuously differentiable; (ii) $G_i^{F_i}$'s derivative is strictly positive on $[b^{c_i}(\underline{a}_i^{F_i}), b^{c_i}(\overline{a}_i^{F_i})]$; and (iii) $G_i^{F_i}(b^{c_i}(\underline{a}_i^{F_i})) > 0$. Then we can ensure the existence of a Bayesian equilibrium in the auction stage and apply their characterization results.

We show that in any equilibrium, there is at most one agent who chooses a costly investment. Consider an arbitrary agent i's incentive to invest. Take any $F_{-i} \in \mathcal{F}_{-i}$, equilibrium bidding strategies $(\beta^{\tilde{F}})_{\tilde{F}\in\mathcal{F}}$ and optimal *ex post* investment strategies $\tilde{a}$. The *ex ante* utility of agent i from choosing $\hat{F}_i \in \mathcal{F}_i$ with $\gamma_i^{c_i}(\hat{F}_i) > 0$ is

$$
\begin{aligned}
& U_i^{FPA}((\hat{F}_i, F_{-i}), \beta^{(\hat{F}_i, F_{-i})}, \tilde{a}) \\
\leq \; & -\gamma_i^{c_i}(\hat{F}_i) + \int_0^{\alpha_i} \left[b^{c_i}(a_i) - b_*\right] \text{Prob}\{i \text{ wins the auction}|a_i\} d\hat{F}_i(a_i) \\
= \; & -\int_0^{\alpha_i} c_i(a_i) \text{Prob}\{i \text{ loses the auction}|a_i\} d\hat{F}_i(a_i) \\
& + \int_0^{\alpha_i} \left[b^{c_i}(a_i) - c_i(a_i) - b_*\right] \text{Prob}\{i \text{ wins the auction}|a_i\} d\hat{F}_i(a_i)
\end{aligned}
$$

where b_* be the minimum value of the winning bid in this first-price auction. Now let $H \equiv \{j \in I | b^{c_j}(0) \geq b^{c_k}(0) \; \forall k \in I\}$ be the set of all agents who have the highest $b^{c_k}(0)$.

Consider $i \notin H$. By the construction of $G_k^{F_k}$, the minimum value of the support of $G_k^{F_k}$ is $b^{c_k}(\underline{a}_k^{F_k})$ for any $k \in I$. Therefore, we can apply Proposition 3 of Maskin and Riley (2000) and Lemma 3 of Maskin and Riley (2003), and obtain $b_* \geq b^{c_i}(0)$.[3] Suppose that $U_i^{FPA}((\hat{F}_i, F_{-i}), \beta^{(\hat{F}_i, F_{-i})}, \tilde{a}) \geq 0$. Then, since $c_i(a_i) > 0$ for any $a_i \in (0, \alpha_i]$ and $b^{c_i}(a_i) - c_i(a_i) = \alpha_i - c_i(\alpha_i) = b^{c_i}(0) \leq b_*$, i must win with probability one for any $a_i \in (0, \alpha_i]$ and $b_* = b^{c_i}(0)$ must hold. But this is a contradiction because if i wins with probability one for a_i with $b^{c_i}(a_i) < \max_{k \in I}\{b^{c_k}(0)\}$, some $j \in H$ could bid higher and win with a positive probability.

Next, consider $i \in H$ when $|H| \geq 2$. By $|H| \geq 2$, we have $b_* \geq b^{c_i}(0)$. Since i must win with probability one for any $a_i \in (0, \alpha_i]$ to have $U_i^{FPA}((\hat{F}_i, F_{-i}), \beta^{(\hat{F}_i, F_{-i})}, \tilde{a}) \geq 0$, there is at most one such agent.

Therefore, we can conclude that in any perfect Bayesian equilibrium, there is at most one agent $i \in H$ who chooses *ex ante* investment $\hat{F}_i \in \mathcal{F}_i$ with $[\underline{a}_i^{\hat{F}_i}, \overline{a}_i^{\hat{F}_i}] \subseteq (0, \alpha_i]$. And for any other agent $j \in I \setminus \{i\}$, the unique optimal choice of *ex ante* investment is $F_j \in \mathcal{F}_j$ with $F_j(0) = 1$. Since the tie is broken by the Vickrey tie-breaking rule, agent i wins with

[3] Lemma 3 of Maskin and Riley (2003) uses a random tie-breaker in the first-price auction, but the proof goes through for the Vickrey tie-breaking rule as well.

probability one by bidding $\max_{j \in I \setminus \{i\}} \{b^{c_j}(0)\}$.[4] And finally, as the optimal choice of agent i's *ex post* investment always maximizes the social welfare, the first-price auction with an investment game achieves full efficiency in any perfect Bayesian equilibrium. □

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[4]If the random tie-breaking rule were used instead, then there could be no Bayesian equilibrium in FPA because agent i cannot win with probability one by bidding $\max_{j \in I \setminus \{i\}} \{b^{c_j}(0)\}$ and she could get strictly better off by bidding slightly higher.