# Electro/Magneto-Sensitive Elastomers and Lagrangian Electro/Magneto-Statics

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## ABSTRACT

Electro/magneto-sensitive elastomers (EMSE) have shown great potentialities as sensors and actuators. Because of the strong electromagnetic-structural coupling and the large deformations they involve, managing the response of such materials should require analytical models and numerical simulation. Mathematical models for EMSE exist. However, their classical formulation turns out to be quite awkward and rather difficult to implement. One of the main concerns is the form of the electromagnetism equations. Indeed, from the continuum physics standpoint, the classical Maxwell's equations obey the Eulerian formalism, which is not well suited for dealing with deformable solids. In this context, this paper presents the Lagrangian electro/magneto-statics and shows the advantages it involves in the study of EMSE.

## 1. INTRODUCTION

Electro/magneto-sensitive elastomers (EMSE) are particular kinds of "smart materials" in which electromagnetic and structural phenomena are strongly coupled. In essence, they are solids which experience large deformations in response to applied electromagnetic fields while, at the same time, alter the electromagnetic fields in response to the deformations undergone.

In the last decade, because of the tremendous advances in the polymer chemistry and the consequent renewed interest by the "smart material and structure" community, a number of EMSE's have been retrieved and discovered [1-2]. At present, due to their inherent multi-physics interaction, researchers have in practice identified EMSE as a candidate technology for building solid-state devices to be used in important applications such as sensors and actuators [1].

Despite their promises, almost all the devices attempted to date, show effective performances rather below expectations. Moreover, their optimization turns out to be rather tricky. Indeed, because of the strong and highly non-linear coupling via multiple physical effects, predicting output and performance of even rather simple EMSE systems becomes an extremely challenging problem that seems to defy intuitive approaches. In this context, the use of theoretical models and engineering simulation tools might be very fruitful. Indeed, these tools could at first provide insights into the phenomenology which drives the EMSE, and then allow for its quantitative estimation.

According to the literature [3], mathematical models for analyzing the behavior of EMSE's exist. In essence, they consist of coupled boundary-value problems built on the combination of the equilibrium equations of structural mechanics and electromagnetism, on the definitions of constitutive relations and on the enforcement of boundary conditions. Although correct, these models rely on a classical formulation that does not lead to a clean realization of the coupling effects and, in particular, does not lead to a straightforward numerical code. Note that this often happens in the context of multi-physics analyses since the full mathematical model is usually built up by simply using methodologies that have been developed only for the study of single-field problems and which indeed may prove difficult in describing the same kind of phenomena when other physical effects are to be dealt with.

As borrowed from classical electromagnetism, which does not, of course, model deformations, from the continuum physics perspective, the usual form of Maxwell's equations which model the electric and magnetic field in an EMSE must be expressed in an Eulerian framework. That is, the classical Maxwell's equations are formulated with respect to the current deformed configuration of the space. However, since EMSE's are highly-deformable solids, they should be best handled by the use of the Lagrangian formalism which, conversely, refers to a fixed, usually undeformed, reference configuration of the space. Indeed, since an EMSE has boundaries which move considerably, the Eulerian formulation involves, in practice, the solution of integral equations whose integration domains depend on the solution of the problem itself. In addition, note that both the boundary conditions and the constitutive equations are usually prescribed on, or best derived in, the reference configuration of the body. Moreover, the use of the reference configuration makes linearization procedures simpler and leads to more readable linearized expressions. This is important, since, in practice, linearization is always required if one wants to cope with the stability analysis or the numerical simulation of an EMSE. However, while Lagrangian structural mechanics is well-established, a Lagrangian form of Maxwell's equations seems to be an almost unknown topic.

For these reasons, in this paper we present a Lagrangian formulation of electro/magneto-statics and propose it as a self-consistent approach for dealing with EMSE's. Indeed, besides the aforementioned advantages, the Lagrangian formalism is shown to provide clearer insights into the coupled effects characterizing EMSE's. In particular, in the Lagrangian framework, the electromagnetic and structural fields turn out to be coupled in a more natural way. That is, like polarization and magnetization, deformation turns out to act as a "source" for electricity and magnetism.

As for the method, for the sake of generality, the derivation of the Lagrangian electro/magneto-statics is carried out in the paper by firstly tackling the Eulerian/Lagrangian conversion of the general equilibrium laws of static physical phenomena. The Lagrangian Maxwell's equations and the involved fields are subsequentially inferred.

## 2. MATERIALS AND METHODS

### 2.1 EMSE and Electro/Magneto-Statics

EMSE's are solids which are characterized by a strong coupling between electromagnetic and structural responses. The phenomenology which governs EMSE's is rather complicated. In practice, several "cross" effects induce such a coupling. First, at the constitutive level, the presence of the basic electromagnetic fields, i.e. the electric field and magnetic induction, and the deformation affect the electromagnetic and mechanical properties of the EMSE such as polarization, magnetization and stress. Second, from a structural standpoint, the presence of electromagnetic forces, which arise from the interaction of matter-related quantities, such as polarization and magnetization with the basic electromagnetic fields, and the presence of stress components which, as said, depend on those fields, induce the EMSE to deform, thus varying the geometry of the space they are contained in. Third, from the electromagnetic standpoint, the polarization and magnetization as well as changes in space geometry modify the basic electromagnetic fields. In this paper we address this latter effect.

It is well known that the electric field, $\mathbf{E}$, and the magnetic induction, $\mathbf{B}$, can be defined by means of the scalar potential, $\Phi$, and the vector potential, $\mathbf{A}$, i.e.

$$\mathbf{E} = -\nabla\Phi \text{ and } \mathbf{B} = \nabla\times\mathbf{A}, \tag{1}$$

where $\nabla$ is the spatial differential operator such that $\nabla\Phi$ is the gradient of the scalar $\Phi$ while $\nabla\times\mathbf{A}$ is the rotational of the vector $\mathbf{A}$. Then, according to modern electromagnetism [4], the electro/magneto-statics in spatial domains containing polarizable and magnetizable matter can be expressed by the set of equations

$$\nabla^2\Phi = -\varepsilon_0^{-1}\left[q - \nabla\cdot\mathbf{P}\right] \text{ and } \nabla\times\nabla\times\mathbf{A} = \mu_0\left[\mathbf{J} + \nabla\times\mathbf{M}\right], \tag{2}$$

where $\varepsilon_0$ and $\mu_0$ are the electric and magnetic permeability of the vacuum which, in the SI-unit system, read as $\varepsilon_0 = 10^7/4\pi c^2$ and $\mu_0 = 4\pi 10^{-2}$, $q$ is the electric free-charge density, while $\mathbf{J}$, $\mathbf{P}$ and $\mathbf{M}$ are, respectively, the electric current, the polarization and the magnetization densities. Note that $\mathbf{J}$, $\mathbf{P}$ and $\mathbf{M}$ are matter-related quantities which account for the electromagnetic response of given materials to the physics environment. Note that the form of Eqs. (2) should be preferred, among the other formulations of electro/magneto-statics, due to issues related to both the practical solution and the correct comprehension of the electromagnetic problem. Indeed, through Eqs. (2), the analysis of electricity and magnetism in spatial domains containing media is recast to the study of an equivalent problem *in vacuo* where the effects related to the presence of matter are taken into account by the introduction of additional "source" terms, such as the polarization charge $-\nabla\cdot\mathbf{P}$ and the magnetization current $\nabla\times\mathbf{M}$.

Being equilibrium laws, Eqs. (2) hold irrespective of material constitution. Essentially, it is through the use of proper constitutive relations, i.e. in the specification of $\mathbf{J}$, $\mathbf{P}$ and $\mathbf{M}$, that Eqs. (2) allow us to deal with electricity and magnetism in presence of given classes of materials. As for the EMSE, its material is considered as being reactive to $\mathbf{E}$, $\mathbf{B}$ and to the deformation gradient $\mathbf{F}$ which describes the state of deformation within the material [5]. While detailed expressions exist for the EMSE, here it suffices to say that, according to Refs. [5] and [6], the constitutive equations must be written in the form

$$\mathbf{J}=\mathbf{J}\left(\mathbf{F}^T\mathbf{F},\mathbf{F}^T\mathbf{E},J\mathbf{F}^{-1}\mathbf{B}\right),\ \mathbf{P}=\mathbf{P}\left(\mathbf{F}^T\mathbf{F},\mathbf{F}^T\mathbf{E},J\mathbf{F}^{-1}\mathbf{B}\right)\text{ and }\mathbf{M}=\mathbf{M}\left(\mathbf{F}^T\mathbf{F},\mathbf{F}^T\mathbf{E},J\mathbf{F}^{-1}\mathbf{B}\right). \tag{3}$$

## 2.2 Eulerian/Lagrangian Conversion of Generalized Static Equilibrium Laws

According to the "classical field theories" of physics [7], all natural events should be described up to the desired degree of completeness by properly chosen equilibrium laws. Generalized static equilibrium laws consist of equations which are able to represent generic physical phenomena in static regimes. Since the fields may be discontinuous and since some physics may have a non-local character, generalized equilibrium laws must be primarily expressed in integral form, and then, if non-local effects are absent, the corresponding differential form and discontinuity conditions can rigorously be obtained by means of the so called "postulate of localization" [8]. Depending on the phenomena, equilibrium laws may be classified into two categories, namely "volume balance laws" and "surface balance laws". They read, respectively, as

$$\int_{\partial\mathscr{V}}\boldsymbol{\tau}\cdot d\mathbf{s}+\int_{\mathscr{V}}\sigma dv=0\text{ and }\oint_{\partial\mathscr{S}}\mathbf{r}\cdot d\mathbf{l}+\int_{\mathscr{S}}\mathbf{h}\cdot d\mathbf{s}=0\,. \tag{4}$$

In the first equation of Eqs. (4), $\sigma$ is the local source within the arbitrary volume $\mathscr{V}$, while $\boldsymbol{\tau}$ is the local flux through $\partial\mathscr{V}$, namely the boundary of $\mathscr{V}$. In addition, in Eq. (4.2), $\mathbf{h}$ is the local source on the arbitrary surface $\mathscr{S}$, while $\mathbf{r}$ is the local flux across $\partial\mathscr{S}$, i.e. the boundary of $\mathscr{S}$. In addition, $dv$, $d\mathbf{s}$ and $d\mathbf{l}$ are, respectively, the infinitesimal volume element, the infinitesimal directed-surface element and the infinitesimal directed-line element of the integration domains. Note that the quantities contained in Eqs. (4) are functions of the position vector $\mathbf{x}$ which describe the geometry of $\mathscr{V}$ and $\mathscr{S}$.

By the use of the Green-Gauss theorems [7] and the postulate of localization [8], the differential formulation of Eqs. (4) follows as

$$\nabla\cdot\boldsymbol{\tau}+\sigma=0 \text{ and } \nabla\times\mathbf{r}+\mathbf{h}=\mathbf{0}. \tag{5}$$

where, of course, $\nabla$ stands for the spatial derivative operator with respect to $\mathbf{x}$ and, therefore, $\nabla\cdot\boldsymbol{\tau}$ is the divergence of the vector $\boldsymbol{\tau}$.

If deformable media are present, the arbitrary surface $\mathscr{S}$ and volume $\mathscr{V}$ may be, or contain, deformable domains. That is, the geometry of $\mathscr{S}$ and $\mathscr{V}$ may change. In such cases, according to continuum mechanics terminology, Eqs. (4) and Eqs. (5) as well as the variables they involve are said to be given in Eulerian form.

Still in the context of deformation, consider the reference volume $\overline{\mathscr{V}}$ and surface $\overline{\mathscr{S}}$ whose geometries are fixed and correspond, respectively, to the undeformed configurations of $\mathscr{V}$ and $\mathscr{S}$. Note that, in opposition to $\mathbf{x}$, it is the position vector $\mathbf{X}$ that describes the geometry of $\overline{\mathscr{V}}$ and $\overline{\mathscr{S}}$. By a change of the integration variables of Eqs. (4), the equilibrium laws can be rewritten in an alternative form, i.e. the Lagrangian form. The exact procedure is as follows.

We introduce of the mapping $\boldsymbol{\chi}$ [9], which gives the motion of the points of the referential configuration, $\mathbf{X}$, to the points of the current configuration, $\mathbf{x}$, namely $\mathbf{x}=\boldsymbol{\chi}(\mathbf{X})$. Hence $\mathscr{V}=\boldsymbol{\chi}\left(\overline{\mathscr{V}}\right)$ and $\mathscr{S}=\boldsymbol{\chi}\left(\overline{\mathscr{S}}\right)$. Second, define the deformation gradient $\mathbf{F}=\partial\boldsymbol{\chi}/\partial\mathbf{X}$ and the Jacobian $J=\det(\mathbf{F})$. Third, recall from Ref. [9] the Eulerian/Lagrangian conversion formulas for line, surface and volume elements, respectively,

$$d\mathbf{l}=\mathbf{F}d\mathbf{L},\ d\mathbf{s}=J\mathbf{F}^{-T}d\mathbf{S},\text{ and } dv=JdV, \tag{6}$$

where, of course, $d\mathbf{L}$, $d\mathbf{S}$, and $dV$ are the infinitesimal Lagrangian directed-line, directed-surface and volume elements. Then, by making use of Eqs. (6), rewrite Eqs. (4) as

$$\int_{\partial\bar{\mathcal{V}}} J\left(\mathbf{F}^{-1}\boldsymbol{\tau}\right)\cdot d\mathbf{S}+\int_{\bar{\mathcal{V}}} J\sigma dV=0 \text{ and } \oint_{\partial\bar{\mathcal{S}}}\left(\mathbf{F}^{T}\mathbf{r}\right)\cdot d\mathbf{L}+\int_{\bar{\mathcal{S}}} J\left(\mathbf{F}^{-1}\mathbf{h}\right)\cdot d\mathbf{S}=0\,. \tag{7}$$

Further, after having defined the following Lagrangian fields

$$\bar{\boldsymbol{\tau}}=J\mathbf{F}^{-1}\boldsymbol{\tau}\,,\ \bar{\sigma}=J\sigma\,,\ \bar{\mathbf{r}}=\mathbf{F}^{T}\mathbf{r} \text{ and } \bar{\mathbf{h}}=J\mathbf{F}^{-1}\mathbf{h}\,, \tag{8}$$

recast Eqs. (7) into the more compact form

$$\int_{\partial\bar{\mathcal{V}}} \bar{\boldsymbol{\tau}}\cdot d\mathbf{S}+\int_{\bar{\mathcal{V}}} \bar{\sigma} dV=0 \text{ and } \oint_{\partial\bar{\mathcal{S}}} \bar{\mathbf{r}}\cdot d\mathbf{L}+\int_{\bar{\mathcal{S}}} \bar{\mathbf{h}}\cdot d\mathbf{S}=0\,. \tag{9}$$

Equations (9) correspond to the integral formulation of the generalized static equilibrium laws in Lagrangian form.

Finally, since Eqs. (9) have the same form as Eqs. (4), application of the same procedure outlined in that case gives the differential formulation of the generalized static equilibrium laws in Lagrangian form which reads as

$$\bar{\nabla}\cdot\bar{\boldsymbol{\tau}}+\bar{\sigma}=0 \text{ and } \bar{\nabla}\times\bar{\mathbf{r}}+\bar{\mathbf{h}}=\mathbf{0}\,, \tag{10}$$

where, as opposed to $\nabla$, $\bar{\nabla}$ stands for the spatial derivative operator with respect to the referential coordinate $\mathbf{X}$.

## 3 RESULTS AND DISCUSSION

### 3.1 The Lagrangian Electro/Magneto-Statics

Proper identification of the generalized fields of Eqs. (5) with the classical Eulerian electromagnetic quantities which appear in Eqs. (2) and the use of Eqs. (8) lead, directly, to the definition of the Lagrangian electromagnetic potentials, basic fields and sources, respectively,

$$\begin{gathered}\bar{\Phi}=\Phi \text{ and } \bar{\mathbf{A}}=\mathbf{F}^{T}\mathbf{A}\,,\\ \bar{\mathbf{E}}=-\bar{\nabla}\bar{\Phi}=\mathbf{F}^{T}\mathbf{E} \text{ and } \bar{\mathbf{B}}=\bar{\nabla}\times\bar{\mathbf{A}}=J\mathbf{F}^{-1}\mathbf{B}\,,\\ \bar{q}=Jq\,,\ \bar{\mathbf{J}}=J\mathbf{F}^{-1}\mathbf{J}\,,\ \bar{\mathbf{P}}=J\mathbf{F}^{-1}\mathbf{P} \text{ and } \bar{\mathbf{M}}=\mathbf{F}^{T}\mathbf{M}\,.\end{gathered} \tag{11}$$

That is, in practice, the Lagrangian field variables may be obtained by the use of the convection operators [8], i.e. $\mathbf{F}^{T}$ and $J\mathbf{F}^{-1}$, to the Eulerian field variables.

Besides, after appropriate manipulation, the Lagrangian Maxwell's equations can be written in the form

$$\bar{\nabla}^2\bar{\mathbf{\Phi}} = -\varepsilon_0^{-1}\left[\bar{q} - \bar{\nabla}\cdot\left(\bar{\mathbf{P}} + \bar{\mathbf{P}}^D\right)\right] \text{ and } \bar{\nabla}\times\bar{\nabla}\times\bar{\mathbf{A}} = \mu_0\left[\bar{\mathbf{J}} + \bar{\nabla}\times\left(\bar{\mathbf{M}} + \bar{\mathbf{M}}^D\right)\right] \quad (12)$$

where $\bar{\mathbf{P}}^D$ and $\bar{\mathbf{M}}^D$ are dummy polarization and magnetization which are representative of the effects which deformation induces on electricity and magnetism. Indeed, they read as

$$\bar{\mathbf{P}}^D = \varepsilon_0\left(J\mathbf{C}^{-1} - \mathbf{1}\right)\bar{\mathbf{E}} \text{ and } \bar{\mathbf{M}}^D = \mu_0^{-1}\left(\mathbf{1} - J^{-1}\mathbf{C}\right)\bar{\mathbf{B}}, \quad (13)$$

where $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ is the right Cauchy-Green deformation tensor and $\mathbf{1}$ is the identity matrix.

That is, through the Lagrangian formalism, the electromagnetic equations in the presence of electro/magneto-sensitive and deformable media can be recast into a more suggestive form in which the presence of matter and its deformation are considered merely as sources for the electricity and magnetism of an equivalent vacuum space whose geometry is known at the outset. This, of course, adds further qualitative and quantitative insights on the electromagnetic-structural coupling of EMSE's.

In addition, from the computational perspective, it is worth mention that comparison between Eqs. (13) and Eqs. (3) shows that both the equations have the same mathematical complexity. Moreover, as for $\bar{\mathbf{J}}$, $\bar{\mathbf{P}}$ and $\bar{\mathbf{M}}$, through simple manipulation of Eqs. (3), it is easy to find that

$$\bar{\mathbf{J}} = \bar{\mathbf{J}}\left(\mathbf{C},\bar{\mathbf{E}},\bar{\mathbf{B}}\right),\ \bar{\mathbf{P}} = \bar{\mathbf{P}}\left(\mathbf{C},\bar{\mathbf{E}},\bar{\mathbf{B}}\right) \text{ and } \bar{\mathbf{M}} = \bar{\mathbf{M}}\left(\mathbf{C},\bar{\mathbf{E}},\bar{\mathbf{B}}\right). \quad (14)$$

As a result, Eqs. (2) and Eqs. (12) also turn out to have the same mathematical complexity. Therefore, in the context of the study of EMSE's, where electromagnetism is coupled with structural mechanics in such a way that the geometry of the domain of integration of Eqs. (2) depends on the solution of the coupled problem itself, it is evident that the Lagrangian formulation of electromagnetism, in which, conversely, the geometry of the integration domains is fixed and known, should be of great computational advantage.

Moreover, since it is well-known that in the case of solids the structural problem is best solved in its Lagrangian form, the introduction of Lagrangian electromagnetism may allow the combined physics of an EMSE to be solved via monolithic algorithms, which effectively treat the combined system as a unique problem. This is very important, since to date, because of the use of mixed formulations, with Eulerian electromagnetism and Lagrangian structural mechanics, commercially available multi-physics simulation packages, Refs. [10-11], can only be used on involuted iterative-sequential solution schemes, which, at each iteration, at first update the integration domain of the Eulerian balance laws, then re-mesh the updated integration domain by using a mesh-morphing algorithm, and finally solve for the electromagnetic and the structural problems in a sequential fashion by treating the two physical effects as being fully decoupled. Note that, apart from thermodynamic motivations, due to issues arising from the consistency of the solution [8], monolithic algorithms should be preferred to sequential approaches, especially when the problem is highly coupled and the coupling is highly non-linear as in the case of an EMSE.

## 4. CONCLUSIONS

Summarizing, in the paper we have presented a Lagrangian form of electro/magneto-statics to be used for the study of electricity and magnetism in the presence of “smart materials” such as electro/magneto-sensitive elastomers (EMSE).

In these contexts, the Lagrangian formulation of electromagnetism has been shown to lead both to clearer insights into the electromagnetic-structural interactions and simpler methods for their expression, manipulation and solution.

As for the theoretical insights, the Lagrangian formulation sheds light on the electromagnetic-structural coupling which is inherent in the equilibrium laws of electromagnetism whenever deformable media are present. Specifically, the effect of media deformation on electricity and magnetism is shown to be qualitatively and quantitatively comparable to that of polarization and magnetization.

In practice, the Lagrangian electromagnetism formulation allows easier determination of the boundary conditions, more efficient definition of the constitutive equations, simpler application of linearization procedures and more straightforward algorithms for the numerical integration of the boundary value problem. In addition, the Lagrangian electromagnetism allows the full physics of an EMSE to be written in a unique and proper Lagrangian framework, allowing for the use of monolithic solvers which, in the context of EMSE, should be better suited than the iterative-sequential algorithms on which, to date, commercially available simulation packages rely.

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