Spontaneous parametric down-conversion photon sources are scalable in the asymptotic limit for boson-sampling

Keith R. Motes,[1] Jonathan P. Dowling,[2,3] and Peter P. Rohde[1,*]

[1]*Centre for Engineered Quantum Systems, Department of Physics & Astronomy,*
Macquarie University, Sydney NSW 2113, Australia
[2]*Hearne Institute for Theoretical Physics and Department of Physics & Astronomy,*
Louisiana State University, Baton Rouge, LA 70803
[3]*Computational Science Research Center, Beijing 100084, China*
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Boson-sampling has emerged as a promising avenue towards post-classical optical quantum computation, and numerous elementary demonstrations have recently been performed. Spontaneous parametric down-conversion (SPDC) is the mainstay for single-photon state preparation, the technique employed in most optical quantum information processing implementations to-date. Here we present a simple architecture for boson-sampling based on multiplexed SPDC sources and demonstrate that the architecture is limited only by the post-selection detection efficiency assuming that other errors, such as spectral impurity, dark counts, and interferometric instability are negligible. For any given number of input photons, there exists a minimum detector efficiency that allows post selection. If this efficiency is achieved, photon-number errors in the SPDC sources are sufficiently low as to guarantee correct boson-sampling most of the time. In this scheme the required detector efficiency must increase exponentially in the photon number. Thus, we show that idealised SPDC sources will not present a bottleneck for future boson-sampling implementations. Rather, photodetection efficiency is the limiting factor and thus future implementations may continue to employ SPDC sources.

I. INTRODUCTION

Linear optics quantum computing (LOQC) [1–3] is a promising route towards scalable universal quantum computing [4]. The first architecture, presented by Knill, Laflamme & Milburn (KLM) [1], demonstrated that scalable quantum computation is possible using only single-photon sources, photodetection, quantum memory and fast-feedforward. However, the physical resource requirements are daunting, and large-scale LOQC appears distant. Since the advent of KLM, numerous simplifications have been suggested, significantly reducing physical resource requirements [5, 6], but nonetheless require technologies such as quantum memory and fast-feedforward that are not presently available.

Recently, Aaronson & Arkiphov [7] presented an alternate linear optical scheme, known as boson-sampling. This scheme is believed to implement a classically hard algorithm for a specific task, but will likely not be universal for quantum computation. In this model, only single-photon state preparation, passive linear optics (beam-splitters and phase-shifters), and photodetection are required, doing away with the more challenging requirements of fast-feedforward and quantum memory. The system's Hilbert space scales exponentially with the physical resources. Gard *et al.* present an elementary argument from a quantum optics perspective as to why boson-sampling scales exponentially [8].

The technology to implement boson-sampling is, for the larger part, available today, making boson-sampling an attractive route towards a type of non-universal optical quantum information processing. Recently, numerous experimental groups have begun implementing elementary demonstrations of boson-sampling using only a few photons [9–13].

While boson-sampling is one of the first non-trivial computational problems solvable with a linear optical interferometer using Fock-state inputs, it is likely not the last. Boson-sampling is a completely new quantum computational scheme that has yet to be fully explored and understood. Fully understanding boson-sampling may present us with new computational problems not accessible by classical computers. Furthermore, the exponentially large Hilbert space and computational complexity associated with such interferometers will likely lead to further breakthroughs in the closely related fields of quantum optical metrology, imaging, and sensing [14, 15].

In this paper we show that large scale boson-sampling can be implemented provided that detection efficiencies, which must increase exponentially with photon number, are sufficient to guarantee post-selection with high probability. Increasing input photon number will thus yield a larger required detection efficiency.

Spontaneous parametric down-conversion (SPDC) has become the mainstay for single-photon state preparation, is widely used in optical quantum information processing, and was employed in all of the recent experimental boson-sampling implementations. A pressing question for future larger-scale implementations is scalability. Scalability in this context refers to increasing the input photon number into the boson-sampling device provided that the error in the single photon photo-detectors, which scales expo-

nentially with input photon number, is sufficiently low to
ensure successful implementation of boson-sampling most
of the time. That is, what are the limitations and require-
ments on physical resources to implement a scalable de-
vice? In particular, will SPDC sources suffice, or will we
have to transition to other photon source technologies?
The issue of scalability of SPDC sources in the context
of boson-sampling was recently discussed by Lund *et al.*
[16].

We consider a general architecture for the experimen-
tal implementation of boson-sampling, where multiplexed
SPDC sources are employed for state preparation. We
show that in such an architecture the device is limited
only by the post-selection probability. In other words, the
architecture is scalable provided that detector efficiencies
are sufficiently high to enable post-selected computation.
In this regime, the quality of current SPDC states is suf-
ficient to enable large-scale boson-sampling. Thus, it is
photodetection, not SPDC sources, that provide the bot-
tleneck to larger-scale demonstrations.

II.   THE BOSON-SAMPLING MODEL

We begin by preparing an m-mode state, in which
the first n modes are initialised with single-photon Fock
states and the remainder in the vacuum state,

$$|\psi\rangle_{\text{in}} = |1_1, \ldots, 1_n, 0_{n+1}, \ldots, 0_m\rangle$$
$$= \hat{a}_1^\dagger \ldots \hat{a}_n^\dagger |0_1, \ldots, 0_m\rangle, \qquad (1)$$

where $\hat{a}_i^\dagger$ is the photon creation operator in the ith mode,
and $m = O(n^2)$. This state is manipulated via a passive
linear optics network which implements a unitary map
on the photon creation operators,

$$\hat{a}_i^\dagger \to \sum_{j=1}^m U_{ij}\hat{a}_j^\dagger, \qquad (2)$$

where U is an $m \times m$ unitary matrix. It was shown by
Reck *et al.* [17] that any U can be efficiently constructed
using $O(m^2)$ linear optics elements.

In an occupation-number representation, the output
state is of the form,

$$|\psi\rangle_{\text{out}} = \sum_S \gamma_S |n_1^{(S)}, \ldots, n_m^{(S)}\rangle, \qquad (3)$$

where S are the different photon number configurations,
the number of which grows exponentially with the num-
ber of photons, as $|S| = \binom{n+m-1}{n}$, and $n_i^{(S)}$ is the number
of photons in mode i associated with configuration S.

Finally, we perform number-resolved photodetection
[18] on the output distribution, obtaining a sample from
the distribution $P(S) = |\gamma_S|^2$. The experiment is re-
peated many times, building up statistics of the output
distribution. It was shown by Aaronson & Arkhipov [7]
that this sampling problem likely cannot be efficiently

simulated classically. The intuitive explanation for this
supposed classical hardness is that each of the ampli-
tudes γ_S is proportional to an $n \times n$ matrix permanent.
Permanents are believed to be classically hard to calcu-
late, residing in the complexity class #**P**-complete, the
class of polynomial time counting problems. The boson-
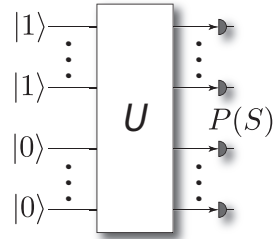sampling model is illustrated in Fig. 1.

FIG. 1: The boson-sampling model. The input state is pre-
pared, comprising a number of single-photon Fock states and
vacuum states. The input state passes through a passive linear
optics network U comprising beamsplitters and phase-shifters.
Finally, the experiment is repeated many times, and each time
the output photon number statistics are sampled from $P(S)$
via coincidence number-resolving photodetection.

Boson-sampling is not believed to be capable of effi-
ciently simulating full quantum computation. Nonethe-
less, it is a relatively simple scheme that can likely rival
classical computers for certain tasks, thus it is an attrac-
tive post-classical quantum computation scheme. It was
shown by Rohde & Ralph that boson-sampling may im-
plement a computationally hard algorithm even in the
presence of high levels of loss [19] and mode-mismatch
[20], although formal hardness proofs are still lacking.

III.   EXPERIMENTAL ARCHITECTURE FOR
BOSON-SAMPLING

Given that SPDC is the most widely used and read-
ily accessible source for single-photon state preparation,
we will present a simple architecture for boson-sampling
based on SPDC sources. In an ideal boson-sampling im-
plementation one would employ deterministic photon
sources that produce exactly one photon on demand.
SPDC sources, on the other hand, coherently prepare
photon pairs in two modes with a correlated Poisson
probability distribution. By measuring one of the modes
and post-selecting upon detecting one photon in that
mode, a single photon is guaranteed to appear in the
other mode. This method provides us with a probabilis-
tic, but heralded single-photon source. It is critical that
each photon is heralded to ensure a pure set of Fock-state
inputs.

Specifically, the two-mode state prepared by a SPDC
is of the form,

$$|\psi\rangle^{\text{SPDC}} = \sum_s \lambda_s |s, s\rangle, \qquad (4)$$

and the photon number probability distribution is given by [21],

$$P^{\text{SPDC}}(s) = |\lambda_s|^2 = \frac{\tanh^{2s} r}{\cosh^2 r}, \tag{5}$$

where s is the photon number (per mode) and r is the squeezing parameter. Thus, the SPDC source most often emits the vacuum state, and sometimes higher-order pairs with exponentially decreasing probability. For small squeezing parameters the higher-order terms can be made small, yielding a heralded source that produces single pairs with high probability.

To herald a single photon, we detect one arm of a single SPDC source using an inefficient number-resolving photodetector. Such a detector can be characterised by the conditional probability of detecting t photons given that s photons were present. For a simple inefficient detector this is given by,

$$P_{\text{D}}(t|s) = \binom{s}{t} \eta^t (1-\eta)^{s-t}, \tag{6}$$

where η is the detection efficiency. Thus, in the presence of loss, the detector exhibits ambiguity in the measured photon number, sometimes detecting fewer photons than were present. Dark counts, the other dominant source of imperfection in photodetection, could also be incorporated into the model, but this effect can be made very small with time-gating.

We specifically consider heralded SPDC states, using just one mode of the SPDC for the computation rather than both, to ensure that the state entering U closely approximates Eq. 1. Without the heralding, the SPDC state is Gaussian, which is inconsistent with the boson-sampling model and not known to implement a classically hard algorithm [22, 23].

Combining Eqs. 5 & 6 we obtain the probability of detecting t photons in the heralding arm of a single SPDC source,

$$\begin{aligned} P_{\text{D}}^{\text{SPDC}}(t) &= \sum_{i \geq t} P_{\text{D}}(t|i) P^{\text{SPDC}}(i) \\ &= \sum_{i \geq t} \binom{i}{t} \eta^t (1-\eta)^{i-t} P^{\text{SPDC}}(i). \end{aligned} \tag{7}$$

Thus the probability of detecting a single photon in the heralding arm is simply,

$$P_{\text{D}}^{\text{SPDC}}(1) = \sum_{i \geq 1} i \, \eta (1-\eta)^{i-1} P^{\text{SPDC}}(i). \tag{8}$$

We will operate N such heralded sources in parallel, where $N \gg n$. The probability that at least n of the SPDC sources successfully herald is given by,

$$P_{\text{prep}}(n) = \sum_{i \geq n} \binom{N}{i} [P_{\text{D}}^{\text{SPDC}}(1)]^i [1 - P_{\text{D}}^{\text{SPDC}}(1)]^{N-i}. \tag{9}$$

In the limit of large N this asymptotes to unity,

$$\lim_{N \to \infty} P_{\text{prep}}(n) = 1. \tag{10}$$

The asymptotic behaviour of P_{prep} is illustrated in Fig. 2. Clearly, with a sufficiently large number of SPDC sources operating in parallel, we are guaranteed to successfully herald the required n single photons.
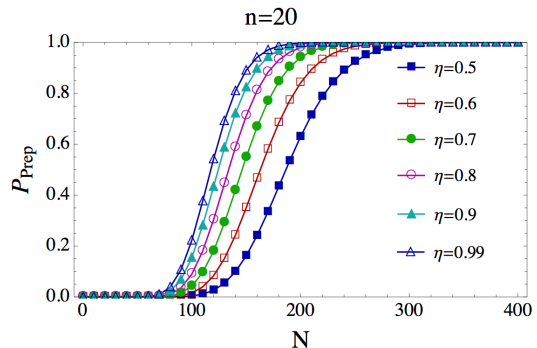

FIG. 2: Asymptotic behaviour of the state preparation success probability as a function of the number of SPDC sources, N, and detector efficiency, η, in the case where we are required to successfully herald $n = 20$ photons. In the limit of large N, P_{prep} approaches unity.

Having successfully heralded at least n SPDC sources, we employ a dynamic multiplexer [24] to route n of the heralded states to the first n modes of the boson-sampling interferometer U. We will assume the multiplexer is ideal in our analysis, although losses could be absorbed into the detector efficiency. Experimental progress has been made recently in developing active multiplexers [25, 26].

Following the unitary network, number-resolving photodetection is applied. Because the photodetectors do not have unit efficiency we must post-select on events where all n photons are detected. The post-selection probability scales as,

$$P_{\text{post}}(n) = \eta^n. \tag{11}$$

Thus, the required detection efficiency exponentially asymptotes to unity for large n. This necessitates that future large-scale boson-sampling implementations will require extremely high efficiency photodetectors.

The full architecture is illustrated in Fig. 3. Note that the multiplexer is *critical* to the operation of the device. Without the multiplexer we still have high likelihood of sampling from at least an n-photon input distribution. However, every time the device is run we are likely to sample from a different permutation of the vacuum and single photon states at the input, making it impossible to perform sampling on a consistent input. Thus, the multiplexing ensures that the input state is consistently of the form of Eq. 1 if the photodetectors have perfect efficiency. The realistic case of inefficient photodetectors is presented next.
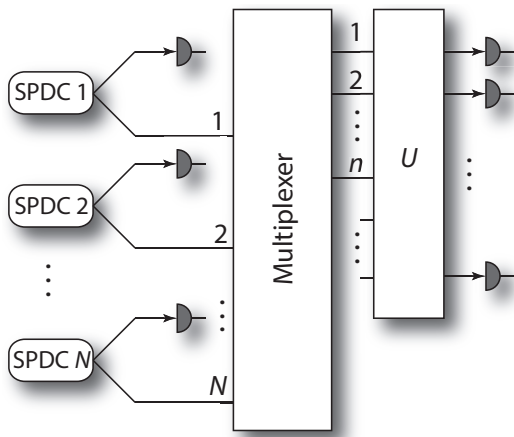
FIG. 3: Architecture for boson-sampling with SPDC sources. N sources operate in parallel, each heralded by an inefficient single-photon number-resolving detection. It is assumed that $N \gg n$, which guarantees that at least n photons will be heralded. The multiplexer dynamically routes the successfully heralded modes to the first n modes of the unitary network U. Finally, photodetection is performed and the output is post-selected on the detection on all n photons.

IV. SCALABILITY OF THE ARCHITECTURE

Having described a general architecture for boson-sampling based on SPDC sources, the pressing question is its scalability. The obvious scaling issue arises from Eq. 11, whereby the photodetection efficiency must be exponentially close to unity. Unless error correction mechanisms are introduced, this scaling is inevitable and post-selection is the only avenue to guarantee successful operation of the device. However, no error correction has been described in the context of boson-sampling. Thus, we will focus on post-selected operation of the device, and address the question as to whether the device acts correctly in that context.

In the described architecture, the dominant error source is incorrect heralding of the SPDC states. In the limit of perfect detectors we are guaranteed to have prepared single-photon states. However, inefficient detectors introduce ambiguity in the heralding, creating a situation where higher-order photon number terms are perceived as single photon terms. For example, if a single photon is lost via detection inefficiency, the two photon state will be interpreted as a single photon state. This will corrupt the input state to the interferometer, yielding an input state different than Eq. 1.

For a single detector, the probability that we have prepared the s-photon Fock state, given that the detector

has outcome t, is given by Bayes' rule,

$$
\begin{aligned}
P_{\text{corr}}(s|t) &= \frac{P_{\text{D}}(t|s)P^{\text{SPDC}}(s)}{P_{\text{D}}^{\text{SPDC}}(t)} \\
&= \frac{\binom{s}{t}(1-\eta)^{s-t}\tanh^{2s}r}{\sum_{i \geq t}\binom{i}{t}(1-\eta)^{i-t}\tanh^{2i}r}.
\end{aligned}
\tag{12}
$$

We are interested in the case where we herald a single photon. Thus,

$$
P_{\text{corr}}(1|1) = [1 - (1-\eta)\tanh^2 r]^2.
\tag{13}
$$

$P_{\text{corr}}(1|1)$ can be interpreted as the conditional probability that we have prepared the correct single photon state given that heralding was successful. For small pump powers ($r \approx 0$) the *unconditional* probability of detecting a single photon approaches zero, although the *conditional* probability approaches unity since there are negligible higher photon-number contributions.

The probability that a single photon is correctly heralded n times in parallel, thereby preparing the n copies of a single photon Fock state, is,

$$
\begin{aligned}
P_{\text{par}}(n) &= [P_{\text{corr}}(1|1)]^n \\
&= [1 - (1-\eta)\tanh^2 r]^{2n}.
\end{aligned}
\tag{14}
$$

We will require that, given n heralded SPDC states, upon post-selection we correctly detect exactly n photons the majority of the time. We will arbitrarily require that $P_{\text{post}}(n) > \epsilon$, where ϵ is the lower bound on the probability that n single photons are successfully detected at the output of the boson-sampling device. This puts a lower bound on the required photodetection efficiency of,

$$
\eta = \sqrt[n]{\epsilon}.
\tag{15}
$$

Next we will assume that all photodetectors in the architecture have the same efficiency. Thus, we obtain that the probability of correctly preparing all n photons via post-selected SPDC is,

$$
P_{\text{par}}(n) = [1 + (\sqrt[n]{\epsilon} - 1)\tanh^2 r]^{2n}.
\tag{16}
$$

In the limit of large n (i.e. large instances of boson-sampling), this asymptotes to,

$$
\lim_{n \to \infty} P_{\text{par}}(n) = \epsilon^{2\tanh^2 r}.
\tag{17}
$$

For small r this approaches unity, and in the limit of large r to ϵ^2. Thus, for $\epsilon = 1/2$, in the worst case scenario, we are sampling from the correct distribution in $1/4$ of the trials. This is shown in Fig. 4.

Eq. 17 specifies the asymptotic probability of sampling from the correct input distribution, given that post-selection was successful. For small squeezing we sample from the correct input distribution most of the time, due to the lower probability of higher-order terms occurring. Thus, for experimentally realistic SPDC sources, provided that detector efficiencies are sufficiently high to
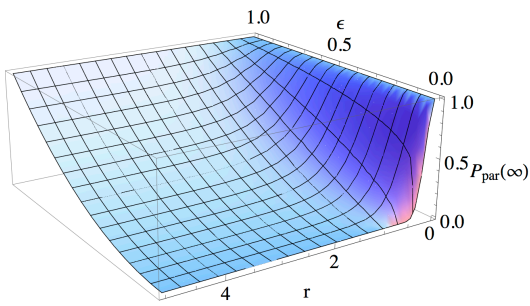
FIG. 4: (Color online) Probability that we are sampling from the correct input distribution in the limit of large n, obtained from Eq. 17, plotted against the SPDC squeezing parameter r.

enable post-selection, we have a high likelihood of correct boson-sampling and SPDC photon-number errors are negligible.

Conversely, we could require that $P_{\text{par}} > \epsilon'$ from Eq. 14, where ϵ' is the lower bound on the probability that a single photon is correctly heralded n times in parallel before entering the multiplexer. Solving this for η yields,

$$\eta = 1 + (\sqrt[2n]{\epsilon'} - 1) \coth^2 r. \qquad (18)$$

From Eq. 11 we obtain an expression for the post-selection probability under the condition that we require a certain fidelity on the SPDC heralding,

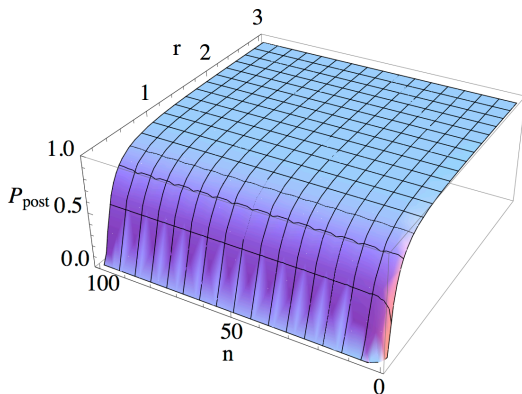$$P_{\text{post}}(n) = [1 + (\sqrt[2n]{\epsilon'} - 1) \coth^2 r]^n. \qquad (19)$$

FIG. 5: (Color online) The post-selection probability P_{post} from Eq. 19 presented as a function of the squeezing parameter r and n single photons being correctly heralded in parallel before entering the multiplexer. Here we assume a fidelity of $\epsilon' = 0.9$.

Fig. 5 illustrates $P_{\text{post}}(n)$ as a function of the squeezing parameter r and the number of successfully routed

photons n. We observe that for large n, post-selection is highly likely to succeed if the SPDC state preparation was successful to within error $\epsilon' = 0.9$. We observe that in the limit of large n and experimentally realistic values of $r \approx 1/2$, boson-sampling using $N \gg n$ SPDC sources is scalable.

V. CONCLUSION

We presented a simple architecture for boson-sampling via multiplexed SPDC sources. We demonstrated that the SPDCs do not limit the scalability of the architecture. Rather, the single-photon detectors, whose efficiencies must increase exponentially with input photon number, limit the scalability. That is provided that detection efficiencies are sufficiently high to enable post-selected operation, the SPDCs will produce Fock states of sufficient fidelity to implement correct boson-sampling with high probability. Conversely, if detection efficiencies are sufficiently high to guarantee SPDC heralding with high fidelity, post-selection will succeed with high probability.

Thus, SPDC sources are a viable photon source technology for future large-scale demonstrations of boson-sampling, and experimentalists should prioritise improving detection efficiencies and developing single-photon multiplexing technologies. Additionally, existing SPDC sources will likely need significant improvement to increase squeezing purity and mode-matching.

While post-selection guarantees correct operation of a boson-sampling device, the required detection efficiencies scale unfavourably. Thus, future work should further address the question as to whether lossy boson-sampling is computationally hard [19], as this could significantly reduce physical resource requirements. Other error models, such as mode-mismatch [20], should also be further investigated.

The analysis presented could be applied to other post-selected linear optics protocols employing SPDCs as heralded Fock state sources.

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