Tripartite-to-bipartite Entanglement Transformation by Stochastic Local Operations and Classical Communication and the Structure of Matrix Spaces

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We study the problem of transforming a tripartite pure state to a bipartite one using stochastic local operations and classical communication (SLOCC). It is known that the tripartite-to-bipartite SLOCC convertibility is characterized by the *maximal Schmidt rank* of the given tripartite state, i.e. the largest Schmidt rank over those bipartite states lying in the support of the reduced density operator. In this paper, we further study this problem and exhibit novel results in both multi-copy and asymptotic settings, about properties of the maximal Schmidt rank, utilizing powerful results from the structure of matrix spaces.

In the multi-copy regime, we observe that the maximal Schmidt rank is strictly super-multiplicative, i.e. the maximal Schmidt rank of the tensor product of two tripartite pure states can be strictly larger than the product of their maximal Schmidt ranks. We then provide a full characterization of those tripartite states whose maximal Schmidt rank is strictly super-multiplicative when taking tensor product with itself. Notice that such tripartite states admit strict advantages in tripartite-to-bipartite SLOCC transformation when multiple copies are provided.

In the asymptotic setting, we focus on determining the tripartite-to-bipartite SLOCC entanglement transformation rate. Computing this rate turns out to be equivalent to computing the *asymptotic maximal Schmidt rank* of the tripartite state, defined as the regularization of its maximal Schmidt rank. Despite the difficulty caused by the super-multiplicative property, we provide explicit formulas for evaluating the asymptotic maximal Schmidt ranks of two important families of tripartite pure states, by resorting to certain results of the structure of matrix spaces, including the study of matrix semi-invariants. These formulas turn out to be powerful enough to give a sufficient and necessary condition to determine whether a given tripartite pure state can be transformed to the bipartite maximally entangled state under SLOCC, in the asymptotic setting. Applying the recent progress on the non-commutative rank problem, we can verify this condition in deterministic polynomial time.

I. INTRODUCTION

As a key concept in quantum mechanics, entanglement plays a central role in quantum information processing. It is the resource responsible for the quantum computational speed-up, quantum communication, quantum cryptography and so on. In the well-known bipartite case, there is no doubt that the bipartite maximally entangled state plays an important role in quantum information theory, since it is usually sufficient to perform many quantum information processing tasks, such as quantum teleportation [1] and superdense coding [2]. Unfortunately, in practice,

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it becomes very difficult to preserve the bipartite pure entanglement as the two systems may interact with other systems (e.g. the environment), thus changing the situation to the multipartite setting. Consequently, a very natural question to ask is, how many bipartite pure entangled states can be distilled from a multipartite state, by means of *local operations and classical communication* (LOCC)? Previous works on this problem introduce various concepts such as the entanglement of assistance [3–5], the localizable entanglement [6–8], the concurrence of assistance [9], the random state entanglement [10], the entanglement of collaboration [11, 12] and the entanglement of combing [13]. These concepts have been shown vastly useful in other areas of quantum information theory, including the study of environment-assisted capacity of quantum channels [14], unital quantum channels, and the quantum Birkhoff's theorem [15].

A slightly different setting is also of great interest. Roughly speaking, assume parties A, B and C share a tripartite pure state. Their goal is to recover some bipartite pure entanglement with a nonzero probability by LOCC, with the help of C. Such protocols usually refer to *stochastic local operations and classical communication* (SLOCC), which has been widely used to study entanglement classification [16, 17] and entanglement transformation [18–20]. The advantage of using SLOCC over LOCC is that SLOCC operations admit a simpler mathematical structure [21]. It is known that the SLOCC bipartite entanglement transformation can be simply characterized by the *Schmidt rank* [22, 23]. The Schmidt rank of a bipartite state $|\phi\rangle$ is the minimum number of product states needed to express it and is denoted by $srk(|\phi\rangle)$. Then $|\phi\rangle$ can be transformed to another bipartite state $|\psi\rangle$, symbolically expressed as $|\phi\rangle \xrightarrow{\text{SLOCC}} |\psi\rangle$, if and only if $srk(|\phi\rangle) \geq srk(|\psi\rangle)$. For SLOCC tripartite entanglement transformations, the *tensor ranks* of tripartite states are nonincreasing under SLOCC, providing an important entanglement monotone [18].

In these contexts, besides characterizing the feasibility of transformations, one may also consider such problems from the algorithmic viewpoint. One important problem is to find efficient algorithms, which, given (the classical description of) two multipartite states, decides one can be transformed to the other using SLOCC. This perspective can be made precise via computational complexity theory. It is well-known that computing the Schmidt rank of a bipartite state (equivalent to computing the rank of a matrix) admits a deterministic polynomial-time algorithm, which can be used to determine the bipartite SLOCC convertibility. On the other hand, computing the tensor rank of a tripartite state is NP-hard [24], and based on this, Chitambar, Duan and Shi [18] have shown that deciding the SLOCC convertibility for tripartite states is also NP-hard in general.

Going back to the multipartite-to-bipartite cases, Chitambar, Duan and Shi [25] have shown that deciding the multipartite-to-bipartite SLOCC convertibility is equivalent to the *polynomial identity testing* (PIT) problem, which is one of the most important problems in theoretical computer science with many applications, such as in perfect matching [26], multiset equality testing [27] and primality testing [28]. Utilizing the Schwartz-Zippel lemma [29, 30], PIT admits a polynomial-time *randomized* algorithm. However, whether a *deterministic* polynomial-time algorithm for PIT exists is still open. Restricting to the tripartite-to-bipartite SLOCC convertibility, it is equivalent to the *symbolic determinant identity testing* (SDIT) problem [26, 31], which asks to compute the rank of a given matrix with entries being linear forms over the complex field and is equivalent to PIT for *weakly-skew arithmetic circuits* [32]. The equivalence can be seen as follows: The tripartite-to-bipartite SLOCC convertibility can be determined by the so called *maximal Schmidt rank* [25], denoted by $msrk(\cdot)$, which is the highest Schmidt rank over those bipartite states lying in the support of the reduced density operator $\mathrm{Tr}_{\mathrm{C}}(|\Psi_{\mathrm{ABC}}\rangle\langle\Psi_{\mathrm{ABC}}|)$ shared by A and B. A tripartite state $|\Psi_{\mathrm{ABC}}\rangle$ can be transformed to a bipartite state $|\phi_{\mathrm{AB}}\rangle$, if and only if the maximal Schmidt rank of $|\Psi_{\mathrm{ABC}}\rangle$ is at least the Schmidt rank of $|\phi_{\mathrm{AB}}\rangle$. Computing the maximal Schmidt rank is exactly equivalent to computing the rank of a given matrix with entries being linear forms over the complex field. Due to this connection, determine the tripartite-to-bipartite SLOCC convertibility can be solved by polynomial-time randomized algorithms for SDIT (e.g. [33]). Interestingly, under plausible computational assumptions, SDIT must also admit a deterministic polynomial-time algorithm [34]. It is believed that to devise such an algorithm would be difficult, as it implies strong circuit lower bounds which seem beyond the current techniques [35].

From the information-theoretic perspective, it is natural to consider asymptotic tripartite-to-bipartite SLOCC transformations. Given n copies of $|\Psi_{\mathrm{ABC}}\rangle$, let $m(n)$ be the maximum number of copies of $|\psi_{\mathrm{AB}}\rangle$ which can be obtained by SLOCC. Then in the asymptotic setting, we are interested in computing the ratio $m(n)/n$ as n goes to infinity, denoted by $R(|\Psi_{\mathrm{ABC}}\rangle, |\psi_{\mathrm{AB}}\rangle)$, which is known as the *SLOCC entanglement transformation rate* (e.g. see Ref. [36, 37]). By defining the *asymptotic maximal Schmidt rank* of a tripartite state as the regularization of its maximal Schmidt rank, the SLOCC transformation rate equals the logarithm of the asymptotic maximal Schmidt rank of the given tripartite state (where the base of the logarithm is the Schmidt rank of the given bipartite state). Asymptotic SLOCC transformations have also been considered in the bipartite and tripartite settings, which lead to the concepts of the asymptotic Schmidt rank and the asymptotic tensor rank, respectively. The asymptotic Schmidt rank equals Schmidt rank itself, as the Schmidt rank is multiplicative, i.e. for bipartite states $|\psi\rangle$ and $|\phi\rangle$, the Schmidt rank of $|\psi\rangle \otimes |\phi\rangle$ equals the product of the Schmidt ranks of $|\psi\rangle$ and $|\phi\rangle$. On the other hand, the tensor rank is not multiplicative [18], which makes the asymptotic tensor rank notoriously difficult to evaluate.

In this paper, we systematically study the tripartite-to-bipartite SLOCC entanglement transformations, in both multi-copy and asymptotic setting. We first illustrate the super-multiplicativity of maximal Schmidt rank, by constructing a tripartite state $|\Psi_{\mathrm{ABC}}\rangle$ satisfying $msrk(|\Psi_{\mathrm{ABC}}\rangle^{\otimes 2}) > msrk(|\Psi_{\mathrm{ABC}}\rangle)^2$. Then we provide a characterization of those tripartite states whose maximal Schmidt ranks are strictly increasing on average under tensor product. Notice that such tripartite states admit strict advantages in tripartite-to-bipartite SLOCC transformation with multiple copies. Interestingly, except for those degenerated cases, this phenomenon holds for all tripartite states of which their maximal Schmidt ranks are not full. In the asymptotic setting, one of the most interesting questions is deciding whether the $d \otimes d$ maximally entangled state $|\Phi_{\mathrm{AB}}\rangle := \frac{1}{\sqrt{d}} \sum_{0 \le i \le d-1} |i_{\mathrm{A}}\rangle |i_{\mathrm{B}}\rangle$ can be obtained from a given tripartite state $|\Psi_{\mathrm{ABC}}\rangle$ by SLOCC asymptotically, i.e. $R(|\Psi_{\mathrm{ABC}}\rangle, |\Phi_{\mathrm{AB}}\rangle) = 1$. Guided by the structure theory of matrix spaces, we exhibit explicit formulas to compute the asymptotic maximal Schmidt ranks of a large family of tripartite states. To obtain one of the formulas, we resort to certain results from invariant theory, specifically from matrix semi-invariants. While the use of invariant theory in entanglement theory is common, to the best of our knowledge, this is the first time that results from matrix semi-invariants are utilized to study SLOCC transformations. Based on these formulas, we settle the question by providing a full characterization of those states that can achieve so. Interestingly, this characterization is algorithmically effective, i.e. there exist deterministic polynomial-time algorithms to determine whether this condition holds for a given tripartite state [38, 39].

Organization. In Section II, we present preliminaries about SLOCC transformations, and some background knowledge of the structure of matrix spaces. In Section III, we construct tripartite states of which their maximal Schmidt ranks are strictly super-multiplicative, and provide a full characterization for those tripartite states that satisfy this property. In Section IV, we explicitly compute the asymptotic maximal Schmidt ranks of a large family of tripartite states and exhibit a sufficient and necessary result to determine whether a tripartite state can be transformed to the bipartite maximally entangled state by SLOCC, in an asymptotic setting. We close in Section V with a brief conclusion.

II. PRELIMINARIES AND BACKGROUNDS

A. Preliminaries

We use $\mathcal{H}_d^{\mathrm{A}}$, $\mathcal{H}_d^{\mathrm{B}}$ and $\mathcal{H}_d^{\mathrm{C}}$ to denote d-dimensional Hilbert spaces (the underlying field is the complex field $\mathbb{C}$) associated with parties A, B and C, respectively. When there is no confusion, we assume $\mathcal{H}^{\mathrm{A}}$ and $\mathcal{H}^{\mathrm{B}}$ have the same dimension (d), and use $\{|0\rangle, \ldots, |d-1\rangle\}$ to denote the computational basis of a d-dimensional Hilbert spaces. For any bipartite pure state $|\psi_{\mathrm{AB}}\rangle$, which is a unit vector in $\mathcal{H}^{\mathrm{A}} \otimes \mathcal{H}^{\mathrm{B}}$, $srk(|\psi_{\mathrm{AB}}\rangle)$ denotes the Schmidt rank of $|\psi_{\mathrm{AB}}\rangle$, which is the minimal number of product states required to linearly span $|\psi_{\mathrm{AB}}\rangle$. For a tripartite pure state $|\Phi_{\mathrm{ABC}}\rangle \in \mathcal{H}^{\mathrm{A}} \otimes \mathcal{H}^{\mathrm{B}} \otimes \mathcal{H}^{\mathrm{C}}$, let $\rho_{\mathrm{AB}}^{\Phi} = \mathrm{Tr}_{\mathrm{C}}(|\Phi_{\mathrm{ABC}}\rangle\langle\Phi_{\mathrm{ABC}}|)$ be the reduced density operator shared by A and B. The mixed state $\rho_{\mathrm{AB}}^{\Phi}$ admits a representation as $\sum_{i=1}^{n} p_i |\psi_i\rangle\langle\psi_i|$, where $\langle\psi_i|\psi_j\rangle = \delta_{ij}$ and $p_i > 0$. The "subnormalized" eigenstates $\{|\tilde{\psi_i}\rangle = \sqrt{p_i}\,|\psi_i\rangle\}_{i=1,\ldots,n}$ span (with respect to complex numbers) the space $\mathrm{supp}(\rho_{\mathrm{AB}}^{\Phi})$, which is called the support of $\rho_{\mathrm{AB}}^{\Phi}$. The *maximal Schmidt rank* of a tripartite pure state $|\Phi_{\mathrm{ABC}}\rangle$ is defined by

$$msrk(|\Phi_{\mathrm{ABC}}\rangle) := \max\{srk(|\phi_{\mathrm{AB}}\rangle) : |\phi_{\mathrm{AB}}\rangle \in \mathrm{supp}(\rho_{\mathrm{AB}}^{\Phi})\}. \tag{1}$$

In the rest of this paper we focus on transforming tripartite *pure* states to bipartite *pure* states by SLOCC, which can be characterized by the following.

Theorem 1 (Chitambar, Duan and Shi [25]) *$|\Phi_{\mathrm{ABC}}\rangle$ can be transformed to $|\psi_{\mathrm{AB}}\rangle$ by means of SLOCC if and only if $msrk(|\Phi_{\mathrm{ABC}}\rangle) \geq srk(|\psi_{\mathrm{AB}}\rangle)$.*

The SLOCC protocol for Theorem 1 as proposed in [25] takes the following form: firstly C makes a measurement on his part of $|\Phi_{\mathrm{ABC}}\rangle$ and broadcasting the result to A and B; then there exists an SLOCC protocol by which A and B can convert their state to $|\psi_{\mathrm{AB}}\rangle$. This "one-way" protocol coincide with the one in the entanglement of assistance [4]. It is also natural to consider the protocol in the entanglement of collaboration [12], which allows two-way communications between A and B on one side, and C on the other side, as follows: before C make measurements, A and B can do measurements on their own systems, and broadcast their outcomes to C. It is known that in the LOCC setting, such two-way communications are necessary for some tripartite-to-bipartite transformations to happen with probability 1 [11]. On the other hand, in the SLOCC setting, Chitambar, Duan and Shi [25] have shown that $|\Phi_{\mathrm{ABC}}\rangle$ can be transformed to $|\psi_{\mathrm{AB}}\rangle$ by means of SLOCC if and only if it can be done by a "one-way" protocol.

Theorem 1 settles the finite-copy case, but leaves the asymptotic setting open, which is natural and important from the information theoretic perspective. In the asymptotic setting, the ability to transform a tripartite pure state $|\Psi_{\mathrm{ABC}}\rangle$ to a bipartite pure state $|\psi_{\mathrm{AB}}\rangle$ is characterized by the *SLOCC entanglement transformation rate* (e.g. see Ref. [36, 37]), defined as follows:

$$R(|\Psi_{\mathrm{ABC}}\rangle, |\psi_{\mathrm{AB}}\rangle) := \sup_{n\geq 1}\left\{\frac{1}{n}\max\{m : |\Psi_{\mathrm{ABC}}\rangle^{\otimes n} \stackrel{\mathrm{SLOCC}}{\longrightarrow} |\psi_{\mathrm{AB}}\rangle^{\otimes m}\}\right\}. \tag{2}$$

Notice that $\max\{m : |\Psi_{\mathrm{ABC}}\rangle^{\otimes n} \stackrel{\mathrm{SLOCC}}{\longrightarrow} |\psi_{\mathrm{AB}}\rangle^{\otimes m}\} = \lfloor \log_{srk(|\psi_{\mathrm{AB}}\rangle)} msrk(|\Psi_{\mathrm{ABC}}\rangle^{\otimes n})\rfloor$ for every fixed n. Define the *asymptotic maximal Schmidt rank* of $|\Psi_{\mathrm{ABC}}\rangle$ as

$$msrk^{\infty}(|\Psi_{\mathrm{ABC}}\rangle) := \sup_{n\geq 1}\sqrt[n]{msrk(|\Psi_{\mathrm{ABC}}\rangle^{\otimes n})}. \tag{3}$$

Then the SLOCC entanglement transformation rate of $|\Psi_{\mathrm{ABC}}\rangle$ and $|\psi_{\mathrm{AB}}\rangle$ can be evaluated by

$$R(|\Psi_{\mathrm{ABC}}\rangle, |\psi_{\mathrm{AB}}\rangle) = \log_{srk(|\psi_{\mathrm{AB}}\rangle)} msrk^{\infty}(|\Psi_{\mathrm{ABC}}\rangle).$$

Essentially, we can replace taking supremum "$\sup_{n\geq 1}$" by taking limit "$\lim_{n\to\infty}$", as shown in the following lemma:

Lemma 2 *$msrk^{\infty}(|\Psi_{\mathrm{ABC}}\rangle)$ is finite for all $|\Psi_{\mathrm{ABC}}\rangle \in \mathcal{H}^{\mathrm{A}} \otimes \mathcal{H}^{\mathrm{B}} \otimes \mathcal{H}^{\mathrm{C}}$. Moreover,*

$$msrk^{\infty}(|\Psi_{\mathrm{ABC}}\rangle) = \lim_{n\to\infty} \sqrt[n]{msrk(|\Psi_{\mathrm{ABC}}\rangle^{\otimes n})}. \tag{4}$$

Proof We shall utilize the following lemma:

Lemma 3 (Lemma in appendix A of Ref. [40]) *Suppose $c_1, c_2, \ldots$ is a nonnegative sequence such that $c_n \leq kn$ for some $k \geq 0$, and $c_m + c_n \leq c_{m+n}$ for all m and n. Then $\lim_{n\to\infty} \frac{c_n}{n}$ exists and is finite.*

Let $c_n = \log_2 msrk(|\Psi_{\mathrm{ABC}}\rangle^{\otimes n})$ and $d = \min\{\dim(\mathcal{H}^{\mathrm{A}}), \dim(\mathcal{H}^{\mathrm{B}})\}$. Choosing $k = \log_2 d$, it is easy to see $c_n \leq n \log_2 d$, as $msrk(|\Psi_{\mathrm{ABC}}\rangle^{\otimes n}) \leq d^n$. On the other hand, to prove $c_m + c_n \leq c_{m+n}$, notice that $srk(|\psi\rangle)srk(|\phi\rangle) = srk(|\psi\rangle \otimes |\phi\rangle)$ holds for any bipartite state $|\psi\rangle$ and $|\phi\rangle$. Then it is easy to see that $mrk(|\Psi_{\mathrm{ABC}}\rangle^{\otimes m} \otimes |\Psi_{\mathrm{ABC}}\rangle^{\otimes n}) \geq mrk(|\Psi_{\mathrm{ABC}}\rangle^{\otimes m})mrk(|\Psi_{\mathrm{ABC}}\rangle^{\otimes n})$, which leads to $c_m + c_n \leq c_{m+n}$. This ensures that $\lim_{n\to\infty} \frac{1}{n} \log_2 msrk(|\Psi_{\mathrm{ABC}}\rangle^{\otimes n}) = \lim_{n\to\infty} \log_2 \sqrt[n]{msrk(|\Psi_{\mathrm{ABC}}\rangle^{\otimes n})}$ exists and is finite by lemma [40]. On the other hand, by Fekete's Lemma [41], the condition that $c_m + c_n \leq c_{m+n}$ for all m and n derives that $\sup_{n\geq 1} \frac{c_n}{n} = \lim_{n\to\infty} \frac{c_n}{n}$. This concludes the proof. □

It is clear that computing the maximal and asymptotic maximal Schmidt rank are physically worthwhile. From the algorithmic perspective, computing the maximal Schmidt rank of a given tripartite state only admits a randomized polynomial-time algorithm [33], since computing the maximal Schmidt rank is equivalent to computing the rank of a matrix whose entries are linear forms [26]. We explain how this equivalence works as this sets the stage for our work as well. Let the space of linear operators from $\mathcal{H}_n^{\mathrm{B}}$ to $\mathcal{H}_m^{\mathrm{A}}$ be $\mathcal{L}(\mathcal{H}_n^{\mathrm{B}}, \mathcal{H}_m^{\mathrm{A}})$. The linear map $\mathrm{vec} : \mathcal{H}_m^{\mathrm{A}} \otimes \mathcal{H}_n^{\mathrm{B}} \to \mathcal{L}(\mathcal{H}_n^{\mathrm{B}}, \mathcal{H}_m^{\mathrm{A}})$ is defined by $\mathrm{vec}(|i\rangle \otimes |j\rangle) = |i\rangle \langle j|$, where $\{|i\rangle : i = 0, \ldots, m-1\}$ and $\{|j\rangle : j = 0, \ldots, n-1\}$ form orthogonal basis of $\mathcal{H}_m^{\mathrm{A}}$ and $\mathcal{H}_n^{\mathrm{B}}$, respectively. vec is a linear isomorphism between $\mathcal{L}(\mathcal{H}_n^{\mathrm{B}}, \mathcal{H}_m^{\mathrm{A}})$ and $\mathcal{H}_m^{\mathrm{A}} \otimes \mathcal{H}_n^{\mathrm{B}}$. Given a bipartite state $|\psi_{\mathrm{AB}}\rangle$, its Schmidt rank equals the rank of $\mathrm{vec}(|\psi_{\mathrm{AB}}\rangle)$. For a tripartite pure state $|\Psi_{\mathrm{ABC}}\rangle \in \mathcal{H}_m^{\mathrm{A}} \otimes \mathcal{H}_n^{\mathrm{B}} \otimes \mathcal{H}_d^{\mathrm{C}}$, we define

$$M(\Psi_{\mathrm{ABC}}) := \mathrm{vec}[\mathrm{supp}(\rho_{\mathrm{AB}}^{\Psi})] = \mathrm{span}\{\mathrm{vec}(|\psi_{\mathrm{AB}}\rangle) : |\psi_{\mathrm{AB}}\rangle \in \mathrm{supp}(\rho_{\mathrm{AB}}^{\Psi})\}, \tag{5}$$

where the linear span is taken over the complex field $\mathbb{C}$. Notice that $M(\Psi_{\mathrm{ABC}})$ is a linear space of linear operators over $\mathbb{C}$. Equivalently, after fixing a basis of $\mathcal{H}_m^{\mathrm{A}} \otimes \mathcal{H}_n^{\mathrm{B}}$, it is a linear space of $m \times n$ matrices over $\mathbb{C}$, which is also called an $m \times n$ *matrix space*. Thus, computing the maximal Schmidt rank of the tripartite state $|\Psi_{\mathrm{ABC}}\rangle$ is equivalent to compute the largest rank over matrices in the matrix space $M(\Psi_{\mathrm{ABC}})$. We define the *maximal rank* and the *asymptotic maximal rank* of a $m \times n$ matrix space $\mathcal{S}$ as

$$mrk(\mathcal{S}) := \max\{rank(E) : E \in \mathcal{S}\}, \;\; mrk^{\infty}(\mathcal{S}) := \sup_{n\geq 1} \sqrt[n]{mrk(\mathcal{S}^{\otimes n})} = \lim_{n\to\infty} \sqrt[n]{mrk(\mathcal{S}^{\otimes n})},$$

where the second equation can be proved using the same argument in lemma 2. It is straightforward to see that, for $|\Psi_{\mathrm{ABC}}\rangle \in \mathcal{H}_m^{\mathrm{A}} \otimes \mathcal{H}_n^{\mathrm{B}} \otimes \mathcal{H}_d^{\mathrm{C}}$, we have

$$msrk(|\Psi_{\mathrm{ABC}}\rangle) = mrk(M(\Psi_{\mathrm{ABC}})), \;\; msrk^{\infty}(|\Psi_{\mathrm{ABC}}\rangle) = mrk^{\infty}(M(\Psi_{\mathrm{ABC}})).$$

In the above, we have reformulated the tripartite-to-bipartite SLOCC transformation problem, in both the finite-copy setting and the asymptotic setting, as computing the maximal ranks and asymptotic ranks of matrix spaces. Therefore, we need to recall some basic properties of matrix spaces. More background knowledge will be covered in the next subsection. Let the space of all

$m \times n$ matrices over the complex field be $M(m \times n, \mathbb{C})$, and let $M(d, \mathbb{C}) := M(d \times d, \mathbb{C})$. The computational basis of $M(m \times n, \mathbb{C})$ is denoted by $\{|i\rangle \langle j| : 0 \leq i \leq m-1, 0 \leq j \leq n-1\}$. We use $\mathbf{0}_{m\times n}$ to denote the zero matrix in $M(m \times n, \mathbb{C})$ (or $\mathbf{0}$ when there is no confusion), and I_d to denote the identity matrix in $M(d, \mathbb{C})$. We use $\mathcal{S} \leq \mathcal{S}'$ to denote that $\mathcal{S}$ is a subspace of $\mathcal{S}'$. Two matrix spaces $\mathcal{S}, \mathcal{S}' \leq M(m \times n, \mathbb{C})$ are *equivalent*, if there exist invertible matrices $P \in M(m, \mathbb{C})$ and $Q \in M(n, \mathbb{C})$, such that $\mathcal{S} = P\mathcal{S}'Q = \{PSQ : S \in \mathcal{S}'\}$. It is easy to see that equivalent matrix spaces have the same maximal rank. A matrix space $\mathcal{S} \leq M(d, \mathbb{C})$ is *non-singular*, if it contains at least one full-rank matrix. Otherwise we say $\mathcal{S}$ is *singular*. One important structure for matrix spaces is the following so-called *shrunk subspace*:

Definition 4 *Given $\mathcal{S} \leq M(d, \mathbb{C})$, a subspace $U \leq \mathbb{C}^d$ is called a* shrunk subspace *of $\mathcal{S}$, if $\dim(U) > \dim(\mathcal{S}(U))$, where $\mathcal{S}(U) := \mathrm{span}\{\cup_{E\in\mathcal{S}}\{E|u\rangle : |u\rangle \in U\}\}$.*

In fact, this definition is reminiscent of the *shrunk subset* as in the famous Hall's marriage theorem [42]. Recall that for a bipartite graph $G = (L \cup R, E)$ where $|L| = |R|$, Hall's marriage theorem states that G has a perfect matching if and only if G does not have shrunk subset, that is a subset $S \subseteq L$ such that $|S| > |N(S)|$ where $N(S)$ denotes the set of neighbours of S. Getting back to the matrix space setting, it is clear that if $\mathcal{S}$ has shrunk subspaces, then $\mathcal{S}$ must be singular. However, unlike in the bipartite graph setting, it is not true that any singular matrix space has shrunk subspaces. For instance, the 3×3 skew symmetric matrix space $\mathrm{span}\{|i\rangle\langle j| - |j\rangle\langle i| : 0 \leq i < j \leq 2\} \leq M(3, \mathbb{C})$ is singular, and has no shrunk subspace. Given a $d \times d$ matrix space $\mathcal{S}$ which has a shrunk subspace U, it admits a particular form when we transform it with appropriate base changes. In the following, we present this form and introduce an important family of matrix spaces, the *maximal-compression matrix spaces*.

Suppose $\dim(U) = d - q$ and $\dim(\mathcal{S}(U)) = p$. By definition 4, we know that $p + q < d$. Fix bases for U and $\mathcal{S}(U)$ so that $U = \mathrm{span}\{|\alpha_q\rangle, \ldots, |\alpha_{d-1}\rangle\}$ and $\mathcal{S}(U) = \mathrm{span}\{|\beta_0\rangle, \ldots, |\beta_{p-1}\rangle\}$. Extend them to full bases of $\mathbb{C}^d$, i.e. $\mathbb{C}^d = \mathrm{span}\{|\alpha_0\rangle, \ldots, |\alpha_{q-1}\rangle, |\alpha_q\rangle, \ldots, |\alpha_{d-1}\rangle\} = \mathrm{span}\{|\beta_0\rangle, \ldots, |\beta_{p-1}\rangle, |\beta_p\rangle, \ldots, |\beta_{d-1}\rangle\}$. Let Q_1 and Q_2 be invertible matrices transforming the original bases to the above two bases, and let $\mathcal{S}' = Q_1\mathcal{S}Q_2^{-1}$. It is easy to verify that every matrix E' in $\mathcal{S}'$ is of the following block form:

$$E' = \left[\begin{array}{c|c} A_{p\times q} & B_{p\times(d-q)} \\ \hline C_{(d-p)\times q} & D_{(d-p)\times(d-q)} \end{array}\right]_{d\times d},$$

where $A_{p\times q} \in \mathrm{span}\{|\beta_i\rangle\langle\alpha_j| : 0 \leq i \leq p-1,\ 0 \leq j \leq q-1\}$, $B_{p\times(d-q)} \in \mathrm{span}\{|\beta_i\rangle\langle\alpha_j| : 0 \leq i \leq p-1,\ q \leq j \leq d-1\}$, $C_{(d-p)\times q} \in \mathrm{span}\{|\beta_i\rangle\langle\alpha_j| : p \leq i \leq d-1,\ 0 \leq j \leq q-1\}$ and $D_{(d-p)\times(d-q)} \in \mathrm{span}\{|\beta_i\rangle\langle\alpha_j| : p \leq i \leq d-1,\ q \leq j \leq d-1\}$. Notice that $D_{(d-p)\times(d-q)} = \mathbf{0}_{(d-p)\times(d-q)}$ for any $E' \in \mathcal{S}'$, since any matrix $E \in \mathcal{S}$ maps U into a subspace of $\mathcal{S}(U)$. Therefore, we conclude that any matrix in $\mathcal{S}$ is transformed by Q_1 and Q_2 to the following form:

$$\left[\begin{array}{c|c} A_{p\times q} & B_{p\times(d-q)} \\ \hline C_{(d-p)\times q} & \mathbf{0}_{(d-p)\times(d-q)} \end{array}\right]_{d\times d}. \tag{6}$$

Given p, q satisfying $p+q < d$, all matrices of form 6 span a matrix space, denoted by $\mathcal{A}(p, q, d)$. Clearly, $\mathcal{A}(p, q, d)$ has shrunk subspaces. In particular, any $d \times d$ matrix space $\mathcal{S}$ with shrunk subspace U, satisfying $\dim(U) = d - q$ and $\dim(\mathcal{S}(U)) = p$, is a subspace of $\mathcal{A}(p, q, d)$ after the above transforming procedure. In this sense we may call $\mathcal{A}(p, q, d)$ a maximal-compression matrix space. A formal definition is as follows.

Definition 5 *Let $p, q \in \mathbb{N}$. The $d \times d$ matrix space $\mathcal{A}(p,q,d)$ is the matrix space spanned by those elementary matrices whose nonzero entry lies either in the first p rows, or in the first q columns, i.e.*

$$\mathcal{A}(p,q,d) := \mathrm{span}\{\{|i\rangle\langle j| : 0 \le i \le p-1,\ 0 \le j \le d-1\} \cup \{|i\rangle\langle j| : p \le i \le d-1,\ 0 \le j \le q-1\}\}.$$

The maximal rank of $\mathcal{A}(p,q,d)$ equals $\min\{p+q,d\}$. Moreover, if $p+q<d$, we say $\mathcal{A}(p,q,d)$ is a maximal-compression matrix space.

This definition can be generalized to the rectangular matrix spaces. Let $\mathcal{A}(p,q,m,n)$ be the $m \times n$ matrix space spanned by the first p rows and q columns of elementary matrices, i.e.

$$\mathcal{A}(p,q,m,n) := \mathrm{span}\{\{|i\rangle\langle j| : 0 \le i \le p-1,\ 0 \le j \le n-1\} \cup \{|i\rangle\langle j| : p \le i \le m-1,\ 0 \le j \le q-1\}\}.$$

Then the maximal rank of $\mathcal{A}(p,q,m,n)$ equals $\min\{p+q,m,n\}$. And we say $\mathcal{A}(p,q,m,n)$ is a maximal-compression matrix space if $p+q<\min\{m,n\}$.

B. Results on shrunk subspaces

In this subsection we first review some mathematical results concerning shrunk subspaces. We then introduce recent progress on algorithms to decide the existence of shrunk subspaces.

Shrunk subspaces emerge in several mathematical areas. The first appearance of shrunk subspaces seems to be in T. G. Room's treatise on determinants in 1930's [43]. We mentioned in Section II A that shrunk subspaces can be viewed as a linear algebraic analogue of shrunk subsets as in the Hall's marriage theorem, which in turn is a basic result in combinatorics [44]. In matroid theory, Lovász observed that the intersection of two linear matroids naturally leads to shrunk subspaces [45].

Shrunk subspaces appear in non-commutative algebra as follows. Suppose $\mathcal{S} \le M(d,\mathbb{F})$ is a matrix space over some field $\mathbb{F}$ and spanned by $\{T_1,\ldots,T_m\} \subset M(d,\mathbb{F})$. Let $\{x_1,\ldots,x_m\}$ be a set of non-commuting variables, and form a matrix $T = x_1T_1 + \cdots + x_mT_m$ whose entries are linear forms in x_i's. Matrices of this type have been studied in non-commutative algebra in the context of the free skew field since the 1970's [46]. The rank of such a matrix over the free skew field, which was named *non-commutative rank* and denoted by $ncrk(\cdot)$, was shown to be the minimum r such that there exists a subspace $U \le \mathbb{F}^d$ with $\dim(U) - \dim(\mathcal{S}(U)) = d-r$ [47]. Thus, $ncrk(\mathcal{S}) < d$ if and only if $\mathcal{S}$ have shrunk subspaces.

Another way to reach the concept of shrunk subspaces is to consider matrix spaces with maximal ranks bounded from above [48]. Characterizing those matrix spaces is known to be a difficult problem; in fact, such matrix spaces basically correspond to certain torsion-free sheaves on projective spaces [48]. To make progress on this topic, one approach is to consider certain "witnesses" that can serve as an upper bound on the maximal rank. As mentioned, shrunk subspaces can be used as such witnesses. Specifically, if a matrix space $\mathcal{S} \le M(d,\mathbb{F})$ has a shrunk subspace U with $\dim(U) - \dim(\mathcal{S}(U)) = c > 0$, then it is clear that $mrk(\mathcal{S}) \le d-c$.

On the other hand, an interesting result by Fortin and Reutenauer showed the following:

Theorem 6 (Corollary 2. in Ref. [47]) *Let $\mathcal{S}$ be a matrix space in $M(d,\mathbb{F})$. Then*

$$mrk(\mathcal{S}) \le ncrk(\mathcal{S}) \le 2mrk(\mathcal{S}).$$

An important characterization of matrix spaces with shrunk subspaces comes from invariant theory. Consider the group action of $(A,C) \in \mathrm{SL}(d) \times \mathrm{SL}(d)$ on $M(d,\mathbb{F})^{\oplus m}$ by sending $(B_1,\ldots,B_m)$ to $(AB_1C^T,\ldots,AB_mC^T)$. This induces an action on the ring of polynomial functions on $M(d,\mathbb{F})^{\oplus m}$. Let $R(d,m)$ be the ring of those polynomials invariant under this action.

$R(d, m)$ is called *the ring of matrix semi-invariants* (for matrices of size $d \times d$) [39, 49]. The common zeros of the homogeneous polynomials of positive degrees in $R(d, m)$, denoted as $N(R(d, m))$, is referred to as the *nullcone* of this invariant ring in the invariant theory literature. The link to those matrix spaces which have shrunk subspaces is the following result from invariant theory, proved using the celebrated Hilbert-Mumford criterion.

Theorem 7 ([50, 51]) *$(B_1, \ldots, B_m)$ is in $N(R(d, m))$ if and only if $\mathrm{span}\{B_1, \ldots, B_m\}$ has a shrunk subspace.*

Therefore, matrix spaces with shrunk subspaces are characterized by those polynomials in $R(d, m)$.

Invariant theory also helps to certify those matrix spaces with no shrunk subspaces. That is, given a matrix space $\mathcal{S} \leq M(d, \mathbb{F})$, if $\mathcal{S}$ does not have shrunk subspace, we would like to present a short witness to certify this fact. For example, if $\mathcal{S}$ contains a full-rank matrix A, then exhibiting A is enough to certify that $\mathcal{S}$ does not contain shrunk subspaces. However, as mentioned, it is possible that a matrix space has neither full-rank matrices nor shrunk subspaces. To resolve this difficulty, we first recall what polynomials in $R(d, m)$ look like. This task is usually resolved in the so-called first fundamental theorem for $R(d, m)$.

Theorem 8 ([51–54]) *Every homogeneous polynomial in $R(d, m)$ is of degree kd for some $k \in \mathbb{N}$, and is a linear combination of polynomials of the form $\det(X_1 \otimes A_1 + \cdots + X_m \otimes A_m)$ where the X_i's are $d \times d$ variable matrices, and the A_i's are $k \times k$ matrices over $\mathbb{F}$.*

Theorem 8 motivates the following definition. For a matrix space $\mathcal{S} \leq M(d, \mathbb{F})$, the kth *blow-up* of $\mathcal{S}$ is $\mathcal{S}^{[k]} = \mathcal{S} \otimes M(k, \mathbb{F})$. It is clear that, if $\mathcal{S}$ possesses a shrunk subspace, then $\mathcal{S}^{[k]}$ has a shrunk subspace for any positive integer k. On the other hand, if $\mathcal{S} = \mathrm{span}\{B_1, \ldots, B_m\}$ does not possess a shrunk subspace, then it is not in the nullcone of $R(d, m)$ (Theorem 7). This implies that there exists some $A_1, \ldots, A_m \in M(k, \mathbb{F})$ such that $\det(B_1 \otimes A_1 + \cdots + B_m \otimes A_m) \neq 0$ (Theorem 8), which just says that $\mathcal{S}^{[k]}$ contains a non-singular matrix. To see that k is finite is classical: by Hilbert's basis theorem, $N(R(d, m))$ can be defined by finitely many polynomials, therefore k is also finite. Recently, exciting progress suggests that k can be taken to be no more than $d - 1$ as long as $|\mathbb{F}|$ is large enough [55]; see also [56] for a simpler proof of $k \leq d + 1$. Summarizing the above we have

Theorem 9 ([55]) *Suppose $|\mathbb{F}| = d^{\Omega(1)}$, where $\Omega(1)$ is asymptotic in d. If $\mathcal{S} \leq M(d, \mathbb{F})$ does not have shrunk subspace, then for some $k \leq d - 1$, $\mathcal{S} \otimes M(k, \mathbb{F})$ contains a full-rank matrix.*

The full-rank matrix as in Theorem 9 then serves as a short witness for $\mathcal{S}$ to have no shrunk subspace. We can also easily formulate an algorithmic problem around shrunk subspaces as follows.

Problem 10 *Given a matrix space $\mathcal{S} \leq M(d, \mathbb{F})$, decide whether $\mathcal{S}$ has a shrunk subspace $U \leq \mathbb{F}^d$.*

This problem is known as the non-commutative rank problem [47], the non-commutative rational identity testing problem [57], and the non-commutative Edmonds' problem [58]. We adopt the non-commutative rank problem as in [47]. This name choice is due to the connection with the free skew field as mentioned above. Recent advances imply that this problem can be solved deterministically in polynomial time.

Theorem 11 (Ref. [38, 39]) *There exists a deterministic polynomial-time algorithm to decide whether a given matrix space $\mathcal{S} \leq M(d, \mathbb{F})$ has full non-commutative rank or not when $|\mathbb{F}|$ is large enough.*

III. MULTI-COPY TRANSFORMATION

In this section, we discuss the super-multiplicativity of the maximal Schmidt rank. We say a map $f : \mathcal{H}_d \to \mathbb{R}$ is super-multiplicative, if for any $|\psi\rangle$, $|\phi\rangle \in \mathcal{H}_d$, $f(|\psi\rangle \otimes |\phi\rangle) \geq f(|\psi\rangle) \times f(|\phi\rangle)$. Clearly, the maximal Schmidt rank, as well as many other information theoretic quantities, are super-multiplicative. A special case of the super-multiplicative maps is the multiplicative maps, i.e. those maps $f : \mathcal{H}_d \to \mathbb{R}$ satisfy $f(|\psi\rangle \otimes |\phi\rangle) = f(|\psi\rangle) \times f(|\phi\rangle)$ for any $|\psi\rangle, |\phi\rangle \in \mathcal{H}_d$. Multiplicative maps are important in quantum in formation theory, as they provides computable asymptotic quantities. For instance, the *negativity* [59] is an important computable entanglement measure, which provides an upper bound on distillable entanglement. Similarly, several multiplicative quantities have been constructed to provide efficiently computable bounds on distillable entanglement [60], entanglement cost [61], and classical and quantum capacity of quantum channels [62–65].

On the other hand, many information-theoretic quantities are not multiplicative, but strictly super-multiplicative. A *strictly super-multiplicative map* $f : \mathcal{H}_d \to \mathbb{R}$ is super-multiplicative, and there exist $|\psi\rangle$, $|\phi\rangle \in \mathcal{H}_d$ satisfying $f(|\psi\rangle \otimes |\phi\rangle) > f(|\psi\rangle) \times f(|\phi\rangle)$. This property causes much difficulty to compute the corresponding asymptotic counterparts, and yields many intriguing phenomenons. One example is that, two quantum channels with vanishing quantum capacities can have nonzero quantum capacity when used together [66]. This phenomenon not only illustrates the super-additivity of quantum capacity (which becomes super-multiplicativity before taking logarithm), but also illustrated that quantum capacity can be "super-activated". In the rest, we show that the maximal Schmidt rank is strictly super-multiplicative.

Theorem 12 *Let d be an odd number and*

$$|\Psi^d_{\mathrm{ABC}}\rangle := \sqrt{\frac{2}{d(d-1)}} \sum_{0\leq i<j\leq d-1} (|i\rangle |j\rangle - |j\rangle |i\rangle) \otimes |\psi_{ij}\rangle \in \mathcal{H}^{\mathrm{A}}_d \otimes \mathcal{H}^{\mathrm{B}}_d \otimes \mathcal{H}^{\mathrm{C}}_{d^2},$$

where $\{|i\rangle : 0 \leq i \leq d-1\}$, $\{|j\rangle : 0 \leq j \leq d-1\}$ and $\{|\psi_{ij}\rangle : 0 \leq i, j \leq d-1\}$ are sets of orthogonal basis of $\mathcal{H}^{\mathrm{A}}_d$, $\mathcal{H}^{\mathrm{B}}_d$ and $\mathcal{H}^{\mathrm{C}}_{d^2}$, respectively. We have

$$msrk(|\Psi^d_{\mathrm{ABC}}\rangle) = d-1,\ msrk(|\Psi^d_{\mathrm{ABC}}\rangle^{\otimes 2}) = d^2 > (d-1)^2. \tag{7}$$

In fact, any tripartite state $|\Phi^d_{\mathrm{ABC}}\rangle$ with $M(\Phi^d_{\mathrm{ABC}}) = \mathrm{span}\{|i\rangle\langle j| - |j\rangle\langle i| : 0 \leq i < j \leq d-1\}$, which is the $d \times d$ skew-symmetric matrix space, satisfies equation (7).

Proof It is known that for odd d, the maximal rank of the $d \times d$ skew-symmetric matrix space is $d-1$ [47]. We now show $mrk(M(\Phi^d_{\mathrm{ABC}})) = d^2$. Specifically, let $E_{i,j} = |i\rangle\langle j| - |j\rangle\langle i| \in M(\Phi^d_{\mathrm{ABC}})$ for $0 \leq i < j \leq d-1$. We claim that

$$P := \sum_{0\leq i<j\leq d-1} E_{i,j} \otimes E_{i,j}$$

has rank d^2. Notice that P is in the block matrix form:

$$P = \begin{pmatrix} \mathbf{0} & E_{0,1} & \cdots & E_{0,d-1} \\ -E_{0,1} & \mathbf{0} & \cdots & E_{1,d-1} \\ \vdots & \vdots & \ddots & \vdots \\ -E_{0,d-1} & -E_{1,d-1} & \cdots & \mathbf{0} \end{pmatrix}.$$

We will prove that $\mathrm{Ker}(P) = \{0\}$. Let $|\alpha\rangle = \sum_{i,j=0}^{d-1} x_i(j)\,|i\rangle\,|j\rangle$ such that $P\,|\alpha\rangle = 0$, where $x_i(j)$ are variables. Denote $|\alpha_i\rangle = \sum_{j=0}^{d-1} x_i(j)\,|j\rangle$. Then for $1 \le k \le d-2$, we have:

$$-\sum_{i=0}^{k-1} E_{i,k}\,|\alpha_i\rangle + \sum_{i=k+1}^{d-1} E_{k,i}\,|\alpha_i\rangle = 0.$$

For $k = 0$, we have $\sum_{i=1}^{d-1} E_{0,i}\,|\alpha_i\rangle = 0$ and for $k = d-1$, we have $\sum_{i=0}^{d-2} E_{i,d-1}\,|\alpha_i\rangle = 0$. Noticing that $E_{j,k}\,|\alpha_i\rangle = x_i(k)\,|j\rangle - x_i(j)\,|k\rangle$, we can rewrite the equations to:

$$\sum_{i=1}^{d-1}(x_i(0)\,|i\rangle - x_i(i)\,|0\rangle) = 0,$$
$$-\sum_{i=0}^{k-1}(x_i(k)\,|i\rangle - x_i(i)\,|k\rangle) + \sum_{i=k+1}^{d-1}(x_i(i)\,|k\rangle - x_i(k)\,|i\rangle) = 0, \quad k = 1,\ldots,d-2$$
$$\sum_{i=0}^{d-2}(x_i(d-1)\,|i\rangle - x_i(i)\,|d-1\rangle) = 0.$$

These yield that $x_i(j) = 0$ for all $0 \le i \ne j \le d-1$ and $m_k = \sum_{i\ne k} x_i(i) = 0$ for $k = 0,\ldots,d-1$. Notice that $m_k - m_{k+1} = x_{k+1}(k+1) - x_k(k) = 0$ for $k = 0,\ldots,d-2$ and $m_{d-1} - m_0 = x_0(0) - x_{d-1}(d-1) = 0$. We derive $x_k(k) = 0$ for $k = 0,\ldots,d-1$. Thus $|\alpha\rangle = 0$ is the only solution for $P\,|\alpha\rangle = 0$, hence $\mathrm{rank}(P) = d^2$. □

Theorem 12 also implies that $|\Psi^d_{\mathrm{ABC}}\rangle$ cannot be transformed to the $d \otimes d$ maximally entangled state by means of SLOCC, but can do so with two copies. In the multi-copy regime, those tripartite states $|\Psi_{\mathrm{ABC}}\rangle$ satisfying $msrk(|\Psi_{\mathrm{ABC}}\rangle^{\otimes 2}) > msrk(|\Psi_{\mathrm{ABC}}\rangle)^2$ are of great interest, as such states allow for more advantages in the tripartite-to-bipartite SLOCC transformations when multiple copies are provided. Interestingly, these tripartite states can be characterized by the following theorem, based on the structure of their corresponding matrix spaces:

Theorem 13 *Given a tripartite state $|\Psi_{\mathrm{ABC}}\rangle \in \mathcal{H}^{\mathrm{A}}_{d_1} \otimes \mathcal{H}^{\mathrm{B}}_{d_2} \otimes \mathcal{H}^{\mathrm{C}}_{d_3}$, let $\mathcal{S} = M(\Psi_{\mathrm{ABC}}) \le M(d_1 \times d_2, \mathbb{C})$. Then $msrk(|\Psi_{\mathrm{ABC}}\rangle^{\otimes 2}) > msrk(|\Psi_{\mathrm{ABC}}\rangle)^2$ if and only if*

1. *$mrk(\mathcal{S}) < \dim[\mathrm{Im}(\mathcal{S})]$, and*

2. *$mrk(\mathcal{S}) < d_2 - \dim[\mathrm{Ker}(\mathcal{S})]$,*

where the image and kernel of a matrix space $\mathcal{S}$ are defined as $\mathrm{Im}(\mathcal{S}) := \mathrm{span}\{\cup_{E\in\mathcal{S}}\mathrm{Im}(E)\}$ and $\mathrm{Ker}(\mathcal{S}) := \cap_{E\in\mathcal{S}}\mathrm{Ker}(E)$.

Remark Theorem 13 indicates that the inequality $msrk(|\Psi_{\mathrm{ABC}}\rangle^{\otimes 2}) > msrk(|\Psi_{\mathrm{ABC}}\rangle)^2$ holds for almost all tripartite states whose maximal Schmidt rank is not full, as the two conditions in theorem 13 put only degenerate restrictions on such states.
Proof Equivalently, we consider when $mrk(\mathcal{S}^{\otimes 2}) > mrk(\mathcal{S})^2$. Without loss of generality, we assume $d_1 \le d_2$. The following observation from Ref. [67] is useful to estimate the rank of linear combinations of two matrices:

Lemma 14 (Lemma 2.2 in Ref. [67]) *Given two matrices $X, Y \in M(d_1 \times d_2, \mathbb{C})$. If $Y\,\mathrm{Ker}(X) \not\le \mathrm{Im}(X)$, then $\mathrm{rank}(X{+}rY) > \mathrm{rank}(X)$ except for at most $\mathrm{rank}(X) + 1$ elements $r \in \mathbb{C}$.*

We first prove the sufficiency. Notice that a matrix space $\mathcal{S}$ satisfying the above two conditions must be singular. Choose $E \in \mathcal{S}$ with the highest rank, i.e. $rank(E) = mrk(\mathcal{S}) < d_1$. Define the following two matrix spaces:

$$\mathcal{X} := \{F \in \mathcal{S} : \mathrm{Im}(F) \leq \mathrm{Im}(E)\}, \;\; \mathcal{Y} := \{F \in \mathcal{S} : \mathrm{Ker}(E) \leq \mathrm{Ker}(F)\}.$$

We claim that $\mathcal{X}$ and $\mathcal{Y}$ are two proper subspaces of $\mathcal{S}$. Otherwise, assuming $\mathcal{X} = \mathcal{S}$, we have $mrk(\mathcal{S}) = rank(E) = \dim[\mathrm{Im}(E)] = \dim[\mathrm{Im}(\mathcal{S})]$, which is a contradiction. If $\mathcal{Y} = \mathcal{S}$, we have $\dim[\mathrm{Ker}(\mathcal{S})] = \dim[\mathrm{Ker}(E)] = d_2 - rank(E) = d_2 - mrk(\mathcal{S})$, which is also a contradiction. Now we can choose $E' \in \mathcal{S}$ such that $E' \notin \mathcal{X} \cup \mathcal{Y}$, i.e. $\mathrm{Im}(E') \not\leq \mathrm{Im}(E)$ and $\mathrm{Ker}(E) \not\leq \mathrm{Ker}(E')$.

Then we claim that there exists $r \in \mathbb{C}$ such that $rank(E \otimes E + rE' \otimes E') > rank(E \otimes E)$. Notice that $\mathrm{Ker}(E \otimes E) = \mathrm{span}\{\mathrm{Ker}(E) \otimes \mathbb{C}^{d_2}, \mathbb{C}^{d_2} \otimes \mathrm{Ker}(E)\}$ and $\mathrm{Im}(E \otimes E) = \mathrm{Im}(E) \otimes \mathrm{Im}(E)$. We are going to show $(E' \otimes E')(\mathrm{Ker}(E) \otimes \mathbb{C}^{d_2}) \not\leq \mathrm{Im}(E) \otimes \mathrm{Im}(E)$. Since E has the highest rank in $\mathcal{S}$, $rank(E{+}rE') \leq rank(E)$ for any $r \in \mathbb{C}$. By Lemma 14, we see that $E'\mathrm{Ker}(E) \leq \mathrm{Im}(E)$. Since $\mathrm{Ker}(E) \not\leq \mathrm{Ker}(E')$, we can choose a non-zero vector $|v\rangle \in \mathrm{Ker}(E)$ such that $0 \neq E' |v\rangle \in \mathrm{Im}(E)$. Moreover, since $\mathrm{Im}(E') \not\leq \mathrm{Im}(E)$, we can find a vector $|u\rangle$ such that $0 \neq E' |u\rangle \notin \mathrm{Im}(E)$. Setting $|v\rangle \otimes |u\rangle \in \mathrm{Ker}(E) \otimes \mathbb{C}^{d_2} \leq \mathrm{Ker}(E \otimes E)$, we have $0 \neq E' |v\rangle \otimes E' |u\rangle \notin \mathrm{Im}(E) \otimes \mathrm{Im}(E)$. Then by Lemma 14, there exists $r \in \mathbb{C}$ such that:

$$mrk(\mathcal{S}^{\otimes 2}) \geq rank(E \otimes E + rE' \otimes E') > rank(E \otimes E) = mrk(\mathcal{S})^2.$$

For the necessity, we shall show that if $\mathcal{S}$ does not satisfy either one of those two conditions, $mrk(\mathcal{S}^{\otimes 2}) = mrk(\mathcal{S})^2$. When $\mathcal{S}$ is non-singular, this equation always holds. Now give a singular matrix space $\mathcal{S} \leq M(d_1 \times d_2, \mathbb{C})$ satisfying $mrk(\mathcal{S}) = \dim[\mathrm{Im}(\mathcal{S})]$, and $mrk(\mathcal{S}) = d' < d_1$. Fix bases of $\mathrm{Im}(\mathcal{S})$, say $\{v_1, \ldots, v_{d'}\}$ and extend them to $\{v_1, \ldots, v_{d'}, v_{d'+1}, \ldots, v_{d_1}\}$, which are full bases of $\mathbb{C}^{d_1}$. For any matrix $F \in \mathcal{S}$, we have $\mathrm{Im}(F) \leq \mathrm{span}\{v_1, \ldots, v_{d'}\}$. Then any matrix $F \in \mathcal{S}$ can be expressed by this basis in the following form:

$$F = \begin{pmatrix} F_{d' \times d_2} \\ \mathbf{0}_{(d_1 - d') \times d_2} \end{pmatrix},$$

where $F_{d' \times d_2}$ is a $d' \times d_2$ matrix and the rest part of F are all zero. Thus we can derive $mrk(\mathcal{S}^{\otimes 2}) \leq mrk(\mathcal{S})^2$ since any matrix in $\mathcal{S} \otimes \mathcal{S}$ will have at most ${d'}^2$ non-zero rows. The other case is similar, and we conclude the proof. □

IV. ASYMPTOTIC TRANSFORMATIONS

Theorem 12 implies the super-multiplicativity of the maximal Schmidt rank. And theorem 13 indicates how the structure of matrix spaces plays a role in this topic. In this section, we initiate the study of the asymptotic maximal (Schmidt) rank. Utilizing certain results from the structure of matrix spaces, including some relevant results from invariant theory, we are able to present explicit formulas to calculate this asymptotic quantity for two important families of matrix spaces (tripartite states), which would be difficult to compute without the help of these tools. Our results are summarized in the following theorem:

Theorem 15 *Given a matrix space $\mathcal{S} \leq M(d, \mathbb{C})$,*

1. *If $\mathcal{S}$ does not have shrunk subspace, we have $\frac{1}{2}d^n \leq mrk(\mathcal{S}^{\otimes n}) \leq d^n$. Thus $mrk^{\infty}(\mathcal{S}) = d$.*

2. *If $\mathcal{S} = \mathcal{A}(p, q, d)$ satisfying $p + q < d$, then*

$$mrk^{\infty}(\mathcal{S}) = d \max\{2^{-D(1-\alpha||p')}, 2^{-D(\alpha||q')}\}, \tag{8}$$

where $p' = \frac{p}{d}$, $q' = \frac{q}{d}$, $\alpha = \frac{\log_2(d-q)-\log_2 p}{\log_2((d-p)(d-q))-\log_2(pq)}$ *and* $D(a||b) := a\log_2\frac{a}{b} + (1-a)\log_2\frac{1-a}{1-b}$.

Proof of theorem 15. 1. Recall that if $\mathcal{S}$ does not have shrunk subspace, $ncrk(\mathcal{S}) = d$. Then by theorem 6, $\frac{1}{2}d \leq mrk(\mathcal{S}) \leq d$. Thus, we first prove that, given two matrix spaces without shrunk subspaces, there tensor product still has no shrunk subspaces.

Lemma 16 *Given two matrix spaces $\mathcal{S}_1 \leq M(d_1, \mathbb{C})$ and $\mathcal{S}_2 \leq M(d_2, \mathbb{C})$ which have no shrunk subspace. Then $\mathcal{S}_1 \otimes \mathcal{S}_2 \leq M(d_1 d_2, \mathbb{C})$ has no shrunk subspace.*

Proof Recall that by theorem 9, for $i = 1, 2$, if $\mathcal{S}_i \leq M(d_i, \mathbb{C})$ has no shrunk subspace, there exists $k_i \leq d_i - 1$, such that $mrk(\mathcal{S} \otimes M(k_i, \mathbb{C})) = k_i d_i$. Now consider the matrix space $\mathcal{S}_1 \otimes \mathcal{S}_2 \otimes M(k_1 k_2, \mathbb{C})$, which is isomorphic to $\mathcal{S}_1 \otimes M(k_1, \mathbb{C}) \otimes \mathcal{S}_2 \otimes M(k_2, \mathbb{C})$. Since $\mathcal{S}_1$ and $\mathcal{S}_2$ have no shrunk subspace, we can find matrices $A_i \in \mathcal{S}_i \otimes M(k_i, \mathbb{C})$ such that $\text{rank}(A_i) = k_i d_i$ for $i = 1, 2$. Then we have $mrk(\mathcal{S}_1 \otimes \mathcal{S}_2 \otimes M(d_1 d_2, \mathbb{C})) = \text{rank}(A_1 \otimes A_2) = k_1 k_2 d_1 d_2$. By theorem 7 and theorem 8, $\mathcal{S}_1 \otimes \mathcal{S}_2$ cannot have shrunk subspaces. □

By this lemma, we derive that for $\mathcal{S} \leq M(d, \mathbb{C})$ satisfying $ncrk(\mathcal{S}) = d$, $ncrk(\mathcal{S}^{\otimes n}) = d^n$ for $n \in \mathbb{N}$. By theorem 6, we have $\frac{1}{2}d^n \leq mrk(\mathcal{S}^{\otimes n}) \leq d^n$, and $mrk^{\infty}(\mathcal{S}) = d$ follows. □

To prove theorem 15. 2, we first explicitly compute the maximal rank of the tensor product of two maximal-compression matrix spaces:

Lemma 17 *Given two maximal-compression matrix spaces $\mathcal{A}_1 = \mathcal{A}(p_1, q_1, m_1, n_1)$ and $\mathcal{A}_2 = \mathcal{A}(p_2, q_2, m_2, n_2)$ (satisfying $p_1 + q_1 < \min\{m_1, n_1\}$ and $p_2 + q_2 < \min\{m_2, n_2\}$), we have:*

$$mrk(\mathcal{A}_1 \otimes \mathcal{A}_2) = p_1 p_2 + \min\{(n_1 - q_1) q_2, p_1(m_2 - p_2)\} + \min\{(m_1 - p_1) p_2, q_1(n_2 - q_2)\} + q_1 q_2. \tag{9}$$

Proof It is convenient to view $\mathcal{A}(p, q, m, n)$ as symbolic matrix P, of which the entries are filled with 0 and $*$. More precisely, the (i, j)th entry of the symbolic matrix is $*$ if and only if there exist matrices in $\mathcal{A}(p, q, m, n)$ such that the (i, j)th entry of which is non-zero. Due to the structure of $\mathcal{A}(p, q, m, n)$, these $*$s can be assigned arbitrary complex numbers independently. Moreover, it is easy to see that, the symbolic matrix of $\mathcal{A}(p_1, q_1, m_1, n_1) \otimes \cdots \otimes \mathcal{A}(p_k, q_k, m_k, n_k)$ is $P_1 \otimes \cdots \otimes P_k$, where P_i is the symbolic matrix of $\mathcal{A}(p_i, q_i, m_i, n_i)$ for $i = 1, \ldots, k$ and the multiplication rule of $\{0, *\}$ is $0 \times 0 = 0$, $0 \times * = * \times 0 = 0$, $* \times * = *$.

With this definition, we firstly write down the symbolic matrix P of $\mathcal{A}_1 \otimes \mathcal{A}_2$:

$$P = \begin{pmatrix} A_{1,1} & \cdots & A_{1,q_1} & A_{1,q_1+1} & \cdots & A_{1,n_1} \\ \vdots & P_0 & \vdots & \vdots & P_1 & \vdots \\ A_{p_1,1} & \cdots & A_{p_1,q_1} & A_{p_1,q_1+1} & \cdots & A_{p_1,n_1} \\ A_{p_1+1,1} & \cdots & A_{p_1+1,q_1} & & & \\ \vdots & P_2 & \vdots & & \mathbf{0} & \\ A_{m_1,1} & \cdots & A_{m_1,q_1} & & & \end{pmatrix}, \tag{10}$$

where $A_{i,j}$ is the symbolic matrix of $\mathcal{A}_2$ for all possible i and j, and the rest block of size $(m_1-p_1)m_2 \times (n_1-q_1)n_2$ are all zero. Denote the upper left block by P_0, the upper right block by P_1 and the lower left block by P_2. We will show that, after properly rearranging rows and columns, P_0, P_1 and P_2 become symbolic matrices of $\mathcal{A}(p, q, m, n)$ with different parameters. This can be done by the follows: In P_1, we move all columns with more than p_1p_2 $*$s to the left and move all rows with more than $(n_1-q_1)q_2$ $*$s to the top. In P_2, we move all rows with more than q_1q_2 $*$s to the top and move all columns with more than $(m_1-p_1)p_2$ $*$s to the top. These row and column rearrangements are equivalent to left and right multiplying with invertible matrices $Q_1 \in M(m_1m_2,\mathbb{C})$ and $Q_2 \in M(n_1n_2,\mathbb{C})$, respectively. More precisely, let $P' = \begin{pmatrix} P'_0 & P'_1 \\ P'_2 & \mathbf{0} \end{pmatrix}$ be the symbolic matrix of $\mathcal{A}' = Q_1(\mathcal{A}_1 \otimes \mathcal{A}_2)Q_2$. Then P'_1 is the symbolic matrix of $\mathcal{A}'_1 = \mathcal{A}(p_1p_2,(n_1-q_1)q_2,p_1m_2,(n_1-q_1)n_2)$ and P'_2 is the symbolic matrix of $\mathcal{A}'_2 = \mathcal{A}((m_1-p_1)p_2,q_1q_2,(m_1-p_1)m_2,q_1n_2)$. Moreover, it is easy to verify that P'_0 is the symbolic matrix of $\mathcal{A}'_0 = \mathcal{A}(p_1p_2,q_1q_2,p_1m_2,q_1n_2)$.

Then we prove that $mrk(\mathcal{A}_1 \otimes \mathcal{A}_2) = mrk(\mathcal{A}'_1) + mrk(\mathcal{A}'_2)$. Firstly, we show that there exists $P' = \begin{pmatrix} P'_0 & P'_1 \\ P'_2 & \mathbf{0} \end{pmatrix} \in \mathcal{A}'$ satisfying $rank(P') = mrk(\mathcal{A}') = mrk(\mathcal{A}_1 \otimes \mathcal{A}_2)$, $rank(P'_1) = mrk(\mathcal{A}'_1)$ and $rank(P'_2) = mrk(\mathcal{A}'_2)$. Notice that there always exists $P'' \in \mathcal{A}'$ with $rank(P'') = mrk(\mathcal{A}')$, $R' = \begin{pmatrix} R'_0 & R'_1 \\ R'_2 & \mathbf{0} \end{pmatrix} \in \mathcal{A}'$ with $rank(R'_1) = mrk(\mathcal{A}'_1)$ and $R'' = \begin{pmatrix} R''_0 & R''_1 \\ R''_2 & \mathbf{0} \end{pmatrix} \in \mathcal{A}'$ with $rank(R''_2) = mrk(\mathcal{A}'_2)$. We claim that there exist $\alpha,\beta,\gamma \in \mathbb{C}$, such that $P' = \alpha P'' + \beta R' + \gamma R''$ is what we need. To see this, consider the matrix $xP'' + yR' + zR''$, where x,y,z are variables. As $rank(P'') = mrk(\mathcal{A}') = r$, there exists an $r \times r$ submatrix of P'' with rank r. Let f_1 be the determinant of the corresponding submatrix in $xP'' + yR' + zR''$. f_1 is a nonzero homogeneous polynomial in $\mathbb{C}[x,y,z]$ of degree r. Similarly, let $s = mrk(\mathcal{A}'_1)$ and $t = mrk(\mathcal{A}'_2)$. Then there exists an $s \times s$ (resp. $t \times t$) submatrix of $xP'' + yR' + zR''$ in the upper right (resp. lower left) part, such that, if we denote its determinant by f_2 (resp. f_3), then f_2 (resp. f_3) is a nonzero homogeneous polynomial in $\mathbb{C}[x,y,z]$ of degree s (resp. t). Since $f = f_1f_2f_3$ is a nonzero polynomial in $\mathbb{C}[x,y,z]$, there exists $(\alpha,\beta,\gamma) \in \mathbb{C}^3$ such that $f(\alpha,\beta,\gamma) \neq 0$. Such (α,β,γ) then translates to our desired conditions for $\alpha P'' + \beta R' + \gamma R''$.

Take such $P' = \begin{pmatrix} P'_0 & P'_1 \\ P'_2 & \mathbf{0} \end{pmatrix} \in \mathcal{A}'$. Since $p_1 + q_1 < \min\{m_1,n_1\}$ and $p_2 + q_2 < \min\{m_2,n_2\}$, we have $p_1p_2 < (n_1-q_1)(n_2-q_2)$ and $q_1q_2 < (m_1-p_1)(m_2-p_2)$. Then submatrix in the upper right part of P'_1 has full row rank p_1p_2, and the lower left part of P'_2 has full column rank q_1q_2. For any $P'_0 \in \mathcal{A}(p_1p_2,q_1q_2,p_1m_2,q_1n_2)$, we can use the upper right part of P'_1 to clear the first p_1p_2 rows of P'_0 without changing the rank of P'. Similarly, we can use the lower left part of P'_2 to clear the first q_1q_2 columns of P'_0 without changing the rank of P'. After these row and column operations, P' is transformed to $\begin{pmatrix} \mathbf{0} & P'_1 \\ P'_2 & \mathbf{0} \end{pmatrix}$. This then shows that

$$\begin{aligned} &mrk(\mathcal{A}_1 \otimes \mathcal{A}_2) = rank(P') = rank(P'_1) + rank(P'_2) \\ =&p_1p_2 + \min\{(n_1-q_1)q_2, p_1(m_2-p_2)\} + \min\{(m_1-p_1)p_2, q_1(n_2-q_2)\} + q_1q_2. \end{aligned} \tag{11}$$

□

Let us examine an example to illustrate the above procedure. Consider $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{A}(1,1,3,3) = \mathrm{span}\{|0\rangle\langle 0|, |0\rangle\langle 1|, |0\rangle\langle 2|, |1\rangle\langle 0|, |2\rangle\langle 0|\}$, where $\{|0\rangle, |1\rangle, |2\rangle\}$ is the computational basis of $\mathcal{H}_3$, and the linear span is taken over $\mathbb{C}$. It is easy to see that $mrk(\mathcal{A}(1,1,3,3)) = 2$. We show how to use lemma 17 to compute the maximal rank of $\mathcal{A}(1,1,3,3)^{\otimes 2}$. Firstly, we exchange rows and columns to obtain an equivalent matrix space $\mathcal{A}'$ of $\mathcal{A}(1,1,3,3)^{\otimes 2}$. This can be done by choosing $Q = |00\rangle\langle 00| + |01\rangle\langle 01| + |02\rangle\langle 02| + |10\rangle\langle 10| + |11\rangle\langle 20| + |12\rangle\langle 12| + |20\rangle\langle 11| + |21\rangle\langle 21| + |22\rangle\langle 22|$. Then $\mathcal{A}' = Q\mathcal{A}(1,1,3,3)^{\otimes 2}Q$ follows. More specifically, let P be the symbolic matrix in $\mathcal{A}(1,1,3,3)^{\otimes 2}$. Multiplying Q with P form the left exchanges the 5th row with the 7th row of P; multiplying Q with P from the right exchanges the 5th column with the 7th column. So letting

$P' = QPQ$, we have

$$P = \begin{pmatrix} * & * & * & * & * & * & * & * & * \\ * & 0 & 0 & * & 0 & 0 & * & 0 & 0 \\ * & 0 & 0 & * & 0 & 0 & * & 0 & 0 \\ * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow P' = \begin{pmatrix} * & * & * & * & * & * & * & * & * \\ * & 0 & 0 & * & * & 0 & 0 & 0 & 0 \\ * & 0 & 0 & * & * & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Denote P_0 to be its submatrix of size 3×3 in the upper left corner, P_1 to be its submatrix of size 3×6 in the upper right corner, P_2 to be its submatrix of size 6×3 in the lower left corner. It is easy to see that P_0 is the symbolic matrix of $\mathcal{A}(1, 1, 3, 3)$, P_1 is the symbolic matrix of $\mathcal{A}(1, 2, 3, 6)$, and P_2 is the symbolic matrix of $\mathcal{A}(2, 1, 6, 3)$. Applying Lemma 17, $mrk(\mathcal{A}(1, 1, 3, 3)^{\otimes 2}) = 1+2+2+1 = 6$, which can be verified easily by looking at the form of P'.

This example also shows that the matrix spaces formed by the anti-diagonal blocks of P', namely $\mathcal{A}(1, 2, 3, 6)$ and $\mathcal{A}(2, 1, 6, 3)$, may not form maximal-compression matrix spaces, as $1 + 2 = \min\{3, 6\}$. (Recall that for $\mathcal{A}(p, q, m, n)$ to be a maximal-compression matrix space we need $p + q < \min\{m, n\}$.) Thus lemma 17 cannot be directly applied to capture a general formula when taking tensor product multiple times. Fortunately, for the case that $m = n = d$, we can evaluate the maximal rank of the Nth tensor powers of a maximal-compression matrix space by the following:

Lemma 18 *Given a maximal-compression matrix space $\mathcal{A}(p, q, d)$ and an integer $N \geq 0$, the maximal rank of $\mathcal{A}(p, q, d)^{\otimes N+1}$ equals*

$$\sum_{k=0}^{N} \binom{N}{k} \Big(\min\{p^{N-k+1}(d-p)^k, q^k(d-q)^{N-k+1}\} + \min\{q^{k+1}(d-q)^{N-k}, p^{N-k}(d-p)^{k+1}\} \Big). \quad (12)$$

Proof The proof idea is as follows: First, we use induction to show that the symbolic matrix of $\mathcal{A}(p, q, d)^{\otimes N}$ is in upper-anti-block-diagonal form, after appropriate row and column rearrangements. Notice that all these block matrices either equals zero matrix, or form $\mathcal{A}(p, q, m, n)$s with different parameters. Second, we explicitly compute the maximal rank of those anti-diagonal $\mathcal{A}(p, q, m, n)$s. Combining these two observations, we use the similar techniques used in lemma 17 to prove this lemma.

The structure of matrices in $\mathcal{A}(p, q, d)^{\otimes N}$ can be shown by the following:

Observation 19 *For $N \geq 1$, there exist invertible matrices $Q_1 \in M(d^N, \mathbb{C})$ and $Q_2 \in M(d^N, \mathbb{C})$, such that the symbolic matrix P of $\mathcal{A}' = Q_1 \mathcal{A}(p, q, d)^{\otimes N} Q_2$ is of upper-anti-block-diagonal form:*

$$P = \begin{pmatrix} P_{0,2^{N-1}-1} & \cdots & P_{0,l} & \cdots & P_0 \\ \vdots & \iddots & \iddots & \iddots & \mathbf{0} \\ P_{l,2^{N-1}-1} & \iddots & P_l & \iddots & \mathbf{0} \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ P_{2^{N-1}-1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix}. \quad (13)$$

1. *For the anti-diagonal block matrices, label them as $P_0, \ldots, P_{2^{N-1}-1}$. For an integer $l = 0, \ldots, 2^{N-1}-1$, let $h(l)$ be the hamming weight of l, i.e. the number of $1'$s in the binary expansion of l. Then P_l is the symbolic matrix of*

$$\mathcal{A}_l = \mathcal{A}(p^{N-h(l)}(d-p)^{h(l)}, q^{h(l)+1}(d-q)^{N-h(l)-1}, p^{N-h(l)-1}(d-p)^{h(l)}d, q^{h(l)}(d-q)^{N-h(l)-1}d).$$

2. *For the upper-left block matrices, label them as $P_{u,v}$ for $u, v \in \{0, 2^{N-1}-1\}$, where u is the label of anti-diagonal block matrix P_u on the right of $P_{u,v}$ and v is the label of anti-diagonal block matrix P_v below $P_{u,v}$. If $h(u) \geq h(v)$, $P_{u,v} = \mathbf{0}$. Otherwise $P_{u,v}$ is the symbolic matrix of*

$$\mathcal{A}_{u,v} = \mathcal{A}(p^{N-h(u)}(d-p)^{h(u)}, q^{h(v)+1}(d-q)^{N-h(v)-1}, p^{N-h(u)-1}(d-p)^{h(u)}d, q^{h(v)}(d-q)^{N-h(v)-1}d).$$

Proof [Proof of Observation 19] We show the observation holds by induction on N. It holds for $N = 1$ trivially. Assume for $\mathcal{A}(p,q,d)^{\otimes N}$, observation 19 holds. We consider $\mathcal{A}(p,q,d)^{\otimes N+1} = \mathcal{A}(p,q,d)^{\otimes N} \otimes \mathcal{A}(p,q,d)$. Firstly, let $\mathcal{A}_1 = (Q_1 \otimes I_d)\mathcal{A}(p,q,d)^{\otimes N+1}(Q_2 \otimes I_d) = \mathcal{A}' \otimes \mathcal{A}(p,q,d)$, where Q_1 and Q_2 are the matrices from the inductive hypothesis. Let P_l be the symbolic matrix of the lth anti-diagonal block. It is sufficient to examine $P_l \otimes P$, where P is the symbolic matrix of $\mathcal{A}(p,q,d)$. By the proof of lemma 17, there exist two invertible matrices Q_l^1 and Q_l^2, such that $Q_l^1 P_l \otimes P Q_l^2 = \begin{pmatrix} P_0 & P_1 \\ P_2 & \mathbf{0} \end{pmatrix}$, where P_1 is the symbolic matrix of $\mathcal{A}_{l0} = \mathcal{A}(p^{N-h(l)+1}(d-p)^{h(l)}, q^{h(l)+1}(d-q)^{N-h(l)}, p^{N-h(l)}(d-p)^{h(l)}d, q^{h(l)}(d-q)^{N-h(l)}d)$, and P_2 is the symbolic matrix of $\mathcal{A}_{l1} = \mathcal{A}(p^{N-h(l)}(d-p)^{h(l)+1}, q^{h(l)+2}(d-q)^{N-h(l)-1}, p^{N-h(l)-1}(d-p)^{h(l)+1}d, q^{h(l)+1}(d-q)^{N-h(l)-1}d)$, where $l0$ and $l1$ denote the N-bit strings in which the first $(N-1)$-bit strings equal the binary expansion of l. Moreover, we observe that $\mathcal{A}_{l0}$ and $\mathcal{A}_{l1}$ remain as the anti-diagonal blocks in $\mathcal{A}_1' = \overline{Q_l^1}\mathcal{A}_1\overline{Q_l^2}$ ($\overline{Q_l^i}$ is the enlarged matrix of Q_l^i for $i = 1, 2$). Then the first fact in observation 19 follows since $h(l0) = h(l)$ and $h(l1) = h(l) + 1$.

For the second fact, for given $u, v \in \{0, \ldots, 2^{N-1}-1\}$, $u \neq v$ and $h(u) < h(v)$, we examine $\mathcal{A}_{u,v} \otimes \mathcal{A}(p,q,d)$. By induction hypothesis,

$$\mathcal{A}_{u,v} = \mathcal{A}(p^{N-h(u)}(d-p)^{h(u)}, q^{h(v)+1}(d-q)^{N-h(v)-1}, p^{N-h(u)-1}(d-p)^{h(u)}d, q^{h(v)}(d-q)^{N-h(v)-1}d).$$

Notice that $\mathcal{A}_{u,v}$ has the same "full" rows as that of $\mathcal{A}_u$ and has the same "full" columns as that of $\mathcal{A}_v$. Here a "full" row (resp. column) means the corresponding row (column) of the symbolic matrix contains $*$ only. Denote the row rearrangements of $\mathcal{A}_u \otimes \mathcal{A}(p,q,d)$ by R_u and the column rearrangements of $\mathcal{A}_v \otimes \mathcal{A}(p,q,d)$ by C_v. These two operations will also rearrange the rows and columns of the symbolic matrix of $\mathcal{A}_{u,v} \otimes \mathcal{A}(p,q,d)$, respectively. For simplicity, we assume $\mathcal{A}_u = \mathcal{A}(p_1, q_1, m_1, n_1)$ and $\mathcal{A}_v = (p_2, q_2, m_2, n_2)$, then $\mathcal{A}_{u,v} = \mathcal{A}(p_1, q_2, m_1, n_2)$. Let $P_{u,v}$ be the symbolic matrix of $\mathcal{A}_{u,v} \otimes \mathcal{A}(p,q,d)$, $P_{u,v}$ has the block matrix form

$$P_{u,v} = \begin{pmatrix} A_{1,1} & \cdots & A_{1,q_2} & A_{1,q_2+1} & \cdots & A_{1,n_2} \\ \vdots & P_0 & \vdots & \vdots & P_1 & \vdots \\ A_{p_1,1} & \cdots & A_{p_1,q_2} & A_{p_1,q_2+1} & \cdots & A_{p_1,n_2} \\ A_{p_1+1,1} & \cdots & A_{p_1+1,q_2} & & & \\ \vdots & P_2 & \vdots & & \mathbf{0} & \\ A_{m_1,1} & \cdots & A_{m_1,q_2} & & & \end{pmatrix}, \tag{14}$$

where $A_{i,j}$ are symbolic matrix of $\mathcal{A}(p,q,d)$ for all possible (i,j). Then R_u moves all rows with more than $(n_2 - q_2)q$ $*$s in P_1 and all rows with more than $q_2 q$ $*$s in P_2 to the top of them. To see this, notice that all rows with more than $(n_2 - q_2)q$ $*$s in P_1 is determined by those "full" rows in $\mathcal{A}_{u,v}$, which are exact those rows in $\mathcal{A}_u$; all those rows with more than $q_2 q$ $*$s in P_2 is determined by those "full" rows in $\mathcal{A}(p,q,d)$, and are also those rows in $\mathcal{A}_u$. Similarly, C_v moves all columns with more than $p_1 p$ $*$s in P_1 and all columns with more than $(m_1 - p_1)p$ $*$s in P_2 to the left. Let $P'_{u,v} = \begin{pmatrix} P'_0 & P'_1 \\ P'_2 & \mathbf{0} \end{pmatrix}$ denotes the symbolic matrix after the row and column rearrangement of $P_{u,v}$. We can then conclude that P'_0 is the symbolic matrix of $\mathcal{A}^*_0 = \mathcal{A}(p^{N-h(u)+1}(d-p)^{h(u)}, q^{h(v)+2}(d-q)^{N-h(v)-1}, p^{N-h(u)}(d-p)^{h(u)}d, q^{h(v)+1}(d-q)^{N-h(v)-1}d)$, P'_1 is the symbolic matrix of $\mathcal{A}^*_1 = \mathcal{A}(p^{N-h(u)+1}(d-p)^{h(u)}, q^{h(v)+1}(d-q)^{N-h(v)}, p^{N-h(u)}(d-p)^{h(u)}d, q^{h(v)}(d-q)^{N-h(v)}d)$ P'_2 is the symbolic matrix of $\mathcal{A}^*_2 = \mathcal{A}(p^{N-h(u)}(d-p)^{h(u)+1}, q^{h(v)+2}(d-q)^{N-h(v)-1}, p^{N-h(u)-1}(d-p)^{h(u)+1}d, q^{h(v)+1}(d-q)^{N-h(v)-1}d)$. Also, $\mathcal{A}^*_0$ will be relabeled as $\mathcal{A}_{u0,v1}$, $\mathcal{A}^*_1$ will be relabeled as $\mathcal{A}_{u0,v0}$ and $\mathcal{A}^*_2$ will be relabeled as $\mathcal{A}_{u1,v1}$ according to their corresponding anti-block terms.

To see the second statement in observation 19 holds for $N+1$, we only need to show that if $h(u) \geq h(v)$, $P_{u,v} = \mathbf{0}$ ($\mathcal{A}_{u,v} = \{\mathbf{0}\}$). For $u, v \in \{0, \dots, 2^N - 1\}$ and $u \neq v$, let $u = u'b$ and $v = v'c$, where $u', v' \in \{0, \dots, 2^N - 1\}$ equal the first $(N-1)$-bit strings of the binary expansion of u and v, and $b, c \in \{0,1\}$ are variables. If $h(u) \geq h(v)$ derives that either $h(u'0) \geq h(v'0)$, $h(u'0) \geq h(v'1) = h(v') + 1$ or $h(u'1) \geq h(v'1)$, it will imply that $h(u') \geq h(v')$. By the induction hypothesis, $P_{u',v'} = \mathbf{0}$ and $P_{u,v} = \mathbf{0}$ holds clearly. Otherwise, if $h(u) \geq h(v)$ derives $h(u'1) \geq h(v'0)$ and $P_{u',v'}$ is nonzero, we can also observe that $P_{u'1,v'0}$ equals $\mathbf{0}$, as it is the lower right part of P^*. This concludes the proof. □

Now we focus on those anti-diagonal forms, and compute the maximal rank of $\mathcal{A}_l$ for $l = 0, \dots, 2^{N-1} - 1$:

Observation 20 *Let $h(l) = k$, the maximal rank of $\mathcal{A}_l = \mathcal{A}(p^{N-k+1}(d-p)^k, q^{k+1}(d-q)^{N-k}, p^{N-k}(d-p)^k d, q^k(d-q)^{N-k}d)$ equals*

$$\min\{p^{N-k+1}(d-p)^k, q^k(d-q)^{N-k+1}\} + \min\{q^{k+1}(d-q)^{N-k}, p^{N-k}(d-p)^{k+1}\}. \tag{15}$$

Proof [Proof of observation 20] Notice that the rank of $\mathcal{A}(p^{N-k+1}(d-p)^k, q^{k+1}(d-q)^{N-k}, p^{N-k}(d-p)^k d, q^k(d-q)^{N-k}d)$ equals

$$\min\{p^{N-k+1}(d-p)^k + q^{k+1}(d-q)^{N-k}, p^{N-k}(d-p)^k d, q^k(d-q)^{N-k}d\}.$$

If $p^{N-k}(d-p)^k \leq q^k(d-q)^{N-k}$, we only need to compare $p^{N-k+1}(d-p)^k + q^{k+1}(d-q)^{N-k}$ and $p^{N-k}(d-p)^k d$. Note that

$$p^{N-k}(d-p)^k d - (p^{N-k+1}(d-p)^k + q^{k+1}(d-q)^{N-k}) = p^{N-k}(d-p)^{k+1} - q^{k+1}(d-q)^{N-k}.$$

We further distinguish two cases. When $p^{N-k}(d-p)^{k+1} \geq q^{k+1}(d-q)^{N-k}$, we take $p^{N-k+1}(d-p)^k + q^{k+1}(d-q)^{N-k}$. When $p^{N-k}(d-p)^{k+1} < q^{k+1}(d-q)^{N-k}$, we take $p^{N-k}(d-p)^k d = p^{N-k+1}(d-p)^k + p^{N-k}(d-p)^{k+1}$. These two cases then can be unified in the following equation

$$\begin{aligned} &\min\{p^{N-k+1}(d-p)^k + q^{k+1}(d-q)^{N-k}, p^{N-k}(d-p)^k d, q^k(d-q)^{N-k}d\} \\ =&p^{N-k+1}(d-p)^k + \min\{q^{k+1}(d-q)^{N-k}, p^{N-k}(d-p)^{k+1}\}. \end{aligned} \tag{16}$$

If $p^{N-k}(d-p)^k > q^k(d-q)^{N-k}$, similarly, we obtain

$$\begin{aligned} &\min\{p^{N-k+1}(d-p)^k + q^{k+1}(d-q)^{N-k}, p^{N-k}(d-p)^k d, q^k(d-q)^{N-k}d\} \\ =&\min\{p^{N-k+1}(d-p)^k, q^k(d-q)^{N-k+1}\} + q^{k+1}(d-q)^{N-k}. \end{aligned} \tag{17}$$

Notice that, $p^{N-k}(d-p)^k \leq q^k(d-q)^{N-k}$ implies $p^{N-k+1}(d-p)^k \leq pq^k(d-q)^{N-k} < q^k(d-q)^{N-k+1}$, where the second inequality uses $p+q<d$, since $\mathcal{A}(p,q,d)$ is maximal-compression. Similarly, $p^{N-k}(d-p)^k > q^k(d-q)^{N-k}$ implies $q^{k+1}(d-q)^{N-k} < qp^{N-k}(d-p)^k < p^{N-k}(d-p)^{k+1}$. This observation allows us to combine equation (16) and equation (17) to obtain a unified equation for the maximal rank of $\mathcal{A}(p^{N-k+1}(d-p)^k, q^{k+1}(d-q)^{N-k}, p^{N-k}(d-p)^k d, q^k(d-q)^{N-k}d)$ as

$$\min\{p^{N-k+1}(d-p)^k, q^k(d-q)^{N-k+1}\} + \min\{q^{k+1}(d-q)^{N-k}, p^{N-k}(d-p)^{k+1}\}. \tag{18}$$

□

Finally, we combine observations 19 and 20 to prove that $mrk(\mathcal{A}(p,q,d)^{\otimes N+1}) = \sum_{l=0}^{2^N-1} mrk(\mathcal{A}_l)$. Then equation 12 follows. Let $\mathcal{A}'$ be the matrix space which is obtained after applying row and column rearrangements described in observation 19 to $\mathcal{A}(p,q,d)^{\otimes N}$. Choose a matrix $P \in \mathcal{A}'$ of the form as shown in equation (13) with $rank(P) = mrk(\mathcal{A}')$, we can assume $rank(P_l) = mrk(\mathcal{A}_l)$ for $0 \leq l \leq 2^N - 1$, using an analogous argument as in lemma 17. Let $\lambda = \log_2 \frac{d-p}{q}$, $\mu = \log_2 \frac{d-q}{p}$, $\alpha = \frac{\mu}{\lambda+\mu}$. Notice that

$$k \leq \lfloor \alpha N + \alpha - 1 \rfloor \Leftrightarrow p^{N-k}(d-p)^{k+1} \leq q^{k+1}(d-q)^{N-k}$$

and

$$k \leq \lfloor \alpha N + \alpha \rfloor \Leftrightarrow p^{N-k+1}(d-p)^k \leq q^k(d-q)^{N-k+1}.$$

Let $N' = \lfloor \alpha N + \alpha \rfloor = \lfloor \alpha N + \alpha - 1 \rfloor + 1$. For any $l \in \{l : h(l) \leq N'-1\}$, P_l has full row rank. For any $l \in \{l : h(l) \geq N'+1\}$, P_l has full column rank. Now we claim that, for any upper-anti-block-diagonal matrices $P_{u,v}$, where $u \neq v$ and $u, v \in \{2^N - 1\}$, there exist row and column operations which convert P into the matrix which only has anti-diagonal blocks. By observation 19, we only need to consider those $P_{u,v}$ satisfying $h(u) < h(v)$. In this case, either $h(u) \leq N'-1$, or $h(v) \geq N'+1$. If $h(u) \leq N'-1$, we can use P_u to clear $P_{u,v}$, since P_u is anti-diagonal, on the right of $P_{u,v}$, and P_u has full row rank. The other case is similar. These yield that $mrk(\mathcal{A}(p,q,d)^{\otimes N+1}) = \sum_{l=0}^{2^N-1} mrk(\mathcal{A}_l)$, which, together with equation (18), allow us to conclude the proof. □

Now we are ready to compute the asymptotic maximal rank for maximal-compression matrix spaces. We restate theorem 15 (2) here:

Theorem 15. 2, restated. If $\mathcal{S} = \mathcal{A}(p,q,d)$ satisfying $p+q<d$, then

$$mrk^\infty(\mathcal{S}) = d \max\{2^{-D(1-\alpha||p')}, 2^{-D(\alpha||q')}\}, \tag{19}$$

where $p' = \frac{p}{d}$, $q' = \frac{q}{d}$, $\alpha = \frac{\log_2(d-q) - \log_2 p}{\log_2((d-p)(d-q)) - \log_2(pq)}$ and $D(a||b) := a\log_2 \frac{a}{b} + (1-a)\log_2 \frac{1-a}{1-b}$.

Proof: Let $\lambda = \log_2 \frac{d-p}{q}$, $\mu = \log_2 \frac{d-q}{p}$, $\alpha = \frac{\log_2(d-q) - \log_2 p}{\log_2((d-p)(d-q)) - \log_2(pq)} = \frac{\mu}{\lambda+\mu}$ and $N' = \lfloor \alpha N + \alpha \rfloor$ as discussed in lemma 18. We can rewrite equation (12) explicitly as the following:

$$\begin{aligned} mrk(\mathcal{A}(p,q,d)^{\otimes(N+1)}) &= \sum_{k=0}^{N'-1} \binom{N}{k} p^{N-k}(d-p)^k d + \sum_{k=N'+1}^{N} \binom{N}{k} q^k (d-q)^{N-k} d \\ &+ \binom{N}{N'} (p^{N-N'+1}(d-p)^{N'} + q^{N'+1}(d-q)^{N-N'}) \\ &= \sum_{k=0}^{N'} \binom{N}{k} p^{N-k}(d-p)^k d + \sum_{k=N'}^{N} \binom{N}{k} q^k (d-q)^{N-k} d \\ &- \binom{N}{N'} (p^{N-N'}(d-p)^{N'+1} + q^{N'}(d-q)^{N-N'+1}). \end{aligned} \tag{20}$$

Let $p' = \frac{p}{d}$ and $q' = \frac{q}{d}$, we have $p' + q' < 1$. The above quantity is upper and lower bounded by

$$mrk(\mathcal{A}(p,q,d)^{\otimes(N+1)}) \leq d^{N+1}(\sum_{k=0}^{N'} \binom{N}{k} {p'}^{N-k}(1-p')^k + \sum_{k=0}^{N-N'} \binom{N}{k} {q'}^{N-k}(1-q')^k); \tag{21}$$

$$mrk(\mathcal{A}(p,q,d)^{\otimes(N+1)}) \geq d^{N+1}(\sum_{k=0}^{N'-1} \binom{N}{k} {p'}^{N-k}(1-p')^k + \sum_{k=0}^{N-N'-1} \binom{N}{k} {q'}^{N-k}(1-q')^k). \tag{22}$$

We shall use the following inequalities:

Lemma 21 (Lemma 4.7.2 in Ref. [68]) *For $N' < Np$, we have:*

$$\frac{1}{\sqrt{2N}} 2^{-ND(\frac{N'}{N}||p)} \leq \sum_{k=0}^{N'} \binom{N}{k} p^k (1-p)^{N-k} \leq 2^{-ND(\frac{N'}{N}||p)}. \tag{23}$$

To apply lemma 21 to prove equation (19), we need $Nq' < N' < N(1-p')$ holds for sufficiently large N. We first prove the following:

Lemma 22 *Let p', q' and α be defined as above. $q' < \alpha < 1 - p'$.*

Proof By expressing α explicitly in terms of p' and q', we need to prove

$$q' < \frac{\log_2 \frac{1-q'}{p'}}{\log_2 \frac{1-q'}{p'} + \log_2 \frac{1-p'}{q'}}, \quad 1 - p' > \frac{\log_2 \frac{1-q'}{p'}}{\log_2 \frac{1-q'}{p'} + \log_2 \frac{1-p'}{q'}}. \tag{24}$$

This is equivalent to show

$$(1-q')^{1-q'} {q'}^{q'} > {p'}^{1-q'} (1-p')^{q'}, \quad (1-p')^{1-p'} {p'}^{p'} > (1-q')^{p'} {q'}^{1-p'}. \tag{25}$$

Consider the function $f(x,y) = x^y (1-x)^{1-y}$ with $x, y \in (0,1)$. The partial derivative in x is

$$\frac{\partial}{\partial x} f(x,y) = \frac{x^{y-1}(y-x)}{(1-x)^y}.$$

For any fixed y, $\max_{x \in (0,1)} f(x,y) = f(y,y)$. Then inequality (25) holds by choosing $x = 1-p', y = q'$ and $x = 1 - q', y = p'$. □

Recall $N' = \lfloor \alpha N + \alpha \rfloor$. To ensure that $Nq' < N' < N(1-p')$, it is sufficient to satisfy that $\alpha + \frac{\alpha}{N} < 1 - p'$ and $q' < \alpha - \frac{1-\alpha}{N}$. Since α, p', and q' are fixed, these can be achieved as long as $N > \max\{\frac{\alpha}{1-p'-\alpha}, \frac{1-\alpha}{\alpha - q'}\} > 0$.

Applying the upper bound in lemma 21 to inequality (21), we obtain

$$mrk(\mathcal{A}(p,q,d)^{\otimes(N+1)}) \leq d^{N+1}(2^{-ND(\frac{N'}{N}||1-p')} + 2^{-ND(1-\frac{N'}{N}||1-q')}). \tag{26}$$

Notice that $-D(a||p)$ is increasing for $0 < a < p$, and $\alpha(N+1) - 1 \leq \lfloor \alpha(N+1) \rfloor \leq \alpha(N+1)$. We can replace $\frac{N'}{N}$ by $\alpha + \frac{\alpha}{N}$, and $1 - \frac{N'}{N}$ by $1 - \alpha + \frac{1-\alpha}{N}$, which gives that

$$mrk(\mathcal{A}(p,q,d)^{\otimes(N+1)}) \leq d^{N+1}(2^{-ND(\alpha+\frac{\alpha}{N}||1-p')} + 2^{-ND(1-\alpha+\frac{1-\alpha}{N}||1-q')}). \tag{27}$$

Let N go to infinity. Since L^p norm converges L^∞ norm when $p \to +\infty$, we have

$$mrk^\infty(\mathcal{A}(p,q,d)) \leq d \max\{2^{-D(\alpha||1-p')}, 2^{-D(1-\alpha||1-q')}\}. \tag{28}$$

Similarly, applying the lower bound in lemma 21 to inequality (22), we obtain

$$\begin{aligned} mrk(\mathcal{A}(p,q,d)^{\otimes(N+1)}) &\geq \frac{d^{N+1}}{(N+1)^2}(2^{-ND(\frac{N'-1}{N}||1-p')} + 2^{-ND(1-\frac{N'-1}{N}||1-q')}) \\ &\geq \frac{d^{N+1}}{(N+1)^2}(2^{-ND(\alpha+\frac{\alpha-2}{N}||1-p')} + 2^{-ND(1-\alpha+\frac{1-\alpha}{N}||1-q')}). \end{aligned} \tag{29}$$

The second inequality holds since $\frac{N'-1}{N} \geq \alpha + \frac{\alpha-2}{N}$ and $1 - \frac{N'-1}{N} \geq 1 - \alpha + \frac{1-\alpha}{N}$. Thus we have

$$mrk^{\infty}(\mathcal{A}(p,q,d)) \geq d \max\{2^{-D(\alpha||1-p')}, 2^{-D(1-\alpha||1-q')}\}. \tag{30}$$

Since $D(a||b) = D(1-a||1-b)$, combining inequalities (28) and (30), we have

$$mrk^{\infty}(\mathcal{A}(p,q,d)) = d \max\{2^{-D(1-\alpha||p')}, 2^{-D(\alpha||q')}\}. \tag{31}$$

□

Theorem 15 provides explicit formulas for computing the asymptotic maximal Schmidt rank of those tripartite pure states whose corresponding matrix spaces have no shrunk subspace, or are maximal-compression. Taking one step further, we now consider whether the asymptotic maximal Schmidt rank of a tripartite state is full. The physical interpretation of this problem is, whether the maximally entangled state can be obtained from the given tripartite state by means of SLOCC asymptotically. In the following, we show that this problem is equivalent to the non-commutative rank problem (problem 10):

Theorem 23 *$|\Psi_{\mathrm{ABC}}\rangle \in \mathcal{H}_d^{\mathrm{A}} \otimes \mathcal{H}_d^{\mathrm{B}} \otimes \mathcal{H}_{d'}^{\mathrm{C}}$ can be transformed to the $d \otimes d$ maximally entangled state in $\mathcal{H}_d^{\mathrm{A}} \otimes \mathcal{H}_d^{\mathrm{B}}$ by means of SLOCC asymptotically, if and only if $M(\Psi_{\mathrm{ABC}})$ does not have shrunk subspace.*

Proof Let $\mathcal{S} = M(\Psi_{\mathrm{ABC}}) \leq M(d,\mathbb{C})$. It is equivalent to prove $mrk^{\infty}(\mathcal{S}) = d$ if and only if $\mathcal{S}$ has no shrunk subspace.

For the necessary part, we have shown $mrk^{\infty}(\mathcal{S}) = d$ if $\mathcal{S}$ has no shrunk subspace, by theorem 15.1.

For the sufficient part, we prove that, any $d \times d$ matrix space $\mathcal{S}$ which has shrunk subspaces admits $mrk^{\infty}(\mathcal{S}) < d$. Let $U \leq \mathbb{C}^d$ be a shrunk subspace of $\mathcal{S}$ satisfying $\dim(U) = d - q$ and $\dim(\mathcal{S}(U)) = p$. With some proper change of bases, $\mathcal{S}$ is a subspace of the maximal-compression matrix space $\mathcal{A}(p,q,d)$. By theorem 15.2 and lemma 22, we have

$$mrk^{\infty}(\mathcal{A}(p,q,d)) = d \max\{2^{-D(1-\alpha||p')}, 2^{-D(\alpha||q')}\},$$

and $q' < \alpha < 1 - p'$. We can derive $mrk^{\infty}(\mathcal{A}(p,q,d)) < d$, which leads to $mrk^{\infty}(\mathcal{S}) < d$. □

In addition, by theorem 11, there exist deterministic polynomial-time algorithms for determining whether a given matrix space has shrunk subspaces or not. Thus, determining whether the asymptotic maximal Schmidt rank of a tripartite state is full is algorithmically effective, i.e.

Corollary 24 *Given a tripartite state $|\Psi_{\mathrm{ABC}}\rangle \in \mathcal{H}_d^{\mathrm{A}} \otimes \mathcal{H}_d^{\mathrm{B}} \otimes \mathcal{H}_{d'}^{\mathrm{C}}$, there exist deterministic polynomial-time algorithms to determine whether $|\Psi_{\mathrm{ABC}}\rangle$ can be transformed to the $d \otimes d$ maximally entangled state by means of SLOCC with rate 1, in an asymptotic setting.*

V. CONCLUSIONS

In this paper, we exhibit novel results on tripartite-to-bipartite SLOCC transformations, in multi-copy and asymptotic settings. We first construct a tripartite pure state $|\Psi^d_{\mathrm{ABC}}\rangle$ satisfying $msrk(|\Psi^d_{\mathrm{ABC}}\rangle^{\otimes 2}) > msrk(|\Psi^d_{\mathrm{ABC}}\rangle)^2$, which implies that the maximal Schmidt rank is strictly super-multiplicative. It also illustrates that, although one copy of $|\Psi^d_{\mathrm{ABC}}\rangle$ cannot be transformed to the bipartite maximally entangled state by SLOCC, one can do so with two copies. We then provide a full characterization for those tripartite states whose maximal Schmidt rank increase on average under tensor product, by considering the structure of their corresponding matrix spaces. In fact, these tripartite states can be viewed as having advantages in tripartite-to-bipartite SLOCC transformations with multiple copies. Interestingly, except for the degenerated case, this phenomenon holds for all tripartite states of which their maximal Schmidt ranks are not full.

In the asymptotic setting, we consider evaluating the tripartite-to-bipartite entanglement transformation rate of a given tripartite pure state and bipartite pure state. Notably, it equals the logarithm of the asymptotic maximal Schmidt rank of the given tripartite state (where the base of the logarithm is the Schmidt rank of the given bipartite state). Nevertheless, the latter is in general difficult to compute due to its super-multiplicativity. To get around this difficulty, we apply certain results related to the structure of matrix space, including the study of matrix semi-invariants, to obtain explicit formulas which compute the asymptotic maximal Schmidt ranks for a large family of tripartite states. Furthermore, we investigate the asymptotic convertibility to the bipartite maximally entangled state, and show its equivalence to the non-commutative rank problem, introduced in Ref. [47]. Based on the recent progress on this problem [38, 39], there exist deterministic polynomial-time algorithms to decide whether a tripartite state can be transformed to the maximally entangled state by SLOCC, asymptotically.

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