Application of polynomial chaos expansions to analytical models of friction oscillators

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ABSTRACT

Despite past substantial research efforts, the prediction of brake squeal propensity remains a largely unresolved problem. The standard practice to predict the brake squeal propensity is to analyse dynamic instabilities using the complex eigenvalue analysis. However, it is well known that not every predicted unstable vibration mode will lead to squeal and vice-versa. Owing to nonlinearity and problem complexity (e.g. operating conditions), treating brake squeal with uncertainty seems appealing. Another indicator of brake squeal propensity, not often used, is based on negative dissipated energy. In this study, uncertainty analysis induced by polynomial chaos expansions is examined for 1-dof and 4-dof friction models. Results are compared with dissipated energy calculations and standard complex eigenvalue analysis. The potential of this approach for the prediction of brake squeal propensity is discussed.

1 INTRODUCTION

Brake squeal originates from friction-induced vibrations with complex underlying relationships and poses a major concern to automotive industries owing to customer dissatisfaction and related warranty claims (Kinkaid et al., 2013). Mechanisms known to cause brake squeal are stick-slip, sprag-slip, instantaneous modes, negative gradient of the friction coefficient with respect to relative velocity, gyroscopic effects and damping itself, mode coupling, surface waves, moving loads and parametric resonances (Kinkaid et al., 2003, Chen, 2009, Oberst, 2011) Problem complexity poses serious modelling issues especially when it comes to contact and friction laws (Butlin and Woodhouse, 2009). Geometrical modifications of pads have been shown to be highly related to the degree of nonlinearity of a brake system (Oberst and Lai, 2011a) and microphone test data of a squealing brake system shows that deterministic chaos is one route to instability in disc brake squeal (Oberst and Lai, 2011b). Simulation tools for predicting brake squeal based only on structural vibrations are available as frequency and time domain methods (Ouyang., 2005). The complex eigenvalue analysis (CEA) in the frequency domain is the most popular method implemented in commercial software packages owing to its ease of application and interpretation. The calculation of acoustic radiation with friction contact between the pad and the rotor is rarely tackled. However, guidelines on how to perform proper numerical analysis in the frequency domain to calculate structural vibrations and acoustic radiation of simple models of brake systems are given by (Oberst et al., 2013) for the first time.

While the CEA predicts vibration modes stability which are interpreted as potential squeal frequencies, neither the sign nor the magnitude of the real part of the complex eigenvalue are indications that audible squeal would occur. Another method is to examine the energy values such as the kinetic energy or dissipated energy (Hoffmann and Gaul, 2004, Oberst, 2011). Negative dissipated energy by friction can be interpreted as energy provided to the system by friction such as feed-in energy or viscous work (Guan and Huang, 2003, Pappiniemi, 2007, Oberst, 2011). Negative dissipated energy may therefore complement the CEA not to detect vibration instabilities but to focus on prediction of increased squeal propensity.

One major problem in the prediction of brake squeal propensity is that the instability of vibration modes predicted by the CEA have been shown not to be a necessary condition and the CEA may only be capable to predict the onset of squeal of some types of instabilities (Shin et al., 2002, Massi et al., 2007, Sinou, 2010). Due to the problems complexity, the transient character of squeal and nonlinearities involved (Oberst, 2011), the incorporation of uncertainty into brake squeal propensity prediction should be considered (Hoffmann and Gaul, 2008, Oberst and Lai, 2010, Oberst 2011).

A comprehensive review on numerous methods for incorporating uncertainty into modelling has been presented by Soize, (2013). One conventional way of incorporating uncertainties involves establishing governing equations with random components, then solving them using the Monte Carlo method. The Monte Carlo method features sampling from populations and usually a large number of samples is required to achieve convergence. The use of “Polynomial Chaos expansions” (PC) is a non-sampling method for faster calculations of a stochastic problem. It was firstly introduced by Wiener (1938) and has been shown by Cameron and Martin (1947) to converge in a L^2 sense (quadratically) for any second-order stochastic process, compared to $1/\sqrt{n}$ for Monte Carlo methods (Rubinstein and Kroese, 2007) which renders the application of the polynomial chaos method suitable for finite element simulations. The polynomial chaos has been recently applied to investigate how uncertainties affect the stability of brake models both analytically (Nechack et al., 2011) and numerically (Sarrouy et al., 2013). However, it has yet not being reported how uncertainties influence the squeal propensity of a model from the energy point of view as a brake system might be stable but still get a higher squeal propensity.

The objective of this study is to examine the use of the uncertainty analysis on CEA and friction work calculations (e.g., the work done by the friction force) in predicting unstable vibration modes and squeal propensity in a linear system. For this purpose, polynomial chaos expansions (Galerkin and the collocation method benchmarked against the Monte Carlo simulation and deterministic (Det) results) are incorporated into the equations of motion of a sinusoidally forced one degree of freedom (DOF) and a 4-DOF friction oscillator. The friction coefficient and viscous damping are treated as random variables. Numerical validity is established and instabilities are quantified by comparing the probabilities and median values of real part of the complex eigenvalues and friction work as squeal prediction and indicator respectively over stiffness as a bifurcation parameter.

2 ANALYTICAL MODELS AND NUMERICAL METHODS

The analytical models of a sinusoidally forced 1-DOF and a 4-DOF friction oscillator with random friction and viscous damping coefficients are described. The Monte Carlo method and the use of polynomial chaos expansions using the Galerkin and the collocation method are introduced to the analytical models

2.1 The sinusoidally forced 1-DOF friction oscillator

The model of the sinusoidally forced 1-DOF friction oscillator proposed by Hinrichs et al. (1998) is shown in Figure 1(a).

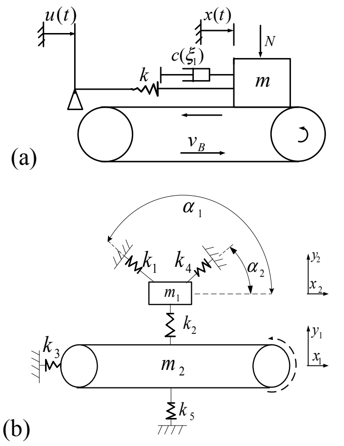


Figure 1. Friction oscillators (a) Hinrichs et al. (1998); (b) Papinniemi (2007)

A slider connected with a spring and a dashpot is pushed by an external normal force against a belt. The friction in the interface of the slider and the belt is a Coulomb-Amonton type. Only the motion of the slider in the *x*-direction is considered without the slider being able to lift off from the belt (Oberst et al., 2013). The equation of motion for the slider is:

$$m\ddot{x}+c(\xi_1)\dot{x}+kx=-Nf(\xi_2)\operatorname{sgn}(v_B-\dot{x})+ku_0\sin\omega t \quad (1)$$

Here, m is the mass, x is the displacement of the slider, $dx/dt=(\cdot)$ denotes the derivative with respect to time, c is the viscous damping coefficient, k is the spring's stiffness, F is an external normal force acting on the slider, f is the friction coefficient, sgn is the signum function, v_B is the velocity of the belt, u_0 is the amplitude of the displacement $u(t)$, ω is the circular frequency of the excitation and t is the time. Let ξ_1 and ξ_2 be two i.i.d. (independent and identically distributed random variables), then uncertainty associated with the viscous damping and the friction coefficient can be denoted as $c(\xi_1)$ and $f(\xi_2)$ respectively.

However, it is difficult to obtain a closed form solution for Equation (1) with a signum function (discontinuity) (von Wagner, 2004). The value of the signum function would be a constant if the velocity of the belt is fast enough (Nechak et al., 2012), so that the direction of the relative velocity between the slider and the belt would not change and a 'steady-sliding condition' is established (Bajer et al., 2004). The approximate threshold velocity of the belt has been estimated numerically as given in the Appendix so that equation (1) can be reduced as:

$$m\ddot{x}+c(\xi_1)\dot{x}+kx=-Nf(\xi_2)+ku_0\sin\omega t \quad (1b)$$

2.2 The 4-DOF friction oscillator

The 1-DOF system in section 2.1 serves to illustrate how the uncertainty would influence dynamics of the slider only in the *x*-direction. The 4-DOF system proposed by (Papinniemi, 2007) shown in Figure 1b may be used to investigate how uncertainties influence the interplay between different DOFs. The equation of free vibration for the 4 DOF system can be written as (Papinniemi, 2007):

$$\mathbf{M}\ddot{\mathbf{X}}+\mathbf{K}\mathbf{X}=\mathbf{0} \text{ with } \mathbf{X}=[x_1,y_1,x_2,y_2]^{\mathrm{T}} \quad (2)$$

2.3 Numerical Methods

2.3.1 Monte Carlo method

The Monte Carlo method (MC) relies on repeated sampling from the population to achieve a reasonable accuracy. Based on the central limit theorem (Casella and Berger, 1990), the average of a large number of independent samples tends to converge to the mathematical expectation of the population. Taking the estimation of the mean of the real part of the complex eigenvalues as an example, the procedure of MC simulation involves (Rubinstein and Kroese, 2007):

1. Generate random damping and friction coefficient from the population $c(\xi_1)$ and $f(\xi_2)$, respectively.
2. Perform computations to extract eigenvalues using Equation (2) with random damping and friction coefficients.
3. Calculate the statistical properties of the results and treat them as the estimators for statistical population descriptors.

2.3.2 Polynomial chaos

Any second order stochastic process can be approximated by the sum of a series of expansions of polynomials. Let $X\{t;\xi\}$ be a stochastic process. This process may then be expanded by the polynomial chaos expansions as done by (Sepahvand et al., 2012):

$$X\{t;\xi\}=\sum_{i=0}^{\infty}x_i(t)\Phi_i(\xi) \quad (3)$$

where ξ is a vector of an independent random variable with a known probability density function $p(\xi)$. The expansion coefficients are referred to as $x_i(t)$. Φ_i is a family of orthogonal polynomials with the following relationship:

$$\langle\Phi_i,\Phi_j\rangle=\int_\Omega\Phi_i(\xi)\Phi_j(\xi)p(\xi)d\xi=\langle\Phi_i^2\rangle\delta_{ij} \quad i,j=0,1,\ldots,\infty \quad (4)$$

where δ_{ij} is the Kronecker delta, $\langle\cdot,\cdot\rangle$ is the ensemble average and Ω is the compact support of the probability space defined by ξ. Since $p(\xi)$ and Φ_i are known and chosen by the user, the key step of using polynomial chaos expansions, is therefore the calculation of the expansion coefficients $x_i(t)$. Basically, there are two numerical methods for calculating the expansion coefficients which are namely the Garlerkin method and the collocation method. We take the 1-DOF friction oscillator model (Figure 1(a)) as an example to illustrate the application of the two numerical methods. The damping and friction coefficients in the one DOF system are assumed to be distributed normally by using the linear transformations:

$$c(\xi_1) = \mu_c + \sigma_c \xi_1 \tag{5}$$

$$f(\xi_2) = \mu_f + \sigma_f \xi_2 \tag{6}$$

where ξ_1 and ξ_2 denote two i.i.d. standard normal random variables, and μ denote the distribution means, and σ denote the standard deviations for $c(\xi_1)$ and $f(\xi_2)$ respectively.

With equations (5) and (6), the response of the 1-DOF system becomes stochastic and can be expanded using the polynomial chaos expansions as described by equation (3). Practically, the order of the polynomial chaos has to be truncated to a finite order *N;* the trade-off is slightly reduced accuracy depending on of which order is *X.*

$$X(t,\xi_1,\xi_2) = \sum_{i=0}^{N} x_i(t) H_i(\xi_1,\xi_2) \tag{7}$$

H_i represents the i^{th} Hermite polynomial (probabilistic type) with *N* being the truncation order. Substituting equations (5), (6), (7) into (1b) yields the equation of motion considering uncertainties.

$$\begin{aligned} & m\sum_{i=0}^{N}\ddot{x}_i(t)H_i(\xi_1,\xi_2)+(c_0+c_1\xi_1)\sum_{i=0}^{N}\dot{x}_i(t)H_i(\xi_1,\xi_2)+ \\ & +k\sum_{i=0}^{N}x_i(t)H_i(\xi_1,\xi_2)=-F(\mu_0+\mu_1\xi_2)+ku_0\sin\omega t \end{aligned} \tag{8}$$

2.3.2.1 Galerkin method

The Galerkin method is based on the orthogonality of polynomials (Sepahvand et al., 2012) and represents a mathematical rigorous way of solving stochastic differential equations. By multiplying both sides of equation (8) with H_i and calculating the ensemble average by making use of the orthogonality in equation (4), one can solve the following equation with using an appropriate solver for differential equations (e.g. ODE45 in Matlab).

$$\mathbf{M}\ddot{\mathbf{X}}(t,\xi_1,\xi_2)+\mathbf{C}\dot{\mathbf{X}}(t,\xi_1,\xi_2)+\mathbf{K}\mathbf{X}(t,\xi_1,\xi_2)=\mathbf{F}_1+\mathbf{F}_2\sin\omega t \tag{9}$$

where **M**, **C**, **K**, $\mathbf{F_1}$ and $\mathbf{F_2}$ are the resultant matrices; the solution of equation (9) represent the expansion coefficients(c.f. Equation (8), more details of the resultant matrices are in the Appendix).

2.3.2.2 Collocation method

The coefficients of the expansion equation (7) could also be determined as follows:

$$x_i(t) = \frac{\langle X(t,\xi_1,\xi_2), H_i(\xi_1,\xi_2)\rangle}{\langle H_i(\xi_1,\xi_2), H_i(\xi_1,\xi_2)\rangle}, \quad i = 1,2,\ldots,n \tag{10}$$

Equation (10) has *n* unknowns $x_i(t)$ to be determined. Therefore, it needs *n* samples from the population to generate *n=m* realizations of $X(t,\xi_1,\xi_2)$ as the system would be otherwise for n<m underdetermined. Then the expansion coefficients are estimated by using a least square method (Tatang et al., 1997). These samples are termed the collocation points. It should be noted that the accuracy of $x_i(t)$ largely depends on the quantity and quality of the samples. It has been suggested that the number of samples should be at least twice the number of unknowns in $x_i(t)$ and the roots of the chosen polynomials should be the collocation points (Huang et al., 2007).

3 NUMERICAL RESULTS AND DISCUSSION

The Monte Carlo method (MC) and the polynomial chaos expansions (PC) are applied to solve the response of the analytical oscillators respectively. The following numerical values are applied to the 1-DOF system: *m*=1 kg, *k*=2 Nm^{-1}, *F*=10 N, x_0=0.2 m and to the 4-DOF system: m_1=0.2 kg, m_2=1 kg, $k_{i=2,\ldots,5}$=1 Nm^{-1}, α_1 =150^0, α_2=30^0. The viscous damping in the 1-DOF friction oscillator and the friction coefficients in both oscillators are assumed to be normally i.i.d. with $c(\xi_1) \sim N(\mu,\sigma^2) = N(0.5, 0.01)$, $f(\xi_2) \sim N(0.4, 0.01)$

3.1 Collocation methods

The collocation method is more convenient to use than the Galerkin method because it does not rely on orthogonality. It is also faster than the MC method (L^2 convergence) as the number of repeated computations can be reduced owing to the of choice of the collocation points, which is at random for the MC method The collocation method is therefore more suitable for large-scale stochastic finite element problems than the Galerkin method (Huang et al., 2007).

To assess the performance of the collocation method, the coefficients of expansions calculated by the collocation method are compared to the Galerkin method (Sepahvand et al., 2012). Figure 2 depicts the comparison of the coefficients of the first four expansion coefficients of the collocation method (Equation (10)) with the results of the Galerkin method (Equation (9)) and shows that all modes agree reasonably well after the steady state is reached.

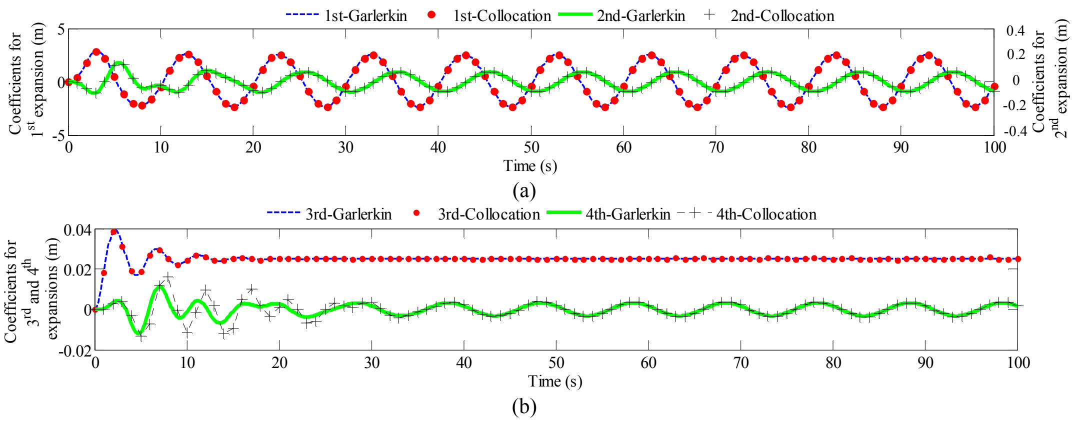


Figure 2. Coefficients of the first four expansions for the displacement of silder in 1-DOF (PC order = 4)

3.2 Accuracy of polynomial chaos expansions

By substituting the expansion coefficients into Equation (7) a stochastic response is synthesised and statistical moments (mean or variance) can be obtained. It is expected that the first and the second statistical moments converge to those calculated by the MC method. Also, the deterministic result (i.e. the values of damping and friction coefficient remain constant with c=0.5 Nm/s, f=0.4) and the mean value of responses are shown in Figure 3 to be identical to the MC and PC.

Figure 4 indicates the calculated variance applying the Galerkin and the collocation method is close to the variance of the Monte Carlo method once the steady state is reached. Comparing the performance of the three numerical methods depicted in Figures 3 and 4, it is revealed that the variance differs more than the mean. These discrepancies would probably be more prominent when of statistical moments higher than order two are compared (Cameron, 1947).

3.3 Real parts of the complex eigenvalues

The displacements of the 1-DOF and 4-DOF systems for free vibration may be expressed as:

$$\mathrm{x} = \mathbf{v}e^{(\lambda + i\omega)t} \tag{11}$$

v is the complex eigenvector, i is the imaginary number $\sqrt{-1}$, λ and ω are the real and imaginary part of a complex eigenvalue respectively. A CEA which gives a positive real part (divergence of trajectory) indicates a system's status of stability. As a positive real part would only be caused here by an asymmetric stiffness matrix, the 1-DOF system without sign function (switching nonlinearity) would always be stable; therefore only the non-viscously damped 4-DOF friction oscillator is discussed.

3.4 Stochastic friction work

Friction between two contacting objects may dissipate energy (damping effect) but also could generate energy; if enough energy is produced and damping is overcome, the system will be driven into instability. So fed back energy increases the propensity of a system to squeal. This energy is generated by the combined effect of the friction and the phase difference between the relative displacement and the relative velocity of contacting objects (Guan and Huang, 2003, Papinniemi, 2007).

The 4-DOF friction oscillator (no viscous damping, see Figure 1(b)) is a good example to illustrate the relationship between the work due to friction and instabilities in a linear dynamic system. The relative velocity and relative displacement of m_1 and m_2 in the x and y directions can be expressed respectively as

$$\dot{X}\{t;\xi_2\} = \dot{x}_1\{t;\xi_2\} - \dot{x}_2\{t;\xi_2\} = A_x(\xi_2)\omega(\xi_2)\cos(\omega(\xi_2)t + \theta_x(\xi_2)) \tag{12}$$

$$Y\{t;\xi_2\} = y_1\{t;\xi_2\} - y_2\{t;\xi_2\} = A_y(\xi_2)\sin(\omega(\xi_2)t + \theta_y(\xi_2)) \tag{13}$$

The work done by friction in an arbitrary time interval $[t_1, t_2]$ is can be calculated according to Papinniemi, (2007):

$$\begin{aligned} W\{t,\xi_2\} &= \int_{t_1}^{t_2} k_2 f(\xi_2) Y(t,\xi_2)\dot{X}(t,\xi_2)dt = \frac{1}{4}k_2 f(\xi_2)A_x(\xi_2)A_y(\xi_2) \\ &\big[\cos(2\omega(\xi_2)t_1 + \theta_x(\xi_2) + \theta_y(\xi_2)) - \cos(2\omega(\xi_2)t_2 + \theta_x(\xi_2) + \theta_y(\xi_2)) \\ &+ 2\omega(\xi_2)\sin(\theta_y(\xi_2) - \theta_x(\xi_2))(t_2 - t_1)\big] \end{aligned} \tag{14}$$

where $A_x(\xi_2)$, $A_y(\xi_2)$ and $\theta_x(\xi_1)$, $\theta_y(\xi_1)$ are stochastic amplitudes and stochastic phases of the relative velocity (which points always in the same direction) and the displacement of the slider mass m_1 and the belt mass m_2 respectively. Let t_1, t_2 be $(n-1)T$ and nT, respectively, where T denotes period. By substituting $(n\text{-}1)T$ and nT into equation (14), the friction work in the n^{th} cycle is given by:

$$W^N(\xi_2) = k_2 f(\xi_2)A_x(\xi_2)A_y(\xi_2)\pi\sin(\theta_y(\xi_2) - \theta_x(\xi_2)) \tag{15}$$

Equation (15) shows that (i) the sign of the friction work is phase dependent, (ii) friction may cause the value calculated by the sin function to be non-zero; having no friction results in arguments of the sine function of either 0^0 or 180^0.

To assess the ability of the friction work for detecting higher squeal propensity, the median of the stochastic real part of the complex eigenvalue and the friction work of the 1st and 2nd mode are compared with their deterministic values as shown in Figure 5 for various values of the stiffness constant k_1. The results of the 3$^{\text{rd}}$ and 4$^{\text{th}}$ mode are zero and not presented here. The results for the PC agree very well with those of the MC method, the deterministic values for the real part of the complex eigenvalue and the friction work (Figure 5). In this 4-DOF linear oscillator, only the 1st mode is predicted to have a positive friction work (Figure 5b) and a positive λ (Figure 5a) for values of k_1 between 1.29 and 2.41 N/m.

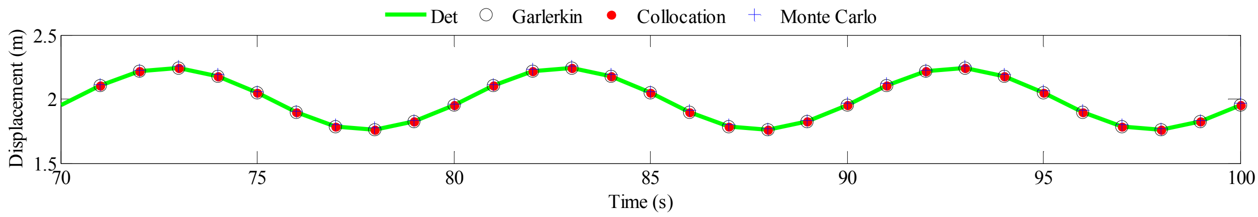


Figure 3. Deterministic displacement and mean of the stochastic displacement for the slider in the 1-DOF

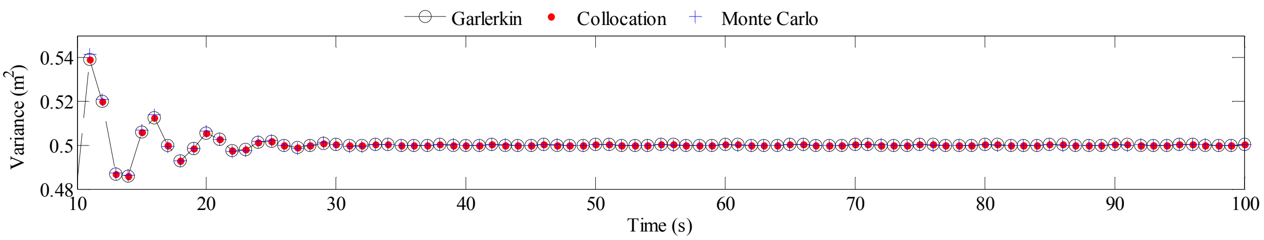
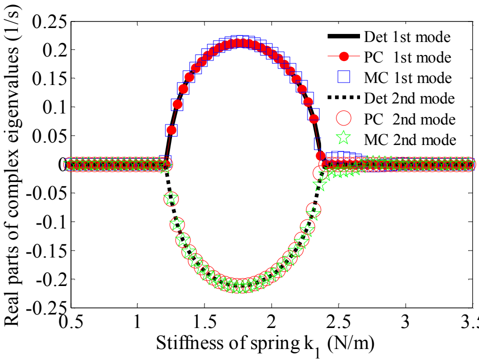
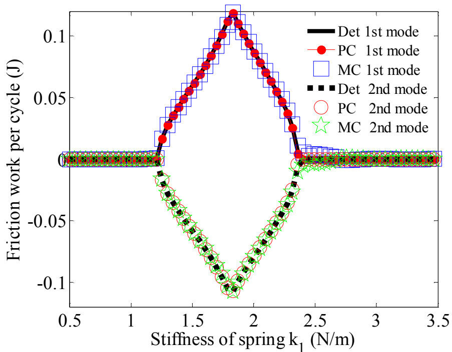


Figure 4. Variance of the stochastic displacement for the slider in the 1-DOF



(a) Real part of complex eigenvalue of 1st and 2nd mode



(b) Friction work per cycle of 1st and 2nd mode

Figure 5. Deterministic and median of λ and W^N for 4-DOF (MC samples: 1E4, PC order =14)

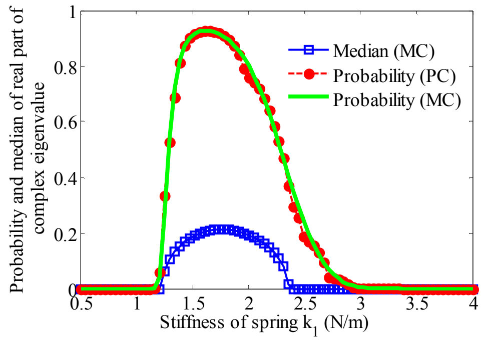
A histogram and an empirical cumulative distribution function (CDF) of the friction work when k_1 equals 2.2 N/m is given in the Appendix to show how the probability is estimated. The relationship between the probabilities of generating a positive real part of the complex eigenvalue and positive friction work per cycle with k_1 for the 1st mode is shown in Figure 6a and 6b respectively. The probability of positive real part and positive friction work for values of k_1 between 1.16 N/m and 3.12 N/m is above 0.16%. The probability of positive real part of CEA and positive friction work corresponding to the other three modes are all zero and not presented here for brevity.

4 CONCLUSIONS

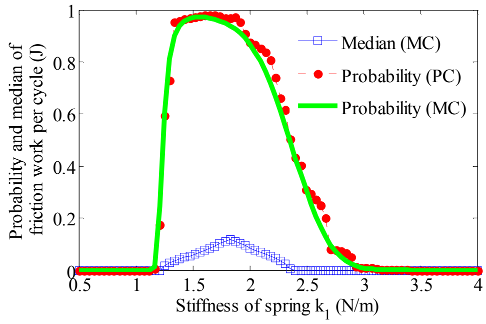
An uncertainty analysis incorporating the method of polynomial chaos expansions has been applied to compute the complex eigenvalues and the stochastic friction work of two linear friction oscillators to illustrate its potential in predicting unstable vibration modes and squeal propensity with potential applications to automotive brake systems. .

To avoid the complexity due to a discontinuous nonlinearity (signum function), the condition of steady sliding is numerically estimated. Based on the 1-DOF oscillator, the collocation method for polynomial chaos expansions has been shown to be a robust alternative compared to the Galerkin method, both showing good agreement with the Monte Carlo simulation. Following the stochastic complex eigenvalue and friction work were calculated using the polynomial chaos expansions and the collocation method. The feasibility of using the friction work to detect higher squeal propensity is explored and two main conclusions can be drawn

(1) Compared with the complex eigenvalue analysis, the friction work is equally capable of estimating instabilities in the linear undamped 4-DOF system.



(a)



(b)

Figure 6. Probability and Median of positive λ and W^N of 1st mode for 4-DOF (MC samples: 1E4, PC order =14)

(2) Uncertainty could enlarge the instability region. The deterministic analysis shows no instabilities if the stiffness k_1 is greater than or equal to 2.5 N/m, while both the stochastic complex eigenvalue and friction work shows that it still has 30% probability of generating instability / higher squeal propensity at k_1 = 2.5 N/m.

As brake squeal is essentially a nonlinear problem, future work should be focussed on the analysis of combining nonlinearities with uncertainties.

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APPENDIX

(a) The matrices and vectors in equation (9) can be obtained by using orthogonality of Hermite polynomials.

$$\mathrm{M}=m\times\mathrm{diag}[\iint_{\Omega}H_0(\xi)W(\xi)d\xi,\iint_{\Omega}H_1(\xi)W(\xi)d\xi, \cdots\iint_{\Omega}H_N(\xi)W(\xi)d\xi] \tag{16}$$

$$\mathrm{C}=\mathrm{diag}[\iint_{\Omega}(c_0+c_1\xi_1)H_0(\xi)W(\xi)d\xi,\iint_{\Omega}(c_0+c_1\xi_1)H_1(\xi)W(\xi)d\xi, \cdots\iint_{\Omega}(c_0+c_1\xi_1)H_N(\xi)W(\xi)d\xi] \tag{17}$$

$$\mathrm{K}=\mathrm{k}\times\mathrm{diag}[\iint_{\Omega}H_0(\xi)W(\xi)d\xi,\cdots,\iint_{\Omega}H_N(\xi)W(\xi)d\xi] \tag{18}$$

$$\mathrm{F}_1=[\iint_{\Omega}-F(\mu_0+\mu_1\xi_2)H_0(\xi)W(\xi)d\xi,\cdots \iint_{\Omega}-F(\mu_0+\mu_1\xi_2)H_N(\xi)W(\xi)d\xi]^{\mathrm{T}} \tag{19}$$

$$\mathrm{F}_2=\mathrm{ku}_0\times\mathrm{diag}[\iint_{\Omega}H_0(\xi)W(\xi)d\xi,\cdots \iint_{\Omega}H_N(\xi)W(\xi)d\xi] \tag{20}$$

b) The solution of equation (1) should be identical to that of equation (21) if the velocity of the belt exceeds a threshold speed. As shown in Figure 7, 0.16 m/s appears to be the threshold speed of the belt. Since the velocity of the belt is equal to 0.16 m/s, no stick-slip motion occurs anymore, the solution of equation (1) is closely matched by that of equation (21).

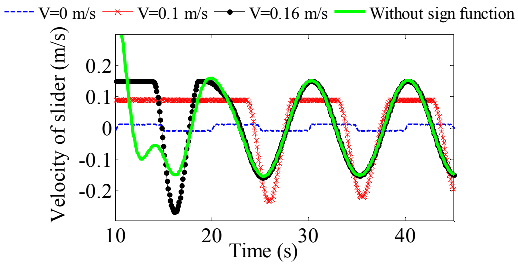
$$m\ddot{x}+c(\xi_1)\dot{x}+kx=-Nf(\xi_2)+ku_0\sin\omega t \tag{21}$$

The threshold speed for the 4-DOF oscillator can be estimated in the same manner and the result is presented in Figure7 (b).

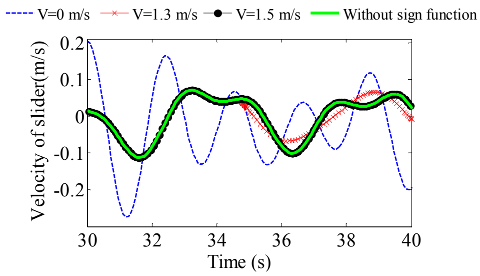
(c) 10,000 samples and 50 bins were used to draw the histogram in Figure 8 (a). The numerical value for generating Figure 8(a) is exactly as presented in Section 3. The probability of generating positive friction work (W^N) can be estimated by the CDF as:

$$\begin{aligned} P(W^N > 0) &= 1 - P(W^N < 0) \\ &= 1 - F(W^N = 0) = 1 - 0.2891 = 0.7109 \end{aligned} \qquad (22)$$

The probability of the positive real part of the complex eigenvalue is calculated in the same manner. It is shown in Figure 8(a) that the zero friction work takes the largest proportion. This feature can also be observed in Figure 9 (b), which shows the area of zero friction work (white) occupying considerable proportion. A similar characteristic is shown in the contour plot of the real part of the complex eigenvalue (Figure 9 (a)).

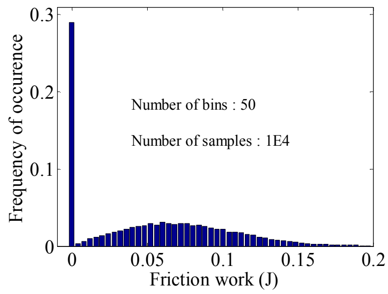


(a)

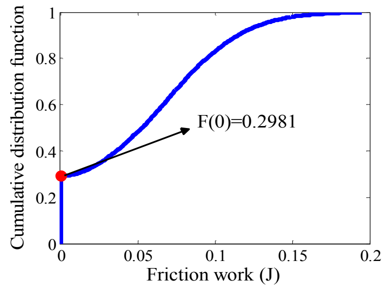


(b)

Figure 7. Estimating the threshold speed for eliminating sign function in the 1 and 4 –DOF oscillators (V is the speed of the belt).

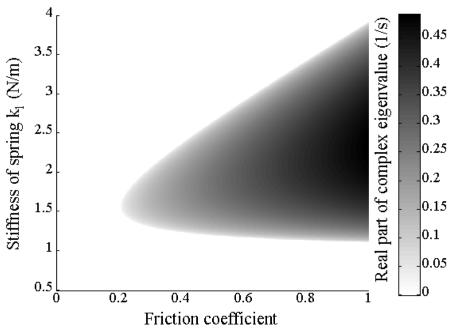


(a)

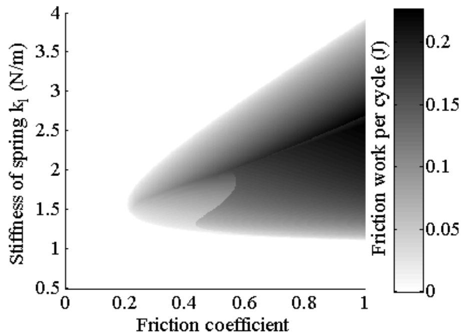


(b)

Figure 8. The distribution of stochastic friction work for the 1st mode when $k_1 = 2.2$ (N/m)



(a)



(b)

Figure 9. Contour plot of deterministic real part of complex eigenvalue and friction work for 1st mode