PROFITABILITY OF TIME SERIES MOMENTUM

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Date: October 1, 2014.

Acknowledgement: We are grateful to valuable comments from Carol Alexander (the editor) and two anonymous referees. We would like to thank Youwei Li and the participants of the 2011 Guangzhou Conference on Nonlinear Economic Dynamics and Financial Market Modelling, the 2011 San Francisco Conference on Computing in Economics and Finance, the 2011 Ancona Annual Workshop on Economic Heterogeneous Interacting Agents, the 2011 Sydney Conference on Quantitative Methods in Finance, the 2012 Prague Conference on Computing in Economics and Finance, the 2013 Reykjavik Annual Workshop on the Economic Science with Heterogeneous Interacting Agents and the 2013 Vancouver Conference on Computing in Economics and Finance for helpful comments. Financial support for He from the Australian Research Council (ARC) under Discovery Grant (DP130103210) is gratefully acknowledged. The usual caveats apply.

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Abstract. We propose a continuous-time heterogeneous agent model consisting of fundamental, momentum, and contrarian traders to explain the significant time series momentum. We show that the performance of momentum strategy is determined by both time horizon and the market dominance of momentum traders. Specifically, when momentum traders are more active in the market, momentum strategies with short (long) time horizons stabilize (destabilize) the market, and meanwhile the market under-reacts (over-reacts) in short-run (long-run). This provides profit opportunity for time series momentum strategies with short horizons and reversal with long horizons. When momentum traders are less active in the market, they always lose. The results provide an insight into the profitability of time series momentum documented in recent empirical studies.

Key words: Time series momentum, profitability, market stability, stochastic delay differential equations.

JEL Classification: C62, D53, D84, G12

1. Introduction

This paper studies time series momentum and its profitability in financial markets. Time series momentum investigated recently in Moskowitz, Ooi and Pedersen (2012) characterizes a strong positive predictability of a security's own past returns. For a large set of futures and forward contracts, Moskowitz et al. (2012) find a time series momentum or "trend" effect based on past 12 month excess returns persists for 1 to 12 months that partially reverses over longer time horizons. This effect based purely on a security's own past returns is related to, but different from, the cross-sectional momentum phenomenon studied extensively in the literature. Through return decomposition, Moskowitz et al. (2012) argue that positive autocovariance is the main driving force for time series momentum and cross-sectional momentum effects, while the contribution of serial cross-correlations and variation in mean returns is small. This paper introduces a model to provide an explanation on the profitability of time series momentum over short horizons and reversal over longer horizons.

To explain the time series momentum, we introduce a simple continuous-time asset pricing model consisting of three types of agents based on typical fundamental, momentum, and contrarian trading strategies. Fundamental agents trade based on the expectation of mean-reversion of market price to the fundamental price; while momentum and contrarian agents trade respectively based on the continuation and reverse of the past price trends over different time horizons. The market price is determined via a market maker mechanism. The model, characterized by a stochastic delay integro-differential system, provides a unified approach to examine the impact of different time horizons of momentum and contrarian strategies on market stability and profitability of these strategies. We show that profitability is closely related to the activity of momentum traders and market stability. In particular, we show that: (i) momentum trading destabilizes the market, while contrarian trading stabilizes the market; (ii) the profitability of momentum strategies is related positively to the activity of momentum traders and negatively to the time horizon used for estimating the price trend; (iii) when momentum traders are more active in the market, the market price under-reacts in short-run and over-reacts in long-run, leading to profitability of momentum strategies with short horizons and loss with longer horizons. The analysis provides an insight into the profitability of time series momentum documented in Moskowitz et al. (2012).

The size and apparent persistence of momentum profits have attracted considerable attention. De Bondt and Thaler (1985) and Lakonishok, Shleifer and Vishny (1994) find supporting evidence on the profitability of contrarian strategies for a holding period of 3-5 years based on the past 3 to 5 year returns. In contrast, Jegadeesh and Titman (1993, 2001) among many others, find supporting evidence on the profitability of momentum strategies for a holding period of 3-12 months based on the returns over past 3-12 months.[1] It is clearly that the time horizons and holding periods play crucial roles in the performance of contrarian and momentum strategies. Many theoretical studies have tried to explain the momentum,[2] however, as argued in Griffin, Ji and Martin (2003), "*the comparison is in some sense unfair since no time horizon is specified in most behavioral models*". This paper provides a uniform treatment on various time horizons used in momentum and contrarian trading strategies and develops an intuitive and parsimonious financial market model of heterogeneous agents in a continuous-time framework to study the impact of different time horizons on the market. To our knowledge, this is the first

[1]In addition to individual stock momentum, Moskowitz and Grinblatt (1999) show industry momentum for a holding period of 1 to 12 months based on past 1 to 12 months and long-run reversals. George and Hwang (2004) find the momentum in price levels by investigating 52-week high. Recently, Novy-Marx (2012) find the term-structure momentum that is primarily driven by firm's performance 12 to 7 months prior to portfolio formation. The evidence has been extended to commodity futures markets (Miffre and Rallis 2007), international markets (Antoniou, Lam and Paudyal 2007) and different asset classes (Asness, Moskowitz and Pedersen 2013).

[2]Among which, the three-factor model of Fama and French (1996) can explain long-run reversal but not short-run momentum. Daniel, Hirshleifer and Subrahmanyam (1998)'s model with single representative agent and Hong and Stein (1999)'s model with different trader types attribute the under- and overreaction to overconfidence and biased self-attribution. Sagi and Seasholes (2007) present a growth option model to identify observable firm-specific attributes that drive momentum. Recently, Vayanos and Woolley (2013) show the slow-moving capital can also generate momentum.

paper to analyze a financial market model with all three types of fundamental, momentum, and contrarian strategies with specified time horizons in a continuous-time framework.

The state of the market is also a critically important factor that affects the profitability as shown in Griffin et al. (2003) and Lou and Polk (2013).[3] Different investment strategies play different roles in market stability and have different implications on market states. Intuitively, momentum strategies are based on the hypothesis of under-reaction with the expectation that the future price will follow the price trend. Consequently the strategies tend to destabilise the market price when momentum traders are more active in the market. While contrarian strategies are based on the hypothesis of overreaction with the expectation that the future price will go against the price trend. Therefore the strategies can stabilize the market when contrarian traders are more active in the market. However, the joint impact of both strategies on market stability can be complicated, depending on their activities in the market. We show that (i) when market is dominated by fundamental and contrarian traders, the market is stabilizing and momentum strategies do not generate profit; (ii) when the activity of momentum traders is "balanced" by the activities of fundamental and contrarian traders, there is a significant overreaction in short horizons and hence momentum trading is not profitable; (iii) when market is dominated by momentum traders, the market is destabilized and can under-react in short-run but over-react in long-run. The results are consistent with the "crowded trading" proposed by Lou and Polk (2013) that "*the underreaction or overreaction characteristic of momentum is time-varying, crucially depending on the size of the*

[3]Cooper, Gutierrez and Hameed (2004) find that short-run (6 months) momentum strategies make profits in up market and lose in down market, but the up-market momentum profits reverse in the long-run (13-60 months). Hou, Peng and Xiong (2009) find momentum strategies with short time horizon (1 year) are not profitable in down market, but return significant profits in up market. Similar results of profitability are also reported in Chordia and Shivakumar (2002) that commonly using macroeconomic instruments related to the business cycle can generate positive returns to momentum strategies during expansionary periods and negative returns during recessions.

momentum crowd". In addition, we find that, with momentum crowd, the momentum trading leads to gain for the strategies with short horizons and loss for the strategies with longer horizons.

This paper is closely related to the literature on the use of technical trading rules. Despite the efficient market hypothesis of financial markets in the academic finance literature (Fama 1970), the use of technical trading rules based on past returns, in particular momentum and contrarian strategies, still seems to be widespread amongst financial market practitioners (Allen and Taylor 1990). The profitability of these strategies and their consistency with the efficient market hypothesis have been investigated extensively in the literature.[4] Recently, Zhu and Zhou (2009) demonstrate that technical analysis, especially the moving average rules, can be a valuable learning tool in general under model or parameter uncertainty. Different from the above studies, which examine profitability by directly applying technical trading rules to the financial data without impacting market price; this paper however shows profitability can also come from the impact of technical trading strategies on market price. Therefore, the momentum strategies can be self-fulfilling.

The modeling approach used in this paper is different from the traditional approach. In the traditional continuous-time asset pricing literature, the underlying processes are Markov. The market price determined by market equilibrium is consistent with the Markov processes. However, when modelling time series momentum that depends on history prices, the underlying processes become non-Markov. This makes it difficult to have consistence between the optimal demands driven by utility maximization and the equilibrium price process. Therefore this paper follows the literature of heterogeneous agent models (HAMs) and focuses on behavior aspects of investors. Over the last three decades, empirical evidence, unconvincing justifications of the assumption of unbounded rationality and the recognition of the relevance of investor psychology have led to the incorporation of heterogeneous and boundedly rational behaviors of investors, such as trend chasing and switching, into asset pricing and financial market modelling. HAMs consider financial markets as

[4]See for example, Frankel and Froot (1986), Neftci (1991), Brock, Lakonishok and LeBaron (1992), Neely, Weller and Dittmar (1997) and Allen and Karjalainen (1999).

expectation feedback systems where asset price fluctuations can be driven by an endogenous mechanism with heterogeneity and bounded rationality. By considering two types of traders, typically fundamentalists and trend followers, Beja and Goldman (1980) and Chiarella (1992) among many others have shown that interaction of agents with heterogeneous expectations may lead to market instability. More significantly, Brock and Hommes (1997, 1998) introduce the concept of an adaptively rational equilibrium in a discrete-time framework. Agents adapt their beliefs over time by choosing from different predictors or expectation functions based upon their past performance (such as realized profits). Such boundedly rational behavior of agents can lead to chaotic market price. HAMs have also been extended to explain the excessive and asymmetric volatility by incorporating the reflection effect of the prospect theory (Park 2014). Most of the HAMs are in discrete-time rather than continuous-time setup. Continuous-time HAMs on asset price dynamics have been developed recently (see Di Guilmi, He and Li 2014). We refer readers to Li (2014) for a discussion on the advantages of the continuous-time models. Overall, these models have successfully explained several market features (such as market booms and crashes, deviations of the market price from the fundamental price), the stylized facts (such as skewness, kurtosis, volatility clustering and fat tails of returns) and the power-law behavior. Different from the extant HAMs, the focus of this paper is on the mechanism of generating the momentum profitability.

This paper is organized as follows. We first present some empirical evidence on the time series momentum in financial market index in Section 2. Section 3 proposes a stochastic HAM in continuous time with time delays to incorporate fundamental, momentum and contrarian traders. To better understand the model, Section 4 follows the standard approach in HAMs and focuses on the dynamics of the underlying deterministic model to examine the impact of these strategies, in particular the different time horizons, on market stability. Section 5 examines the stochastic model numerically and investigates the connection between market stability and profitability. Section 6 concludes. All the proofs and extensions of the model are included in Appendices.

2. Time Series Momentum of the S&P 500

We first provide some evidence on time series momentum in the S&P 500. Most momentum literature is cross-sectional. The time series momentum is explored recently in Moskowitz et al. (2012) who show that a security's own past returns have strong positive predictability for its future return among almost five dozen diverse futures and forward contracts. Similar to Moskowitz et al. (2012), we apply the momentum strategy based on the standard moving average rules (MA) to the monthly data of the total return index of the S&P 500 from Jan. 1988 to Dec. 2012 obtained from *Datastream*.

We first define the trading signal for momentum trading. Let $P(t)$ be the log (cum dividend) price of a stock index at time t. The trading signal can be defined by

$$\begin{aligned} S_t^{(1)} :=& sign\Big(P_t - \frac{P_{t-1} + \cdots + P_{t-m}}{m}\Big) \\ =& sign\Big(\frac{1}{m}\big[m\Delta P_{t-1} + (m-1)\Delta P_{t-2} + \cdots + \Delta P_{t-m}\big]\Big), \end{aligned} \tag{2.1}$$

which is a decaying weighted average of past return over a horizon of m-month.

The mean profit of a momentum strategy with m-month horizon and n-month holding period ($m, n = 1, 2, \cdots, 60$) is calculated as follows. The strategy is to long (short) one unit of index for n months when the trading signal is positive (negative). Hence, at each time t (except for $t < n-1$), we have n long/short positions in the index. The average (log) excess return of the momentum strategy at time t is calculated by the average excess monthly returns of the n positions in the index,

$$\left[\frac{1}{n}\sum_{k=1}^{n} S_{t-k}^{(i)}\right] \times \big(\Delta P_t - r_{f,t}\big), \quad i = 1, 2, \tag{2.2}$$

where $r_{f,t}$ is the 1 month Treasury bill rate.

With the trading signal defined by (2.1), Table 2.1 reports the annualized (log) excess returns of the momentum strategies for the S&P 500 with horizon (m) and holding (n) periods from 1 to 60 months. It shows that the momentum strategies are profitable for time horizons and holding periods up to 3 years. In particular, the profits become significant (up to about 7% p.a.) for time horizons from 6 months to 3 years and holding periods from 1 to 12 months. Fig. 2.1 reports the t-statistic

TABLE 2.1. The annualized percentage (log) excess returns of the momentum strategies (2.1) for the S&P 500 with horizon (m) and holding (n) from 1 to 60 months period. Note: *, **, *** denote the significance at 10%, 5% and 1% levels, respectively.

$m \setminus n$	1	3	6	12	24	36	48	60
1	2.63	3.77**	1.99	3.43***	2.28**	1.96**	1.68*	1.34
3	1.38	2.91	3.36*	3.80**	3.40**	2.92**	2.50*	2.29*
6	6.03**	5.01*	4.62**	4.39**	3.25*	2.45	2.21	2.02
12	7.52**	6.54**	5.93**	5.00**	2.95	2.23	2.18	2.25
24	6.57**	7.87***	6.16**	5.03*	3.08	2.37	2.30	2.55
36	6.72**	6.76**	5.55*	3.47	2.38	2.08	2.21	2.76
48	4.34	2.07	1.52	1.22	0.67	1.15	1.47	2.30
60	1.05	0.66	-0.56	-0.44	0.06	0.72	1.17	2.27

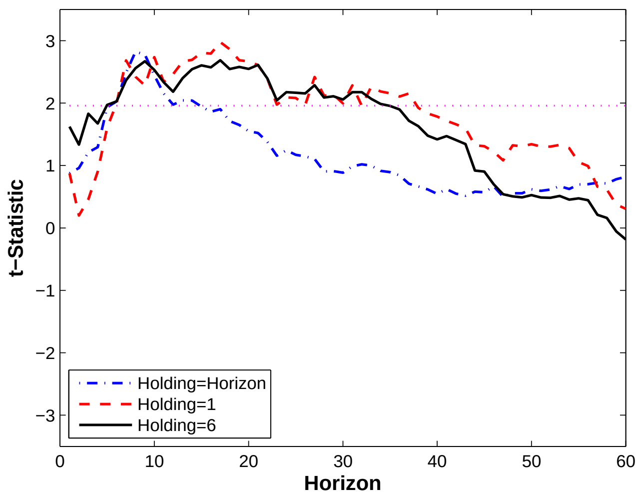


FIGURE 2.1. The t-statistic of the average excess return of the momentum strategies the S&P 500 based on (2.1) for time horizon from 1 to 60 months periods and holding periods equal to horizon ($n = m$), 1 month ($n = 1$) and 6 month periods ($n = 6$) respectively.

of the excess return of the momentum strategies investing in the S&P 500 for time horizons from 1 to 60 months and holding period equals to the time horizon ($n = m$), 1 month ($n = 1$) and 6 months ($n = 6$) respectively. It shows that the momentum strategies are significantly profitable for short holding periods from 1 to 6 months and time horizons from 6 to 30 months with the corresponding t-statistics being above 1.96, the critical value at 95% confidence level.

Table 2.2. The Sharpe ratio of the momentum strategies (2.1) for the S&P 500 with horizon (m) and holding (n) from 1 to 60 months period.

$m \setminus n$	1	3	6	12	24	36	48	60
1	0.050	**0.128**	0.093	**0.187**	**0.134**	**0.114**	0.099	0.078
3	0.026	0.070	0.106	**0.137**	**0.138**	**0.122**	0.105	0.095
6	**0.116**	0.109	**0.118**	**0.123**	0.105	0.083	0.075	0.068
12	**0.145**	**0.135**	**0.129**	**0.116**	0.077	0.062	0.060	0.061
24	**0.126**	**0.159**	**0.131**	0.114	0.074	0.061	0.058	0.064
36	**0.129**	**0.135**	0.117	0.076	0.056	0.051	0.054	0.068
48	0.083	0.042	0.032	0.026	0.016	0.027	0.035	0.055
60	0.020	0.014	-0.012	-0.010	0.001	0.017	0.027	0.000

We also report the Sharpe ratio of the strategies to adjust for risk, which is defined as the ratio of the mean excess return on the (managed) portfolio and the standard deviation of the portfolio return. If a strategy's Sharpe ratio exceeds the market Sharpe ratio, the active portfolio dominates the market portfolio (in an unconditional mean-variance sense). For empirical applications, the (ex post) Sharpe ratio is usually estimated as the ratio of the sample mean of the excess return on the portfolio and the sample standard deviation of the portfolio return. The average monthly return on the total return index of the S&P 500 over the period January 1988–December 2012 is 0.76% with an estimated (unconditional) standard deviation of 4.30%. The Sharpe ratio of the market index is 0.108. The Sharpe ratio of the strategy based on (2.1) and (2.2) is documented in Table 2.2. Tables 2.1 and 2.2 are very consistent. Specifically, when Table 2.1 shows a momentum strategy with certain time horizon and holding period generates significantly positive excess return, Table 2.2 shows this strategy can also outperform the market according to the Sharpe ratio.

Alternatively, motivated by Moskowitz et al. (2012), we also consider the trading signal defined by

$$S_t^{(2)} := sign\Big(\frac{1}{m}\big[(\Delta P_{t-1} - r_{f,t-1}) + (\Delta P_{t-2} - r_{f,t-2}) + \cdots + (\Delta P_{t-m} - r_{f,t-m}\big]\Big), \quad (2.3)$$

which is an equally weighted average of excess returns over the past m periods. With the trading signal defined by (2.3), similar results are obtained and reported in Table A.1 and Fig. A.1 in Appendix A.

3. The Model

In this section, we establish an asset pricing model of single risky asset to characterize the time series momentum. The modelling approach follows closely to the current HAM framework. By considering the financial market as an expectations feedback mechanism, Chiarella (1992), Lux (1995) and Brock and Hommes (1997, 1998) were amongst the first to have shown that the interaction of agents with heterogeneous expectations may lead to market instability. By incorporating bounded rationality and heterogeneity, HAMs have successfully explained the complexity of market price behavior, market booms and crashes, and long deviations of the market price from the fundamental price. They show great potentials in generating the stylized facts (such as skewness, kurtosis, volatility clustering and fat tails in returns), and various power laws (such as the long memory in return volatility) observed in financial markets. We refer readers to Hommes (2006), LeBaron (2006), Lux (2009) and Chiarella, Dieci and He (2009) for surveys of the recent developments in this literature.

To examine the effect of time horizons, instead of using a discrete-time setup, we consider a continuous-time setup in this paper with fundamentalists who trade according to fundamental analysis and momentum and contrarian traders who trade differently based on price trend calculated from moving averages of historical prices over different time horizons. As in Beja and Goldman (1980), the market price is arrived at via a market maker scenario.[5] To focus on price dynamics, we motivate the excess demand functions of the three different types of traders directly, rather than deriving them from utility maximization of their portfolio investments. These

[5]As presented in O'Hara (1995), the Walrasian scenario, even though widely used in economic analysis, only plays a part in one real market (the market for silver in London).

demand functions are also consistent those in the discrete-time HAMs literature derived from heterogeneous expectations and utility maximization.[6]

3.1. **Fundamental Traders.** Let $P(t)$ and $F(t)$ denote the log (cum dividend) price and (log) fundamental value $F(t)$, respectively of a risky asset at time t. The fundamental traders believe that the market price[7] $P(t)$ is mean-reverting to the fundamental price $F(t)$, which can be estimated based on some fundamentals. They buy (sell) the stock when the current price $P(t)$ is below (above) the fundamental price $F(t)$ of the stock. For simplicity, we assume that the excess demand of the fundamental traders, $D_f(t)$ at time t, is proportional to the deviation of the market price $P(t)$ from the fundamental value $F(t)$, namely,

$$D_f(t) = \beta_f\big(F(t) - P(t)\big), \tag{3.1}$$

where $\beta_f > 0$ is a constant, measuring the speed of mean-reversion of $P(t)$ to $F(t)$, which may be weighted by the risk tolerance of the traders. For simplicity, we assume that the fundamental return follows a pure white noise process:

$$dF(t) = \sigma_F dW_F(t), \ F(0) = \bar{F}, \tag{3.2}$$

where $\sigma_F > 0$ represents the volatility of the fundamental return and $W_F(t)$ is a standard Wiener process.

3.2. **Momentum and Contrarian Traders.** Both momentum and contrarian traders trade based on their estimated market price trends, although they behave differently. Momentum traders believe that future market price follows a price trend $u_m(t)$. When the current market price is above the trend, they expect future market price to rise and therefore they take a long position of the risky asset; otherwise, they

[6]In the traditional approach in the continuous-time literature, the usual way is to first specify a price process and then derive the optimal demand functions. The parameters in the price process are then determined by market clearing conditions. Because of the Markov property of the underlying processes, the price process and utility maximization are consistent. However, when modelling time series momentum, we do not have such consistency due to the non-Markov property of the underlying process in general. Therefore, the demand functions are motivated from behavioral aspects in the paper.

[7]For convenience, the price is referred to the log price in this paper, unless specified otherwise.

take a short position. Different from the momentum traders, contrarians believe that future market price goes opposite to a price trend $u_c(t)$. When the current market price is above the trend, they expect future market price to decline and therefore they take a short position of the risky asset; otherwise, they take a long position.

The price trend used for the momentum traders and contrarians can be different in general. Among various price trends used in practice, the standard moving average (MA) rules with different time horizons are the most popular ones,

$$u_i(t) = \frac{1}{\tau_i} \int_{t-\tau_i}^{t} P(s)ds, \qquad i = m, c, \tag{3.3}$$

where the time delay $\tau_i \geq 0$ represents the time horizon of the MA.[8] We therefore assume that the excess demand of the momentum traders and contrarians are given, respectively, by

$$D_m(t) = g_m\big(P(t) - u_m(t)\big), \qquad D_c(t) = g_c\big(u_c(t) - P(t)\big), \tag{3.4}$$

where the S-shaped demand function $g_i(x)$ for $i = m, c$ satisfies

$$g_i(0) = 0, \qquad g_i'(x) > 0, \qquad g_i'(0) = \beta_i > 0, \qquad x g_i''(x) < 0, \quad \text{for } x \neq 0, \tag{3.5}$$

and parameter β_i represents the extrapolation rate of the price trend when the market price deviation from the trend is small. Notice the trading signal of the strategy (3.4) is consistent with (2.1). In the following discussion, we take $g_i(x) = \tanh(\beta_i x)$, which satisfies condition (3.5).[9]

[8]The price trend $u_i(t)$ can be regarded as the logarithm of the geometric mean of market price over the past τ_i periods. Zhu and Zhou (2009) show that little performance differences emerge in their paper with the use of geometric MA and arithmetic MA. In particular, $u_i(t) \to P(t)$ as $\tau_i \to 0$, implying that the price trend is given by the current price.

[9] Chiarella (1992) provides an explanation for the increasing and bounded S-shaped excess demand function. For example, traders may seek to allocate a fixed amount of wealth between the risky asset and a bond so as to maximize their expected utility of consumption. The demand becomes bounded due to wealth constraints. From behaviorial point of view, traders may become cautious when the deviation of the market price from the price trend is large. This together leads approximately to an S-shaped increasing excess demand function. For consistence, we also study the cases when both demand functions are linear or S-shaped. We find that adding/dropping the S-shaped demands does not affect the local stability presented in Section 4; however imposing S-shaped demand function to the fundamentalists leads to a more "unstable" market in the sense

3.3. **Market Price via a Market Maker.** Assume net zero supply in the risky asset and let α_f, α_m and α_c be the market population fractions of the fundamental, momentum, and contrarian traders, respectively, with $\alpha_f + \alpha_m + \alpha_c = 1$ and $\alpha_i > 0$ for $i = f, m, c$.[10] Then the aggregate market excess demand for the risky asset, weighted by the population weights, is given by $\alpha_f D_f(t) + \alpha_m D_m(t) + \alpha_c D_c(t)$. Following Beja and Goldman (1980), Kyle (1985) and Farmer and Joshi (2002), we assume that the price $P(t)$ at time t is set via a market maker mechanism and adjusted according to the aggregate excess demand, that is,

$$dP(t) = \mu[\alpha_f D_f(t) + \alpha_m D_m(t) + \alpha_c D_c(t)]dt + \sigma_M dW_M(t), \tag{3.6}$$

where $\mu > 0$ represents the speed of the price adjustment by the market maker, $W_M(t)$ is a standard Wiener process capturing the random excess demand process driven by either noise traders or liquidity traders, and $\sigma_M \geq 0$ is constant. $W_M(t)$ is assumed to be independent of the Wiener process for the fundamental price $W_F(t)$.[11] Based on Eqs. (3.1)-(3.6), the market price of the risky asset is determined by

$$\begin{aligned} dP(t) =& \mu\Big[\alpha_f \beta_f\big(F(t) - P(t)\big) + \alpha_m \tanh\Big(\beta_m\big(P(t) - \frac{1}{\tau_m}\int_{t-\tau_m}^{t} P(s)ds\big)\Big) \\ &+ \alpha_c \tanh\Big(-\beta_c\big(P(t) - \frac{1}{\tau_c}\int_{t-\tau_c}^{t} P(s)ds\big)\Big)\Big]dt + \sigma_M dW_M(t), \end{aligned} \tag{3.7}$$

where the fundamental price $F(t)$ is defined by (3.2). Therefore, the asset price dynamics is determined by the stochastic delay integro-differential equation (3.7).

As argued in Lo, Mamaysky and Wang (2000), "*The general goal of technical analysis is to identify regularities in the time series of prices by extracting nonlinear patterns from noisy data. Implicit in this goal is the recognition that some price*

that the amplitude of price fluctuation becomes greater when the original system is unstable. On the profitability, when both demand functions are either linear or nonlinear, we find that the profit level can be different, however the conclusion on the profitability does not change.

[10]To simplify the analysis, we first assume that the market fractions are constant. When agents are allowed to switch among different strategies based on some fitness measure (see He and Li 2012), the market fractions become time-varying. An analysis of this extension is given in Appendices D and E.

[11]The two Wiener processes can be correlated. We refer readers to He and Li (2012) for related discussion on the impact of the correlation on the price behavior and the stylized facts in financial market.

movements are significant—they contribute to the formation of a specific pattern—and others are merely random fluctuations to be ignored." We are interested in the connection between market stability and profitability of the trading strategies. Given the complex structure of the nonlinear model, we follow the standard approach in the HAM literature and combine the stability analysis of the underlying deterministic model with numerical simulation of the stochastic model. By ignoring the noises, Section 4 aims to recognize the patterns of the mean values in the noisy price system, which underlies the profitability mechanism of momentum strategies studied in Section 5. The stability analysis provides an insight into the effect of the interaction and activities of different types of traders on market stability. It helps us to understand the relation between different states of market stability and profitability of trading strategies. Note that it is the interaction of deterministic dynamics and noise processes that provides a complete picture of the price dynamics of the full stochastic model. In the following section, we first examine the stability of the corresponding deterministic delay integro-differential equation model.

4. Market Stability

By assuming a constant fundamental price $F(t) \equiv \bar{F}$ and no market noise $\sigma_M = 0$, system (3.7) becomes a deterministic delay integro-differential equation, which represents the process of the mean value of the market return:

$$\begin{aligned}\frac{dP(t)}{dt} =&\mu\Big[\alpha_f\beta_f\big(\bar{F}-P(t)\big)+\alpha_m\tanh\Big(\beta_m\big(P(t)-\frac{1}{\tau_m}\int_{t-\tau_m}^{t}P(s)ds\big)\Big)\\ &+\alpha_c\tanh\Big(-\beta_c\big(P(t)-\frac{1}{\tau_c}\int_{t-\tau_c}^{t}P(s)ds\big)\Big)\Big]. \end{aligned} \tag{4.1}$$

It is easy to see that $P(t)=\bar{F}$ is the unique steady state price of the system (4.1). We therefore call $P=\bar{F}$ the *fundamental steady state.*

In this section, we study the dynamics of the deterministic model (4.1) by focusing on the local stability of the fundamental steady state. Denote $\gamma_i = \mu\alpha_i\beta_i$ ($i = f, m, c$), which characterize the activity of type-i traders.[12] In general, the dynamics

[12]Intuitively, the speed of the price adjustment μ of the market maker measures the activity across the market. Both the population size α_i and behaviour activity β_i qualify the trading behaviour of type-i traders. Therefore, γ_i measures the activity or dominance of type i traders.

depend on the behavior of fundamental, momentum, contrarian traders, market maker, and time horizons. To understand the impact, we first consider two special cases where only momentum traders or contrarians are involved.

4.1. **The Stabilizing Role of the Contrarians.** Contrarian trading strategies are based on the hypothesis of market overreaction. Intuitively, contrarians can induce market stability. To support the intuition, we consider a market with the fundamental and contrarian traders only, that is, $\alpha_m = 0$. In this case the system (4.1) reduces to

$$\frac{dP(t)}{dt} = \mu\Big[\alpha_f\beta_f\big(\bar{F} - P(t)\big) + \alpha_c \tanh\Big(- \beta_c\big(P(t) - \frac{1}{\tau_c}\int_{t-\tau_c}^{t} P(s)ds\big)\Big)\Big]. \tag{4.2}$$

The following proposition confirms the stabilizing role of the contrarians.[13]

Proposition 4.1. *The fundamental steady state price $P = \bar{F}$ of the system* (4.2) *is asymptotically stable for all $\tau_c \geq 0$.*

Proposition 4.1 shows that the market consisting of fundamental and contrarian investors is always stable, and the result is independent of the time horizon and extrapolation of the contrarians.[14]

4.2. **The Destabilizing Role of the Momentum Traders.** Momentum trading strategies based on the hypothesis of market under-reaction are aimed to explore the opportunities of market price continuity. Intuitively, when the market is dominated by fundamental traders, the market is expected to reflect the fundamental price and then the impact of the momentum traders on market stability can be very limited. However, when the market is dominated by the momentum traders, the extrapolation of the market price continuity can have significant impact on market stability.

[13] All the proofs can be found in Appendix B.

[14] Note that this result is different from that in discrete-time HAMs, in which market can become unstable when activity of contrarians is strong, see for example, Chiarella and He (2002). This difference is due to the continuous adjustment of the market price. The impact of any strong activity from the contrarians becomes insignificant over a small time increment. Hence the time horizon used to form the MA becomes more irrelevant in this case.

To explore the impact, we now consider a market consisting of the fundamentalists and momentum traders only, that is $\alpha_c = 0$. In this case, system (4.1) reduces to

$$\frac{dP(t)}{dt} = \mu\Big[\alpha_f\beta_f\big(\bar{F} - P(t)\big) + \alpha_m \tanh\Big(\beta_m\big(P(t) - \frac{1}{\tau_m}\int_{t-\tau_m}^{t} P(s)ds\big)\Big)\Big], \qquad (4.3)$$

and the price dynamics can be described by the following proposition.

Proposition 4.2. *The fundamental steady state price $P = \bar{F}$ of the system* (4.3) *is*

(i) *asymptotically stable for all $\tau_m \geq 0$ when $\gamma_m < \frac{\gamma_f}{1+a}$;*

(ii) *asymptotically stable for either $0 \leq \tau_m < \tau^*_{m,l}$ or $\tau_m > \tau^*_{m,h}$ and unstable for $\tau^*_{m,l} < \tau_m < \tau^*_{m,h}$ when $\frac{\gamma_f}{1+a} \leq \gamma_m \leq \gamma_f$; and*

(iii) *asymptotically stable for $\tau_m < \tau^*_{m,l}$ and unstable for $\tau_m > \tau^*_{m,l}$ when $\gamma_m > \gamma_f$.*

Here $a = \max\{-\sin x/x; x > 0\} (\approx 0.2172)$, $\tau^*_{m,1} = 2\gamma_m/(\gamma_f - \gamma_m)^2$, $\tau^*_{m,l} (< \tau^*_{m,1})$ is the minimum positive root of equation

$$f(\tau_m) := \frac{\tau_m}{\gamma_m}(\gamma_f - \gamma_m)^2 - \cos\left[\sqrt{2\gamma_m\tau_m - (\gamma_f - \gamma_m)^2\tau_m^2}\right] - 1 = 0, \qquad (4.4)$$

and $\tau^*_{m,h} (\in (\tau^*_{m,l}, \tau^*_{m,1}))$ is the maximum among all the roots of (4.4) *which are less than $\tau^*_{m,1}$.*

Proposition 4.2 shows that the impact of the time horizon used in forming the MA for the momentum traders depends on γ_m and γ_f, which measure the activity or dominance of the momentum and fundamental traders, respectively. On the one hand, when the fundamental traders dominate momentum traders (so that $\gamma_m < \gamma_f/(1+a)$), the market is always stable and time horizon plays no role in market stability. On the other hand, when momentum traders dominate fundamental traders (so that $\gamma_m > \gamma_f$), the market is stable when time horizon is small (so that $\tau_m < \tau^*_{m,l}$), but becomes unstable when the time horizon is large (so that $\tau_m > \tau^*_{m,l}$). In fact, the difference between price and the price trend based on the MA becomes insignificant when the time horizon is small and a strong activity from the momentum traders has very limited impact on market stability, yielding the stability for small time horizon. However, due to the smoothness of the MA when the time horizon is longer, the difference can become significant, which, together with a strong activity from the momentum traders, makes the market unstable. When the activity of the trend followers is balanced by that of the fundamental traders (so that

$\gamma_f/(1+a) \le \gamma_m \le \gamma_f$), the market is stable when the time horizon is either short (so that $\tau_m < \tau^*_{m,l}$) or longer (so that $\tau_m > \tau^*_{m,h}$), but becomes unstable with medium time horizon (so that $\tau^*_{m,l} < \tau_m < \tau^*_{m,h}$). This is an unexpected result. Intuitively, when time horizon is short, the price trend follows the price closely, which limits the trading opportunity for the momentum traders. When horizon is longer, the price trend becomes insensitive to the price changes. However, due to the balanced activity from the fundamental traders, the extrapolation activity of the momentum traders is limited. Therefore, in both cases, the market becomes stable.

4.3. **The Joint Impact of Momentum and Contrarian Trading.** The previous analysis shows the different role of the time horizon used in the MA by either the contrarians or momentum traders. We analyze the market stability when both strategies are employed in the market. For simplicity, we consider $\tau_m \equiv \tau_c := \tau$ in the rest of the paper and leave the general case with different τ_m and τ_c in Appendix C. It is found that this special case can well reflect the impact of different types of traders' activities on the stability and further on their profitability. Let $\tau^*_1 = 2(\gamma_m - \gamma_c)/(\gamma_f - \gamma_m + \gamma_c)^2$, and $\tau^*_l (< \tau^*_1)$ and $\tau^*_h (\in (\tau^*_l, \tau^*_1))$ be the minimum and maximum positive roots which are less than τ^*_1, respectively, of the equation

$$h(\tau) := \frac{\tau}{\gamma_m - \gamma_c}(\gamma_f - \gamma_m + \gamma_c)^2 - \cos\left[\sqrt{2(\gamma_m - \gamma_c)\tau - (\gamma_f - \gamma_m + \gamma_c)^2\tau^2}\right] - 1 = 0.$$

In this case, the market stability of the system (4.1) can be characterized by the following proposition.[15]

Proposition 4.3. *If $\tau_m \equiv \tau_c := \tau$, then the fundamental steady state price $P = \bar{F}$ of the system* (4.1) *is*

(1) *asymptotically stable for all $\tau \ge 0$ when $\gamma_m < \gamma_c + \frac{\gamma_f}{1+a}$;*

[15]Notice time horizon plays an important role in the asset price dynamics. Only very few models have addressed this important issue. One related paper is Chiarella, He and Hommes (2006), who study a discrete-time HAM and show that an increase of the time horizon used by momentum traders can destabilize the market. Proposition 4.3 verifies this argument in continuous time. We refer readers to He and Li (2012) and Di Guilmi, He and Li (2014) for more comparisons on the role of time horizon between continuous-time and discrete-time HAMs, including local stability, distribution of market fraction, the comovements of market fraction and market price, and some financial stylized facts.

(2) *asymptotically stable for either* $0 \leq \tau < \tau_l^*$ *or* $\tau > \tau_h^*$ *and unstable for* $\tau_l^* < \tau < \tau_h^*$ *when* $\gamma_c + \frac{\gamma_f}{1+a} \leq \gamma_m \leq \gamma_c + \gamma_f$*; and*

(3) *asymptotically stable for* $\tau < \tau_l^*$ *and unstable for* $\tau > \tau_l^*$ *when* $\gamma_m > \gamma_c + \gamma_f$.

Despite the activity of both momentum and contrarian traders, Proposition 4.3 shares the same message to Proposition 4.2 with respect to the joint impact of the time horizons and the activity of the momentum traders on market stability, except that the activity of the fundamental traders in Proposition 4.2 is measured jointly by the activities of the fundamental and contrarian traders in Proposition 4.3. Given the stabilizing nature of the contrarian strategy indicated in Proposition 4.1, this is not unexpected. The three conditions

$$(1): \gamma_m < \gamma_c + \frac{\gamma_f}{1+a}, \qquad (2): \gamma_c + \frac{\gamma_f}{1+a} \leq \gamma_m \leq \gamma_c + \gamma_f, \qquad (3): \gamma_m > \gamma_c + \gamma_f$$

in Proposition 4.3 characterize three different states of market stability, which have different implications to the profitability of momentum trading strategy. For convenience, market state k is referred to condition (k) for $k = 1, 2, 3$ in the following analysis.

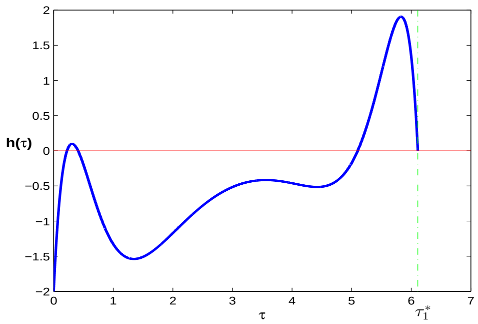
To illustrate the price dynamics in different market state, we now conduct numerical analysis.[16] For market state 1, the fundamental price is stable, independent of the time horizon. For market state 2, Fig. 4.1 (a) illustrates the three values[17] $\tau_l^* \approx 0.23$, $\tau_3^* \approx 0.41$, and $\tau_h^* \approx 5.10$. Correspondingly, Fig. 4.1 (b) shows that the fundamental steady state price $P = \bar{F}$ is stable when $\tau \in [0, \tau_l^*) \cup (\tau_h^*, \infty)$ and unstable when $\tau \in (\tau_l^*, \tau_h^*)$. The stability switches twice.[18] For market state 3, Fig. 4.2 (a) illustrates the first (Hopf bifurcation) value $\tau_l^* \approx 0.22$, which leads to stable

[16]The numerical results in this paper (except for the Appendices D and E) are based on $\alpha_f = 0.3$, $\alpha_m = 0.4$, $\alpha_c = 0.3$, $\mu = 5$ and $\bar{F} = 1$, unless specified otherwise.

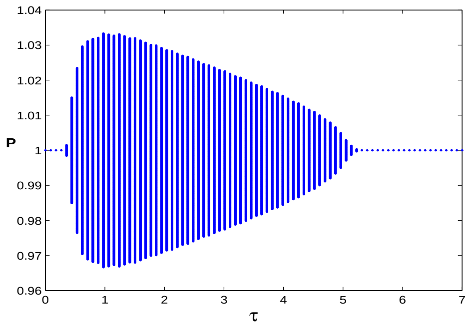
[17]These values are called Hopf bifurcation values in stability theory (see Hale 1997), meaning that the steady state loses the stability at these values and bifurcates to periodic cycles around the steady state, as illustrated by the price bifurcation plot with respect to the time horizon τ.

[18]Simulations (not reported here) show that the speed of the convergence when the fundamental steady state becomes stable after switching from instability as τ increases is very slow, although $\bar{F}$ is stable. The properties on the number of bifurcations and the stability switching are further illustrated in Fig. B.1 in Appendix B.4. There are some interesting properties on the nature of bifurcations related to Proposition 4.3, including the number of bifurcations, stability switching

limit cycles for $\tau > \tau_l^*$, as shown in Fig. 4.2 (b). The stability switches only once at τ_l^*.

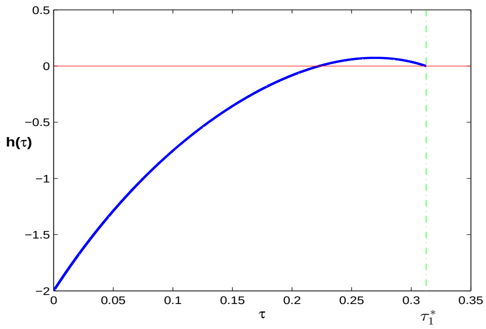


(a) The function $h(\tau)$

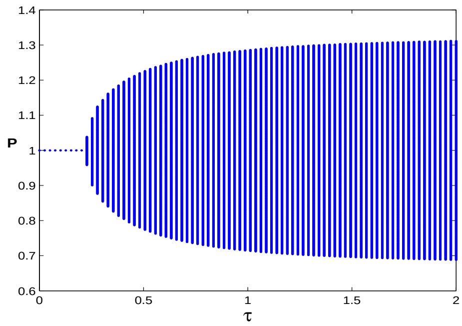


(b) Price bifurcation

FIGURE 4.1. (a) The function $h(\tau)$; (b) the corresponding bifurcation diagram for market state 2. Here $\gamma_f = 20$, $\gamma_m = 22.6$ and $\gamma_c = 5$.



(a) The function $h(\tau)$



(b) Price bifurcation

FIGURE 4.2. (a) The function $h(\tau)$; (b) the bifurcation diagram of the market price for market state 3. Here $\gamma_f = 2$, $\gamma_m = 20$ and $\gamma_c = 10$.

The above numerical analysis clearly illustrates that dependence of the market price dynamics on the time horizon is different in different market states. We show in the following section that the market states also have different implications on the underreaction/overreaction and momentum profitability. We complete the discussion of this section by considering a very special case when $\alpha_m = \alpha_c$, $\beta_m = \beta_c$

and the dependence of the bifurcation values on the parameters. We provide a detailed analysis in Appendix B.4.

and $\tau_m = \tau_c$, that is the momentum and the contrarian traders have the same population, extrapolation rate and time horizon. In this case, system (4.1) reduces to $dP(t)/dt = \gamma_f(\bar{F} - P(t))$. The destabilizing effect of momentum traders is completely offset by contrarians, which leads to the global stability of the fundamental price.

5. Momentum Profitability

This section numerically examines the profitability of the time series momentum trading strategies. We show that the profitability is closely related to the market states defined according to the stability analysis in Section 4. In particular, we show that, in market state 3, the momentum strategy is profitable when the time horizon is short and unprofitable when the time horizon is long. In other market states, the strategy is not profitable for any time horizon. We also provide some explanation to the profitability mechanism through autocorrelation and time series analysis.

As in Section 4, we focus on the special case when momentum and contrarian traders use the same time horizon and holding period τ. The profit is calculated using a buy-and-hold strategy on the number and position determined by the demand function of the trading strategy.[19] It follows from Eq. (3.4) that the excess demands of momentum and contrarian traders with time horizon τ are given, respectively, by

$$\begin{aligned} D_m(t) &= \tanh\Big(\beta_m\big(P(t) - \frac{1}{\tau}\int_{t-\tau}^{t} P(s)ds\big)\Big), \\ D_c(t) &= \tanh\Big(-\beta_c\big(P(t) - \frac{1}{\tau}\int_{t-\tau}^{t} P(s)ds\big)\Big). \end{aligned} \tag{5.1}$$

Based on buy and hold strategy, the realized spot profits of fundamental, momentum, contrarian traders, and the market maker at time t can be calculated by

$$U_i(t) = D_i(t)\big(P(t+\tau) - P(t)\big), \qquad i = f, m, c, M, \tag{5.2}$$

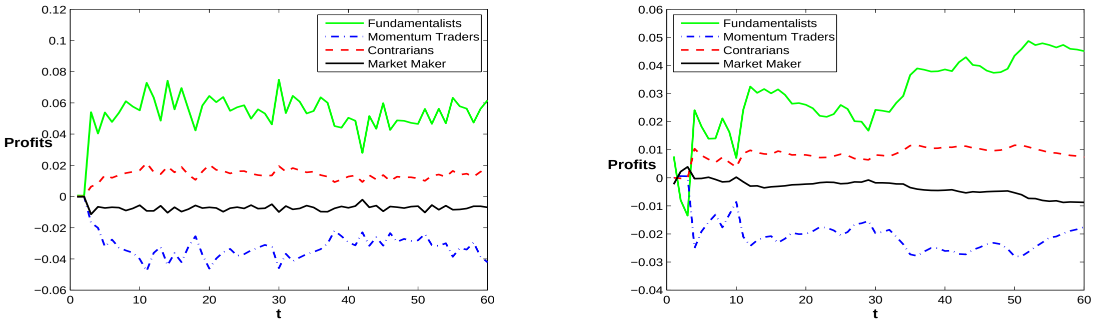
where the excess demand of the fundamental strategy $D_f(t)$ is defined by Eq. (3.1) and the excess demand of the market maker is given by $D_M(t) = -\big(\alpha_f D_f(t) + \alpha_m D_m(t) + \alpha_c D_c(t)\big)$, which is based on the liquidity provided to clean the market.

[19] Alternatively, the profit can be calculated based on buy-and-hold strategy on one unit of position taking, as in Section 2. We find that this does not affect the profitability results obtained in this section.

In addition, we also calculate the average accumulated profit yielded over a time interval $[t_0, t]$ by

$$\bar{U}_i(t) = \frac{1}{t - t_0} \int_{t_0}^{t} D_i(s)\big(P(s+\tau) - P(s)\big)ds, \qquad i = f, m, c, M. \tag{5.3}$$

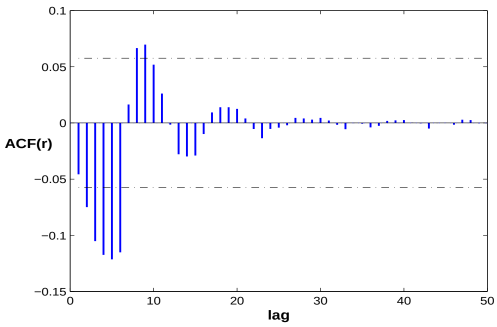
We now examine the profitability in different market states. In the rest of the paper, the time unit is one year and the time step Δt is one month. Given 14.9% annually standard deviation of the log return for the S&P 500 index used in Section 2, we choose $\sigma_M = 0.15$ for the annual market volatility and $\sigma_F = 0.1$ for the annual volatility of the fundamental price.



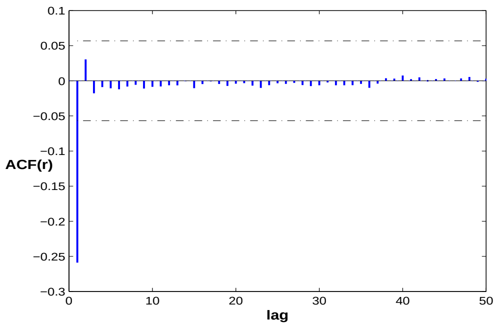
(a) Average spot profits (b) Average accumulated profits

FIGURE 5.1. (a) The average spot profits of trading strategies based on 1000 simulations; (b) the average accumulated profits based on a typical simulation for market state 1. Here $\gamma_f = 15$, $\gamma_m = 15$ and $\gamma_c = 3$ and $\tau = 0.5$.

5.1. **State 1.** In market state 1, the market is dominated jointly by the fundamental and contrarian traders (so that $\gamma_m < \gamma_c + \gamma_f/(1+a)$). In this case, the stability of the fundamental price of the underlying deterministic model is independent of the time horizon. Based on 1,000 simulations, Fig. 5.1 (a) reports the average spot profits of different strategies and Fig. 5.1 (b) illustrates the average accumulated profits based on a typical simulation. They show that the contrarian and fundamental strategies



(a) $\tau = 0.5$



(b) $\tau = 3$

FIGURE 5.2. The average ACs of market return based on 1000 simulations for market state 1 with (a) $\tau = 0.5$ and (b) $\tau = 3$. Here $\gamma_f = 15$, $\gamma_m = 15$ and $\gamma_c = 3$.

are profitable, but not the momentum strategy and the market maker.[20] Note that the amounts of profit/loss are small, which is underlined by the stable market price.

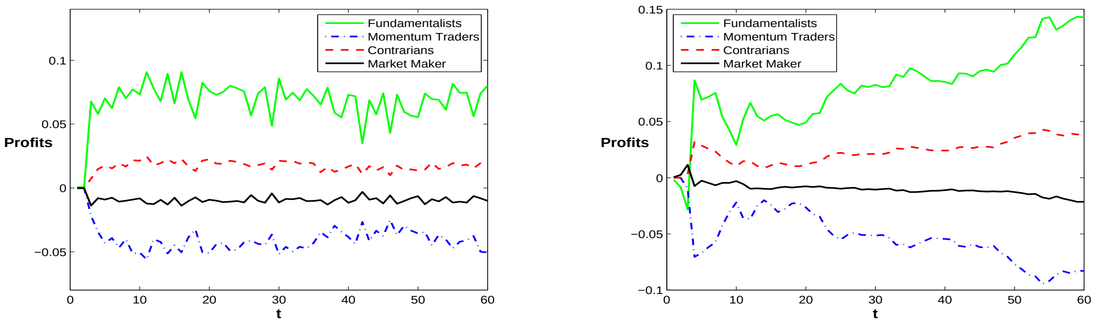
To understand the mechanism of the profitability, we present the average return autocorrelations (ACs) based on 1000 simulations in Fig. 5.2 for $\tau = 0.5$ in (a) and $\tau = 3$ in (b).[21] It shows some significant and negative ACs for small lags and insignificant ACs for large lags. This indicates market overreaction in short-run and hence the fundamental and contrarian trading can generate significant profits. There is no significant and positive ACs, indicating no market under-reaction, and hence the momentum trading is not profitable.

5.2. **State 2.** In market state 2, the momentum traders are active, but their activities are balanced by the fundamental and contrarian traders (so that $\gamma_c + \gamma_f/(1+a) \leq \gamma_m \leq \gamma_c + \gamma_f$). In this case, the stability of the underlying deterministic model is

[20]Notice the profits of market maker is slightly negative by providing liquidity. But in reality, market maker can make profits on the transaction costs and bid-ask spread. We also examine two scenarios with medium and high transaction costs of 0.5% and 1% of the value of the trade respectively. We find that market maker can make significant profits in these cases (not reported here). We would like to thank a referee for this remark.

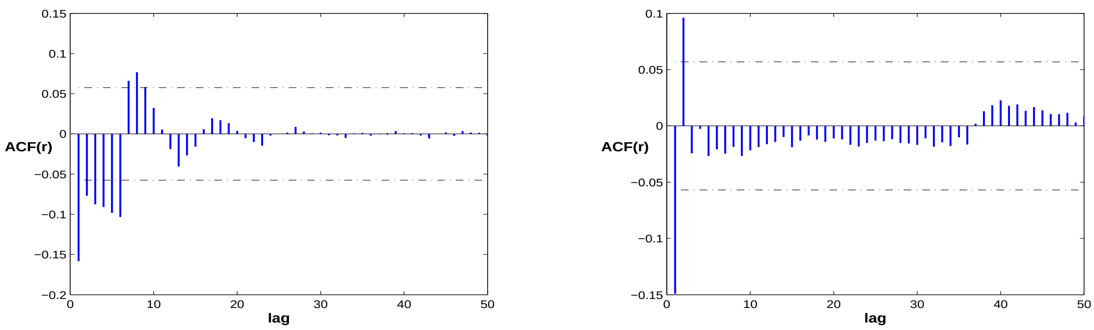
[21]Recently, Zhou and Zhu (2014) show that the moving average rules can explain the time series momentum in an equilibrium model by examining the autocovariance of return for different time horizons.

illustrated in Fig. 4.1, showing that the fundamental price is stable for either short or longer time horizons, but unstable for medium time horizons. With the same parameters used in Fig. 4.1, we illustrate the profitability of the different trading strategies in Fig. 5.3. It shows that the fundamental and contrarian trading strategies are profitable, but not the momentum traders and the market maker. Further simulations (not reported here) show the same result with different time horizons, although the losses/profits increase as time horizon increases.



(a) Average spot profits (b) Average accumulated profits

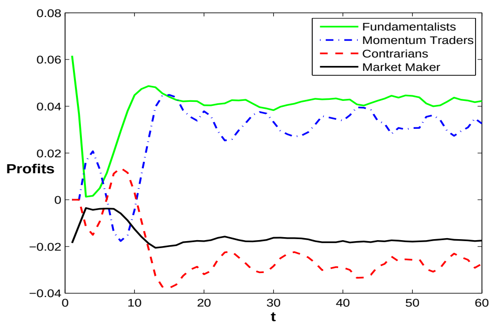
FIGURE 5.3. (a) The average spot profits based on 1,000 simulations; (b) the average accumulated profits based on a typical simulation for market state 2. Here $\gamma_f = 20$, $\gamma_m = 22.6$, $\gamma_c = 5$ and $\tau = 0.5$.



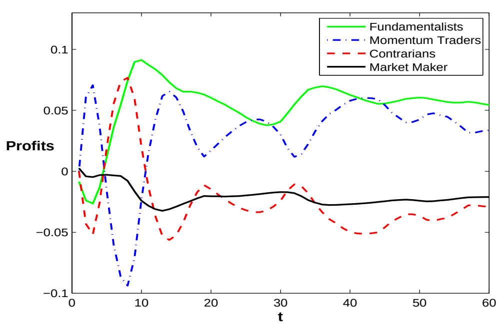
(a) $\tau = 0.5$ (b) $\tau = 3$

FIGURE 5.4. The average ACs of market return based on 1000 simulations for market state 2 with (a) $\tau = 0.5$ and (b) $\tau = 3$.

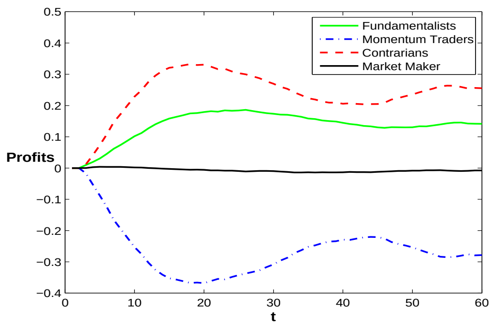
As in market state 1, we also calculate the return ACs with the same set of parameters as Fig. 4.1. Fig. 5.4 presents the average ACs based on 1000 simulations for time horizon $\tau = 0.5$ in (a) and $\tau = 3$ in (b), showing some significantly negative ACs, in particular for $\tau = 0.5$, over short lags. This indicates the profitability of the fundamental and contrarian trading due to market overreaction, but not for the momentum trading. Therefore, both states 1 and 2 lead to the same conclusion on the profitability, although the amount of profit/loss in state 2 is higher than in state 1.



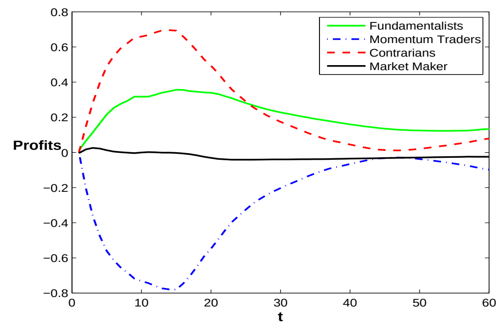
(a) Average spot profits for $\tau = 0.5$



(b) Average accumulated profits for $\tau = 0.5$



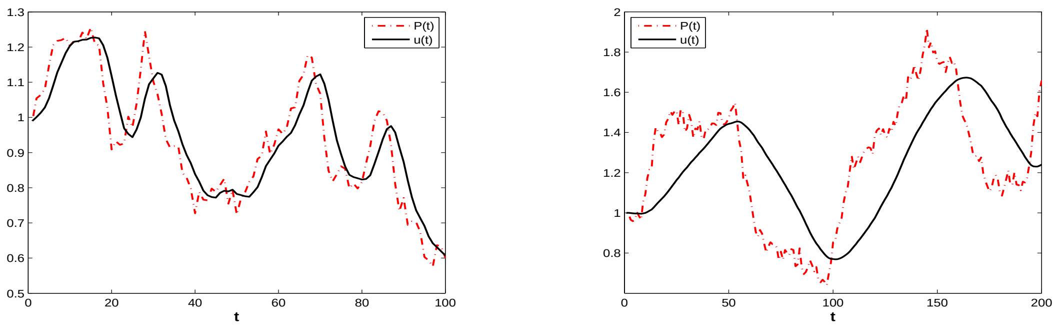
(c) Average spot profits for $\tau = 3$



(d) Average accumulated profits for $\tau = 3$

FIGURE 5.5. The average spot profits based on 1000 simulations for (a) $\tau = 0.5$ and (c) $\tau = 3$ and the average accumulated profits based on a typical simulation for (b) $\tau = 0.5$ and (d) $\tau = 3$ for market state 3. Here $\gamma_f = 2$, $\gamma_m = 20$ and $\gamma_c = 10$.

5.3. **State 3.** In market state 3, the market is dominated by the momentum traders (so that $\gamma_m > \gamma_c + \gamma_f$). The stability of the underlying deterministic model is illustrated in Fig. 4.2, showing that the fundamental price is stable for short horizons, but unstable for longer horizons. With the same set of parameters in Fig. 4.2, we report the profitability of the different trading in Fig. 5.5. It shows clearly that, for short horizon $\tau = 0.5$, the fundamental and momentum trading strategies are profitable, but not the contrarians, as illustrated in Figs. 5.5 (a) and (b). However, for longer horizon $\tau = 3$, the fundamental and contrarian strategies are profitable, but not the momentum traders, see Figs. 5.5 (c) and (d).

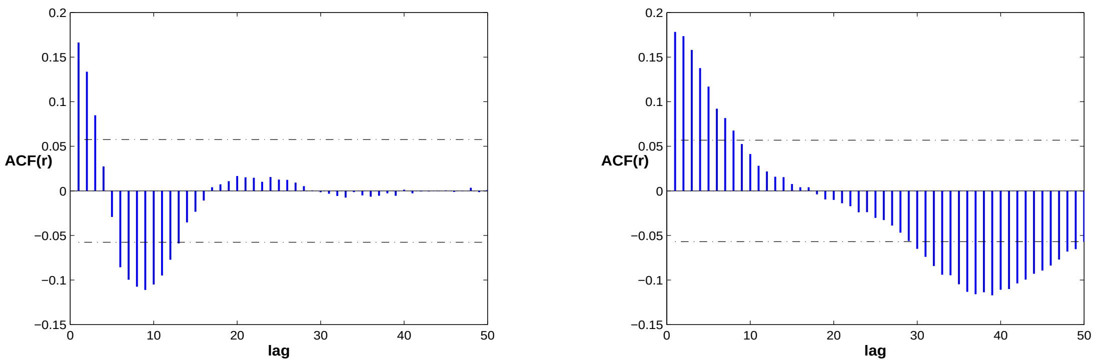


(a) Time series for $\tau = 0.5$

(b) Time series for $\tau = 3$

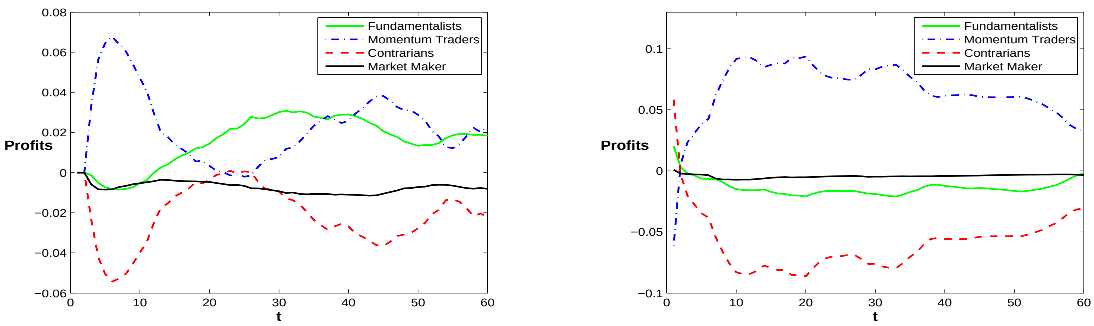
FIGURE 5.6. Time series of price $P(t)$ and price trend $u(t)$ for (a) $\tau = 0.5$ and (b) $\tau = 3$.

To explore the profit opportunity of the momentum trading with different time horizons, we plot the time series of the price and price trend in Fig. 5.6 (a) for $\tau = 0.5$ and Fig. 5.6 (b) for $\tau = 3$, based on the same simulation in Fig. 5.5 (b) and (d), respectively. There are two interesting observations. (i) For short horizon $\tau = 0.5$, the market price fluctuates due to the unstable steady state of the underlying deterministic system. When the market price increases, the price trend follows the market price closely and increases too, as illustrated in Fig. 5.6 (a). This implies that, with short holding period, the momentum trading strategy is profitable by taking long positions. Similarly, when the market price declines, the price trend follows. Hence the momentum trading is profitable by taking short positions. Therefore, the momentum trading is profitable (except for the starting periods of sudden changes in the price tendency). (ii) For longer horizon $\tau = 3$, the market price fluctuates widely due to the unstable fundamental value of the underlying deterministic system. The relation between market price and price trend is similar to the case for the short horizon, as illustrated in Fig. 5.6 (b). However, a longer horizon makes the price trend less sensitive to the changes in price. Also, since the holding period is also longer, the momentum trading mis-matches the profitability opportunity. For example, when the market price reaches a peak at $t \approx 50$ (months), which is higher than the trend, the momentum traders take a long position in the stock. After holding the stocks for 3 years, they sell at a much lower price at $t \approx 86$, implying a loss from the momentum strategy. This illustrates that, with longer horizon, the momentum trading is not profitable.



(a) $\tau = 0.5$ (b) $\tau = 3$

FIGURE 5.7. The average ACs of market return based on 1000 simulations for market state 3 with (a) $\tau = 0.5$ and (b) $\tau = 3$.



(a) Average spot profits (b) Average accumulated profits

FIGURE 5.8. (a) The average spot profits based on 1,000 simulations; (b) the average accumulated profits based on a typical simulation for market state 3 with 3 years horizon and 0.5 year holding period.

To provide further insight into the profitability mechanism, we calculate the return ACs and present the results in Fig. 5.7. It shows clearly the market under-reaction in short run and overreaction in long run, characterized by significantly positive ACs for short lags and negative ACs for long lags for both short and longer horizons. With the short horizon and holding period, the momentum trading is profitable due to the under-reaction in short-run (Fig. 5.7 (a)). However with the long time horizon, the momentum trading is no longer profitable for long holding period due to the overreaction in long-run (Fig. 5.7 (b)), although it can be profitable with short holding period due to the under-reaction illustrated in Fig. 5.7 (b), which is verified in Fig. 5.8 with 3 years horizon and 6 month holding period. This result is consistent with Lou and Polk (2013).

Table 5.1. The annualized percentage (log) excess returns of momentum strategy (2.1) for the time series generated from the model in market state 3 with horizon (m) and holding (n) from 1 to 60 months period. Note: *, **, *** denote the significance at 10%, 5% and 1% levels, respectively.

$m \setminus n$	1	3	6	12	24	36	48	60
1	1.23	3.30	3.05**	3.09***	1.71**	0.44	-0.17	-0.42
3	2.78	3.21	4.73**	4.16**	2.68**	0.25	-0.42	-0.55
6	2.48	3.85	5.74**	5.09**	2.29	-0.37	-1.41	-1.15
12	6.89**	7.87**	8.12***	5.91**	1.48	-1.78	-2.50*	-1.99
24	9.92***	9.89***	7.40**	2.89	-2.71	-4.82**	-3.82*	-2.14
36	8.84**	5.43	2.16	-1.79	-6.22**	-7.57***	-5.44**	-3.34*
48	5.00	2.41	-0.01	-3.66	-7.76***	-8.52***	-5.91***	-3.57*
60	2.50	-0.03	-2.01	-5.25	-9.00***	-9.19***	-6.15***	-3.63*

It would be interesting to see if the model is able to replicate the time series momentum profit explored for the S&P 500 in Section 2 based on the momentum strategies (2.1) and (2.3). Table 5.1 reports the annual excess returns of various momentum trading strategies based on (2.1) investing in the model generated data in market state 3 for time horizon and holding period from 1 to 60 months. Fig. 5.9 reports the corresponding t-statistic of the average excess return of the momentum strategies for time horizon from 1 to 60 months periods and holding period equals to horizon, 1 month and 6 month periods respectively. Similar results based on trading

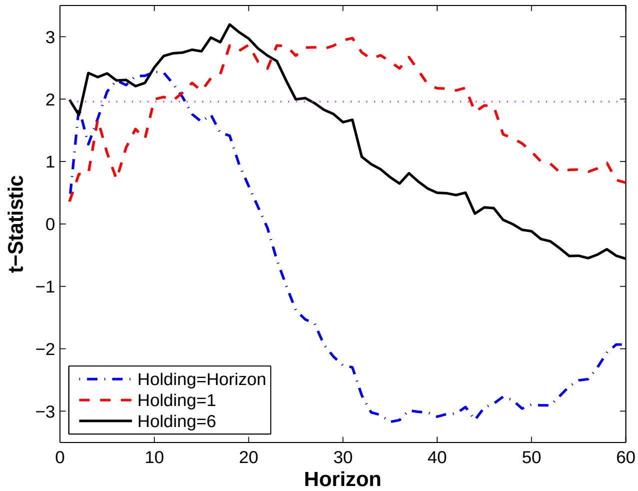


FIGURE 5.9. The t-Statistic of the average excess return of the momentum strategy (2.1) investing in the model generated data in market state 3 for time horizon from 1 to 60 months periods and holding equal to horizon ($n = m$), 1 month ($n = 1$) and 6 month periods ($n = 6$) respectively.

strategy (2.3) are reported in Table A.2 and Fig. A.2 in Appendix A. We see that both the profit and t-statistic patterns generated from the model are very similar to the S&P 500 reported in Section 2. The results are consistent with Moskowitz et al. (2012) who find that the time series momentum strategy with 12 months horizon and 1 month holding is the most profitable among others.

To complete this section, we add the following remarks. (i) The analysis of this paper focuses on the same time horizon and holding period. An extension to different time horizon and holding period is presented in Appendix C. (ii) Simulations (not reported here) show that the level of profitability of momentum (contrarian) strategy is positively (negatively) related to β_m and negatively (positively) related to β_c. Also, the level of profitability of both momentum and contrarian strategies is positive related to the price adjustment speed μ. (iii) The time horizon τ can affect the profitability greatly. Recall that the stability of the system depends on $\gamma_i = \mu\alpha_i\beta_i$ $(i = f, m, c)$ and τ completely and the profitability is closely related to the market states. (iv) When investors switch their trading strategies based on some fitness functions, we extend the model in Appendices D and E and show that the profits/losses can be enhanced due to the switching among different trading strategies.

6. Conclusion

Based on market underreaction and overreaction hypotheses, momentum and contrarian strategies are widely used by financial market practitioners and their profitability has been extensively investigated by academics. However, most behavioral models do not specify the time horizon, which plays a crucial role in the performance of momentum and contrarian strategies. Following the recent development in the heterogeneous agent models literature, this paper proposes a continuous-time heterogeneous agent model of investor behavior consisting of fundamental, contrarian, and momentum strategies. The underlying stochastic delay integro-differential equation of the model provides a unified approach to deal with different time horizons of momentum and contrarian strategies. By examining their impact on market stability explicitly and analyzing the profitability numerically, this paper examines the profitability of the time series momentum trading strategies. We show that the profitability is closely related to the market states defined by the stability of the underlying deterministic model. In particular, we show that, in market state 3 where the momentum traders dominate the market, the momentum strategy is profitable when the time horizon is short and unprofitable when the time horizon is long. In other market states, the strategy is not profitable for any time horizon. We also provide some explanation to the profitability mechanism through autocorrelation patterns and the under-reaction and overreaction hypotheses. In addition, we show that the momentum strategy works in the stock index.

Although the model proposed in this paper is very simple, it provides an insight into the time series momentum documented in recent empirical literature. As we discussed in the introduction, the time series momentum plays a very important role in explaining cross-sectional momentum, which had been widely researched in the literature. Motivated by the results obtained in this paper, one can extend the market of one risky asset to one with many risky assets so that the profitability of portfolios constructed from momentum and contrarian strategies can be examined. We would expect the same mechanism can be used to explain cross-sectional momentum. In addition, it has been shown that volatility can affect the autocorrelations in returns and hence affect profitability and even trading volume. This could be examined by using the setup in this paper. We leave these for future research.

Appendix A. Time Series Momentum Profit

Table A.1. The annualized percentage (log) excess returns of the momentum strategies (2.3) for the S&P 500 with horizon (m) and holding (n) from 1 to 60 months period. Note: *, **, *** denote the significance at 10%, 5% and 1% levels, respectively.

$m \setminus n$	1	3	6	12	24	36	48	60
1	1.64	3.61**	1.61	3.00***	2.01**	2.01**	1.55*	1.12
3	1.15	2.15	2.88	3.34**	2.67*	1.88	1.57	1.25
6	4.21	4.39*	5.47**	4.67**	2.74	1.77	1.67	1.37
12	9.24***	7.81***	6.72**	5.22**	2.83	1.82	1.70	1.82
24	7.20**	6.92**	5.47*	3.81	2.28	1.68	8 1.83	2.68
36	3.98	4.50	2.80	1.58	0.72	0.93	1.55	2.97
48	1.76	0.14	-1.19	-1.94	-1.59	-0.43	1.30	2.51
60	-2.55	-4.24	-4.84	-3.86	-2.11	0.07	1.80	2.74

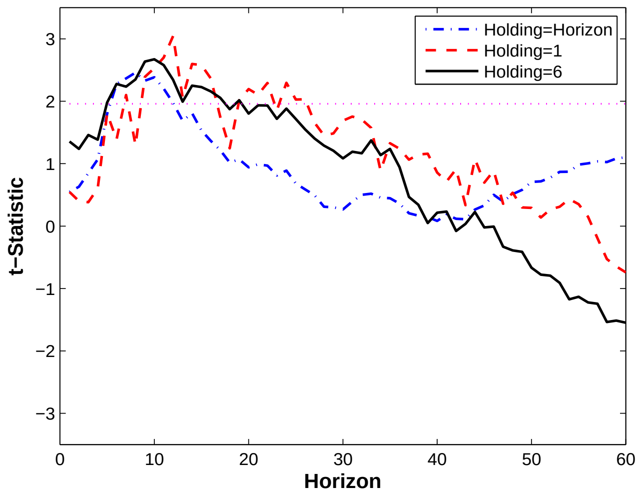


Figure A.1. The t-Statistic of the average excess return of the momentum strategies (2.3) investing the S&P 500 for time horizon from 1 to 60 months periods and holding equal to horizon ($n = m$), 1 month ($n = 1$) and 6 month periods ($n = 6$) respectively.

TABLE A.2. The annualized percentage (log) excess returns of the momentum strategies (2.3) for the time series generated from the model in market state 3 with horizon (m) and holding (n) from 1 to 60 months period. Note: *, **, *** denote the significance at 10%, 5% and 1% levels, respectively.

$m \setminus n$	1	3	6	12	24	36	48	60
1	1.81	3.53	3.28**	3.20***	1.78**	0.44	-0.17	-0.42
3	3.38	3.28	4.93**	4.56***	2.46*	0.33	-0.28	-0.32
6	6.35*	6.57**	6.02**	5.46**	1.75	-0.72	-1.63	-1.20
12	6.86**	7.29**	7.09**	4.74*	0.58	-2.50	-2.95**	-2.64**
24	5.22	5.28	3.94	0.82	-3.02	-4.85**	-3.97*	-2.46
36	1.32	0.24	-1.61	-4.91	-7.96***	-7.81***	-5.39**	-3.23
48	-1.46	-2.69	-4.30	-6.57**	-8.66***	-7.61***	-4.99**	-3.28
60	-2.42	-5.80	-6.47*	-6.31*	-6.97**	-5.83*	-4.14	-2.71

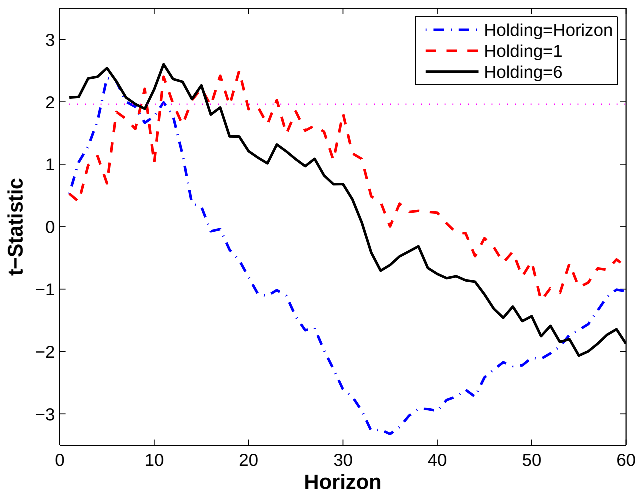


FIGURE A.2. The t-Statistic of the average excess return of the momentum strategy (2.3) investing in the model generated data in market state 3 for time horizon from 1 to 60 months periods and holding equal to horizon ($n = m$), 1 month ($n = 1$) and 6 month periods ($n = 6$) respectively.

Appendix B. Proofs and Remarks of the Deterministic Model

The characteristic equation of the system (4.1) at the fundamental steady state $P = \bar{F}$ is given by[22]

$$\lambda + \gamma_f - \gamma_m + \gamma_c + \frac{\gamma_m}{\lambda\tau_m}(1 - e^{-\lambda\tau_m}) - \frac{\gamma_c}{\lambda\tau_c}(1 - e^{-\lambda\tau_c}) = 0. \tag{B.1}$$

For delay integro-differential equation, the eigenvalue analysis can be complicated.

B.1. **Proof of Proposition 4.1.** The characteristic equation (B.1) reduces to

$$\lambda + \gamma_f + \gamma_c - \frac{\gamma_c}{\lambda\tau_c}(1 - e^{-\lambda\tau_c}) = 0, \tag{B.2}$$

which has no zero eigenvalue. The root of (B.2) has negative real part $-\gamma_f$ when $\tau_c \to 0$. Let $\lambda = i\omega(\omega > 0)$ be a root of Eq. (B.2). Substituting it into Eq. (B.2) and separating the real and imaginary parts yield

$$\omega^2\tau_c - \gamma_c(\cos\omega\tau_c - 1) = 0, \qquad \omega\tau_c(\gamma_f + \gamma_c) - \gamma_c \sin\omega\tau_c = 0,$$

which lead to

$$\omega^2\tau_c^2 + 2\tau_c\gamma_c + \tau_c^2(\gamma_f + \gamma_c)^2 = 0, \tag{B.3}$$

However equation (B.3) cannot be true for $\tau_c > 0$, hence $\lambda \neq i\omega$.

It is known that, as τ_c varies, the sum of the multiplicities of roots of Eq. (B.2) in the open right half-plane can change only if a root appears on or crosses the imaginary axis (see Ruan and Wei 2003 and Li and Wei 2009). Therefore, all roots of Eq. (B.2) have negative real parts for all $\tau_c \geq 0$. This implies the local stability of the system (4.2).

B.2. **Proof of Proposition 4.2.** The characteristic equation (B.1) collapses to

$$\lambda + \gamma_f - \gamma_m + \frac{\gamma_m}{\lambda\tau_m}(1 - e^{-\lambda\tau_m}) = 0, \tag{B.4}$$

which has no zero eigenvalue. Substituting $\lambda = i\omega(\omega > 0)$ into Eq. (B.4) and separating the real and imaginary parts yield

$$\omega^2\tau_m + \gamma_m(\cos\omega\tau_m - 1) = 0, \qquad \omega\tau_m(\gamma_f - \gamma_m) + \gamma_m \sin\omega\tau_m = 0. \tag{B.5}$$

[22]It is known (see Hale 1997) that the stability is characterised by the eigenvalues of the characteristic equation of the system at the steady state.

Let $a = \max\{-\sin x/x; x > 0\}(\approx 0.2172)$. When $\gamma_m < \gamma_f/(1+a)$, the two functions $y_1 := \frac{\gamma_m-\gamma_f}{\gamma_m}x$ and $y_2 := \sin x$ have no intersection for $x > 0$, hence the second equation in (B.5) cannot hold and Eq. (B.4) has no pure imaginary root. Correspondingly, Eq. (B.4) has no root appearing on the imaginary axis. In addition, Eq. (B.4) has only one negative eigenvalue when $\tau_m \to 0$. Therefore, all roots of Eq. (B.4) have negative real parts for all $\tau_m \geq 0$ when $\gamma_m < \gamma_f/(1+a)$, which leads to the local stability of the system (4.3).

Next, we consider the case of $\gamma_m \geq \gamma_f/(1+a)$. If follows from Eq. (B.5) that

$$\omega^2 + (\gamma_f - \gamma_m)^2 - \frac{2\gamma_m}{\tau_m} = 0. \tag{B.6}$$

When $\tau_m > \tau^*_{m,1} := \frac{2\gamma_m}{(\gamma_f-\gamma_m)^2}$, Eq. (B.6) has no solution, implying that $\lambda = i\omega$ is not an eigenvalue. Hence there is no stability switching for $\tau_m > \tau^*_{m,1}$. Substituting $\lambda = \Re\{\lambda\} + i\Im\{\lambda\}$ into Eq. (B.4) and separating the real and imaginary parts yield

$$\begin{aligned}
&\Re^2\{\lambda\} + \Im^2\{\lambda\} + (\gamma_f - \gamma_m)\Re\{\lambda\} + \frac{\gamma_m}{\tau_m}(1 - e^{-\Re\{\lambda\}\tau_m}\cos\Im\{\lambda\}) = 0,\\
&2\Re\{\lambda\}\Im\{\lambda\} + (\gamma_f - \gamma_m)\Im\{\lambda\} + \frac{\gamma_m}{\tau_m}e^{-\Re\{\lambda\}\tau_m}\sin\Im\{\lambda\}\tau_m = 0.
\end{aligned} \tag{B.7}$$

When $\tau_m \to \infty$, if there exists a root λ with $\Re\{\lambda\} > 0$, then (B.7) reduces to

$$\begin{aligned}
&\Re^2\{\lambda\} + \Im^2\{\lambda\} + (\gamma_f - \gamma_m)\Re\{\lambda\} = 0,\\
&2\Re\{\lambda\}\Im\{\lambda\} + (\gamma_f - \gamma_m)\Im\{\lambda\} = 0,
\end{aligned} \tag{B.8}$$

which hold only when $\gamma_m > \gamma_f$. Note that (B.8) cannot hold with $\Re\{\lambda\} = 0$ since (B.4) has no zero eigenvalue. Therefore, (B.4) has at least one root with positive real part for $\gamma_m > \gamma_f$ and all roots with negative real parts for $\gamma_m \leq \gamma_f$ when $\tau_m \to \infty$. So the fundamental steady state of system (4.3) is asymptotically stable for $\gamma_m \leq \gamma_f$ and unstable for $\gamma_m > \gamma_f$ when $\tau_m > \tau^*_{m,1}$. However, if $\tau_m < \tau^*_{m,1}$, by substituting Eq. (B.6) into the first equation of (B.5) we have

$$\frac{\tau_m}{\gamma_m}(\gamma_f - \gamma_m)^2 - \cos\left[\sqrt{2\gamma_m\tau_m - (\gamma_f - \gamma_m)^2\tau_m^2}\right] - 1 = 0. \tag{B.9}$$

Let $\tau^*_{m,l}$ be the minimum positive root of (B.9). Then all the eigenvalues of Eq. (B.4) have negative real parts when $0 \leq \tau_m < \tau^*_{m,l}$ and Eq. (B.4) has a pair of pure imaginary roots when $\tau_m = \tau^*_{m,l}$. In addition, it can be verified that $\Delta(\tau^*_{m,l}) := \frac{d\Re\{\lambda(\tau_m)\}}{d\tau_m}\,|_{\tau_m=\tau^*_{m,l}} \neq 0$. So $P = \bar{F}$ undergoes a Hopf bifurcation at $\tau_m = \tau^*_{m,l}$.

Furthermore, the stability switching happens only once when $\gamma_m > \gamma_f$ and only twice when $\gamma_f/(1+a) \le \gamma_m \le \gamma_f$. In fact, the stability switching[23] at a bifurcation value τ_m^* depends on the sign of $\Delta(\tau_m^*) := \frac{d\Re\{\lambda(\tau_m)\}}{d\tau_m}\ |_{\tau_m=\tau_m^*}$. An increase in τ_m near the bifurcation value τ_m^* may result in a switching of the steady state from stable to unstable when $\Delta(\tau_m^*) > 0$ and from unstable to stable when $\Delta(\tau_m^*) < 0$. For a Hopf bifurcation value τ_m^*, we have $\Delta(\tau_m^*) := \frac{d\Re\{\lambda\}}{d\tau_m}\ |_{\tau_m=\tau_m^*} = \frac{\Im^2\{\lambda\}\left(2\gamma_m-\gamma_f-\tau_m^*(\gamma_f-\gamma_m)^2\right)}{\tau_m^*\left(\left(\gamma_f-\gamma_m+\gamma_m\cos\Im\{\lambda\}\tau_m^*\right)^2+\left(2\Im\{\lambda\}-\gamma_m\sin\Im\{\lambda\}\tau_m^*\right)^2\right)}$. Let $\tau_{m,2}^* := \frac{2\gamma_m-\gamma_f}{(\gamma_f-\gamma_m)^2}\ (< \tau_{m,1}^*)$. Then $sign(\Delta(\tau_m^*)) > 0$ for $\tau_m^* < \tau_{m,2}^*$ and $sign(\Delta(\tau_m^*)) < 0$ for $\tau_m^* > \tau_{m,2}^*$, implying that an unstable fundamental steady state cannot become stable as τ_m varies within $(\tau_{m,l}^*, \tau_{m,2}^*)$ and a stable fundamental steady state cannot become unstable as τ_m varies within $(\tau_{m,2}^*, \infty)$. When $\gamma_m > \gamma_f$, it has been proved that $P = \bar{F}$ is stable for $\tau_m < \tau_{m,l}^*$ and unstable for either τ_m in some right neighborhood of $\tau_{m,l}^*$ or $\tau_m > \tau_{m,1}^*$. Hence the stability switches only once at $\tau_{m,l}^*$, implying that $P = \bar{F}$ is unstable for $\tau_m > \tau_{m,l}^*$. Let $\tau_{m,h}^*$ be the largest of the roots of Eq. (B.9) that are less than $\tau_{m,1}^*$. When $\gamma_f/(1+a) \le \gamma_m \le \gamma_f$, $P = \bar{F}$ is stable for either $\tau_m < \tau_{m,l}^*$ or $\tau_m > \tau_{m,1}^*$. Due to $\tau_{m,h}^*$ is a Hopf bifurcation and $\Delta(\tau_{m,h}^*) < 0$, $P = \bar{F}$ is unstable for τ in some left neighborhood of $\tau_{m,h}^*$. Hence the stability switches only twice at $\tau_{m,l}^*$ and $\tau_{m,h}^*$, implying that $P = \bar{F}$ is unstable for $\tau_{m,l}^* < \tau_m < \tau_{m,h}^*$ and stable for either $\tau_m < \tau_{m,l}^*$ or $\tau_m > \tau_{m,h}^*$. This completes the proof.

B.3. **Proof of Proposition 4.3.** The characteristic equation (B.1) becomes

$$\lambda + \gamma_f - \gamma_m + \gamma_c + \frac{\gamma_m - \gamma_c}{\lambda\tau}(1 - e^{-\lambda\tau}) = 0. \tag{B.10}$$

Substituting $\lambda = i\omega(\omega > 0)$ into Eq. (B.10) and separating the real and imaginary parts yield

$$\begin{aligned} &\omega^2\tau + (\gamma_m - \gamma_c)(\cos\omega\tau - 1) = 0, \\ &\omega\tau(\gamma_f - \gamma_m + \gamma_c) + (\gamma_m - \gamma_c)\sin\omega\tau = 0. \end{aligned} \tag{B.11}$$

[23]For simplicity, we arbitrarily assume that the bifurcating periodic solutions are stable and can be globally extended, which can be observed in the numerical simulations. We refer to He et al (2009) for the computation of stability and the proof of global existence for the periodic solutions.

We first consider the case of $\gamma_m \le \gamma_c$. In this case, the first equation of (B.11) cannot hold, meaning that equation (B.10) has no pure imaginary root. Note that (B.10) has no zero eigenvalue and the root of (B.10) is negative when $\tau \to 0$. Hence all the roots of (B.10) have negative real parts for $\tau \ge 0$, leading to the local stability of the steady state.

Second, we consider the case of $\gamma_m > \gamma_c$. In this case, if $\gamma_c < \gamma_m < \gamma_c + \gamma_f/(1+a)$ then the second equation of (B.11) cannot hold, implying that $\lambda \neq i\omega$. However, if $\gamma_m \ge \gamma_c + \gamma_f/(1+a)$, similar discussion to Appendix A2, we have the local stability for $\gamma_m \le \gamma_f + \gamma_c$ and instability for $\gamma_m > \gamma_f + \gamma_c$ when $\tau > \tau_1^* := \frac{2(\gamma_m - \gamma_c)}{(\gamma_f - \gamma_m + \gamma_c)^2}$. When $\tau < \tau_l^*$, where τ_l^* is the minimum positive roof of the following equation $\frac{\tau}{\gamma_m - \gamma_c}(\gamma_f - \gamma_m + \gamma_c)^2 - \cos\left[\sqrt{2(\gamma_m - \gamma_c)\tau - (\gamma_f - \gamma_m + \gamma_c)^2\tau^2}\right] - 1 = 0$, all the eigenvalues of Eq. (B.10) have negative real parts. When $\tau = \tau_l^*$, Eq. (B.10) has a pair of purely imaginary roots.

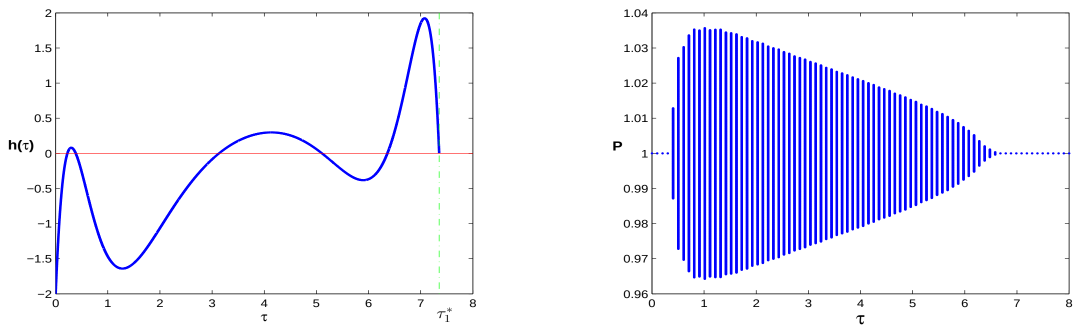
Therefore, the stability switching happens only once when $\gamma_m > \gamma_c + \gamma_f$ and only twice when $\gamma_c + \gamma_f/(1+a) \le \gamma_m < \gamma_c + \gamma_f$, and consequently completes the proof.

B.4. **Some Remarks on Proposition 4.3.** These remarks provide some properties on the nature of bifurcations related to Proposition 4.3, including the number of bifurcations, stability switching and the dependence of the bifurcation values on the parameters of the model.

First, it follows from the proof in Appendix A3 that all the roots of $h(\tau)$ except $\tau = \tau_1^*$ are Hopf bifurcation values. Note that $h(\tau_1^*) = 0$. However, we know that $\omega = 0$ if and only if $\tau = \tau_1^*$. Hence τ_1^* is not a bifurcation value.

Second, when $\gamma_c + \gamma_f/(1+a) \le \gamma_m < \gamma_c + \gamma_f$, the number of bifurcations defined by $h(\tau^*) = 0$ is odd. Indeed, it follows from $h'(\tau_1^*) = \frac{-\gamma_f[2(\gamma_m - \gamma_c) - \gamma_f]}{\gamma_m - \gamma_c} < 0$ that $h(\tau_1^* - 0) > 0$. Note that $h(0) < 0$, $h(\tau)$ is continuous and $y = h(\tau)$ is not tangent to $y = 0$ when $\mid \frac{\gamma_f}{\gamma_m - \gamma_c} - 1 \mid \neq \frac{2}{(1+2k)\pi}$, $k = 0, 1, 2, \cdots$. Therefore if $\mid \frac{\gamma_f}{\gamma_m - \gamma_c} - 1 \mid \neq \frac{2}{(1+2k)\pi}$, then $h(\tau)$ has odd roots when $\tau \in (0, \tau_1^*)$, that is, the number of the Hopf bifurcation that the fundamental steady state price $P = \bar{F}$ undergoes in the interval $(0, \tau_1^*)$ must be odd. Furthermore, the number of the Hopf bifurcation that the fundamental steady state price $P = \bar{F}$ can undergo in the interval $(0, \tau_1^*)$ increases when $\gamma_f + \gamma_c \to \gamma_m$. In fact, we have $h'(\tau) =$

$\frac{\gamma_m-\gamma_c-(\gamma_f-\gamma_m+\gamma_c)^2\tau}{\sqrt{2(\gamma_m-\gamma_c)\tau-(\gamma_f-\gamma_m+\gamma_c)^2\tau^2}}\sin\sqrt{2(\gamma_m-\gamma_c)\tau-(\gamma_f-\gamma_m+\gamma_c)^2\tau^2}$. When $\gamma_f+\gamma_c\to\gamma_m$, $\max_\tau\left\{\sqrt{2(\gamma_m-\gamma_c)\tau-(\gamma_f-\gamma_m+\gamma_c)^2\tau^2}\right\}\to\infty$, hence the sign of $h'(\tau)$ can change many times. This implies that the number of roots of $h(\tau)$ increases in this case. Despite the facts that the number of bifurcations defined by $h(\tau^*)=0$ is odd and the number of the Hopf bifurcation increases when $\gamma_f+\gamma_c\to\gamma_m$, Proposition 4.3 shows that the stability switches only twice. This is verified numerically in Fig. 4.1 and Fig. B.1. In Fig. 4.1 (a), there are three Hopf bifurcation values, while in Fig. B.1 (a), there are five bifurcation values. However, the stability switches only twice in Fig. 4.1 (b) and Fig. B.1 (b).



(a) The function $h(\tau)$ (b) Price bifurcation

FIGURE B.1. (a) The function $h(\tau)$; (b) the bifurcation diagram of the market price. Here $\gamma_f=20$, $\gamma_m=22.8$ and $\gamma_c=5$.

Finally, the first bifurcation value τ_l^* depends on the population fractions, the extrapolation rates and the speed of the price adjustment. It increases as γ_f or γ_c increase, or γ_m decreases, however it is always bounded away from zero and infinity. In fact, when $\gamma_m\geq\gamma_c+\gamma_f/(1+a)$, let $x=\sqrt{2(\gamma_m-\gamma_c)\tau-(\gamma_f-\gamma_m+\gamma_c)^2\tau^2}$. Solving τ then leads to $\tau(x)=\frac{\gamma_m-\gamma_c}{(\gamma_f-\gamma_m+\gamma_c)^2}-\sqrt{\frac{(\gamma_m-\gamma_c)^2}{(\gamma_f-\gamma_m+\gamma_c)^4}-\frac{x^2}{(\gamma_f-\gamma_m+\gamma_c)^2}}$. Note that $\tau=0$ implies $x=0$ and $x(\tau)$ is an increasing function of τ. Hence the first bifurcation value τ_l^* corresponds the minimum positive root x_l^* of the following function $h(x)=-\sqrt{1-\frac{(\gamma_f-\gamma_m+\gamma_c)^2x^2}{(\gamma_m-\gamma_c)^2}}-\cos x=0$. It can be shown that $\frac{\pi}{2}<x_l^*<\pi$ and while $\mid\frac{\gamma_f-\gamma_m+\gamma_c}{\gamma_m-\gamma_c}\mid$ decreases, x_l^* increases, implying that τ_l^* increases. Therefore, when $\gamma_c+\gamma_f/(1+a)\leq\gamma_m\leq\gamma_c+\gamma_f$, the first bifurcation value τ_l^* increases as either γ_f or γ_c decrease, or γ_m increases. When $\gamma_m>\gamma_c+\gamma_f$, the first bifurcation value τ_l^* increases as either γ_f or γ_c increase, or γ_m decreases. Furthermore, let $x_{min} = \{\sqrt{1-ax^2} + \cos x = 0 \mid \frac{\pi}{2} < x < \pi\}(\approx 2.5536)$. Because of $\frac{\gamma_f - \gamma_m + \gamma_c}{\gamma_m - \gamma_c} < a$, we have $x_{min} < x_l^* < \pi$, implying $\tau(x_{min}) < \tau_l^* < \tau(\pi)$, where

$$\tau(x_{min}) = \frac{\gamma_m - \gamma_c}{(\gamma_f - \gamma_m + \gamma_c)^2} - \sqrt{\frac{(\gamma_m - \gamma_c)^2}{(\gamma_f - \gamma_m + \gamma_c)^4} - \frac{x_{min}^2}{(\gamma_f - \gamma_m + \gamma_c)^2}},$$

$$\tau(\pi) = \frac{\gamma_m - \gamma_c}{(\gamma_f - \gamma_m + \gamma_c)^2} - \sqrt{\frac{(\gamma_m - \gamma_c)^2}{(\gamma_f - \gamma_m + \gamma_c)^4} - \frac{\pi^2}{(\gamma_f - \gamma_m + \gamma_c)^2}}.$$

Appendix C. The General Case with Any Positive τ_m and τ_c

In the general case, the market stability of the system (4.1) can be characterized by the following proposition.

Proposition C.1. *The fundamental steady state price of the system* (4.1) *is*

(i) *asymptotically stable for all* $\tau_m, \tau_c \geq 0$ *when* $\gamma_m < \gamma_c + \frac{\gamma_f}{1+a}$*;*

(ii) *asymptotically stable for either* $0 \leq \tau_m, \tau_c < \tau_l^*$ *or* $\tau_m, \tau_c > \tau_h^*$ *when* $\gamma_c + \frac{\gamma_f}{1+a} \leq \gamma_m \leq \gamma_c + \gamma_f$*; and*

(iii) *asymptotically stable for* $\tau_m, \tau_c < \tau_l^*$ *when* $\gamma_m > \gamma_c + \gamma_f$.

Proof. We first consider the case of $\gamma_m \leq \gamma_c + \gamma_f/(1+a)$.[24] Suppose there exist $\tau_m^{(1)} \geq 0$ and $\tau_c^{(1)} \geq 0$ such that the fundamental steady state $P = \bar{F}$ of system (4.1) is unstable for the delay pair $(\tau_m, \tau_c) = \big(\tau_m^{(1)}, \tau_c^{(1)}\big)$. Without loss of generality, assume $\tau_m^{(1)} > \tau_c^{(1)}$. Proposition 4.3 implies $P = \bar{F}$ is stable when $(\tau_m, \tau_c) = \big(\tau_c^{(1)}, \tau_c^{(1)}\big)$. If $P = \bar{F}$ is stable when $(\tau_m, \tau_c) = \big(\frac{\tau_m^{(1)}+\tau_c^{(1)}}{2}, \tau_c^{(1)}\big)$, then let $(\tau_m^{(2)}, \tau_c^{(2)}) = \big(\tau_m^{(1)}, \frac{\tau_m^{(1)}+\tau_c^{(1)}}{2}\big)$. Otherwise, let $(\tau_m^{(2)}, \tau_c^{(2)}) = \big(\frac{\tau_m^{(1)}+\tau_c^{(1)}}{2}, \tau_c^{(1)}\big)$. So $P = \bar{F}$ is stable when $(\tau_m, \tau_c) = \big(\tau_c^{(2)}, \tau_c^{(2)}\big)$ and unstable when $(\tau_m, \tau_c) = \big(\tau_m^{(2)}, \tau_c^{(2)}\big)$. Repeating the above process, we have a sequence of nested closed intervals $[\tau_c^{(1)}, \tau_m^{(1)}] \supset [\tau_c^{(2)}, \tau_m^{(2)}] \supset [\tau_c^{(3)}, \tau_m^{(3)}] \supset \cdots$ and $\lim_{n\to\infty}(\tau_m^{(n)} - \tau_c^{(n)}) = 0$.[25] By the nested interval theorem, there exists a $\tau^{(\infty)} \in [\tau_c^{(n)}, \tau_m^{(n)}]$ such that $\tau_m^{(n)} \to \tau^{(\infty)}$ as $n \to \infty$. So $P = \bar{F}$ is unstable when $(\tau_m, \tau_c) = \big(\tau^{(\infty)}, \tau^{(\infty)}\big)$, which contradicts Proposition 4.3. Therefore, $P = \bar{F}$ is stable for all $\tau_m, \tau_c \geq 0$ when $\gamma_m \leq \gamma_c + \gamma_f/(1+a)$.

Similarly, items (ii) and (iii) can be proved.

□

Simulations (not reported here) show that if momentum traders do not dominate the market ($\gamma_m \leq \gamma_c + \gamma_f$), then momentum traders always lose no matter how long time horizons are used, and contrarians can make profits when τ_m and τ_c are large,

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[24] Assume arbitrarily again that the stable periodic solutions bifurcating from the Hopf bifurcation can be extended with respect to the time horizons.

[25] If $\frac{\tau_m^{(n)}+\tau_c^{(n)}}{2}$ is a bifurcation value, then by the definition of bifurcation, we can choose a proper value close to it as $\tau_m^{(n+1)}$ (or $\tau_c^{(n+1)}$) such that $P = \bar{F}$ is stable when $(\tau_c, \tau_m) = \big(\tau_c^{(n+1)}, \tau_c^{(n+1)}\big)$ and unstable when $(\tau_c, \tau_m) = \big(\tau_c^{(n+1)}, \tau_m^{(n+1)}\big)$.

and lose when τ_c is small and τ_m is large. If momentum traders dominate the market ($\gamma_m > \gamma_c + \gamma_f$), then, for any given $\tau_c > 0$, momentum strategy is profitable when τ_m is small and unprofitable when τ_m is big; and the profitabilities for contrarians are opposite for any given $\tau_m > 0$. These results are consistent with the analysis in Section 5.

Appendix D. Population Evolution between Momentum and Contrarian Traders

To focus on the impact of time horizons, we consider a special case of fixed market fractions in previous sections, which have shown that the time horizons and the joint impact of different traders play very important roles in the stability of market price and profitability. In this section we investigate the impact of population evolution on the market price and profitability. The switching mechanism follows the modelling in He and Li (2012).

Let $q_f(t)$, $q_m(t)$ and $q_c(t)$ be the market fractions of fundamentalists, momentum traders and contrarians respectively. We first suppose there is no switching between fundamentalists and chartists and choose constant market fraction of fundamentalists $q_f(t) = \alpha_f$. Assume the market fractions of the two kinds of chartists have a fixed component and a time varying component. Let m_m and m_c be the fixed proportions of momentum and contrarian traders who stay with their strategy over time, respectively. Then $1 - \alpha_f - m_m - m_c$ is the proportion of chartists who may switch from one strategy to the other: we denote them as *switching or adaptively rational* chartists. Among switching chartists, denote by $n_m(t)$ and $n_c(t) = 1 - n_m(t)$ the proportions of momentum and contrarian traders at time t, respectively. Therefore, $q_m(t) = m_m + (1 - \alpha_f - m_m - m_c)n_m(t)$ and $q_c(t) = m_c + (1 - \alpha_f - m_m - m_c)n_c(t)$. The net profits of the momentum and contrarian strategies over a short time interval $[t - dt, t]$ can be measured respectively by $\pi_m(t)dt = D_m(t)dP(t) - C_m dt$ and $\pi_c(t)dt = D_c(t)dP(t) - C_c dt$, where $C_m,\ C_c \geq 0$ are constant costs of the strategies per unit time. To measure performance of the strategies, we introduce a cumulated profits by $U_i(t) = \eta \int_{-\infty}^{t} e^{-\eta(t-s)}\pi_i(s)ds$, $i = m, c$, where $\eta > 0$ represents a decay parameter of the historical profits. That is the performance is defined by a cumulated net profit of the strategy decaying exponentially over all past time.[26] Consequently, $dU_i(t) = \eta\big[\pi_i(t) - U_i(t)\big]dt,\ i = m, c$. Following Hofbauer and Sigmund (1998) (Chapter 7), the evolution dynamics of the market populations are governed by $dn_i(t) = \beta n_i(t)[dU_i(t) - d\bar{U}(t)]$, $i = m, c$, where $d\bar{U}(t) = n_m(t)dU_m(t) + n_c(t)dU_c(t)$ is the average performance of the two strategies and the switching intensity $\beta > 0$ is a constant, measuring the intensity of choice. In particular, if $\beta = 0$, there is no switching between strategies, while for $\beta \to \infty$ all agents switch immediately to the better strategy.

To sum up, by letting $U(t) = U_m(t) - U_c(t)$, $\pi(t) = \pi_m(t) - \pi_c(t)$ and $C = C_m - C_c$, the market price of the risky asset is determined according to the following stochastic delay integro-differential system

$$\begin{cases} dP(t) = \mu\Big[q_f(t)D_f(t) + q_m(t)D_m(t) + q_c(t)D_c(t)\Big]dt + \sigma_M dW_M(t), \\ dU(t) = \eta\Big[\pi(t) - U(t)\Big]dt, \end{cases} \tag{D.1}$$

where

$$\begin{aligned} &q_f(t) = \alpha_f, \qquad q_m(t) = m_m + (1 - \alpha_f - m_m - m_c)n_m(t), \\ &q_c(t) = m_c + (1 - \alpha_f - m_m - m_c)\big(1 - n_m(t)\big), \qquad n_m(t) = \frac{1}{1 + e^{-\beta U(t)}}, \\ &D_f(t) = \beta_f\big(F(t) - P(t)\big), \qquad D_m(t) = \tanh\Big(\beta_m\big(P(t) - \frac{1}{\tau_m}\int_{t-\tau_m}^{t} P(s)ds\big)\Big), \\ &D_c(t) = \tanh\Big(-\beta_c\big(P(t) - \frac{1}{\tau_c}\int_{t-\tau_c}^{t} P(s)ds\big)\Big), \\ &\pi(t) = \mu\big[q_f(t)D_f(t) + q_m(t)D_m(t) + q_c(t)D_c(t)\big]\big[D_m(t) - D_c(t)\big] - C. \end{aligned}$$

[26]Hommes, Kiseleva, Kuznetsov and Verbic (2012) investigate the impact of time horizons in the fitness measure for switching on market stability. Different from the discrete-time HAMs, the time horizons do not affect the local stability and bifurcation analysis in the continuous-time HAMs. This is due to the fact that they are in higher order terms and they affect the nonlinear dynamics, rather than the dynamics of the linearized system.

D.1. **Dynamics of the Deterministic Model.** The deterministic skeleton of (D.1) is given by

$$\begin{cases} \dfrac{dP}{dt} = \mu\Big[q_f(t)D_f(t) + q_m(t)D_m(t) + q_c(t)D_c(t)\Big]dt, \\ \dfrac{dU}{dt} = \eta\Big[\pi(t) - U(t)\Big], \end{cases} \tag{D.2}$$

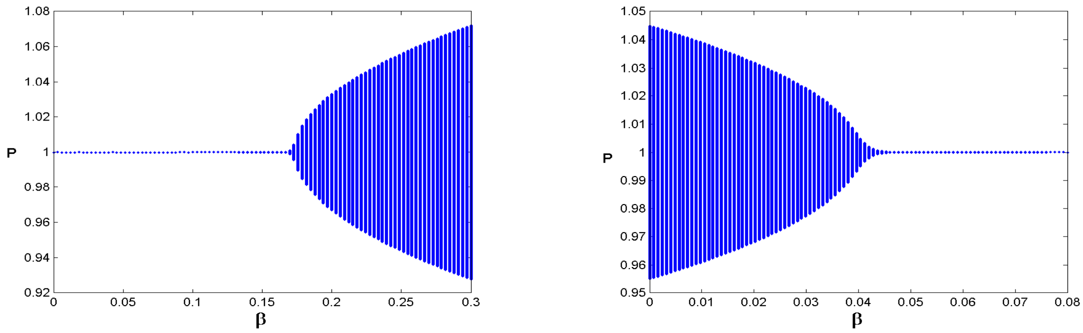
whose steady state is $(P, U) = (\bar{F}, -C)$, consisting of the constant fundamental price and the strategy cost disparity.

If there is no intensity of choice, that is $\beta = 0$, then the system (D.2) reduces to the constant population model (4.1) with the constant market population fractions of the three kinds of agents $(\alpha_f, \alpha_m, \alpha_c) = (\alpha_f, \frac{1-\alpha_f+m_m-m_c}{2}, \frac{1-\alpha_f-m_m+m_c}{2})$. For the case of $\beta > 0$, at the fundamental steady state, the proportions of the switching momentum and contrarian traders are $\frac{1}{1+e^{\beta C}} := n_m^*$ and $\frac{1}{1+e^{-\beta C}} := n_c^*$ respectively, and hence the market fractions of momentum and contrarian traders become $q_m(t) = m_m + \frac{1-\alpha_f-m_m-m_c}{1+e^{\beta C}} := \alpha_m^*$ and $q_c(t) = m_c + \frac{1-\alpha_f-m_m-m_c}{1+e^{-\beta C}} := \alpha_c^*$ respectively. Obviously, when $C = 0$, $n_m^* = n_c^* = \frac{1}{2}$ for any β. This makes sense because the difference in profits is zero at the fundamental steady state. However, if $C > 0$, that is costs for momentum strategy exceed the costs for contrarian trading rules, then there are more contrarians than momentum traders among the switching chartists at the fundamental steady state, i.e., $n_c^* \geq n_m^*$. (If $C < 0$, then $n_c^* \leq n_m^*$.) Furthermore, when $C > 0$, an increase in β leads to a decrease in n_m^*, the fraction using the expensive momentum strategy. This makes economic sense. There is no point in paying any cost at a fundamental steady state for a trading strategy that yields no extra profit at that fundamental steady state. As intensity of choice β increases, the mass on the most profitable strategy in net terms increases.

We still use γ_i, $i = f, m, c$ to characterize the activity of type-i agent, where $\gamma_f = \mu\alpha_f\beta_f$, $\gamma_m = \mu\alpha_m^*\beta_m$ and $\gamma_c = \mu\alpha_c^*\beta_c$. Then the characteristic equation of the system (D.2) at the fundamental steady state $(P, U) = (\bar{F}, -C)$ is given by

$$(\lambda + \eta)\Big(\lambda + \gamma_f - \gamma_m + \gamma_c + \frac{\gamma_m}{\lambda\tau_m}(1 - e^{-\lambda\tau_m}) - \frac{\gamma_c}{\lambda\tau_c}(1 - e^{-\lambda\tau_c})\Big) = 0. \tag{D.3}$$

Notice $\eta > 0$ and the second multiplication factor of Eq. (D.3) shares the same form as the characteristic equation (B.1) except for the expression of γ_m and γ_c. So the price dynamics of the system (D.2) can be characterized by Proposition C.1.



(a) $\beta_f = 4.7$, $\beta_m = 7.5$, $\beta_c = 6.7$ and $C = -2$ (b) $\beta_f = 3.33$, $\beta_m = 7.5$, $\beta_c = 6.67$ and $C = 2$

FIGURE D.1. Price bifurcation with respect to β for (a) $C < 0$ and (b) $C > 0$.

Simulations show that the population evolution can enlarge the period and oscillation amplitude of the market price (not reported here). We choose $\alpha_f = 0.3$, $m_m = 0.3$, $m_c = 0.2$, $\mu = 10$, $\eta = 0.5$, $\tau_m = 1.2$, $\tau_c = 1.2$ and $\bar{F} = 1$. When $\beta = 0$, we have $\gamma_m < \gamma_c + \gamma_f/(1+a)$ and Proposition C.1 (i) shows that the steady state of the system (D.2) is stable for all $\tau_m, \tau_c \geq 0$. However, one can verify that $\gamma_m > \gamma_c + \gamma_f/(1+a)$ when the intensity of choice β is greater than 0.11. Proposition C.1 (ii) and (iii) demonstrate that the steady state is unstable when $\tau_m, \tau_c \in (\tau_l^*, \tau_h^*)$. The results are illustrated in Fig. D.1 (a). On the other hand, when $C > 0$, an increase in the intensity of choice β may stabilize the unstable market price as shown in Fig. D.1 (b). When the intensity of choice is small ($\beta < 0.12$), the market price is unstable. With the increase in β, the market price becomes stable. Therefore, the population evolution has a conditional impact on the market stability.

D.2. **Profitability.** For small switching intensity β, numerical simulations (not reported here) on the profitability in this case coincide with the profitability results of the no switching model (3.7) that (*i*) the fundamentalists profit and market maker loses in general. (*ii*) When momentum traders dominate the market, they profit for small time horizon but lose for big time horizon; contrarians with long time horizon can profit but lose with short time horizon. (*iii*) When momentum traders do not dominate the market, contrarian strategy can always profit but momentum strategy always loses. But simulations also show that the switching can enlarge the profits and losses by choosing the same parameters (the market fraction parameters being chosen to satisfy $\alpha_j = \alpha_j^*$, $j = m, c$) for the no switching model (3.7) and switching model (D.1).

Appendix E. Population Evolution among Fundamentalist, Momentum and Contrarian Traders

Let $q_f(t) = m_f + (1 - m_f - m_m - m_c)n_f(t)$ where m_f is the fixed proportion of fundamentalists who stay with their strategy over time and $n_f(t)$ is the proportion of fundamentalists among the switching traders. The technique of modelling population evolution among fundamentalist, momentum and contrarian traders in this section is the same as previous section. Then the market price of the risky asset is determined according to the following stochastic delay integro-differential system

$$\begin{cases} dP(t) = \mu\Big[q_f(t)D_f(t) + q_m(t)D_m(t) + q_c(t)D_c(t)\Big]dt + \sigma_M dW_M(t), \\ dU_1(t) = \eta\Big[\pi_1(t) - U_1(t)\Big]dt, \\ dU_2(t) = \eta\Big[\pi_2(t) - U_2(t)\Big]dt, \end{cases} \tag{E.1}$$

where

$$q_f(t) = 1 - q_m(t) - q_c(t), \qquad q_m(t) = m_m + (1 - m_f - m_m - m_c)n_m(t),$$
$$q_c(t) = m_c + (1 - m_f - m_m - m_c)n_c(t), \qquad n_m(t) = \frac{1}{1 + e^{\beta U_1(t)} + e^{\beta(U_1(t) - U_2(t))}},$$
$$n_c(t) = \frac{1}{1 + e^{\beta U_2(t)} + e^{\beta(U_2(t) - U_1(t))}},$$
$$\pi_1(t) = \mu\big[q_f(t)D_f(t) + q_m(t)D_m(t) + q_c(t)D_c(t)\big]\big[D_f(t) - D_m(t)\big] - C_1,$$
$$\pi_2(t) = \mu\big[q_f(t)D_f(t) + q_m(t)D_m(t) + q_c(t)D_c(t)\big]\big[D_f(t) - D_c(t)\big] - C_2.$$

The steady state of the deterministic part of the system (E.1) is $(P, U_1, U_2) = (\bar{F}, -C_1, -C_2)$ and the dynamics can be also characterized by Proposition C.1.

The profitability property is consistent with that in Appendix D.

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