D-optimal orthogonal array minus t run designs

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Abstract

This paper considers the design performance of orthogonal arrays in which one or more runs are missing at random. We focus on orthogonal arrays of index unity and on the 18 run ternary arrays.

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1 Introduction

Orthogonal arrays (OAs) are a class of fractional factorial designs (FFDs), and are optimal according to a range of optimality criteria. We investigate whether all OAs with the same parameters are equally good if there is a possibility that one or more runs will be missing at random from the OA. We summarise earlier work on this problem below.

A design which performs well in the absence of one or more runs is said to be *robust to missing runs*. Herzberg (1982) gives Box (1953) the credit for the introduction of the term robust generally into statistics. In the context of designs she says that robustness can focus either on investigating how well a design optimal under one criterion performs under another, or on the construction of designs which guard against particular short-comings, say the consequences of missing runs when fitting a model. It is this second context which we will investigate in this paper.

Ghosh (1982b) developed four equivalent conditions, all based on properties of the model matrix, which allow one to determine whether or not a design would be robust to the loss of any t runs. Let $t_{\max}$ denote the largest t such that any t runs can be missing but the model is still estimable while there is at least one set of $t+1$ runs for which the model is not estimable. The ideas in Ghosh (1982b) are extended in MacEachern et al. (1995) to give an upper bound for $t_{\max}$ and this bound is easily calculated from the model matrix. Ghosh (1982a) defined the information contained in run $\mathbf{a}$ of a fractional factorial design to be $\mathbf{a}'(X'X)^{-1}\mathbf{a}$, where X is the model matrix for the model of interest. He shows that in a 2^m factorial design all runs with the same Hamming weight (the number of non-zero entries) contain the same amount of information, for instance. The information that Ghosh associates with each run is the leverage of that run in a regression setting.

Tanco et al. (2013) considered the robustness performance for estimating the full second-order model when runs are dropped from the 9 designs with m ternary factors that they have chosen to focus on. These designs have been specifically developed for estimating the second-order model and vary greatly in terms of robustness (summarised in their Table 2) and also in terms of the number of runs, N, required (for example, when $m = 3$, N varies from 14 to 18; when $m = 7$, N varies from 40 to 82).

Akhtar and Prescott (1986) developed a minimax loss criterion (a criterion which minimises the maximum loss), based on D-efficiency, in the context of central composite designs. This work was adapted and extended by Ahmad and Gilmour (2010) in the context of subset designs. In the ternary context, for example, subset designs (Gilmour (2006)) have all runs with Hamming weight r appearing equally often. Ahmad et al. (2012) construct augmented pairs minimax loss designs from Plackett-Burman plans where the design construction is based on the minimax loss criterion given by Akhtar and Prescott (1986).

da Silva a et al. (2016) use a compound criterion to decide which designs are robust to missing runs when fitting a second order model. They discuss the consequences of missing runs which contain high leverage, but also observe that designs which only contain points of low leverage "will perform poorly in terms of estimation precision". They therefore extend their compound measure to include an indicator of "leverage uniformity". They develop an exchange algorithm to construct designs which satisfy the compound criteria that they have developed.

In this paper we investigate design performance when a small subset of runs is missing from an orthogonal array, and we want to estimate a main effects only model. A run may be missing because no response is observed for that run, or the response observed may be considered suspect in some sense (e.g., be considered to be an outlier) due to unforeseen circumstances, or the results may be being collected sequentially and time delays mean that not all runs can be completed. D-optimality and model estimation performance is closely linked to the number of positions in which the missing runs have different levels, that is, the Hamming distance properties of the missing runs, just as it is when runs are adjoined (Bird and Street (2016)). Thus we will first focus on these distance properties for runs within OAs of index unity, extending the results in Srivastava et al. (1991). Then we investigate all 18 run combinatorially non-isomorphic ternary OAs of strength 2 and determine which are the best to use when up to 4 runs might be lost during the course of the experiment. We also determine $t_{\max}$ for these designs.

In the following section we define the notation that we will use in this paper. In Section 3 we define the matrix to be optimised and we then give some theoretical results for D-optimal OA minus t runs designs. In Section 4 we consider missing runs in OAs of index unity, and in Section 5 we determine when the theoretical results can be applied in practice in the context of 18 run ternary OAs of strength 2 missing t run designs. We finish with a brief discussion in Section 6.

2 Notation

An *asymmetric orthogonal array* OA$[N; s_1, s_2, \ldots, s_m; S]$ is a $N \times k$ array with elements from $Z_{s_i} = \{0, 1, \ldots, s_i - 1\}$ in column i such that any $N \times S$ subarray has each S-tuple appearing as a row an equal number of times. Such an array is said to have *strength* S. In this paper we assume that there are k distinct s_i, that there are m_i factors with s_i levels, $1 \leq i \leq k$ so $m = \sum_{i=1}^{k} m_i$ and that all OAs are of strength 2. We denote an OA with N runs by writing OA$[N, s_1^{m_1} \times s_2^{m_2} \times \ldots \times s_k^{m_k}]$. When $k = 1$ and $N = s^2$ the OA is said to be of *index unity.*

We will let $\mathbf{r}_i$, $1 \leq i \leq t$, be the runs that are missing by chance from the OA.

We let I_s be the identity matrix of order s and $\mathbf{1}_s$ be the $s \times 1$ vector with all elements equal to 1. Since we want to estimate the main effects only model, we replace each level of each factor with appropriate entries from a set of orthogonal polynomials, as this will be appropriate for quantitative factors. The results on optimality that we obtain are independent of the representation that we use, and so for qualitative factors other sets of orthogonal contrasts, which may be more natural, can be used instead. (In Section 5 we only discuss qualitative factors.) For the orthogonal polynomial coding we follow the approach of Chatzopoulos et al. (2011) and define P_s to be a contrast matrix of order $(s-1) \times s$ that satisfies $P_s P_s' = sI_{s-1}$ and $P_s \mathbf{1}_s = 0$. Then for each run $\mathbf{r}$ of the complete factorial we associate a row vector with first entry 1 and in which each level of each factor in $\mathbf{r}$ is replaced by the transpose of the corresponding column of the matrix P_{s_i}. We denote this extended row vector by $\mathbf{q}$ and we observe that each extended row vector has $\alpha = 1 + \sum_i (s_i - 1) m_i$ entries in it. The following example illustrates these ideas.

Example 2.1 Let $s_1 = 3$ and $m = m_1 = 4$. Then $P_3 = \begin{pmatrix} -\sqrt{\frac{3}{2}} & 0 & \sqrt{\frac{3}{2}} \\ \frac{1}{\sqrt{2}} & -\sqrt{2} & \frac{1}{\sqrt{2}} \end{pmatrix}$. If $\mathbf{r} = (0, 2, 1, 0)$, say, then the corresponding value of $\mathbf{q} = (1, -\sqrt{\frac{3}{2}}, \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}, \frac{1}{\sqrt{2}}, 0, -\sqrt{2}, -\sqrt{\frac{3}{2}}, \frac{1}{\sqrt{2}})$ which has 1+(3-1)4=9=α entries in it.

We use Q for the $N \times \alpha$ matrix of the extended row vectors associated with the runs in the orthogonal array. So Q is the model matrix for the main effects only model. Thus $M = Q'Q$ is the information matrix of the orthogonal array and hence we know that $M = NI_\alpha$.

3 The information matrix of the altered design

Suppose that t runs are missing from an OA$[N, s_1^{m_1} \times s_2^{m_2} \times \ldots \times s_k^{m_k}]$. Then we can partition the runs of Q as $\left[\begin{array}{c} Q_B \\ B \end{array}\right]$ where B is a $t \times \alpha$ matrix that contains the t row vectors associated with the missing runs. Thus we can write the information matrix as $Q'Q = Q'_B Q_B + B'B$. Since $Q'Q = NI_\alpha$, the information matrix for the design with t runs missing is $M_B = Q'_B Q_B = NI_\alpha - B'B$.

As we are interested in D-optimality, we seek to maximise the determinant of M_B, $|M_B|$. Applying Theorem 18.1.1 from Harville (1997) we have that

$$\begin{aligned} |M_B| &= |NI_\alpha||-I_t||(-I_t)^{-1} + B\,(NI_\alpha)^{-1}\,B'| \\ &= (-1)^t N^{\alpha - t}|BB' - NI_t|. \end{aligned} \tag{3.1}$$

As $(-1)^t N^{\alpha-t}$ is constant for all designs in the class of competing designs, that is, all OA$[N, s_1^{m_1} \times s_2^{m_2} \times \ldots \times s_k^{m_k}]$ with t runs missing, we will use Ω_B to denote $BB' - NI_t$, the matrix whose determinant is to be maximised. As it is the inner product of the pairs of missing rows of Q that gives rise to the off-diagonal entries of Ω_B, we now focus on these values.

When $t = 1$, $|M_B| = -N^{\alpha-1}(\alpha - N) = N^\alpha - \alpha N^{\alpha-1}$, and so the absence of any single run of the design, equivalently row of Q, results in designs which are equally good. This has been shown before, and so, at least in terms of D-optimality, the initial OA is immaterial in the case of a single missing run. For larger values of t, design performance can be different for different OAs with the same parameter values. We discuss this in detail in Sections 4 and 5.

3.1 A bound on the determinant

Let $\mathbf{b}_x$ be the xth row in B, the matrix of the row vectors of the missing runs. As $\mathbf{b}_x.\mathbf{b}'_x = \alpha$ for all runs in the complete factorial, all the diagonal entries of Ω_B are $\alpha - N$, and hence the trace of Ω_B is $(\alpha - N)t$. Let $(\lambda_1, \lambda_2, \ldots, \lambda_t)$ be the eigenvalues of Ω_B. Then we have $\sum_{i=1}^t \lambda_i = (\alpha - N)t$. Using the arithmetic-geometric mean inequality we see that $\prod_{i=1}^t \lambda_i \leq (\alpha - N)^t$ and hence $|\Omega_B| \leq (\alpha - N)^t$. This upper bound on $|\Omega_B|$ is realised when all the eigenvalues are the same. Thus the design with t runs missing is D-optimal if the inner product of the row vectors associated with any pair of missing runs is 0.

We will begin by considering the case when $t = 2$ runs are missing from the OA. Let these two runs be $\mathbf{r}_1$ and $\mathbf{r}_2$ with corresponding rows in B of $\mathbf{b}_1$ and $\mathbf{b}_2$. Rather than calculate the Hamming distance between $\mathbf{r}_1$ and $\mathbf{r}_2$, we instead calculate the Hamming distance within each of the sets of m_i factors with the the same value of s_i. Hence we let d_i be the Hamming distance between the m_i factors with s_i levels in $\mathbf{r}_1$ and $\mathbf{r}_2$, $1 \leq i \leq k$, and we write $d_H(\mathbf{r}_1, \mathbf{r}_2) = (d_1, d_2, \ldots, d_k)$. Then, as in Equation (3.1) of Bird and Street (2016), $\mathbf{b}_1.\mathbf{b}'_2 = 1 + \sum_{i=1}^k [(s_i - 1)m_i - s_i d_i]$ and so $|\Omega_B|$ will equal its theoretical upper bound for any set of d_i such that $\mathbf{b}_1.\mathbf{b}'_2$ is equal to 0. When no set of d_i exists which satisfies this condition, $|\Omega_B|$ will be maximised by minimising abs$(\mathbf{b}_1.\mathbf{b}'_2)$, the absolute value of $\mathbf{b}_1.\mathbf{b}'_2$.

In the following section, we consider what happens when sets of t runs are missing from an OA$[s^2, s^m]$, as these arrays have no repeated pairs of levels between any pairs of factors, and this allows us to say a lot about the possible form of BB'. In Section 5, we study all OA$[18, 3^m]$, to determine the best realisable values of abs$(\mathbf{b}_1.\mathbf{b}'_2)$ for each OA, and the 18 run OAs give an idea of the wide range of behaviour that can be seen.

4 Missing runs from an OA$[s^2, s^m]$

In an OA$[s^2, s^m]$ any two runs are of Hamming distance m or $m-1$, since any ordered pair of levels appears exactly once when any sub-array of two distinct columns is considered. The number of runs at Hamming distance $m-1$ from any given run is $m(s-1)$ and hence the number of runs at Hamming distance m is $s^2 - 1 - m(s-1) = (s-1)(s+1-m)$. Thus when $m = s+1$ all pairs are at Hamming distance $m - 1 = s$.

If one run is missing from an OA$[s^2, s^m]$ then the resulting array has an information matrix with determinant equal to $(-1)N^{\alpha-1}(\alpha-N) = s^{2(\alpha-1)}(s^2-\alpha)$ and so all designs perform equally well according to the criterion of D-optimality, as we saw above. Of course if $m = s+1$, then $\alpha = s^2$ and the model cannot be estimated if any runs are missing from an OA$[s^2, s^m]$.

Consider what happens when $t = 2$ runs are missing from an OA$[s^2, s^m]$. We denote the missing runs by $\mathbf{r}_x$ and $\mathbf{r}_y$, with $\mathbf{b}_x$ and $\mathbf{b}_y$ as the corresponding rows of B. We know that

$$|M_B| = (-1)^2 s^{2(\alpha-2)}|BB' - s^2 I_2| = s^{2(\alpha-2)}((\alpha - s^2)^2 - (\mathbf{b}_y\mathbf{b}_x')^2),$$

and so the design performance depends on abs($\mathbf{b}_{\mathrm{y}}\mathbf{b}_{\mathrm{x}}'$). The only two possibilities for abs($\mathbf{b}_{\mathrm{y}}\mathbf{b}_{\mathrm{x}}'$) are $m-1$, when $d_H(\mathbf{r}_x, \mathbf{r}_y) = m$, and abs(s + 1 − m), when $d_H(\mathbf{r}_x, \mathbf{r}_y) = m-1$. Thus we have that

$$|M_B| = \begin{cases} s^{2(\alpha-2)}s(m-s)(2+sm-s^2-2m) & \text{when } d_H(\mathbf{r}_x, \mathbf{r}_y) = m, \\ s^{2(\alpha-2)}s(m-s-1)(2+sm+s-s^2-2m) & \text{when } d_H(\mathbf{r}_x, \mathbf{r}_y) = m-1. \end{cases}$$

Comparing the determinants, we see that it is better if a pair of runs with Hamming distance $m-1$ is missing when $2m > s+2$, and a pair of runs at Hamming distance m when $2m < s+2$. When $2m = s+2$ it makes no difference to the determinant of the information matrix which pair of runs are missing.

When three runs are missing, again it is only the actual distances that matter, since the location of the off-diagonal entries in a matrix of order 3 is immaterial when evaluating the deteminant. When four or more runs are missing then the structure of the runs, as well as the actual Hamming distances, both impact on the determinant of the design. For instance, consider the OA$[16,4^3]$ given in Table 4.1. The runs 1, 2, 10 and 16 have pairwise Hamming distances $\{2, 2, 2, 3, 3, 3\}$, the same as the runs 1, 3, 9 and 16, yet if the first set of four runs are missing then the main effects model can still be estimated but if the second set of four runs are missing then the main effects model can not be estimated. For the OA in Table 4.1, the omission of any set of three runs results in a set of 13 runs from which the main effects model can be estimated. If four runs are missing then 60 of these sets result in a set of 12 runs from which the model can not be estimated, and hence for this design $t_{\max} = 3$. It is interesting that there are also 80 sets of four runs which if missing result in sets of 12 runs which are equi-information. One such set is runs 1, 2, 9 and 10.

Table 4.1: Transpose of OA$[16, 4^3]$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	0	0	0	1	1	1	1	2	2	2	2	3	3	3	3
0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3
0	1	2	3	1	0	3	2	2	3	1	0	3	2	0	1

The results in this section also apply directly to designs in which the levels of one, or more, factors are replaced by the rows of a saturated, symmetric OA (e.g., a factor with 4 levels is replaced by the runs of an OA$[4, 2^3]$ resulting in an OA$[16, 2^3 \times 4^{m-1}]$). If, however, any of the factors with the smaller number of levels are removed from the OA then the number of possibilities for the Hamming distances, and hence for the values of $|M_B|$, increase.

5 Symmetric OAs of larger index

When an OA does not have index unity then there can be a wide range of Hamming distances between the runs of the OA, from 0, if there are repeated runs, to m, if there are no levels in common. As we have seen above, the properties of an OA with missing runs depends on the Hamming distance between the missing runs. Thus we need to know the distance structure of the runs within an OA to be able to make comments about its performance when runs are missing. This structure does not depend only on the parameters of the OA, and so we need to identify all non-isomorphic designs for a given set of parameter values, and determine the distance structure of the runs within each isomorphism class, before we can make recommendations about the best design to use. Hence in this section we will assume that all factors are qualitative so that we can limit ourselves to combinatorial isomorphism, where the designs have been completely enumerated for the design parameters that we are interested in, rather than geometric isomorphism.

There are 23,275 different parameter values for OAs with at most 100 runs (Bird and Street (2016)) and many of these have a very large number of isomorphism classes. For instance there are 1,470,157 classes of OA[24,2^8] (Eendebak and Schoen (2010)). Thus we have chosen to illustrate the issues involved by focusing on the symmetric 18 run ternary OAs. These designs are much used in practice as they are fairly small designs that allow factors to have more than two levels. Schoen (2009) has enumerated all combinatorially non-isomorphic OA$[18, 3^m]$, and found that there are 4, 12, 10, 8 and 3 isomorphism classes for $m = 3$, 4, 5, 6 and 7, respectively. We will investigate all OA$[18, 3^m]$, $3 \leq m \leq 7$, for their performance when runs are missing. The designs can be found in Bird and Street (2017).

We have enumerated the Hamming distance between all $\binom{18}{2} = 153$ pairs of runs for a representative design from each class. The distributions of these Hamming distances for $m = 3$ and $m = 4$, along with the associated $\mathbf{b}_1.\mathbf{b}_2'$, are given in Tables 5.1 and 5.2 respectively. Analogous results for $m = 5$, 6 and 7 can be found in Tables 5.3, 5.4 and 5.5. The D-efficiency of each OA with two runs missing is given in the column headed Eff$_B$. The D-efficiency of each OA is calculated relative to the largest determinant that could be realised if the pair of missing runs had inner product 0. This is called the theoretical bound.

Table 5.1: Distribution of Hamming distances between all pairs of runs for each of the 4 OA$[18, 3^3]$ combinatorial isomorphism classes

	Hamming		**Count by Class**			
$\mathbf{b}_1.\mathbf{b}_2'$	**Distance**	**Eff$_B$**†	**1**	**2**	**3**	**4**
1	2	99.88%	90	84	81	108
−2	3	99.52%	42	44	45	36
4	1	97.99%	18	24	27	–
7	0	92.85%	3	1	–	9

† *Efficiency of OA minus two run design compared to theoretical bound*

Table 5.2: Distribution of Hamming distances between all pairs of runs for each of the 12 OA$[18, 3^4]$ combinatorial isomorphism classes

	Hamming		**Count by Class**					
$\mathbf{b}_1.\mathbf{b}_2'$	**Distance**	**Eff$_B$**†	**1**	**2**	**3**	**4**	**5**	**6**
0	3	100%	92	104	99	87	81	114
3	2	98.70%	36	24	27	39	45	18
−3	4	98.70%	20	16	18	22	24	12
6	1	93.68%	4	8	9	5	3	6
9	0	*NA*§	1	1	–	–	–	3
	Hamming		**Count by Class**					
$\mathbf{b}_1.\mathbf{b}_2'$	**Distance**	**Eff$_B$**†	**7**	**8**	**9**	**10**	**11**	**12**
0	3	100%	80	81	78	99	72	144
3	2	98.70%	48	45	48	27	54	–
−3	4	98.70%	24	24	25	18	27	–
6	1	93.68%	–	3	2	9	–	–
9	0	*NA*§	1	–	–	–	–	9

† *Efficiency of OA minus two run design compared to theoretical bound*
§ Ω_B *is singular*

The choice of OA will depend on the objectives of the experimenter. For example, when $m = 3$, class 4 has the most pairs of runs with the best realisable $\mathbf{b}_1.\mathbf{b}_2'$, so if we intend to maximise our chance of obtaining an optimal design after a pair of runs has failed, we can do so by using a design from this class. Furthermore, if we suspect that more than two runs may fail, this class is more likely to contain a set of runs that pairwise realise the best $\mathbf{b}_1.\mathbf{b}_2'$. However, this class also has the most pairs of runs with the worst realisable $\mathbf{b}_1.\mathbf{b}_2'$. Thus, this class also maximises our chance of realising the worst outcome. Hence, a trade-off needs to be made between minimising the probability of obtaining a less favourable Hamming distance and maximising the probability of obtaining the most favourable Hamming distance. So, class 3, for example, might be preferred as it does not contain any pairs of runs that realise the worst $\mathbf{b}_1.\mathbf{b}_2'$, but this choice would be made at the expense of the probability of realising the best $\mathbf{b}_1.\mathbf{b}_2'$, which appears the least number of times in this class.

In Table 5.2 we can see that different classes need not have different distributions of Hamming distances. For example, when $m = 4$, classes 3 and 10 are 'essentially the same' from our perspective as they have the same distribution of Hamming distances, as do classes 5 and 8. The trade-off we described earlier between maximising the probability of obtaining the best $\mathbf{b}_1.\mathbf{b}_2'$ versus minimising the probability of obtaining the worst $\mathbf{b}_1.\mathbf{b}_2'$ is well-illustrated in class 12 of the OA$[18, 3^4]$ designs. This class has considerably more instances of 100% efficiency than any other class, yet it also has the highest number of the worst realisable Hamming distance, which in this case means that for an OA with two runs missing the model will not be estimable. While a loss in efficiency may be considered undesirable yet tolerable in some circumstances, the inability to estimate the model will clearly never be acceptable. It is advisable to avoid classes 1, 2, 6, 7 and 12 as from all these a singular information matrix could be obtained with two missing runs; that is, $t_{\max} = 1$ for these classes.

We note that for $m = 3$ and $m = 4$, some classes have pairs of runs with a Hamming distance of 0. This means that the OA contains repeated runs.

When $m > 4$, there are no designs with repeated runs, as we can see from Tables 5.3, 5.4 and 5.5. We note that when $m = 5$, classes 3 and 5 are essentially the same in terms of the distribution of Hamming distances, as are classes 6 and 8. When $m = 6$, classes 1, 2, 4 and 5 all have the same distribution of Hamming distances and they are also the only classes that do not realise the worst $\mathbf{b}_1.\mathbf{b}_2'$. When $m = 7$ the distribution of Hamming distances are the same for all three classes.

Table 5.3: Distribution of Hamming distances between all pairs of runs for each of the 10 OA$[18, 3^5]$ combinatorial isomorphism classes

	Ham.		**Count by Class**									
$\mathbf{b}_1.\mathbf{b}_2'$	**Dist.**	**Eff**$_B$ †	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
−1	4	99.82%	72	66	81	75	81	63	57	63	99	45
2	3	99.23%	63	69	54	60	54	72	78	72	36	90
−4	5	96.47%	9	11	6	8	6	12	14	12	–	18
5	2	93.72%	9	7	12	10	12	6	4	6	18	–

† *Efficiency of OA minus two run design compared to theoretical bound*

Table 5.4: Distribution of Hamming distances between all pairs of runs for each of the 8 OA$[18, 3^6]$ combinatorial isomorphism classes

	Ham.		**Count by Class**							
$\mathbf{b}_1.\mathbf{b}_2'$	**Dist.**	**Eff**$_B$ †	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
1	4	99.69%	81	81	93	81	81	99	111	135
−2	5	98.67%	54	54	42	54	54	36	24	–
4	3	92.44%	18	18	14	18	18	12	8	–
−5	6	*NA*§	–	–	4	–	–	6	10	18

† *Efficiency of OA minus two run design compared to theoretical bound*

§ Ω_B *is singular*

Table 5.5: Distribution of Hamming distances between all pairs of runs for each of the 3 OA$[18, 3^7]$ combinatorial isomorphism classes

	Ham.		**Count by Class**		
$\mathbf{b}_1.\mathbf{b}_2'$	**Dist.**	**Eff**$_B$ †	**1**	**2**	**3**
0	5	100%	108	108	108
3	4	*NA*§	27	27	27
−3	6	*NA*§	18	18	18

† *Efficiency of OA minus two run design compared to theoretical bound*

§ Ω_B *is singular*

When $t = 3$, only the values of the distances matter, since the position of the off-diagonal elements in Ω_B does not change the value of the determinant. Thus we need only consider the unique sets of triples of realisable values of $\mathbf{b}_x.\mathbf{b}_y'$.

Tables A1, A2, A3, A4 and A5 in the appendix give all possible sets of pairwise Hamming distances for $t = 3$ missing runs for each combinatorial isomorphism class. In these tables, we introduce the notation $\{\mathbf{b}\}$ to denote the ordered tuple of values $\{\mathbf{b}_1.\mathbf{b}_2', \mathbf{b}_1.\mathbf{b}_3', \ldots, \mathbf{b}_{t-1}.\mathbf{b}_t'\}$, although in practice for $t = 3$ we have presented the values within each tuple in ascending order. The ith element in $\{\mathbf{b}\}$ is associated with the ith element in the vector of pairwise Hamming distances.

Tables A6 to A14 in the appendix give all possible sets of pairwise Hamming distances for $t = 4$ missing runs for each combinatorial isomorphism class. We order the entries in $\{\mathbf{b}\}$ lexicographically in these tables as we can no longer ignore the location of the off-diagonal entries in Ω_B. The rows of these tables are ordered by the efficiency of the designs.

As we mentioned earlier, $t_{\max}$ is the largest t such that any t runs can be missing but the model is still estimable, while there is at least one set of $t + 1$ runs for which the model is not estimable. The values for t_{max} for the 18 run ternary arrays are given below.
$m = 3$: 5, 5, 5, 3
$m = 4$: 1, 1, 5, 5, 5, 1, 1, 5, 3, 3, 3, 1
$m = 5$: 3, 3, 3, 3, 3, 3, 3, 3, 3, 3
$m = 6$: 3, 3, 1, 3, 3, 1, 1, 1
$m = 7$: 1, 1, 1.

For designs with $m = 7$, for instance, any one run can be mssing from the design and the main effects model is still estimable. For all three classes of designs there are 45 pairs of runs (see Table 5.5) which, when deleted, result in 16 run arrays from which the model can not be estimated. In all three designs these pairs of runs can be used to divide the 18 runs into three sets of 6 runs; if any two runs from the same set are missing then the model is not estimable. Design performance is also the same across these three classes when three runs are missing.

6 Discussion

In this paper we have shown that if a pair of runs is missing from an OA then the ability of the OA to estimate a main effects only model is hampered least when the pair of runs minimises the absolute value of the inner product of the corresponding rows of the model matrix as we have defined it. This idea holds true for sets of three and four missing runs as well. An enumeration of combinatorial isomorphism classes is necessary to be able to recommend the best OA with a given set of parameters for designs of index greater than unity and with qualitative factors.

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Appendix

Design performance with $t = 3$ runs missing

Table A1: Distribution of pairwise Hamming distances between all sets of $t = 3$ runs for each of the 4 OA$[18, 3^3]$ combinatorial isomorphism classes

	Hamming		**Count by Class**			
{b}	**Distance**	**Eff**$_B$†	**1**	**2**	**3**	**4**
$\{1, 1, 1\}$	$\{2, 2, 2\}$	99.62%	144	120	108	216
$\{-2, 1, 1\}$	$\{3, 2, 2\}$	99.32%	288	240	216	432
$\{-2, -2, 1\}$	$\{3, 3, 2\}$	98.81%	72	96	108	–
$\{-2, -2, -2\}$	$\{3, 3, 3\}$	98.71%	12	8	6	24
$\{1, 1, 4\}$	$\{2, 2, 1\}$	97.63%	72	96	108	–
$\{-2, 1, 4\}$	$\{3, 2, 1\}$	97.51%	108	144	162	–
$\{-2, -2, 4\}$	$\{3, 3, 1\}$	96.47%	36	48	54	–
$\{1, 4, 4\}$	$\{2, 1, 1\}$	95.09%	36	48	54	–
$\{1, 1, 7\}$	$\{2, 2, 0\}$	92.24%	36	12	–	108
$\{-2, -2, 7\}$	$\{3, 3, 0\}$	90.23%	12	4	–	36

† *Efficiency of OA minus three run design compared to theoretical bound*

Table A2: Distribution of pairwise Hamming distances between all sets of $t = 3$ runs for each of the 12 OA$[18, 3^4]$ combinatorial isomorphism classes

	Hamming		**Count by Class**					
{b}	**Distance**	**Eff$_B$**†	**1**	**2**	**3**	**4**	**5**	**6**
$\{0,0,0\}$	$\{3,3,3\}$	100.00%	200	272	222	150	114	372
$\{0,0,3\}$	$\{3,3,2\}$	98.70%	168	144	162	186	198	108
$\{-3,0,0\}$	$\{4,3,3\}$	98.70%	96	96	108	108	108	72
$\{0,3,3\}$	$\{3,2,2\}$	97.25%	84	48	54	90	108	36
$\{-3,-3,0\}$	$\{4,4,3\}$	97.25%	20	16	18	22	24	12
$\{-3,0,3\}$	$\{4,3,2\}$	97.25%	120	96	108	132	144	72
$\{-3,-3,-3\}$	$\{4,4,4\}$	96.72%	–	–	–	–	–	–
$\{-3,3,3\}$	$\{4,2,2\}$	96.72%	24	–	–	24	36	–
$\{3,3,3\}$	$\{2,2,2\}$	94.35%	12	–	–	12	18	–
$\{-3,-3,3\}$	$\{4,4,2\}$	94.35%	12	–	–	12	18	–
$\{0,0,6\}$	$\{3,3,1\}$	93.68%	24	48	54	30	18	36
$\{-3,0,6\}$	$\{4,3,1\}$	91.38%	16	32	36	20	12	24
$\{0,3,6\}$	$\{3,2,1\}$	91.38%	24	48	54	30	18	36
$\{0,0,9\}$	$\{3,3,0\}$	NA§	16	16	–	–	–	48

	Hamming		**Count by Class**					
{b}	**Distance**	**Eff$_B$**†	**7**	**8**	**9**	**10**	**11**	**12**
$\{0,0,0\}$	$\{3,3,3\}$	100.00%	128	114	92	222	48	672
$\{0,0,3\}$	$\{3,3,2\}$	98.70%	192	198	204	162	216	–
$\{-3,0,0\}$	$\{4,3,3\}$	98.70%	96	108	120	108	144	–
$\{0,3,3\}$	$\{3,2,2\}$	97.25%	120	108	120	54	144	–
$\{-3,-3,0\}$	$\{4,4,3\}$	97.25%	24	24	16	18	–	–
$\{-3,0,3\}$	$\{4,3,2\}$	97.25%	144	144	144	108	144	–
$\{-3,-3,-3\}$	$\{4,4,4\}$	96.72%	–	–	2	–	6	–
$\{-3,3,3\}$	$\{4,2,2\}$	96.72%	48	36	42	–	54	–
$\{3,3,3\}$	$\{2,2,2\}$	94.35%	24	18	20	–	24	–
$\{-3,-3,3\}$	$\{4,4,2\}$	94.35%	24	18	24	–	36	–
$\{0,0,6\}$	$\{3,3,1\}$	93.68%	–	18	12	54	–	–
$\{-3,0,6\}$	$\{4,3,1\}$	91.38%	–	12	8	36	–	–
$\{0,3,6\}$	$\{3,2,1\}$	91.38%	–	18	12	54	–	–
$\{0,0,9\}$	$\{3,3,0\}$	NA§	16	–	–	–	–	144

† *Efficiency of OA minus three run design compared to theoretical bound*

§ Ω_B *is singular*

Table A3: Distribution of pairwise Hamming distances between all sets of $t = 3$ runs for each of the 10 OA$[18, 3^5]$ combinatorial isomorphism classes

{b}	Hamming Distance	Eff_B †	Count by Class				
			1	**2**	**3**	**4**	**5**
$\{-1,-1,-1\}$	$\{4,4,4\}$	99.48%	72	64	114	106	114
$\{-1,-1,2\}$	$\{4,4,3\}$	98.70%	216	184	240	208	240
$\{-1,2,2\}$	$\{4,3,3\}$	98.42%	216	236	156	176	156
$\{2,2,2\}$	$\{3,3,3\}$	96.92%	42	62	36	56	36
$\{-4,-1,-1\}$	$\{5,4,4\}$	96.24%	36	28	24	16	24
$\{-4,2,2\}$	$\{5,3,3\}$	95.51%	18	38	12	32	12
$\{-4,-1,2\}$	$\{5,4,3\}$	94.31%	72	88	48	64	48
$\{-1,2,5\}$	$\{4,3,2\}$	92.92%	72	56	96	80	96
$\{-4,-4,-1\}$	$\{5,5,4\}$	92.41%	–	4	–	4	–
$\{-1,-1,5\}$	$\{4,4,2\}$	92.41%	36	28	60	52	60
$\{-4,-4,-4\}$	$\{5,5,5\}$	91.87%	–	–	–	–	–
$\{-4,2,5\}$	$\{5,3,2\}$	90.03%	18	14	12	8	12
$\{2,2,5\}$	$\{3,3,2\}$	86.77%	18	14	12	8	12
$\{-1,5,5\}$	$\{4,2,2\}$	81.47%	–	–	6	6	6

{b}	Hamming Distance	Eff_B †	Count by Class				
			6	**7**	**8**	**9**	**10**
$\{-1,-1,-1\}$	$\{4,4,4\}$	99.48%	60	60	60	198	60
$\{-1,-1,2\}$	$\{4,4,3\}$	98.70%	168	112	168	288	–
$\{-1,2,2\}$	$\{4,3,3\}$	98.42%	246	284	246	36	360
$\{2,2,2\}$	$\{3,3,3\}$	96.92%	72	88	72	24	120
$\{-4,-1,-1\}$	$\{5,4,4\}$	96.24%	24	16	24	–	–
$\{-4,2,2\}$	$\{5,3,3\}$	95.51%	48	62	48	–	90
$\{-4,-1,2\}$	$\{5,4,3\}$	94.31%	96	124	96	–	180
$\{-1,2,5\}$	$\{4,3,2\}$	92.92%	48	32	48	144	–
$\{-4,-4,-1\}$	$\{5,5,4\}$	92.41%	6	4	6	–	–
$\{-1,-1,5\}$	$\{4,4,2\}$	92.41%	24	16	24	108	–
$\{-4,-4,-4\}$	$\{5,5,5\}$	91.87%	–	2	–	–	6
$\{-4,2,5\}$	$\{5,3,2\}$	90.03%	12	8	12	–	–
$\{2,2,5\}$	$\{3,3,2\}$	86.77%	12	8	12	–	–
$\{-1,5,5\}$	$\{4,2,2\}$	81.47%	–	–	–	18	–

† *Efficiency of OA minus three run design compared to theoretical bound*

Table A4: Distribution of pairwise Hamming distances between all sets of $t = 3$ runs for each of the 8 OA$[18, 3^6]$ combinatorial isomorphism classes

{b}	Hamming Distance	Eff_B †	Count by Class							
			1	**2**	**3**	**4**	**5**	**6**	**7**	**8**
$\{1,1,1\}$	$\{4,4,4\}$	98.88%	120	120	184	120	120	216	324	540
$\{-2,1,1\}$	$\{5,4,4\}$	98.22%	216	216	216	216	216	216	144	–
$\{-2,-2,-2\}$	$\{5,5,5\}$	96.72%	30	30	10	30	30	–	–	–
$\{-2,-2,1\}$	$\{5,5,4\}$	95.85%	180	180	132	180	180	108	72	–
$\{-2,1,4\}$	$\{5,4,3\}$	90.87%	180	180	132	180	180	108	72	–
$\{1,1,4\}$	$\{4,4,3\}$	88.88%	72	72	72	72	72	72	48	–
$\{-2,4,4\}$	$\{5,3,3\}$	81.68%	18	18	6	18	18	–	–	–
$\{-5,1,1\}$	$\{6,4,4\}$	*NA*§	–	–	48	–	–	72	138	270
$\{-5,-2,-2\}$	$\{6,5,5\}$	*NA*§	–	–	12	–	–	18	12	–
$\{-5,4,4\}$	$\{6,3,3\}$	*NA*§	–	–	4	–	–	6	4	–
$\{-5,-5,-5\}$	$\{6,6,6\}$	*NA*§	–	–	–	–	–	–	2	6

† *Efficiency of OA minus three run design compared to theoretical bound*

§ Ω_B *is singular*

Table A5: Distribution of pairwise Hamming distances between all sets of $t = 3$ runs for each of the 3 OA$[18, 3^7]$ combinatorial isomorphism classes

	Hamming		**Count by Class**		
$\{\mathbf{b}\}$	**Distance**	$\mathbf{Eff}_B$ †	**1**	**2**	**3**
$\{0, 0, 0\}$	$\{5, 5, 5\}$	100.00%	216	216	216
$\{-3, 0, 0\}$	$\{6, 5, 5\}$	*NA*§	216	216	216
$\{0, 0, 3\}$	$\{5, 5, 4\}$	*NA*§	324	324	324
$\{-3, 3, 3\}$	$\{6, 4, 4\}$	*NA*§	54	54	54
$\{-3, -3, -3\}$	$\{6, 6, 6\}$	*NA*§	6	6	6

† *Efficiency of OA minus three run design compared to theoretical bound*

§ Ω_B *is singular*

Design performance with $t = 4$ runs missing

Table A6: Distribution of pairwise Hamming distances between all sets of $t = 4$ runs for each of the 4 OA$[18, 3^3]$ combinatorial isomorphism classes

	Hamming		Count by Class			
{b}	**Distance**	**Eff**$_B$†	**1**	**2**	**3**	**4**
$\{1,1,1,1,1,1\}$	$\{2,2,2,2,2,2\}$	99.18%	24	23	18	–
$\{-2,1,1,1,1,1\}$	$\{3,2,2,2,2,2\}$	98.96%	528	336	270	1296
$\{-2,1,1,1,1,-2\}$	$\{3,2,2,2,2,3\}$	98.71%	156	96	81	432
$\{-2,-2,1,1,1,1\}$	$\{3,3,2,2,2,2\}$	98.51%	204	204	162	–
$\{-2,-2,1,-2,1,1\}$	$\{3,3,2,3,2,2\}$	98.45%	96	48	36	288
$\{-2,-2,1,1,-2,1\}$	$\{3,3,2,2,3,2\}$	98.07%	144	168	162	–
$\{-2,-2,-2,1,1,1\}$	$\{3,3,3,2,2,2\}$	97.80%	24	48	72	–
$\{1,1,1,1,1,4\}$	$\{2,2,2,2,2,1\}$	97.06%	72	96	108	–
$\{-2,1,1,1,4,1\}$	$\{3,2,2,2,1,2\}$	97.06%	264	312	324	–
$\{-2,1,1,1,4,-2\}$	$\{3,2,2,2,1,3\}$	96.96%	132	132	108	–
$\{-2,-2,4,1,1,1\}$	$\{3,3,1,2,2,2\}$	96.80%	60	60	54	–
$\{-2,-2,1,1,1,4\}$	$\{3,3,2,2,2,1\}$	96.58%	168	264	324	–
$\{-2,-2,1,1,-2,4\}$	$\{3,3,2,2,3,1\}$	96.42%	12	48	81	–
$\{-2,-2,1,4,1,1\}$	$\{3,3,2,1,2,2\}$	96.08%	120	120	108	–
$\{-2,-2,1,4,-2,1\}$	$\{3,3,2,1,3,2\}$	95.78%	120	168	216	–
$\{-2,-2,-2,-2,1,4\}$	$\{3,3,3,3,2,1\}$	95.61%	72	72	54	–
$\{-2,1,4,4,1,-2\}$	$\{3,2,1,1,2,3\}$	95.19%	9	18	27	–
$\{1,1,4,4,1,1\}$	$\{2,2,1,1,2,2\}$	94.87%	6	18	27	–
$\{-2,1,1,4,4,1\}$	$\{3,2,2,1,1,2\}$	94.61%	48	84	108	–
$\{-2,-2,4,4,1,1\}$	$\{3,3,1,1,2,2\}$	94.50%	24	72	108	–
$\{-2,1,1,1,4,4\}$	$\{3,2,2,2,1,1\}$	94.50%	144	192	216	–
$\{-2,-2,1,4,-2,-2\}$	$\{3,3,2,1,3,3\}$	94.32%	–	12	27	–
$\{1,1,1,1,4,4\}$	$\{2,2,2,2,1,1\}$	94.32%	60	60	54	–
$\{-2,-2,1,4,1,4\}$	$\{3,3,2,1,2,1\}$	93.59%	120	120	108	–
$\{-2,-2,4,4,-2,1\}$	$\{3,3,1,1,3,2\}$	93.39%	12	24	27	–
$\{-2,1,4,4,1,4\}$	$\{3,2,1,1,2,1\}$	92.24%	36	48	54	–
$\{-2,-2,-2,1,4,4\}$	$\{3,3,3,2,1,1\}$	91.72%	12	36	54	–
$\{-2,1,1,1,1,7\}$	$\{3,2,2,2,2,0\}$	91.36%	72	24	–	216
$\{1,1,1,1,1,7\}$	$\{2,2,2,2,2,0\}$	91.36%	96	30	–	324
$\{1,1,4,4,1,4\}$	$\{2,2,1,1,2,1\}$	91.22%	12	36	54	–
$\{-2,-2,4,4,-2,-2\}$	$\{3,3,1,1,3,3\}$	90.76%	6	6	–	–
$\{1,1,4,1,4,4\}$	$\{2,2,1,2,1,1\}$	90.61%	12	16	18	–
$\{-2,-2,1,7,1,1\}$	$\{3,3,2,0,2,2\}$	89.63%	96	24	–	432
$\{1,1,4,7,1,1\}$	$\{2,2,1,0,2,2\}$	89.01%	24	12	–	–
$\{-2,-2,4,7,1,1\}$	$\{3,3,1,0,2,2\}$	88.36%	24	12	–	–
$\{-2,-2,-2,1,1,7\}$	$\{3,3,3,2,2,0\}$	88.36%	24	12	–	–
$\{-2,-2,-2,-2,-2,7\}$	$\{3,3,3,3,3,0\}$	87.68%	6	–	–	36
$\{1,4,4,4,4,1\}$	$\{2,1,1,1,1,2\}$	86.22%	6	3	–	–
$\{-2,-2,1,7,-2,-2\}$	$\{3,3,2,0,3,3\}$	86.22%	6	6	–	–
$\{1,1,7,7,1,1\}$	$\{2,2,0,0,2,2\}$	82.74%	3	–	–	27
$\{-2,-2,4,7,-2,-2\}$	$\{3,3,1,0,3,3\}$	80.62%	6	–	–	–
$\{-2,-2,7,7,-2,-2\}$	$\{3,3,0,0,3,3\}$	*NA*§	–	–	–	9

† *Efficiency of OA minus four run design compared to theoretical bound*

§ Ω_B *is singular*

Table A7: Part 1 of the distribution of pairwise Hamming distances between all sets of $t = 4$ runs for each of the first 6 (out of 12) OA$[18, 3^4]$ combinatorial isomorphism classes

{b}	Hamming Distance	Eff$_B$†	Count by Class 1	2	3	4	5	6
$\{0,0,0,0,0,0\}$	$\{3,3,3,3,3,3\}$	100.00%	220	400	270	116	66	726
$\{-3,0,0,0,0,0\}$	$\{4,3,3,3,3,3\}$	98.70%	132	160	144	116	66	144
$\{0,0,0,0,0,3\}$	$\{3,3,3,3,3,2\}$	98.70%	228	240	216	186	156	216
$\{-3,0,0,0,0,-3\}$	$\{4,3,3,3,3,4\}$	97.42%	8	16	18	18	6	–
$\{0,0,3,3,0,0\}$	$\{3,3,2,2,3,3\}$	97.42%	14	24	27	27	24	–
$\{-3,0,0,0,0,3\}$	$\{4,3,3,3,3,2\}$	97.42%	16	32	90	88	114	–
$\{0,0,0,0,3,3\}$	$\{3,3,3,3,2,2\}$	97.25%	296	192	216	284	300	216
$\{-3,-3,0,0,0,0\}$	$\{4,4,3,3,3,3\}$	97.25%	72	64	72	64	84	72
$\{-3,0,0,0,3,0\}$	$\{4,3,3,3,2,3\}$	97.25%	448	448	396	352	312	432
$\{-3,0,3,0,3,0\}$	$\{4,3,2,3,2,3\}$	96.72%	48	–	–	48	60	–
$\{-3,0,3,3,0,0\}$	$\{4,3,2,2,3,3\}$	95.79%	32	32	90	106	144	–
$\{-3,0,0,0,3,-3\}$	$\{4,3,3,3,2,4\}$	95.79%	20	16	54	50	96	–
$\{-3,-3,0,0,0,3\}$	$\{4,4,3,3,3,2\}$	95.79%	32	32	36	44	36	–
$\{0,0,3,3,0,3\}$	$\{3,3,2,2,3,2\}$	95.79%	52	48	54	82	102	–
$\{-3,0,0,0,3,3\}$	$\{4,3,3,3,2,2\}$	95.79%	72	64	72	88	108	–
$\{-3,-3,3,0,0,0\}$	$\{4,4,2,3,3,3\}$	95.59%	44	48	54	54	48	36
$\{-3,0,0,3,3,0\}$	$\{4,3,3,2,2,3\}$	95.59%	96	96	108	108	84	72
$\{0,0,3,0,3,3\}$	$\{3,3,2,3,2,2\}$	95.59%	20	16	18	18	12	12
$\{-3,-3,3,0,0,3\}$	$\{4,4,2,3,3,2\}$	95.19%	32	–	–	32	48	–
$\{-3,0,3,3,3,0\}$	$\{4,3,2,2,2,3\}$	95.19%	64	–	–	64	120	–
$\{-3,0,3,0,3,3\}$	$\{4,3,2,3,2,2\}$	95.19%	32	–	–	44	60	–
$\{-3,-3,0,-3,0,3\}$	$\{4,4,3,4,3,2\}$	95.19%	–	–	–	–	–	–
$\{-3,0,0,3,3,-3\}$	$\{4,3,3,2,2,4\}$	95.19%	32	–	–	28	36	–
$\{-3,0,3,3,0,3\}$	$\{4,3,2,2,3,2\}$	94.57%	32	32	36	40	48	–
$\{-3,-3,0,0,-3,3\}$	$\{4,4,3,3,4,2\}$	94.57%	8	16	18	18	6	–
$\{-3,3,3,3,3,-3\}$	$\{4,2,2,2,2,4\}$	94.35%	2	–	–	–	3	–
$\{-3,0,3,3,3,-3\}$	$\{4,3,2,2,2,4\}$	94.35%	8	–	–	14	12	–
$\{0,0,0,3,3,3\}$	$\{3,3,3,2,2,2\}$	94.35%	24	–	–	20	36	–
$\{-3,-3,0,3,0,0\}$	$\{4,4,3,2,3,3\}$	94.35%	24	–	–	28	36	–
$\{0,3,3,3,3,0\}$	$\{3,2,2,2,2,3\}$	93.68%	24	–	–	18	27	18
$\{-3,-3,0,0,3,3\}$	$\{4,4,3,3,2,2\}$	93.68%	28	16	18	26	36	36
$\{0,0,0,0,0,6\}$	$\{3,3,3,3,3,1\}$	93.68%	40	80	72	36	18	72
$\{-3,0,3,3,0,-3\}$	$\{4,3,2,2,3,4\}$	93.68%	28	24	9	15	3	36
$\{-3,-3,0,3,0,3\}$	$\{4,4,3,2,3,2\}$	92.46%	40	–	–	40	60	–
$\{-3,-3,-3,0,0,3\}$	$\{4,4,4,3,3,2\}$	92.46%	8	–	–	8	12	–
$\{-3,-3,3,3,0,0\}$	$\{4,4,2,2,3,3\}$	92.46%	20	–	–	16	18	–
$\{-3,0,3,3,3,3\}$	$\{4,3,2,2,2,2\}$	92.46%	64	–	–	40	72	–
$\{-3,-3,0,3,-3,0\}$	$\{4,4,3,2,4,3\}$	92.46%	16	–	–	16	24	–
$\{0,0,3,3,3,3\}$	$\{3,3,2,2,2,2\}$	92.46%	44	–	–	44	78	–
$\{-3,-3,3,3,0,3\}$	$\{4,4,2,2,3,2\}$	92.46%	32	–	–	32	48	–
$\{-3,-3,0,3,-3,3\}$	$\{4,4,3,2,4,2\}$	92.46%	16	–	–	12	24	–
$\{0,0,3,6,0,0\}$	$\{3,3,2,1,3,3\}$	92.46%	8	16	45	33	27	–

†*Efficiency of OA minus four run design compared to theoretical bound*

Table A8: Part 2 of the distribution of pairwise Hamming distances between all sets of $t = 4$ runs for each of the first 6 (out of 12) OA$[18, 3^4]$ combinatorial isomorphism classes

	Hamming		Count by Class					
{b}	**Distance**	**Eff$_B$**†	**1**	**2**	**3**	**4**	**5**	**6**
$\{-3, 0, 0, 3, 3, 3\}$	$\{4, 3, 3, 2, 2, 2\}$	92.46%	32	–	–	44	36	–
$\{0, 0, 0, 0, 3, 6\}$	$\{3, 3, 3, 3, 2, 1\}$	91.38%	112	224	198	90	42	216
$\{-3, 0, 0, 0, 6, 0\}$	$\{4, 3, 3, 3, 1, 3\}$	91.38%	64	128	144	72	48	144
$\{-3, -3, -3, -3, 3, 3\}$	$\{4, 4, 4, 4, 2, 2\}$	90.20%	–	–	–	–	–	–
$\{-3, -3, 3, 3, -3, 3\}$	$\{4, 4, 2, 2, 4, 2\}$	90.20%	–	–	–	2	6	–
$\{-3, 3, 3, 3, 3, 3\}$	$\{4, 2, 2, 2, 2, 2\}$	90.20%	–	–	–	6	6	–
$\{-3, 0, 0, 0, 3, 6\}$	$\{4, 3, 3, 3, 2, 1\}$	89.88%	16	32	72	48	36	–
$\{0, 0, 3, 6, 0, 3\}$	$\{3, 3, 2, 1, 3, 2\}$	89.88%	16	32	36	36	24	–
$\{-3, -3, 0, 0, 0, 6\}$	$\{4, 4, 3, 3, 3, 1\}$	89.88%	16	32	36	24	12	–
$\{-3, 0, 3, 6, 0, 0\}$	$\{4, 3, 2, 1, 3, 3\}$	89.88%	16	32	36	24	12	–
$\{-3, 0, 0, 0, 6, 3\}$	$\{4, 3, 3, 3, 1, 2\}$	88.87%	8	32	36	16	12	–
$\{-3, 0, 0, 0, 6, -3\}$	$\{4, 3, 3, 3, 1, 4\}$	88.87%	4	16	18	14	–	–
$\{-3, 0, 3, 6, 3, 0\}$	$\{4, 3, 2, 1, 2, 3\}$	88.87%	16	–	–	16	12	–
$\{0, 0, 3, 3, 0, 6\}$	$\{3, 3, 2, 2, 3, 1\}$	88.87%	4	16	45	29	21	–
$\{0, 0, 3, 0, 3, 6\}$	$\{3, 3, 2, 3, 2, 1\}$	88.51%	20	48	54	26	12	36
$\{-3, -3, 6, 0, 0, 0\}$	$\{4, 4, 1, 3, 3, 3\}$	88.51%	4	16	18	6	–	12
$\{-3, 0, 3, 6, 0, 3\}$	$\{4, 3, 2, 1, 3, 2\}$	88.51%	40	32	36	36	24	72
$\{-3, 0, 0, 3, 6, 0\}$	$\{4, 3, 3, 2, 1, 3\}$	88.51%	24	96	108	36	24	72
$\{-3, -3, 0, 0, -3, 6\}$	$\{4, 4, 3, 3, 4, 1\}$	88.51%	4	–	–	–	6	12
$\{-3, 0, 3, 3, 0, 6\}$	$\{4, 3, 2, 2, 3, 1\}$	88.51%	20	32	18	10	6	36
$\{0, 0, 6, 6, 0, 0\}$	$\{3, 3, 1, 1, 3, 3\}$	87.76%	4	8	9	3	–	–
$\{0, 3, 3, 3, 3, 3\}$	$\{3, 2, 2, 2, 2, 2\}$	87.36%	–	–	–	6	6	–
$\{-3, -3, 3, 3, -3, 0\}$	$\{4, 4, 2, 2, 4, 3\}$	87.36%	4	–	–	4	6	–
$\{-3, -3, 0, 3, 3, 3\}$	$\{4, 4, 3, 2, 2, 2\}$	87.36%	4	–	–	4	18	–
$\{-3, 0, 3, 6, 0, -3\}$	$\{4, 3, 2, 1, 3, 4\}$	85.12%	8	16	18	6	6	–
$\{0, 3, 3, 3, 6, 0\}$	$\{3, 2, 2, 2, 1, 3\}$	85.12%	8	16	18	6	–	–
$\{0, 0, 6, 3, 3, 3\}$	$\{3, 3, 1, 2, 2, 2\}$	83.51%	4	–	–	4	6	–
$\{-3, 0, 6, 6, 0, 3\}$	$\{4, 3, 1, 1, 3, 2\}$	83.51%	–	16	18	4	–	–
$\{0, 0, 6, 6, 0, 3\}$	$\{3, 3, 1, 1, 3, 2\}$	83.51%	–	–	9	3	3	–
$\{-3, -3, 0, 3, 0, 6\}$	$\{4, 4, 3, 2, 3, 1\}$	83.51%	8	–	–	8	–	–
$\{-3, -3, 6, 3, 0, 0\}$	$\{4, 4, 1, 2, 3, 3\}$	83.51%	4	–	–	4	6	–
$\{3, 3, 3, 3, 3, 3\}$	$\{2, 2, 2, 2, 2, 2\}$	*NA*§	2	–	–	–	–	–
$\{-3, -3, 3, 3, -3, -3\}$	$\{4, 4, 2, 2, 4, 4\}$	*NA*§	–	–	–	–	–	–
$\{-3, -3, -3, 3, 3, 3\}$	$\{4, 4, 4, 2, 2, 2\}$	*NA*§	–	–	–	–	–	–
$\{0, 0, 0, 0, 0, 9\}$	$\{3, 3, 3, 3, 3, 0\}$	*NA*§	60	72	–	–	–	246
$\{0, 0, 3, 9, 0, 0\}$	$\{3, 3, 2, 0, 3, 3\}$	*NA*§	36	24	–	–	–	54
$\{0, 3, 6, 6, 3, 0\}$	$\{3, 2, 1, 1, 2, 3\}$	*NA*§	2	4	–	–	–	9
$\{-3, 0, 0, 0, 0, 9\}$	$\{4, 3, 3, 3, 3, 0\}$	*NA*§	20	16	–	–	–	36
$\{-3, 0, 6, 6, 0, -3\}$	$\{4, 3, 1, 1, 3, 4\}$	*NA*§	–	–	–	–	–	6
$\{0, 0, 6, 9, 0, 0\}$	$\{3, 3, 1, 0, 3, 3\}$	*NA*§	4	8	–	–	–	18
$\{0, 0, 9, 9, 0, 0\}$	$\{3, 3, 0, 0, 3, 3\}$	*NA*§	–	–	–	–	–	3

† *Efficiency of OA minus four run design compared to theoretical bound*

§ Ω_B *is singular*

Table A9: Part 1 of the distribution of pairwise Hamming distances between all sets of $t = 4$ runs for each of the last 6 (out of 12) OA$[18, 3^4]$ combinatorial isomorphism classes

	Hamming		**Count by Class**					
{b}	**Distance**	**Eff$_B$**†	**7**	**8**	**9**	**10**	**11**	**12**
$\{0,0,0,0,0,0\}$	$\{3,3,3,3,3,3\}$	100.00%	72	60	32	297	–	2016
$\{-3,0,0,0,0,0\}$	$\{4,3,3,3,3,3\}$	98.70%	96	90	80	108	72	–
$\{0,0,0,0,0,3\}$	$\{3,3,3,3,3,2\}$	98.70%	192	159	114	162	–	–
$\{-3,0,0,0,0,-3\}$	$\{4,3,3,3,3,4\}$	97.42%	6	18	24	–	54	–
$\{0,0,3,3,0,0\}$	$\{3,3,2,2,3,3\}$	97.42%	18	33	42	–	72	–
$\{-3,0,0,0,0,3\}$	$\{4,3,3,3,3,2\}$	97.42%	12	102	121	162	180	–
$\{0,0,0,0,3,3\}$	$\{3,3,3,3,2,2\}$	97.25%	360	282	296	324	288	–
$\{-3,-3,0,0,0,0\}$	$\{4,4,3,3,3,3\}$	97.25%	72	60	52	108	–	–
$\{-3,0,0,0,3,0\}$	$\{4,3,3,3,2,3\}$	97.25%	432	306	328	324	288	–
$\{-3,0,3,0,3,0\}$	$\{4,3,2,3,2,3\}$	96.72%	96	66	90	–	144	–
$\{-3,0,3,3,0,0\}$	$\{4,3,2,2,3,3\}$	95.79%	48	132	124	162	144	–
$\{-3,0,0,0,3,-3\}$	$\{4,3,3,3,2,4\}$	95.79%	24	60	80	108	72	–
$\{-3,-3,0,0,0,3\}$	$\{4,4,3,3,3,2\}$	95.79%	48	48	32	–	–	–
$\{0,0,3,3,0,3\}$	$\{3,3,2,2,3,2\}$	95.79%	72	126	136	–	216	–
$\{-3,0,0,0,3,3\}$	$\{4,3,3,3,2,2\}$	95.79%	96	108	124	–	144	–
$\{-3,-3,3,0,0,0\}$	$\{4,4,2,3,3,3\}$	95.59%	24	54	24	54	–	–
$\{-3,0,0,3,3,0\}$	$\{4,3,3,2,2,3\}$	95.59%	48	96	60	108	–	–
$\{0,0,3,0,3,3\}$	$\{3,3,2,3,2,2\}$	95.59%	24	6	12	18	–	–
$\{-3,-3,3,0,0,3\}$	$\{4,4,2,3,3,2\}$	95.19%	48	48	32	–	–	–
$\{-3,0,3,3,3,0\}$	$\{4,3,2,2,2,3\}$	95.19%	144	108	124	–	144	–
$\{-3,0,3,0,3,3\}$	$\{4,3,2,3,2,2\}$	95.19%	48	72	86	–	144	–
$\{-3,-3,0,-3,0,3\}$	$\{4,4,3,4,3,2\}$	95.19%	–	–	24	–	72	–
$\{-3,0,0,3,3,-3\}$	$\{4,3,3,2,2,4\}$	95.19%	48	36	34	–	72	–
$\{-3,0,3,3,0,3\}$	$\{4,3,2,2,3,2\}$	94.57%	60	48	58	–	72	–
$\{-3,-3,0,0,-3,3\}$	$\{4,4,3,3,4,2\}$	94.57%	12	18	6	–	–	–
$\{-3,3,3,3,3,-3\}$	$\{4,2,2,2,2,4\}$	94.35%	12	–	3	–	9	–
$\{-3,0,3,3,3,-3\}$	$\{4,3,2,2,2,4\}$	94.35%	–	21	20	–	–	–
$\{0,0,0,3,3,3\}$	$\{3,3,3,2,2,2\}$	94.35%	48	24	26	–	–	–
$\{-3,-3,0,3,0,0\}$	$\{4,4,3,2,3,3\}$	94.35%	48	42	64	–	144	–
$\{0,3,3,3,3,0\}$	$\{3,2,2,2,2,3\}$	93.68%	48	18	30	27	36	–
$\{-3,-3,0,0,3,3\}$	$\{4,4,3,3,2,2\}$	93.68%	48	30	20	54	–	–
$\{0,0,0,0,0,6\}$	$\{3,3,3,3,3,1\}$	93.68%	–	24	12	54	–	–
$\{-3,0,3,3,0,-3\}$	$\{4,3,2,2,3,4\}$	93.68%	48	15	18	–	36	–
$\{-3,-3,0,3,0,3\}$	$\{4,4,3,2,3,2\}$	92.46%	72	60	88	–	144	–
$\{-3,-3,-3,0,0,3\}$	$\{4,4,4,3,3,2\}$	92.46%	12	12	8	–	–	–
$\{-3,-3,3,3,0,0\}$	$\{4,4,2,2,3,3\}$	92.46%	36	18	22	–	–	–
$\{-3,0,3,3,3,3\}$	$\{4,3,2,2,2,2\}$	92.46%	144	48	52	–	–	–
$\{-3,-3,0,3,-3,0\}$	$\{4,4,3,2,4,3\}$	92.46%	24	24	16	–	–	–
$\{0,0,3,3,3,3\}$	$\{3,3,2,2,2,2\}$	92.46%	84	90	92	–	144	–
$\{-3,-3,3,3,0,3\}$	$\{4,4,2,2,3,2\}$	92.46%	96	48	80	–	144	–
$\{-3,-3,0,3,-3,3\}$	$\{4,4,3,2,4,2\}$	92.46%	48	18	24	–	–	–
$\{0,0,3,6,0,0\}$	$\{3,3,2,1,3,3\}$	92.46%	–	15	18	81	–	–

† *Efficiency of OA minus four run design compared to theoretical bound*

Table A10: Part 2 of the distribution of pairwise Hamming distances between all sets of $t = 4$ runs for each of the last 6 (out of 12) OA$[18, 3^4]$ combinatorial isomorphism classes

{b}	**Hamming Distance**	**Eff**$_B$†	**Count by Class** 7	8	9	10	11	12
$\{-3,0,0,3,3,3\}$	$\{4,3,3,2,2,2\}$	92.46%	60	60	74	–	144	–
$\{0,0,0,0,3,6\}$	$\{3,3,3,3,2,1\}$	91.38%	–	60	24	162	–	–
$\{-3,0,0,0,6,0\}$	$\{4,3,3,3,1,3\}$	91.38%	–	36	24	216	–	–
$\{-3,-3,-3,-3,3,3\}$	$\{4,4,4,4,2,2\}$	90.20%	–	–	6	–	18	–
$\{-3,-3,3,3,-3,3\}$	$\{4,4,2,2,4,2\}$	90.20%	–	6	5	–	18	–
$\{-3,3,3,3,3,3\}$	$\{4,2,2,2,2,2\}$	90.20%	–	12	15	–	36	–
$\{-3,0,0,0,3,6\}$	$\{4,3,3,3,2,1\}$	89.88%	–	24	24	108	–	–
$\{0,0,3,6,0,3\}$	$\{3,3,2,1,3,2\}$	89.88%	–	24	24	–	–	–
$\{-3,-3,0,0,0,6\}$	$\{4,4,3,3,3,1\}$	89.88%	–	18	12	–	–	–
$\{-3,0,3,6,0,0\}$	$\{4,3,2,1,3,3\}$	89.88%	–	18	12	–	–	–
$\{-3,0,0,0,6,3\}$	$\{4,3,3,3,1,2\}$	88.87%	–	12	10	–	–	–
$\{-3,0,0,0,6,-3\}$	$\{4,3,3,3,1,4\}$	88.87%	–	12	8	–	–	–
$\{-3,0,3,6,3,0\}$	$\{4,3,2,1,2,3\}$	88.87%	–	18	10	–	–	–
$\{0,0,3,3,0,6\}$	$\{3,3,2,2,3,1\}$	88.87%	–	9	14	81	–	–
$\{0,0,3,0,3,6\}$	$\{3,3,2,3,2,1\}$	88.51%	–	12	8	54	–	–
$\{-3,-3,6,0,0,0\}$	$\{4,4,1,3,3,3\}$	88.51%	–	–	–	18	–	–
$\{-3,0,3,6,0,3\}$	$\{4,3,2,1,3,2\}$	88.51%	–	24	12	108	–	–
$\{-3,0,0,3,6,0\}$	$\{4,3,3,2,1,3\}$	88.51%	–	12	12	108	–	–
$\{-3,-3,0,0,-3,6\}$	$\{4,4,3,3,4,1\}$	88.51%	–	–	–	18	–	–
$\{-3,0,3,3,0,6\}$	$\{4,3,2,2,3,1\}$	88.51%	–	12	4	–	–	–
$\{0,0,6,6,0,0\}$	$\{3,3,1,1,3,3\}$	87.76%	–	3	–	–	–	–
$\{0,3,3,3,3,3\}$	$\{3,2,2,2,2,2\}$	87.36%	–	3	6	–	–	–
$\{-3,-3,3,3,-3,0\}$	$\{4,4,2,2,4,3\}$	87.36%	–	6	4	–	–	–
$\{-3,-3,0,3,3,3\}$	$\{4,4,3,2,2,2\}$	87.36%	–	12	10	–	–	–
$\{-3,0,3,6,0,-3\}$	$\{4,3,2,1,3,4\}$	85.12%	–	–	–	–	–	–
$\{0,3,3,3,6,0\}$	$\{3,2,2,2,1,3\}$	85.12%	–	6	–	–	–	–
$\{0,0,6,3,3,3\}$	$\{3,3,1,2,2,2\}$	83.51%	–	6	4	–	–	–
$\{-3,0,6,6,0,3\}$	$\{4,3,1,1,3,2\}$	83.51%	–	–	1	–	–	–
$\{0,0,6,6,0,3\}$	$\{3,3,1,1,3,2\}$	83.51%	–	–	–	27	–	–
$\{-3,-3,0,3,0,6\}$	$\{4,4,3,2,3,1\}$	83.51%	–	6	2	–	–	–
$\{-3,-3,6,3,0,0\}$	$\{4,4,1,2,3,3\}$	83.51%	–	6	4	–	–	–
$\{3,3,3,3,3,3\}$	$\{2,2,2,2,2,2\}$	NA§	5	–	–	–	–	–
$\{-3,-3,3,3,-3,-3\}$	$\{4,4,2,2,4,4\}$	NA§	3	–	3	–	9	–
$\{-3,-3,-3,3,3,3\}$	$\{4,4,4,2,2,2\}$	NA§	4	–	–	–	–	–
$\{0,0,0,0,0,9\}$	$\{3,3,3,3,3,0\}$	NA§	48	–	–	–	–	1008
$\{0,0,3,9,0,0\}$	$\{3,3,2,0,3,3\}$	NA§	48	–	–	–	–	–
$\{0,3,6,6,3,0\}$	$\{3,2,1,1,2,3\}$	NA§	–	–	–	–	–	–
$\{-3,0,0,0,0,9\}$	$\{4,3,3,3,3,0\}$	NA§	24	–	–	–	–	–
$\{-3,0,6,6,0,-3\}$	$\{4,3,1,1,3,4\}$	NA§	–	–	–	9	–	–
$\{0,0,6,9,0,0\}$	$\{3,3,1,0,3,3\}$	NA§	–	–	–	–	–	–
$\{0,0,9,9,0,0\}$	$\{3,3,0,0,3,3\}$	NA§	–	–	–	–	–	36

† *Efficiency of OA minus four run design compared to theoretical bound*

§ Ω_B *is singular*

Table A11: Part 1 of the distribution of pairwise Hamming distances between all sets of $t = 4$ runs for each of the 10 OA$[18, 3^5]$ combinatorial isomorphism classes

{**b**}	**Hamming Distance**	**Eff**$_B$†	**Count by Class** 1	2	3	4	5	6	7	8	9	10
{−1, −1, −1, −1, −1, −1}	{4, 4, 4, 4, 4, 4}	99.04%	–	12	24	54	24	21	27	18	126	45
{−1, −1, −1, −1, −1, 2}	{4, 4, 4, 4, 4, 3}	98.10%	198	117	336	192	336	87	60	60	576	–
{−1, −1, −1, −1, 2, 2}	{4, 4, 4, 4, 3, 3}	97.59%	144	164	144	248	144	144	104	192	216	–
{−1, −1, 2, −1, 2, 2}	{4, 4, 3, 4, 3, 3}	97.59%	108	138	144	144	144	150	216	168	–	360
{−1, −1, 2, 2, −1, 2}	{4, 4, 3, 3, 4, 3}	97.19%	360	310	240	160	240	288	208	216	–	–
{−1, 2, 2, 2, 2, −1}	{4, 3, 3, 3, 3, 4}	97.05%	72	76	–	49	–	84	136	117	9	270
{−1, −1, 2, 2, −1, −1}	{4, 4, 3, 3, 4, 4}	96.92%	63	39	102	60	102	36	–	60	180	–
{−1, −1, 2, 2, 2, 2}	{4, 4, 3, 3, 3, 3}	96.04%	180	204	96	96	96	204	168	120	–	–
{−4, −1, −1, −1, −1, −1}	{5, 4, 4, 4, 4, 4}	95.88%	18	14	24	8	24	6	8	12	–	–
{−1, −1, −1, 2, 2, 2}	{4, 4, 4, 3, 3, 3}	95.57%	72	66	120	144	120	66	48	96	144	–
{−4, −1, −1, −1, −1, 2}	{5, 4, 4, 4, 4, 3}	95.07%	18	6	–	–	–	6	–	–	–	–
{−4, −1, 2, −1, 2, −1}	{5, 4, 3, 4, 3, 4}	94.90%	36	34	24	64	24	42	16	48	–	–
{−4, −1, 2, −1, 2, 2}	{5, 4, 3, 4, 3, 3}	94.73%	36	30	–	–	–	30	24	24	–	–
{−1, 2, 2, 2, 2, 2}	{4, 3, 3, 3, 3, 3}	94.55%	90	153	78	132	78	201	288	240	36	540
{−4, 2, 2, 2, 2, −1}	{5, 3, 3, 3, 3, 4}	94.55%	–	28	6	16	6	36	76	72	–	180
{−4, −1, −1, −1, 2, −1}	{5, 4, 4, 4, 3, 4}	93.80%	108	82	96	64	96	66	40	72	–	–
{−4, −1, 2, 2, 2, −1}	{5, 4, 3, 3, 3, 4}	93.20%	108	128	96	80	96	144	128	72	–	–
{−4, −1, −1, −1, 2, 2}	{5, 4, 4, 4, 3, 3}	93.20%	144	122	96	32	96	102	80	96	–	–
{−4, 2, 2, 2, 2, 2}	{5, 3, 3, 3, 3, 3}	92.78%	–	16	–	16	–	18	16	–	–	–
{−4, −1, −1, 2, 2, −1}	{5, 4, 4, 3, 3, 4}	92.11%	72	86	48	104	48	102	176	108	–	360
{−4, −1, 2, 2, 2, 2}	{5, 4, 3, 3, 3, 3}	92.11%	36	144	–	144	–	204	360	240	–	720
{−4, −1, 2, 2, −1, 2}	{5, 4, 3, 3, 4, 3}	92.11%	72	90	48	48	48	108	168	120	–	360
{−1, −1, 2, −1, 2, 5}	{4, 4, 3, 4, 3, 2}	92.11%	72	34	48	40	48	30	16	24	72	–
{−4, 2, 2, 2, 2, −4}	{5, 3, 3, 3, 3, 5}	91.87%	–	2	–	8	–	6	8	–	–	–
{−1, −1, 2, 2, −1, 5}	{4, 4, 3, 3, 4, 2}	91.87%	18	18	48	72	48	24	–	72	216	–
{−1, 2, 2, 2, 5, −1}	{4, 3, 3, 3, 2, 4}	91.38%	72	66	96	48	96	54	48	–	–	–
{−1, −1, 2, 5, −1, 2}	{4, 4, 3, 2, 4, 3}	91.13%	144	104	240	176	240	84	56	120	144	–
{−4, −4, 2, −1, 2, 2}	{5, 5, 3, 4, 3, 3}	91.13%	–	8	–	8	–	12	8	12	–	–
{−1, −1, −1, −1, −1, 5}	{4, 4, 4, 4, 4, 2}	91.13%	18	20	72	80	72	24	20	30	180	–
{−1, −1, −1, −1, 2, 5}	{4, 4, 4, 4, 3, 2}	91.13%	108	98	240	176	240	78	56	48	576	–
{−1, −1, 5, 2, 2, 2}	{4, 4, 2, 3, 3, 3}	90.59%	36	38	48	80	48	30	32	48	144	–
{−4, −4, 2, −1, −1, 2}	{5, 5, 3, 4, 4, 3}	90.32%	–	16	–	16	–	24	16	24	–	–
{−4, −4, −1, −1, −1, 2}	{5, 5, 4, 4, 4, 3}	90.32%	–	16	–	16	–	24	16	24	–	–
{−4, −1, 2, 2, −1, −1}	{5, 4, 3, 3, 4, 4}	90.32%	18	42	–	48	–	36	48	48	–	–
{−4, −1, −1, 2, 2, −4}	{5, 4, 4, 3, 3, 5}	90.32%	–	18	–	–	–	18	24	24	–	–
{2, 2, 2, 2, 2, 2}	{3, 3, 3, 3, 3, 3}	89.73%	–	8	6	8	6	6	8	–	–	–
{−4, −1, 2, −1, 2, 5}	{5, 4, 3, 4, 3, 2}	89.42%	–	6	24	–	24	6	–	–	–	–
{−4, −1, −1, −1, 2, −4}	{5, 4, 4, 4, 3, 5}	89.10%	18	6	–	–	–	6	–	–	–	–
{−4, −1, 2, −1, 5, 2}	{5, 4, 3, 4, 2, 3}	89.10%	36	22	–	16	–	18	16	24	–	–
{−4, 2, 2, 2, 5, −1}	{5, 3, 3, 3, 2, 4}	89.10%	–	22	–	16	–	30	16	24	–	–
{−1, −1, 2, 5, −1, −1}	{4, 4, 3, 2, 4, 4}	88.77%	18	12	12	24	12	12	–	–	72	–
{−4, −4, −1, −4, 2, 2}	{5, 5, 4, 5, 3, 3}	88.77%	–	–	–	–	–	–	30	–	–	90
{−4, −1, 2, −1, 5, −1}	{5, 4, 3, 4, 2, 4}	88.77%	36	18	48	–	48	6	–	–	–	–
{−4, −1, 2, 2, 2, −4}	{5, 4, 3, 3, 3, 5}	88.77%	–	11	–	8	–	15	26	24	–	90
{−4, −1, −1, −1, −1, 5}	{5, 4, 4, 4, 4, 2}	88.77%	18	6	12	–	12	–	–	–	–	–
{−4, −1, 2, 2, 5, −1}	{5, 4, 3, 3, 2, 4}	87.70%	72	34	48	16	48	18	16	24	–	–
{−4, −1, −1, 2, 2, 2}	{5, 4, 4, 3, 3, 3}	87.70%	36	36	24	–	24	36	24	24	–	–
{−4, −1, 5, 2, 2, 2}	{5, 4, 2, 3, 3, 3}	87.70%	36	34	48	16	48	30	16	24	–	–
{−4, −1, 2, 2, −1, 5}	{5, 4, 3, 3, 4, 2}	87.70%	36	22	24	16	24	18	16	–	–	–
{−4, −4, 2, −1, −1, −1}	{5, 5, 3, 4, 4, 4}	87.32%	–	8	–	8	–	12	8	12	–	–
{−1, −1, 2, 2, 2, 5}	{4, 4, 3, 3, 3, 2}	86.03%	108	80	96	32	96	60	32	48	–	–

† *Efficiency of OA minus four run design compared to theoretical bound*

Table A12: Part 2 of the distribution of pairwise Hamming distances between all sets of $t = 4$ runs for each of the 10 OA$[18, 3^5]$ combinatorial isomorphism classes

	Hamming		**Count by Class**									
$\{\mathbf{b}\}$	**Distance**	$\mathbf{Eff}_B$ †	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$\{-4,-4,2,-1,-1,5\}$	$\{5,5,3,4,4,2\}$	85.56%	–	8	–	8	–	12	8	12	–	–
$\{-4,-4,-1,-1,2,2\}$	$\{5,5,4,4,3,3\}$	85.56%	–	4	–	4	–	6	4	6	–	–
$\{-4,-1,-1,-1,2,5\}$	$\{5,4,4,4,3,2\}$	85.56%	–	8	–	32	–	12	8	24	–	–
$\{-4,-1,2,2,5,-4\}$	$\{5,4,3,3,2,5\}$	85.56%	–	4	–	4	–	6	4	6	–	–
$\{-4,2,2,2,5,-4\}$	$\{5,3,3,3,2,5\}$	85.56%	9	3	–	–	–	–	–	–	–	–
$\{-4,-1,5,2,2,-1\}$	$\{5,4,2,3,3,4\}$	85.56%	36	30	–	24	–	30	24	36	–	–
$\{-1,-1,2,5,2,2\}$	$\{4,4,3,2,3,3\}$	85.56%	36	38	–	32	–	30	32	48	–	–
$\{-1,-1,5,5,-1,2\}$	$\{4,4,2,2,4,3\}$	85.56%	18	7	–	16	–	9	4	–	72	–
$\{-1,2,5,5,2,-1\}$	$\{4,3,2,2,3,4\}$	85.06%	9	3	12	12	12	–	–	12	36	–
$\{-4,2,2,2,5,2\}$	$\{5,3,3,3,2,3\}$	83.35%	18	16	24	16	24	18	16	24	–	–
$\{-1,2,2,2,5,2\}$	$\{4,3,3,3,2,3\}$	81.20%	36	22	–	16	–	30	16	24	–	–
$\{-4,-1,2,2,2,5\}$	$\{5,4,3,3,3,2\}$	81.20%	–	10	–	16	–	6	16	24	–	–
$\{-1,-1,5,5,2,2\}$	$\{4,4,2,2,3,3\}$	80.34%	–	6	24	–	24	6	–	–	–	–
$\{-1,-1,2,5,-1,5\}$	$\{4,4,3,2,4,2\}$	80.34%	–	–	48	48	48	–	–	–	144	–
$\{-1,-1,2,-1,5,5\}$	$\{4,4,3,4,2,2\}$	80.34%	–	–	24	24	24	–	–	–	72	–
$\{-4,2,2,2,2,5\}$	$\{5,3,3,3,3,2\}$	80.34%	9	3	–	–	–	6	–	–	–	–
$\{-1,-1,-1,2,2,5\}$	$\{4,4,4,3,3,2\}$	80.34%	36	12	24	–	24	12	–	–	–	–
$\{-4,-1,2,2,5,2\}$	$\{5,4,3,3,2,3\}$	75.43%	–	6	–	–	–	6	–	–	–	–
$\{-4,2,5,5,2,2\}$	$\{5,3,2,2,3,3\}$	75.43%	9	3	–	–	–	–	–	–	–	–
$\{-4,2,5,5,2,-4\}$	$\{5,3,2,2,3,5\}$	*NA*§	–	–	3	–	3	–	–	–	–	–
$\{-1,2,2,2,2,5\}$	$\{4,3,3,3,3,2\}$	*NA*§	–	4	6	4	6	–	4	6	–	–
$\{-4,-1,2,2,-1,-4\}$	$\{5,4,3,3,4,5\}$	*NA*§	9	7	12	4	12	9	19	6	–	45
$\{-1,-1,5,5,-1,-1\}$	$\{4,4,2,2,4,4\}$	*NA*§	–	2	15	5	15	–	2	3	9	–
$\{-1,-1,-1,-1,5,5\}$	$\{4,4,4,4,2,2\}$	*NA*§	–	–	6	6	6	–	–	–	18	–
$\{-1,-1,5,5,-1,5\}$	$\{4,4,2,2,4,2\}$	*NA*§	–	–	6	6	6	–	–	–	18	–

† *Efficiency of OA minus four run design compared to theoretical bound*

Table A13: Distribution of pairwise Hamming distances between all sets of $t = 4$ runs for each of the 8 OA$[18, 3^6]$ combinatorial isomorphism classes

	Hamming		**Count by Class**							
{b}	**Distance**	**Eff**$_B$ †	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
$\{-2,1,1,1,1,1\}$	$\{5,4,4,4,4,4\}$	97.20%	108	108	180	108	108	180	216	–
$\{1,1,1,1,1,1\}$	$\{4,4,4,4,4,4\}$	97.20%	90	90	188	90	90	267	503	1215
$\{-2,1,1,1,1,-2\}$	$\{5,4,4,4,4,5\}$	96.72%	72	72	72	72	72	96	–	–
$\{-2,-2,1,1,1,1\}$	$\{5,5,4,4,4,4\}$	95.07%	432	432	480	432	432	456	432	–
$\{-2,-2,-2,-2,1,1\}$	$\{5,5,5,5,4,4\}$	92.99%	216	216	72	216	216	–	–	–
$\{-2,-2,1,1,-2,1\}$	$\{5,5,4,4,5,4\}$	92.99%	216	216	144	216	216	144	–	–
$\{-2,-2,-2,-2,-2,1\}$	$\{5,5,5,5,5,4\}$	92.15%	36	36	12	36	36	–	–	–
$\{-2,-2,1,-2,1,4\}$	$\{5,5,4,5,4,3\}$	88.88%	144	144	48	144	144	–	–	–
$\{-2,1,1,1,4,-2\}$	$\{5,4,4,4,3,5\}$	88.88%	144	144	96	144	144	96	–	–
$\{-2,-2,4,1,1,1\}$	$\{5,5,3,4,4,4\}$	87.37%	216	216	192	216	216	180	120	–
$\{-2,-2,-2,1,1,1\}$	$\{5,5,5,4,4,4\}$	87.37%	72	72	64	72	72	60	40	–
$\{-2,-2,1,1,-2,4\}$	$\{5,5,4,4,5,3\}$	87.37%	216	216	144	216	216	96	96	–
$\{-2,1,1,1,4,1\}$	$\{5,4,4,4,3,4\}$	87.37%	432	432	480	432	432	456	432	–
$\{-2,1,1,1,1,4\}$	$\{5,4,4,4,4,3\}$	85.46%	72	72	72	72	72	96	–	–
$\{-2,-2,4,1,1,4\}$	$\{5,5,3,4,4,3\}$	78.53%	144	144	48	144	144	–	–	–
$\{-2,1,1,4,4,-2\}$	$\{5,4,4,3,3,5\}$	78.53%	72	72	24	72	72	–	–	–
$\{-2,-2,1,1,1,4\}$	$\{5,5,4,4,4,3\}$	78.53%	144	144	96	144	144	96	–	–
$\{-2,1,4,4,1,1\}$	$\{5,4,3,3,4,4\}$	78.53%	72	72	48	72	72	48	–	–
$\{-2,-2,1,1,-2,-2\}$	$\{5,5,4,4,5,5\}$	*NA*§	45	45	27	45	45	12	24	–
$\{1,1,1,1,1,4\}$	$\{4,4,4,4,4,3\}$	*NA*§	36	36	60	36	36	60	72	–
$\{-5,1,1,1,1,1\}$	$\{6,4,4,4,4,4\}$	*NA*§	–	–	156	–	–	270	624	1620
$\{-5,1,1,1,1,-2\}$	$\{6,4,4,4,4,5\}$	*NA*§	–	–	72	–	–	72	144	–
$\{-5,-2,1,-2,1,1\}$	$\{6,5,4,5,4,4\}$	*NA*§	–	–	72	–	–	108	72	–
$\{-5,-2,1,-2,1,-2\}$	$\{6,5,4,5,4,5\}$	*NA*§	–	–	48	–	–	72	48	–
$\{-2,1,4,4,1,-2\}$	$\{5,4,3,3,4,5\}$	*NA*§	45	45	27	45	45	12	24	–
$\{-5,-2,-2,-2,-2,1\}$	$\{6,5,5,5,5,4\}$	*NA*§	–	–	12	–	–	18	12	–
$\{-5,1,1,1,1,4\}$	$\{6,4,4,4,4,3\}$	*NA*§	–	–	24	–	–	24	48	–
$\{-2,-2,1,-2,4,4\}$	$\{5,5,4,5,3,3\}$	*NA*§	18	18	6	18	18	–	–	–
$\{-5,-2,1,-2,1,4\}$	$\{6,5,4,5,4,3\}$	*NA*§	–	–	24	–	–	36	24	–
$\{-5,1,1,1,1,-5\}$	$\{6,4,4,4,4,6\}$	*NA*§	–	–	6	–	–	15	39	135
$\{-2,1,4,4,4,-2\}$	$\{5,4,3,3,3,5\}$	*NA*§	18	18	6	18	18	–	–	–
$\{-5,1,4,1,4,1\}$	$\{6,4,3,4,3,4\}$	*NA*§	–	–	24	–	–	36	24	–
$\{-5,1,4,1,4,-2\}$	$\{6,4,3,4,3,5\}$	*NA*§	–	–	24	–	–	36	24	–
$\{-5,-2,4,-2,4,1\}$	$\{6,5,3,5,3,4\}$	*NA*§	–	–	12	–	–	18	12	–
$\{-5,-5,1,-5,1,1\}$	$\{6,6,4,6,4,4\}$	*NA*§	–	–	–	–	–	–	30	90

† *Efficiency of OA minus four run design compared to theoretical bound*

§ Ω_B *is singular*

Table A14: Distribution of pairwise Hamming distances between all sets of $t = 4$ runs for each of the 3 OA$[18, 3^7]$ combinatorial isomorphism classes

	Hamming		Count by Class		
{b}	**Distance**	$\mathbf{Eff}_B$ †	**1**	**2**	**3**
$\{0, 0, 0, 0, 0, 3\}$	$\{5, 5, 5, 5, 5, 4\}$	$NA^§$	972	972	972
$\{-3, 0, 0, 0, 0, 0\}$	$\{6, 5, 5, 5, 5, 5\}$	$NA^§$	648	648	648
$\{0, 0, 3, 3, 0, 0\}$	$\{5, 5, 4, 4, 5, 5\}$	$NA^§$	243	243	243
$\{-3, 0, 0, 0, 0, 3\}$	$\{6, 5, 5, 5, 5, 4\}$	$NA^§$	324	324	324
$\{-3, 0, 0, 0, 0, -3\}$	$\{6, 5, 5, 5, 5, 6\}$	$NA^§$	108	108	108
$\{-3, 0, 3, 0, 3, 0\}$	$\{6, 5, 4, 5, 4, 5\}$	$NA^§$	648	648	648
$\{-3, -3, 0, -3, 0, 0\}$	$\{6, 6, 5, 6, 5, 5\}$	$NA^§$	72	72	72
$\{-3, 3, 3, 3, 3, -3\}$	$\{6, 4, 4, 4, 4, 6\}$	$NA^§$	27	27	27
$\{-3, -3, 3, -3, 3, 3\}$	$\{6, 6, 4, 6, 4, 4\}$	$NA^§$	18	18	18

† *Efficiency of OA minus four run design compared to theoretical bound*

§ Ω_B *is singular*