Matching with Aggregate Externalities

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Abstract

Certain aggregate externalities, like those due to knowledge and public goods, do not change very much in response to changes in two individuals' actions. Thus, individuals rationally regard the level of the externality as fixed in their negotiations with each other. We leverage this observation to develop a general framework for the existence of stable matchings in moderately sized one-to-one matching games, and we characterize intuitive restrictions on preferences that are sufficient for existence.

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1 Introduction

Most of the two-sided matching literature neglects the role of externalities. In these models, workers and firms only care about the identities of their partners, and *not* the identities of the other players' partners. Yet, people often care about others' partners. For instance, a worker may care about the total amount of economic activity since more employment means a larger tax base and more public goods–e.g., more parks and better schools. Likewise, a firm may care about the quality of its competitor's workers since a higher-quality staff engages in more experimentation and produces more knowledge, which improves the competitors' efficiencies, as well as the efficiency of the firm itself since knowledge (eventually) spills over.

Beginning with Sasaki and Toda [19], a number of authors (see the next subsection) have sought to incorporate externalities into the classic two-sided matching model. Usually, their

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models assume that players have preferences over the set of all matchings and are rational, in that they can foresee the effects of their own actions on others' actions. While this rationality assumption is economically reasonable, it is also technically burdensome and has inhibited the development of simple sufficient conditions for the existence of a stable matching. Our goal is not to relax or remove it, but rather to build a simple model where it "does not matter" and give straightforward, economically motivated conditions that guarantee existence.

Our key insight is that certain "aggregate" externalities, such as those due to knowledge and certain public goods, only change slightly in response to changes in any two individuals' actions in moderately sized markets. Thus, individuals do not need to be excessively rational; instead, they may *rationally* ignore the effects of their actions on the externality. To illustrate, suppose that Abby, a mother of two young children, is considering whether to move to Flagstaff, Arizona, for a new, higher-paying job. Her decision depends, in part, on the quality of Flagstaff's public schools–if the schools are poor and her pay increase is insufficient to send her kids to private schools, then she may choose to keep her current job. If Abby is rational in our sense, then she looks at recent test scores and graduation rates to determine the quality of Flagstaff's schools. However, if Abby is rational in the sense of Sasaki and Toda [19], then she predicts the quality of Flagstaff's schools after accounting for the effects of the additional school funding she would provide via taxes. Yet, it is intuitive that the additional funding provided by one reasonably paid worker in a town of 70,500 workers is negligible.[1] Hence, the school qualities arrived at by both types of Abby are very close and both types make the same decision. Therefore, Abby can rationally ignore the effect of her decision on school quality. We establish this rational-ignorance result formally in Lemma 1.

We leverage this insight to develop a general framework for the existence of stable matchings in moderately sized one-to-one matching markets with aggregate externalities. We first establish a fixed point characterization of the existence problem. Our primary tool is an "Auxiliary (Matching) Game" where players regard the level of the externality as fixed. We use these games to define a correspondence whose set of fixed points is equal to the set of stable matchings in the full game (see Proposition 1). We also develop an abstract sufficient condition for the existence of such fixed points (see Proposition 2).

We next make use of Proposition 2 to show that stable matchings exist, provided one of two intuitive and economically motivated restrictions on preferences hold. Our restrictions focus on non-additive positive externalities, where the externality is an increasing function of the number of employed workers. The first restriction is motivated by (industrial) knowledge. It stipulates that workers and firms who are skilled enough to be matched at low levels of

[1] The employment count is as of June 2015 and comes from the Bureau of Labor Statistics.

knowledge are "individually rational" to each other at higher levels of knowledge. That is, additional knowledge, which is produced by the economic activities of workers and firms, enhances productivity. The second restriction is motivated by public goods. It stipulates that as the number of employed workers increases, players find it less appealing to quit the market and move elsewhere. That is, more public goods (e.g., new parks and better teachers), which are funded by the economic activities of workers and firms, make the community "nicer" (e.g., by providing places to relax and by improving school quality). We prove the first and second conditions are sufficient for the existence of stable matchings in Propositions 3 and 4.

The next subsection discusses the related literature. Section 2 gives the model and solution concept. Section 3 lays out the framework for existence. Section 4 develops our conditions for positive externalities. Section 5 gives three illustrative examples. Section 6 concludes.

Related Literature

Our work is an extension of the matching with externalities literature, which began with Sasaki and Toda [19]. As mentioned above, Sasaki and Toda model externalities by assuming players have preferences over the set of all matchings. Since the reactions of the rest of the players to a deviating pair affects the deviators' utilities, the deviators needs to "estimate" the (set of) matchings that will result from their block. Sasaki and Toda assume these estimates are exogenous and record them with "estimation functions." They show that the only estimation function that is compatible with a stable matching is the one that estimates all matchings are possible. Hafalir [11] endogenizes these estimation functions and shows that if players estimate in a "sophisticated" way, then (i) stable matchings exist and (ii) players may estimate that not all matchings are possible.

More recently, Mumcu and Saglam [15] and Bando [3] have also considered matching markets with externalities. In contrast to Sasaki and Toda's [19] strong rationality requirement, these papers use variants of a simple, intuitive rationality requirement: that a deviating pair holds the actions of all other players constant when considering their block.[2] Mumcu and Saglam [15] consider a one-to-one matching market and show that a stable matching exists if there is a subset of matchings $\mathcal{V}$ such that each player's $|\mathcal{V}|$ most favored matchings are in $\mathcal{V}$; notice that this condition is on the whole preference profile rather than individual preferences.

Bando [3] considers a many-to-one matching market where firms care about the workers hired by their rivals, while workers only care about the identity of the firms that hire them. He shows that a stable matching exists if four conditions hold: (i) substitutability, (ii) positive

[2] Alternatively, that a deviation does not trigger a response among the rest of the players.

externalities, which requires that a firm does better as its rivals employ more workers, (iii) increasing choice, which requires the set of workers an unmatched firm chooses from a set of unmatched workers expands as the number of workers hired by rivals increases, and (iv) no external effect by an unchosen worker, which requires that if a firm does not choose a worker w from a set W when the firm is inactive, then the firm's choice from C is unaffected when w is removed from W and assigned to another firm.

In another recent work, Pycia and Yenmez [17] consider a general many-to-many matching market with contracts and externalities. They establish existence of stable matchings under a condition that extends the classical substitutes condition to allow for externalities. Although they consider a "one-shot" definition of stability that is similar to Mumcu and Saglam [15] and Bando [3], they remark how "far-sighted" notions (e.g., those of Sasaki and Toda [19]) can be embedded with a suitable selection of the choice functions. They establish existence of side-optimal stable matchings, even though the standard fixed point techniques do not apply in their setup.[3] On top of the technical differences in our approaches–and more importantly–we differ in our economic emphasis by focusing on aggregate externalities and by developing simple sufficient conditions crafted for our environment.

In an applied paper, Baccara et al. [2] consider an office assignment problem with externalities and transferable utility. Using a linear programming problem, they show a stable matching exists when externalities are additive and only depend on the number of peers in close proximity to a player's assigned office. Uetake and Watanabe [20] consider an industry game where (i) players make entry and merger decisions and (ii) each firm's profit is strictly decreasing in the total number of operating firms. They establish existence of stable matchings in this setup via generalizing the Deferred Acceptance algorithm by taking the "estimated number of operating firms" into account.[4] We contribute to these papers via our focus on aggregate externalities and our technical approach.

2 The Model and Stability

In this section, we first develop our model and then we give and motivate our stability concept.

Model

We consider a one-to-one matching market with two finite groups of players: workers

[3]Fixed point methods are routinely used in the theory of matching without externalities; notable examples include Adachi [1] and Fleiner [9].

[4]There is also a literature that examines externalities due to peer effects among workers matched to the same firm or preferences of couples; see, for instance, Dutta and Masso [6], Echenique and Yenmez [7], Klaus and Klijn [12], Kojima et al. [13], and Pycia [16].

$\mathcal{W} = \{1, \ldots, W\}$ and firms $\mathcal{F} = \{W+1, \ldots, N\}$, with $N > W > 0$. Let $\mathcal{N} = \mathcal{W} \cup \mathcal{F}$ denote the set of players. We write w for an arbitrary worker, f for an arbitrary firm, and i for an arbitrary player. Each player may either be single or may be matched to a member of the opposite group. A **matching** is a function, ϕ, that specifies each player's match, i.e., $\phi : \mathcal{N} \to \mathcal{N}$ such that: (i) for each worker w, $\phi(w) \in \mathcal{F} \cup \{w\}$; (ii) for each firm f, $\phi(f) \in \mathcal{W} \cup \{f\}$; and (iii) for each worker w and each firm f, $\phi(w) = f \iff w = \phi(f)$. We write $\Phi = \{\phi : \mathcal{N} \to \mathcal{N} | \, \phi \text{ is a matching}\}$ for the finite set of all matchings. A player is single if she is matched to herself and is partnered/employed if she is matched to a member of the opposite group. Also, for each $\phi \in \Phi$, let $C(\phi) = |\{w | \phi(w) \neq w\}|$ denote the number of couples/employed workers in ϕ, where $|\cdot|$ denotes the cardinality of a set.

Each matching ϕ is associated with a certain level of (aggregate) externality $e(\phi)$, where $e : \Phi \to \mathbb{R}$ is the "externality function." For example, $e(\phi)$ may record the amount of knowledge or school quality in the matching ϕ. Let $E = \cup_{\phi \in \Phi}\{e(\phi)\}$ be the set of possible levels of externality; E is a finite subset of $\mathbb{R}$ since Φ is finite. Players care about whom they are matched with and the level of the externality. Formally, each player i has payoff $u_i : \mathcal{M}_i \times E \to \mathbb{R}$, where $\mathcal{M}_i = \mathcal{F} \cup \{i\}$ when $i \in \mathcal{W}$ and $\mathcal{M}_i = \mathcal{W} \cup \{i\}$ when $i \in \mathcal{F}$. We write $u_i(\phi)$ for player i's payoff to matching ϕ, i.e., $u_i(\phi) \equiv u_i(\phi(i), e(\phi))$.

Note that our framework does not (yet) restrict the "dimensionality" of the externality or require that players be affected by the externality in the same way. In fact, it allows them to have arbitrary preferences over the set of all matchings Φ.[5] It is only in Section 4 that we will restrict preferences.

STABILITY

We say that a matching ϕ is **individually rational** if no player wants to exit the market and become single, assuming the externality level is unchanged, i.e., if $u_i(\phi) \geq u_i(i, e(\phi))$ for each player i. We say that a worker w and a firm f **block** a matching ϕ if they both do strictly better by matching with each other instead of following ϕ, assuming the level of externality remains unchanged, i.e., if $u_w(f, e(\phi)) > u_w(\phi)$ and $u_f(w, e(\phi)) > u_f(\phi)$. We say that a matching ϕ is **stable** if it is individually rational and it is not blocked by any worker and firm. Stable matchings are our solution concept.

There are two rationalizations for our notions of individual rationality and blocking. The first rationalization is the one we discussed in the introduction: the actions of any individual worker or firm or of any pair of a worker and a firm have only a small effect on the externality because the market is moderately sized.[6] Thus, (i) individuals can *rationally*

[5]To see this, suppose that player i has a preference $\tilde{u}_i : \Phi \to \mathbb{R}$. To each matching $\phi \in \Phi$, assign a unique natural number $n(\phi)$. Then, we may embed i's preference in our game by setting $u_i(\phi(i), n(\phi)) = \tilde{u}_i(\phi)$ for each $\phi \in \Phi$ and $u_i(\cdot) = 0$ everywhere else.

[6]We would like to emphasize that the market does not have to be large in order for individuals and pairs

ignore the effects of their exits when thinking about leaving and (ii) a worker and a firm can *rationally* ignore the effects of their matching when negotiating with each other. In other words, when deviating, a player or a pair of players find the effects of a partner change much more important than the small externality change brought about by their deviation, so they choose to exit or block regardless of whether they consider the small externality change. We comment on this below. The second rationalization is one that could work even for small markets: players may be boundedly rational and ignore the effects of their individual actions on the level of externality because (i) it is excessively costly for them to compute these effects or (ii) they simply lack the ability to do so.

To develop the first rationalization more formally, let $\delta^\star$ be the maximum amount by which any individual or any pair of a worker and a firm can change the externality.[7] In moderately sized markets, we find the following conditions intuitive.

Condition 1. The maximum change in a player's utility from a change in the externality of not more than $\delta^\star$ is not more than $\epsilon^\star$. Formally, for all y and y' in E such that $\|y - y'\| \leq \delta^\star$, all $i \in \mathcal{N}$, and all $j \in \mathcal{M}_i$, we have $\|u_i(j, y) - u_i(j, y')\| \leq \epsilon^\star$.

Condition 2. For a fixed level of the externality, a player's utility varies by more than $2\epsilon^\star$ across different partners. In particular, for all $y \in E$, all $i \in \mathcal{N}$, and all distinct j and k in $\mathcal{M}_i \backslash \{i\}$, we have $\|u_i(j, y) - u_i(k, y)\| > 2\epsilon^\star$. This requirement also extends to the outside option when its value is affected by the externality, i.e., for all $y \in E$, all $i \in \mathcal{N}$, all $j \in \mathcal{M}_i \backslash \{i\}$, we have $\|u_i(j, y) - u_i(i, y)\| > 2\epsilon^\star$ if $\|u_i(i, y) - u_i(i, y')\| > 0$ for some $y' \in E$ with $\|y - y'\| \leq \delta^\star$.

In moderately sized markets, $\delta^\star$ and $\epsilon^\star$ are small numbers. A matching market has **small individual and pair effects on externalities** if Conditions 1 and 2 hold.[8]

For a moment, suppose that all players are rational and can foresee the consequences of their actions. Then, player i wants to exit and become single if and only if it does strictly better after accounting for the effect of its action on the externality, i.e., if $u_i(\phi) < u_i(i, e(\phi_i))$, where ϕ_i is the matching obtained from ϕ when i separates from its partner in ϕ and matches with itself. Thus, ϕ is individually rational if and only if $u_i(\phi) \geq u_i(i, e(\phi_i))$ for each player

to have small effects on the externality. Indeed, in Section 5, we give an example of "small effects" with only 200 players–a small number by comparison to the tens of thousands of people who work in a typical city.

[7]Since an individual can only change the externality by exiting and since a worker and a firm can only change the externality by separating from their current partners (if any) and forming a new partnership with each other, $\delta^\star = \max\{\delta_{IR}, \delta_B\}$, with $\delta_{IR} = \max_{(\phi,i)\in\Phi\times\mathcal{N}} |e(\phi_i) - e(\phi)|$ and $\delta_B = \max_{(\phi,w,f)\in\Phi\times\mathcal{W}\times\mathcal{F}} |e(\phi_{wf}) - e(\phi)|$, where ϕ_i is the matching obtained from ϕ when player i separates from its partner in ϕ and matches with itself and where ϕ_{wf} is the matching obtained from ϕ when worker w and firm f separate from their partners in ϕ and match with each other. (To be clear, the former partner of i is single in ϕ_i and the former partners of w and f are single in ϕ_{wf}.)

[8]See Section 5 for examples of markets with small individual and pair effects on externalities.

i; we call this **prudent individual rationality**. Likewise, a worker w and a firm f want to block a matching ϕ if and only if they do strictly better after accounting the effect of their actions on the externality, i.e., if $u_w(f, e(\phi_{wf})) > u_w(\phi)$ and $u_f(w, e(\phi_{wf})) > u_f(\phi)$, where ϕ_{wf} is the matching obtained from ϕ when w and f separate from their partners in ϕ and match with each other; we call this a **prudent block**.

We say that a matching is **prudently stable** if it is prudently individually rational and no worker and firm prudent block it. The next lemma rationalizes our definition of blocking.

Lemma 1. Stability and Prudent Stability.
If a matching market has small individual and pair effects on externalities, then a matching is stable if and only if it is prudently stable.

Proof. We will show that (i) a matching ϕ is prudently individually rational if and only if it is individually rational and (ii) a worker w and a firm f prudently block ϕ if and only if they block ϕ. It follows that ϕ is stable if and only if it is prudently stable.

We first show (i). Suppose that ϕ is prudently individually rational, so $u_i(\phi) \geq u_i(i, e(\phi_i))$ for each player i. Then, Condition 2 gives $u_i(\phi) > u_i(i, e(\phi)) + 2\epsilon^\star$ if there is $y' \in E$ with $\|e(\phi) - y'\| \leq \delta^\star$ such that $\|u_i(i, e(\phi)) - u_i(i, y')\| > 0$. (The condition does not imply $u_i(\phi) + 2\epsilon^\star < u_i(i, e(\phi))$, for if it did, then $u_i(\phi) + \epsilon^\star < u_i(i, e(\phi_i))$ by Condition 1, which is a contradiction.) If there is such a y', then we are done. If there is no such y', then $\|u_i(i, e(\phi)) - u_i(i, e(\phi_i))\| = 0$, so $u_i(\phi) \geq u_i(i, e(\phi))$ and ϕ is individually rational.

As to the converse, suppose ϕ is individually rational, so $u_i(\phi) \geq u_i(i, e(\phi))$ for each player i. Again Condition 2 gives $u_i(\phi) > u_i(i, e(\phi)) + 2\epsilon^\star$ if there is a $y' \in E$ with $\|e(\phi) - y'\| \leq \delta^\star$ such that $\|u_i(i, e(\phi)) - u_i(i, y')\| > 0$. If there is such a y', then Condition 1 gives $u_i(\phi) > u_i(i, e(\phi_i)) + \epsilon^\star$. If there is no such y', then $u_i(i, e(\phi)) = u_i(i, e(\phi_i))$, so $u_i(\phi) \geq u_i(i, e(\phi_i))$ and ϕ is prudently individually rational.

We next show (ii). Suppose that worker w and firm f prudently block matching ϕ, i.e., $u_w(f, e(\phi_{wf})) > u_w(\phi)$ and $u_f(w, e(\phi_{wf})) > u_f(\phi)$. Focus on w. Condition 2 gives $u_w(f, e(\phi_{wf})) > u_w(\phi(w), e(\phi_{wf})) + 2\epsilon^\star$. (The condition does not imply $u_w(f, e(\phi_{wf})) + 2\epsilon^\star < u_w(\phi(w), e(\phi_{wf}))$, for if it did, then $u_w(f, e(\phi_{wf})) + \epsilon^\star < u_w(\phi)$ by Condition 1, which is a contradiction.) Two applications of Condition 1 give $u_w(f, e(\phi)) > u_w(\phi)$. An analogous argument gives $u_f(w, e(\phi)) > u_f(\phi)$, so w and f block.

As to the converse, suppose that w and f block the matching ϕ, i.e., $u_w(f, e(\phi)) > u_w(\phi)$ and $u_f(w, e(\phi)) > u_f(\phi)$. Focus on w. Condition 2 gives $u_w(f, e(\phi)) > u_w(\phi) + 2\epsilon^\star$, so Condition 1 gives $u_w(f, e(\phi_{wf})) > u_w(\phi) + \epsilon^\star$ by Condition 1. An analogous argument gives $u_f(w, e(\phi_{wf})) > u_f(\phi)$, so w and f prudently block ϕ. □

3 Framework for Existence

In this section, we first develop a fixed point characterization of the existence of stable matchings. Subsequently, we develop a useful sufficient condition for existence.

Our characterization is based on an "Auxiliary Game." For each possible level of externality $y \in E$, we define an **Auxiliary Matching Game**, $G(y)$, where players regard the externality as fixed at y. Formally, each player i's payoff in $G(y)$ is $\hat{u}_i(j) = u_i(j, y)$ for each $j \in \mathcal{M}_i$. We say that a matching ϕ is **auxiliary individually rational** (or **a-rational**) in $G(y)$ if $\hat{u}_i(\phi(i)) \geq \hat{u}_i(i)$ for each player i, and we say that a matching ϕ is **auxiliary blocked** (or **a-blocked**) in $G(y)$ if there is a worker w and a firm f with $\hat{u}_f(w) > \hat{u}_f(\phi(f))$ and $\hat{u}_w(f) > \hat{u}_w(\phi(w))$. A matching ϕ is **auxiliary stable** (or **a-stable**) if it is a-rational and not a-blocked in $G(y)$. Let $\mathcal{S}(y)$ denote the set of a-stable matchings in $G(y)$. Since the Auxiliary Game is without externalities, $\mathcal{S}(y)$ is nonempty by Gale and Shapley's [10] Deferred Acceptance algorithm.

For each $y \in E$, let $T(y) \equiv \{e(\phi) | \phi \in \mathcal{S}(y)\}$ be the set of externalities associated with the a-stable matchings of $G(y)$. The next proposition uses this correspondence to give a fixed point characterization of the existence question.

Proposition 1. A Fixed Point Characterization of the Existence of Stable Matching. *There is a stable matching $\phi^\star$ if and only if there is a $y^\star$ such that $y^\star \in T(y^\star)$. Moreover, $y^\star = e(\phi^\star)$.*

Proof. Let $\phi^\star$ be a stable matching. We need to show that $y^\star = e(\phi^\star)$ is a fixed point of T. It suffices to show that $\phi^\star \in \mathcal{S}(y^\star)$ since then $y^\star \in e(\mathcal{S}(y^\star)) = T(y^\star)$. We argue that $\phi^\star \in \mathcal{S}(y^\star)$ by contradiction. If not, then $\phi^\star$ is (i) not a-rational or (ii) is a-blocked in $G(y^\star)$. If (i), then there is a player i for whom $\hat{u}_i(\phi^\star(i)) < \hat{u}_i(i)$. By construction of the payoffs, $\hat{u}_i(\phi^\star(i)) < \hat{u}_i(i) \iff u_i(\phi^\star(i), y^\star) < u_i(i, y^\star) \iff u_i(\phi^\star) < u_i(i, e(\phi^\star))$, so $\phi^\star$ is not individually rational. If (ii), then there is a worker w and a firm f such that $\hat{u}_w(f) > \hat{u}_w(\phi^\star(w))$ and $\hat{u}_f(w) > \hat{u}_f(\phi^\star(f))$, implying $u_w(f, e(\phi^\star)) > u_w(\phi^\star)$ and $u_f(w, e(\phi^\star)) > u_f(\phi^\star)$, so $\phi^\star$ is blocked. Both cases contradict the stability of $\phi^\star$.

As to the converse, let $y^\star$ be a fixed point of T. Since $y^\star \in e(\mathcal{S}(y^\star))$ it must be that there is a matching $\phi^\star \in \mathcal{S}(y^\star)$ with $e(\phi^\star) = y^\star$. We argue that $\phi^\star$ is stable by contradiction. If not, then $\phi^\star$ is (i) not individually rational or (ii) is blocked. If (i), then there is a player i for whom $u_i(\phi^\star) < u_i(i, e(\phi^\star))$, implying $\hat{u}_i(\phi^\star(i)) < \hat{u}_i(i)$, so $\phi^\star$ is not a-rational in $G(y^\star)$. If (ii), then there is a worker w and a firm f such that $u_w(f, e(\phi^\star)) > u_w(\phi^\star)$ and $u_f(w, e(\phi^\star)) > u_f(\phi^\star)$, implying $\hat{u}_w(f) > \hat{u}_w(\phi^\star(w))$ and $\hat{u}_f(w) > \hat{u}_f(\phi^\star(f))$, so $\phi^\star$ is a-blocked in $G(y^\star)$. Both cases contradict the a-stability of $\phi^\star$. □

In light of Proposition 1, we only need to show that T has a fixed point. Since E is discrete, we cannot employ traditional continuity arguments. Instead, we rely on Tarski's Fixed Point Theorem. We say that a function $f : E \to \mathbb{R}$ is a **selection** of T if $f(y) \in T(y)$ for all $y \in E$.

Proposition 2. A Sufficient Condition for Existence.
If there is a selection f of T that is increasing, then there is a stable matching.

Proof. Suppose that f is increasing. Since f takes E into itself (as it is a selection of T) and since E is a trivially complete lattice (as it is a finite subset of $\mathbb{R}$), Tarski's fixed point theorem gives f has a fixed point $y^\star$, which is also a fixed point of T. Existence now follows from Proposition 1. $\square$

While the antecedents of Proposition 2 are only sufficient for existence, it is easy to construct examples where there is no stable matching when T is "decreasing"; see Section 5 for one such example. It is generally non-trivial to ensure T is increasing. The reason is that the evolution of T depends, in a highly non-linear way, on both (i) the players' individual and joint responses to the externality shift and (ii) how the externality function e evaluates these responses. Indeed, we spend the rest of this paper developing economically intuitive conditions under which T is increasing.

4 Positive Externalities

We consider two economic environments with non-additive positive externalities.[9] In both environments, we assume that the level of externality $e(\phi)$ is increasing in the number of employed workers (i.e., the number of couples). In particular, we suppose $e(\phi) = g(C(\phi))$, where $g : \mathbb{R} \to \mathbb{R}$ is an increasing function.[10] (Recall that $C(\phi) = |\{w|\phi(w) \neq w\}|$.) Moreover, we assume that, for each level of externality in E, players have *strict* preferences over their potential matches.

[9] It is almost immediate that if the externality is additive, i.e., if $u_i(\phi(i), y) = \tilde{u}_i(\phi(i)) + h_i(y)$ for some $\tilde{u}_i : \mathcal{M}_i \to \mathbb{R}$ and $h_i : \mathbb{R} \to \mathbb{R}$ for each player i, then a stable matching exists. Simply, for all y and y' in E, we have $\mathcal{S}(y) = \mathcal{S}(y')$ when the externality is additive, so the set of stable matchings is equal to $\mathcal{S}(y)$ per Proposition 1 and, thus, is non-empty.

[10] We focus on this case for expositional simplicity only. Our results readily generalize. For instance, we can allow the externality to depend on the number of employed workers of a particular type–as is the case for knowledge, where the number of employed "superstars" increases knowledge more quickly than the number of employed "regular" workers. We can also allow the externality to depend on the number of partnered firms of a particular type, or even on the number of workers of a particular type who are employed by firms of a particular type.

4.1 Knowledge

Consider an economic environment in which there is learning-by-doing, i.e., in which increases in economic activity, which we model via increases in the number of employed workers, cause workers and firms to create and share new (industrial) knowledge. This new knowledge (i) enhances production and economic activity *and* (ii) shifts players' preferences. For instance, the knowledge may concern the procedures for mass-producing a certain type of product–e.g., a light bulb. At a low level of knowledge, everyone knows how to build a light bulb; they just do not know how to do it quickly or in volume. Thus, firms may want to hire high-skill workers who are adaptable and are good at experimenting with different methods of production. As more light bulbs are produced, the firms learn which production methods are most effective, begin to implement them, and slow their experimentation. Hence, firms' preferences over workers may change. Instead of preferring the expensive high-skill workers, firms may begin to prefer less expensive, low-skill workers who are ill-suited to experimentation, but are good at learning and implementing single method of production. Likewise, the high-skill worker's preferences may shift. They may begin to prefer working for smaller, "start up" firms where their adaptability will be put to good use.

We find it reasonable to impose the following requirement on how preferences change as knowledge increases. Let $\mathcal{A}(y) = \{w \in \mathcal{W} | \phi(w) \neq w \text{ for all } \phi \in \mathcal{S}(y)\}$ and let $\mathcal{B}(y) = \{f \in \mathcal{F} | \phi(f) \neq f \text{ for all } \phi \in \mathcal{S}(y)\}$ be the sets of workers and firms that are partnered in $G(y)$.

Condition 3. Let $y \leq y'$ be points in E. For each firm $f \in \mathcal{B}(y)$, we have $u_f(w, y') \geq u_f(f, y')$ for each $w \in \mathcal{A}(y)$. For each worker $w \in \mathcal{A}(y)$, we have $u_w(f, y') \geq u_w(w, y')$ for each $f \in \mathcal{B}(y)$.

The first part of the condition stipulates that if a worker is employed at a low level of knowledge y, then all firms that are partnered at y (in the corresponding Auxiliary Game) find her individually rational at any higher level of knowledge y'. In the context of our light bulb example, this means that, as knowledge increases, partnered firms always find high-skill workers individually rational, even though they may begin to (strictly) prefer lower skill workers. The rationale is that knowledge enhances the productivity of high-skill workers and increases their value to partnered firms, so it remains profitable for these firms to employ high-skill workers; it is just that more knowledge allows these firms to employ cheaper low-skill workers instead and achieve even greater profits. The second part of the condition stipulates that if a firm is partnered at a low level of knowledge y, then all workers that are employed at y find it individually rational at any higher level of knowledge y'. For our light bulb example, this means that, as knowledge increases, high-skill workers are willing to work for established firms, even though they may begin to (strictly) prefer start-ups due to changes in the nature of the working environment at established firms.

When Condition 3 holds, we have the following result.

Proposition 3. Existence of Stable Matchings with Knowledge.
If Condition 3 holds, then a stable matching exists.

Proof. Since preferences are strict for each $y \in E$, the number of couples in every a-stable matching in $\mathcal{S}(y)$ is the same. (In fact, the same set of players is partnered in every a-stable matching by Theorem 2.22 of Roth and Sotomayor [18].) Hence, T is a function. By Proposition 2, we only need to show that it is increasing.

We argue that T is increasing by contradiction. Let y and y' be distinct points in E with $y < y'$, and suppose that $T(y) > T(y')$. Let $\phi \in \mathcal{S}(y)$ and $\phi' \in \mathcal{S}(y')$. Since $T(y) > T(y')$, we must have that $|\mathcal{A}(\phi)| > |\mathcal{A}(\phi')|$ since $g(\cdot)$ is increasing. Then, there is a $w \in \mathcal{A}(\phi)$ and a $f \in \mathcal{B}(\phi)$ who are both single in ϕ': if not, then the total number of partnered players could not have decreased. We claim w and f form a blocking pair. Since $w \in \mathcal{A}(y)$ and $f \in \mathcal{B}(y)$, we have f is a-rational to w and w is a-rational to f when externality is y'. Thus, w and f a-block ϕ' in $G(y')$ since they strictly prefer each other to being single (per strict preferences), a contradiction of the stability of ϕ'. $\square$

4.2 Public Goods

It is often the case that a city's public goods (e.g., the number of parks, the quantity and quality of its public schools and libraries, and even the effectiveness of its emergency services) are increasing in tax receipts and local economic activity. It goes, almost without saying, that the workers and firms who live in the city value such public goods and prefer ever-higher levels of them. Thus, as economic activity increases, which we model via increases in the number of employed workers, each player's payoff to living in the city increases, making its outside option, quitting and exiting the city, less attractive.

We model this economic environment with the following condition.

Condition 4. For each player i, (i) $u_i(i, y) = 0$ for all levels of the externality y, (ii) $u_i(j, y)$ is (weakly) increasing in y for each player $j \in \mathcal{M}_i \backslash \{i\}$, and (iii) for any distinct j and k in $\mathcal{M}_i \backslash \{i\}$, we have $u_i(j, y) > u_i(k, y)$ implies $u_i(j, y') > u_i(k, y')$ for all y and y' in E.[11]

The condition stipulates that each player i's payoff to being single is normalized to zero (and

[11] This condition is a generalization of the classic "preference extension" considered by Roth and Sotomayor [18] in which each player adds unacceptable partners to the end of its list of acceptable partners. Roth and Sotomayor (and the studies cited therein) only extend one side's preferences because they are interested in obtaining welfare comparative statics and such results are notoriously difficult to obtain when both sides' preferences are simultaneously extended. Since our interest instead lies in the size of the matching, we are able to extend both sides' preferences simultaneously.

is unaffected by the externality), that i's payoff to potential partners is increasing in the externality (so i's opportunity cost of exiting the market is increasing), and that i's strict preferences over possible partners are unchanged by shifts in the externality. When this condition holds, we have the following result.

Proposition 4. Existence of Stable Matchings with Public Goods.
If Condition 4 holds, then a stable matching exists.

Proof. Since preferences are strict, for each $y \in E$ the same set of workers is partnered in every element of $\mathcal{S}(y)$, so $T(y)$ is a function. By Proposition 2, it suffices to show that $T(y)$ is an increasing function, we argue this directly.

Let y and y' be two points in E with $y \leq y'$. Let $\phi_{\mathcal{F}}$ be the firm optimal a-stable matching in $G(y)$, where the payoff profile is $\{\{u_w(\cdot, y)\}_{w \in \mathcal{W}}, \{u_f(\cdot, y)\}_{f \in \mathcal{F}}\}$. (This a-stable matching exists since preferences are strict; see Theorem 2 of Gale and Shapley [10].) We begin by shifting the workers' preferences from $\{u_w(\cdot, y)\}_{w \in \mathcal{W}}$ to $\{u_w(\cdot, y')\}_{w \in \mathcal{W}}$, while holding the firms' preferences constant at $\{u_f(\cdot, y)\}_{f \in \mathcal{F}}$. This induces an intermediate matching without externalities game, with payoff profile $\{\{u_w(\cdot, y')\}_{w \in \mathcal{W}}, \{u_f(\cdot, y)\}_{f \in \mathcal{F}}\}$. Let $\phi^I_{\mathcal{F}}$ be the firm optimal "stable" matching under this profile.[12] Since each worker's individually rational set of partners expands because of this shift, Theorem 2.24 of Roth and Sotomayor [18] implies that

$$u_f(\phi^I_{\mathcal{F}}(f), y) \geq u_f(\phi_{\mathcal{F}}(f), y) \text{ for each firm } f. \tag{1}$$

Equation (1) implies that $C(\phi_{\mathcal{F}}) \leq C(\phi^I_{\mathcal{F}})$. For each firm f that is partnered in $\phi_{\mathcal{F}}$,

$$u_f(\phi^I_{\mathcal{F}}(f), y) \geq u_f(\phi_{\mathcal{F}}(f), y) > u_f(f, y) = 0,$$

where the weak inequality follows from equation (1), the strict inequality follows from strict preferences and the a-rationality of $\phi_{\mathcal{F}}$ in $G(y)$, and the equality is due to the hypothesis. Thus, f is partnered in $\phi^I_{\mathcal{F}}$: if f were single in $\phi^I_{\mathcal{F}}$, then we would have $u_f(\phi^I_{\mathcal{F}}, y) = 0$, so $0 > 0$, a contradiction. Hence, $C(\phi_{\mathcal{F}}) \leq C(\phi^I_{\mathcal{F}})$.

Next, we hold the workers' preferences constant at $\{u_w(\cdot, y')\}_{w \in \mathcal{W}}$ and shift the firms' preferences from $\{u_f(\cdot, y)\}_{f \in \mathcal{F}}$ to $\{u_f(\cdot, y')\}_{f \in \mathcal{F}}$. This induces the Auxiliary Game $G(y')$, with payoff profile $\{\{u_w(\cdot, y')\}_{w \in \mathcal{W}}, \{u_f(\cdot, y')\}_{f \in \mathcal{F}}\}$. Let $\phi^I_{\mathcal{W}}$ be the worker optimal stable matching in the intermediate game and let $\phi_{\mathcal{W}}$ be the worker optimal a-stable in $G(y')$. Since each firm's individually rational set of partners expands as a result of this shift, we have $u_w(\phi_{\mathcal{W}}(w), y') \geq u_w(\phi^I_{\mathcal{W}}(w), y')$ for each worker w. Thus, $C(\phi^I_{\mathcal{W}}) \leq C(\phi_{\mathcal{W}})$.

[12] A matching ϕ is stable in the intermediate game if (i) $u_w(\phi(w), y') \geq u_w(w, y')$ for each worker w and $u_f(\phi(f), y) \geq u_f(f, y)$ for each firm f, and (ii) there is no worker w and firm f with $u_w(f, y') > u_w(\phi(w), y')$ and $u_f(w, y) > u_f(\phi(f), y)$.

Since preferences are strict in the intermediate game, $C(\phi^I_{\mathcal{W}}) = C(\phi^I_{\mathcal{F}})$. Thus, $C(\phi_{\mathcal{F}}) \leq C(\phi_{\mathcal{W}})$, which implies $T(y) \leq T(y')$ since $g(\cdot)$ is increasing. $\square$

Remark. While Conditions 3 and 4 both ensure that a stable matching exists, they differ in a key respect: Condition 4 requires that the externality does not shift players' strict preferences over the other side of the market, whereas Condition 3 allows for this.

5 Examples

We give three examples to illustrate our results. The first example shows how one may apply Condition 4 to ensure that a stable matching exists and may use the fact that T is increasing to compute a stable matching. The second extends the first example to illustrate Condition 3. The third example shows that a stable matching need not exist in a moderately sized market with small individual and peer effects on externalities when the conditions we developed do not hold.

Example 1. Existence and Computation of a Stable Matching with Conditions 1, 2, and 4.

Setup. Suppose there are a hundred workers and firms, i.e., $\mathcal{W} = \{1, \ldots, 100\}$ and $\mathcal{F} = \{101, \ldots, 200\}$. Let the externality function be $e(\phi) = 2\theta C(\phi)$, where $\theta > 0$. It is readily verified that $\delta^\star = 2\theta$ and that $E = \{0, \theta 2, \theta 4, \ldots, \theta 200\}$. For simplicity, we assume that workers and firms have a common preference over the other side at each level of externality; however, workers differ in their opportunity costs of working. Specifically, for each worker w, $u_w(\phi, y) = (201 - \phi(w) + y - 2w)\mathbb{I}(\phi(w) \neq w)$, where $\mathbb{I}(\phi(w) \neq w)$ is an indicator function that equals one if w is not single and equals zero otherwise. At each level of externality y, (i) workers prefer lower-indexed firms to higher-indexed firms and (ii) higher-indexed workers have higher opportunity costs to working. Also, each worker's value of being partnered is increasing in the externality. For each firm f, $u_f(\phi, y) = (101 - \phi(f))\mathbb{I}(\phi(f) \neq f)$, where $\mathbb{I}(\phi(f) \neq f)$ is an indicator function that equals one if f is not single and equals zero otherwise. Hence, firms do not care about the externality and all firms prefer lower-indexed workers to higher-indexed workers.

Small Individual and Pair Effects on Externalities. When $\theta < 1/2$, Conditions 1 and 2 hold. Since the firms' payoffs are constant in the externality, Conditions 1 and 2 trivially hold for them. Thus, we only need to verify that the conditions hold for the workers. Consider worker w, and let f and f' be two distinct firms. Since (i) $||u_w(f, y) - u_w(f', y)|| \geq 1$ for all $y \in E$, (ii) $||u_w(f, y) - u_w(f, y')|| = ||y - y'||$, (iii) $\theta < 1/2$, and (iv) $u_i(i, y) = 0$ for each player i, the condition holds with $\delta^\star = \epsilon^\star = 2\theta < 1$. Hence, individuals only have a small effect on the externality. (That said, in the unique stable matching, there is a substantial level of externality.)

Existence of a Stable Matching. A stable matching exists because Condition 3 holds. Simply, for each player i, (i) $u_i(i, y) = 0$, (ii) i's payoff to being partnered is increasing in y, and (iii) i's strict preferences over possible partners do not change with the level of the externality y. Hence, Proposition 4 establishes that there is a stable matching.

Computation of a Stable Matching. Our goal is to find a stable matching. Since $T(y)$ is an increasing function (a fact we established in the Proof of Proposition 4), we can find it by iterating T from 0 until we hit a fixed point. We do this now, while assuming $\theta = 1/4$ for simplicity.

Preliminary In $G(y)$, each side has a common preferences over the other side, so workers and firms must be assortatively matched. Hence, in the unique a-stable matching: worker 1 is matched to firm 101, worker 2 is matched to firm 102, and so on until worker w'_y is matched to firm $100 + w'_y$, where w_y is the highest-indexed worker who finds matching individually rational, i.e.,

$$w'_y = \max_{\leq}\{\tilde{w} \in \mathcal{W} \,|\, 0 \leq 201 - (100 + \tilde{w}) + y - 2\tilde{w}\} = \min\{\lfloor \frac{101 + y}{3} \rfloor, 100\},$$

where $\lfloor \cdot \rfloor$ is the floor function. The externality is $w'_y/2$.

Iteration Described in the following table.

Step 1	Compute $T(0)$: $T(0) = \frac{1}{2}w'_0 = \frac{1}{2}\min\{\lfloor \frac{101}{3} \rfloor, 100\} = 16.5$
Step 2	Compute $T(16.5)$: $T(16.5) = \frac{1}{2}w'_{16.5} = \frac{1}{2}\min\{\lfloor \frac{101+16.5}{3} \rfloor, 100\} = 19.5$
Step 3	Compute $T(19.5)$: $T(19.5) = \frac{1}{2}w'_{19.5} = \frac{1}{2}\min\{\lfloor \frac{101+19.5}{3} \rfloor, 100\} = 20$
Step 4	Compute $T(20)$: $T(20) = \frac{1}{2}w'_{20} = \frac{1}{2}\min\{\lfloor \frac{101+20}{3} \rfloor, 100\} = 20$

Thus, our fixed point externality is $y^\star = 20$, so our preliminary step tells us that it is stable for workers 1 to 40 to match assortatively to firms 101 to 140,[13] while workers 41 to 100 and firms 141 to 200 are single.[14] $\triangle$

Example 2. Existence and Computation of a Stable Matching with Conditions 1, 2, and 3.

Setup. We maintain the same setup as with Example 1 with $\theta = 1/4$, save that we edit

[13] On the subject of assortative sorting, Becker [4] and Legros and Newman [14] are the classic references for matching games without externalities, while Chade and Eeckhout [5] develop formal results for a transferable utility environment with externalities.

[14] The point $y^\star = 20$ is the minimal fixed point of T. To find the maximal fixed point, we start the iteration from 50 ($= 200/4$) instead of 0. Doing this, we find that the maximal fixed point of T is also $y^\star = 20$. Hence, the stable matching described above is unique.

the preferences of firm 101 as follows:

$$u_{101}(\phi, y) = \begin{cases} 101 - \phi(f) & \text{if } \phi(f) \neq f \text{ and } y \leq 15 \\ 101 - \phi(f) + (1+k)\mathbb{I}(\phi(101) = 2) & \text{if } \phi(f) \neq f \text{ and } y > 15 \\ 0 & \text{if } \phi(f) = f, \end{cases}$$

where $\mathbb{I}(\cdot)$ is an indicator function and $k > 0$ is a small number. In words, firm 101 has the same preferences as in Example 1 when the externality is less than or equal to 15. However, when the externality exceeds 15, then worker 2 becomes the firm's most preferred worker and worker 1 becomes the second-most preferred worker. (Observe that Condition 4 does not hold because of this preference reversal.)

Existence of a Stable Matching. A stable matching exists by Proposition 3 since Condition 3 holds. To establish that Condition 3 holds, we first need to characterize $\mathcal{S}(y)$. Then, we will leverage this characterization and the notion of a-rationality to show the condition holds.

We first characterize $\mathcal{S}(y)$. When $y \leq 15$, recall from Example 1 that, in the unique a-stable matching, workers 1 to w'_y are matched assortatively to firms 101 to $100 + w'_y$, where w'_y is as defined in Example 1, and all other players are single. When $y > 15$, it is easily verified that, in the unique a-stable matching, worker 2 is matched to firm 101, worker 1 is matched to firm 102, worker w is matched to firm $100 + w$ for all $w \in \{3, \ldots, w'_y\}$, and all other players are single.

Now we establish that Condition 3 holds. Let $y \leq y'$ be points in E. Since each firm's set of individually rational workers is unaffected by the externality, the first part of Condition 3 holds. Thus, we only need to show that the second part holds as well. Let w be a worker in $\mathcal{A}(y)$ and let f be a firm in $\mathcal{B}(y)$,[15] we need to show $u_w(f, y') \geq u_w(w, y')$, equivalently, $f \leq 201 + y' - 2w$. To these ends, let $\phi \in \mathcal{S}(y)$. Since $f \in \mathcal{B}(y)$, we have $f \leq 100 + w'_y = \phi(w'_y)$. Since $\phi(w'_y)$ is a-rational to worker w'_y in $G(y)$, we obtain $\phi(w'_y) \leq 201 + y - 2w'_y$. Thus, $f \leq 201 + y - 2w'_y$. Since $y \leq y'$ and since $w \leq w'_y$, we have $f \leq 201 + y' - 2w$ and Condition 3 holds.

Computation of a Stable Matching. As in Example 1, we exploit the monotonicity of $T(\cdot)$ to find a stable matching. Starting from $T(0)$, we compute that the lowest fixed point of T is 20 and that the associated stable matching is for worker 2 and firm 101 to match, worker 1 and firm 102 to match, and for workers 3 to 40 to match assortatively to firms 103 to 140, while everyone else is single. △

Example 3. Non-Existence of a Stable Matching with Conditions 1 and 2.

Setup. Let $\mathcal{W} = \{1, \ldots, 100\}$, let $\mathcal{F} = \{101, \ldots, 200\}$, and let the externality function be

[15] Observe that $\mathcal{A}(y) = \{1, \ldots, w'_y\}$ and $\mathcal{B}(y) = \{100, \ldots, 100 + w'_y\}$.

$e(\phi) = \frac{1}{2}C(\phi)$. It is readily verified that $\delta^\star = \frac{1}{2}$ and that $E = \{\frac{0}{2}, \frac{1}{2}, \frac{2}{2}, \ldots, \frac{100}{2}\}$.

For simplicity, we assume that the workers and firms have common preferences. Specifically, for each worker w, $u_w(\phi, y) = (201 - \phi(w) - 0.853\, y)\mathbb{I}(\phi(w) \neq w)$. And for each firm f, $u_f(\phi, y) = (101 - \phi(f))\mathbb{I}(\phi(f) \neq f)$. Notice that the externality is negative since it makes the workers worse off (e.g., pollution) and that the firms do not care about the externality.

Small Individual and Pair Effects on Externalities. Conditions 1 and 2 hold. Since the firms do not care about the externality, the conditions trivially hold for them. Thus, we focus on the workers. Consider worker w and let f and f' be two distinct firms. Since (i) $||u_w(f, y) - u_w(f', y)|| \geq 1$ for all $y \in E$, (ii) $||u_w(f, y) - u_w(f, y')|| = 0.835\, ||y - y'||$, and (iii) $\delta^\star = \frac{1}{2}$, and (iv) $u_i(i, y) = 0$ for each player i and $y \in E$, Conditions 1 and 2 hold with $\epsilon^\star = 0.4175 = \frac{0.835}{2}$.

Nonexistence. There is no stable matching. We will establish this by showing that $T(\cdot)$ has no fixed point. Given this, Proposition 1 implies there is no stable matching.

We begin by establishing a closed-form expression for $T(\cdot)$. Let $y \in E$. Since each side has a common preference, it is readily verified that the unique a-stable matching is for worker 1 to match with firm 101, worker 2 to match with firm 102, and so on until worker w'_y matches to firm $100 + w'_y$, where w_y is the highest-indexed worker who finds matching individually rational, i.e., $w'_y = \min\{\lfloor 101 - 0.853\, y \rfloor, 100\}$. Thus, $T(y) = \frac{1}{2}\min\{\lfloor 101 - 0.853\, y \rfloor, 100\}$.

Consider the related function $Q(y) = T(y) - y$. Clearly, (i) $y^\star$ is a fixed point of T if and only if $Q(y^\star) = 0$ and (ii) $Q(y)$ is decreasing in y. Computation shows that $Q(35) = 0.5$ and $Q(35.5) = -0.5$. Since E contains no elements strictly between 35 and 35.5, facts (i) and (ii) imply there is no fixed point of T. $\triangle$

6 Conclusion

Motivated by many real-world examples (e.g., emergency services, school quality, and knowledge) where the effect of an individual worker and firm on the externality is minimal, we introduce and formally analyze a matching with externalities game where players can rationally regard externalities as fixed while considering a block. We develop a fixed point approach to the problem of existence and we demonstrate that the set of stable matchings is nonempty when the economy exhibits natural, positive externalities: knowledge and public goods. We would like to note that while we motivate Condition 3 with knowledge and Condition 4 with public goods, both conditions can be motivated on other economic grounds–e.g., Condition 3 might reflect improvements in safety regulations in an inherently dangerous industry and Condition 4 might reflect positive spillovers from improvements in private-homes aesthetics.

We close with three observations. First, our conditions for non-additive positive externalities can be easily adapted to ensure that as a parameter increases, the number of employed workers increases in externality-free matching markets. Second, our existence conditions are only sufficient. Much work remains in terms of finding necessary conditions and finding other economically meaningful sufficient conditions. Third, our results readily extend to matching games with (continuous) contracts and thus to matching with transfers; see Fisher [8] for the link between these games and ours. We conjecture that they also extend to many-to-one matching games; though one would need to appropriately modify the classic substitutes condition and ensure that mass hirings or firings by firms have a sufficiently small effect on the externality. We leave the formal extension for future work.

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