Semidefinite programming converse bounds for classical communication over quantum channels

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***Abstract*—We study the classical communication over quantum channels when assisted by no-signalling (NS) and PPT-preserving (PPT) codes. We first show that both the optimal success probability of a given transmission rate and one-shot ϵ-error capacity can be formalized as semidefinite programs (SDPs) when assisted by NS or NS∩PPT codes. Based on this, we derive SDP finite blocklength converse bounds for general quantum channels, which also reduce to the converse bound of Polyanskiy, Poor, and Verdú for classical channels. Furthermore, we derive an SDP strong converse bound for the classical capacity of a general quantum channel: for any code with a rate exceeding this bound, the optimal success probability vanishes exponentially fast as the number of channel uses increases. In particular, applying our efficiently computable bound, we derive improved upper bounds to the classical capacity of the amplitude damping channels and also establish the strong converse property for a new class of quantum channels.**

I. Introduction

The reliable transmission of classical information via noisy quantum channels is central to quantum information theory. The classical capacity of a noisy quantum channel is the highest rate at which it can transmit classical information reliably over asymptotically many uses of the channel. The Holevo-Schumacher-Westmoreland (HSW) theorem [1], [2], [3] gives a full characterization of the classical capacity of quantum channels:

$$C(\mathcal{N}) \coloneqq \sup_{n\geq 1} \frac{1}{n}\chi(\mathcal{N}^{\otimes n}), \tag{1}$$

where $\chi(\mathcal{N})$ is the Holevo capacity of $\mathcal{N}$ given by $\chi(\mathcal{N}) \coloneqq \max_{\{(p_i,\rho_i)\}} H\left(\sum_i p_i \mathcal{N}(\rho_i)\right) - \sum_i p_i H(\mathcal{N}(\rho_i))$, $\{(p_i,\rho_i)\}_i$ is an ensemble of quantum states on A and $H(\sigma) = -\operatorname{Tr}\sigma\log\sigma$ is the von Neumann entropy of quantum state. For a general quantum channel, our understanding of the classical capacity is still limited. The work of Hastings [4] shows that the Holevo capacity is generally not additive, thus the regularization in Eq. (1) is necessary in general. Since the complexity of computing the Holevo capacity is NP-complete [5], the regularized Holevo capacity of a general quantum channel is notoriously difficult to calculate. Even for the qubit amplitude damping channel, the classical capacity remains unknown.

The converse part of the HSW theorem states that if the communication rate exceeds the capacity, then the error probability of any coding scheme cannot approach to zero in the limit of many channel uses. This kind of "weak" converse suggests the possibility for one to increase communication rates by allowing an increased error probability. A *strong converse property* leaves no such room for the trade-off, i.e., the error probability necessarily converges to one in the limit of many channels uses whenever the rate exceeds the capacity of the channel. For classical channels, the strong converse property for classical capacity is established by Wolfowitz [6]. For quantum channels, the strong converse property for classical capacity is confirmed for several classes of channels [7], [8], [9], [10], [11]. Unfortunately, for a general quantum channel, less is known about the strong converse property of classical capacity and it remains open whether this property holds for all quantum channels. A *strong converse bound* for the classical capacity is a quantity such that the success probability of transmitting classical messages vanishes exponentially fast as the number of channel uses increases if the rate of communication exceeds this quantity.

Another fundamental problem, of both theoretical and practical interest, is the trade-off between the channel uses, communication rate and error probability in the non-asymptotic (or finite blocklength) regime. Note that one only needs to study one-shot communication over the channel since it can correspond to a finite blocklength. Also, one can study the asymptotic capacity via the finite blocklength approach. The study of finite blocklength regime has recently attracted great interest in classical information theory (e.g., [12], [13]) as well as in quantum information theory (see [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25] for a partial list). For classical channels, Polyanskiy, Poor, and Verdú [12] derive the finite blocklength converse bound via hypothesis testing. For classical-quantum channels, the one-shot converse and achievability bounds are given in [26], [16], [18]. Recently, the one-shot converse bounds for entanglement-assisted and unassisted codes were given in [15], which generalizes the hypothesis testing approach in [12] to quantum channels.

To gain insights into the intractable problem of evaluating the capacities of quantum channels, a natural approach is to study the performance of extra free resources in the coding scheme. This scheme, called a *code*, is equivalently a bipartite operation performed jointly by the sender Alice and the receiver Bob to assist the communication [22], [27]. The *PPT-preserving codes*, i.e. the PPT-preserving bipartite operations, include all operations that can be implemented by local operations and classical communication (LOCC) and were introduced to study entanglement distillation in an early paper by Rains [28]. The *no-signalling (NS) codes* refer to the bipartite quantum operations with the no-signalling constraints, which arise in the research of the relativistic causality of quantum operations [29], [30], [31]. Recently these general codes have been used to study the classical communication over classical channels [32], [33], and zero-error classical communication [27], [34] and quantum communication [22] over quantum channels. Our work follows this approach and focuses on the classical communication over quantum channels assisted by NS and NS∩PPT codes. We show SDP finite blocklength converse and strong converse bounds for the classical communication over any quantum channels.

II. Preliminaries

In the following, we will frequently use symbols such as A (or A') and B (or B') to denote (finite-dimensional) Hilbert spaces associated with Alice and Bob, respectively. We use d_A to denote the dimension of system A. The set of linear operators over A is denoted by $\mathcal{L}(A)$. Note that for a linear operator $R \in \mathcal{L}(A)$, we define $|R| = \sqrt{R^\dagger R}$, where $R^\dagger$ is the adjoint operator of R, and the trace norm of R is given by $\|R\|_1 = \operatorname{Tr}|R|$. The operator norm $\|R\|_\infty$ is defined as the maximum eigenvalue of $|R|$. A deterministic quantum operation (quantum channel) $\mathcal{N}$ from A' to B is simply a completely positive (CP) and trace-preserving (TP) linear map from $\mathcal{L}(A')$ to $\mathcal{L}(B)$. The Choi-Jamiołkowski matrix of $\mathcal{N}$ is given by $J_\mathcal{N} = \sum_{ij} |i_A\rangle\langle j_A| \otimes \mathcal{N}(|i_{A'}\rangle\langle j_{A'}|)$, where $\{|i_A\rangle\}$ and $\{|i_{A'}\rangle\}$ are orthonormal bases on isomorphic Hilbert spaces A and A', respectively. A positive semidefinite operator $E \in \mathcal{L}(A \otimes B)$ is said to be a positive partial transpose operator (or simply PPT) if $E^{T_B} \geq 0$, where T_B means the partial transpose with respect to the party B, i.e., $(|ij\rangle\langle kl|)^{T_B} = |il\rangle\langle kj|$. As shown in [28], a bipartite operation $\Pi(A_iB_i \to A_oB_o)$ is PPT-preserving if and only if its Choi-Jamiołkowski matrix $Z_{A_iB_iA_oB_o}$ is PPT.

The constraints of PPT and NS can be characterized in a mathematically tractable way. A bipartite operation $\Pi(A_iB_i \to A_oB_o)$ is no-signalling and PPT-preserving if and only if its Choi-Jamiołkowski matrix $Z_{A_iB_iA_oB_o}$ satisfies [22]:

$$\begin{aligned}
Z_{A_iB_iA_oB_o} &\geq 0, && \text{(CP)}\\
\operatorname{Tr}_{A_oB_o} Z_{A_iB_iA_oB_o} &= \mathbb{1}_{A_iB_i}, && \text{(TP)}\\
Z^{T_{B_iB_o}}_{A_iB_iA_oB_o} &\geq 0, && \text{(PPT)}\\
\operatorname{Tr}_{A_o} Z_{A_iB_iA_oB_o} &= \frac{\mathbb{1}_{A_i}}{d_{A_i}} \otimes \operatorname{Tr}_{A_oA_i} Z_{A_iB_iA_oB_o}, && (A \not\to B)\\
\operatorname{Tr}_{B_o} Z_{A_iB_iA_oB_o} &= \frac{\mathbb{1}_{B_i}}{d_{B_i}} \otimes \operatorname{Tr}_{B_oB_i} Z_{A_iB_iA_oB_o}, && (B \not\to A)
\end{aligned}$$

where the five lines correspond to characterize that Π is completely positive, trace-preserving, PPT-preserving, no-signalling from A to B, no-signalling from B to A, respectively. The no-signalling codes is also studied in [27].

Semidefinite programming [35] is a subfield of convex optimization and is a powerful tool in quantum information theory with many applications (e.g., [15], [22], [27], [28], [36], [37], [38], [39]). In this work, we use CVX [40] and QETLAB [41] to solve the SDPs in this work.

III. Communication assisted by NS and PPT codes

A. Semidefinite programs for optimal success probability

Suppose Alice wants to send the classical message labeled by $\{1, \ldots, m\}$ to Bob using the composite channel $\mathcal{M} = \Pi \circ \mathcal{N}$, see Fig. 1 for details. Then the input register of $\mathcal{M}$ can be considered to be classical. After the action of $\mathcal{E}$ and $\mathcal{N}$, the message results in quantum state at Bob's side. Bob then performs a POVM with m outcomes on the resulting quantum state. The POVM is a component of the operation $\mathcal{D}$. Since the results of the POVM and the input messages are both classical, we assume that $\mathcal{M}$ is with classical registers throughout this paper. If the outcome $k \in \{1, \ldots, m\}$ happens, he concludes that the message with label k was sent. Let Ω be some class of codes. The average success probability of the code Π and the Ω-class code are defined as follows.

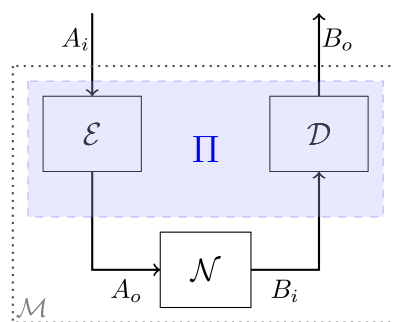

Fig. 1. Bipartite operation $\Pi(A_iB_i \to A_oB_o)$ is equivalently the coding scheme ($\mathcal{E}$,$\mathcal{D}$) with free extra resources, such entanglement or no-signalling correlations. The whole operation is to emulate a noiseless classical (or quantum) channel $\mathcal{M}(A_i \to B_o)$ using a given noisy quantum channel $\mathcal{N}(A_o \to B_i)$ and the bipartite operation Π.

Definition 1 *The average success probability of $\mathcal{N}$ to transmit m messages assisted with the code Π is defined by*

$$f(\mathcal{N}, \Pi, m) = \frac{1}{m} \sum_{k=1}^{m} \operatorname{Tr} \mathcal{M}(|k\rangle\langle k|)|k\rangle\langle k|, \tag{2}$$

where $\mathcal{M} \equiv \Pi \circ \mathcal{N}$ and $\{|k\rangle\}$ is the computational basis in system A_i. Furthermore, the optimal average success probability of $\mathcal{N}$ to transmit m messages assisted with Ω-class code is defined by

$$f_\Omega(\mathcal{N}, m) = \sup_{\Pi} f(\mathcal{N}, \Pi, m), \tag{3}$$

where the maximum is over the codes in class Ω.

We now define the Ω-assisted classical capacity of a quantum channel as below:

$$C_\Omega(\mathcal{N}) := \sup\{r : \lim_{n\to\infty} f_\Omega(\mathcal{N}^{\otimes n}, 2^{rn}) = 1\}. \tag{4}$$

As described above, one can simulate a channel $\mathcal{M}$ with the channel $\mathcal{N}$ and code Π, where Π is a bipartite CPTP operation from A_iB_i to A_oB_o which is no-signalling (NS) and PPT-preserving (PPT). In this work, we shall also consider other classes of codes, such as entanglement-assisted (EA) code, unassisted (UA) code, and we use Ω to denote the specific class of codes in this paper. Let $\mathcal{M}(A_i \to B_o)$ denote the resulting composition channel of Π and $\mathcal{N}$, written $\mathcal{M} = \Pi \circ \mathcal{N}$. Based on the results in [42], the Choi-Jamiołkowski matrix of $\mathcal{M}$ [22] is given by $J_{\mathcal{M}} = \operatorname{Tr}_{A_oB_i}(J_{\mathcal{N}}^T \otimes \mathbb{1}_{A_iB_o})Z_{A_iA_oB_iB_o}$.

We are now able to derive the one-shot characterization of classical communication over general quantum channels assisted by NS (or NS∩PPT) codes.

Theorem 2 *For a given quantum channel $\mathcal{N}$, the optimal success probability of $\mathcal{N}$ to transmit m messages assisted by NS∩PPT codes is given by*

$$
\begin{aligned}
f_{\text{NS}\cap\text{PPT}}(\mathcal{N}, m) = \max \operatorname{Tr} J_{\mathcal{N}} F_{AB} \\
\text{s.t. } & 0 \le F_{AB} \le \rho_A \otimes \mathbb{1}_B, \\
& \operatorname{Tr}\rho_A = 1, \operatorname{Tr}_A F_{AB} = \mathbb{1}_B/m, \\
& 0 \le F_{AB}^{T_B} \le \rho_A \otimes \mathbb{1}_B \text{ (PPT)}.
\end{aligned} \tag{5}
$$

Similarly, when assisted by NS codes, one can remove the PPT constraint to obtain the optimal success probability.

Proof Sketch. (The full proof can be found in [43].) We first use the Choi-Jamiołkowski representations of quantum channels to refine the average success probability. Without loss of generality, we assume that A_i and B_o are classical registers with size m, i.e., the inputs and outputs are $\{|k\rangle_{A_i}\}_{k=1}^m$ and $\{|k'\rangle_{B_i}\}_{k'=1}^m$, respectively. For some NS∩PPT code Π, the Choi-Jamiołkowski matrix of $\mathcal{M} = \Pi \circ \mathcal{N}$ is given by $J_{\mathcal{M}} = \sum_{ij} |i\rangle\langle j|_{A_i} \otimes \mathcal{M}(|i\rangle\langle j|_{A_i'})$, where A_i' is isometric to A_i. Then, we can refine $f(\mathcal{N}, \Pi, m)$ to

$$
\begin{aligned}
f(\mathcal{N}, \Pi, m) &= \frac{1}{m}\sum_{k=1}^m \operatorname{Tr}[|k\rangle\langle k|_{A_i} \otimes \mathcal{M}(|k\rangle\langle k|_{A_i'})|k\rangle\langle k|_{B_o}] \\
&= \frac{1}{m}\operatorname{Tr} J_{\mathcal{M}} \sum_{k=1}^m |kk\rangle\langle kk|_{A_iB_o}.
\end{aligned}
$$

Then, denoting $D_{A_iB_o} = \sum_{k=1}^m |kk\rangle\langle kk|_{A_iB_o}$, we have

$$
f_{\text{NS}\cap\text{PPT}}(\mathcal{N}, m) = \max_{\mathcal{M}=\Pi\circ\mathcal{N}} \frac{1}{m} \operatorname{Tr}(J_{\mathcal{M}} D_{A_iB_o}),
$$

where $\mathcal{M} = \Pi \circ \mathcal{N}$ and Π is any feasible NS∩PPT bipartite operation . (See FIG. 1 for the implementation of $\mathcal{M}$.) Noting that $J_{\mathcal{M}} = \operatorname{Tr}_{A_oB_i}(J_{\mathcal{N}}^T \otimes \mathbb{1}_{A_iB_o})Z_{A_iA_oB_iB_o}$, we can further simplify $f(\mathcal{N}, m)$. Then, we exploit symmetry to simplify the optimization of $f(\mathcal{N}, m)$ over all possible codes, i.e., $Z_{A_iA_oB_iB_o}$ can be rewritten as

$$
Z_{A_iA_oB_iB_o} = F_{A_oB_i} \otimes D_{A_iB_o} + E_{A_oB_i} \otimes (\mathbb{1} - D_{A_iB_o}),
$$

for some operators $E_{A_oB_i}$ and $F_{A_oB_i}$. Thus, the objective function can be simplified to $\operatorname{Tr} J_{\mathcal{N}}^T F$.

Finally, we impose the no-signalling and PPT-preserving constraints to obtain the optimal average success probability as showed in Eq. (5). It is worthing noting that $f_{\text{NS}}(\mathcal{N}, m)$ can be obtained by removing the PPT constraint and it corresponds with the optimal NS-assisted channel fidelity in [22].

B. One-shot ϵ-error capacity

For given $0 \le \epsilon < 1$, the *one-shot ϵ-error classical capacity assisted by Ω-class codes* is defined as

$$
C_{\Omega}^{(1)}(\mathcal{N}, \epsilon) := \sup\{\log \lambda : 1 - f_{\Omega}(\mathcal{N}, \lambda) \le \epsilon\}. \tag{6}
$$

We derive the one-shot ϵ-error capacity as follows.

Theorem 3 *For given channel $\mathcal{N}$ and error threshold ϵ, the one-shot ϵ-error* NS∩PPT-assisted capacity is given by

$$
\begin{aligned}
C_{\text{NS}\cap\text{PPT}}^{(1)}(\mathcal{N}, \epsilon) = -\log \min \eta \\
\text{s.t. } & 0 \le F_{AB} \le \rho_A \otimes \mathbb{1}_B, \\
& \operatorname{Tr}\rho_A = 1, \operatorname{Tr}_A F_{AB} = \eta \mathbb{1}_B, \\
& \operatorname{Tr} J_{\mathcal{N}} F_{AB} \ge 1 - \epsilon, \\
& 0 \le F_{AB}^{T_B} \le \rho_A \otimes \mathbb{1}_B \text{ (PPT)},
\end{aligned} \tag{7}
$$

Similarly, when assisted by NS codes, one can remove the PPT constraint to obtain the optimal success probability.

Proof omitted (see [43]).

Since no-signalling-assisted codes are potentially stronger than the entanglement-assisted codes, $C_{\text{NS}}^{(1)}(\mathcal{N}, \epsilon)$ and $C_{\text{NS}\cap\text{PPT}}^{(1)}(\mathcal{N}, \epsilon)$ can provide the converse bounds of classical communication for entanglement-assisted and unassisted codes, respectively. We further compare our one-shot ϵ-error capacities with the previous SDP converse bounds derived by the quantum hypothesis testing technique in [15]. To be specific, for a given channel $\mathcal{N}(A \to B)$ and error thresold ϵ, Matthews and Wehner [15] establish that

$$
C_{\text{E}}^{(1)}(\mathcal{N}, \epsilon) \le I_{\epsilon}^{\textbf{All}}(\mathcal{N}) \text{ and } C^{(1)}(\mathcal{N}, \epsilon) \le I_{\epsilon}^{\textbf{PPT}}(\mathcal{N}),
$$

with $I_{\epsilon}^{\textbf{All}}(\mathcal{N}) = \max_{\rho_A} \min_{\sigma_B} D_H^{\epsilon}((id_{A'} \otimes \mathcal{N})(\rho_{A'A}) \| \rho_{A'} \otimes \sigma_B)$ and $I_{\epsilon}^{\textbf{PPT}}(\mathcal{N}) = \max_{\rho_A} \min_{\sigma_B} D_{H,PPT}^{\epsilon}((id_{A'} \otimes \mathcal{N})(\rho_{A'A}) \| \rho_{A'} \otimes \sigma_B)$. Here, $\rho_{A'A} = (\mathbb{1}_{A'} \otimes \rho_A^{\frac{1}{2}})\Phi_{A'A}(\mathbb{1}_{A'} \otimes \rho_A^{\frac{1}{2}})$ is a purification of ρ_A and $\rho_{A'} = \operatorname{Tr}_A \rho_{A'A}$. Moreover, $D_H^{\epsilon}(\rho_0 \| \rho_1)$ is the hypothesis testing relative entropy [16], [15] and $D_{H,PPT}^{\epsilon}(\rho_0 \| \rho_1)$ is the similar quantity with a PPT constraint on the POVM.

Interestingly, even when we allow stronger assistances (NS or $\text{NS} \cap \text{PPT}$ codes), the one-shot ϵ-error capacities are still smaller than or equal to the SDP converse bounds in [15]. (Note that EA ⊂ NS). And the inequalities can be strict.

Proposition 4 *For a given channel $\mathcal{N}$ and error threshold ϵ,*

$$
C_{\text{NS}}^{(1)}(\mathcal{N}, \epsilon) \le I_{\epsilon}^{\textbf{\textit{All}}}(\mathcal{N}), \tag{8}
$$

$$
C_{\text{NS}\cap\text{PPT}}^{(1)}(\mathcal{N}, \epsilon) \le I_{\epsilon}^{\textbf{\textit{PPT}}}(\mathcal{N}). \tag{9}
$$

In particular, both inequalities can be strict for some quantum channels such as the amplitude damping channels.

Proof and examples can be found in [43]. This means that we can use $C^{(1)}_{\mathrm{NS}\cap\mathrm{PPT}}(\mathcal{N},\epsilon)$ and $C^{(1)}_{\mathrm{NS}}(\mathcal{N},\epsilon)$ to provide better SDP converse bounds for entanglement-assisted and unassisted codes, respectively.

IV. Strong converse bound for classical capacity

A. An SDP strong converse bound

It is well known that evaluating the classical capacity of a general channel is extremely difficult. To the best of our knowledge, the only known nontrivial strong converse bound is the entanglement-assisted capacity [44] and there is also computable upper bound derived from entanglement measures [45]. In this section, we derive an SDP strong converse bound for the classical capacity of a general quantum channel. Our bounds are efficiently computable and do not depend on any special properties of the channel. We further show that for some quantum channels, our bound is strictly smaller than the entanglement-assisted capacity and the previous bound in [45].

Theorem 5 *For any quantum channel $\mathcal{N}$,*

$$C(\mathcal{N}) \le C_{\mathrm{NS}\cap\mathrm{PPT}}(\mathcal{N}) \le C_\beta(\mathcal{N}) = \log\beta(\mathcal{N}),$$

where

$$\begin{aligned}\beta(\mathcal{N}) = \min \operatorname{Tr} S_B \quad \textit{s.t.} \ & -R_{AB} \le J_{\mathcal{N}}^{T_B} \le R_{AB},\\ & -\mathbb{1}_A\otimes S_B \le R_{AB}^{T_B} \le \mathbb{1}_A \otimes S_B.\end{aligned} \tag{10}$$

In particular, when the communication rate exceeds $C_\beta(\mathcal{N})$, the error probability goes to one exponentially fast as the number of channel uses increases.

We outline the proof sketch here. The first step is to introduce a subadditive upper bound $f^+(\mathcal{N},m)$ on $f_{\mathrm{NS}\cap\mathrm{PPT}}(\mathcal{N},m)$. Then, the n-shot error probability satisfies that $\epsilon_n = 1 - f_{\mathrm{NS}\cap\mathrm{PPT}}(\mathcal{N}^{\otimes n}, 2^{rn}) \le 1 - f^+(\mathcal{N},2^r)^n$. Finally, we show that for any $2^r > \beta(\mathcal{N})$, it holds that $f^+(\mathcal{N},2^r) < 1$, which means ϵ_n will go to one exponentially fast as n increases. The detailed proof can be found in [43].

B. Amplitude damping channel

For the amplitude damping channel $\mathcal{N}_\gamma^{AD} = \sum_{i=0}^1 E_i \cdot E_i^\dagger$ $(0\le\gamma\le 1)$ with $E_0 = |0\rangle\langle 0| + \sqrt{1-\gamma}|1\rangle\langle 1|$ and $E_1 = \sqrt{\gamma}|0\rangle\langle 1|$, the Holevo capacity is solved in [46]. However, its classical capacity remains unknown. The only known nontrivial upper bound was established in [45]. As an application of Theorem 5, we show a simple strong converse bound for the classical capacity of the amplitude damping channel, which improves the previously best upper bound in [45].

Theorem 6 *For amplitude damping channel $\mathcal{N}_\gamma^{AD}$,*

$$C(\mathcal{N}_\gamma^{AD}) \le C_\beta(\mathcal{N}_\gamma^{AD}) = \log(1+\sqrt{1-\gamma}).$$

The idea is to apply the bound C_β to the amplitude damping channel. Full proof can be found in [43]. We compare our bound with the previous upper bound [45] and lower bound [46] in FIG. 2. It is clear that our bound provides a tighter bound to the classical capacity than the previous bound [45].

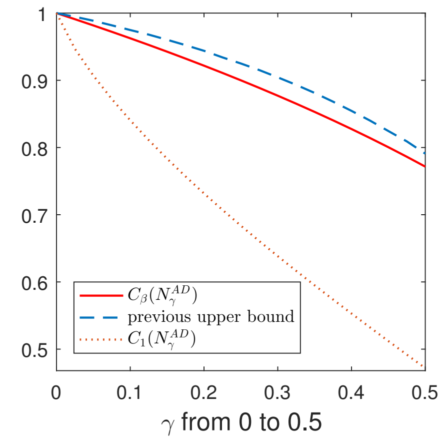

Fig. 2. The solid line depicts $C_\beta(\mathcal{N}_\gamma^{AD})$, the dashed line depicts the previous bound of $C(\mathcal{N}_\gamma^{AD})$ [45], and the dotted line depicts the lower bound [46]. Our bound is tighter than the previous bound in [45].

C. Strong converse property for new channels

In [47], a class of qutrit-to-qutrit channels was introduced to show the gap between quantum Lovász number and entanglement-assisted zero-error classical capacity. It turns out that this class of channels also has strong converse property for classical capacity. To be specific, the channel from register A to B is given by $\mathcal{N}_\alpha(\rho) = E_0\rho E_0^\dagger + E_1\rho E_1^\dagger$ $(0<\alpha\le\pi/4)$ with $E_0 = \sin\alpha|0\rangle\langle 1| + |1\rangle\langle 2|$ and $E_1 = \cos\alpha|2\rangle\langle 1| + |1\rangle\langle 0|$.

Proposition 7 *For $\mathcal{N}_\alpha$ $(0<\alpha\le\pi/4)$, we have that*

$$C(\mathcal{N}_\alpha) = C_{\mathrm{NS}\cap\mathrm{PPT}}(\mathcal{N}_\alpha) = C_\beta(\mathcal{N}_\alpha) = 1.$$

Proof omitted (see [43]).

V. Conclusions and Discussions

In summary, we have obtained the optimal success probabilities of transmitting classical information assisted by NS or NS∩PPT codes. Based on this, we have also derived the one-shot ϵ-error NS-assisted and NS∩PPT-assisted capacities. In particular, all of these one-shot characterizations are in the form of SDPs. Remarkably, the one-shot NS-assisted and NS∩PPT-assisted) ϵ-error capacities provide an improved finite blocklength estimation of the classical communication than the previous quantum hypothesis testing converse bounds in [15].

Furthermore, in the asymptotic regime, we have derived an SDP strong converse bound for the classical capacity of a general quantum channel, which can be strictly smaller than the entanglement-assisted capacity. As an example, we have shown an improved upper bound of the classical capacity of the qubit amplitude damping channel. Moreover, we have proved that the strong converse property holds for the classical capacity of a new class of quantum channels.

It would also be interesting to study how to implement the NS and PPT-preserving codes. It is also interesting to improve the strong converse bound based on the optimal success probability assisted by NS∩PPT codes.

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