A Passivity Preserving Frequency-Weighted Model Order Reduction Technique

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***Abstract*—Frequency-weighted model order reduction techniques aim to yield a reduced order model whose output matches that of the original system in the emphasized frequency region. However, passivity of the original system is only known to be preserved in the single-sided weighted case. A frequency-weighted model order reduction technique is proposed which guarantees the passive reduced models in the double-sided weighted case. A set of easily computable error bound expressions are also presented.**

***Index Terms*—Error bound, Frequency-weighted Gramians, Model order reduction, Passivity.**

I. Introduction

MODEL order reduction (MOR) has been the attention of researchers for the past few decades. The aim of MOR is to find a fairly accurate lower order approximate model which retains the essential properties of the original system like stability, passivity and input-output behaviour [1]-[5]. Truncated balanced realization (TBR) [6] is one of the most popular MOR techniques due to its accuracy, preservation of stability and easily computable *a priori* error bound.

In the context of MOR of *RLC* networks, integrated circuits, control systems and interconnected circuits, passivity is an important property to be preserved because non-passive reduced order model (ROM) may yield nonphysical behaviour by generating energy at high frequencies and resulting in erratic time-domain behaviours. Passivity implies stability but the opposite is not true. Philips *et al.* [7] extended TBR [6] to preserve passivity of the original system and also presented the corresponding error bound expressions.

In many practical applications, it is desirable that the output of ROM matches that of the original system in some specified frequency region [8]. Enns [9] presented a frequency-weighted generalization of TBR [6] for that purpose. However, stability is only guaranteed in the case of single-sided frequency weighting with no *a priori* error bound provided. Several modifications exist in the literature like [10]-[13] to ensure stability in the double-sided weighted case and error bounds are also derived. These techniques [10]-[13], however, do not guarantee passivity of the ROM.

Heydari and Pedram [14] proposed a frequency-weighted generalization of Philips *et al.'s* technique [7]. Similarly, Enns' technique [9] and its modifications [10]-[11] are generalized in [15] which claimed to yield the guaranteed passive ROMs. However, it is pointed in [16] that these techniques [14]-[15] guarantee the passivity only in the single-sided weighted case. Many approximation criteria in MOR are double-sided weighted criteria, particularly, when the model to be reduced is part of a closed-loop system and preservation of the closed-loop behaviour is important [17]. Frequency-weighted MOR (FWMOR) techniques are used to satisfy these criteria and preservation of passivity in the double-sided weighted case is, therefore, critical.

To the authors' best knowledge, there is no FWMOR technique in the literature so far which guarantees the passivity in the double-sided weighted case. In this paper, an FWMOR technique is proposed which preserves passivity of the original system both in the single and double-sided weighted cases. A set of easily computable error bound expressions are also derived. Numerical examples are presented to show the efficacy of proposed technique.

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II. Background

Consider an n^{th} order positive-real linear time invariant system

$$G(s) = C(sI - A)^{-1}B + D$$

where $\{A, B, C, D\}$ is its minimal state-space realization, $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^{m\times m}$.

A. Philips et al.'s Technique [7]

The controllability Gramian-like matrix P_a and the observability Gramian-like matrix Q_a of the system $\{A, B, C, D\}$ are the solution of following Lur'e equations:

$$AP_a + P_aA^T = -K_cK_c^T \quad (1)$$

$$P_aC^T - B = -K_cJ_c^T \qquad J_cJ_c^T = D + D^T \quad (2)$$

$$A^TQ_a + Q_aA = -K_o^TK_o \quad (3)$$

$$Q_aB - C^T = -K_o^TJ_o \qquad J_o^TJ_o = D + D^T \quad (4)$$

The transformation matrix T_a is calculated such that $T_a^TQ_aT_a = T_a^{-1}P_aT_a^{-T} = diag\{\xi_1, \xi_2, \cdots, \xi_n\}$ and $\xi_1 \geq \xi_2 \geq \cdots \geq \xi_n$. ROM is obtained by applying the transformation on the original system and truncating the transformed system up to the desired order.

Remark 1: The Lur'e equations in (1)-(4) can be solved using Riccati equation based algorithms in [18].

B. Heydari and Pedram's Technique (HPT) [14]

Let $V(s)$ and $W(s)$ be the positive-real transfer functions of the input and output frequency weights respectively.

$$V(s) = \hat{C}_v(sI - \hat{A}_v)^{-1}\hat{B}_v + \hat{D}_v$$
$$W(s) = \hat{C}_w(sI - \hat{A}_w)^{-1}\hat{B}_w + \hat{D}_w$$

Then, the augmented systems are represented as the following:

$$G(s)V(s) = \left[\begin{array}{c|c} A_i & B_i \\ \hline C_i & D_i \end{array}\right] = \left[\begin{array}{cc|c} A & B\hat{C}_v & B\hat{D}_v \\ 0 & \hat{A}_v & \hat{B}_v \\ \hline C & D\hat{C}_v & D\hat{D}_v \end{array}\right]$$
$$W(s)G(s) = \left[\begin{array}{c|c} A_o & B_o \\ \hline C_o & D_o \end{array}\right] = \left[\begin{array}{cc|c} A & 0 & B \\ \hat{B}_w C & \hat{A}_w & \hat{B}_w D \\ \hline \hat{D}_w C & \hat{C}_w & \hat{D}_w D \end{array}\right]$$

Heydari and Pedram [14] considered $D = 0$. The controllability Gramian-like matrix P_i and observability Gramian-like matrix Q_o of the augmented systems are the solution of following Lur'e equations:

$$A_iP_i + P_iA_i^T = -K_{c,i}K_{c,i}^T \qquad P_iC_i^T = B_i \tag{5}$$
$$A_o^TQ_o + Q_oA_o = -K_{o,o}^TK_{o,o} \qquad Q_oB_o = C_o^T \tag{6}$$

The matrices P_i, Q_o, $K_{c,i}$ and $K_{o,o}$ can be partitioned as

$$P_i = \begin{bmatrix} \bar{P}_{11} & \bar{P}_{12} \\ \bar{P}_{12}^T & \bar{P}_{22} \end{bmatrix} \quad Q_o = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} \\ \bar{Q}_{12}^T & \bar{Q}_{22} \end{bmatrix}$$
$$K_{c,i} = \begin{bmatrix} \bar{K}_{c,1} \\ \bar{K}_{c,2} \end{bmatrix} \qquad K_{o,o} = \begin{bmatrix} \bar{K}_{o,1} \\ \bar{K}_{o,2} \end{bmatrix}$$

The frequency-weighted controllability Gramian-like matrix $\bar{P}_{11}$ and frequency-weighted observability Gramian-like matrix $\bar{Q}_{11}$, corresponding to the original system, can be expressed as

$$A\bar{P}_{11} + \bar{P}_{11}A^T = -X \qquad \bar{P}_{11}C^T = B\hat{D}_v \tag{7}$$
$$A^T\bar{Q}_{11} + \bar{Q}_{11}A = -Y \qquad \bar{Q}_{11}B = C^T\hat{D}_w^T \tag{8}$$

where

$$X = B\hat{C}_v\bar{P}_{12}^T + \bar{P}_{12}\hat{C}_v^TB^T + \bar{K}_{c,1}\bar{K}_{c,1}^T \tag{9}$$
$$Y = C^T\hat{B}_w^T\bar{Q}_{12}^T + \bar{Q}_{12}\hat{B}_wC + \bar{K}_{o,1}^T\bar{K}_{o,1} \tag{10}$$

Inspired by Wang *et al.'s* technique [11], Heydari and Pedram [14] replaced the generally indefinite symmetric matrices X and Y with their positive semidefinite approximations $\hat{K}_c\hat{K}_c^T$ and $\hat{K}_o^T\hat{K}_o$ respectively to ensure passivity (similar to the ensured stability in Wang *et al.* [11]) where $\hat{K}_c = U|S|^{1/2}$ and $\hat{K}_o = |R|^{1/2}V^T$. R, S, U and V are obtained by the eigenvalue decomposition of X and Y i.e. $X = USU^T$ and $Y = VRV^T$ where $S = diag\{s_1, s_2, \cdots, s_n\}$ and $R = diag\{r_1, r_2, \cdots, r_n\}$.

$$AP_h + P_hA^T = -\hat{K}_c\hat{K}_c^T \qquad P_hC^T = \hat{B} \tag{11}$$
$$A^TQ_h + Q_hA = -\hat{K}_o^T\hat{K}_o \qquad \bar{Q}_hB = \hat{C}^T \tag{12}$$

The transformation matrix T_h is calculated such that $T_h^TQ_hT_h = T_h^{-1}P_hT_h^{-T} = \Sigma = diag\{\bar{\xi}_1, \bar{\xi}_2, \cdots, \bar{\xi}_n\}$ and $\bar{\xi}_1 \geq \bar{\xi}_2 \geq \cdots \geq \bar{\xi}_n \geq 0$. The r^{th} order ROM $G_r(s)$ is obtained by applying the transformation on the original system and truncating the transformed system up to the desired order. The following error bound holds for HPT [14]:

$$||W(s)(G(s) - G_r(s))V(s)||_\infty \leq 2||W(s)L||_\infty||KV(s)||_\infty \sum_{k=r+1}^{n} \bar{\xi}_k \tag{13}$$

where

$$L = CV diag\{|r_1|^{-\frac{1}{2}}, |r_2|^{-\frac{1}{2}}, \cdots, |r_j|^{-\frac{1}{2}}, 0, \cdots, 0\}$$
$$K = diag\{|s_1|^{-\frac{1}{2}}, |s_2|^{-\frac{1}{2}}, \cdots, |s_i|^{-\frac{1}{2}}, 0, \cdots, 0\}U^TB$$
$$i = rank[X] \text{ and } j = rank[Y].$$

C. Discussion

It is shown by Muda *et al.* in [16] that HPT [14] does not guarantee the passivity in the double-sided weighted case. The reason is that P_h and Q_h are obtained from two different systems i.e. $\{A, \hat{B}, C\}$ and $\{A, B, \hat{C}\}$ respectively and a transformation matrix T_h is obtained which balances P_h and Q_h. Then T_h is applied on the original system $\{A, B, C\}$ and therefore T_h does not balance P_a and Q_a. Consequently, the results of Philips *et al.* [7] for passivity assurance can not be used and the resulting ROM may be non-passive as shown in [16]. Although not explicitly mentioned by Muda *et al.* [16], the techniques in [15] do not guarantee the passivity either for the same reason. *"Extended Wang's technique"* in [15] is actually similar to HPT [14]. It is also shown in [16] that the passivity is guaranteed in the single-sided weighted case for HPT [14] because in this case either P_a or Q_a is diagonalized and thus, the results of Philips *et al.* [7] for passivity assurance can be used. The techniques in [15] also guarantee the passivity in the single-sided weighted case for the same reason.
It is also claimed in [16] that HPT [14] does not even guarantee the stability and therefore, the error bound expression in (13) is not valid. However, we will now show that this criticism is incorrect.

Lemma 1: HPT [14] yields stable ROMs and the error bound expression in (13) is also valid.
Proof: Consider the triple $\{A_t, B_t, C_t\} = \{T_h^{-1}AT_h, T_h^{-1}\hat{K}_c, \hat{K}_oT_h\}$. It can easily be noted that this triple satisfy the following Lyapunov equations:

$$A_t\Sigma + \Sigma A_t^T = -B_tB_t^T \qquad A_t^T\Sigma + \Sigma A_t = -C_t^TC_t$$

where $\Sigma = T_h^TQ_hT_h = T_h^{-1}P_hT_h^{-T} = diag\{\bar{\xi}_1, \bar{\xi}_2, \cdots, \bar{\xi}_n\}$ and $\bar{\xi}_1 \geq \bar{\xi}_2 \geq \cdots \geq \bar{\xi}_n \geq 0$. It can then be concluded from Appendix C of [19] that $A_r = \mathcal{P}_rA_t\mathcal{P}_r^T$ is Hurwitz, where $\mathcal{P}_r = \begin{bmatrix} I_r & 0 \end{bmatrix}$ is the truncating projection matrix. Note that this A_r is also the A-matrix of the ROMs. Thus, ROMs are guaranteed to be stable.

This result is, in fact, not new and can be found at several places in the literature like [19]-[21]. Majority of the stability preserving FWMOR techniques [10]-[13] rely on this result for stability assurance. We agree with Muda *et al.* [16] that the Lur'e equations of ROM yielded by HPT [14]

do not reduce to Lyapunov equations but this is not the only way to prove the stability of ROM.
An important shortcoming of the techniques in [14]-[15] (which is not pointed out in [16]) is needed to be discussed first before proceeding to the actual solution. The FWMOR techniques in [14]-[15] use the augmented systems comprising of a series connection of the original and weighting systems. However, the augmented system may not be passive even if both the original and weighting systems are passive because the series interconnection of passive systems, in contrast with the parallel and feedback interconnection, is not always passive (see for instance [22]). Thus, $W(s)G(s)$ and $G(s)V(s)$ satisfy (5)-(6) (with $P_i \geq 0$ and $Q_o \geq 0$) only when the augmented realizations are passive which is not always the case. These techniques [14]-[15] are only guaranteed to work for a special case when the augmented systems are passive.

III. Main Work

It is customary in the reduction of large scale *RLC* networks that at the first stage Krylov methods are applied because these are computationally efficient. However, the ROMs yielded by Krylov methods are not compact. Therefore, in the next stage TBR [6] is used to obtain a compact ROM [7], [23]. Inspired by this staging concept, it is proposed to consider the double-sided weighted problem as a sequence of two single-sided weighted problems. In the first stage, the states which have the least share in the energy transfer within the frequency region emphasized by the input frequency weight are truncated. In the second stage, the states which have the least share in the energy transfer within the frequency region emphasized by the output frequency weight are truncated. The sequence of the stages can be swapped. The obtained ROM is passive due to the fact that passivity is guaranteed in the single-sided weighted case [16].
A passivity preserving mixed balancing algorithm is presented in [24] which requires one Lyapunov equation and one Riccati equation to be solved. Inspired by [24], it is proposed to calculate the weighted Gramians from the Lyapunov equations and unweighted Gramians-like matrices from the Lur'e equations. In this way, the condition on the frequency weights and augmented system to be passive is removed. This is particularly useful in the controller reduction problems when the plant is not passive. The weights and consequently the augmented system in this scenario may not be passive even if the controller is passive [17]. Therefore, the proposed technique is more general and have a wider range of applicability.

The controllability Gramian of the input augmented system $G(s)V(s)$ is the solution of following Lyapunov equation:

$$A_iP_{ci} + P_{ci}A_i^T + B_iB_i^T = 0 \qquad (14)$$

P_{ci} can be partitioned as $P_{ci} = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}$. Block (1,1) of (14) can be written as

$$AP_{11} + P_{11}A^T + X_1 = 0 \qquad (15)$$

where

$$X_1 = B\hat{C}_vP_{12}^T + P_{12}\hat{C}_v^TB^T + B\hat{D}_v\hat{D}_v^TB^T \qquad (16)$$

The frequency-weighted controllability Gramian P_u is the solution of following Lyapunov equation

$$AP_u + P_uA^T + B_uB_u^T = 0 \qquad (17)$$

where the fictitious input matrix $B_u = \bar{U}_1\bar{S}_1^{\frac{1}{2}}$. The matrices $\bar{U}_1$ and $\bar{S}_1$ are obtained from the eigenvalue decomposition of symmetric indefinite matrix X_1 (as done in [12]) i.e. $X_1 = \bar{U}\bar{S}\bar{U}^T = \begin{bmatrix} \bar{U}_1 & \bar{U}_2 \end{bmatrix} \begin{bmatrix} \bar{S}_1 & 0 \\ 0 & \bar{S}_2 \end{bmatrix} \begin{bmatrix} \bar{U}_1^T \\ \bar{U}_2^T \end{bmatrix}$ where $\bar{S}_1 = diag\{\bar{s}_1, \bar{s}_2, \cdots, \bar{s}_l\}$, $\bar{s}_1 \geq \bar{s}_2 \geq \cdots \geq \bar{s}_l > 0$ and l is the number of positive eigenvalues of X_1.
Let $Y_1 = K_o^TK_o$ where K_o is defined in (3). Then, the eigenvalue decomposition of Y_1 is $Y_1 = \bar{V}\bar{R}\bar{V}^T$ where $\bar{R} = diag\{\bar{r}_1, \bar{r}_2, \cdots, \bar{r}_n\}$. The fictitious output matrix C_f can be defined as $C_f = \bar{R}^{\frac{1}{2}}\bar{V}^T$. Q_a (in (3)) can be rewritten as a solution of following Lyapunov equation:

$$A^TQ_a + Q_aA + C_f^TC_f = 0 \qquad (18)$$

P_u and Q_a are the controllability Gramian and observability Gramian respectively of the system $\{A, B_u, C_f, D\}$ similar to FWBT problems [10]-[13] (and similar to [12] in particular). The transformation matrix T_1 is calculated such that $T_1^TQ_aT_1 = T_1^{-1}P_uT_1^{-T} = \bar{\Sigma} = diag\{\bar{\sigma}_1, \bar{\sigma}_2, \cdots, \bar{\sigma}_n\}$ and $\bar{\sigma}_1 \geq \bar{\sigma}_2 \geq \cdots \geq \bar{\sigma}_n$. The transformed system $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\} = \{T_1^{-1}AT_1, T_1^{-1}B, CT_1, D\}$ is then given by

$$\left[\begin{array}{c|c} T_1^{-1}AT_1 & T_1^{-1}B \\ \hline CT_1 & D \end{array}\right] = \left[\begin{array}{cc|c} A_{r_1} & A_{12} & B_{r_1} \\ A_{21} & A_{22} & B_2 \\ \hline C_{r_1} & C_2 & D \end{array}\right]. \qquad (19)$$

Then $G_{r_1}(s) = C_{r_1}(sI - A_{r_1})^{-1}B_{r_1} + D$ is the r_1^{th} ($r_1 < r$) order ROM such that $A_{r_1} \in \mathbb{R}^{r_1 \times r_1}, B_{r_1} \in \mathbb{R}^{r_1 \times m}$ and $C_{r_1} \in \mathbb{R}^{m \times r_1}$.

Theorem 1: The following *a priori* error bound holds for this stage of reduction if $rank \begin{bmatrix} B_u & B \end{bmatrix} = rank \begin{bmatrix} B_u \end{bmatrix}$ and $rank \begin{bmatrix} C_f \\ K_c \end{bmatrix} = rank \begin{bmatrix} C_f \end{bmatrix}$:

$$||(G(s) - G_{r_1}(s))V(s)||_\infty \leq 2||L_1||_\infty||K_1V(s)||_\infty \sum_{k=r_1+1}^{n} \bar{\sigma}_k$$

where

$$L_1 = C\bar{V}diag\{\bar{r}_1^{-\frac{1}{2}}, \bar{r}_2^{-\frac{1}{2}}, \cdots, \bar{r}_{\bar{j}}^{-\frac{1}{2}}, 0, \cdots, 0\}$$
$$K_1 = diag\{\bar{s}_1^{-\frac{1}{2}}, \bar{s}_2^{-\frac{1}{2}}, \cdots, \bar{s}_{\bar{i}}^{-\frac{1}{2}}, 0, \cdots, 0\}\bar{U}^TB$$
$$\bar{i} = rank[X_1] \text{ and } \bar{j} = rank[Y_1].$$

Proof: The proof is similar to the error bound expressions in [11]-[13] and hence omitted.

Proposition 1: ROM $\{A_{r_1}, B_{r_1}, C_{r_1}, D\}$ is passive.
Proof: $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}$ is passive since it is just the similarity

transformation of $\{A, B, C, D\}$ and therefore, it satisfies the following Lur'e equations such that $\bar{\Sigma} \geq 0$ [7].

$$\bar{A}^T\bar{\Sigma} + \bar{\Sigma}\bar{A} = -\bar{K}_o^T\bar{K}_o \tag{20}$$

$$\bar{\Sigma}\bar{B} - \bar{C}^T = -\bar{K}_o^T J_o \qquad J_o^T J_o = D + D^T \tag{21}$$

$\bar{\Sigma}$ can be partitioned as $diag\{\bar{\Sigma}_{r_1}, \bar{\Sigma}_{(n-r_1)}\}$ where $\bar{\Sigma}_{r_1} = diag\{\bar{\sigma}_1 \geq \bar{\sigma}_2 \geq \cdots \geq \bar{\sigma}_{r_1}\}$. Block(1,1) of (20)-(21) can be written as

$$A_{r_1}^T\bar{\Sigma}_{r_1} + \bar{\Sigma}_{r_1}A_{r_1} = -\bar{K}_{r_1}^T\bar{K}_{r_1}$$

$$\bar{\Sigma}_{r_1}B_{r_1} - C_{r_1}^T = -\bar{K}_{r_1}^T J_o \qquad J_o^T J_o = D + D^T$$

$\{A_{r_1}, B_{r_1}, C_{r_1}, D\}$ satisfies positive-real lemma since $\bar{\Sigma}_{r_1} \geq 0$ (as $\bar{\Sigma} \geq 0$). Hence, $G_{r_1}(s)$ is passive.

Consider the output augmented system $W(s)G_{r_1}(s)$ which is represented in the state-space form as:

$$W(s)G_{r_1}(s) = \left[\begin{array}{c|c} \tilde{A}_o & \tilde{B}_o \\ \hline \tilde{C}_o & \tilde{D}_o \end{array}\right] = \left[\begin{array}{cc|c} A_{r_1} & 0 & B_{r_1} \\ \hat{B}_w C_{r_1} & \hat{A}_w & \hat{B}_w D \\ \hline \hat{D}_w C_{r_1} & \hat{C}_w & \hat{D}_w D \end{array}\right]$$

The observability Gramian of the output augmented system $W(s)G_{r_1}(s)$ is the solution of following Lyapunov equation:

$$\tilde{A}_o^T Q_{oo} + Q_{oo}\tilde{A}_o + \tilde{C}_o^T\tilde{C}_o = 0 \tag{22}$$

Q_{oo} can be partitioned as $Q_{oo} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix}$. Block (1,1) of (22) can be written as

$$A_{r_1}^T Q_{11} + Q_{11}A_{r_1} + Y_2 = 0 \tag{23}$$

where

$$Y_2 = C_{r_1}^T\hat{B}_w^T Q_{12}^T + Q_{12}\hat{B}_w C_{r_1} + C_{r_1}^T\hat{D}_w^T\hat{D}_w C_{r_1} \tag{24}$$

The frequency-weighted observability Gramian Q_u is the solution of following Lyapunov equation

$$A_{r_1}^T Q_u + Q_u A_{r_1} + C_u^T C_u = 0 \tag{25}$$

where the fictitious output matrix $C_u = \tilde{R}_1^{\frac{1}{2}}\tilde{V}_1^T$. The matrices $\tilde{V}_1$ and $\tilde{R}_1$ are obtained from the eigenvalue decomposition of symmetric indefinite matrix Y_2 (as done in [12]) i.e. $Y_2 = \tilde{V}\tilde{R}\tilde{V}^T = \begin{bmatrix}\tilde{V}_1 & \tilde{V}_2\end{bmatrix}\begin{bmatrix}\tilde{R}_1 & 0 \\ 0 & \tilde{R}_2\end{bmatrix}\begin{bmatrix}\tilde{V}_1^T \\ \tilde{V}_2^T\end{bmatrix}$ where $\tilde{R}_1 = diag\{\tilde{r}_1, \tilde{r}_2, \cdots, \tilde{r}_{\tilde{l}}\}$, $\tilde{r}_1 \geq \cdots \geq \tilde{r}_{\tilde{l}} > 0$ and $\tilde{l}$ is the number of positive eigenvalues of Y_2.

The controllability Gramian-like matrix $\tilde{P}_a$ of the system $G_{r_1}(s)$ is the solution of following Lur'e equation:

$$A_{r_1}\tilde{P}_a + \tilde{P}_a A_{r_1}^T = -\tilde{K}_c\tilde{K}_c^T \tag{26}$$

$$\tilde{P}_a C_{r_1}^T - B_{r_1} = -\tilde{K}_c\tilde{J}_c^T \qquad \tilde{J}_c\tilde{J}_c^T = D + D^T \tag{27}$$

Similar to the previous stage, let $X_2 = \tilde{K}_c\tilde{K}_c^T$. Then, the eigenvalue decomposition of X_2 is $X_2 = \tilde{U}\tilde{S}\tilde{U}^T$ where $\tilde{S} = diag\{\tilde{s}_1, \tilde{s}_2, \cdots, \tilde{s}_n\}$. The fictitious input matrix B_f can be defined as $B_f = \tilde{U}\tilde{S}^{\frac{1}{2}}$. $\tilde{P}_a$ (in (26)) can be rewritten as a solution of following Lyapunov equation:

$$A_{r_1}\tilde{P}_a + \tilde{P}_a A_{r_1} + B_f B_f^T = 0 \tag{28}$$

The transformation matrix T_2 is calculated such that $T_2^T Q_u T_2 = T_2^{-1}\tilde{P}_a T_2^{-T} = \tilde{\Sigma} = diag\{\tilde{\sigma}_1, \tilde{\sigma}_2, \cdots, \tilde{\sigma}_{r_1}\}$ and $\tilde{\sigma}_1 \geq \tilde{\sigma}_2 \geq \cdots \geq \tilde{\sigma}_{r_1}$. The transformed system $\{\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}\} = \{T_2^{-1}A_{r_1}T_2, T_2^{-1}B_{r_1}, C_{r_1}T_2, D\}$ is then given by

$$\left[\begin{array}{c|c} T_2^{-1}A_{r_1}T_2 & T_2^{-1}B_{r_1} \\ \hline C_{r_1}T_2 & D \end{array}\right] = \left[\begin{array}{cc|c} A_r & A_{r_{12}} & B_r \\ A_{r_{21}} & A_{r_{22}} & B_{r_2} \\ \hline C_r & C_{r_2} & D \end{array}\right] \tag{29}$$

where $G_r(s) = C_r(sI - A_r)^{-1}B_r + D$ is the r^{th} order ROM such that $A_r \in \mathbb{R}^{r\times r}, B_r \in \mathbb{R}^{r\times m}$ and $C_r \in \mathbb{R}^{m\times r}$. The following error bound holds for this stage of reduction if $rank\begin{bmatrix}B_f & \tilde{K}_c\end{bmatrix} = rank\begin{bmatrix}B_f\end{bmatrix}$ and $rank\begin{bmatrix}C_u \\ C_{r_1}\end{bmatrix} = rank\begin{bmatrix}C_u\end{bmatrix}$:

$$||W(s)(G_{r_1}(s) - G_r(s))||_\infty \leq 2||W(s)L_2||_\infty||K_2||_\infty \sum_{k=r+1}^{r_1}\tilde{\sigma}_k$$

where

$$L_2 = C_{r_1}\tilde{V}diag\{\tilde{r}_1^{-\frac{1}{2}}, \tilde{r}_2^{-\frac{1}{2}}, \cdots, \tilde{r}_{\tilde{j}}^{-\frac{1}{2}}, 0, \cdots, 0\}$$

$$K_2 = diag\{\tilde{s}_1^{-\frac{1}{2}}, \tilde{s}_2^{-\frac{1}{2}}, \cdots, \tilde{s}_{\tilde{i}}^{-\frac{1}{2}}, 0, \cdots, 0\}\tilde{U}^T B_{r_1}$$

$$\tilde{i} = rank[X_2] \text{ and } \tilde{j} = rank[Y_2].$$

The overall error bound can be expressed as:

$$\begin{aligned}
&||W(s)(G(s) - G_r(s))V(s)||_\infty = \\
&\quad ||W(s)(G(s) - G_{r_1}(s) + G_{r_1}(s) - G_r(s))V(s)||_\infty \leq \\
&||W(s)||_\infty||(G(s) - G_{r_1}(s))V(s)||_\infty + \\
&\quad ||W(s)(G_{r_1}(s) - G_r(s))||_\infty||V(s)||_\infty \leq \\
&2||W(s)||_\infty||L_1||_\infty||K_1V(s)||_\infty \sum_{k=r_1+1}^{n}\bar{\sigma}_k + \\
&\quad 2||V(s)||_\infty||W(s)L_2||_\infty||K_2||_\infty \sum_{k_1=r+1}^{r_1}\tilde{\sigma}_{k_1}
\end{aligned}$$

Theorem 2: ROM $\{A_r, B_r, C_r, D\}$ is passive.

Proof: Proof is similar to that of Proposition 1 and hence omitted.

Algorithm 1: Steps to calculate ROM $\{A_r, B_r, C_r, D\}$

1: Compute P_u and Q_a using (14)-(17) and (3)-(4) respectively.
2: Compute the Cholesky factor R_u of P_u i.e. $P_u = R_u^T R_u$.
3: Compute the singular value decomposition of $R_u Q_a R_u^T$ such that $R_u Q_a R_u^T = U_u\bar{\Sigma}^2 U_u^T$.
4: Compute T_1 as $T_1 = R_u^T U_u\bar{\Sigma}^{-1/2}$.
5: $G_{r_1}(s)$ is obtained using (19).
6: Compute $\tilde{P}_a$ and Q_u using (26)-(27) and (22)-(25) respectively.
7: Compute the Cholesky factor $\tilde{R}_a$ of $\tilde{P}_a$ i.e. $\tilde{P}_a = \tilde{R}_a^T\tilde{R}_a$.
8: Compute the singular value decomposition of $\tilde{R}_a Q_u\tilde{R}_a^T$ such that $\tilde{R}_a Q_u\tilde{R}_a^T = \tilde{U}_a\tilde{\Sigma}^2\tilde{U}_a^T$.
9: Compute T_2 as $T_2 = \tilde{R}_a^T\tilde{U}_a\tilde{\Sigma}^{-1/2}$.
10: The r^{th} order ROM is obtained using (29).

Remark 2: The sequence of the stages can be swaped, i.e. the first stage can be output-weighted stage and the second stage can be input-weighted stage or vice versa.

Remark 3: ROMs yielded by the proposed technique are not unique. Therefore, the parameter r_1 ($r < r_1 < n$) can be adjusted to achieve the best results. In the single-sided weighted case, $r_1 = r$ (for input weighting only) or $r_1 = n$ (for output weighting only).

IV. Numerical Examples

In this section, two passive models of a wire comprising of 100 and 150 repeated *RLC* sections are considered which produce a 200^{th} and 300^{th} order system respectively [23]. The input and output weighting systems considered are

$$V(s) = \frac{10}{(s+10)} \qquad W(s) = \frac{(s+0.01)}{(s+0.05)}$$

The models are reduced using HPT [14], Extended Enns' technique (EET) [15], Extended Lin and Chiu's technique (ELCT) [15] and the proposed technique. Although HPT [14], EET [15] and ELCT [15] do not guarantee the passive ROMs, these do not always yield non-passive models. r_1 is not changed during the experiment for a fair comparison i.e. $r_1 = 100$ in the first case and $r_2 = 150$ in the second case. Table I compares the weighted error $||W(s)(G(s) - G_r(s))V(s)||_\infty$ (magnified by the factor of 10^5) of the ROMs yielded by these techniques. As shown, the proposed technique guarantees passivity of the ROMs without any considerable increase in error.

TABLE I: Error Comparison

Example	Order	$\|\|W(s)(G(s)-G_r(s))V(s)\|\|_\infty$ $\times 10^5$			
		HPT [14]	EET [15]	ELCT [15]	Proposed
1	11	1.1988	0.7133	0.7710	0.3773
	12	0.0651	0.0653	0.0702	0.0328
	13	0.0571	0.0524	0.0581	0.0325
	14	0.0250	0.0152	0.0164	0.0070
	15	0.0012	0.0171	0.0033	0.0080
2	16	0.0016	0.0138	0.0052	0.0077
	17	0.0018	0.0353	0.0044	0.0075
	18	0.0017	0.0136	0.0044	0.0075
	19	0.0017	0.0135	0.0044	0.0076
	20	0.0017	0.0135	0.0043	0.0076

V. Conclusion

In this paper, some limitations in the existing FWMOR techniques are first highlighted. Then, an FWMOR technique is proposed which considers the double-sided weighted problem as a sequence of two single-sided problems and addresses these limitations. Easily computable error bound expressions are also presented. It is shown with the help of numerical examples that the proposed technique compares well in accuracy with the existing FWMOR techniques (which do not guarantee the passivity). However, the ROMs yielded by the proposed technique are guaranteed to be passive.

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