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Determining periodic orbits via nonlinear filtering and recurrence spectra in the presence of noise

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Abstract

In nonlinear dynamical systems the determination of stable and unstable periodic orbits as part of phase space prediction is problematic in particular if perturbed by noise. Fourier spectra of the time series or its autocorrelation function have shown to be of little use if the dynamic process is not strictly wide-sense stationary or if it is nonlinear. To locate unstable periodic orbits of a chaotic attractor in phase space the least stable eigenvalue can be determined by approximating locally the trajectory via linearisation. This approximation can be achieved by employing a Gaussian kernel estimator and minimising the summed up distances of the measured time series i.e. its estimated trajectory (e.g. via Levenberg-Marquardt). Noise poses a significant problem here. The application of the Wiener-Khinchin theorem to the time series in combination with recurrence plots, i.e. the Fourier transform of the recurrence times or rates, has been shown capable of detecting higher order dynamics (period-2 or period-3 orbits), which can fail using classical Fourier-based methods. However little is known about its parameter sensitivity, e.g. with respect to the time delay, the embedding dimension or perturbations.

Here we provide preliminary results on the application of the recurrence time spectrum by analysing the Hénon and the Rössler attractor. Results indicate that the combination of recurrence time spectra with a nonlinearly filtered plot of return times is able to estimate the unstable periodic orbits. Owing to the use of recurrence plot based measures the analysis is more robust against noise than the conventional Fourier transform.

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Keywords: Phase space prediction; unstable periodic orbits; Wiener-Khinchin theorem; recurrence plot analysis

1. Introduction

The detection of periodicity and associated limit cycles is one key element in describing a structure's dynamic behaviour for reasons of stability and control [1,2]. Often a look at the power spectrum (or other Fourier transform-based applications) is sufficient. However, in nonlinear systems, a classical Fourier analysis does often not provide all information as clearly as required, in particular if the data are noisy and if only short time series are available [3]. Real life nonlinear system behaviour is often related to instabilities which are, however, manifold and can be encountered in

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many technical applications as well as in nature including electrical and in mechanical systems (pulsed lasers [1]), in electronic meta-materials [4], thin, elastic structures [5,6], friction-induced vibrations [7–9]), or in biology, chemistry or geology (e.g. population dynamics of predator-prey relationships [10], stirred tank reactors [11]). In practical applications, random additive and/or multiplicative noise (*higher dimensional dynamics*) can couple to the dynamics and may change its time evolution [12–14].

For proof of concepts studies simplified analytical benchmark systems in form of nonlinear algebraic equations (e.g. Logistic map, Hénon map) or nonlinear differential equations (e.g. Rössler or Lorentz system) are commonly employed of [2]. Unique mathematical measures or invariants are sought which describe the complex dynamics. An important type of invariant subset is that of an unstable periodic orbit (UPO). However, its quantification is inherently difficult.

Dominant UPOs can be used to represent the skeleton of an attracting *invariant* set the chaotic attractor[15,16]. This skeleton could then be used, as suggested by Gilmore and LeFranc in 2002 [16] to calculate other invariant measures such as linking numbers or rotation rates and combined with other classical invariants (dimension estimates, Lyapunov exponents) to generate a template as blue print of the dynamics.

Pierson and Moss used a periodically forced Van der Pol oscillator and a bistable, first order time delay system with noise [15] to determine the presence of UPOs. However, only statistical estimators were used to determine whether UPOs are actually present, neither the locations, nor the frequencies, nor the recurrence times of the UPOs were provided [15]. Schmelcher and Diakonos [17] used Voronoi diagrams to detect UPOs in low-dimensional chaotic systems, however, without considering noise, an essential requirement for the analysis of real-life data [1,14,15]. Relatively novel is the approach to quantify invariant sets based on the analysis of recurrent states in dynamical systems by using recurrence plots (RP) [18] and their quantification measures [19].

In 2007 Zbilut and Marwan have shown using the Wiener-Khinchin theorem that the power spectrum of recurrent lines in a RP (recurrence rate spectrum) is equivalent to the classical power spectrum but better suited to detect higher order harmonics of nonlinear dynamical systems [3]. Complementary to this study we test whether the recurrence rate spectra and recurrent times probabilities can be used to retrieve information about unstable periodic orbits of the noisy Hénon or Rössler system.

2. Models and Methods

Models. The Hénon map and the Rössler attractor [2] are employed here. The Hénon attractor is calculated using

$$\textbf{(a)}\ X_{i+1} = 1 - aX_i^2 + Y_i, \text{ and } \textbf{(b)}\ Y_{i+1} = bX_i \tag{1}$$

with initial values of $X_0 = 0.2$, $Y_0 = 0.2$, and $b = 0.3$. Two vectors $(\mathbf{X}, \mathbf{Y})$ of length $n = 5{,}000$ are generated (sampling frequency of one cycle per iteration). The first 2,000 samples are discarded as they could represent transient states [20]. The parameter a is used to control the system to obtain either period−1 ($a = 0.9$), period−2 ($a = 1.0$), period−4 ($a = 1.035$), or chaotic ($a = 1.4$) dynamics.

The Rössler system [2] is calculated using

$$\textbf{(a)}\ \frac{dx_1}{dt} = -x_2 - x_3\ , \ \textbf{(b)}\ \frac{dx_2}{dt} = x_1 + \alpha x_2 \text{ and, } \textbf{(c)}\ \frac{dx_3}{dt} = \beta + x_3(x_1 - \gamma). \tag{2}$$

with the parameters $\alpha = 0.2$, $\beta = 0.2$ and either $\gamma = 2.5$ (period−1), $\gamma = 3.3$ (period−2), $\gamma = 4.0$ (period−4), or $\gamma = 7.5$ (chaotic regime) [2]. As initial values $[x_1(0), x_2(0), x_3(0)] = [14.5, 0, 0.1]$ are taken; the oscillation frequency is about $\frac{1}{2\pi}$ [21]. The Rössler system of ordinary differential equations (ODEs) is solved for 10 s (time steps of 1E-2 s resolution) using Matlab's ode45 by setting the relative and absolute tolerance to 1E-5. The first 75,000 samples are cut off to remove transients and the following16,000 samples are sampled down to 25 Hz. Each time series is normalised with the maximum absolute amplitude found within the four different dynamic regimes of the Rössler system and multiplied by 0.762, the maximum amplitude found in the Y_i component of the Hénon attractor.

As surrogate model for real-life data an observable $\tilde{X}_n$ which is contaminated with additive, uniformly white noise ($\mathbf{wn} \in [0, 1]$) is generated (cf. [13,14]). The noise level (power of 0 dB relative to 1 W) is weighted with p

$$\tilde{\mathbf{X}} = \mathbf{X} + p \cdot \mathbf{wn} \circ \mathbf{1} \tag{3}$$

so that in the Welch power spectrum estimate of the period−4 regime all subharmonics would lie within the noise floor ($p = 0\%$, $p = 1\%$ and $p = 10\%$ or 25% for the Hénon map and the Rössler system). In equation 3, $\circ$ represents the Hadarmard product; **1** is a vector which contains only ones.

Methods. To visualise the effect of noise on (un)stable periodic orbits, the trajectory in phase space is plotted and data is described using histograms. From the original data of the Hénon map 2D histograms (55 bins) are generated. For the Rössler attractor the minimum and maximum phase space diameter in each direction are segmented into $60 \times 60 \times 60$ cubic neighbourhoods to determine the relative frequency of occurrence of a trajectory meeting these areas/volumes. The seven largest relative frequencies were identified and their centre coordinates were mapped back onto the original attractor. This statistical approach delivers relative frequencies and showcases the effect of noise on unstable periodic orbits (cf. Supplementary Information, SI).

However, in real life situations, usually single measurements are obtained and it is sought to reconstruct the phase spacing using reconstruction techniques. Therefore, a single component is chosen as observable $\tilde{\mathbf{X}}$; Welch's power spectral densities with and without noise ($n = 4,000$ samples) are estimated [3] (window size 2,048 samples, overlap of 50%, 2,047 FFT lines). Delay vectors $\tilde{\mathbf{X}}$ are generated by determining a (i) time delay τ and (ii) an embedding dimension m which are calculated using the averaged cross-mutual information and the false nearest neighbour algorithm [1,2]. With these embedding parameters a recurrence matrix can be setup,

$$R(i, j) = \Theta(\epsilon - \|\tilde{X}(i) - \tilde{X}(j)\|) \in \{0, 1\}, \text{ for } \forall i, j = 1...N \tag{4}$$

Here, $R(i, j)$ stands for one single element in the recurrence matrix, Θ describes the Heavyside function and$\tilde{X}(i) = X_i$ represents a component of the delay-embedded state space vector, with $\tilde{X}(j) = X_j = \tilde{X}(i + \tau)$ being delay vectors with ϵ as arbitrary threshold value. The neighbourhood size ϵ is chosen as a fraction of the maximum phase space diameter (D) [19].

The Hénon system without noise with period−1, the period−2, the period−4 and chaotic dynamics was embedded in a phase space of dimension $m = 2, 2, 2, 2$ and a delay of $\tau = 1, 1, 1, 6$ (with noise $m = 7, 7, 7, 7$, $\tau = 1, 1, 1, 8$).

For the noise-free case we chose ϵ to be 1.5% and 25% if noise was considered. For the period−1, period−2, period−4 and chaotic dynamics of the noise-free Rössler system we estimated $m = 2, 2, 2, 3$ and $\tau = 136, 152, 136, 152$ ($\epsilon = 7.5\%$); with noise the embedding parameters changed to $m = 7, 7, 7, 7$ and $\tau = 136, 144, 144, 152$ ($\epsilon = 50\%$). For the calculation of the recurrence rate we used moving windows of length $l = 4$ and $l = 300$ for the Hénon and the Rössler system, respectively.

For correctly embedded dynamics, a recurrence plot displays patterns of diagonals with rather few vertical or horizontal line structures [19]. Diagonal lines correspond to recurrent dynamical states, whereas vertical lines indicate trapped dynamics. In case the maximum norm $\|\cdot\|_\infty$ is used as distance function, the recurrence plot matrix is quadratic. From line structures in the RP, histograms and quantification measures are estimated. Here we use the recurrence rate (RR: density of recurrent states in a recurrence plot $\propto$ probability that a certain dynamic state repeats itself) and the mean recurrence time (RT: mean time between recurrent states) which are calculated as

$$\textbf{(a)}\ \text{RR} = \frac{1}{N^2} \sum_{i,j}^{N} \mathbf{R}(i, j), \text{ and } \textbf{(b)}\ \text{RT} = frac \sum_{k=1}^{N} kp(k) \sum_{k=1}^{N} p(k), \tag{5}$$

with N being the length of the trajectory, $p(k)$ being the mean of the vertical lines of length k [22]. To generate RT the observation vector is divided into shorter segments of length L for which each ML is calculated. Here $L = N$ is chosen so that the period can be exactly determined and a bar diagram can be generated. Using the neighbourhood criterion, close-by trajectories are captured as recurrent states and rather the probability of capturing a certain state is estimated via its relative frequency.

The signal's power is calculated using a FFT according to [3] considering the average squared distances and linking them over the auto-covariance function to the power spectral density (details [3] & SI). The contamination with noise is however only a first test on performance of the recurrence plot based measures compared to the classical power spectra; in a realistic situation the observable might noisy and filtering is required. Therefore we use the nonlinear filter filtering algorithm *ghkss* [1] which is based on projecting the dynamics on a low dimensional manifold which houses the attracting set and thereby correcting the coordinates by retracting the measurement points onto the true trajectory, cf. [23].

3. Results

In case of the Hénon map a contamination with 1% noise makes it already very difficult to localise the first three dominant UPOs and their coordinates, and becomes impossible for noise levels greater than 5% (see SI Fig. 1 & 2). However, using recurrence plot based measures show encouraging results up to the maximum noise level of 10%: in the RR spectrum the dominant frequencies are still visible and the probabilities of the RT of 'true' UPOs remain the highest.

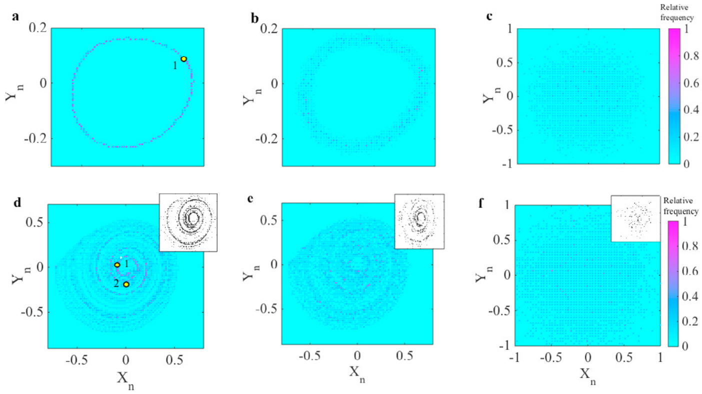

Fig. 1. Frequency of occurrence relative to the maximum value of the Rössler attractor within the (**a**-**c**) period−1 regime ($\gamma = 2.5$) and the (**d**-**f**) chaotic regime ($\gamma = 7.5$); (**a**, **d**) 0%, (**b**, **e**) 1%, and (**c**, **f**) 25% noise are added; dominant periodic orbits, if identified, are marked and numbered

Figure 1 shows the *histograms* of the period−1 and the chaotic dynamics of the Rössler system with 0%, 1% and 25% noise. The periodic orbit (period−1 regime, upper row) is easily identified in the noise-free case, but already with 1% noise, the trajectory becomes blurry and the calculation of relative frequencies (similar to box counting) does not help much anymore. A similar picture is given for the chaotic dynamics (lower row of subfigures); in the noise-free case only the first two dominant unstable periodic orbits can be distinguished, as soon as noise is added, it becomes visibly hard to extract information by merely using relative frequencies.

Fig. 2 depicts the time series, Welch's power spectrum estimate, the recurrence spectrum and the probability of recurrence times for period−1, period−2, period−4 and chaotic dynamics of the noise-free and 25% noise-contaminated Rössler system. The spectra ($\mathbf{a_2}$ - $\mathbf{d_2}$) are significantly lifted up due to noise which adds energy to the system, making it hard to see the subharmonics in the Welch spectrum, which is not the case for the recurrence spectrum ($\mathbf{a_3}$ - $\mathbf{d_3}$).

In Fig. ($\mathbf{a_4}$ - $\mathbf{d_4}$) the probability of recurrence times are depicted: the noise free case clearly shows the dominant UPO having the largest probability, however if the time trace is contaminated with 25% no UPOs are distinguishable anymore. The reason for that lies in the ϵ which was kept constant at about 0.25% of D: contrary to the Hénon map, the trajectories move very close to each other and diverge rendering an adaptive neighbourhood more suitable (e.g. fixed amount of neighbours [19]).

However, for the chaotic case the dominant unstable periodic orbit is still picked up with a return time of 5.25 s whereas the second unstable periodic orbit returns at about 10.04 s (Fig. 2($\mathbf{d_4}$)); both of these frequencies are also visible in the recurrence spectrum below a newly emerging major peak of at about 20 Hz but less clearly visible in its associated Welch spectrum (($\mathbf{b_4}$ and $\mathbf{c_4}$).

Contamination with noise does not necessary allow us to reliably identify more than two unstable periodic orbits as shown for the map and the continuous system. It is therefore suggested here to nonlinearly filter the signal ([1,14, 23]) to see whether after filtering some of the old properties of the recurrence rate spectrum or the recurrence time probability can be recovered. The Rössler attractor with 25% noise has been chosen as a worst case scenario, with results being depicted in Fig. 3. The dynamics were embedded within a phase space of dimension $m = 7$, delay $\tau = 30$, and projected onto an attractor of dimension $q = 3$ within five iterations.

Fig. 3 depicts four different plots with the (**a**) time trace, (**b**) the Welch spectrum, (**c**) the recurrence spectrum, and($\mathbf{d_1}$-$\mathbf{d_3}$) the recurrence time plots.

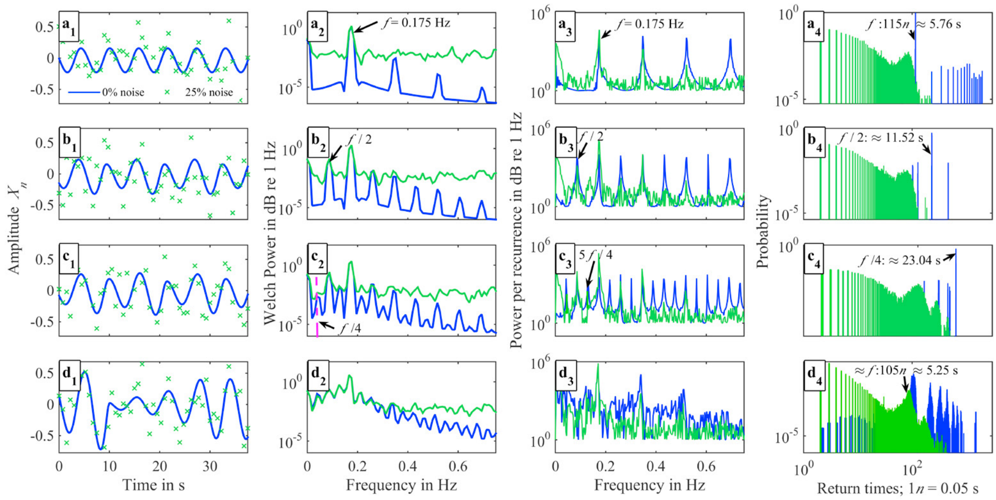

Fig. 2. The Rössler attractor with and without noise in the (**a**) period-1, (**b**) period-2, (**c**) period-4, and the (**d**) chaotic regime; the first column (**1**) provides the time series, the second column (**2**) gives its Welch power spectral density estimate, (**3**) shows the recurrence rate spectrum and the fourth column (**c**) shows the recurrence time probabilities.

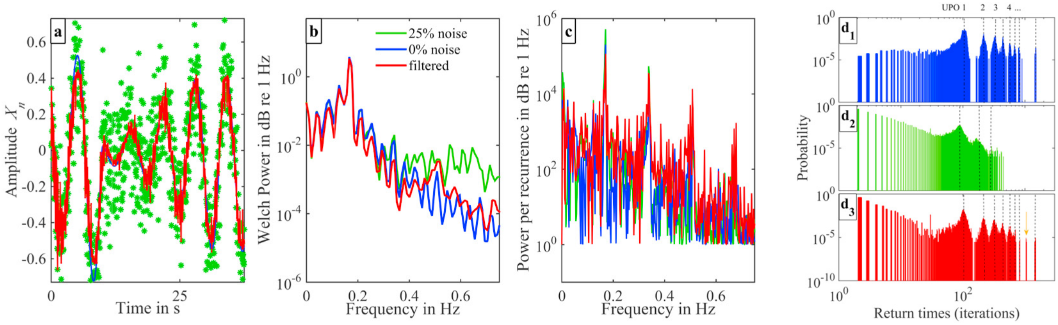

Fig. 3. (**a**) Time trace, (**b**) the Welch spectrum, (**c**) recurrence spectrum, and (**d_1** - **d_3**) recurrence time plots for Rössler attractor with and without noise and filtered in the chaotic regime; vertical lines in **d** indicate correspondence with the unstable periodic orbits detected with 0% noise.

After filtering (**a**) only small residuals are distinguishable which is also indicated by the high frequency noise reduction in the Welch spectrum (**b**)).

The recurrence spectrum looks rather unchanged. However, comparing the plots of the recurrence time probabilities indicates that filtering allows recovering all of the higher order periods while with noise only up to three unstable periodic orbits (dashed vertical lines) show up. However, a spurious period appears (indicated by arrow) and which is attributed to residual noise.

4. Conclusions

Analysing the Hénon map and the Rössler system highlights the potential of using RP based measures to identify (un-)stable periodic orbits in the presence of noise; by using simply channel noise (additive, uncorrelated noise) the attractor remains preserved. While the classical Welch power spectral estimate or simply counting relative frequencies (box counting) becomes difficult to interpret for short and noisy time series the recurrence rate spectra remain finer resolved and highlight all harmonics. The recurrence time probability plot gives both the recurrence times and the probabilities that the dynamics return to the same trajectory. For the noise free Hénon map, all periodic orbits are identified, whereas some small deviations of less than 2% occur for the Rössler system. Filtering the time series using a nonlinear projective filter recovers most of the features of the recurrence time probability plots which, in combination with recurrence rate spectra, facilities the identification of the most important unstable periodic orbits. Whether the method suggested here also provides sensible results in the case of the analysis of experimental data or whether it can be used in connection with dynamical systems which are intrinsically noisy, i.e. which include simultaneously higher and lower dimensional processes (random dynamical systems), or which are influenced by dynamic noise or coupled multiplicatively to noise (and which can produce fake UPOs [13]), needs to be studied in more detail in the future.

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Supplementary Information

Additional Supplementary Information can be found online under doi:10.13140/RG.2.2.10420.86401.

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