Estimation of Cusp Location of Stochastic Processes: a Survey

S. Dachian[1], N. Kordzakhia[2], Yu.A. Kutoyants[3] and A. Novikov[4]

[1]University of Lille, Lille, France
[2]Macquarie University, Sydney, Australia
[3]Le Mans University, Le Mans, France
[1,3]National Research University "MPEI", Moscow, Russia
[3]Tomsk State University, Tomsk, Russia
[4]University of Technology Sydney, Australia

Abstract

We present a review of some recent results on estimation of location parameter for several models of observations with cusp-type singularity at the change point. We suppose that the cusp-type models fit better to the real phenomena described usually by change point models. The list of models includes Gaussian, inhomogeneous Poisson, ergodic diffusion processes, time series and the classical case of i.i.d. observations. We describe the properties of the maximum likelihood and Bayes estimators under some asymptotic assumptions. The asymptotic efficiency of estimators are discussed as well and the results of some numerical simulations are presented. We provide some heuristic arguments which demonstrate the convergence of log-likelihood ratios in the models under consideration to the fractional Brownian motion.

Key words: *Change-point models, cusp-type singularity, inhomogeneous Poisson processes, Diffusion processes, Maximum likelihood and Bayes estimators, fractional Brownian motion.*

1 Models of observations

We consider several models of observations having a cusp-type singularity at the point of location parameter. Such models can be considered as a natural alternative or extension to the change-point (in space and in time) models with jumps in characteristics. The list of models of observations included in this paper contains the case of independent identically distributed random variables (i.i.d. r.v.s, the change point in space), a signal in white Gaussian noise (the change point in time), Poisson process (the change point in time), the ergodic diffusion process (the change point in space) and perturbed dynamical system (the change point in space).

For general illustration of our approaches we start with the model of signal observed with a small additive white Gaussian noise:

$$\mathrm{d}X_t = S(\vartheta, t)\,\mathrm{d}t + \varepsilon \mathrm{d}W_t, \qquad X_0 = 0, \quad 0 \le t \le T. \tag{1}$$

Here $S(\vartheta, \cdot) \in \mathcal{L}_2[0, T]$ is the signal, $\vartheta \in \Theta = (\alpha, \beta)$ is an unknown parameter and W_t is a standard Wiener process. The parameter $\varepsilon > 0$ is supposed to be small. We are interested in the following problem: how the asymptotic properties of estimators depend on the order of regularity of the signal $S(\vartheta, t)$? The special attention will be paid to the case of models with *cusp-type* singularity.

Note that the case of the observations $X^T = (X(t), 0 \le t \le T = n\tau)$ of type (1) with τ-periodic signal $S(\vartheta, t)$ (the period τ is supposed to be known) and $\varepsilon = 1$ can be reduced to the previous model by setting

$$X_t = \frac{1}{n}\sum_{j=1}^{n}\left[X(\tau(j-1)+t) - X(\tau(j-1))\right], \qquad 0 \le t \le \tau.$$

Now this averaging process satisfies (1) with $\varepsilon = n^{-1/2}$ and another Wiener process (see also Section 1.3 for the case of an inhomogeneous Poisson process with τ-periodic intensity function).

To estimate the parameter ϑ and to describe the asymptotic properties of the estimators as $\varepsilon \to 0$ (a *small noise* asymptotics) we shall use the likelihood ratio function

$$V\left(\vartheta, X^T\right) = \exp\left\{\frac{1}{\varepsilon^2}\int_0^T S(\vartheta, t)\,\mathrm{d}X_t - \frac{1}{2\varepsilon^2}\int_0^T S(\vartheta, t)^2\,\mathrm{d}t\right\}, \quad \vartheta \in \Theta. \tag{2}$$

In this paper we shall discuss the asymptotic properties of the maximum likelihood estimator (MLE) $\hat\vartheta_\varepsilon$ and Bayes estimator (BE) $\tilde\vartheta_\varepsilon$ for the quadratic loss function. These estimators are defined by the relations

$$V(\hat\vartheta_\varepsilon, X^T) = \sup_{\vartheta\in\Theta} V\left(\vartheta, X^T\right), \qquad \tilde\vartheta_\varepsilon = \frac{\int_\alpha^\beta \vartheta p(\vartheta)\, V\left(\vartheta, X^T\right)\mathrm{d}\vartheta}{\int_\alpha^\beta p(\vartheta)\, V\left(\vartheta, X^T\right)\mathrm{d}\vartheta}.$$

We suppose that the density $p(\cdot)$ of the prior distribution for ϑ is a positive continuous function.

It is known that if the signal $S(\vartheta,t)$ is *smooth* w.r.t. ϑ, then the MLE and BE are asymptotically normal and asymptotically efficient with the rate ε, i.e.

$$\varepsilon^{-1}(\hat\vartheta_\varepsilon-\vartheta)\Longrightarrow\zeta,\qquad \varepsilon^{-1}(\tilde\vartheta_\varepsilon-\vartheta)\Longrightarrow\zeta,\qquad \zeta\sim\mathcal{N}\left(0,\mathrm{I}(\vartheta)^{-1}\right),$$

where $\mathrm{I}(\vartheta)$ is the Fisher information [8]. The arrow $\Longrightarrow$ means the convergence in distribution.

In contrast to the above smooth case, in the following classical change-point model of observations

$$\mathrm{d}X_t=\mathbb{1}_{\{t\geq\vartheta\}}\mathrm{d}t+\varepsilon\mathrm{d}W_t,\qquad X_0=0,\quad 0\leq t\leq T, \tag{3}$$

the MLE $\hat\vartheta_\varepsilon$ and BE $\tilde\vartheta_\varepsilon$ have the rate of convergence ε^2 (see [9]), i.e.

$$\varepsilon^{-2}\left(\hat\vartheta_\varepsilon-\vartheta\right)\Longrightarrow\hat\xi,\qquad \varepsilon^{-2}\left(\tilde\vartheta_\varepsilon-\vartheta\right)\Longrightarrow\tilde\xi,$$

where the random variables $\hat\xi$ and $\tilde\xi$ are defined by the relations

$$Z(\hat\xi)=\sup_{u\in\mathcal{R}}Z(u),\qquad \tilde\xi=\frac{\int_{-\infty}^{\infty}uZ(u)\,\mathrm{d}u}{\int_{-\infty}^{\infty}Z(u)\,\mathrm{d}u}. \tag{4}$$

Here $Z(u)=\exp\{\gamma W(u)-\frac{|u|}{2}\gamma^2\}$, $u\in\mathcal{R}$, $W(\cdot)$ is a two-sided Wiener process, γ is a constant (to be defined explicitly below). The uniqueness with probability 1 of the solution of the first equation was shown in [18].

Typically, real physical phenomena and technical devices have a transition phase from one state to another which can be described in many ways, for example, with use a smooth function (signal) having a very large Fisher information. The important question is what happens if the regularity of the signal is different of the supposed one, see some results in this direction in [14]. We consider here another model with the signals having *cusp*-type singularities. As alternative to (3) one can consider (1) with, for example, the signal

$$S(\vartheta,t)=\frac{1}{2}\left(1+\mathrm{sgn}\,(t-\vartheta)\left|\frac{t-\vartheta}{\delta}\right|^{\kappa}\right)\mathbb{1}_{\{|t-\vartheta|\leq\delta\}}+\mathbb{1}_{\{t>\vartheta+\delta\}}. \tag{5}$$

Here $\delta>0$ is some small parameter and $\kappa\in(0,\frac{1}{2})$. We suppose that $\vartheta\in(\alpha,\beta)$, where $\alpha>\delta$ and $\beta<T-\delta$. This signal is a continuous function and for small δ and κ it can be a good $\mathcal{L}_2\,[0,T]$ approximation for the signal

in (3). Note that when the Fisher information does not exist, the problem of estimation ϑ is singular.

The examples of the corresponding curves of signals are given in Figure 1.

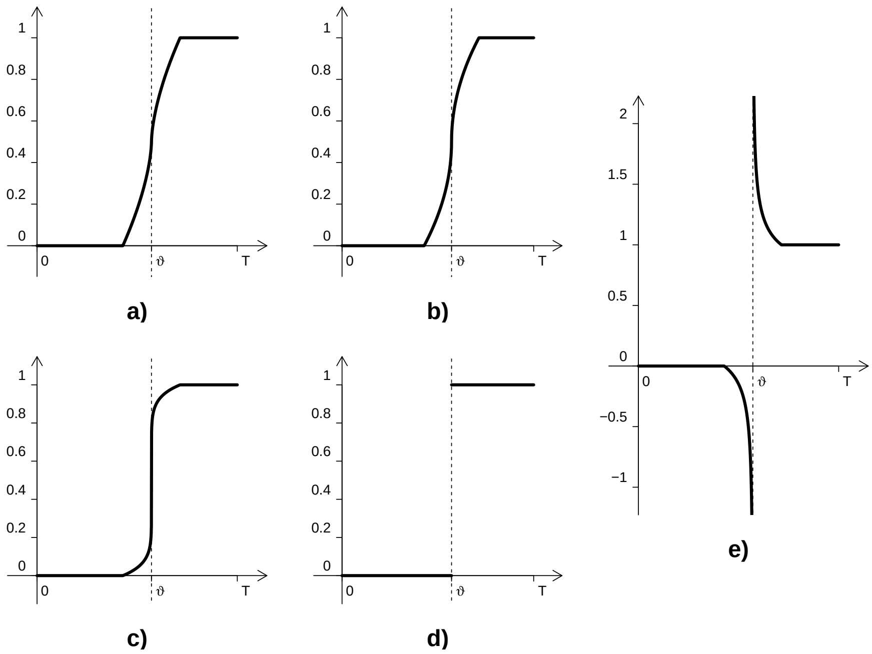

Figure 1: **a)** $\kappa = \frac{5}{8}$, **b)** $\kappa = \frac{1}{2}$, **c)** $\kappa = \frac{1}{8}$, **d)** $\kappa = 0$, **e)** $\kappa = -\frac{3}{8}$.

The asymptotic properties of the MLE and BE for ϑ under the assumptions (1) and (5) with $\kappa \in (-\frac{1}{2}, \frac{1}{2})$ are as follows:

$$\varepsilon^{-\frac{2}{2\kappa+1}} \left(\hat{\vartheta}_\varepsilon - \vartheta\right) \Longrightarrow \hat{\xi}, \qquad \varepsilon^{-\frac{2}{2\kappa+1}} \left(\tilde{\vartheta}_\varepsilon - \vartheta\right) \Longrightarrow \tilde{\xi}, \tag{6}$$

where the random variables $\hat{\xi}$ and $\tilde{\xi}$ are defined in (4) with

$$Z\left(u\right) = \exp\left\{\gamma W^H\left(u\right) - \frac{\left|u\right|^{2H}}{2}\gamma^2\right\}, \qquad u \in \mathcal{R}. \tag{7}$$

Here $H = \kappa + \frac{1}{2}$ is the *Hurst parameter*,

$$\gamma = \frac{1}{2\delta^\kappa \Gamma_*}, \qquad \Gamma_*^2 = \int_{-\infty}^{\infty} \left[\operatorname{sgn}\left(s-1\right)\left|s-1\right|^\kappa - \operatorname{sgn}\left(s\right)\left|s\right|^\kappa\right]^2 \mathrm{d}s$$

and $W^H(\cdot)$ is a fractional Brownian motion (fBm), i.e. $W^H(u), u \in \mathcal{R}$, is a Gaussian process, $W^H(0) = 0, \mathbf{E}W^H(u) = 0$ with the covariance function

$$\mathbf{E}W^H(u_1) W^H(u_2) = \frac{1}{2}\left(|u_1|^{2H} + |u_2|^{2H} - |u_1 - u_2|^{2H}\right). \tag{8}$$

Note that it was first A.N. Kolmogorov [11] who discussed this type of Gaussian processes under the equivalent assumption to (8):

$$\mathbf{E}|W^H(u_1) - W^H(u_2)|^2 = |u_1 - u_2|^{2H}.$$

The convergence in distribution, convergence of moments of these estimators and the fact of the asymptotic efficiency of the BE were established in [1] (for $\kappa \in \left(0, \frac{1}{2}\right)$) and in [12] (for $\kappa \in \left(-\frac{1}{2}, 0\right)$).

The rate of convergence of mean square error of the MLE and BE in the model (1) with signal (5) essentially depends on κ.

We have the following asymptotics of the mean square error

a) *Smooth signal,* $\kappa > \frac{1}{2}$: $\quad \mathbf{E}_\vartheta\left(\hat{\vartheta}_\varepsilon - \vartheta\right)^2 \sim \varepsilon^2$ (see [8]).

b) *Smooth signal with* $\kappa = \frac{1}{2}$: $\quad \mathbf{E}_\vartheta\left(\hat{\vartheta}_\varepsilon - \vartheta\right)^2 \sim \frac{\varepsilon^2}{\ln\frac{1}{\varepsilon}}$.

c) *Continuous signal with cusp* $\kappa \in \left(0, \frac{1}{2}\right)$: $\quad \mathbf{E}_\vartheta\left(\hat{\vartheta}_\varepsilon - \vartheta\right)^2 \sim \varepsilon^{\frac{4}{2\kappa+1}}$. *Note that* $2 < \frac{4}{2\kappa+1} < 4$ (see [1]).

d) *Discontinuous signal,* $\kappa = 0$: $\quad \mathbf{E}_\vartheta\left(\hat{\vartheta}_\varepsilon - \vartheta\right)^2 \sim \varepsilon^4$ (see [9]).

e) *Discontinuous signal with cusp* $\kappa \in \left(-\frac{1}{2}, 0\right)$: $\quad \mathbf{E}_\vartheta\left(\hat{\vartheta}_\varepsilon - \vartheta\right)^2 \sim \varepsilon^{\frac{4}{2\kappa+1}}$. *In this case* $\frac{4}{2\kappa+1} > 4$ (see [12]) .

Here $A_\varepsilon \sim B_\varepsilon$ means that there exist constants $0 < c \leq C$ such that $c \leq \frac{A_\varepsilon}{B_\varepsilon} \leq C$ as $\varepsilon \to 0$. It must be noted that if $\kappa \leq -\frac{1}{2}$, then the probability measures generated by the observation X^T are singular for all different values ϑ and hence the parameter ϑ can be estimated without error. To illustrate this fact, suppose that $S(\vartheta, t) = |t - \vartheta|^\kappa$ and $\kappa \in \left(-1, -\frac{1}{2}\right)$. Let $\kappa_* \in \left(-\frac{1}{2}, 0\right)$ such that $\kappa + \kappa_* \leq -1$ and the integral

$$J(\vartheta) = \int_0^T |s - \vartheta|^{\kappa_*} \,\mathrm{d}X_s = \int_0^T |s - \vartheta|^{\kappa_*} |s - \vartheta_0|^\kappa \,\mathrm{d}s + \varepsilon \int_0^T |s - \vartheta|^{\kappa_*} \,\mathrm{d}W_s.$$

It is easy to see that here the stochastic integral above is always finite with probability one and the ordinary integral diverges at the point $\vartheta = \vartheta_0$. Using this property we can construct an estimator of ϑ_0 without error. For example, ϑ_0 is solution of the equation $J(\vartheta)^{-1} = 0$.

The case **b)** can be treated similar to **a)**, using the same method as in [8].

The goal of this work is to present the review of the range of models with cusp-type singularities and to describe the asymptotic properties of the MLE and BE. Below we consider the i.i.d. random variables $X_1, \ldots, X_n$ with a marginal density having the cusp-type singularity (as $n \to \infty$), the ergodic diffusion processes $X_t, 0 \le t \le T$ with the trend coefficient having cusp-type singularity (as $T \to \infty$), τ-periodic Poisson process $X_t, 0 \le t \le n\tau$ with the intensity function having a cusp-type singularity (as $n \to \infty$), the diffusion processes with small diffusion coefficient ε^2 and with the trend coefficient having cusp-type singularity as $\varepsilon \to 0$. In all such models the normalized likelihood ratio processes converge to the process (7) with some constant γ. The proofs of week convergence of the likelihood ratio processes to the exponent of fBm W^H are rather tedious. Therefore, in this survey, we present only heuristic arguments showing why convergence (7) should hold. The detailed proofs can be found in the cited papers.

For illustrating our approaches to studying MLE and BE we start from (1) with the specified signal (5) under the assumption $\varepsilon \to 0$. The general technique to study for such models was developed by Ibragimov and Khasminskii in series of papers and can be found in their fundamental monograph [10]. To use this technique we introduce the normalized likelihood ratio process

$$Z_\varepsilon(u) = \frac{V(\vartheta_0 + \varphi_\varepsilon u, X)}{V(\vartheta_0, X)}, \qquad u \in \mathbb{U}_\varepsilon = \left(\frac{\alpha - \vartheta_0}{\varphi_\varepsilon}, \frac{\beta - \vartheta_0}{\varphi_\varepsilon} \right),$$

where we denoted ϑ_0 the true value and the normalizing function $\varphi_\varepsilon = \varepsilon^{\frac{1}{H}}$. In all problems under consideration in this paper we represent the log-likelihood ratios as follows

$$\ln Z_\varepsilon(u) = \mathbb{A}_\varepsilon(u) - \mathbb{B}_\varepsilon(u) + o(1)$$

demonstrating the following convergences

$$\mathbb{A}_\varepsilon(u) \Longrightarrow \gamma W^H(u), \qquad \mathbb{B}_\varepsilon(u) \longrightarrow \frac{|u|^{2H}}{2}\gamma^2,$$

where $\longrightarrow$ means convergence in probability. The particular forms of $\mathbb{A}_\varepsilon(u)$ and $\mathbb{B}_\varepsilon(u)$ in the different problems are different and here we give a general symbolic representation. We would like to show the universality of this local structure for rather different models of observations without providing full technical details of the proofs.

Suppose that we have already proved the weak convergence to the process $Z(u)$ in (7)

$$Z_\varepsilon(\cdot) \Longrightarrow Z(\cdot).$$

Then the limit distributions of the estimators can be obtained as follows. For the MLE $\hat\vartheta_\varepsilon$ we can write

$$\begin{aligned}
\mathbf{P}_{\vartheta_0}\left(\frac{\hat\vartheta_\varepsilon-\vartheta_0}{\varphi_\varepsilon}<x\right) &= \mathbf{P}_{\vartheta_0}\left(\hat\vartheta_\varepsilon<\vartheta_0+\varphi_\varepsilon x\right)\\
&= \mathbf{P}_{\vartheta_0}\left\{\sup_{\vartheta<\vartheta_0+\varphi_\varepsilon x} V\left(\vartheta,X^T\right) > \sup_{\vartheta\geq\vartheta_0+\varphi_\varepsilon x} V\left(\vartheta,X^T\right)\right\}\\
&= \mathbf{P}_{\vartheta_0}\left\{\sup_{\vartheta<\vartheta_0+\varphi_\varepsilon x} \frac{V\left(\vartheta,X^T\right)}{V\left(\vartheta_0,X^T\right)} > \sup_{\vartheta\geq\vartheta_0+\varphi_\varepsilon x} \frac{V\left(\vartheta,X^T\right)}{V\left(\vartheta_0,X^T\right)}\right\}\\
&= \mathbf{P}_{\vartheta_0}\left\{\sup_{u<x,u\in\mathbb{U}_\varepsilon} Z_\varepsilon(u) > \sup_{u\geq x,u\in\mathbb{U}_\varepsilon} Z_\varepsilon(u)\right\}\\
&\longrightarrow \mathbf{P}_{\vartheta_0}\left\{\sup_{u<x} Z(u) > \sup_{u\geq x} Z(u)\right\} = \mathbf{P}_{\vartheta_0}\left(\hat\xi<x\right).
\end{aligned}\tag{9}$$

For the BE $\tilde\vartheta_\varepsilon$ with the change of variable $\vartheta_u=\vartheta_0+\varphi_\varepsilon u$ we have

$$\begin{aligned}
\tilde\vartheta_\varepsilon &= \frac{\int\theta p(\theta)V\left(\theta,X^T\right)\mathrm{d}\theta}{\int p(\theta)V\left(\theta,X^T\right)\mathrm{d}\theta} = \vartheta_0+\varphi_\varepsilon\frac{\int_{\mathbb{U}_\varepsilon} up(\theta_u)V\left(\theta_u,X^T\right)\mathrm{d}u}{\int_{\mathbb{U}_\varepsilon} p(\theta_u)V\left(\theta_u,X^T\right)\mathrm{d}u}\\
&= \vartheta_0+\varphi_\varepsilon\frac{\int_{\mathbb{U}_\varepsilon} up(\theta_u)\frac{V\left(\theta_u,X^T\right)}{V\left(\vartheta_0,X^T\right)}\mathrm{d}u}{\int_{\mathbb{U}_\varepsilon} p(\theta_u)\frac{V\left(\theta_u,X^T\right)}{V\left(\vartheta_0,X^T\right)}\mathrm{d}u} = \vartheta_0+\varphi_\varepsilon\frac{\int_{\mathbb{U}_\varepsilon} up(\theta_u)Z_\varepsilon(u)\mathrm{d}u}{\int_{\mathbb{U}_\varepsilon} p(\theta_u)Z_\varepsilon(u)\mathrm{d}u}.
\end{aligned}$$

Hence

$$\frac{\tilde\vartheta_\varepsilon-\vartheta_0}{\varphi_\varepsilon} = \frac{\int_{\mathbb{U}_\varepsilon} up(\theta_u)Z_\varepsilon(u)\mathrm{d}u}{\int_{\mathbb{U}_\varepsilon} p(\theta_u)Z_\varepsilon(u)\mathrm{d}u} \Longrightarrow \frac{\int_{\mathcal{R}} uZ(u)\mathrm{d}u}{\int_{\mathcal{R}} Z(u)\mathrm{d}u} = \tilde\xi. \tag{10}$$

Thus we have (formally) verified (6).

Moreover, in all such models under consideration the following lower bound for the mean-square errors of all normalized estimators $\bar\vartheta_\varepsilon$ holds:

$$\lim_{\delta\to0}\varliminf_{\varepsilon\to0}\sup_{|\vartheta-\vartheta_0|\leq\delta}\mathbf{E}_\vartheta\left(\frac{\bar\vartheta_\varepsilon-\vartheta}{\varphi_\varepsilon}\right)^2 \geq \mathbf{E}_{\vartheta_0}(\tilde\xi^2)$$

(see, e.g. [1]). This implies that an estimator ϑ^*_ε is asymptotically efficient (i.e. optimal) if for all $\vartheta_0\in\Theta$ we have

$$\lim_{\delta\to0}\lim_{\varepsilon\to0}\sup_{|\vartheta-\vartheta_0|\leq\delta}\mathbf{E}_\vartheta\left(\frac{\vartheta^*_\varepsilon-\vartheta}{\varphi_\varepsilon}\right)^2 = \mathbf{E}_{\vartheta_0}(\tilde\xi^2).$$

Note that if we put $\varphi_\varepsilon = \varepsilon^{\frac{1}{H}}\gamma^{-\frac{1}{H}}$, then

$$Z_\varepsilon\left(\cdot\right) \Longrightarrow Z_0\left(\cdot\right),$$

where $Z_0\left(\cdot\right)$ coincides with $Z\left(\cdot\right)$ in (7) with $\gamma = 1$.

Let $\hat{\xi}_0$ and $\tilde{\xi}_0$ be defined by (4) with $Z\left(\cdot\right)$ replaced by $Z_0\left(\cdot\right)$. Obviously, the distributions of $\hat{\xi}_0$ and $\tilde{\xi}_0$ do not depend on ϑ_0 and we obtain the following relations: $\hat{\xi} = \gamma^{-\frac{1}{H}}\hat{\xi}_0$ and $\tilde{\xi} = \gamma^{-\frac{1}{H}}\tilde{\xi}_0$. In particular, $\mathbf{E}_{\vartheta_0}(\tilde{\xi}^2) = \gamma^{-\frac{2}{H}}\mathbf{E}(\tilde{\xi}_0^2)$.

It is of interest to compare the limit variances of $\hat{\vartheta}_\varepsilon$ and $\tilde{\vartheta}_\varepsilon$ for the different values of $H = \kappa + \frac{1}{2}$. In [17] it was shown via numerical simulations that the limit values $\mathbf{E}(\hat{\xi}_0^2)$ could be essentially larger than $\mathbf{E}(\tilde{\xi}_0^2)$. The results are presented in Figure 2 for $H \in (0.4, 1]$ or, correspondingly, $\kappa \in (-0.1, 0.5]$. In Figure 3 we present the densities of the random variables $\hat{\xi}_0$ and $\tilde{\xi}_0$ obtained by the numerical simulations in [12]. Note that on Panel B: $H = 0.5$ the solid line shows the analytic curve for the density of MLE, this is the only case where the density is known in an analytic form, see [12] for details.

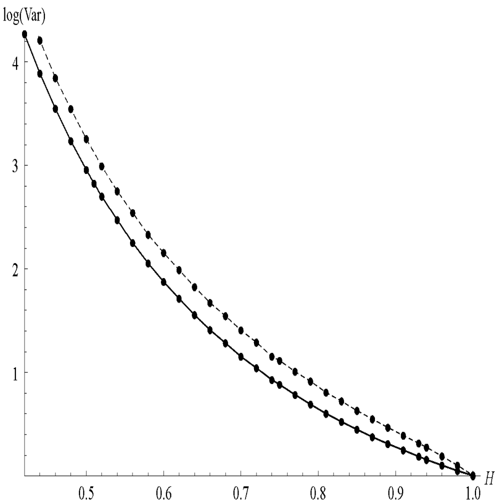

Figure 2: Limit curves of the values $\ln\mathbf{E}(\hat{\xi}_0^2) > \ln\mathbf{E}(\tilde{\xi}_0^2)$.

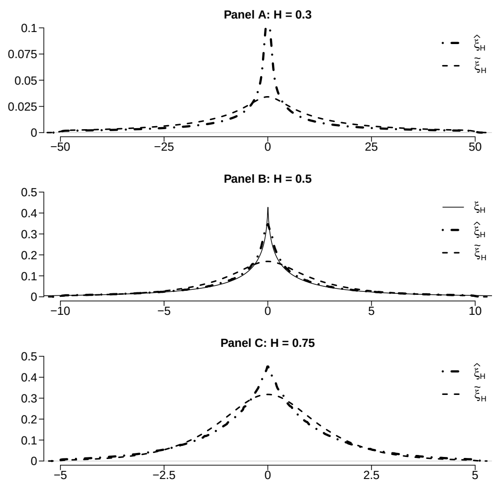

Figure 3: Densities of the MLE and BE for $H = 0.3, 0.5, 0.75$.

Below we consider several models of observations with the cusp-type singularity which lead to the same limit likelihood ratio process (7). To demonstrate this we shall use the following representations of the fBm

$$W^H\left(u\right) = \Gamma_*^{-1}\int_{-\infty}^{\infty}\Big[\mathrm{sgn}\left(v-u\right)\left|v-u\right|^\kappa - \mathrm{sgn}\left(v\right)\left|v\right|^\kappa\Big]\mathrm{d}W\left(v\right).$$

Here, obviously, $W^H(0)=0$, $\mathbf{E}W^H(u)=0$. Denoting $s_{v,u}=\operatorname{sgn}(v-u)$ and $g_{v,u}=s_{v,u}\left|v-u\right|^{\kappa}-s_{v,0}\left|v\right|^{\kappa}$ we can write

$$\begin{aligned}\mathbf{E}W^H(u_1)\,W^H(u_2)&=\Gamma_*^{-2}\int_{-\infty}^{\infty}g_{v,u_1}g_{v,u_2}\mathrm{d}v\\&=\frac{1}{2\Gamma_*^2}\int_{-\infty}^{\infty}g_{v,u_1}^2\mathrm{d}v+\frac{1}{2\Gamma_*^2}\int_{-\infty}^{\infty}g_{v,u_2}^2\mathrm{d}v-\frac{1}{2\Gamma_*^2}\int_{-\infty}^{\infty}\left[g_{v,u_2}-g_{v,u_1}\right]^2\mathrm{d}v\\&=\frac{1}{2}\left(\left|u_1\right|^{2H}+\left|u_2\right|^{2H}-\left|u_2-u_1\right|^{2H}\right).\end{aligned}$$

Hence the process $W^H(\cdot)$ satisfies (8). Here we used the elementary identity $2ab=a^2+b^2-(a-b)^2$ and changed the variables $v=su_1$, $v=su_2$, $v=u_1+s(u_2-u_1)$.

We would like to mention here that a more general representation of fBm is found in [12].

1.1 Independent random variables

We suppose that the observations $X^n=(X_1,\ldots,X_n)$ have the marginal density function

$$f(x-\vartheta)=h(x-\vartheta)\exp\left\{a\operatorname{sgn}(x-\vartheta)\left|x-\vartheta\right|^{\kappa}\right\},\qquad \vartheta\in\Theta=(\alpha,\beta).$$

The function $f(x)$ has at the point $x=0$ the cusp-type singularity $\kappa\in\left(0,\frac{1}{2}\right)$. The constant $a\neq 0$ and the function $h(\cdot)$ are known, $h(0)>0$. We suppose that the function $h(\cdot)$ is continuously differentiable. Our goal is to describe the behavior of the MLE $\hat\vartheta_n$ and BE $\tilde\vartheta_n$. According to (9)–(10) we describe the asymptotics of the normalized likelihood ratio process

$$Z_n(u)=\frac{V(\vartheta_0+\varphi_n u,X^n)}{V(\vartheta_0,X^n)},\qquad u\in\mathbb{U}_n=\left(\frac{\alpha-\vartheta_0}{\varphi_n},\frac{\beta-\vartheta_0}{\varphi_n}\right),$$

where $\varphi_n=n^{-\frac{1}{2\kappa+1}}$. In particular, we have to verify the convergence of $Z_n(u)$ to $Z(u)$ defined in (7). Let us see how the fBm appears in the limit log-likelihood ratio, using just heuristic arguments. Below we do it for $u\geq 0$. Let us denote $f_{u,x}=f(X_j-\vartheta_0-\varphi_n u)$, then we have

$$\begin{aligned}\ln Z_n(u)&=\sum_{j=1}^{n}\ln\frac{f(X_j-\vartheta_0-\varphi_n u)}{f(X_j-\vartheta_0)}=\sum_{j=1}^{n}\ln\frac{f_{u,X_j}}{f_{0,X_j}}\\&=\sum_{j=1}^{n}\ln\left(1+\frac{f_{u,X_j}-f_{0,X_j}}{f_{0,X_j}}\right)\\&=\sum_{j=1}^{n}\frac{f_{u,X_j}-f_{0,X_j}}{f_{0,X_j}}-\frac{1}{2}\sum_{j=1}^{n}\left(\frac{f_{u,X_j}-f_{0,X_j}}{f_{0,X_j}}\right)^2+o(1),\end{aligned}$$

where we used Taylor expansion $\ln(1+x) = x - \frac{x^2}{2} + o(x^2)$.

Introduce the further notations: $g(u,x) = as_{x,u}|x-\vartheta_0-\varphi_n u|^\kappa$ and $s_{x,u} = \operatorname{sgn}(x-\vartheta_0-\varphi_n u)$. Using the expansion $e^x - 1 = x + o(x)$ we can write

$$\sum_{j=1}^n \left(\frac{f_{u,X_j}-f_{0,X_j}}{f_{0,X_j}}\right)^2 = \sum_{j=1}^n \left[g(u,X_j)-g(0,X_j)\right]^2(1+o(1)).$$

Recall the relation: for any continuous function $g(x)$ we have

$$\frac{1}{n}\sum_{j=1}^n g(X_j) = \int_{-\infty}^{\infty} g(x)\,\mathrm{d}\hat{F}_n(x), \qquad \hat{F}_n(x) = \frac{1}{n}\sum_{j=1}^n \mathbb{I}_{\{X_j<x\}},$$

where $\hat{F}_n(x)$ is empirical distribution function. Let us denote

$$\mathbb{B}_n(u) = \frac{1}{2}\sum_{j=1}^n \left[g(u,X_j)-g(0,X_j)\right]^2.$$

Below $y = x-\vartheta_0, y = v\varphi_n, v = su$:

$$\begin{aligned}\mathbb{B}_n(u) &= n\frac{1}{2n}\sum_{j=1}^n \left[g(u,X_j)-g(0,X_j)\right]^2 = \frac{n}{2}\int_{\mathcal{R}}\left[g(u,x)-g(0,x)\right]^2\mathrm{d}\hat{F}_n(x)\\ &\approx \frac{n}{2}\int_{\mathcal{R}}\left[g(u,x)-g(0,x)\right]^2\mathrm{d}F(x-\vartheta_0)\\ &\approx \frac{a^2n}{2}\int_{\mathcal{R}}\left[\operatorname{sgn}(y-\varphi_n u)|y-\varphi_n u|^\kappa - \operatorname{sgn}(y)|y|^\kappa\right]^2 f(y)\,\mathrm{d}y\\ &= \frac{a^2n\varphi_n^{2\kappa+1}}{2}\int_{\mathcal{R}}\left[\operatorname{sgn}(v-u)|v-u|^\kappa - \operatorname{sgn}(v)|v|^\kappa\right]^2 f(v\varphi_n)\,\mathrm{d}v\\ &= \frac{a^2h(0)|u|^{2\kappa+1}}{2}\int_{\mathcal{R}}\left[\operatorname{sgn}(s-1)|s-1|^\kappa - \operatorname{sgn}(s)|s|^\kappa\right]^2\mathrm{d}s.\end{aligned}$$

Hence

$$\mathbb{B}_n(u) \longrightarrow \frac{a^2h(0)|u|^{2\kappa+1}}{2}\Gamma_*^2 = \frac{|u|^{2\kappa+1}}{2}\gamma^2.$$

It is known that $B_n(x) = \sqrt{n}\left(\hat{F}_n(x) - F_{\vartheta_0}(x)\right) \Rightarrow B(F_{\vartheta_0}(x))$, where $B(t)$, $t\in[0,1]$ is a Brownian bridge, $B(t) = W(t) - tW(1)$, $t\in[0,1]$. Recall that

$\mathbf{E}_{\vartheta_0}\frac{f_{u,X_j}-f_{0,X_j}}{f_{0,X_j}}=0$. Hence we can write formally

$$\begin{aligned}\mathbb{A}_n\left(u\right)&=\sqrt{n}\int_{\mathcal{R}}\frac{f_{u,x}-f_{0,x}}{f_{0,x}}\mathrm{d}B_n\left(x\right)\approx\sqrt{n}\int_{\mathcal{R}}\frac{f_{u,x}-f_{0,x}}{f_{0,x}}\mathrm{d}W_n\left(F_{\vartheta_0}\left(x\right)\right)\\&\quad-\sqrt{n}W_n\left(1\right)\int_{\mathcal{R}}\frac{f_{u,x}-f_{0,x}}{f_{0,x}}\mathrm{d}F_{\vartheta_0}\left(x\right)\\&=\sqrt{n}\int_{\mathcal{R}}\left[g\left(u,x\right)-g\left(0,x\right)\right]\mathrm{d}W_n\left(F_{\vartheta_0}\left(x\right)\right)+o\left(1\right).\end{aligned}$$

Below once more $y=x-\vartheta_0, y=v\varphi_n$ and $s_{y,u}=\operatorname{sgn}\left(y-u\right)$:

$$\begin{aligned}\mathbb{A}_n\left(u\right)&=a\sqrt{n}\int_{\mathcal{R}}\left[s_{y,u}\left|y-\varphi_nu\right|^{\kappa}-s_{y,0}\left|y\right|^{\kappa}\right]\mathrm{d}W_n\left(F\left(y\right)\right)+o\left(1\right)\\&=a\sqrt{f\left(0\right)n}\varphi_n^{\kappa+\frac{1}{2}}\int_{\mathcal{R}}\left[\operatorname{sgn}\left(v-u\right)\left|v-u\right|^{\kappa}-\operatorname{sgn}\left(v\right)\left|v\right|^{\kappa}\right]\mathrm{d}w_n\left(v\right)+o\left(1\right)\\&\Longrightarrow a\sqrt{h\left(0\right)}\int_{\mathcal{R}}\left[\operatorname{sgn}\left(v-u\right)\left|v-u\right|^{\kappa}-\operatorname{sgn}\left(v\right)\left|v\right|^{\kappa}\right]\ \mathrm{d}W\left(v\right)=\gamma W^H\left(u\right).\end{aligned}$$

Here we used the relations

$$w_n\left(v\right)=\frac{W_n\left(F\left(\varphi_nv\right)\right)-W_n\left(F\left(0\right)\right)}{\sqrt{f\left(0\right)\varphi_n}}\Longrightarrow W\left(v\right).$$

Therefore

$$Z_n\left(u\right)=e^{\mathbb{A}_n\left(u\right)-\mathbb{B}_n\left(u\right)+o\left(1\right)}\Longrightarrow Z\left(u\right)=e^{\gamma W^H\left(u\right)-\frac{\left|u\right|^{2\kappa+1}}{2}\gamma^2}.$$

The MLE $\hat\vartheta_n$ and the BE $\tilde\vartheta_n$ are consistent, have different limit distributions

$$n^{\frac{1}{2\kappa+1}}\left(\hat\vartheta_n-\vartheta_0\right)\Longrightarrow\hat\xi,\qquad Z(\hat\xi)=\sup_u Z\left(u\right),$$

$$n^{\frac{1}{2\kappa+1}}\left(\tilde\vartheta_n-\vartheta_0\right)\Longrightarrow\tilde\xi,\qquad \tilde\xi=\frac{\int uZ\left(u\right)\mathrm{d}u}{\int Z\left(u\right)\mathrm{d}u},$$

their moments converge and the BE are asymptotically efficient:

$$\lim_{\delta\to0}\lim_{n\to\infty}\sup_{\left|\vartheta-\vartheta_0\right|\leq\delta}n^{\frac{2}{2\kappa+1}}\mathbf{E}_{\vartheta}\left(\tilde\vartheta_n-\vartheta\right)^2=\mathbf{E}_{\vartheta_0}(\tilde\xi^2).$$

For the proof see [10], Section 6.4, where this case is called *singularity of order* 2κ *of the second type.*

Remark 1. The so-called generalized Gaussian distribution with the density of the form

$$f\left(x-\vartheta\right)=h\left(x-\vartheta\right)\exp\left\{-a\left|x-\vartheta\right|^{\kappa}\right\},\qquad\vartheta\in\Theta=\left(\alpha,\beta\right),\kappa>0,$$

was discussed by H.Daniels [4] for the case $\kappa>\frac{1}{2}$. The case $\kappa\in(0,\frac{1}{2})$ was first studied by Prakasa Rao [19].

1.2 Signal in white Gaussian noise

Consider the observations $X^T = (X_t, 0 \leq t \leq T)$ of the stochastic process satisfying the equation

$$\mathrm{d}X_t = \left[a\,\mathrm{sgn}\,(t-\vartheta)\,|t-\vartheta|^k + h\,(t-\vartheta)\right]\mathrm{d}t + \varepsilon \mathrm{d}W_t, \qquad 0 \leq t \leq T. \quad (11)$$

Here $X_0 = 0$, the parameters $a \neq 0, \kappa \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ and function $h\,(\cdot)$ are known and we have to estimate the parameter ϑ. The function $h\,(\cdot)$ is continuously differentiable and $\{W_t, 0 \leq t \leq T\}$ is a standard Wiener process. The model defined by (1) and (5) is similar to (11). We assume the T is fixed and consider the case $\varepsilon \to 0$ (*small noise asymptotics*).

The unknown parameter $\vartheta \in \Theta = (\alpha, \beta)$, where $0 < \alpha < \beta < T$ and the likelihood ratio function is defined in (2), where the signal is of the form $S\,(\vartheta, t) = a\,\mathrm{sgn}\,(t-\vartheta)\,|t-\vartheta|^k + h\,(t-\vartheta)$. We have to verify the convergence

$$Z_\varepsilon\,(u) = \frac{V\left(\vartheta_0 + \varphi_\varepsilon u, X^T\right)}{V\left(\vartheta_0, X^T\right)} \Longrightarrow Z\,(u) = \exp\left\{\gamma W^H\,(u) - \frac{|u|^{2H}}{2}\gamma^2\right\}. \quad (12)$$

Here we set $\varphi_\varepsilon = \varepsilon^{\frac{1}{\kappa+\frac{1}{2}}}$ and $\gamma = a\Gamma_*$.

We shall demonstrate the validity of (12) for $u > 0$. Below we denoted $s_{t,u} = \mathrm{sgn}\,(t - \vartheta_0 - \varphi_\varepsilon u)$. We have

$$\begin{aligned}\ln Z_\varepsilon\,(u) = \frac{a}{\varepsilon}\int_0^T &\left[s_{t,u}\,|t-\vartheta_0-\varphi_\varepsilon u|^\kappa - s_{t,0}\,|t-\vartheta_0|^\kappa\right]\mathrm{d}W_t \\ &- \frac{a^2}{2\varepsilon^2}\int_0^T \left[s_{t,u}\,|t-\vartheta_0-\varphi_\varepsilon u|^\kappa - s_{t,0}\,|t-\vartheta_0|^\kappa\right]^2\mathrm{d}t + o\,(1)\,,\end{aligned}$$

because

$$\frac{1}{\varepsilon^2}\int_0^T \left[h\,(t-\vartheta_0-\varphi_\varepsilon u) - h\,(t-\vartheta_0)\right]^2\mathrm{d}t \leq C\frac{\varphi_\varepsilon^2}{\varepsilon^2} \leq C\,\varepsilon^{\frac{2}{\kappa+\frac{1}{2}}-2} \longrightarrow 0.$$

Further,

$$\begin{aligned}\mathbb{B}_\varepsilon\,(u) &= \frac{a^2}{2\varepsilon^2}\int_0^T \left[s_{t,u}\,|t-\vartheta_0-\varphi_\varepsilon u|^\kappa - s_{t,0}\,|t-\vartheta_0|^\kappa\right]^2\mathrm{d}t \\ &= \frac{a^2}{2\varepsilon^2}\int_{-\vartheta_0}^{T-\vartheta_0}\left[\mathrm{sgn}\,(y-\varphi_\varepsilon u)\,|y-\varphi_\varepsilon u|^\kappa - \mathrm{sgn}\,(y)\,|y|^\kappa\right]^2\mathrm{d}y \\ &= \frac{a^2\varphi_\varepsilon^{2\kappa+1}}{2\varepsilon^2}\int_{-\frac{\vartheta_0}{\varphi_\varepsilon}}^{\frac{T-\vartheta_0}{\varphi_\varepsilon}}\left[\mathrm{sgn}\,(v-u)\,|v-u|^\kappa - \mathrm{sgn}\,(v)\,|v|^\kappa\right]^2\mathrm{d}v \\ &= \frac{a^2\,|u|^{2\kappa+1}}{2}\int_{-\frac{\vartheta_0}{u\varphi_\varepsilon}}^{\frac{T-\vartheta_0}{u\varphi_\varepsilon}}\left[\mathrm{sgn}\,(s-1)\,|s-1|^\kappa - \mathrm{sgn}\,(s)\,|s|^\kappa\right]^2\mathrm{d}s \longrightarrow \frac{|u|^{2\kappa+1}}{2}\gamma^2,\end{aligned}$$

where we did the change of variables $t=y+\vartheta_0, y=\varphi_\varepsilon v$ and $v=su$. The similar change of variables in stochastic integral gives us the following limit

$$\begin{aligned}
\mathbb{A}_\varepsilon\left(u\right) &= \frac{a}{\varepsilon}\int_{0}^{T}\left[s_{t,u}\left|t-\vartheta_0-\varphi_\varepsilon u\right|^\kappa - s_{t,0}\left|t-\vartheta_0\right|^\kappa\right]{\rm d}W_t\\
&= \frac{a}{\varepsilon}\int_{-\vartheta_0}^{T-\vartheta_0}\left[{\rm sgn}\left(y-\varphi_\varepsilon u\right)\left|y-\varphi_\varepsilon u\right|^\kappa - {\rm sgn}\left(y\right)\left|y\right|^\kappa\right]{\rm d}\tilde W_y\\
&= \frac{a\varphi_\varepsilon^{\kappa+\frac{1}{2}}}{\varepsilon}\int_{-\frac{\vartheta_0}{\varphi_\varepsilon}}^{\frac{T-\vartheta_0}{\varphi_\varepsilon}}\left[{\rm sgn}\left(v-u\right)\left|v-u\right|^\kappa - {\rm sgn}\left(v\right)\left|v\right|^\kappa\right]{\rm d}\tilde{\tilde W}_v\\
&= a\int_{-\frac{\vartheta_0}{\varphi_\varepsilon}}^{\frac{T-\vartheta_0}{\varphi_\varepsilon}}\left[{\rm sgn}\left(v-u\right)\left|v-u\right|^\kappa - {\rm sgn}\left(v\right)\left|v\right|^\kappa\right]{\rm d}\tilde{\tilde W}_v \Longrightarrow a\Gamma_* W^H\left(u\right).
\end{aligned}$$

Therefore we verified the convergence (12) and the MLE $\hat\vartheta_\varepsilon$ and the BE $\tilde\vartheta_\varepsilon$ are consistent, have limit distributions The MLE $\hat\vartheta_n$ and the BE $\tilde\vartheta_n$ are consistent, have different limit distributions

$$\begin{aligned}
&\varepsilon^{-\frac{1}{\kappa+\frac{1}{2}}}\left(\hat\vartheta_\varepsilon-\vartheta_0\right)\Longrightarrow\hat\xi, \qquad Z(\hat\xi)=\sup_u Z\left(u\right),\\
&\varepsilon^{-\frac{1}{\kappa+\frac{1}{2}}}\left(\tilde\vartheta_\varepsilon-\vartheta_0\right)\Longrightarrow\tilde\xi, \qquad \tilde\xi=\frac{\int uZ\left(u\right){\rm d}u}{\int Z\left(u\right){\rm d}u},
\end{aligned}$$

their moments converge and the BE are asymptotically efficient. For the proofs see [1].

1.3 Inhomogeneous Poisson processes

Suppose that we observe a trajectory of an inhomogeneous Poisson process $X^T=\left(X_t, 0\leq t\leq T\right)$ with τ-periodic intensity function $\lambda\left(t-\vartheta\right)$ admitting the representation

$$\lambda\left(t-\vartheta\right)=a\,{\rm sgn}\left(t-\vartheta\right)\left|t-\vartheta\right|^\kappa+h\left(t-\vartheta\right), \qquad 0\leq t\leq\tau,$$

on the first period and periodically continued on the whole real line. Here $\vartheta\in\Theta=\left(\alpha,\beta\right)$, $0<\alpha<\beta<\tau$. The function $\lambda\left(t\right)>0$, $t\in\left[0,\tau\right]$ and the parameter, $\kappa\in\left(0,\frac{1}{2}\right)$ are known. For simplicity we assume that $T=n\tau$ and study asymptotics $n\rightarrow\infty$.

The likelihood function is (see [16])

$$V\left(\vartheta,X^T\right)=\exp\left\{\int_{0}^{T}\ln\lambda\left(t-\vartheta\right){\rm d}X_t-n\int_{0}^{\tau}\left[\lambda\left(t-\vartheta\right)-1\right]{\rm d}t\right\}, \quad \vartheta\in\Theta.$$

We have to show that the normalized ($\varphi_n = n^{-\frac{1}{2\kappa+1}}$) likelihood ratio process converges

$$Z_n\left(u\right)=\frac{V\left(\vartheta_0+\varphi_n u,X^T\right)}{V\left(\vartheta_0,X^T\right)}\Longrightarrow Z\left(u\right)=\exp\left\{\gamma W^H\left(u\right)-\frac{\left|u\right|^{2H}}{2}\gamma^2\right\}. \quad (13)$$

Here $\gamma = ah\left(0\right)^{-\frac{1}{2}}\Gamma_*$ with the same Γ_* as before.

Let us introduce the random processes

$$X_j\left(t\right)=X_{\{\tau(j-1)+t\}}-X_{\{\tau(j-1)\}},\qquad 0\leq t\leq\tau,\quad j=1,\ldots,n,$$
$$W_n\left(t\right)=\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\left[X_j\left(t\right)-\Lambda\left(\vartheta_0,t\right)\right],\quad \Lambda\left(\vartheta_0,t\right)=\int_0^t\lambda\left(s-\vartheta_0\right)\mathrm{d}s,$$

and denote $\vartheta_u = \vartheta_0 + \varphi_n u$, $s_{t,u} = \mathrm{sgn}\left(t-\vartheta_u\right)$. Then using the relations $\ln\left(1+x\right) = x + o\left(x\right), x - \ln\left(1+x\right) = \frac{x^2}{2} + o\left(x^2\right)$ and

$$\frac{\lambda\left(t-\vartheta_u\right)}{\lambda\left(t-\vartheta_0\right)}=1+\frac{\lambda\left(t-\vartheta_u\right)-\lambda\left(t-\vartheta_0\right)}{\lambda\left(t-\vartheta_0\right)}$$
$$=1+\frac{as_{t,u}\left|t-\vartheta_u\right|^\kappa-as_{t,0}\left|t-\vartheta_0\right|^\kappa}{\lambda\left(t-\vartheta_0\right)}+\frac{h\left(t-\vartheta_u\right)-h\left(t-\vartheta_0\right)}{\lambda\left(t-\vartheta_0\right)}$$

we can write

$$\ln Z_n\left(u\right)=\sum_{j=1}^{n}\int_0^\tau\ln\left(\frac{\lambda\left(t-\vartheta_u\right)}{\lambda\left(t-\vartheta_0\right)}\right)\left[\mathrm{d}X_j\left(t\right)-\lambda\left(t-\vartheta_0\right)\mathrm{d}t\right]$$
$$-n\int_0^\tau\left[\frac{\lambda\left(t-\vartheta_u\right)}{\lambda\left(t-\vartheta_0\right)}-1-\ln\left(\frac{\lambda\left(t-\vartheta_u\right)}{\lambda\left(t-\vartheta_0\right)}\right)\right]\lambda\left(t-\vartheta_0\right)\mathrm{d}t$$
$$=\sum_{j=1}^{n}\int_0^\tau\left(\frac{\lambda\left(t-\vartheta_u\right)-\lambda\left(t-\vartheta_0\right)}{\lambda\left(t-\vartheta_0\right)}\right)\left[\mathrm{d}X_j\left(t\right)-\lambda\left(t-\vartheta_0\right)\mathrm{d}t\right]$$
$$-\frac{n}{2}\int_0^\tau\frac{\left(\lambda\left(t-\vartheta_u\right)-\lambda\left(t-\vartheta_0\right)\right)^2}{\lambda\left(t-\vartheta_0\right)}\mathrm{d}t+o\left(1\right)$$
$$=a\sum_{j=1}^{n}\int_0^\tau\left(\frac{s_{t,u}\left|t-\vartheta_u\right|^\kappa-s_{t,0}\left|t-\vartheta_0\right|^\kappa}{a\,s_{t,0}\left|t-\vartheta_0\right|^\kappa+h\left(t-\vartheta_0\right)}\right)\left[\mathrm{d}X_j\left(t\right)-\lambda\left(t-\vartheta_0\right)\mathrm{d}t\right]$$
$$-\frac{na^2}{2}\int_0^\tau\frac{\left(s_{t,u}\left|t-\vartheta_u\right|^\kappa-s_{t,0}\left|t-\vartheta_0\right|^\kappa\right)^2}{a\,s_{t,0}\left|t-\vartheta_0\right|^\kappa+h\left(t-\vartheta_0\right)}\;\mathrm{d}t+o\left(1\right),$$

because

$$n\int_0^\tau\frac{\left(h\left(t-\vartheta_u\right)-h\left(t-\vartheta_0\right)\right)^2}{\lambda\left(t-\vartheta_0\right)}\mathrm{d}t\leq Cn\varphi_n^2=C\,n^{1-\frac{2}{2\kappa+1}}\longrightarrow 0.$$

Below we change the variables $t=y+\vartheta_0, y=\varphi_n v, v=su$

$$\begin{aligned}\mathbb{B}_n\left(u\right)&=\frac{na^2}{2}\int_0^\tau\frac{\left(s_{t,u}\left|t-\vartheta_0-\varphi_n u\right|^\kappa-s_{t,0}\left|t-\vartheta_0\right|^\kappa\right)^2}{a\,s_{t,0}\left|t-\vartheta_0\right|^\kappa+h\left(t-\vartheta_0\right)}\,{\rm d}t\\&=\frac{na^2}{2}\int_{-\vartheta_0}^{\tau-\vartheta_0}\frac{\left({\rm sgn}\left(y-\varphi_n u\right)\left|y-\varphi_n u\right|^\kappa-{\rm sgn}\left(y\right)\left|y\right|^\kappa\right)^2}{a\,{\rm sgn}\left(y\right)\left|y\right|^\kappa+h\left(y\right)}\,{\rm d}y\\&=\frac{na^2\varphi_n^{2\kappa+1}}{2}\int_{-\frac{\vartheta_0}{\varphi_n}}^{\frac{\tau-\vartheta_0}{\varphi_n}}\frac{\left({\rm sgn}\left(v-u\right)\left|v-u\right|^\kappa-{\rm sgn}\left(v\right)\left|v\right|^\kappa\right)^2}{a\varphi_n^\kappa{\rm sgn}\left(v\right)\left|v\right|^\kappa+h\left(v\varphi_n^\kappa\right)}\,{\rm d}v\\&=\frac{a^2\left|u\right|^{2\kappa+1}}{2h\left(0\right)}\int_{-\frac{\vartheta_0}{u\varphi_n}}^{\frac{\tau-\vartheta_0}{u\varphi_n}}\left({\rm sgn}\left(s-1\right)\left|s-1\right|^\kappa-{\rm sgn}\left(s\right)\left|s\right|^\kappa\right)^2{\rm d}s+o\left(1\right)\\&\longrightarrow\frac{a^2\left|u\right|^{2\kappa+1}}{2h\left(0\right)}\Gamma_*^2=\frac{\left|u\right|^{2\kappa+1}}{2}\,\gamma^2.\end{aligned}$$

The same change of variables ($t=\vartheta_0+\varphi_n v$) in the stochastic integral provide us the relations

$$\begin{aligned}\mathbb{A}_n\left(u\right)&=a\sqrt{n}\int_0^\tau\left(\frac{s_{t,u}\left|t-\vartheta_0-\varphi_n u\right|^\kappa-s_{t,0}\left|t-\vartheta_0\right|^\kappa}{a\,s_{t,0}\left|t-\vartheta_0\right|^\kappa+h\left(t-\vartheta_0\right)}\right){\rm d}W_n\left(t\right)\\&=\frac{a\sqrt{n}\varphi_n^{\kappa+\frac12}}{\sqrt{h\left(0\right)}}\int_{-\frac{\vartheta_0}{\varphi_n}}^{\frac{\tau-\vartheta_0}{\varphi_n}}\left({\rm sgn}\left(v-u\right)\left|v-u\right|^\kappa-{\rm sgn}\left(v\right)\left|v\right|^\kappa\right){\rm d}w_n\left(v\right)\\&\Longrightarrow\frac{a}{\sqrt{h\left(0\right)}}\int_{-\infty}^{\infty}\left({\rm sgn}\left(v-u\right)\left|v-u\right|^\kappa-{\rm sgn}\left(v\right)\left|v\right|^\kappa\right){\rm d}W\left(v\right)=\gamma\,W^H\left(u\right),\end{aligned}$$

where

$$w_n\left(v\right)=\frac{W_n\left(\vartheta_0+\varphi_n v\right)-W_n\left(\vartheta_0\right)}{\sqrt{\varphi_n}}\Longrightarrow W\left(v\right).$$

Therefore we have the convergence (13). The MLE $\hat\vartheta_n$ and the BE $\tilde\vartheta_n$ are consistent, have limit distributions

$$n^{\frac{1}{2\kappa+1}}\left(\hat\vartheta_n-\vartheta_0\right)\Longrightarrow\hat\xi,\qquad\qquad n^{\frac{1}{2\kappa+1}}\left(\tilde\vartheta_n-\vartheta_0\right)\Longrightarrow\tilde\xi,$$

the moments of these estimators converge and the BE are asymptotically efficient. For the proofs see [2].

1.4 Ergodic diffusion process

Consider the observations $X^T=\left(X_t,0\leq t\leq T\right)$ of the ergodic diffusion process

$${\rm d}X_t=\left[a\,{\rm sgn}\left(X_t-\vartheta\right)\left|X_t-\vartheta\right|^\kappa+h\left(X_t-\vartheta\right)\right]{\rm d}t+{\rm d}W_t,\quad X_0.\qquad(14)$$

Here $a \neq 0$, $\kappa \in \left(0, \frac{1}{2}\right)$, the function $h(\cdot)$ is known and has bounded derivative. Moreover we suppose that the function $S(x) = a \operatorname{sgn}(x)|x|^\kappa + h(x)$ is such that the conditions $\mathcal{ES}$, $\mathcal{EM}$ and $\mathcal{A}_0(\Theta)$ in [13] are fulfilled. For example, these conditions are fulfilled if $h(x) = -bx$ with $b > 0$. These conditions provide the existence and uniqueness of the solution of this equation, the existence of finite invariant measure with the density

$$f(\vartheta, x) = f(x - \vartheta) = G \exp\left\{2 \int_0^{x-\vartheta} [a \operatorname{sgn}(z)|z|^\kappa + h(z)] \,\mathrm{d}z\right\}$$

and finiteness of all polynomial moments. Here $G > 0$ is the normalizing constant. The likelihood function is (see [16])

$$V\left(\vartheta, X^T\right) = \exp\left\{\int_0^T S(X_t - \vartheta)\mathrm{d}X_t - \frac{1}{2}\int_0^T S(X_t - \vartheta)^2 \mathrm{d}t\right\}, \qquad \vartheta \in \Theta.$$

For the normalized likelihood ratio process we have to show the convergence

$$Z_T(u) = \frac{V\left(\vartheta_0 + \varphi_T u, X^T\right)}{V\left(\vartheta_0, X^T\right)} \Longrightarrow Z(u) = \exp\left\{\gamma W^H(u) - \frac{|u|^{2H}}{2}\gamma^2\right\}. \quad (15)$$

Here $\varphi_T = T^{-\frac{1}{2\kappa+1}}$ and $\gamma = a\Gamma_* G^{1/2}$.

Let us see once more how the fBm $W^H(u)$ appears in this limit likelihood ratio. Denote $\vartheta_u = \vartheta_0 + \varphi_T u$, $g(x) = a \operatorname{sgn}(x)|x|^\kappa$ and write

$$\begin{aligned}
\ln Z_T(u) &= \int_0^T \left(S(X_t - \vartheta_u) - S(X_t - \vartheta_0)\right) \mathrm{d}W_t \\
&\qquad - \frac{1}{2}\int_0^T \left(S(X_t - \vartheta_u) - S(X_t - \vartheta_0)\right)^2 \mathrm{d}t \\
&= \int_0^T \left(g(X_t - \vartheta_u) - g(X_t - \vartheta_0)\right) \mathrm{d}W_t \\
&\qquad - \frac{1}{2}\int_0^T \left(g(X_t - \vartheta_u) - g(X_t - \vartheta_0)\right)^2 \mathrm{d}t + o(1)
\end{aligned}$$

because

$$\int_0^T \left(h(X_t - \vartheta_u) - h(X_t - \vartheta_0)\right)^2 \mathrm{d}t \leq C\, u^2 \varphi_T^2\, T = C\, u^2 T^{-\frac{2}{2\kappa+1}+1} \longrightarrow 0.$$

Let us denote $\Lambda_T(x)$ the *local time* of the diffusion process (14) and put $f_T^\circ(x) = 2T^{-1}\Lambda_T(x)$. Recall that $f_T^\circ(x)$ is the *local time estimator* of the invariant density. This estimator is consistent $\left(\text{i.e. } f_T^\circ(x) \to f(\vartheta_0, x)\right)$ and

$\sqrt{T}$-asymptotically normal. Recall that for any continuous function $H(x)$ we have

$$\frac{1}{T}\int_0^T H(X_t)\,\mathrm{d}t = \int_{-\infty}^{\infty} H(x)\, f_T^\circ(x)\,\mathrm{d}x \longrightarrow \int_{-\infty}^{\infty} H(x)\, f(\vartheta_0, x)\,\mathrm{d}x.$$

We have

$$\begin{aligned}
\mathbb{B}_T(u) &= \frac{a^2}{2}\int_0^T \left[s_{X_t,u}\left|X_t-\vartheta_u\right|^\kappa - s_{X_t,0}\left|X_t-\vartheta_0\right|^\kappa\right]^2 \mathrm{d}t\\
&= \frac{a^2 T}{2}\int_{-\infty}^{\infty}\left[s_{x,u}\left|x-\vartheta_u\right|^\kappa - s_{x,0}\left|x-\vartheta_0\right|^\kappa\right]^2 f_T^\circ(x)\,\mathrm{d}x\\
&= \frac{a^2 T}{2}\int_{-\infty}^{\infty}\left[\mathrm{sgn}\left(y-\varphi_T u\right)\left|y-\varphi_T u\right|^\kappa - \mathrm{sgn}\left(y\right)\left|y\right|^\kappa\right]^2 f_T^\circ(\vartheta_0+y)\,\mathrm{d}y\\
&= \frac{a^2 T\varphi_T^{2\kappa+1}}{2}\int_{-\infty}^{\infty}\left[\mathrm{sgn}\left(v-u\right)\left|v-u\right|^\kappa - \mathrm{sgn}\left(v\right)\left|v\right|^\kappa\right]^2 f_T^\circ(\vartheta_0+\varphi_T v)\,\mathrm{d}v\\
&= \frac{a^2 \left|u\right|^{2\kappa+1}}{2}\int_{-\infty}^{\infty}\left[\mathrm{sgn}\left(s-1\right)\left|s-1\right|^\kappa - \mathrm{sgn}\left(s\right)\left|s\right|^\kappa\right]^2 f_T^\circ(\vartheta_0+\varphi_T us)\,\mathrm{d}s\\
&\longrightarrow \frac{\left|u\right|^{2\kappa+1}}{2} a^2\Gamma_*^2 f\left(\vartheta_0,\vartheta_0\right) = \frac{\left|u\right|^{2\kappa+1}}{2} a^2\Gamma_*^2 G = \frac{\left|u\right|^{2\kappa+1}}{2}\,\gamma^2. \qquad (16)
\end{aligned}$$

Let us denote

$$\mathbb{A}_T(u) = a\int_0^T\left[\mathrm{sgn}\left(X_t-\vartheta_u\right)\left|X_t-\vartheta_u\right|^\kappa - \mathrm{sgn}\left(X_t-\vartheta_0\right)\left|X_t-\vartheta_0\right|^\kappa\right]\,\mathrm{d}W_t.$$

From the convergence (16) and the central limit theorem for stochastic integrals we obtain

$$\mathbb{A}_T(u) \Longrightarrow \gamma W^H(u).$$

More detailed analysis allows verify the convergence of the finite-dimensional distributions

$$\Big(\mathbb{A}_T(u_1),\ldots,\mathbb{A}_T(u_k)\Big) \Longrightarrow \Big(\gamma W^H(u_1),\ldots,\gamma W^H(u_k)\Big).$$

Therefore we have (15).

For the MLE $\hat\vartheta_T$ and the BE $\tilde\vartheta_T$ we have the convergences

$$T^{\frac{1}{2\kappa+1}}\left(\hat\vartheta_T-\vartheta_0\right)\Longrightarrow \hat\xi, \qquad\qquad T^{\frac{1}{2\kappa+1}}\left(\tilde\vartheta_T-\vartheta_0\right)\Longrightarrow \tilde\xi.$$

Once more we have the convergence of all polynomial moments and the BE are asymptotically efficient. For the detailed proof see [3], [13]. Note that the case $\kappa\in\left(-\frac{1}{2},0\right)$ was discussed in [7].

1.5 Dynamical system with small noise

Suppose that the observed process $X^T = (X_t, 0 \leq t \leq T)$ is a solution of the stochastic differential equation

$$\mathrm{d}X_t = \left[a \,\mathrm{sgn}\left(X_t - \vartheta\right)\left|X_t - \vartheta\right|^{\kappa} + h\left(X_t - \vartheta\right)\right]\mathrm{d}t + \varepsilon \mathrm{d}W_t, \quad 0 \leq t \leq T,$$

where the initial value $X_0 = x_0$ is deterministic, $a > 0$, $\kappa \in \left(0, \frac{1}{2}\right)$ and the function $h\left(\cdot\right)$ is known and has bounded derivative. Moreover we suppose that the function $S\left(x\right) = a \,\mathrm{sgn}\left(x\right)\left|x\right|^{\kappa} + h\left(x\right) > 0$ for all x. We have to estimate ϑ and describe the asymptotic ($\varepsilon \to 0$) properties of the MLE $\hat{\vartheta}_\varepsilon$ and BE $\tilde{\vartheta}_\varepsilon$. The likelihood ratio function is (see [16])

$$V\left(\vartheta, X^T\right) = \exp\left\{\int_0^T \frac{S\left(X_t - \vartheta\right)}{\varepsilon^2}\mathrm{d}X_t - \int_0^T \frac{S\left(X_t - \vartheta\right)^2}{2\varepsilon^2}\mathrm{d}t\right\}, \quad \vartheta \in \Theta.$$

The set Θ will be defined below.

We have to verify the convergence of the normalized likelihood ratio

$$Z_\varepsilon\left(u\right) = \frac{V\left(\vartheta_0 + \varphi_\varepsilon u, X^T\right)}{V\left(\vartheta_0, X^T\right)} \Longrightarrow Z\left(u\right) = \exp\left\{\gamma W^H\left(u\right) - \frac{\left|u\right|^{2H}}{2}\gamma^2\right\}.$$

Here $\varphi_\varepsilon = \varepsilon^{\frac{1}{\kappa + \frac{1}{2}}}$ and $\gamma = a\Gamma_* h\left(0\right)^{-1/2}$.

The stochastic process X_t converges uniformly on $t \in [0, T]$ to $x_t = x_t\left(\vartheta\right)$ — solution of the ordinary differential equation

$$\frac{\mathrm{d}x_t}{\mathrm{d}t} = a \,\mathrm{sgn}\left(x_t - \vartheta\right)\left|x_t - \vartheta_0\right|^{\kappa} + h\left(x_t\right), \quad x_0, \quad 0 \leq t \leq T.$$

Suppose that $\vartheta \in (\alpha, \beta)$, where $\alpha > x_0$ and $\beta < \inf_{\{\alpha < \theta\}} x_T\left(\vartheta\right)$.

We can write

$$\begin{aligned}
\mathbb{B}_\varepsilon\left(u\right) &= \frac{1}{2\varepsilon^2}\int_0^T \left(S\left(X_t - \vartheta_u\right) - S\left(X_t - \vartheta_0\right)\right)^2\mathrm{d}t \\
&= \frac{a^2}{2\varepsilon^2}\int_0^T \left(s_{X_t,u}\left|X_t - \vartheta_u\right|^{\kappa} - s_{X_t,0}\left|X_t - \vartheta_0\right|^{\kappa}\right)^2\mathrm{d}t + o\left(1\right) \\
&= \frac{a^2}{2\varepsilon^2}\int_0^T \left(s_{x_t,u}\left|x_t - \vartheta_0 - \varphi_\varepsilon u\right|^{\kappa} - s_{x_t,0}\left|x_t - \vartheta_0\right|^{\kappa}\right)^2\mathrm{d}t + o\left(1\right) \\
&= \frac{a^2}{2\varepsilon^2}\int_0^T \frac{\left(s_{x_t,u}\left|x_t - \vartheta_0 - \varphi_\varepsilon u\right|^{\kappa} - s_{x_t,0}\left|x_t - \vartheta_0\right|^{\kappa}\right)^2}{S\left(x_t - \vartheta_0\right)}\mathrm{d}\left(x_t - \vartheta_0\right) + o\left(1\right) \\
&= \frac{a^2}{2\varepsilon^2}\int_{x_0 - \vartheta_0}^{x_T - \vartheta_0} \frac{\left(\mathrm{sgn}\left(y - \varphi_\varepsilon u\right)\left|y - \varphi_\varepsilon u\right|^{\kappa} - \mathrm{sgn}\left(y\right)\left|y\right|^{\kappa}\right)^2}{a \,\mathrm{sgn}\left(y\right)\left|y\right|^{\kappa} + h\left(y\right)}\mathrm{d}y + o\left(1\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{a^2\varphi_\varepsilon^{2\kappa+1}}{2\varepsilon^2}\int_{-\frac{\vartheta_0-x_0}{\varphi_\varepsilon}}^{\frac{x_T-\vartheta_0}{\varphi_\varepsilon}} \frac{\left(\operatorname{sgn}(v-u)\left|v-u\right|^\kappa-\operatorname{sgn}(v)\left|v\right|^\kappa\right)^2}{a\operatorname{sgn}(v)\left|v\right|^\kappa\varphi_\varepsilon^\kappa+h\left(v\varphi_\varepsilon\right)}{\rm d}v+o\left(1\right)\\
&\longrightarrow \frac{a^2}{2h\left(0\right)}\int_{-\infty}^{\infty}\left(\operatorname{sgn}(v-u)\left|v-u\right|^\kappa-\operatorname{sgn}(v)\left|v\right|^\kappa\right)^2{\rm d}v=\frac{\left|u\right|^{2\kappa+1}}{2}\gamma^2.
\end{aligned}
$$

Using the same change of variables as above we have

$$
\begin{aligned}
\mathbb{A}_\varepsilon\left(u\right) &= \frac{1}{\varepsilon}\int_0^T\left(S\left(X_t-\vartheta_u\right)-S\left(X_t-\vartheta_0\right)\right){\rm d}W_t\\
&= \frac{a}{\varepsilon}\int_0^T\left[s_{X_t,u}\left|X_t-\vartheta_0-\varphi_\varepsilon u\right|^\kappa-s_{X_t,0}\left|X_t-\vartheta_0\right|\right]{\rm d}W_t+o\left(1\right)\\
&= \frac{a}{\varepsilon}\int_0^T\left[s_{x_t,u}\left|x_t-\vartheta_0-\varphi_\varepsilon u\right|^\kappa-s_{x_t,0}\left|x_t-\vartheta_0\right|\right]{\rm d}W_t+o\left(1\right)\\
&\Longrightarrow \frac{a}{\sqrt{h\left(0\right)}}\int_{-\infty}^{\infty}\left[\operatorname{sgn}(v-u)\left|v-u\right|^\kappa-\operatorname{sgn}(v)\left|v\right|^\kappa\right]{\rm d}W\left(v\right).
\end{aligned}
$$

As above this leads to the following limit distributions for the MLE $\hat\vartheta_\varepsilon$ and BE $\tilde\vartheta_\varepsilon$

$$
\varepsilon^{-\frac{1}{\kappa+\frac12}}\left(\hat\vartheta_\varepsilon-\vartheta_0\right)\Longrightarrow\hat\xi,\qquad\qquad \varepsilon^{-\frac{1}{\kappa+\frac12}}\left(\tilde\vartheta_\varepsilon-\vartheta_0\right)\Longrightarrow\tilde\xi,
$$

convergence of moments of these estimators and the asymptotic efficiency of the BE. For the full proofs see [15].

2 Discussion

The proofs presented above can be applied to the other models of observations. For example, suppose that we have a nonlinear stationary time series

$$
X_{j+1}=a\left|X_j-\vartheta\right|^\kappa+h\left(X_j-\vartheta\right)+\varepsilon_j,\quad j=1,\ldots,n
$$

with i.i.d. noise $\left(\varepsilon_j\right)_{j\geq1}$. Assume the density $q\left(x\right)$ of the random variable ε_j and the function $h\left(x\right)>0$ are sufficiently smooth functions. The likelihood function is

$$
V\left(\vartheta,X^n\right)=\prod_{j=1}^{n-1}q\left(X_{j+1}-a\left|X_j-\vartheta\right|^\kappa-h\left(X_j-\vartheta\right)\right),\qquad \vartheta\in\Theta.
$$

Introduce the notation: $q_{j,u}=q\left(X_{j+1}-a\left|X_j-\vartheta_u\right|^\kappa-h\left(X_j-\vartheta_u\right)\right)$, ϑ_0 is the true value, $\vartheta_u=\vartheta_0+\varphi_n u$ and $g_{j,u}=a\left|X_j-\vartheta_u\right|^\kappa$. Then following the same steps as in the i.i.d. case above the normalized log-likelihood can be written as follows

$$\begin{aligned}
\ln Z_n\left(u\right) &= \sum_{j=1}^{n-1} \ln \frac{q_{j,u}}{q_{j,0}} = \sum_{j=1}^{n-1} \ln \left[1+\frac{q_{j,u}-q_{j,0}}{q_{j,0}}\right] \\
&= \sum_{j=1}^{n-1} \frac{q_{j,u}-q_{j,0}}{q_{j,0}} - \frac{1}{2}\sum_{j=1}^{n-1}\left(\frac{q_{j,u}-q_{j,0}}{q_{j,0}}\right)^2 + o\left(1\right) \\
&= \sum_{j=1}^{n-1} \frac{g_{j,u}-g_{j,0}}{q_{j,0}} q'\left(\varepsilon_j\right) - \frac{1}{2}\sum_{j=1}^{n-1}\left(\frac{g_{j,u}-g_{j,0}}{q_{j,0}}\right)^2 q'\left(\varepsilon_j\right)^2 + o\left(1\right) \\
&= a\sum_{j=1}^{n-1} \frac{\left|X_j-\vartheta_0-\varphi_n u\right|^\kappa - \left|X_j-\vartheta_0\right|^\kappa}{q\left(\varepsilon_j\right)} q'\left(\varepsilon_j\right) \\
&\qquad - \frac{a^2}{2}\sum_{j=1}^{n-1}\left(\frac{\left|X_j-\vartheta_0-\varphi_n u\right|^\kappa - \left|X_j-\vartheta_0\right|^\kappa}{q\left(\varepsilon_j\right)}\right)^2 q'\left(\varepsilon_j\right)^2 + o\left(1\right) \\
&= \mathbb{A}_n\left(u\right) - \mathbb{B}_n\left(u\right) + o\left(1\right).
\end{aligned}$$

For nonlinear regression models with cusp-type singularity the properties of estimators were studied in [20], [5] and [6].

Another interesting problem to discuss is the estimation of the other parameters of the model. For example, consider the simplest model

$$\mathrm{d}X_t = a\left|t-b\right|^\kappa \mathrm{d}t + \varepsilon \mathrm{d}W_t, \quad X_0 = 0, \quad 0 \leq t \leq T.$$

Remind that the parameter $\varepsilon \in (0,1)$ can be estimated without error as follows. By Itô formula for X_t^2 we have for any $t \in (0,T]$

$$X_t^2 = 2\int_0^t X_s \mathrm{d}X_s + \varepsilon^2 t, \quad \text{and} \quad \varepsilon^2 = t^{-1}X_t^2 - 2t^{-1}\int_0^t X_s \mathrm{d}X_s.$$

The problem of estimation $\vartheta = (a,\kappa)$ is regular and the MLE and BE of this parameter are consistent and asymptotically normal with the regular rate ε (see, e.g. [8]). There is no difficulty to describe the behavior of the MLE and BE in the case $\vartheta = (a,b)$, where $\kappa \in \left(0,\frac{1}{2}\right)$. It can be shown that, say, the MLE $\hat\vartheta_\varepsilon = (\hat a_\varepsilon, \hat b_\varepsilon)$ has the following limit distribution

$$\varepsilon^{-1}\left(\hat a_\varepsilon - a_0\right) \Longrightarrow \zeta, \qquad \varepsilon^{-\frac{1}{\kappa+\frac{1}{2}}}\left(\hat b_\varepsilon - b_0\right) \Longrightarrow \hat\xi,$$

where ζ (Gaussian) and $\hat\xi$ are independent random variables.

The problem of estimation $\vartheta = (b, \kappa)$ is technically more complicate because the rate of convergence of the estimator $\hat{b}_\varepsilon$ depends on the unknown parameter κ. It seems that the general results from the monograph [10] can not be applied directly here and this problem requires a special study.

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