

Irreversibility of Asymptotic Entanglement Manipulation Under Quantum Operations Completely Preserving Positivity of Partial Transpose

Xin Wang^{1*} and Runyao Duan^{1,2†}

¹*Centre for Quantum Software and Information,
Faculty of Engineering and Information Technology,*

University of Technology Sydney, NSW 2007, Australia and

²*UTS-AMSS Joint Research Laboratory for Quantum Computation and Quantum Information Processing,
Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China*

We demonstrate the irreversibility of asymptotic entanglement manipulation under quantum operations that completely preserve the positivity of partial transpose (PPT), resolving a major open problem in quantum information theory. Our key tool is a new efficiently computable additive lower bound for the asymptotic relative entropy of entanglement with respect to PPT states, which can be used to evaluate the entanglement cost under local operations and classical communication (LOCC). We find that for any rank-two mixed state supporting on the $3 \otimes 3$ antisymmetric subspace, the amount of distillable entanglement by PPT operations is strictly smaller than one entanglement bit (ebit) while its entanglement cost under PPT operations is exactly one ebit. As byproduct, we find that for this class of states, both the Rains' bound and its regularization, are strictly less than the asymptotic relative entropy of entanglement. So, in general, there is no unique entanglement measure for the manipulation of entanglement by PPT operations. We further show a computable sufficient condition for the irreversibility of entanglement distillation by LOCC (or PPT) operations.

Introduction: In quantum information science, the resource theory of entanglement studies the interconvertibilities of entanglement under restricted classes of allowed operations. The irreversibility is crucial to this entanglement resource theory and it was sometimes argued to be the difference between entanglement and thermodynamics, as the Carnot cycle is reversible. When local operations and classical communication (LOCC) is available, the manipulation of entanglement is irreversible in the finite-copy regime. More precisely, the amount of pure entanglement that can be distilled from a finite number of copies of a state ρ is usually strictly smaller than the amount of pure entanglement needed to prepare the same number of copies of ρ [1]. Surprisingly, in the asymptotic limit of an arbitrarily large number of copies of the bipartite pure states, this process is shown to be reversible [2]. But for mixed states, this asymptotic reversibility does not hold anymore [3–7]. In particular, one requires a positive rate of pure states to generate the bound entanglement by LOCC [3, 9], while it is well known that no pure state can be distilled from it [8].

Various approaches have been considered to enlarge the class of operations to ensure reversible interconversion of entanglement in the asymptotic regime. A natural candidate is the class of quantum operations that completely preserve positivity of partial transpose (PPT) [10], which include all quantum operations that can be implemented by LOCC. A remarkable result is that any state with a nonpositive partial transpose (NPT) is distillable under this class of operations [11]. This suggests the possibility of reversibility under PPT operations, and there are examples of mixed states which can be reversibly converted into pure states in the asymptotic setting, e.g. the class of antisymmetric states of ar-

bitrary dimension [12]. It is noteworthy that for tripartite states, Ishizaka and Plenio [13] showed that asymptotic entanglement manipulation is irreversible. However, for bipartite states, the reversibility under PPT operations remained unknown so far since there were no further examples. Recently, a reversible theory of entanglement considering all asymptotically non-entangling transformations was studied in Refs. [14, 15] and the unique entanglement measure is identified as the asymptotic relative entropy of entanglement. A more general reversible framework for quantum resource theories was recently introduced in Ref. [16].

When the unit of pure entanglement is set to be the standard $2 \otimes 2$ Bell pair $1/\sqrt{2}(|00\rangle + |11\rangle)$, or entanglement bit (ebit), two fundamental ways of entanglement manipulation are well known, namely, entanglement distillation and entanglement dilution [1, 2]. These two tasks also naturally raise two fundamental entanglement measures: distillable entanglement and entanglement cost [1]. To be specific, distillable entanglement is the highest rate at which one can obtain Bell pairs from the given state under allowed operations, while entanglement cost is the lowest rate for converting Bell pairs to the given state. It is worth noting that if one can show a gap between the distillable entanglement and entanglement cost under PPT operations, then it will lead to the irreversibility of asymptotic entanglement manipulation. However, this problem is still very hard since for general mixed states it is highly nontrivial to evaluate these two measures both of which are given by a limiting procedure.

In this Letter, we demonstrate that irreversibility still exists in the asymptotic entanglement manipulation under PPT operations, which resolves a long-standing open problem in quantum information theory [12, 17, 18]. Our

approach is to show a gap between the regularized Rains' bound and the asymptotic relative entropy of entanglement [19] with respect to PPT states, which also resolves another open problem in Ref. [20]. More precisely, we introduce an additive semidefinite programming (SDP) lower bound for the asymptotic relative entropy of entanglement with respect to PPT states. With this lower bound, we are able to show that the PPT-entanglement cost of any rank-two state supporting on the $3 \otimes 3$ antisymmetric subspace is exactly one ebit while its PPT-distillable entanglement is strictly smaller than one. As a corollary, we show that there is no unique entanglement measure under PPT operations. This means that entanglement theory under PPT operations differs from thermodynamics, since in the second law of thermodynamics, the entropy uniquely determines whether a state is adiabatically accessible from another. We also give a sufficient condition to efficiently verify the irreversibility of entanglement distillation by LOCC (or PPT) operations. A general class of states are constructed to illustrate this phenomenon, see FIG. 1.

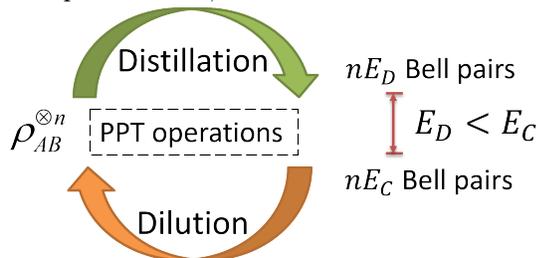


FIG. 1: The amount of Bell pairs distilled from the state is insufficient to reconstruct the given state under PPT operations in the asymptotic regime.

Before we present our main results, let us review some notations and preliminaries. We will use symbols such as A (or A') and B (or B') to denote (finite-dimensional) Hilbert spaces associated with Alice and Bob, respectively. The set of linear operators over A is denoted by $\mathcal{L}(A)$. For a linear operator R over a Hilbert space, we define $|R| = \sqrt{R^\dagger R}$ and the trace norm $\|R\|_1 = \text{Tr} |R|$, where R^\dagger is the Hermitian conjugate of R . The operator norm $\|R\|_\infty$ is defined as the maximum eigenvalue of $|R|$. A deterministic quantum operation \mathcal{N} from A' to B is simply a completely positive and trace-preserving (CPTP) linear map from $\mathcal{L}(A')$ to $\mathcal{L}(B)$. A positive semidefinite operator $E_{AB} \in \mathcal{L}(A \otimes B)$ is said to be PPT if $E_{AB}^{T_B} \geq 0$, where T_B means the partial transpose over the system B , i.e., $(|i_A j_B\rangle\langle k_A l_B|)^{T_B} = |i_A l_B\rangle\langle k_A j_B|$.

The task of entanglement distillation aims at obtaining maximally entangled states such as Bell pairs from less-entangled bipartite states. Imagine that Alice and Bob share a large supply of identically prepared state, and they want to convert these states to high fidelity Bell pairs using Ω operation. (We use Ω to represent one of LOCC or PPT through out the paper.) The *distillable entanglement* $E_{D,\Omega}$ of ρ quantifies the optimal rate r of

converting $\rho^{\otimes n}$ to rn Bell pairs with an arbitrarily high fidelity in the limit of large n . The reverse task is entanglement dilution. At this time, Alice and Bob share a large supply of Bell pairs and they are to convert rn Bell pairs to n high fidelity copies of the desired state $\rho^{\otimes n}$. The *entanglement cost* $E_{C,\Omega}$ quantifies the optimal rate r of converting rn Bell pairs to $\rho^{\otimes n}$ with an arbitrarily high fidelity in the limit of large n .

For simplicity, we denote $E_{D,PPT}$ and $E_{C,PPT}$ as E_D and E_C , respectively. For entanglement cost, Hayden, Horodecki and Terhal [22] proved that $E_{C,LOCC}$ equals to the regularized entanglement of formation [1] while the similar result is not true for PPT operations. For distillable entanglement, the best known bound is the Rains' bound [10] and it is reformulated in Ref. [23] as the following convex optimization problem:

$$R(\rho) = \min S(\rho|\sigma) \text{ s.t. } \sigma \geq 0, \|\sigma^{T_B}\|_1 \leq 1. \quad (1)$$

In this formula, $S(\rho|\sigma) = \text{Tr}(\rho \log \rho - \rho \log \sigma)$ denotes the quantum relative entropy, where we take $\log \equiv \log_2$ throughout the paper. The regularized Rains' bound, i.e., $R^\infty(\rho) = \inf_{n \geq 1} R(\rho^{\otimes n})/n$, was first introduced in Ref. [32]. Very recently we showed that the Rains' bound is not additive even for a class of two-qubit states [24]. The regularized Rains' bound is thus a better upper bound for the distillable entanglement.

The PPT-relative entropy of entanglement (REE) [25–27] is defined by

$$E_R(\rho) = \min S(\rho|\sigma) \text{ s.t. } \sigma, \sigma^{T_B} \geq 0, \text{Tr} \sigma = 1.$$

And the asymptotic PPT-relative entropy of entanglement is given by $E_R^\infty(\rho) = \inf_{n \geq 1} E_R(\rho^{\otimes n})/n$. It was shown in Ref. [32] that the asymptotic REE is indeed a lower bound to the PPT-entanglement cost. Then, for a general quantum state ρ , it always holds that

$$E_D(\rho) \leq R^\infty(\rho) \leq E_R^\infty(\rho) \leq E_C(\rho). \quad (2)$$

And it has been open for years whether any of these inequalities could be strict.

The main contribution of this Letter is to show that the second inequality is strict for a class of rank-two states supporting on the $3 \otimes 3$ antisymmetric subspace. As the first example, let us consider $\rho_v = \frac{1}{2}(|v_1\rangle\langle v_1| + |v_2\rangle\langle v_2|)$ with

$$|v_1\rangle = 1/\sqrt{2}(|01\rangle - |10\rangle), |v_2\rangle = 1/\sqrt{2}(|02\rangle - |20\rangle).$$

The projection onto $\text{supp}(\rho_v)$ is $P_v = |v_1\rangle\langle v_1| + |v_2\rangle\langle v_2|$. In Ref. [33], Chitambar and one of us showed that this state can be transformed into some $2 \otimes 2$ pure entangled state by a suitable separable operation while no finite-round LOCC protocol can do that. Here we show that

$$E_D(\rho_v) = R^\infty(\rho_v) < E_R^\infty(\rho_v) = E_C(\rho_v). \quad (3)$$

That means the asymptotic entanglement manipulation of ρ_v under PPT operations is irreversible, thus resolving a long-standing open problem in quantum information theory [12, 17, 18]. Furthermore, it also answers another open problem in Ref. [20] by showing a nonzero gap between the regularized Rains' bound and the asymptotic REE of ρ_v . The proofs are clear from Propositions 2 and 3 below.

An SDP lower bound for $E_R^\infty(\rho)$: Our key tool is an efficiently computable additive lower bound for the asymptotic REE. In the one-copy case, we need to do some relaxations of the minimization of $S(\rho||\sigma)$ with respect to PPT states. Note that the support of a state ρ , denoted by $\text{supp}(\rho)$, is a subspace spanned by the eigenvectors of ρ with positive eigenvalues. Let $D(\rho) = \{\rho' : \text{supp}(\rho') \subseteq \text{supp}(\rho)\}$ be the set of quantum states whose supports are contained in that of ρ , and let Γ be the set of PPT states. We can first relax the minimization of $S(\rho||\sigma)$ to the smallest relative entropy distance between $D(\rho)$ and the set Γ . See FIG. 2 below for an intuitive illustration of the ideas.

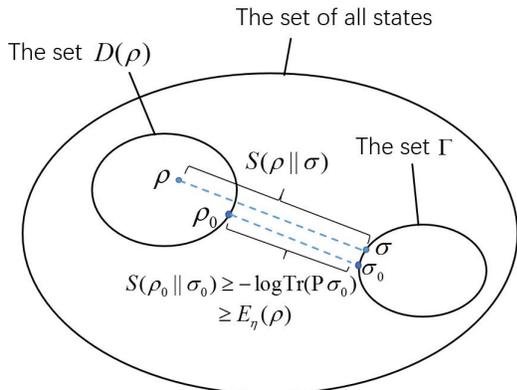


FIG. 2: The PPT-relative entropy of entanglement is defined as the smallest quantum relative entropy from ρ to the state σ taken from the set of PPT states Γ . Assume that ρ_0 and σ_0 give the smallest quantum relative entropy from $D(\rho)$ to Γ . It is clear that $E_R(\rho) = S(\rho||\sigma) \geq S(\rho_0||\sigma_0)$ and we show $S(\rho_0||\sigma_0) \geq -\log \text{Tr} P \sigma_0 \geq E_\eta(\rho)$ in Proposition 1, where P is the projection onto the support of ρ . This lower bound $E_\eta(\rho)$ is powerful since it still works in the asymptotic setting due to its additivity under tensor product.

Then applying properties of quantum relative entropy, we can further relax the problem to minimizing $-\log \text{Tr} P \sigma$ over all PPT states σ , where P is the projection onto $\text{supp}(\rho)$. Noting that this is SDP-computable, we can use SDP techniques to obtain the following bound

$$E_\eta(\rho) = \max -\log \|Y^{TB}\|_\infty, \text{ s.t. } -Y \leq P^{TB} \leq Y. \quad (4)$$

Interestingly, $E_\eta(\cdot)$ is additive under tensor product, i.e.,

$$E_\eta(\rho_1 \otimes \rho_2) = E_\eta(\rho_1) + E_\eta(\rho_2),$$

so we can overcome the difficulty of estimating the regularised relative entropy of entanglement. The additivity of $E_\eta(\cdot)$ can be proved by utilizing the duality theory of SDP [21, 34]. The detailed proof can be found in the supplementary material. This E_η can be efficiently computed since SDP can be solved by efficient algorithms [35] and it can also be implemented via CVX [36] and QET-LAB [37]. In particular, E_η becomes a computable lower bound for the entanglement cost under LOCC (or PPT) operations.

Proposition 1 For any state ρ ,

$$E_R^\infty(\rho) \geq E_\eta(\rho). \quad (5)$$

Consequently, $E_{C,LOCC}(\rho) \geq E_C(\rho) \geq E_\eta(\rho)$.

Proof Firstly, let us introduce a CPTP map by $\mathcal{N}(\tau) = P\tau P + (\mathbb{1} - P)\tau(\mathbb{1} - P)$. Then for $\rho_0 \in D(\rho)$ and $\sigma_0 \in \Gamma$, we have that

$$\begin{aligned} S(\rho_0||\sigma_0) &\geq S(\mathcal{N}(\rho_0)||\mathcal{N}(\sigma_0)) \\ &= S(\rho_0||P\sigma_0 P / \text{Tr} P\sigma_0 P) - \log \text{Tr} P\sigma_0 \\ &\geq -\log \text{Tr} P\sigma_0, \end{aligned} \quad (6)$$

where the first inequality is from the monotonicity of quantum relative entropy [38, 39] and the second inequality is due to the non-negativity of quantum relative entropy. Therefore,

$$\min_{\sigma \in \Gamma} S(\rho||\sigma) \geq \min_{\rho_0 \in D(\rho), \sigma_0 \in \Gamma} S(\rho_0||\sigma_0) \geq \min_{\sigma_0 \in \Gamma} -\log \text{Tr} P\sigma_0.$$

This step transforms the problem to SDP problems and it can also be proved via the min-relative entropy in [40].

Secondly, utilizing the weak duality of SDP [21] (see the supplementary material for details), we have that

$$\begin{aligned} \max_{\sigma_0 \in \Gamma} \text{Tr} P\sigma_0 &\leq \min t \text{ s.t. } Y^{TB} \leq t\mathbb{1}, P^{TB} \leq Y \\ &\leq \min t \text{ s.t. } -t\mathbb{1} \leq Y^{TB} \leq t\mathbb{1}, -Y \leq P^{TB} \leq Y \\ &= \min \|Y^{TB}\|_\infty, \text{ s.t. } -Y \leq P^{TB} \leq Y. \end{aligned}$$

Thus,

$$E_R(\rho) \geq -\log \max_{\sigma_0 \in \Gamma} \text{Tr} P\sigma_0 \geq E_\eta(\rho).$$

Finally, noting that $E_\eta(\rho)$ is additive, we have that

$$\begin{aligned} E_R^\infty(\rho) &= \inf_{n \geq 1} E_R(\rho^{\otimes n})/n \\ &\geq \inf_{n \geq 1} E_\eta(\rho^{\otimes n})/n = E_\eta(\rho). \end{aligned}$$

By Eq. (2), we have $E_C(\rho) \geq E_\eta(\rho)$. \square

PPT-entanglement cost of ρ_v : Applying the lower bound $E_\eta(\rho)$, we are now ready to show that the PPT-entanglement cost of ρ_v is still one ebit.

Proposition 2 For state ρ_v , $E_C(\rho_v) = E_R^\infty(\rho_v) = 1$.

Proof Firstly, suppose that $Q = |01\rangle\langle 01| + |10\rangle\langle 10| + |02\rangle\langle 02| + |20\rangle\langle 20|$ and we can prove that $E_\eta(\rho_v) \leq E_R^\infty(\rho_v) \leq 1$ by choosing a PPT state $\tau = Q/4$ such that $S(\rho_v|\tau) = 1$.

Secondly, we are going to prove $E_\eta(\rho_v) \geq 1$. To see this, suppose that

$$Y = \frac{1}{2}(Q + |00\rangle\langle 00| + (|11\rangle + |22\rangle)(\langle 11| + \langle 22|)).$$

Noting that

$$Y - P_v^{T_B} = \frac{1}{2}(|00\rangle + |11\rangle + |22\rangle)(\langle 00| + \langle 11| + \langle 22|),$$

it is clear that $P^{T_B} \leq Y$. Moreover,

$$Y + P_v^{T_B} = Q + \frac{1}{2}(|00\rangle - |11\rangle - |22\rangle)(\langle 00| - \langle 11| - \langle 22|),$$

which means that $P_v^{T_B} \geq -Y$.

Then Y is a feasible solution to the SDP (4) of $E_\eta(\rho_v)$. Thus,

$$E_\eta(\rho_v) \geq -\log \|Y^{T_B}\|_\infty = -\log 1/2 = 1,$$

and we can conclude that $E_\eta(\rho_v) = E_R^\infty(\rho_v) = 1$.

Finally, it is obvious that a standard Bell pair is sufficiently to prepare an exact copy of ρ_v by LOCC. Combining with the above bounds, we have that $1 = E_\eta(\rho_v) \leq E_R^\infty(\rho_v) \leq E_C(\rho_v) \leq E_{C,LOCC}(\rho_v) \leq 1$. \square

It is worth pointing out that our approach to evaluating the PPT-entanglement cost is to combine the lower bound E_η and the upper bound $E_{C,LOCC}$. This result provides a new proof of the entanglement cost of the rank-two $3 \otimes 3$ antisymmetric state in Ref. [41]. Moreover, our result is stronger as it shows that the entanglement cost under PPT operations of this state is still one ebit.

PPT-distillable entanglement of ρ_v : We can evaluate the PPT-distillable entanglement of ρ_v by the upper bound of Rains' bound and the SDP characterization of the one-copy deterministic PPT-distillable entanglement [30].

Proposition 3

$$E_D(\rho_v) = R^\infty(\rho_v) = \log(1 + 1/\sqrt{2}). \quad (7)$$

Proof Firstly, we need to introduce upper and lower SDP bounds to evaluate the distillable entanglement and the regularized Rains' bound. The logarithmic negativity [28, 29] is an upper bound on PPT-distillable entanglement, i.e., $E_N(\rho) = \log \|\rho^{T_B}\|_1$.

The following one-copy deterministic PPT-distillable entanglement was also obtained in Ref. [30, 31],

$$E_{0,D}^{(1)}(\rho) = \max -\log_2 \|R^{T_B}\|_\infty, \quad (8)$$

s.t. $P \leq R \leq \mathbf{1}$.

where P is the projection onto the support of ρ . Clearly $E_{0,D}^{(1)}(\rho)$ is efficiently computable by SDP, and for a general bipartite state ρ we have

$$E_{0,D}^{(1)}(\rho) \leq E_D(\rho) \leq R^\infty(\rho) \leq E_N(\rho),$$

which is very helpful to determine the exact values of PPT-distillable entanglement for some states.

Now it is easy to check that $\|\rho_v^{T_B}\|_1 = 1 + 1/\sqrt{2}$. Then,

$$R^\infty(\rho_v) \leq E_N(\rho_v) \leq \log(1 + 1/\sqrt{2}). \quad (9)$$

On the other hand, let

$$R = (3 - 2\sqrt{2})(|r_1\rangle\langle r_1| + |r_2\rangle\langle r_2|) + P_v$$

with $|r_1\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$ and $|r_2\rangle = (|02\rangle + |20\rangle)/\sqrt{2}$. It is easy to check that $P_v \leq R \leq \mathbf{1}$, which means that R is a feasible solution to SDP (8) of $E_{0,D}^{(1)}(\rho_v)$. Therefore,

$$E_{0,D}^{(1)}(\rho_v) \geq -\log \|R^{T_B}\|_\infty = \log(1 + 1/\sqrt{2}). \quad (10)$$

Finally, combining Eq. (9) and Eq. (10), we have that $E_D(\rho_v) = R^\infty(\rho_v) = \log(1 + 1/\sqrt{2})$. \square

General irreversibility under PPT operations:

We have shown the irreversibility of the entanglement distillation of ρ_v under PPT operations. One can use similar technique to prove this irreversibility for any ρ with spectral decomposition

$$\rho = p|u_1\rangle\langle u_1| + (1-p)|u_2\rangle\langle u_2| \quad (0 < p < 1), \quad (11)$$

where $|u_1\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$, $|u_2\rangle = (|ab\rangle - |ba\rangle)/\sqrt{2}$ and $\langle u_1|u_2\rangle = 0$. Interestingly, it holds that $E_D(\rho) < 1 = E_C(\rho)$. (See the supplementary material). More generally, we can provide a sufficient condition for the irreversibility under PPT operations and construct a general class of such states. For this purpose, we consider an improved version of logarithmic negativity introduced in Ref. [30], namely

$$E_W(\rho) = \min \log \|X^{T_B}\|_1, \text{ s.t. } X \geq \rho.$$

It was shown in Ref. [30] that $E_D(\rho) \leq E_W(\rho) \leq E_N(\rho)$, and the second equality can be strict. It is straightforward to see that if $E_W(\rho) < E_\eta(\rho)$, then $E_D(\rho) < E_C(\rho)$.

Indeed, we can obtain a more specific condition if we use logarithmic negativity E_N instead of E_W . That is, for a bipartite state ρ , if there is a Hermitian matrix Y such that $P_{AB}^{T_B} \pm Y \geq 0$ and $\|\rho^{T_B}\|_1 < \|Y^{T_B}\|_\infty^{-1}$, we have $E_D(\rho) < E_C(\rho)$.

We further show the irreversibility in asymptotic manipulations of entanglement under PPT operations by a class of $3 \otimes 3$ states in defined by $\rho^{(\alpha)} = (|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|)/2$, where $|\psi_1\rangle = \sqrt{\alpha}|01\rangle - \sqrt{1-\alpha}|10\rangle$ and $|\psi_2\rangle = \sqrt{\alpha}|02\rangle - \sqrt{1-\alpha}|20\rangle$ with $0.42 \leq \alpha \leq 0.5$. Then the projection onto the range of $\rho^{(\alpha)}$ is $P_{AB} = |\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|$. One can easily calculate that

$$E_W(\rho^{(\alpha)}) \leq \log \|(\rho^{(\alpha)})^{T_B}\|_1 = \log(1 + \sqrt{2\alpha(1-\alpha)}).$$

We then construct a feasible solution to the dual SDP (4) of $E_\eta(\rho^{(\alpha)})$, i.e., $Y = \alpha(|01\rangle\langle 01| + |02\rangle\langle 02|) + (1 - \alpha)(|10\rangle\langle 10| + |20\rangle\langle 20|) + \sqrt{\alpha(1 - \alpha)}(|00\rangle\langle 00| + |11\rangle\langle 11| + |22\rangle\langle 22| + |11\rangle\langle 22| + |22\rangle\langle 11|)$. It can be checked that $-Y \leq P_{AB}^{TB} \leq Y$ and $\|Y^{TB}\|_\infty \leq 1 - \alpha$. Thus, $E_\eta(\rho^{(\alpha)}) \geq -\log(1 - \alpha)$.

When $0.42 \leq \alpha \leq 0.5$, it is easy to check that $-\log(1 - \alpha) > \log(1 + \sqrt{2\alpha(1 - \alpha)})$. Therefore, $E_D(\rho^{(\alpha)}) \leq E_W(\rho^{(\alpha)}) < E_\eta(\rho^{(\alpha)}) \leq E_C(\rho^{(\alpha)})$.

Discussions We prove that distillable entanglement can be strictly smaller than entanglement cost under PPT operations, which implies the irreversibility of asymptotic entanglement manipulation under PPT operations. In particular, we prove that the PPT-distillable entanglement of any rank-two $3 \otimes 3$ antisymmetric state is strictly smaller than its PPT-entanglement cost. A byproduct is that there is a gap between the regularized Rains' bound and the asymptotic PPT-relative entropy of entanglement. Consequently, there is no unique entanglement measure in general for the asymptotic entanglement manipulation under PPT operations, which indicates that entanglement theory under PPT operations differs from thermodynamics. We also obtain an SDP-computable lower bound for the entanglement cost under both LOCC and PPT operations. Finally, we show an efficiently computable sufficient condition for the irreversibility of entanglement distillation of by LOCC (or PPT) operations.

However, the lower bound E_η for entanglement cost is in general not tight and could be sometimes smaller than distillable entanglement. To see this, consider the state $\sigma_a = P_a/3$ with P_a the projection over the $3 \otimes 3$ antisymmetric subspace. We have $E_\eta(\sigma_a) = \log 3/2 < E_D(\sigma_a) = E_C(\sigma_a) = \log 5/3$ [12]. How to further refine the lower bound E_η remains an interesting problem.

RD would like to thank Andreas Winter for inspirational discussions on the potential gap between regularized Rains' bound and asymptotic REE. The authors also thank Jonathan Oppenheim and Martin Plenio for their helpful comments. This work was partly supported by the Australian Research Council under Grant Nos. DP120103776 and FT120100449.

* Electronic address: xin.wang-8@student.uts.edu.au

† Electronic address: runyao.duan@uts.edu.au

- [1] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, *Phys. Rev. A* **54**, 3824 (1996).
- [2] C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, *Phys. Rev. A* **53**, 2046 (1996).
- [3] G. Vidal and J. I. Cirac, *Phys. Rev. Lett.* **86**, 5803 (2001).
- [4] G. Vidal and J. I. Cirac, *Phys. Rev. A* **65**, 012323 (2001).
- [5] G. Vidal, W. Dür, and J. I. Cirac, *Phys. Rev. Lett.* **89**, 027901 (2002).
- [6] K. G. H. Vollbrecht, R. F. Werner, and M. M. Wolf, *Phys.*

- Rev. A* **69**, 062304 (2004).
- [7] M. F. Cornelio, M. C. de Oliveira, and F. F. Fanchini, *Phys. Rev. Lett.* **107**, 020502 (2011).
- [8] M. Horodecki, P. Horodecki, and R. Horodecki, *Phys. Rev. Lett.* **80**, 5239 (1998).
- [9] D. Yang, M. Horodecki, R. Horodecki, and B. Synak-Radtke, *Phys. Rev. Lett.* **95**, 190501 (2005).
- [10] E. M. Rains, *IEEE Trans. Inf. Theory* **47**, 2921 (2001).
- [11] T. Eggeling, K. G. H. Vollbrecht, R. F. Werner, and M. M. Wolf, *Phys. Rev. Lett.* **87**, 257902 (2001).
- [12] K. Audenaert, M. B. Plenio, and J. Eisert, *Phys. Rev. Lett.* **90**, 027901 (2003).
- [13] S. Ishizaka and M. B. Plenio, *Phys. Rev. A* **71**, 052303 (2005).
- [14] F.G.S.L. Brandão and M. B. Plenio, *Nat. Phys.* **4**, 873 (2008).
- [15] F.G.S.L. Brandão and M. B. Plenio, *Commun. Math. Phys.* **295**, 829 (2010).
- [16] F.G.S.L. Brandão and G. Gour, *Phys. Rev. Lett.* **115**, 070503 (2015).
- [17] M. Horodecki, J. Oppenheim, and R. Horodecki, *Phys. Rev. Lett.* **89**, 240403 (2002).
- [18] M. Plenio, Problem 20 in the list <https://oqp.iqoqi.univie.ac.at/open-quantum-problems>.
- [19] K. Audenaert, J. Eisert, E. Jane, M. B. Plenio, S. Virmani, and B. De Moor, *Phys. Rev. Lett.* **87**, 217902 (2001).
- [20] M. B. Plenio and S. Virmani, *Quantum Inf. Comput.* **7**, 1 (2007).
- [21] L. Vandenberghe and S. Boyd, *SIAM Rev.* **38**, 49 (1996).
- [22] P. M. Hayden, M. Horodecki, and B. M. Terhal, *J. Phys. A: Math. Gen.* **34**, 6891 (2001).
- [23] K. Audenaert, B. De Moor, K. G. H. Vollbrecht, and R. F. Werner, *Phys. Rev. A* **66**, 032310 (2002).
- [24] X. Wang and R. Duan, *Phys. Rev. A* **95**, 062322 (2017).
- [25] V. Vedral, M. B. Plenio, M. A. Rippin, and P. L. Knight, *Phys. Rev. Lett.* **78**, 2275 (1997).
- [26] V. Vedral and M. B. Plenio, *Phys. Rev. A* **57**, 1619 (1998).
- [27] V. Vedral, M. B. Plenio, K. Jacobs, and P. L. Knight, *Phys. Rev. A* **56**, 4452 (1997).
- [28] G. Vidal and R. F. Werner, *Phys. Rev. A* **65**, 032314 (2002).
- [29] M. B. Plenio, *Phys. Rev. Lett.* **95**, 090503 (2005).
- [30] X. Wang and R. Duan, *Phys. Rev. A* **94**, 050301 (2016).
- [31] Kun Fang, Xin Wang, Marco Tomamichel, and Runyao Duan, [arXiv:1706.06221](https://arxiv.org/abs/1706.06221).
- [32] M. Hayashi, *Quantum Information* (Springer, 2006).
- [33] E. Chitambar and R. Duan, *Phys. Rev. Lett.* **103**, 110502 (2009).
- [34] J. Watrous, *Theory of Quantum Information*, University of Waterloo (2011).
- [35] F. Alizadeh, *SIAM J. Optim.* **5**, 13 (1995).
- [36] M. Grant and S. Boyd. CVX: Matlab software for disciplined convex programming, <http://cvxr.com> (2014).
- [37] Nathaniel Johnston, QETLAB: A MATLAB toolbox for quantum entanglement, <http://qetlab.com> (2015).
- [38] G. Lindblad, *Commun. Math. Phys.* **40**, 147 (1975).
- [39] A. Uhlmann, *Commun. Math. Phys.* **54**, 21 (1977).
- [40] N. Datta, *IEEE Trans. Inf. Theory* **55**, 2816 (2009).
- [41] F. Yura, *J. Phys. A: Math. Gen.* **36**, 15 (2003).

Supplemental Material

The additivity of $E_\eta(\rho)$ under tensor product

To see the additivity of $E_\eta(\rho)$, we reformulate it as $E_\eta(\rho) = -\log \eta(\rho)$, where

$$\begin{aligned} \eta(\rho) &= \min t \\ \text{s.t. } & -Y \leq P^{TB} \leq Y, \\ & -t\mathbb{1} \leq Y^{TB} \leq t\mathbb{1}, \end{aligned} \quad (12)$$

where P is the projection onto $\text{supp}(\rho)$.

The dual SDP of $\eta(\rho)$ can be derived by Lagrange multiplier method. It is given by

$$\begin{aligned} \eta(\rho) &= \max \text{Tr} P(V - F)^{TB}, \\ \text{s.t. } & V + F \leq (W - X)^{TB}, \\ & \text{Tr}(W + X) \leq 1, \\ & V, F, W, X \geq 0. \end{aligned} \quad (13)$$

The optimal values of the primal and the dual SDPs above coincide by strong duality. The details about strong duality theorem can be found in [34].

Proposition 4 *For any two bipartite states ρ_1 and ρ_2 , we have that*

$$E_\eta(\rho_1 \otimes \rho_2) = E_\eta(\rho_1) + E_\eta(\rho_2).$$

Proof On one hand, suppose that the optimal solution to SDP (12) of $\eta(\rho_1)$ and $\eta(\rho_2)$ are $\{t_1, Y_1\}$ and $\{t_2, Y_2\}$, respectively. It is easy to see that

$$\begin{aligned} Y_1 \otimes Y_2 + P_1^{TB} \otimes P_2^{TB'} &= \frac{1}{2}[(Y_1 + P_1^{TB}) \otimes (Y_2 + P_2^{TB'}) + (Y_1 - P_1^{TB}) \otimes (Y_2 - P_2^{TB'})] \geq 0, \\ Y_1 \otimes Y_2 - P_1^{TB} \otimes P_2^{TB'} &= \frac{1}{2}[(Y_1 + P_1^{TB}) \otimes (Y_2 - P_2^{TB'}) + (Y_1 - P_1^{TB}) \otimes (Y_2 + P_2^{TB'})] \geq 0. \end{aligned}$$

Then, we have that $-Y_1 \otimes Y_2 \leq P_1^{TB} \otimes P_2^{TB'} \leq Y_1 \otimes Y_2$. Moreover,

$$\|Y_1^{TB} \otimes Y_2^{TB'}\|_\infty \leq \|Y_1^{TB}\|_\infty \|Y_2^{TB'}\|_\infty \leq t_1 t_2,$$

which means that $-t_1 t_2 \mathbb{1} \leq Y_1^{TB} \otimes Y_2^{TB'} \leq t_1 t_2 \mathbb{1}$. Therefore, $\{t_1 t_2, Y_1 \otimes Y_2\}$ is a feasible solution to the SDP (12) of $\eta(\rho_1 \otimes \rho_2)$, which means that

$$\eta(\rho_1 \otimes \rho_2) \leq t_1 t_2 = \eta(\rho_1) \eta(\rho_2). \quad (14)$$

On the other hand, suppose that the optimal solutions to SDP (13) of $\eta(\rho_1)$ and $\eta(\rho_2)$ are $\{V_1, F_1, W_1, X_1\}$ and $\{V_2, F_2, W_2, X_2\}$, respectively. Assume that

$$\begin{aligned} V &= V_1 \otimes V_2 + F_1 \otimes F_2, F = V_1 \otimes F_2 + F_1 \otimes V_2, \\ W &= W_1 \otimes W_2 + X_1 \otimes X_2, X = W_1 \otimes X_2 + X_1 \otimes W_2. \end{aligned}$$

It is easy to see that

$$V + F = (V_1 + F_1) \otimes (V_2 + F_2) \leq (W_1 - X_1)^{TB} \otimes (W_2 - X_2)^{TB'} = (W - X)^{TB B'}$$

and $\text{Tr}(W + X) = \text{Tr}(W_1 + X_1) \otimes (W_2 + X_2) \leq 1$. Thus, $\{V, F, W, X\}$ is a feasible solution to the SDP (13) of $\eta(\rho_1 \otimes \rho_2)$. This means that

$$\eta(\rho_1 \otimes \rho_2) \geq \text{Tr}(P_1 \otimes P_2)(V - F)^{TB B'} = \text{Tr}(P_1 \otimes P_2)((V_1 - F_1)^{TB} \otimes (V_2 - F_2)^{TB'}) = \eta(\rho_1) \eta(\rho_2). \quad (15)$$

Hence, combining Eq. (14) and Eq. (15), it is clear that $\eta(\rho_1 \otimes \rho_2) = \eta(\rho_1) \eta(\rho_2)$, which means that

$$E_\eta(\rho_1 \otimes \rho_2) = E_\eta(\rho_1) + E_\eta(\rho_2).$$

□

Proof of an inequality in Proposition 1 using weak duality of SDP

In the following, we will utilize the weak duality of SDP to show an important inequality in Proposition 1, i.e.,

$$\max_{\sigma_0 \in \Gamma} \text{Tr } P\sigma_0 \leq \min t \quad \text{s.t. } Y^{T_B} \leq t\mathbf{1}, P^{T_B} \leq Y. \quad (16)$$

To see this, we note that $\max_{\sigma_0 \in \Gamma} \text{Tr } P\sigma_0$ is the prime SDP and its dual can be derived by Lagrange multiplier method. To be specific, we associate the operator $G \geq 0$ to the constraint $\sigma_0^{T_B} \geq 0$ and a real multiplier t to the constraint that $\text{Tr } \sigma_0 = 1$. The resulting Lagrangian is

$$\text{Tr } P\sigma_0 + t(1 - \text{Tr } \sigma_0) + \text{Tr } G\sigma_0^{T_B} = t + \text{Tr } \sigma_0(P + G^{T_B} - t\mathbf{1}).$$

The dual SDP is to minimise t subject to

$$P + G^{T_B} - t\mathbf{1} \leq 0, G \geq 0. \quad (17)$$

Let $Y = P^{T_B} + G$, then the dual SDP is to minimise t subject to

$$Y^{T_B} \leq t\mathbf{1}, P^{T_B} \leq Y. \quad (18)$$

Therefore, the prime and dual SDPs are as follows.

$$\textbf{(Primal)} \quad \max \left\{ \text{Tr } P\sigma_0 : \sigma_0 \geq 0, \sigma_0^{T_B} \geq 0, \text{Tr } \sigma_0 = 1 \right\}, \quad (19)$$

$$\textbf{(Dual)} \quad \min \left\{ t : Y^{T_B} \leq t\mathbf{1}, P^{T_B} \leq Y \right\}. \quad (20)$$

Finally, the inequality (16) follows from the weak duality theorem, which states that the value of the dual SDP attained at any dual feasible solution is at least the value of the primal SDP at any primal feasible solution. Interested readers can consult [21, 34] for more details.

Irreversibility for any rank-two $3 \otimes 3$ antisymmetric state

Proposition 5 *For any state ρ with spectral decomposition*

$$\rho = p|u_1\rangle\langle u_1| + (1-p)|u_2\rangle\langle u_2| \quad (0 < p < 1),$$

where

$$|u_1\rangle = (|01\rangle - |10\rangle)/\sqrt{2}, |u_2\rangle = (|ab\rangle - |ba\rangle)/\sqrt{2},$$

it holds that

$$E_D(\rho) < 1 = E_C(\rho).$$

Proof Suppose that $|a\rangle = a_0|0\rangle + a_1|1\rangle + a_2|2\rangle$ and $|b\rangle = b_0|0\rangle + b_1|1\rangle + b_2|2\rangle$. Noting that $\langle u_1|u_2\rangle = 0$, we have $a_0b_1 - a_1b_0 = 0$. Thus, with simple calculation, it is easy to see that

$$|u_2\rangle = [(a_0|0\rangle + a_1|1\rangle) \otimes b_2|2\rangle + a_2|2\rangle \otimes (b_0|0\rangle + b_1|1\rangle) - (b_0|0\rangle + b_1|1\rangle) \otimes a_2|2\rangle - b_2|2\rangle \otimes (a_0|0\rangle + a_1|1\rangle)]/\sqrt{2}.$$

Then, one can simplify $|u_2\rangle$ to

$$|u_2\rangle = [(\cos \theta|0\rangle + \sin \theta|1\rangle) \otimes |2\rangle - |2\rangle \otimes (\cos \theta|0\rangle + \sin \theta|1\rangle)]/\sqrt{2} \quad (0 \leq \theta \leq \pi/2),$$

where θ is determined by $|a\rangle$ and $|b\rangle$. We assume that $P_{AB} = |u_1\rangle\langle u_1| + |u_2\rangle\langle u_2|$.

It is can be calculated that $\|\rho^{T_B}\|_1 < 2$ for any $0 < p < 1$ and $0 \leq \theta \leq \pi/2$, which means that $E_D(\rho) < 1$.

Moreover, let us choose

$$Y = P^{T_B} + \frac{1}{2}(|00\rangle + |11\rangle + |22\rangle)(\langle 00| + \langle 11| + \langle 22|).$$

It is clear that $Y \geq P^{T_B}$ and it can be easily checked that $-Y \leq P^{T_B}$. Thus, Y is a feasible solution to the SDP (12) of $E_\eta(\rho)$, which means that

$$E_\eta(\rho) \geq -\log \|Y^{T_B}\|_\infty = 1.$$

Finally, it is clear that a standard $2 \otimes 2$ maximally entangled state (one ebit) such as $1/\sqrt{2}(|00\rangle + |11\rangle)$ is sufficiently to prepare an exact copy of ρ via LOCC. Combining with the above lower bounds, we have that $1 = E_\eta(\rho) \leq E_R^\infty(\rho) \leq E_C(\rho) \leq E_{C,LOCC}(\rho) \leq 1$. \square