

Compression of Quantum Multi-Prover Interactive Proofs

Zhengfeng Ji

Centre for Quantum Software and Information, School of Software, Faculty of Engineering and Information Technology, University of Technology Sydney, NSW, Australia

October 12, 2016

Abstract

We present a protocol that transforms any quantum multi-prover interactive proof into a nonlocal game in which questions consist of logarithmic number of bits and answers of constant number of bits. As a corollary, this proves that the promise problem corresponding to the approximation of the nonlocal value to inverse polynomial accuracy is complete for QMIP*, and therefore NEXP-hard. This establishes that nonlocal games are provably harder than classical games without any complexity theory assumptions. Our result also indicates that gap amplification for nonlocal games may be impossible in general and provides a negative evidence for the possibility of the gap amplification approach to the multi-prover variant of the quantum PCP conjecture.

1 Introduction

The notion of the efficient proof verification is one of the fundamental concepts in the theory of computing. Proof verification models and corresponding complexity classes ranging from NP, to IP, MIP and PCP greatly enrich the theory of computing. The class NP [14, 41, 31], one of the cornerstones of theoretical computer science, corresponds to the proof verification of a proof string by an efficient deterministic computer. Interactive models of proof verification were first proposed by Babai [4] and Goldwasser, Micali, and Rackoff [20]. It is generalized to the multi-prover setting by Ben-Or, Goldwasser, Kilian and Wigderson [7]. The study of different proof systems through the computational lens has led to a blossom of celebrated results in computational complexity theory and cryptography (e.g., [42, 54, 5, 3, 2, 19, 21]).

The efforts of understanding proof systems in the context of quantum computing have also been fruitful (see e.g., [36, 32, 23, 50, 37, 63, 1, 62, 26, 25, 60, 52]). These interesting results are nicely summarized in the recent survey on quantum proofs by Vidick and Watrous [61]. We emphasize that, in the development of quantum proofs, entanglement has played a dramatic role—it is both the cause of the problems and the key to the solutions as well.

A quantum analog of NP was proposed by Kitaev [36, 37, 1]. In this generalization, a quantum witness state plays the role of the proof string and a polynomial-time quantum computer checks whether the witness state is valid for the input. Kitaev introduces the class QMA of problems that admit efficient verifiable quantum proofs. He also establishes the quantum analog of the Cook-Levin theorem by showing that the local Hamiltonian problem, the natural quantum version of the constraint satisfaction problems, is complete for QMA. As elaborated in [1], the difficulty is to perform local propagation checks on the snapshot states which may be highly entangled. The circuit-to-Hamiltonian construction, the key technique for the quantum Cook-Levin theorem,

demonstrates how one can locally check the propagation of quantum computation, by introducing an extra clock system that entangles with the computational system. A generalization of this construction to the interactive setting will be one of the key ingredients of our result.

Entanglement also has unexpected use in single-prover quantum interactive proof systems, QIP, in which an efficient quantum verifier exchanges quantum messages with a quantum prover before making decisions. Watrous presented a constant-round quantum interactive proof system for PSPACE [62, 35], in which entanglement is exploited to enforce the correct temporal structure in a classical interactive proof for PSPACE. Alternatively, one can view this parallelization technique as dividing the interactive computation into two halves and check either forward or backward from the middle point. This idea also gave rise to a simple public coin characterization of QIP called QMAM [43], which in turn helps in the final proof that $\text{QIP} = \text{PSPACE}$ [26]. The technique can be extended to the multi-prover setting and show that quantum multi-prover interactive proof systems also parallelize to constant-rounds [34]. This will provide a starting point for our work.

This paper is about quantum multi-prover interactive proofs and nonlocal games, the scaled-down version of one-round quantum multi-prover proofs with classical messages. The class of languages that have quantum multi-prover interactive proofs is denoted as QMIP^* . In the multi-prover setting, shared entanglement among the provers becomes the natural focus of the study, a topic that has received continuing interests in physics foundations since 1960's [6, 39, 58, 64, 47, 51]. From the complexity perspective, it is known that, without shared entanglement, or with limited amount of entanglement, the collection of languages that have quantum multi-prover interactive proof systems equals to the classical counterpart, MIP [38] (and, hence, also equals to NEXP [5]). It was pointed out in [12] that provers with shared entanglement may break the soundness condition of a classically sound protocol. One striking example is given by the so-called magic square game [47, 51], which has nonlocal value¹ one even though it corresponds to a system of constraints with no classical solution [12]. Strong evidences are also given in that paper that the entanglement between the players may indeed weaken the power of two-player XOR games.

Several methods have been proposed to control the cheating ability of entangled provers and recover soundness in certain cases. It is proved that approximating the nonlocal value of a multi-player game to inverse-polynomial precision is NP-hard [33, 24], and therefore at least as hard as approximating the classical value [40]. Several natural problems arise from the study of nonlocality, including the binary constraint system game [13], the quantum coloring game [9, 53] and the game corresponding to the Kochen-Specker sets [39], are shown to be NP-hard in [27]. By proving that the multi-linearity test [5] is sound against entangled provers, Ito and Vidick proved the containment of NEXP in MIP^* [25]. This was later improved to the result that three-player XOR games are NP-hard to approximate even to constant precision [60]. Very recently, techniques introduced in [18, 28] allow us to go beyond the NP-hardness type of results and prove that nonlocal games are QMA-hard. The problem of the existence of perfect commuting-operator strategy for binary constraint system games was shown to be undecidable in a recent breakthrough [57]. It is, however, not comparable to the above results mainly because it does not tolerate approximation errors.

In this paper, we significantly improve the understanding of quantum multi-prover interactive proofs and nonlocal games by showing that any quantum multi-prover interactive proof can be *compressed* in the sense that the resulting protocol, a nonlocal game, has one round of classical communication with messages consisting of logarithmic number of bits. It has perfect completeness and an inverse polynomial completeness and soundness gap. Our result is made possible by combining and exploiting the unique features of entanglement that have already led to intriguing

¹The nonlocal value of a multi-player one-round game is the supremum of the probability that entangled players can make the verifier accept.

understandings of quantum proof systems as discussed above.

Theorem 1. *For $r \in \text{poly}$, any problem A that has an r -prover quantum interactive proof, and any instance x of the problem, there exists an $(r + 8)$ -player one-round game and real numbers $s \in 1 - \text{poly}^{-1}(|x|)$, such that*

1. *The questions are classical bit strings of length $O(\log(|x|))$.*
2. *The answers are classical bit strings of length $O(1)$.*
3. *If $x \in A$, then the nonlocal value of the game is 1.*
4. *If $x \notin A$, then the nonlocal value of the game is at most s .*

We mention that a corresponding claim in the classical case does not hold since $\text{MIP} = \text{NEXP}$, the approximation of classical value is in NP , and $\text{NEXP} \neq \text{NP}$ by a diagonalization argument [15]. The approximation problem of nonlocal value is obviously in QMIP^* by designing a multi-prover interactive protocol that sequentially repeats the multi-player game polynomially many times. This observation and Theorem 1 imply that the problem is in fact complete for the class QMIP^* . As NEXP is contained in QMIP^* [25], a direct corollary of Theorem 1 is that approximating the nonlocal value of a multi-player game is NEXP -hard, improving the QMA -hardness result of [28].

Corollary 2. *Given a multi-player one-round game in which the questions are strings of $O(\log n)$ bits and answers are of strings of $O(1)$ bits, it is QMIP^* -complete, and hence NEXP -hard, to approximate the nonlocal value of the game to inverse polynomial precision.*

The same problem for the classical value is obviously in NP . This means that the nonlocal value of multi-player one-round games is provably harder to approximate than the classical value without any complexity theory assumptions.

Our main theorem has the following consequence for the quantum multi-prover interactive proofs with inverse exponential completeness and soundness gap by scaling up the problem size. Let NEEXP be the class of nondeterministic double-exponential time. Let $\text{MIP}^*(r, m, c, s)$ (and $\text{MIP}(r, m, c, s)$) be the class of languages that have r -prover, m -round interactive proofs with a classical polynomial-time verifier, entangled provers (classical provers respectively), completeness c , and soundness s . We mention that $\text{MIP}(\text{poly}, \text{poly}, 1, s) \subseteq \text{NEXP}$ even for $s = 1 - \Omega(\exp(-p(n)))$ as a nondeterministic exponential time machine may first guess all the interactions and compute the value for this interaction to a precision of polynomially many bits.

Corollary 3. *There exists a constant r_0 such that for $r \geq r_0$, there exist choices of soundness $s = 1 - \Omega(\exp(-p(n)))$ where $p(n)$ is some polynomial, such that*

$$\text{NEEXP} \subseteq \text{MIP}^*(r, 1, 1, s),$$

and therefore, by the nondeterministic hierarchy theorem [15],

$$\text{MIP}(r, 1, 1, s) \neq \text{MIP}^*(r, 1, 1, s).$$

Our result also indicates that the gap amplification for nonlocal games may not be possible. For classical multi-player game, one can reduce the inverse polynomial approximation problem of the game value to the constant approximation problem of some derived game, a procedure known as gap amplification. This is an equivalent formulation of the classical PCP theorem and the approach of the alternative proof of the PCP theorem given by Dinur [16]. Whether one can also amplify the gap of nonlocal game in a similar way has been an interesting open problem. Our result implies that it may not be possible at all. If gap amplification works for nonlocal games, then one can

start with any nonlocal game, first perform gap amplification, and then scale up the instance size (assuming that the resulting referee after gap amplification has polylog time) and use our protocol to transform it back into a nonlocal game with eight extra players. This series of transformations will prove that nonlocal games with a constantly many more players are exponentially harder, a situation which does not seem to be plausible. This provides negative evidence for the strong form of the quantum PCP conjecture that asks whether constant approximation of the nonlocal value is as hard as inverse polynomial approximation. It may still be possible to prove, and even using the gap amplification approach for some special nonlocal games with certain structure, that constant approximation to the nonlocal value is QMA-hard, a weaker form of the multi-player variant of quantum PCP conjecture.

Historically in the study of classical proof systems, we have started from NP, generalized it to IP and MIP [42, 54, 5], motivated the study of PCP and come back to NP with the celebrated PCP theorem [3, 2, 16]. Our result indicates that the landscape of quantum proof systems may be very different.

Two important open questions are left open by this work. First, it is an intriguing problem to understand the complexity of constant approximation of the nonlocal value of a multi-player game. Second, it is important to provide upper bounds for the class QMIP*, a problem that remains widely open.

1.1 Techniques and Proof Overview

Our proof is motivated by, and reuses many techniques from, the previous work in [18, 28] but requires several new techniques that we now discuss.

First, we recall that it is crucial in [18, 28] that we encode the quantum witness state with certain quantum error correcting/detecting code and distribute the encoded state among the players so that we can prove rigidity theorems [44, 59, 52, 45] that are helpful to enforce the behavior of the players. This can be thought of as the quantum analog of the oracularization technique [40]. This technique alone, however, does not work anymore when we are dealing with quantum interactive proofs instead of quantum witness states as in the case for QMA for the following reason. In order to check the correct propagation for the provers' step, we will ask the players to simulate the provers' actions, applications of unitary circuits on their private qubits and the message qubits. This will require that the provers' circuits are *transversal* over the underlying quantum code. It is however well known that no quantum error correcting code supports transversal universal quantum computation [66, 17]. To this end, we need to find a different way to encode and distribute the qubits used in the interactive proof.

We introduce extended nonlocal games called propagation games and constraint propagation games. Propagation games exploit the idea of propagation checks in the proof of QMA-completeness for the local Hamiltonian problem and define a corresponding game so that the shared state between the referee and the player, who possess the clock and computation system respectively, must be approximately close to the history state with respect to the player's measurement strategies. The constraint propagation game then adds the constraint checks to the propagation game. The constraints can be of any product form and can represent commutativity and anti-commutativity as special cases. We then define a variant of the constraint propagation game using a constraint system satisfied by the Pauli operators on n qubits of weight k . With this game, we avoid the problem of transversality and obtain rigidity at the same time.

The use of extended nonlocal games for obtaining rigidity provides great flexibility and largely simplifies the structure of the game. This is the reason that constraint propagation games work with single player, and also the reason that we can check the propagation of the provers' step in an

interactive proof system. In particular, an extended nonlocal game defined by the stabilizer of the GHZ state serves as a nonlocal game implementation of the forward-backward checking technique in quantum interactive proofs discussed in the introduction.

Our resulting nonlocal game for QMIP* has perfect completeness. To achieve this, we modify the stabilizer game introduced in [28] so that the new stabilizer game has perfect quantum strategies. An eight-qubit code is used to define the stabilizer game and a much simpler proof of rigidity is provided for it. We also need a proof technique first used in the construction of zero-knowledge proofs for QMA [8], with which we design a propagation verification procedure for the verifier's circuits so that it suffices to measure commuting Pauli operators with X and Z factors only.

Our proof has the following overall structure. First, we generalize Kitaev's circuit-to-Hamiltonian construction for QMA to the interactive setting and turn a quantum multi-prover interactive proof system into an honest player game. This honest player game plays the role of the random checking protocol of the local Hamiltonian problem. We then use the rigidity of the constraint propagation game based on a constraint system satisfied by the Pauli operators to remove the requirement that the players must measure honestly. This gives rise to an extended nonlocal game for QMIP*. Finally, we turn this extended nonlocal game into a nonlocal game by using eight extra players who encode and simulate the Pauli measurements on the quantum system of the referee in the extended nonlocal game.

2 Preliminaries

2.1 Notions

In this paper, a *quantum register* refers to a named collection of qubits that we view as a single unit. Register names are represented by capital letters in a *sans serif* font, such as $X, Y,$ and Z . The associated Hilbert spaces are denoted by the same letters used in the register names in a calligraphic font. For example, $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ are the associated Hilbert spaces of registers $X, Y,$ and Z respectively. To refer to some specific qubits in a register X , we use qubit index followed by the register name. For example, (X, i_1, i_2) represents the i_1 and i_2 -th qubits of register X and the parentheses are omitted when this is used in subscripts. Hilbert spaces named with letter \mathcal{B} are two-dimensional unless stated otherwise.

We use $D(\mathcal{X}), L(\mathcal{X}), \text{Herm}(\mathcal{X}), \text{Pos}(\mathcal{X})$ to denote the set of density operators, bounded linear operators, Hermitian operators and positive semidefinite operators on \mathcal{X} . The adjoint of matrix M is denoted as M^* . For two Hermitian operators $M, N \in \text{Herm}(\mathcal{X})$, we write $M \leq N$ to mean $N - M \in \text{Pos}(\mathcal{X})$. For matrix M , $|M|$ is defined to be $\sqrt{M^*M}$. The operator norm $\|M\|$ of matrix M is the largest eigenvalue of $|M|$. The trace norm $\|M\|_1$ of M is the trace of $|M|$. An operator $R \in \text{Herm}(\mathcal{X})$ is a reflection if $R^2 = \mathbb{1}$. An operator $R \in L(\mathcal{X})$ is a contraction if $\|R\| \leq 1$. An operator $M \in L(\mathcal{X})$ is called traceless if $\text{tr}(M) = 0$.

A positive-operator valued measure (POVM) is described by the collection $\{M^a\}$ for $M^a \in \text{Pos}(\mathcal{X})$. Recall that the Naimark's theorem states that, for any POVM $\{M^a\}$, there exists an isometry

$$V = \sum_a \sqrt{M^a} \otimes |a\rangle,$$

such that

$$M^a = V^*(\mathbb{1} \otimes |a\rangle\langle a|)V.$$

It will be technically convenient to apply the Naimark's theorem and assume without loss of generality that the measurements considered in this paper are projective measurements, and that

each measurement operator has the same rank.

For each reflection R , there naturally associates a two-outcome projective quantum measurement $\{R^a\}$ where

$$R^a = \frac{\mathbb{1} + (-1)^a R}{2},$$

for $a = 0, 1$. Conversely, for any two-outcome projective measurement $\{R^a\}$, there associates a reflection $R = R^0 - R^1$. If the measurement operators have the same rank, the associated reflection is traceless.

For any projective measurement $M = \{M^a\}$ with k -bit outcome, define reflections

$$R_i = \sum_{a \in \{0,1\}^k} (-1)^{a_i} M^a,$$

for $i \in [k]$. The reflections R_1, R_2, \dots, R_k will be called the *derived reflections* of M . It is easy to see that the a projective measurement M with k -bit outcome has a one-to-one correspondence with the tuple of k derived reflections $(R_i)_{i=1}^k$.

For a Hermitian operator $H \in \text{Herm}(\mathcal{X})$, and a subspace $S \subseteq \mathcal{X}$, the restriction of H to S is

$$H|_S = \Pi_S H \Pi_S,$$

where Π_S is the projection onto the space S .

We will refer to the following elementary quantum gates in the paper:

1. The four single-qubit Pauli operators

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

which may also be denoted as $\sigma_0, \sigma_1, \sigma_2$, and σ_3 respectively sometimes.

2. The Hadamard gate

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

3. Two-qubit unitary gates CNOT and SWAP

$$\text{CNOT } |j, k\rangle = |j, j \oplus k\rangle, \quad \text{SWAP } |j, k\rangle = |k, j\rangle, \quad \text{for } j, k \in \{0, 1\}.$$

4. The Toffoli gate

$$\text{TOFFOLI } |j, k, l\rangle = |j, k, l \oplus jk\rangle, \quad \text{for } j, k, l \in \{0, 1\}.$$

For unitary gate U , define $\Lambda_c(U)$ to be the controlled gate

$$\Lambda_c(U) = |0\rangle\langle 0|_c \otimes \mathbb{1} + |1\rangle\langle 1|_c \otimes U.$$

A quantum channel is a physically admissible transformation of quantum states. Mathematically, a quantum channel

$$\mathfrak{E} : L(\mathcal{X}) \rightarrow L(\mathcal{Y})$$

is a completely positive, trace-preserving linear map.

The trace distance of two quantum states $\rho_0, \rho_1 \in D(\mathcal{X})$ is

$$D(\rho_0, \rho_1) \stackrel{\text{def}}{=} \frac{1}{2} \|\rho_0 - \rho_1\|_1.$$

The monotonicity of the trace distance states that for quantum states $\rho_0, \rho_1 \in \mathcal{D}(\mathcal{X})$ and all quantum channel $\mathfrak{E} : \mathcal{L}(\mathcal{X}) \rightarrow \mathcal{L}(\mathcal{Y})$,

$$D(\mathfrak{E}(\rho_0), \mathfrak{E}(\rho_1)) \leq D(\rho_0, \rho_1).$$

For a string x , $|x|$ denotes its length. For a positive integer k , $[k]$ is the abbreviation of the set $\{1, 2, \dots, k\}$. For a set A , $|A|$ denotes the number of elements in A . We use *poly* to denote the collection of polynomially bounded functions of either n or $|x|$ depending on the context. For two complex numbers a, b , we use $a \approx_\epsilon b$ as a shorthand notion for $|a - b| \leq O(\epsilon)$.

For quantum state $\rho \in \mathcal{D}(\mathcal{X})$ and operators $M, N \in \mathcal{L}(\mathcal{X})$, introduce the following notions

$$\text{tr}_\rho(M) = \text{tr}(M\rho), \tag{1a}$$

$$\langle M, N \rangle_\rho = \text{tr}_\rho(M^*N), \tag{1b}$$

$$\|M\|_\rho = \sqrt{\langle M, M \rangle_\rho}. \tag{1c}$$

It is straightforward to verify that $\langle \cdot, \cdot \rangle_\rho$ is a semi-inner-product, $\|\cdot\|_\rho$ is a seminorm and they become an inner product and a norm, respectively, when ρ is a full-rank state. By the Cauchy-Schwarz inequality,

$$\left| \langle M, N \rangle_\rho \right| \leq \|M\|_\rho \|N\|_\rho,$$

or more explicitly,

$$\left| \text{tr}_\rho(M^*N) \right| \leq \left[\text{tr}_\rho(M^*M) \text{tr}_\rho(N^*N) \right]^{1/2}.$$

For state $\rho \in \mathcal{D}(\mathcal{X} \otimes \mathcal{Y})$, and an operator $M \in \mathcal{L}(\mathcal{X})$, we may also write $\text{tr}_\rho(M)$ even though the state ρ and the operator M do not act on the same space. In this case, it is understood that $\text{tr}_\rho(M) = \text{tr}_{\rho_X}(M)$ where ρ_X is the reduced state of ρ on register \mathcal{X} . This is one reason that makes $\text{tr}_\rho(\cdot)$ easy to use as it is not necessary to specify the correct reduced state explicitly all the time.

We use the following version of the gentle measurement lemma [65]:

Lemma 4. *Let $\rho \in \mathcal{D}(\mathcal{X})$ be a state on \mathcal{X} , $\Pi \in \text{Pos}(\mathcal{X})$ a projection and $\epsilon \in [0, 1]$ a real number. If $\langle \Pi, \rho \rangle \geq 1 - \epsilon$, then for $\rho_\Pi = \Pi\rho\Pi / \langle \Pi, \rho \rangle$,*

$$D(\rho, \rho_\Pi) \leq O(\sqrt{\epsilon}).$$

A simple corollary of the gentle measurement lemma is the following.

Lemma 5. *Let $H \in \text{Pos}(\mathcal{X})$ be a Hamiltonian that has a 0-eigenspace S and all other eigenvalues are at least Δ . Let Π be the projection onto the 0-eigenspace S . Let $\rho \in \mathcal{D}(\mathcal{X})$ be a quantum state. If $\text{tr}_\rho(H) \leq \epsilon$, then*

$$D(\rho, \rho_\Pi) \leq (\sqrt{\epsilon/\Delta}),$$

where $\rho_\Pi = \Pi\rho\Pi / \langle \Pi, \rho \rangle$.

2.2 Quantum Multi-Player Proof Systems

An r -prover quantum interactive proof system consists of a verifier V and r provers P_1, P_2, \dots, P_r . The verifier V possesses a private quantum register \mathcal{V} consisting of $q_V \in \text{poly}$ qubits, each prover P_i possesses a private quantum register \mathcal{P}_i . There are also r message registers \mathcal{M}_i each of which contains $q_M \in \text{poly}$ qubits. Before the interaction starts, all qubits in register \mathcal{V} are initialized to $|0\rangle$. The provers are not allowed to communicate after the interaction starts. The interaction consists of $m \in \text{poly}$ alternating turns of the applications of the verifier and the provers' circuits.

The verifier of an m -turn quantum multi-prover interactive proof system is described by a tuple $V = (V^i)_{i=1}^{\lceil (m+1)/2 \rceil}$, where each V^i is a polynomial-time uniformly generated quantum circuit from input x acting on registers V, M_1, \dots, M_r . For $l \in [r]$, the prover P_l for an m -turn quantum multi-prover interactive proof system is described by a tuple $W^l = (W^{i,l})_{i=1}^{\lceil m/2 \rceil}$. For odd m , the provers start by choosing a state $|\psi\rangle \in \mathcal{V} \otimes \mathcal{M}$ then the verifier and the provers applies the circuits $V^1, \otimes_{l=1}^r W^{1,l}, \dots, \otimes_{l=1}^r W^{(m-1)/2,l}, V^{(m+1)/2}$ in order. For even m , the verifier initializes all qubits in M_l to $|0\rangle$ and the verifier and provers apply the circuit $V^1, \otimes_{l=1}^r W^{1,l}, \dots, \otimes_{l=1}^r W^{m/2,l}, V^{(m+2)/2}$. The verifier then measures the first qubit in V , accepts if the outcome is 1 and rejects otherwise. The maximum acceptance probability $\text{MAP}(V)$ for a given verifier circuit V is the maximum of the verifier's acceptance probability for all possible quantum provers described by $(W^l)_{l=1}^r$ and all correctly initialized state.

A language $A \in \text{QMIP}^*(r, m, c, s)$ if and only if there exists an r -prover, m -turn quantum interactive proof systems with verifier V such that the following conditions hold:

1. (Completeness) If $x \in A$, $\text{MAP}(V) \geq c$,
2. (Soundness) If $x \notin A$, $\text{MAP}(V) \leq s$.

Define QMIP^* to be $\text{QMIP}^*(\text{poly}, \text{poly}, 2/3, 1/3)$. If the exchanged messages in a quantum multi-prover interactive proof system are classical while the provers may still share entanglement before the interaction starts, the corresponding complexity class will be denoted as MIP^* . It is now known that $\text{QMIP}^* = \text{MIP}^*$ [52].

2.3 Nonlocal Games and Extended Nonlocal Games

Multi-player games have similar structure as multi-prover interactive proofs with the main difference in the length of the messages. In multi-prover interactive proofs, messages can consist of polynomially many bit (or qubits), while in multi-player games, the messages consist of logarithmic number of bits. In a multi-player one-round game, a referee communicates with two or more players classically in one round. The referee samples questions and sends them out to the players and expects to receive answers back. He then accepts or rejects based on the questions and answers. The players are allowed to agree on a strategy before the game starts, but cannot communicate with each other during the game.

Let there be r players, $(1), (2), \dots, (r)$. Let $\Gamma^{(i)}$ be a finite set of questions for player (i) and $\Lambda^{(i)}$ be a finite set of possible answers from player (i) . An r -player game is defined by a distribution π over $\prod_{i=1}^r \Gamma^{(i)}$ and a function $V : \prod_{i=1}^r \Lambda^{(i)} \times \prod_{i=1}^r \Gamma^{(i)} \rightarrow [0, 1]$, specifying the acceptance probability. By a convexity argument, it suffices to consider the strategy of classical players described by functions $f^{(i)} : \Gamma^{(i)} \rightarrow \Lambda^{(i)}$. The value of the strategy is the acceptance probability

$$\omega = \mathbb{E}_{q \sim \pi} V(a(q), q),$$

for $q = (q_1, q_2, \dots, q_r)$ distributed according to π and $a(q) = (f^{(1)}(q_1), f^{(2)}(q_2), \dots, f^{(r)}(q_r))$. The classical value of the game is the maximum of the values of all classical strategies.

In a nonlocal game, the players are allowed to share an arbitrary entangled state before the game starts. A quantum strategy \mathfrak{G} for the nonlocal game is described by the shared state ρ , the measurements $\{M_{q_i}^{(i)}\}$ that player (i) performs when the question is $q_i \in \Gamma^{(i)}$. The value of the strategy is defined as

$$\omega^*(\mathfrak{G}) = \mathbb{E}_{q \sim \pi} \sum_a \left[\text{tr}_\rho \left(\bigotimes_{i=1}^r M_{q_i}^{(i), a_i} \right) V(a, q) \right],$$

for $a = (a_1, a_2, \dots, a_r)$ and $q = (q_1, q_2, \dots, q_r)$. The nonlocal value of the game is the supremum of the values of all quantum strategies.

Extended nonlocal games were introduced in [29] as an extension of the nonlocal game. It is originally defined in a multi-player setting while we found that it is also interesting to consider extended nonlocal games with a single player. An extended nonlocal game generalizes the nonlocal game in the following sense. The referee possesses a quantum register S but otherwise samples and sends out questions similarly as in a nonlocal game. The players can choose an initial state shared between the referee's register and their private quantum systems. The referee performs measurement on his quantum register and may depend his acceptance also on the measurement outcome. More formally, an r -player extended nonlocal game is defined by a distribution π over $\prod_{i=1}^r \Gamma^{(i)}$ and a function $V : \prod_{i=1}^r \Lambda^{(i)} \times \prod_{i=1}^r \Gamma^{(i)} \rightarrow [0, \mathbb{1}]$, where $[0, \mathbb{1}]$ is the set

$$\{V \in \text{Pos}(\mathcal{S}) \mid V \leq \mathbb{1}\}.$$

A quantum strategy \mathfrak{S} for the extended nonlocal game is described by the shared state ρ , the measurements $\{M_{q_i}^{(i)}\}$ that player (i) performs when the question is $q_i \in \Gamma^{(i)}$. The value of the strategy is defined as

$$\omega^*(\mathfrak{S}) = \mathbb{E}_{q \sim \pi} \sum_a \left[\text{tr}_\rho \left(\bigotimes_{i=1}^r M_{q_i}^{(i), a_i} \otimes V(a, q) \right) \right],$$

for $a = (a_1, a_2, \dots, a_r)$ and $q = (q_1, q_2, \dots, q_r)$. The value of the game is the supremum of the values of all quantum strategies.

2.4 Pauli Operators and Stabilizer Codes

Let \mathbb{P}_n be the group generated by the n -fold tensor product of Pauli operators

$$\mathbb{P}_n = \left\{ e^{i\phi} \bigotimes_{j=1}^n D_j, \text{ for } \phi \in \{0, \pi/2, \pi, 3\pi/2\}, D_j \in \{I, X, Y, Z\} \right\}.$$

The weight of a Pauli operator in \mathbb{P}_n is the number of non-identity tensor factors in it. A Pauli operator is of XZ-form if each tensor factor is one of I , X and Z . For Pauli operator P of the form $(-1)^\tau \bigotimes_j D_j$, the bit $\tau \in \{0, 1\}$ is called the sign bit of the operator.

We present several relevant definitions and facts about the stabilizer codes and refer the reader to the thesis of Gottesman [22] for more details. A stabilizer \mathcal{S} is an abelian subgroup of \mathbb{P}_n not containing $-\mathbb{1}$. It provides a succinct description of a corresponding subspace of $(\mathbb{C}^2)^{\otimes n}$ —the simultaneous $+1$ -eigenspace of the operators in the stabilizer. Let $C(\mathcal{S})$ be the centralizer of \mathcal{S} in \mathbb{P}_n , the set of operators in \mathbb{P}_n that commutes with all operators in \mathcal{S} . The distance of the stabilizer code is d if there is no operator of weight less than d in $C(\mathcal{S}) - \mathcal{S}$. The logical X and Z operators L_X and L_Z are a pair of anti-commuting operators in $C(\mathcal{S}) - \mathcal{S}$.

As a simple example, the operators XXX , ZZI and IZZ generate a stabilizer for the GHZ state. Operators $XXXX$ and $ZZZZ$ generate the stabilizer for the four-qubit quantum error detecting code, which has distance two and encodes two qubits.

2.5 Distance Measures of Quantum Strategies

We review the state-dependent distance measure d_ρ and consistency measure C_ρ of quantum measurements discussed in detail in [28]. We also introduce the notion that an operator approximately

stabilizes a state and prove several related facts. These concepts play an important role in analyzing the behavior of the entangled players and are used extensively in our proofs.

Consider the situation where two players, Alice and Bob, share a quantum state $\rho \in D(\mathcal{X} \otimes \mathcal{Y})$ and Alice measures X with either $\{M_0^a\}$ or $\{M_1^a\}$. The post-measurement states are

$$\rho_i = \sum_a |a\rangle\langle a| \otimes M_i^a \rho (M_i^a)^*, \quad (2)$$

for $i = 0, 1$, respectively, depending on which measurement is performed. By the monotonicity of the trace distance, the difference is bounded by $D(\rho_0, \rho_1)$. As a special case, if Bob measures on Y and then the referee makes the decision, the acceptance probabilities will differ by at most $D(\rho_0, \rho_1)$. The state-dependent distance defined next provides a bound on the distance $D(\rho_0, \rho_1)$. The claim is stated in Lemma 7 whose proof can be found in [28].

Definition 6. For two quantum measurements $M_i = \{M_i^a\}$ with $i = 0, 1$ that have the same set of possible outcomes, define

$$d_\rho(M_0, M_1) \stackrel{\text{def}}{=} \left[\sum_a \|M_0^a - M_1^a\|_\rho^2 \right]^{1/2}. \quad (3)$$

More explicitly,

$$d_\rho(M_0, M_1) = \left[2 - 2 \operatorname{Re} \sum_a \operatorname{tr}_\rho((M_0^a)^* M_1^a) \right]^{1/2}. \quad (4)$$

Lemma 7. Let $M_i = \{M_i^a\}$ for $i = 0, 1$ be two quantum measurements with the same set of possible outcomes, and ρ_i be the post-measurement states in Eq. (2). Then

$$D(\rho_0, \rho_1) \leq d_\rho(M_0, M_1).$$

A direct corollary of the above lemma is that replacing measurement M_0 with M_1 in a strategy for a nonlocal game changes the value by at most $d_\rho(M_0, M_1)$. As this claim works for the general quantum measurement, it generalizes to the special cases such as positive-operator valued measures (POVM), projective measurements and measurements corresponding to reflections.

In analysis of games, the post-measurement state is not important and hence POVMs are the suitable formulation for quantum measurements. It is technically convenient to adopt the following definition for the state-dependent distance between to POVMs $M_i = \{M_i^a\}$ with $i = 0, 1$,

$$d_\rho(M_0, M_1) \stackrel{\text{def}}{=} \inf_{N_i = \{N_i^a\}} d_\rho(\{N_0^a\}, \{N_1^a\})$$

where the infimum is taken over all possible measurement operators N_i^a such that $M_i^a = (N_i^a)^* (N_i^a)$ for all a and $i = 0, 1$. The freedom to use arbitrary measurement operators, instead of $\sqrt{M_i^a}$, provides simpler ways to upper bound the distance.

We focus on the case of reflections which will be extensively used in later sections.

For two reflections R_0, R_1 , let

$$R_i^a = \frac{\mathbb{1} + (-1)^a R_i}{2}$$

be the projective measurement operators correspond to R_i . Define

$$d_\rho(R_0, R_1) \stackrel{\text{def}}{=} d_\rho(\{R_0^a\}, \{R_1^a\}) = [1 - \operatorname{Re} \operatorname{tr}_\rho(R_0 R_1)]^{1/2}.$$

It is easy to verify that d_ρ satisfy the triangle inequality.

Lemma 8. Let M_0, M_1, M_2 be three measurements on state ρ . Then

$$d_\rho(M_0, M_2) \leq d_\rho(M_0, M_1) + d_\rho(M_1, M_2).$$

Next, we recall the consistency measure for two quantum measurements that act on two *different* quantum systems.

Definition 9. Let $\rho \in D(\mathcal{X} \otimes \mathcal{Y})$ be a quantum state on X, Y , let $M = \{M^a\}, N = \{N^a\}$ be two POVMs on registers X, Y respectively having the same set of possible outcomes. Define the consistency of M, N on state ρ as

$$C_\rho(M, N) \stackrel{\text{def}}{=} \sum_a \text{tr}_\rho(M^a \otimes N^a). \quad (5)$$

M and N are called ϵ -consistent on state ρ if $C_\rho(M, N) \geq 1 - \epsilon$.

The consistency of measurements puts strong structural constraints on the strategies of nonlocal game, which greatly simplify the analysis. In this paper, we will be mostly interested in the consistency of two reflections. For two reflections R, S , let $\{R^a\}, \{S^a\}$ be their corresponding projective measurements. Define

$$C_\rho(R, S) \stackrel{\text{def}}{=} C_\rho(\{R^a\}, \{S^a\}) = \frac{1 + \text{tr}_\rho(R \otimes S)}{2}. \quad (6)$$

The condition $\text{tr}_\rho(R \otimes S) \approx_\epsilon 1$, or equivalently, R, S are $O(\epsilon)$ -consistent on ρ , can be thought of as a quantitative way of saying that ρ is approximately stabilized by $R \otimes S$.

More generally, we introduce the notion of ϵ -stabilizer as follows. It is crucial for later applications that we define this concept not only for reflections but for the more general notion of contractions.

Definition 10. Let $R \in L(\mathcal{X})$ be a contraction and $\rho \in D(\mathcal{X})$ be a quantum state. We say that R ϵ -stabilizes ρ if

$$\text{Re tr}_\rho R \geq 1 - \epsilon.$$

Lemma 11. Let $R_0, R_1 \in L(\mathcal{X})$ be two contractions such that $\text{Re tr}_\rho R_i = 1 - \epsilon_i$ for $i = 0, 1$. Then the product $R_0 R_1$ satisfies

$$\text{Re tr}_\rho(R_0 R_1) \geq 1 - \epsilon,$$

for $\epsilon = (\epsilon_0^{1/2} + \epsilon_1^{1/2})^2$. As a special case, if ρ is $O(\epsilon)$ -stabilized by both R_0 and R_1 , it is also $O(\epsilon)$ -stabilized by $R_0 R_1$.

Proof. We first prove that

$$\text{Re tr}_\rho(\mathbb{1} - R_0)(\mathbb{1} - R_1) \geq -2\epsilon_0^{1/2}\epsilon_1^{1/2}. \quad (7)$$

In fact, by Cauchy-Schwarz inequality, the absolute value of the left hand side is at most

$$\left[\text{tr}_\rho((\mathbb{1} - R_0)(\mathbb{1} - R_0)^*) \text{tr}_\rho((\mathbb{1} - R_1)^*(\mathbb{1} - R_1)) \right]^{1/2},$$

which is bounded by $2\epsilon_0^{1/2}\epsilon_1^{1/2}$ using the conditions for contractions R_0, R_1 . By Eq. (7),

$$\text{Re tr}_\rho(R_0 R_1) \geq \text{Re tr}_\rho R_0 + \text{Re tr}_\rho R_1 - 1 - 2\epsilon_0^{1/2}\epsilon_1^{1/2} = 1 - (\epsilon_0^{1/2} + \epsilon_1^{1/2})^2.$$

□

Lemma 12. Let $R \in L(\mathcal{X})$ be a contraction that ϵ -stabilizes state ρ , then for any contraction S ,

$$\operatorname{Re} \operatorname{tr}_\rho(SR) \approx_{\sqrt{\epsilon}} \operatorname{Re} \operatorname{tr}_\rho(S).$$

Proof. The absolute value of the difference of the two terms in the equation is upper bounded by the Cauchy-Schwarz inequality as

$$\left| \operatorname{tr}_\rho(S(\mathbb{1} - R)) \right| \leq \left[\operatorname{tr}_\rho(SS^*) \cdot \operatorname{tr}_\rho((\mathbb{1} - R)^*(\mathbb{1} - R)) \right]^{1/2} \leq O(\sqrt{\epsilon}).$$

□

Lemma 13. Let $R_0, R_1, R_3 \in L(\mathcal{X})$ be contractions such that

$$\operatorname{Re} \operatorname{tr}_\rho(R_0 R_1) \approx_\epsilon 1, \quad \operatorname{Re} \operatorname{tr}_\rho(R_1^* R_2) \approx_\epsilon 1,$$

then

$$\operatorname{Re} \operatorname{tr}_\rho(R_0 R_2) \approx_\epsilon 1.$$

Proof. By Lemma 11, we have

$$\operatorname{Re} \operatorname{tr}_\rho(R_0 R_1 R_1^* R_2) \approx_\epsilon 1.$$

It therefore suffices to prove that

$$\left| \operatorname{tr}_\rho(R_0(\mathbb{1} - R_1 R_1^*) R_2) \right| \leq O(\epsilon).$$

By the Cauchy-Schwarz inequality, the above term is bounded by

$$\left[\operatorname{tr}_\rho(R_0 R_0^* - R_0 R_1 R_1^* R_0^*) \operatorname{tr}_\rho(R_2^* R_2 - R_2^* R_1 R_1^* R_2) \right]^{1/2}.$$

Applying Lemma 11 again to the condition for R_0, R_1 , we have

$$\operatorname{tr}_\rho(R_0 R_1 R_1^* R_0^*) \approx_\epsilon 1.$$

By the fact that R_0 is a contraction, it implies that

$$\operatorname{tr}_\rho(R_0 R_0^* - R_0 R_1 R_1^* R_0^*) \leq O(\epsilon).$$

A similar bound applies to the second term in the square root and this completes the proof. □

In the analysis, it is important to have a quantity characterizing the approximate commutativity and anti-commutativity of two reflections. Two reflections R_0, R_1 is said to be ϵ -commutative on state ρ if

$$\operatorname{Re} \operatorname{tr}_\rho(R_0 R_1 R_0 R_1) \geq 1 - \epsilon, \tag{8}$$

and ϵ -anti-commutative on ρ if

$$\operatorname{Re} \operatorname{tr}_\rho(R_0 R_1 R_0 R_1) \leq \epsilon - 1. \tag{9}$$

We prove the following lemma which roughly says that ϵ -anti-commutative reflections are close to X, Z respectively up to a change of basis. It will be used multiple times in later analysis to establish rigidity theorems.

Lemma 14. Let $\rho \in \mathcal{D}(\mathcal{X})$ be a quantum state and $R_0, R_1 \in \text{Herm}(\mathcal{X})$ be two traceless reflections such that

$$\text{Re tr}_\rho(R_0 R_1 R_0 R_1) \approx_\epsilon -1.$$

There exists a unitary $V \in \text{L}(\mathcal{X}, \mathcal{B} \otimes \mathcal{X}')$ such that $R_1 = V^*(Z \otimes \mathbb{1})V$ and

$$\text{Re tr}_\rho(R_0 V^*(X \otimes \mathbb{1})V) \approx_\epsilon 1.$$

Equivalently,

$$d_\rho(R_0, V^*(X \otimes \mathbb{1})V) \leq O(\sqrt{\epsilon}).$$

The choice of V is independent of state ρ and is determined solely by the operators R_0, R_1 .

The proof of the lemma relies on the Jordan's Lemma.

Lemma 15 (Jordan's Lemma [30]). For any two reflections R_0, R_1 acting on a finite dimensional Hilbert space \mathcal{H} , there exists a decomposition of \mathcal{H} into orthogonal one- and two-dimensional subspaces invariant under both R_0 and R_1 .

Proof of Lemma 14. Using Jordan's Lemma and the condition that the reflections are traceless, one get simultaneous 2-by-2 block diagonalizations of R_0 and R_1 such that each 2-by-2 block is a reflection having both ± 1 eigenvalues. Hence, there is a unitary operator $V \in \text{L}(\mathcal{X}, \mathcal{B} \otimes \mathcal{X}')$ such that²

$$R_1 = V^*(Z \otimes I)V,$$

and

$$R_0 = V^* \sum_l \left[\begin{pmatrix} \cos \theta_l & \sin \theta_l \\ \sin \theta_l & -\cos \theta_l \end{pmatrix} \otimes |l\rangle\langle l| \right] V,$$

where $\theta_l \in [0, \pi]$ and l is the index of the two-dimensional invariant subspaces obtained by Jordan's lemma.

By a direct calculation, we have

$$R_0 R_1 R_0 R_1 = V^* \sum_l \left[\begin{pmatrix} 1 - 2 \sin^2 \theta_l & -2 \cos \theta_l \sin \theta_l \\ 2 \cos \theta_l \sin \theta_l & 1 - 2 \sin^2 \theta_l \end{pmatrix} \otimes |l\rangle\langle l| \right] V.$$

The condition, $\text{Re tr}_\rho(R_0 R_1 R_0 R_1) \approx_\epsilon -1$, then simplifies to

$$\mathbb{E}_l \sin^2 \theta_l \approx_\epsilon 1, \tag{10}$$

where the expectation \mathbb{E}_l is over the probability distribution

$$\text{Pr}(l) = \text{tr}_\rho[V^*(\mathbb{1} \otimes |l\rangle\langle l|)V].$$

We then have,

$$\text{Re tr}_\rho(R_0 V^*(X \otimes \mathbb{1})V) = \mathbb{E}_l \sin \theta_l \geq \mathbb{E}_l \sin^2 \theta_l \approx_\epsilon 1,$$

which completes the proof by the definition of d_ρ for two reflections. \square

²The traceless condition simplifies the discussion here. Otherwise, the dimension of \mathcal{X} may not be even and one has to take V to be an isometry instead of a unitary operator, which will in turn make later discussions more complicated.

2.6 Rigidity Using Extended Nonlocal Games

Rigidity of a nonlocal game states that if the players win the game with probability that is close to optimal, then they have to approximately follow the optimal strategy up to an isometry, including the initialization of a shared state and the application of the quantum measurement for each question. It has found a wide range of applications in self-testing of quantum apparatus [44, 59, 48, 46] and quantum multi-player interactive proofs [52, 28, 49].

We demonstrate that it is easier both to construct extended nonlocal games that have rigidity properties and to establish rigidity for them. Extended nonlocal games are in some sense a variant of the nonlocal games with one honest player, who honestly follows a prescribed measurement strategy. This may explain the reason behind the advantages of the use of the extended nonlocal games.

In [28], the CHSH game was revisited in the framework of stabilizers starting from the fact that the EPR state is stabilized by XX and ZZ . There, one need to rotate the basis for one of players by 45 degree, a mysterious twist that one has to perform in the case for nonlocal games. Consider the following extended nonlocal game for the EPR stabilizer between the referee and one player that literally translates the generators of stabilizer into random questions. The referee possesses a single qubit register B and samples a random bit $q \in \{0, 1\}$, and send it to the player. He then measures X or Z on his qubit for $q = 0$ or $q = 1$ respectively and accepts if and only if the measurement outcome equals the answer bit a from the player. The game value for this simple extended nonlocal game is one, which can be achieved by a player who shares the EPR state and measures X, Z for question 0, 1 respectively. The strategy of the player can be described by the tuple (ρ, \hat{X}, \hat{Z}) where $\rho \in D(\mathcal{B} \otimes \mathcal{R})$ is the state the player chooses, and $\hat{X}, \hat{Z} \in \text{Herm}(\mathcal{R})$ are traceless reflections that describe the player's two-outcome measurements for question 0 and 1 respectively. The value of the strategy is

$$\frac{1}{2} \sum_{D \in \{X, Z\}} \text{tr}_\rho \frac{\mathbb{1} + D \otimes \hat{D}}{2}.$$

If the value is at least $1 - \epsilon$, it then follows that

$$\text{tr}_\rho(D \otimes \hat{D}) \approx_\epsilon 1,$$

for $D = X, Z$. By Lemma 11, we have

$$\text{Re tr}_\rho(XZXZ \otimes \hat{X}\hat{Z}\hat{X}\hat{Z}) \approx_\epsilon 1,$$

which simplifies to

$$\text{Re tr}_\rho(\hat{X}\hat{Z}\hat{X}\hat{Z}) \approx_\epsilon -1.$$

Lemma 14 then establishes the rigidity for this game.

A similar construction based on the stabilizer for the GHZ state is used in Sec. 5 to check the correct propagation of the provers' actions.

3 Stabilizer Games, Redefined

3.1 Stabilizer Games

Stabilizer games were first defined in [28] as an extension of the CHSH game[11]. In this section, we introduce yet another type of stabilizer games. Their advantage over the stabilizer games defined

in [28] is that they have perfect quantum strategies, a property that is crucial to obtain perfect completeness. The analysis of this new stabilizer game is also arguably simpler.

Consider the stabilizer in Fig. 1. It is an eight-qubit code encoding two logical qubits and has distance two, and we will refer to it as the eight-qubit code in this paper. The operators g_1, g_2, \dots, g_6 are the generators for the stabilizer. The operators L_X and L_Z are the logical X, Z operators for one of the logical qubits. One can derive this code by concatenating the $[4, 2, 2]$ code stabilized by $X^{\otimes 4}, Z^{\otimes 4}$ and the $[2, 1, 1]$ code stabilized by $Y^{\otimes 2}$. We note that the construction of the stabilizer game generalizes to other stabilizer codes with generators of XZ -form. It suffices for our purpose to consider the game defined by the eight-qubit code only.

The stabilizer of the eight-qubit code consists of 64 operators, $\prod_{i=1}^6 g_i^{\mu_i}$, for $\mu_i \in \{0, 1\}$. There are 32 of them that have XZ -form. These are the operators when one and only one of μ_1, μ_2 is 1. Let Ξ be the set of these XZ -form operators. Examples of operators in Ξ contain g_1, g_2 and $g_{1,3} = g_1 g_3, g_{2,3} = g_2 g_3$ in Fig. 2. Note that we have listed g_2 before g_1 in Fig. 2 on purpose for reasons to be clear later.

Name	Operator
g_1	XXXXXXXX
g_2	XZXZXZXZ
g_3	YYIIIIII
g_4	I I Y Y I I I I
g_5	I I I I Y Y I I
g_6	I I I I I I Y Y
L_X	XXXXIIII
L_Z	XZII XZII

Figure 1: An eight-qubit stabilizer code used in the stabilizer game in Fig. 3.

Name	Operator
g_2	XZXZXZXZ
g_1	XXXXXXXX
$g_{1,3}$	-ZZXXXXXXXX
$g_{2,3}$	ZZXZXZXZ

Figure 2: Four examples of XZ -form operators in the stabilizer of the eight-qubit code

The stabilizer game considered in this paper is an eight-player XOR game as specified in Fig. 3. We abuse the notion and use the Pauli operators X, Z as labels for the questions. Intuitively, an X question requests that the player measures Pauli X operator on his system and reply with the outcome and similarly for Z questions.

A strategy of the stabilizer is specified by the state ρ shared between the eight players and the measurement the players perform for question indexed by X, Z . We will use $\hat{D}^{(i)}$ to denote the reflections that correspond to the measurement player (i) performs for question $D \in \{X, Z\}$. Without loss of generality, we assume that $\hat{D}^{(i)}$ are traceless reflections. When there is no ambiguity, the player index in the superscript may be omitted.

We prove the following rigidity theorem for the stabilizer game.

Theorem 16. *The nonlocal value of stabilizer game in Fig. 3 is 1. Furthermore, the game has the following rigidity property. Let $\mathfrak{S} = (\rho, \{\hat{D}^{(i)}\})$ be a strategy for the stabilizer game where ρ is the state shared*

Stabilizer Game

Let Ξ be the subset of stabilizer operators of XZ -form for the eight-qubit code. The stabilizer game for the eight-qubit code is the eight-player nonlocal game defined as follows.

1. The referee selects one of the 32 operators from Ξ uniformly at random. Let $D^{(i)} \in \{X, Z\}$, $s \in \{0, 1\}$ be the i -th tensor factor and the sign of the chosen operator respectively.
2. For $i \in [8]$, the referee sends $D^{(i)}$ to player (i) and receive a bit $a^{(i)}$ back;
3. Accepts if $\bigoplus_{i=1}^8 a^{(i)} = s$ and rejects otherwise.

Figure 3: Stabilizer game defined by the eight-qubit code in Fig. 1.

between the players before the game starts and $\hat{D}^{(i)} \in \text{Herm}(\mathcal{R}_i)$ is the traceless reflection corresponding to the measurements the player (i) performs for question $D \in \{X, Z\}$. If the value of strategy \mathfrak{S} is at least $1 - \epsilon$, then there are unitary operators $V_i \in \text{L}(\mathcal{R}_i, \mathcal{B}_i \otimes \mathcal{R}'_i)$ for $i \in [8]$, such that the following properties hold

- For all $i \in [8]$, $\hat{Z}^{(i)} = \check{Z}^{(i)}$ and

$$d_\rho(\hat{X}^{(i)}, \check{X}^{(i)}) \leq O(\sqrt{\epsilon}),$$

where $\check{D}^{(i)} = V_i^*(D \otimes \mathbb{1})V_i$ for $D \in \{X, Z\}$.

- Let Π be the projection to the code space of the eight qubit code, and let V be the unitary operator $\bigotimes_{i=1}^8 V_i$, then

$$\langle \Pi \otimes \mathbb{1}, V\rho V^* \rangle \geq 1 - O(\epsilon),$$

where Π acts on the eight qubits in registers $(\mathcal{B}_i)_{i=1}^8$.

Proof. It is obvious that if the players share an encoded state of the eight qubit code and perform the X, Z measurements to obtain the answers for questions X, Z respectively, the referee accepts with certainty.

For each operator Pauli $P \in \Xi$ of the form

$$P = (-1)^v \bigotimes_{i=1}^8 D^{(i)},$$

where $D^{(i)} \in \{X, Z\}$, define reflection

$$\hat{P} = (-1)^v \bigotimes_{i=1}^8 \hat{D}^{(i)},$$

by replacing the X and Z 's with $\hat{X}^{(i)}$ and $\hat{Z}^{(i)}$ from the strategy respectively. For a strategy $\mathfrak{S} = (\rho, \{\hat{D}^{(i)}\})$, its value of the game can be expressed as

$$\frac{1}{32} \sum_{P \in \Xi} \text{tr}_\rho \frac{\mathbb{1} + \hat{P}}{2}. \quad (11)$$

If strategy \mathfrak{S} has value $1 - \epsilon$, we have

$$\text{tr}_\rho \hat{P} \approx_\epsilon 1,$$

for each operator $P \in \Xi$.

These conditions for the four operators in Fig. 2 and a repeated application of Lemma 11 conclude that

$$\operatorname{Re} \operatorname{tr}_\rho (\widehat{g}_2 \widehat{g}_1 \widehat{g}_{1,3} \widehat{g}_{2,3}) \geq 1 - O(\epsilon).$$

Observing that, as in Fig. 2, all the reflections cancel out for all players except those for player (2), we have

$$\operatorname{Re} \operatorname{tr}_\rho (\widehat{Z}^{(2)} \widehat{X}^{(2)} \widehat{Z}^{(2)} \widehat{X}^{(2)}) \approx_\epsilon -1.$$

Lemma 14 then proves the first item for $i = 2$. A similar argument proves the case for $i = 1$ by considering

$$\operatorname{Re} \operatorname{tr}_\rho (\widehat{g}_2 \widehat{g}_{1,3} \widehat{g}_1 \widehat{g}_{2,3}).$$

The symmetry of the game then completes the proof of the first item for all $i \in [8]$.

To prove the second item, consider strategy $\check{\mathfrak{S}} = (\rho, \{\check{D}^{(i)}\})$ where $\check{D}^{(i)}$ are reflections as defined in the first item of the theorem for $D \in \{X, Z\}$. By the claim in the first item, Lemma 11 and the expression for the game value in Eq. (11), it is easy to see that the value of strategy $\check{\mathfrak{S}}$ is at least $1 - O(\epsilon)$. That is

$$\frac{1}{32} \operatorname{tr}_{\rho'} \sum_{P \in \Xi} P \geq 1 - O(\epsilon)$$

where $\rho' = V\rho V^*$. It is easy to see that the operator $\sum_{P \in \Xi} P$ has the code space as its eigenspace of eigenvalue 32, and all other eigenvalues are at most 0. It then follows that $\sum_{P \in \Xi} P \leq 32\Pi$, where Π is the projection to the code space. A direct calculation then proves the second item of the theorem. \square

3.2 Multi-Qubit Stabilizer Game

In this section, we consider a multi-qubit variant of the stabilizer game called the (n, k) -stabilizer game. It is again an eight-player game and the referee sends questions in the form of measurement instructions to the players. The players are expected to hold a quantum register of n qubits and follow the measurement instructions that encode what quantum measurements the honest players are supposed to perform.

A (n, k) -stabilizer game was defined in [28] where the players receive instructions of at most k single-qubit measurement. The (n, k) -stabilizer game considered here is more general in the sense that the measurement instructions to the player may include a set of k pairwise commuting XZ -form Pauli operators of weight at most k . For example, in the case of $k = 2$, a possible question may be $\{X_1 X_2, Z_1 Z_2\}$, asking the player to measure both $X_1 X_2$ and $Z_1 Z_2$ simultaneously, and reply with the two measurement outcome bits. In this section, we will follow the convention that the subscripts for X, Z are the index for the qubits these operators act on.

Let $\mathbb{P}_{n,k}$ be the set of XZ -form Pauli operators on n qubits of weight at most k . Let $\mathbb{Q}_{n,k}$ be the collection of size- k subsets of $\mathbb{P}_{n,k}$ of pairwise commuting operators, each of which acts on the same set of k qubits. The sizes of $\mathbb{P}_{n,k}$ and $\mathbb{Q}_{n,k}$ are at most polynomial in n for constant k . The measurement specification that the players receive will be either a single Pauli operator $P \in \mathbb{P}_{n,k}$ or a set $Q \in \mathbb{Q}_{n,k}$. In the first case, the player is supposed to measure P and respond with a single bit, while in the latter case, the player is supposed to measure all operators in Q and reply with k outcome bits. Pauli operators of weight one in $\mathbb{P}_{n,1}$ is usually denoted as D_u for qubit index $u \in [n]$ and $D_u \in \{X_u, Z_u\}$.

The (n, k) -stabilizer game is given in Fig. 4. It is easy to see that the nonlocal value of the game equals one, which can be achieved by players who share an correctly encoded state and follow

Multi-Qubit Stabilizer Game

Let $[n]$ be the index of n qubits and let $k \geq 2$ be a constant. The (n, k) -stabilizer game is an eight-player nonlocal game where the referee does the following with equal probability:

1. *Stabilizer Check.* The referee plays the stabilizer game on a randomly selected qubit. That is, he
 - (a) Samples a qubit $u \in [n]$ uniformly at random; samples $D_u^{(i)} \in \{X_u, Z_u\}$ as in the stabilizer game.
 - (b) Sends $D_u^{(i)}$ to player (i) and receive an answer bit $a^{(i)}$.
 - (c) Accepts if the referee for the stabilizer game accepts on questions $D_u^{(i)}$ and answers $a^{(i)}$.
2. *Confusion Check.* The referee plays the stabilizer game but confuses one of the players by hiding the index of the qubit the stabilizer game checks against in a set of qubits.
 - (a) The referee selects a subset $J \subset [n]$ of size k , an index $u \in J$, and a player $t \in [8]$, all uniformly at random. For each qubit $v \in J$, indecently samples questions $D_v^{(i)} \in \{X_v, Z_v\}$ as in the stabilizer game.
 - (b) Sends $D_u^{(i)}$ to player (i) and receives an answer bit $a^{(i)}$ if $i \neq t$; sends $Q = \{D_v^{(t)}\}_{v \in J}$ to player (t) , and receives a k -bit string $b = (b_v)_{v \in J}$. Define $a^{(t)} = b_u$ and $a = (a^{(1)}, a^{(2)}, \dots, a^{(8)})$.
 - (c) Accepts if and only if the referee for the stabilizer game accepts when the questions are D_u and answer bits are a .
3. *Parity Check.* The referee tests the consistency of a multi-qubit XZ-form Pauli measurement and individual X and Z measurements.
 - (a) Samples $P \in \mathbb{P}_{n,k}$ and $t \in [8]$ uniformly at random. Let J be the support of P and $P = \prod_{v \in J} D_v$ for $D_v \in \{X_v, Z_v\}$. If $|J| < k$, randomly chooses $J' \supset J$ of size k and randomly choose $D_v \in \{X_v, Z_v\}$ for $v \in J' \setminus J$.
 - (b) Sends $Q = \{D_v\}_{v \in J'}$ to player (i) and receives k bits $(a_v^{(i)})_{v \in J'}$ if $i \neq t$; sends P to player (t) and receive a bit $a^{(t)}$. Define $a^{(i)} = \bigoplus_{v \in J'} a_v^{(i)}$ for $i \neq t$.
 - (c) Accepts if $\bigoplus_{i=1}^8 a^{(i)} = 0$; rejects otherwise.
4. *Pauli Check.* The referee tests the consistency of answers between multiple and single Pauli questions.
 - (a) Samples $Q \in \mathbb{Q}_{n,k}$, $P \in Q$ and $t \in [8]$ uniformly at random.
 - (b) Sends P to player (i) and receives a bit $a^{(i)}$ if $i \neq t$; sends Q to player (t) and receive a k -bit answer $(a_{P'}^{(t)})_{P' \in Q}$. Define $a^{(t)} = a_P^{(t)}$.
 - (c) Accepts if $\bigoplus_{i=1}^8 a^{(i)} = 0$; rejects otherwise.

Figure 4: Multi-qubit stabilizer game.

measurement specifications honestly. Let \mathcal{R}_i be the state space of player (i) . A strategy for the k -qubit stabilizer game,

$$\mathfrak{S} = (\rho, \{R_P^{(i)}\}, \{M_Q^{(i)}\}),$$

consists of a state $\rho \in D(\otimes_{i=1}^r \mathcal{R}_i)$, reflections $R_P^{(i)}$ the players measure for question P and measurements $M_Q^{(i)}$ with k -bit outcomes for question Q . The superscripts of the measurements indexing the players are sometimes omitted if there will be no ambiguity.

The reflection R_P and measurement M_Q in the strategy will sometimes be denoted as \hat{P} and \hat{Q} respectively. Without loss of generality, it is assumed that the measurements \hat{Q} are projective measurements and each measurement operator has the same rank. For $P \in Q$, define the derived reflections

$$\widehat{P|Q} = \sum_{b \in \{0,1\}^Q} (-1)^{b_P} \hat{Q}^b.$$

By the assumption on measurement \hat{Q} , the derived reflections are traceless.

We prove the following rigidity property of the (n, k) -stabilizer game.

Theorem 17. *For any constant integer $k \geq 2$, there exists a constant $\kappa > 0$ that depends only on k such that the (n, k) -stabilizer game in Fig. 4 has the following rigidity property. For any quantum strategy $\mathfrak{S} = (\rho, \{\hat{P}^{(i)}\}, \{\hat{Q}^{(i)}\})$ that has value at least $1 - \epsilon$, there are isometries $V_i \in L(\mathcal{R}_i, \mathcal{B}_i^{\otimes n} \otimes \mathcal{R}'_i)$, such that the following properties hold*

- For all $i \in [8]$, $P \in \mathbb{P}_{n,k}$, and $Q \in \mathbb{Q}_{n,k}$,

$$d_\rho(\hat{P}^{(i)}, \check{P}^{(i)}) \leq O(n^\kappa \epsilon^{1/\kappa}), \quad (12a)$$

$$d_\rho(\hat{Q}^{(i)}, \check{Q}^{(i)}) \leq O(n^\kappa \epsilon^{1/\kappa}), \quad (12b)$$

where $\check{P}^{(i)} = V_i^*(P \otimes \mathbb{1})V_i$ and $\check{Q}^{(i)}$ is the measurement that first performs isometry V_i and then measures the k Pauli operators in Q .

- Let Π be the projection to the code space of the stabilizer code, V be the isometry $\otimes_{i=1}^r V_i$, then

$$\langle \Pi^{\otimes n} \otimes I, V\rho V^* \rangle \geq 1 - O(n^\kappa \epsilon^{1/\kappa}), \quad (13)$$

where the t -th tensor factor of $\Pi^{\otimes n}$ acts on eight qubits, each of which is the t -th qubit of each player's system after the application of V .

The proof of Theorem 17 relies on the following lemmas.

Lemma 18. *Let $\rho \in D(\mathcal{X} \otimes \mathcal{Y})$ be a quantum state, $R_1, R_2, S_1, S_2 \in \text{Herm}(\mathcal{X})$ be four reflections on \mathcal{X} , and $U_1, U_2 \in \text{Herm}(\mathcal{Y})$ be two reflections on \mathcal{Y} . If S_1, S_2 commute, both R_1, S_1 are ϵ -consistent with U_1 , and both R_2, S_2 are ϵ -consistent with U_2 , then*

$$\text{Re tr}_\rho(R_1 R_2 R_1 R_2) \approx_\epsilon 1. \quad (14)$$

Proof. First, by the commutativity of S_1, S_2 , we have

$$\text{Re tr}_\rho(S_1 S_2 S_1 S_2) = 1.$$

Lemma 11 and the ϵ -consistency of R_1, S_1 with U_1 then imply

$$\text{Re tr}_\rho \left[(S_1 \otimes U_1) ((S_1 S_2 S_1 S_2) \otimes \mathbb{1}_\mathcal{Y}) (R_1 \otimes U_1) \right] \approx_\epsilon 1,$$

which simplifies to

$$\text{Re tr}_\rho(S_2 S_1 S_2 R_1) \approx_\epsilon 1.$$

That is, we can move S_1 in the front to the end and replace it with R_1 without causing too much error in the expression. Repeating similar arguments three more times, we have

$$\operatorname{Re} \operatorname{tr}_\rho(R_1 R_2 R_1 R_2) \approx_\epsilon 1.$$

□

Lemma 19. Let $\mathcal{X}, \mathcal{Y}, \mathcal{B}$ be two-dimensional Hilbert spaces. Let $V \in L(\mathcal{R}, \mathcal{B} \otimes \mathcal{R}')$ be a unitary operator, $R \in L(\mathcal{R})$ be any operator and $|\Phi\rangle$ be the EPR state on $\mathcal{X} \otimes \mathcal{Y}$. Define isometry $W \in L(\mathcal{R}, \mathcal{X} \otimes \mathcal{Y} \otimes \mathcal{R})$ as

$$W = (\mathbb{1} \otimes V^*) \operatorname{SWAP}(|\Phi\rangle \otimes V),$$

where the SWAP acts on \mathcal{X} and \mathcal{B} . Then

$$W^* R W = \frac{1}{4} \sum_{i=0}^3 (V^*(\sigma_i \otimes \mathbb{1})V) R (V^*(\sigma_i \otimes \mathbb{1})V),$$

where $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ are the Pauli operators.

Lemma 20. Let $R_1, R_2, \dots, R_k \in \operatorname{Herm}(\mathcal{Y})$ be k pairwise commuting reflections, $V \in L(\mathcal{X}, \mathcal{Y})$ an isometry, $\rho \in \mathcal{D}(\mathcal{X} \otimes \mathcal{Z})$ a quantum state. Define operators $\check{R}_i = V^* R_i V \in \operatorname{Herm}(\mathcal{X})$. If \check{R}_i has ϵ -consistent reflections on \mathcal{Z} for $i \in [k]$, then

$$\operatorname{Re} \operatorname{tr}_\rho \left[V^* \left(\prod_{i=1}^k R_i \right) V \prod_{i=1}^k \check{R}_i \right] \approx_\epsilon 1.$$

Proof. We prove by induction on k . For $k = 1$, the claim follows from the fact that \check{R}_1 has ϵ -consistent reflections and Lemma 11. We now assume that the claim holds for $k - 1$ and prove it for k . By the induction hypothesis and Lemma 11,

$$\operatorname{Re} \operatorname{tr}_\rho \left[\check{R}_k V^* \left(\prod_{i=1}^{k-1} R_i \right) V \left(\prod_{i=1}^{k-1} \check{R}_i \right) \check{R}_k \right] \approx_\epsilon 1.$$

It therefore suffices to prove that the difference on the left hand sides of the above equation and the equation in the lemma is at most $O(\epsilon)$. The absolute value of the difference can be bounded by Cauchy-Schwarz inequality as follows

$$\begin{aligned} & \left| \operatorname{tr}_\rho \left[V^* \left(R_k (\mathbb{1} - V V^*) \prod_{i=1}^{k-1} R_i \right) V \left(\prod_{i=1}^k \check{R}_i \right) \right] \right| \\ & \leq \left(1 - \operatorname{tr}_\rho \check{R}_k^2 \right)^{1/2} \left[1 - \operatorname{tr}_\rho \left| V^* \left(\prod_{i=1}^{k-1} R_i \right) V \left(\prod_{i=1}^k \check{R}_i \right) \right|^2 \right]^{1/2}. \end{aligned} \tag{15}$$

It follows from the induction hypothesis and Lemma 11 that both terms in the square roots on the right hand side are at most $O(\epsilon)$. □

Lemma 21. Let $M = \{M^a\}$ be a projective measurement of k -bit outcome on quantum register X and R_1, R_2, \dots, R_k be its derived reflections. Let $N = \{N^a\}$ be a projective measurement of k -bit outcome on quantum register Y and S_1, S_2, \dots, S_k be its derived reflections. Let $V \in L(\mathcal{X}, \mathcal{Y})$ be an isometry and

$\rho \in \mathcal{D}(\mathcal{X} \otimes \mathcal{Z})$ a quantum state. For $i \in [k]$, define $\check{S}_i = V^* S_i V \in \text{Herm}(\mathcal{X})$. Let \check{N} be the quantum measurement that measures N after the application of isometry V .

If $\text{Re tr}_\rho(R_i \check{S}_i) \approx_\epsilon 1$ and R_i has ϵ -consistent reflections on \mathcal{Z} for $i \in [k]$, then

$$d_\rho(M, \check{N}) \leq O(\epsilon^{1/2}).$$

Proof. By the definition of measurement \check{N} , we have

$$\begin{aligned} \check{N}^a &= V^* \left[\prod_{i=1}^k \frac{\mathbb{1} + (-1)^{a_i} S_i}{2} \right] V \\ &= \frac{1}{2^k} \sum_{x \in \{0,1\}^k} (-1)^{\langle a, x \rangle} V^* \left(\prod_{i=1}^k S_i^{x_i} \right) V. \end{aligned}$$

This implies that

$$\begin{aligned} \sum_a \text{Re tr}_\rho(M^a \check{N}^a) &= \frac{1}{2^{2k}} \sum_{a,x,y} (-1)^{\langle a, x \oplus y \rangle} \text{Re tr}_\rho \left[\left(\prod_{i=1}^k R_i^{x_i} \right) \left(V^* \prod_{i=1}^k S_i^{y_i} V \right) \right] \\ &= \frac{1}{2^k} \sum_x \text{Re tr}_\rho \left[\left(\prod_{i=1}^k R_i^{x_i} \right) V^* \left(\prod_{i=1}^k S_i^{x_i} \right) V \right]. \end{aligned}$$

Note that in this proof, the superscript of an operator represents the corresponding power of the operator and is not the index for measurement outcome as in the other parts of the paper.

We claim that for all $x \in \{0,1\}^k$, the term in the summand

$$\text{Re tr}_\rho \left[\left(\prod_{i=1}^k R_i^{x_i} \right) V^* \left(\prod_{i=1}^k S_i^{x_i} \right) V \right] \approx_\epsilon 1.$$

This will conclude the proof by the definition of d_ρ for POVMs by choosing $\{VM^a\}$ and

$$\left\{ \left[\prod_{i=1}^k \frac{\mathbb{1} + (-1)^{a_i} S_i}{2} \right] V \right\}$$

as the measurement operators for the two POVMs M and \check{N} respectively. We prove this claim by an induction on k . For $k = 1$, the claim is exactly the condition $\text{Re tr}_\rho(R_1 \check{S}_1) \approx_\epsilon 1$. Assume now the claim holds for $k - 1$, and we prove the case for k .

We have

$$\text{Re tr}_\rho \left[\left(\prod_{i=1}^k R_i^{x_i} \right) V^* \left(\prod_{i=1}^{k-1} S_i^{x_i} \right) V R_k^{x_k} \right] \approx_\epsilon 1, \quad (16a)$$

$$\text{Re tr}_\rho \left[\left(\prod_{i=1}^k R_i^{x_i} \right) V^* \left(\prod_{i=1}^{k-1} S_i^{x_i} \right) V \check{S}_k^{x_k} \right] \approx_\epsilon 1, \quad (16b)$$

where the first approximation follows from Lemma 11 and the induction hypothesis, the second approximation follows from the condition $\text{Re tr}_\rho(R_k \check{S}_k) \approx_\epsilon 1$. Then, by the use of Cauchy-Schwarz inequality as in Eq. (15), it follows that

$$\text{Re tr}_\rho \left[\left(\prod_{i=1}^k R_i^{x_i} \right) V^* \left(\prod_{i=1}^k S_i^{x_i} \right) V \right] \approx_\epsilon 1.$$

□

Proof of Theorem 17. We first prove the claims in Eqs. (12a) and (12b) of the theorem for Pauli operators $D_u \in \mathbb{P}_{n,k}$ of weight one and $Q \in \mathbb{Q}_{n,k}$ containing k such operators acting on k different qubits. For this, we only need to consider the first two tests of the game and the analysis is almost the same as the proof for the multi-qubit stabilize game defined in [28]. We prove the claim for player (t), which suffices to conclude the claims for all players by the symmetry of the game.

In the proof, we will refer to operators with different accents. Reflections with a hat are from the measurement strategy of the players, reflections with a tilde represent intermediate operators, and operators with a check correspond honest players' measurements up to an isometry. Our goal is therefore bound the distance between operators with a hat to those with a check. Keeping this convention in mind may help understanding the proof.

As the strategy \mathfrak{S} has value at least $1 - \epsilon$, the referee rejects in the *Stabilizer Check* with probability at most 4ϵ . It follows that, for $u \in [n]$, strategy $\mathfrak{S}_u = (\rho, \{\hat{D}_u^{(i)}\})$ has value at least $1 - \epsilon_1$ in the eight-qubit stabilizer game in Fig. 3 for $\epsilon_1 = 4n\epsilon$. By the rigidity of the stabilizer game (Theorem 16), there exist unitary operators $V_u \in L(\mathcal{R}_t, \mathcal{B}_u \otimes \tilde{\mathcal{R}}_t)$ for $u \in [n]$ such that

$$\hat{Z}_u = \tilde{Z}_u, \quad \text{Re tr}_\rho(\hat{X}_u \tilde{X}_u) \approx_{\epsilon_1} 1, \quad (17)$$

where $\tilde{D}_u = V_u^*(D \otimes \mathbb{1})V_u$ for $D \in \{X, Z\}$ and $u \in [n]$. Note that we have omitted the superscript for the player index for simplicity.

Define a new strategy \mathfrak{S}'_u which is the same as \mathfrak{S}_u except that reflections \hat{D}_u are replaced with \tilde{D}_u for player (t). It then follows by Lemma 11 and Eq. (17) that strategy \mathfrak{S}'_u has value at least $1 - O(\epsilon_1)$ in the eight-qubit stabilizer game. As $X^{\otimes 8}$ and $Z^{\otimes 8}$ are both in the set Ξ , it follows that \tilde{D}_u and $\otimes_{i \neq t} \hat{D}_u^{(i)}$ are $O(\epsilon_1)$ -consistent.

Similarly, for all $D_u \in Q \in \mathbb{Q}_{n,k}$ where Q contains k Pauli operators of weight one, consider the state ρ , reflections $\hat{D}_u^{(i)}$ for $i \neq t$ and reflection $\widehat{D}_u|Q$ for player (t). By the test in *Confusion Check*, they form a strategy for the stabilizer game with value at least $1 - O(\epsilon_2)$ where $\epsilon_2 = n^k \epsilon$. It follows that $\widehat{D}_u|Q$ and $\otimes_{i \neq t} \hat{D}_u^{(i)}$ are $O(\epsilon_2)$ -consistent.

For any Q that contains D_u, D_v , applying Lemma 18 with \tilde{D}_u, \tilde{D}_v as R_1, R_2 , $\widehat{D}_u|Q, \widehat{D}_v|Q$ as S_1, S_2 , and $\otimes_{i \neq t} \hat{D}_u^{(i)}, \otimes_{i \neq t} \hat{D}_v^{(i)}$ as U_1 and U_2 , we have for all $u \neq v \in [n]$,

$$\text{Re tr}_\rho(\tilde{D}_u \tilde{D}_v \tilde{D}_u \tilde{D}_v) \approx_{\epsilon_2} 1. \quad (18)$$

Let X, Y be quantum registers with n qubits each. For $u \in [n]$, define $|\Phi\rangle_u$ to be the EPR state between the qubits (X, u) and (Y, u) , and define isometry $W_u \in L(\mathcal{R}_t, \mathcal{X}_u \otimes \mathcal{Y}_u \otimes \mathcal{R}_t)$ as

$$W_u = (\mathbb{1} \otimes V_u^*) \text{SWAP}_u(|\Phi\rangle_u \otimes V_u),$$

where SWAP_u is the SWAP gate acting on qubits (X, u) and the first output qubit B_u of V_u .

Define isometry $V \in L(\mathcal{R}_t, \mathcal{X} \otimes \mathcal{Y} \otimes \mathcal{R}_t)$ as the sequential application of W_1, W_2, \dots, W_n ,

$$V = W_n W_{n-1} \cdots W_1. \quad (19)$$

We claim that this choice of V works for the stated claims by taking \mathcal{R}'_t to be $\mathcal{Y} \otimes \mathcal{R}_t$. Define operators

$$\check{D}_u = V^*(D_u \otimes \mathbb{1})V \text{ for } D_u \in \{X_u, Z_u\}. \quad (20)$$

As D_u and W_v commute for all $v > u$, we have

$$\check{D}_u = W_1^* W_2^* \cdots W_{u-1}^* \tilde{D}_u W_{u-1} W_{u-2} \cdots W_1.$$

For each $u \in [n]$, define a quantum channel

$$\mathfrak{T}_u(\rho) = \frac{\rho + \tilde{X}_u \rho \tilde{X}_u + \tilde{Z}_u \rho \tilde{Z}_u + \tilde{X}_u \tilde{Z}_u \rho \tilde{Z}_u \tilde{X}_u}{4}. \quad (21)$$

By a series of applications of Lemma 19,

$$\check{D}_u = \mathfrak{T}_1 \circ \mathfrak{T}_2 \circ \cdots \circ \mathfrak{T}_{u-1}(\tilde{D}_u). \quad (22)$$

We remark that the channel \mathfrak{T}_u is exactly the depolarizing channel with respect to the qubit defined by the anti-commuting pair of reflections \tilde{X}_u and \tilde{Z}_u . The expression in Eq. (22) states that the operators \check{D}_u defined in Eq. (20) using EPR states and SWAP can be constructed by sequentially depolarizing the qubits defined by reflections \tilde{X}_v and \tilde{Z}_v for $v = u-1, u-2, \dots, 1$.

We then claim that for all $v \leq u \in [n]$

$$\text{Re tr}_\rho(\tilde{D}_u \mathfrak{T}_v \circ \mathfrak{T}_{v+1} \circ \cdots \circ \mathfrak{T}_{u-1}(\tilde{D}_u)) \approx_{(u-v)^2 \epsilon_2} 1. \quad (23)$$

We prove the claim by induction on v . For $v = u$, the claim follows from the fact that \tilde{D}_u is a reflection. Assume that the statement holds for some $v > 1$ and we prove the case for $v-1$. Define for simplicity

$$R_v = \mathfrak{T}_v \circ \mathfrak{T}_{v-1} \circ \cdots \circ \mathfrak{T}_{u-1}(\tilde{D}_u),$$

and the induction hypothesis becomes

$$\text{Re tr}_\rho(\tilde{D}_u R_v) \approx_{(u-v)^2 \epsilon_2} 1.$$

For $v-1$, we have by the induction hypothesis, the existence of consistency reflections for \tilde{X}_{v-1} , Eq. (18) and Lemma 11,

$$\text{Re tr}_\rho(\tilde{D}_u \tilde{X}_{v-1} R_v \tilde{X}_{v-1}) \approx_{(u-v+1)^2 \epsilon_2} 1.$$

By a similar argument that adds \tilde{Z}_{v-1} and $\tilde{X}_{v-1} \tilde{Z}_{v-1}$ in the expression, we have

$$\text{Re tr}_\rho(\tilde{D}_u \mathfrak{T}_{v-1}(R_v)) \approx_{(u-v+1)^2 \epsilon_2} 1,$$

which proves the claim in Eq. (23) for $v-1$.

Taking $v = 1$ in Eq. (23), we have for all $u \in [n]$,

$$\text{Re tr}_\rho(\tilde{D}_u \check{D}_u) \approx_{n^2 \epsilon_2} 1.$$

Lemma 11 and Eq. (17) then imply that, for $\epsilon_3 = n^2 \epsilon_2$,

$$\text{Re tr}_\rho(\hat{D}_u \check{D}_u) \approx_{\epsilon_3} 1, \quad (24)$$

which proves the approximation in Eq. (12a) for Pauli operator of weight one by choosing κ large enough.

We then prove Eq. (12b) for the case where $Q \subset \mathbb{P}_{n,1}$, containing commuting Pauli operators of weight one only. First recall that

$$\operatorname{Re} \operatorname{tr}_\rho \left(\widehat{D}_u \otimes \left(\bigotimes_{i \neq t} \widehat{D}_u^{(i)} \right) \right) \approx_{\epsilon_1} 1.$$

By Lemma 11 and the consistency analysis in the beginning of the proof, we have

$$\operatorname{Re} \operatorname{tr}_\rho \left(\widehat{D}_u \widehat{D}_u | Q \right) \approx_{\epsilon_2} 1.$$

By Eq. (24),

$$\operatorname{Re} \operatorname{tr}_\rho \left(\check{D}_u \check{D}_u | Q \right) \approx_{\epsilon_3} 1.$$

With Lemma 21, this proves the approximation in Eq. (12b) for $Q \subseteq \mathbb{P}_{n,1}$.

For the proof of Eq. (12a), we analyze the *Parity Check* part of the game. Consider a strategy \mathfrak{S}' that is the same as \mathfrak{S} but with reflections \widehat{D}_u replaced by \check{D}_u and measurements \widehat{Q} replaced by \check{Q} for $Q \subseteq \mathbb{P}_{n,1}$ for all players. It then follows by the claims proved above and the monotonicity of trace distance that the value of strategy \mathfrak{S}' is at least $1 - O(\epsilon_4)$ for $\epsilon_4 = n^k \epsilon_3^{1/2}$. For $i \in [8]$, and $P = \prod_{u \in J} D_u$, define operators

$$\tilde{P}^{(i)} \stackrel{\text{def}}{=} \prod_{u \in J} \check{D}_u^{(i)}, \quad \check{P}^{(i)} \stackrel{\text{def}}{=} V_i^* (P \otimes \mathbb{1}) V_i,$$

and write \tilde{P} for $\tilde{P}^{(t)}$, \check{P} for $\check{P}^{(t)}$. When strategy \mathfrak{S}' is used, the bit $a^{(i)}$ defined in the game corresponds to the measurement defined by $\check{P}^{(i)}$ for $i \neq t$.

As strategy \mathfrak{S}' has value at least $1 - O(\epsilon_4)$, the test in *Parity Check* implies that for $P \in \mathbb{P}_{n,k}$

$$\operatorname{Re} \operatorname{tr}_\rho \left(\widehat{P} \otimes \left(\bigotimes_{i \neq t} \check{P}^{(i)} \right) \right) \approx_{\epsilon_5} 1, \tag{25}$$

for $\epsilon_5 = |\mathbb{P}_{n,k}| \epsilon_4$. Similarly, the test in *Stabilizer Check* implies that

$$\operatorname{Re} \operatorname{tr}_\rho \left(\check{D}_u \otimes \left(\bigotimes_{i \neq t} \check{D}_u^{(i)} \right) \right) \approx_{\epsilon_5} 1.$$

By Lemma 11, this implies that

$$\operatorname{Re} \operatorname{tr}_\rho \left(\tilde{P} \otimes \left(\bigotimes_{i \neq t} \check{P}^{(i)} \right) \right) \approx_{\epsilon_5} 1.$$

By Lemma 20,

$$\operatorname{Re} \operatorname{tr}_\rho \left(\check{P}^{(i)} \cdot \tilde{P}^{(i)} \right) \approx_{\epsilon_3} 1,$$

for all $i \in [r]$ and therefore, by Lemma 13,

$$\operatorname{Re} \operatorname{tr}_\rho \left(\check{P} \otimes \left(\bigotimes_{i \neq t} \check{P}^{(i)} \right) \right) \approx_{r\epsilon_5} 1. \tag{26}$$

Together with Eq. (25) and this equation, Lemma 13 implies

$$\operatorname{Re} \operatorname{tr}_\rho \left(\widehat{P} \cdot \check{P} \right) \approx_{r\epsilon_5} 1,$$

which completes the proof for Eq. (12a).

We now prove the statement in (12b). Consider a strategy $\check{\mathfrak{S}}$ that is the same as \mathfrak{S} but replaces all reflection \widehat{P} with \check{P} . It then follows that the value of the strategy $\check{\mathfrak{S}}$ is at least $1 - O(n^k r^{1/2} \epsilon_5^{1/2})$. In strategy $\check{\mathfrak{S}}$, the players measures \check{P} for question P . The test in *Pauli Check* implies that for $\epsilon_6 = |\mathcal{Q}_{n,k}| n^k \epsilon_5^{1/2}$

$$\text{Re tr}_\rho \left(\widehat{P|Q} \otimes \left(\bigotimes_{i \neq t} \check{P}^{(i)} \right) \right) \approx_{\epsilon_6} 1.$$

Applying Lemma 13 again to this equation and Eq. (26), we have

$$\text{Re tr}_\rho (\widehat{P|Q} \cdot \check{P}) \approx_{\epsilon_6} 1,$$

which completes the proof for Eq. (12b) using Lemma 21.

The second part of the theorem follows from a similar argument used to prove the second part of Theorem 16. \square

We note that we haven't tried to optimize the dependence of the approximation error on n and ϵ as it is not important for our work. With a more careful analysis, it should be possible to have a much better dependence on these parameters.

4 Propagation Games, Constraint Propagation Games and Rigidity

4.1 Propagation Games

In this section, we define the propagation game, an extended nonlocal game that checks the propagation of a sequences of reflections. Let R_1, R_2, \dots, R_n be n reflections. Let $\mathfrak{R} = (R_{\zeta_i})_{i=1}^N$ be a sequence of reflections with indices $\zeta_i \in [n]$. The propagation game is an extended nonlocal game that checks the propagation of the reflection sequence \mathfrak{R} on the player's system.

Let $(v_i)_{i=0}^N$ be an increasing sequence of integers of length $N + 1$. The propagation graph $G = (V, E)$ of the reflections $\mathfrak{R} = (R_{\zeta_i})_{i=1}^N$ over the vertex sequence $(v_i)_{i=0}^N$ is the labeled graph with vertex set $V = \{v_i \mid i = 0, 1, \dots, N\}$, edge set $E = \{(v_{i-1}, v_i)\}$ and edge labels $\Gamma_e = R_{\zeta_i}$ for $e = (v_{i-1}, v_i)$.

For each propagation graph G , we can define a propagation game as in Fig. 5. In the game, the referee possesses a clock register S with associated Hilbert space \mathbb{C}^V . The question is sampled from the set $[n]$, the index set for the reflections. The player is expected to perform the two-outcome measurement corresponding to the reflection R_j for question j and reply with the measurement outcome. For each edge $e = (u, v) \in E$, define a projective measurement $\Pi_e = \{\Pi_e^0, \Pi_e^1, \Pi_e^2\}$ on S where

$$\begin{aligned} \Pi_e^0 &= \frac{(|u\rangle + |v\rangle)(\langle u| + \langle v|)}{2}, \\ \Pi_e^1 &= \frac{(|u\rangle - |v\rangle)(\langle u| - \langle v|)}{2}, \\ \Pi_e^2 &= \mathbb{1} - \Pi_e^0 - \Pi_e^1. \end{aligned} \tag{27}$$

We prove the following lemma for propagation games. Roughly, it says that the shared state has to be close to a history state of the computation defined by the sequence of the reflections.

Propagation Game

Let $G = (V, E)$ be the propagation graph defined above. The referee selects an edge $e \in E$ uniformly at random and sends the index $j \in [n]$ for edge label $\Gamma_e = R_j$ to the player and receives an answer bit a ; performs the projective measurement Π_e on register S and accepts if the outcome is 2 or is equal to a ; rejects otherwise.

Figure 5: The propagation game between a referee and a player.

Lemma 22. *The propagation game in Fig. 5 has nonlocal value 1. Furthermore, the following property holds. Let $\mathfrak{S} = (\rho, \{\hat{R}_j\})$ be a strategy with $\rho \in \mathcal{D}(\mathcal{S} \otimes \mathcal{R})$ and $\hat{R}_j \in \text{Herm}(\mathcal{R})$. Define $\hat{U} \in \mathcal{U}(\mathcal{S} \otimes \mathcal{R})$,*

$$\hat{U} = \sum_{i=0}^N |v_i\rangle\langle v_i| \otimes \hat{R}_{\zeta_i} \hat{R}_{\zeta_{i-1}} \cdots \hat{R}_{\zeta_1}, \quad (28)$$

and state $|V\rangle \in \mathcal{S}$

$$|V\rangle = \frac{1}{\sqrt{|V|}} \sum_{v \in V} |v\rangle. \quad (29)$$

If strategy \mathfrak{S} has value at least $1 - \epsilon$, then there exists a state $\rho'_R \in \mathcal{D}(\mathcal{R})$ such that

$$\mathcal{D}(\rho, \hat{U}(|V\rangle\langle V| \otimes \rho'_R) \hat{U}^*) \leq O(N^{3/2} \epsilon^{1/2}).$$

Proof. The player chooses an arbitrary state $|\psi\rangle \in \mathcal{R}$ and initialize the registers S and R in the state $U(|V\rangle \otimes |\psi\rangle)$ for

$$U = \sum_{i=0}^N |v_i\rangle\langle v_i| \otimes R_{\zeta_i} R_{\zeta_{i-1}} \cdots R_{\zeta_1},$$

He replies each question $j \in [n]$ with the outcome of the measurement defined by reflection R_j . A direct calculation then concludes the first part of the lemma.

We now proves the second part of the lemma. For edge e , let $\zeta_e \in [n]$ be the index of reflection for the label $\Gamma_e = R_{\zeta_e}$. For a strategy $\mathfrak{S} = (\rho, \{\hat{R}_j\})$, the referee rejects with probability

$$\begin{aligned} & \mathbb{E}_{e \in E(G)} \text{tr}_{\rho} \left(\Pi_e^0 \otimes \frac{\mathbb{1} - \hat{R}_{\zeta_e}}{2} + \Pi_e^1 \otimes \frac{\mathbb{1} + \hat{R}_{\zeta_e}}{2} \right) \\ &= \frac{1}{2N} \text{tr}_{\rho} \sum_{i=1}^N [(|v_{i-1}\rangle\langle v_{i-1}| + |v_i\rangle\langle v_i|) \otimes \mathbb{1} - (|v_{i-1}\rangle\langle v_i| + |v_i\rangle\langle v_{i-1}|) \otimes \hat{R}_{\zeta_i}] \\ &= \frac{1}{2N} \text{tr}_{\rho'} \sum_{i=1}^N (|v_{i-1}\rangle\langle v_{i-1}| + |v_i\rangle\langle v_i| - |v_{i-1}\rangle\langle v_i| - |v_i\rangle\langle v_{i-1}|) \\ &= \frac{1}{2N} \text{tr}_{\rho'} L(G), \end{aligned} \quad (30)$$

where $\rho' = \hat{U}^* \rho \hat{U}$ and the matrix $L(G)$ is the Laplacian matrix of graph G . As the strategy has value at least $1 - \epsilon$, we have

$$\text{tr}_{\rho'} L(G) \leq 2N\epsilon. \quad (31)$$

By standard techniques [56, 10], the matrix $L(G)$ has a zero eigenvalue with eigenvector $|V\rangle$ and all other eigenvalues are at least $1/(N+1)^2$. That is, for projection $\Pi_V = |V\rangle\langle V|$, we have

$$L(G) \geq \frac{\mathbb{1} - \Pi_V}{(N+1)^2}.$$

Together with Eq. (31), this proves that

$$\text{tr}_{\rho'}(\Pi_V) \geq 1 - 2N(N+1)^2\epsilon.$$

By Lemma 4, it follows that

$$D(\rho', |V\rangle\langle V| \otimes \rho'_R) \leq O(N^{3/2}\epsilon^{1/2}),$$

where $\rho'_R \in D(\mathcal{R})$ is defined as

$$\rho'_R = \frac{\langle V|\rho'|V\rangle}{\text{tr}_{\rho'}\Pi_V}.$$

It then follows that

$$D(\rho, \hat{U}(|V\rangle\langle V| \otimes \rho'_R)\hat{U}^*) \leq O(N^{3/2}\epsilon^{1/2}), \quad (32)$$

which completes the proof. \square

We introduce two types of extensions to the propagation game. The first type allows controlled reflections of the form $\Lambda_c(R_j)$ in the sequence of reflections for $j \in [n]$ and c the control qubit index of one of the referee's registers. In the second extension, the referee may confuse the player by asking him to perform k pairwise commuting reflections and answer k output bits, even though the referee is interested in the measurement outcome of one of the reflections. For this type of extension, the corresponding reflection in the reflection sequence will have the form $R_{j|q}$, where $j \in q$ and q a subset of $[n]$ of size k . The reflection sequence now becomes $\mathfrak{U} = (U_i)_{i=1}^N$ where each U_i has one of the following three possible forms:

1. $U_i = R_j$ for some reflection index $j \in [n]$,
2. $U_i = \Lambda_c(R_j)$ for $j \in [n]$ and c the control qubit index,
3. $U_i = R_{j|q}$ for $j \in q$ and q a subset of $[n]$ of size k .

In the propagation graph for the sequence $\mathfrak{U} = (U_i)_{i=1}^N$ over vertices $(v_i)_{i=0}^N$, the label for edge $e = (v_{i-1}, v_i)$ is $\Gamma_e = U_i$.

The propagation game is updated to accommodate the changes accordingly in Fig. 6. The referee possesses two registers, the clock register S as before and a register X containing the control qubits. The question to the player is either a $j \in [n]$ or a set $q \subset [n]$ of size k . The strategy of the player can be described by $\mathfrak{S} = (\rho, \{\hat{R}_j\}, \{M_q\})$ where \hat{R}_j is the reflection corresponding to the measurement the player performs for question j . Measurement $\{M_q\}$ is a projective measurement with k outcome bits for question q . For each $j \in q$, define derived reflections

$$\hat{R}_{j|q} = \sum_{b:q \rightarrow \{0,1\}} (-1)^{b(j)} M_q^b.$$

These k reflections then equivalently characterize the measurement $\{M_q\}$. The strategy can then be equivalently described by

$$\mathfrak{S} = (\rho, \{\hat{R}_j\}, \{\hat{R}_{j|q}\}).$$

Propagation Game (Extended)

Let $G = (V, E)$ be the propagation graph. The referee selects an edge $e \in E$ uniformly at random.

1. If $\Gamma_e = R_j$ for some $j \in [n]$, he sends j to the player and receives an answer bit a ; performs the projective measurement Π_e on register S and accepts if the outcome is 2 or is equal to a ; rejects otherwise.
2. If $\Gamma_e = \Lambda_c(R_j)$, sends j to the player and receives an answer bit a ; performs the projective measurement Π_e on register S and accepts if the outcome $t = 2$ and continue otherwise; measures $Z_{X,c}$ with outcome bit b and rejects if $b = 0, t = 1$ or $b = 1, a \oplus t = 1$; accepts otherwise.
3. Otherwise, for $\Gamma_e = R_{j|q}$, sends q to the player and receives answer $b : q \rightarrow \{0, 1\}$; performs the projective measurement Π_e on register V and accepts if the outcome is 2 or equals to $b(j)$; rejects otherwise.

Figure 6: The propagation game between a referee and a player.

We will refer to this generalization of the propagation game also as the propagation game as there will be no confusion. As Lemma 22, the following lemma holds for this extended version of propagation games.

Lemma 23. *The propagation game with controlled reflections in Fig. 6 has nonlocal value 1. Furthermore, the following property holds. Let $\mathfrak{S} = (\rho, \{\hat{R}_j\}, \{\hat{R}_{j|q}\})$ be a strategy with $\rho \in \mathcal{D}(\mathcal{S} \otimes \mathcal{X} \otimes \mathcal{R})$, $\hat{R}_j \in \text{Herm}(\mathcal{R})$ and $\hat{R}_{j|q}$ the derived reflections from the strategy. Define $\hat{U} \in \mathcal{U}(\mathcal{S} \otimes \mathcal{X} \otimes \mathcal{R})$*

$$\hat{U} = \sum_{i=0}^N |v_i\rangle\langle v_i| \otimes \hat{U}_i \hat{U}_{i-1} \cdots \hat{U}_1, \quad (33)$$

where $\hat{U}_i = \mathbb{1}_{\mathcal{X}} \otimes \hat{R}_j$ for $U_i = R_j$, $\hat{U}_i = \Lambda_{X,c}(\hat{R}_j)$ for $U_i = \Lambda_c(R_j)$ and $\hat{U}_i = \mathbb{1}_{\mathcal{X}} \otimes \hat{R}_{j|q}$ for $U_i = R_{j|q}$. Define state $|V\rangle \in \mathcal{S}$

$$|V\rangle = \frac{1}{\sqrt{|V|}} \sum_{v \in V} |v\rangle.$$

If strategy \mathfrak{S} has value at least $1 - \epsilon$, then there exists a state $\rho'_{XR} \in \mathcal{D}(\mathcal{X} \otimes \mathcal{R})$ such that

$$\mathcal{D}\left(\rho, \hat{U}(|V\rangle\langle V| \otimes \rho'_{XR}) \hat{U}^*\right) \leq O(N^{3/2} \epsilon^{1/2}).$$

Proof. If $\Gamma_e = \Lambda_c(R_j)$, the referee rejects with probability

$$\text{tr}_\rho \left[|0\rangle\langle 0|_{X,c} \otimes \Pi_e^1 \otimes \mathbb{1}_R + |1\rangle\langle 1|_{X,c} \otimes \left(\Pi_e^1 \otimes \frac{\mathbb{1} + \hat{R}_j}{2} + \Pi_e^0 \otimes \frac{\mathbb{1} - \hat{R}_j}{2} \right) \right].$$

If $e = (v_{i-1}, v_i)$, a direct calculation simplifies the above expression to

$$\frac{1}{2} \text{tr}_\rho \left[(|v_i\rangle\langle v_i| + |v_{i-1}\rangle\langle v_{i-1}|) \otimes \mathbb{1}_{XR} - (|v_i\rangle\langle v_{i-1}| + |v_{i-1}\rangle\langle v_i|) \otimes \Lambda_{X,c}(\hat{R}_j) \right].$$

Similarly, for $\Gamma_e = R_{j|q}$, the referee rejects with probability

$$\frac{1}{2} \text{tr}_\rho [(|v_i\rangle\langle v_i| + |v_{i-1}\rangle\langle v_{i-1}|) \otimes \mathbb{1}_R - (|v_i\rangle\langle v_{i-1}| + |v_{i-1}\rangle\langle v_i|) \otimes \widehat{R}_{j|q}].$$

Note that even though the player does not know j when the referee sampled $\Gamma_e = R_{j|q}$, he can nevertheless prepare the initial state that depends on $R_{j|q}$ as the propagation graph of the game is known to the player.

The proof now follows along similar lines as in the proof for Lemma 22. \square

4.2 Constraint Propagation Game

We define an extended nonlocal game, called the constraint propagation game, for any system of product-form reflection constraints. A system of product-form reflection constraints is described by a set of n reflections R_1, R_2, \dots, R_n and a set of m operator identities C_1, C_2, \dots, C_m where each C_i is of the form

$$R_{j_{i,1}} R_{j_{i,2}} \cdots R_{j_{i,n_i}} = (-1)^{\tau_i} \mathbb{1},$$

for $j_{i,1}, j_{i,2}, \dots, j_{i,n_i} \in [n]$ and $\tau_i \in \{0, 1\}$. We will sometimes abuse the notion and also use C_i to denote the product

$$R_{j_{i,1}} R_{j_{i,2}} \cdots R_{j_{i,n_i}},$$

so the constraints become $C_i = (-1)^{\tau_i} \mathbb{1}$. Let $N_i = \sum_{j=1}^i n_j$ be the total number of occurrences of the reflections in C_1, C_2, \dots, C_i . Define $N = N_m$ and $N_0 = 0$ for notational convenience. The number N is referred to as the size of the constraint system.

We say that the constraint system has a *quantum satisfying assignment* if there exists an Hilbert space and n reflections acting on the space such that all constraints are satisfied. These definitions are closely related to but different from the concept of binary constraint system games introduced in [13]. The formulation here allows more flexible ways of specifying relations between reflections and do not enforce commutativity between reflections occurring in the same constraint. For example, we allow the constraints of the form

$$R_1 R_2 R_1 R_2 = \mathbb{1}, \quad R_1 R_2 R_1 R_2 = -\mathbb{1},$$

representing the commutativity and anti-commutativity of two reflections R_1 and R_2 respectively, while they are not explicitly available in binary constraint system games.

For later convenience, we allow the derived reflections in the constraints. In this case, each constraint is now of the form

$$U_{i,1} U_{i,2} \cdots U_{i,n_i} = (-1)^{\tau_i} \mathbb{1},$$

where $U_{i,j}$ is either one of the n reflections R_j or of the form $R_{j|q}$ for $j \in q$ and $q \subset [n]$ of constant size. In an satisfying assignment of the constraint system, we require additionally that the reflections assigned to $R_{j|q}$ and $R_{j'|q}$ for $j, j' \in q$ commute. But we do not require that $R_{j|q}$ is related to R_j in any way at this moment.

Let $\mathfrak{U} = (U_i)_{i=1}^N$ be the sequence of all the reflections in C_1, C_2, \dots, C_m in the order that they occur. More specifically, for $1 \leq i \leq m$ and $N_{i-1} < v \leq N_i$, $U_v = U_{i,v-N_{i-1}}$. Let graph $G_{\text{prop}} = (V_{\text{prop}}, E_{\text{prop}})$ be the propagation graph of the sequence \mathfrak{U} over the vertices $(i)_{i=0}^N$, and graph $G_{\text{cons}} = (V_{\text{cons}}, E_{\text{cons}})$ be the propagation graph of the sequence $((-1)^{\tau_i} \mathbb{1})_{i=1}^m$ over the vertices $(N_i)_{i=0}^m$. Define the constraint graph of the constraint system as $G = (V, E)$ with vertex set

$V = V_{\text{prop}}$ and $E = E_{\text{prop}} \cup E_{\text{cons}}$. More explicitly, the vertex set is $V = \{0, 1, \dots, N\}$, edge sets E_{prop} and E_{cons} are

$$\begin{aligned} E_{\text{prop}} &= \{(i-1, i) \mid 1 \leq i \leq N\}, \\ E_{\text{cons}} &= \{(N_{i-1}, N_i) \mid 1 \leq i \leq m\}. \end{aligned}$$

Edge $e = (i-1, i) \in E_{\text{prop}}$ is labeled by $\Gamma_e = U_i$. Edge $e = (N_{i-1}, N_i) \in E_{\text{cons}}$ is labeled by the operator $\Gamma_e = (-1)^{\tau_i} \mathbb{1}$ on the right hand side of the constraint C_i .

The constraint propagation game defined by the system of reflection constraints is an extended nonlocal game between a referee and a player. The referee holds a quantum register \mathcal{S} associated with Hilbert space $\mathcal{S} = \mathbb{C}^V$ for V being the vertex set of graph G . The referee will either sample and send $j \in [n]$ and expect the player to perform the measurement represented by the corresponding reflection R_j and respond with the measurement outcome, or send $q \subset [n]$ and expect the player to perform all the measurement represented by the corresponding reflection with index in q and send back all the measurement outcomes. The constraint propagation game is given in Fig. 7. Intuitively, in the *Propagation Check* part, the referee checks the propagation of the reflections U_i for $i = 1, 2, \dots, N$ on the state of the player. In the *Constraint Check* part, the referee tests whether the constraints are satisfied assuming correct propagations enforced by the *Propagation Check*. Note that in this part, the referee does not ask any question to the player as the reflections are proportional to identities.

Constraint Propagation Game

Let $G, G_{\text{prop}}, G_{\text{cons}}$ be the constraint graph and two corresponding propagation graphs defined above. The referee does the following with equal probability:

1. *Propagation Check*. Plays the *Propagation Game* specified by the propagation graph G_{prop} .
2. *Constraint Check*. Selects an edge $e \in E_{\text{cons}}$ uniformly at random; measures Π_e on register V and accepts if the outcome is 2 or is equal to the sign τ in $\Gamma_e = (-1)^{\tau} \mathbb{1}$; rejects otherwise.

Figure 7: The constraint propagation game defined by a system of reflection constraints.

We prove the following lemma about constraint propagation games.

Lemma 24. *For any system of reflection constraints that has a quantum assignment, the nonlocal value of the corresponding propagation game is 1. Moreover, for any strategy $\mathfrak{S} = (\rho, \{\hat{R}_j\}, \{\hat{R}_{j|q}\})$ that has value at least $1 - \epsilon$, the constraints C_i 's are approximately satisfied in the following sense. Let N be the size of the constraint system. Let p_0 and $\rho_0 \in \mathcal{D}(\mathcal{R})$ be the probability and the post-measurement state if a computational basis measurement on \mathcal{S} is performed on ρ and the outcome is 0. Define*

$$\hat{C}_i = \hat{U}_{i,1} \hat{U}_{i,2} \cdots \hat{U}_{i,n_i}.$$

Then, there exists a constant κ such that,

$$\text{Re tr}_{\rho_0} \hat{C}_i \approx N^\kappa \epsilon^{1/\kappa} (-1)^{\tau_i}, \quad (34)$$

and

$$p_0 \approx N^\kappa \epsilon^{1/\kappa} \frac{1}{N+1}. \quad (35)$$

Proof. If the constraint system has a quantum assignment, choose \mathcal{R} to be the Hilbert space of the assignment and let $R_j, R_{j|n} \in \text{Herm}(\mathcal{R})$ be the reflections that satisfy all the constraints. The player chooses an arbitrary state $|\psi\rangle \in \mathcal{R}$ and initialize the registers S and R in the state $U(|V\rangle \otimes |\psi\rangle)$ for U and $|V\rangle$ defined as

$$U = \sum_{i=0}^N |v_i\rangle\langle v_i| \otimes U_i U_{i-1} \cdots U_1$$

and $|V\rangle = \sum_{v \in V} |v\rangle / \sqrt{|V|}$. He replies each question $j \in [n]$ with the outcome of the measurement defined by reflection R_j and replies question $q \subseteq [n]$ with the measurement outcomes of $R_{j|q}$ for all $j \in q$. A direct calculation then concludes the first part of the lemma.

We now prove the second part of the lemma. Let $\mathfrak{S} = (\rho, \{\hat{R}_j\}, \{\hat{R}_{j|q}\})$ be a strategy that has value at least $1 - \epsilon$. The referee rejects with probability at most 2ϵ in the *Propagation Check* and, by Lemma 23,

$$D(\rho, \tilde{\rho}) \leq O(\epsilon'), \quad (36)$$

for state $\tilde{\rho} = \hat{U}(|V\rangle\langle V| \otimes \rho'_R) \hat{U}^*$, $\rho'_R \in D(\mathcal{R})$, and $\epsilon' = N^{3/2} \epsilon^{1/2}$.

We now analyze the checking corresponding to the edges in E_{cons} . For edge $e_i = (N_{i-1}, N_i)$, define operator

$$H_i = \frac{1 - (-1)^{\tau_i}}{2} \Pi_{e_i}^0 + \frac{1 + (-1)^{\tau_i}}{2} \Pi_{e_i}^1,$$

which simplifies to,

$$H_i = \frac{1}{2} [|N_{i-1}\rangle\langle N_{i-1}| + |N_i\rangle\langle N_i| - (-1)^{\tau_i} |N_{i-1}\rangle\langle N_i| - (-1)^{\tau_i} |N_i\rangle\langle N_{i-1}|],$$

by the definitions of $\Pi_{e_i}^0$ and $\Pi_{e_i}^1$. Edge e_i is sampled with probability $1/2m$ in the game, and the probability that the referee rejects in this case is at most $2m\epsilon$ as the strategy has value at least $1 - \epsilon$. This implies that $\text{tr}_\rho H_i \leq 2m\epsilon$, and by Eq. (36),

$$\text{tr}_{\tilde{\rho}} H_i \leq O(\epsilon').$$

By the definition of \hat{U} , \hat{C}_i and $\tilde{\rho}$, this simplifies to

$$\text{Re tr}_{\rho'_R} (\hat{C}_1 \hat{C}_2 \cdots \hat{C}_{i-1} \hat{C}_i \hat{C}_{i-1}^* \cdots \hat{C}_1^*) \approx_{N\epsilon'} (-1)^{\tau_i}.$$

For $i_0 \leq i_1 \in [m]$, introduce the notion

$$\hat{C}_{i_0, i_1} = \hat{C}_{i_0} \hat{C}_{i_0+1} \cdots \hat{C}_{i_1},$$

and define $\hat{C}_{i_0, i_1} = \mathbb{1}$ if $i_0 > i_1$. The last equation then becomes, for $i \in [m]$,

$$\text{Re tr}_{\rho'_R} (\hat{C}_{1, i} \hat{C}_{1, i-1}^*) \approx_{N\epsilon'} (-1)^{\tau_i}. \quad (37)$$

Using Lemma 12, we have for $1 \leq j < i$,

$$\text{Re tr}_{\rho'_R} (\hat{C}_{1, j} \hat{C}_i \hat{C}_{1, j}^*) \approx_{\sqrt{N\epsilon'}} \text{Re tr}_{\rho'_R} [(\hat{C}_{1, j} \hat{C}_{1, j-1}^*)^* \hat{C}_{1, j} \hat{C}_i \hat{C}_{1, j}^* (\hat{C}_{1, j} \hat{C}_{1, j-1}^*)] = \text{Re tr}_{\rho'_R} (\hat{C}_{1, j-1} \hat{C}_i \hat{C}_{1, j-1}^*).$$

A repeated application of the above approximation gives

$$\text{Re tr}_{\rho'_R} \hat{C}_i \approx_{m\sqrt{N\epsilon'}} \text{Re tr}_{\rho'_R} (\hat{C}_{1, i-1} \hat{C}_i \hat{C}_{1, i-1}^*) \approx_{N\epsilon'} (-1)^{\tau_i}.$$

This proves the Eq. (34) in the lemma but with state ρ'_R instead of ρ_0 . To complete the proof it suffices to bound the distance between ρ'_R and ρ_0 .

Consider a computational basis measurement on system \mathcal{S} of state $\tilde{\rho}$. By the definition of state $\tilde{\rho}$, it is obvious that the state left on \mathcal{R} will be ρ'_R if the measurement outcome is 0. Let $\tilde{p}_0 = 1/(N+1)$ be the probability that outcome 0 happens. By the monotonicity of the trace distance and Eq. (36), $|p_0 - \tilde{p}_0| \leq \epsilon'$. This proves Eq. (35). Also by the monotonicity of the trace distance and the triangle inequality, we have

$$\begin{aligned} D(\rho_0, \rho'_R) &= \frac{N+1}{2} \|\tilde{p}_0 \rho_0 - \tilde{p}_0 \rho'_R\|_1 \\ &\leq \frac{N+1}{2} [\|\tilde{p}_0 \rho_0 - p_0 \rho_0\|_1 + \|p_0 \rho_0 - \tilde{p}_0 \rho'_R\|_1] \\ &\leq O(N\epsilon'), \end{aligned}$$

which completes the proof for Eq. (34). \square

4.3 Constraint Propagation Game for Multi-Qubit Pauli Operators

Consider the following constraint system satisfied by the Pauli operators of weight k on n qubits. The reflections under consideration will be those in $\mathbb{P}_{n,k}$. Let $P|Q$ be the same as reflection P . These reflections satisfy the constraints as follows:

1. $D_u D_v D_u D_v = \mathbb{1}$ for $u \neq v \in [n]$ and $D_u \in \{X_u, Z_u\}$ and $D_v \in \{X_v, Z_v\}$;
2. $X_u Z_u X_u Z_u = -\mathbb{1}$ for $u \in [n]$;
3. $D_v X_u Z_u X_u Z_u D_v = -\mathbb{1}$ for $u, v \in [n]$, and $D_v \in \{X_v, Z_v\}$;
4. $P \cdot \prod_{v \in J} D_v = \mathbb{1}$ for $P \in \mathbb{P}_{n,k}$, J the support of P and $P = \prod_{v \in J} D_v$;
5. $(P|Q)P = \mathbb{1}$ for $Q \in \mathbb{Q}_{n,k}$ and $P \in Q$.

We refer to this particular constraint system of reflections as the (n, k) -constraint system. It is easy to see that the size $N_{n,k}$ and the number of constraints $m_{n,k}$ of the constraint system are at most polynomially in n for any constant k .

Consider the constraint propagation game of the (n, k) -constraint system. For operators of the form $X_u, Z_u, P, (P|Q)$, we add a hat to denote the corresponding reflection in the player's strategy. For example, \hat{X}_u, \hat{Z}_u and \hat{P} denote the corresponding reflections of the player's strategy when receiving measurement requests of $X_u, Z_u, P \in \mathbb{P}_{n,k}$ respectively. Similarly, $\widehat{P|Q}$ denotes the derived reflection from the player's projective measurement \hat{Q} for the question $Q \in \mathbb{Q}_{n,k}$ and $P \in Q$.

By Lemma 24, we can enforce approximate satisfaction of these constraints on a quantum state. For example, if $(\rho, \{\hat{P}\}, \{\hat{Q}\})$ is a strategy that has value at least $1 - \epsilon$ in the constraint propagation game defined by the (n, k) -constraint system, then for constant κ and $\epsilon' = N_{n,k}^\kappa \epsilon^{1/\kappa}$,

$$\text{tr}_{\rho_0}(\hat{D}_u \hat{D}_v \hat{D}_u \hat{D}_v) \approx_{\epsilon'} 1,$$

and

$$\text{tr}_{\rho_0}(\hat{X}_u \hat{Z}_u \hat{X}_u \hat{Z}_u) \approx_{\epsilon'} -1.$$

These conditions will be helpful to prove rigidity type of results. But unfortunately, we do not know if these conditions are already sufficient enough to establish the existence of an isometry V such that, \hat{P} is close to P under the conjugation of V . To complete the proof, we need to establish these approximation properties not only on state ρ_0 , but also on several other states derived from it.

This is the reason that we will need to consider the following more complicated game defined by the (n, k) -constraint system.

Let $\mathfrak{V}_{n,k} = (V_i)_{i=1}^{N_{n,k}}$ be the sequence of the reflections (derived reflections) of the (n, k) -constraint system. Let $\mathfrak{W} = (W_i)_{i=1}^{N'}$ be the concatenation of sequences

$$\mathfrak{V}_{n,k}, \Lambda_{2i-1}(X_i), \mathfrak{V}_{n,k}, \Lambda_{2i}(Z_i)$$

for $i = 1, 2, \dots, n$. The length of \mathfrak{W} is $N' = 2n(N_{n,k} + 1)$.

A sequence of Pauli operators is called primitive if it consists of XZ-form Pauli operators of weight one and has length at most k . For any $Q \in \mathcal{Q}_{n,k}$, a derived sequence for Q is a sequence of the form $P_i|Q$ for $P_i \in Q$ of length at most k . Let $(\Omega_l)_{l=1}^L$ be the sequence of all sequences that is the concatenation of all possible primitive and derived sequences, including the empty sequence as the first entry Ω_1 . The length of Ω_l is denoted as q_l . For a sequence \mathfrak{R} of reflections, let \mathfrak{R}^* be the sequence of entries in \mathfrak{R} in the reversed order. For $l \in [L]$, let \mathfrak{U}_l be the concatenation of sequences

$$\Omega_l, \mathfrak{W}, \mathfrak{W}^*, \Omega_l^*.$$

The length of \mathfrak{U}_l is $2(N' + q_l)$. Finally, let $\mathfrak{U} = (U_i)_{i=1}^N$ be the concatenation of sequences \mathfrak{U}_l for $l = 1, 2, \dots, L$.

Let N_j be the number of reflections in the first j constraints of the (n, k) -constraint system. For $i = 0, 1, \dots, 2n - 1$, define integer $N_0^i = i(N_{n,k} + 1)$ that marks the vertex index for the start of the $(i + 1)$ -th occurrence of $\mathfrak{V}_{n,k}$ in \mathfrak{W} . For $i = 0, 1, \dots, 2n - 1$, and $j = 0, 1, \dots, m_{n,k}$, define integer $N_j^i = N_0^i + N_j$. For $l \in [L]$, define

$$N_j^{i,l} = \sum_{l'=1}^{l-1} 2(N' + q_{l'}) + q_l + N_j^i.$$

Let graph $G_{\text{prop}} = (V_{\text{prop}}, E_{\text{prop}})$ be the propagation graph of the sequence \mathfrak{U} over the vertex sequence $\{0, 1, \dots, N\}$. For $i = 1, 2, \dots, 2n$, and $l = 1, 2, \dots, L$, let graph $G_{\text{cons}}^{i,l} = (V_{\text{cons}}^{i,l}, E_{\text{cons}}^{i,l})$ be the propagation of $((-1)^{c_i} \mathbb{1})_{j=1}^{m_{n,k}}$ from the right hand sides of the (n, k) -constraint system over vertex sequences $(N_j^{i,l})_{j=0}^{m_{n,k}}$. Finally, define the constraint graph $G = (V, E)$ as $V = V_{\text{prop}}$ and

$$E = E_{\text{prop}} \cup \left(\bigcup_{i=1}^{2n} \bigcup_{l=1}^L E_{\text{cons}}^{i,l} \right).$$

Define the (n, k) -constraint propagation game as in Fig. 8. The (n, k) -constraint propagation game is an extended nonlocal game between a referee and a player. The referee possesses two registers S and X . Register S is a clock register with associated Hilbert space \mathbb{C}^V . The control register X has $2n$ qubits.

Theorem 25. *The (n, k) -constraint propagation game has value 1. Furthermore, there is a constant κ such that for any strategy $\mathfrak{S} = (\rho, \{\hat{P}\}, \{\hat{Q}\})$ that has value at least $1 - \epsilon$, there exists an isometry $V \in \mathcal{L}(\mathcal{R}, \mathcal{B}^{\otimes n} \otimes \mathcal{R}')$ such that the following properties hold*

- For all $P \in \mathbb{P}_{n,k}$

$$d_{\rho_0}(\hat{P}, \check{P}) \leq O(N^\kappa \epsilon^{1/\kappa}), \quad (38)$$

(n, k) -Constraint Propagation Game

The referee does the following with equal probability:

1. *Propagation Check.* Plays the propagation game corresponding to the propagation graph G_{prop} .
2. *Initialization Check.* Measures the register S , accepts if the outcome is not 0 and continues otherwise; samples $i \in [2n]$, measures $X_{X,i}$ and accepts if the outcomes is 0; rejects otherwise.
3. *Constraint Check.* Randomly samples $i \in [2n]$, $l \in [L]$ and an edge $e \in E_{\text{cons}}^{i,l}$; measures Π_e on register S and accepts if the outcome is 2 or is equal to τ for $\Gamma_e = (-1)^\tau \mathbf{1}$; rejects otherwise.

Figure 8: The constraint propagation game for the (n, k) -constraint system of Pauli operators on n qubits of weight k .

and, for all $Q \in \mathcal{Q}_{n,k}$,

$$d_{\rho_0}(\hat{Q}, \check{Q}) \leq O(N^k \epsilon^{1/k}), \quad (39)$$

where $\check{P} = V^*(P \otimes \mathbf{1})V$, and \check{Q} is the measurement of k Pauli operators in Q after the application of isometry V .

- The probability p_0 satisfies

$$p_0 \approx_{N^k \epsilon^{1/k}} \frac{1}{N+1}.$$

In the statement, ρ_0 is the reduced state on register R when the computational basis measurement on S has outcome 0, and p_0 is the probability of outcome 0 when measuring S .

We prove the following lemmas before proving the above theorem.

Lemma 26. Let $R_1, R_2, \dots, R_k \in \text{Herm}(\mathcal{Y})$ be k pairwise commuting reflections, $V \in \text{L}(\mathcal{X}, \mathcal{Y})$ an isometry, $\rho \in \text{D}(\mathcal{X} \otimes \mathcal{Z})$ a quantum state. Define $\check{R}_i = V^* R_i V$. If for all $x \in \{0, 1\}^k$, and $\check{R}_x = \prod_{i=1}^k R_i^{x_i}$,

$$\text{Re tr}_\rho(\check{R}_x^* R_i \check{R}_i \check{R}_x) \approx_\epsilon 1,$$

then

$$\text{Re tr}_\rho \left[V^* \left(\prod_{i=1}^k R_i \right) V \prod_{i=1}^k (\check{R}_i) \right] \approx_{\sqrt{\epsilon}} 1.$$

Proof. By the Cauchy-Schwarz inequality, it is easy to show

$$\text{Re tr}_\rho \left[V^* \left(\prod_{i=1}^k R_i \right) V \prod_{i=1}^k (\check{R}_i) \right] \approx_{\sqrt{\epsilon}} \text{Re tr}_\rho \left[V^* \left(R_k V V^* \prod_{i=1}^{k-1} R_i \right) V \prod_{i=1}^k (\check{R}_i) \right]. \quad (40)$$

By this equation and Lemma 12, we have

$$\text{Re tr}_\rho \left[V^* \left(\prod_{i=1}^k R_i \right) V \prod_{i=1}^k (\check{R}_i) \right] \approx_{\sqrt{\epsilon}} \text{Re tr}_\rho \left[R_k V^* \left(\prod_{i=1}^{k-1} R_i \right) V \prod_{i=1}^{k-1} (\check{R}_i) R_k \right].$$

A repeated application of the above procedure then proves the claim in the lemma. \square

Lemma 27. Let $M = \{M^a\}$ be a projective measurement of k -bit outcome on quantum register \mathcal{X} and R_1, R_2, \dots, R_k be its derived reflections. Let $N = \{N^a\}$ be a projective measurement of k -bit outcome on quantum register \mathcal{Y} and S_1, S_2, \dots, S_k be its derived reflections. Let $V \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ be an isometry and $\rho \in \mathcal{D}(\mathcal{X} \otimes \mathcal{Z})$ a quantum state. For $i \in [k]$, define $\tilde{S}_i = V^* S_i V$. Let \tilde{N} be the quantum measurement that measures N after the application of isometry V .

If for $i \in [k]$, and all $x \in \{0, 1\}^k$, $\text{tr}_\rho(\tilde{R}_x R_i \tilde{S}_i \tilde{R}_x) \approx_\epsilon 1$ where $\tilde{R}_x = \prod R_i^{x_i}$, then

$$d_\rho(M, \tilde{N}) \leq O(\epsilon^{1/4}).$$

Proof. By the first part of the proof for Lemma 21, we have

$$\sum_a \text{tr}_\rho(M^a \tilde{N}^a) = \frac{1}{2^k} \sum_{x \in \{0, 1\}^k} \text{tr}_\rho \left[\left(\prod_{i=1}^k R_i^{x_i} \right) V^* \left(\prod_{i=1}^k S_i^{x_i} \right) V \right].$$

For all $x \in \{0, 1\}^k$, we have

$$\begin{aligned} \text{tr}_\rho \left[\left(\prod_{i=1}^k R_i^{x_i} \right) V^* \left(\prod_{i=1}^k S_i^{x_i} \right) V \right] &\approx_{\sqrt{\epsilon}} \text{tr}_\rho \left[\left(\prod_{i=1}^k R_i^{x_i} \right) V^* \left(\prod_{i=1}^{k-1} S_i^{x_i} \right) V \tilde{S}_k^{x_k} \right] \\ &\approx_{\sqrt{\epsilon}} \text{tr}_\rho \left[R_k^{x_k} \left(\prod_{i=1}^{k-1} R_i^{x_i} \right) V^* \left(\prod_{i=1}^{k-1} S_i^{x_i} \right) V R_k^{x_k} \right]. \end{aligned}$$

The proof completes by repeating the above approximation procedure. \square

Proof of Theorem 25. For $\mathfrak{U} = (U_i)_{i=1}^N$, define a sequence $\mathfrak{U}' = (U'_i)_{i=1}^N$ as follows. Operator U'_i is defined to be P if $U_i = P$ or $U_i = (P|Q)$ for $P \in \mathbb{P}_{n,k}$, $Q \in \mathbb{Q}_{n,k}$, and is defined to be $\Lambda_{X,c}(D_u)$ if $U_i = \Lambda_c(D_u)$ for $D_u \in \{X_u, Z_u\}$ and $u \in [n]$.

Define unitary operator

$$U' = \sum_{i=0}^N |i\rangle\langle i| \otimes U'_i U'_{i-1} \cdots U'_1,$$

and

$$|\Phi\rangle_{\mathcal{X}} = \frac{1}{2^n} \sum_{x \in \{0, 1\}^{2n}} |x\rangle.$$

If the player chooses state $U'(|V\rangle_S |\Phi\rangle_{\mathcal{X}} |\psi\rangle_{\mathcal{R}})$ for some $|\psi\rangle \in \mathcal{R}$ and measure honestly, the referee accepts with certainty. This proves the first part of the theorem.

We now prove the second part of the theorem. Define a sequence $\hat{\mathfrak{U}} = (\hat{U}_i)_{i=1}^N$ as follows. Operators \hat{U}_i is defined to be \hat{P} if $U_i = P$, $\widehat{P|Q}$ if $U_i = (P|Q)$ and $\Lambda_{X,c}(\hat{P})$ if $U_i = \Lambda_c(P)$. For $i_0 < i_1 \in [N]$, introduce the notion

$$\hat{U}_{i_1 \leftarrow i_0} = \hat{U}_{i_1} \hat{U}_{i_1-1} \cdots \hat{U}_{i_0}.$$

Define unitary operator

$$\hat{U} = \sum_{i=0}^N |i\rangle\langle i| \otimes \hat{U}_{i \leftarrow 1},$$

As the strategy \mathfrak{S} has value at least $1 - 3\epsilon$ in the *Propagation Check* part, Lemma 23 implies that there is a state

$$\rho'_0 \in \mathcal{D}(\mathcal{X} \otimes \mathcal{R}),$$

such that

$$D(\rho, \rho'_0) \leq O(\epsilon_1) \quad (41)$$

for $\rho' = \hat{U}(|V\rangle\langle V| \otimes \rho'_0) \hat{U}^*$ and $\epsilon_1 = N^{3/2} \epsilon^{1/2}$.

Define a strategy \mathfrak{S}' that is the same as \mathfrak{S} except that the shared state is ρ' instead of ρ . By monotonicity of the trace distance, strategy \mathfrak{S}' has value at least $1 - O(\epsilon_1^{1/2})$. Define Hamiltonian

$$H_{\text{in}} = \frac{1}{2n} \sum_{i=1}^{2n} |0\rangle\langle 0|_S \otimes \frac{(\mathbb{1} - X)_{X,i}}{2}.$$

The referee rejects \mathfrak{S}' in the *Initialization Check* part with probability

$$\text{tr}_{\rho'}(H_{\text{in}}) = \frac{1}{N} \text{tr}_{\rho'_0} \left[\sum_{i=1}^{2n} \frac{(\mathbb{1} - X)_{X,i}}{2} \right] \leq O(\epsilon_1^{1/2}).$$

Define states $\hat{\rho}_0 \in \mathcal{D}(\mathcal{R})$, $\tilde{\rho}_0 \in \mathcal{D}(\mathcal{X} \otimes \mathcal{R})$ as

$$\hat{\rho}_0 = \frac{\langle \Phi |_{\mathcal{X}} \rho'_0 | \Phi \rangle_{\mathcal{X}}}{\text{tr}_{\rho'_0}(|\Phi\rangle\langle \Phi|_{\mathcal{X}})}, \quad \tilde{\rho}_0 = |\Phi\rangle\langle \Phi|_{\mathcal{X}} \otimes \hat{\rho}_0.$$

By the spectrum of H_{in} and Lemma 5, it follows that

$$D(\rho'_0, \tilde{\rho}_0) \leq O(\epsilon_2), \quad (42)$$

where $\epsilon_2 = n^{1/2} N^{1/2} \epsilon_1^{1/4}$. Define state $\tilde{\rho}$ as

$$\tilde{\rho} = \hat{U}(|V\rangle\langle V| \otimes \tilde{\rho}_0) \hat{U}^*.$$

By Eqs. (41), (42) and the triangle inequality, we have

$$D(\rho, \tilde{\rho}) \leq O(\epsilon_2).$$

For $i = 0, 1, \dots, 2n - 1$ and $l = 1, 2, \dots, L$, define states $\rho_{i,l}, \tilde{\rho}_{i,l} \in \mathcal{D}(\mathcal{X} \otimes \mathcal{R})$,

$$\rho_{i,l} = \frac{\langle N_0^{i,l} | \rho | N_0^{i,l} \rangle}{\text{tr}_{\rho}(|N_0^{i,l}\rangle\langle N_0^{i,l}|)}, \quad \tilde{\rho}_{i,l} = \frac{\langle N_0^{i,l} | \tilde{\rho} | N_0^{i,l} \rangle}{\text{tr}_{\tilde{\rho}}(|N_0^{i,l}\rangle\langle N_0^{i,l}|)}.$$

By the definition of $\tilde{\rho}$, it is easy to see

$$\tilde{\rho}_{i,l} = \hat{U}_{N_0^{i,l} \leftarrow 1} \tilde{\rho}_0 \hat{U}_{N_0^{i,l} \leftarrow 1}^*. \quad (43)$$

As the product of all reflections in \mathfrak{U}_l reduces to $\mathbb{1}$, we have

$$\tilde{\rho}_{0,l} = \tilde{Q}_l^* \tilde{\rho}_0 \tilde{Q}_l, \quad (44)$$

for $\tilde{Q}_l = \hat{P}_1 \hat{P}_2 \cdots \hat{P}_{q_l}$ where $\mathfrak{Q}_l = (P_j)_{j=1}^{q_l}$. We note that, as in the discussion before the construction of the (n, k) -constraint propagation game, states of the form in Eq. (44) are the states on which we need to establish the approximate satisfaction of the (n, k) -constraint system.

Consider strategy $\tilde{\mathfrak{S}}$ which is the same as \mathfrak{S} except that the shared state becomes $\tilde{\rho}$ instead of ρ . The value of $\tilde{\mathfrak{S}}$ is at least $1 - O(\epsilon_2^{1/2})$. Consider the *Constraint Check* part. For $j \in [m_{n,k}]$, if edge $e_j = (N_{j-1}^{i,l}, N_j^{i,l}) \in E_{\text{cons}}^{i,l}$ is sampled, the referee rejects $\tilde{\mathfrak{S}}$ with probability

$$\text{tr}_{\tilde{\rho}} H_j^i \leq O(nLm_{n,k}\epsilon_2^{1/2}),$$

where

$$H_j^{i,l} = \frac{1}{2} \left[(|N_{j-1}^{i,l}\rangle\langle N_{j-1}^{i,l}| + |N_j^{i,l}\rangle\langle N_j^{i,l}|) \otimes \mathbf{1} - (-1)^{\tau_j} (|N_{j-1}^{i,l}\rangle\langle N_j^{i,l}| + |N_j^{i,l}\rangle\langle N_{j-1}^{i,l}|) \right].$$

By the definition of $\tilde{\rho}$ and $\tilde{\rho}_{i,l}$, this is equivalent to

$$\text{Re tr}_{\tilde{\rho}_{i,l}} (\hat{U}_{N_j^{i,l} \leftarrow N_0^{i,l}+1}^* \hat{U}_{N_{j-1}^{i,l} \leftarrow N_0^{i,l}+1}) \approx_{\epsilon_3} (-1)^{\tau_j},$$

for $\epsilon_3 = nLm_{n,k}N\epsilon_2^{1/2}$. This can be further simplified to

$$\text{Re tr}_{\tilde{\rho}_{i,l}} (\hat{C}_1 \hat{C}_2 \cdots \hat{C}_j \hat{C}_{j-1}^* \cdots \hat{C}_1^*) \approx_{\epsilon_3} (-1)^{\tau_j},$$

or equivalently, using the shorthand notion introduced in the proof of Lemma 24,

$$\text{Re tr}_{\tilde{\rho}_{i,l}} (\hat{C}_{1,j} \hat{C}_{1,j-1}^*) \approx_{\epsilon_3} (-1)^{\tau_j}.$$

Using similar arguments as in the proof of Lemma 24, this proves that for $i = 0, 1, \dots, 2n-1$, $l \in [L]$, $j \in [m_{n,k}]$ and $\epsilon_4 = m_{n,k}\epsilon_3^{1/2}$,

$$\text{Re tr}_{\tilde{\rho}_{i,l}} \hat{C}_j \approx_{\epsilon_4} (-1)^{\tau_j}. \quad (45)$$

For $i = 0, 1, \dots, 2n-1$, and $l \in [L]$, define states $\hat{\rho}_{i,l} \in \mathcal{D}(\mathcal{R})$ as

$$\hat{\rho}_{i,l} = \text{tr}_{\mathcal{X}} \tilde{\rho}_{i,l}.$$

As \hat{C}_j is an operator in $\mathcal{U}(\mathcal{R})$, Eq. (45) can be written as

$$\text{Re tr}_{\hat{\rho}_{i,l}} \hat{C}_j \approx_{\epsilon_4} (-1)^{\tau_j}. \quad (46)$$

By the definition of $\tilde{\rho}_{i,l}$ in Eq. (43), we have for $i = 1, 2, \dots, 2n-1$,

$$\tilde{\rho}_{i,l} = \hat{U}_{N_0^{i,l} \leftarrow N_0^{i-1,l}+1} \tilde{\rho}_{i-1,l} \hat{U}_{N_0^{i,l} \leftarrow N_0^{i-1,l}+1}^*.$$

Equivalently, for $i = 1, 2, \dots, n$,

$$\tilde{\rho}_{2i-1,l} = \Lambda_{2i-1}(\hat{X}_i) \hat{C}_{1,m_{n,k}}^* \tilde{\rho}_{2i-2,l} \hat{C}_{1,m_{n,k}} \Lambda_{2i-1}(\hat{X}_i),$$

and for $i = 1, 2, \dots, n-1$,

$$\tilde{\rho}_{2i,l} = \Lambda_{2i}(\hat{Z}_i) \hat{C}_{1,m_{n,k}}^* \tilde{\rho}_{2i-1,l} \hat{C}_{1,m_{n,k}} \Lambda_{2i}(\hat{Z}_i).$$

Taking partial trace over \mathcal{X} on the above two equations, we have for $i = 1, 2, \dots, 2n-1$,

$$\hat{\rho}_{i,l} = \mathfrak{F}_i(\hat{C}_{1,m_{n,k}}^* \hat{\rho}_{i-1,l} \hat{C}_{1,m_{n,k}}),$$

where $\mathfrak{F}_1, \mathfrak{F}_2, \dots, \mathfrak{F}_{2n-1}$ are quantum channels defined as

$$\begin{aligned}\mathfrak{F}_{2i-1}(\rho) &= \frac{\rho + \hat{X}_i \rho \hat{X}_i}{2}, \text{ for } i = 1, 2, \dots, n, \\ \mathfrak{F}_{2i}(\rho) &= \frac{\rho + \hat{Z}_i \rho \hat{Z}_i}{2}, \text{ for } i = 1, 2, \dots, n-1.\end{aligned}$$

Define states $\check{\rho}_{i,l} \in \mathcal{D}(\mathcal{R})$ for $i = 0, 1, \dots, 2n-1$,

$$\check{\rho}_{i,l} = \mathfrak{F}_i \circ \mathfrak{F}_{i-1} \circ \dots \circ \mathfrak{F}_1(\hat{\rho}_{0,l}).$$

We claim that Eq. (46) then implies, for $\epsilon_5 = nm_{n,k}\epsilon_4$,

$$\text{Re tr}_{\check{\rho}_{i,l}}(\hat{C}_j) \approx_{\epsilon_5} (-1)^{\tau_j}. \quad (47)$$

In fact, by Eq. (46) and the definition of $\hat{\rho}_{i,l}$, we have

$$\text{Re tr}_{\hat{\rho}_{i-1,l}}\left(\hat{C}_{1,m_{n,k}} \mathfrak{F}_i(\hat{C}_j) \hat{C}_{1,m_{n,k}}^*\right) \approx_{\epsilon_4} (-1)^{\tau_j}.$$

Using Lemma 11 and Eq. (46), the above approximation is simplified to

$$\text{Re tr}_{\hat{\rho}_{i-1,l}}\left(\mathfrak{F}_i(\hat{C}_j)\right) \approx_{m_{n,k}\epsilon_4} (-1)^{\tau_j}.$$

The claim in Eq. (47) follows by a repeated application of the above procedure and the choice of ϵ_5 .

Lemma 14 applied to Eq. (46) and (47) with constraint $X_u Z_u X_u Z_u = -1$ implies the existence of unitary operators $V_u \in \mathcal{L}(\mathcal{R}, \mathcal{B}_u \otimes \mathcal{R}')$ such that

$$\tilde{Z}_u = \hat{Z}_u,$$

and

$$\begin{aligned}\text{Re tr}_{\hat{\rho}_{i,l}}(\tilde{X}_u \hat{X}_u) &\approx_{\epsilon_4} 1, \\ \text{Re tr}_{\check{\rho}_{i,l}}(\tilde{X}_u \hat{X}_u) &\approx_{\epsilon_5} 1,\end{aligned} \quad (48)$$

where $\tilde{D}_u = V_u^*(D \otimes \mathbb{1})V_u$ for $D \in \{X, Z\}$.

Similarly, using the condition for constraints $D_v X_u Z_u X_u Z_u D_v = -1$, we have

$$\begin{aligned}\text{Re tr}_{\hat{\rho}_{i,l}}(\hat{D}_v \tilde{X}_u \hat{X}_u \hat{D}_v) &\approx_{\epsilon_4} 1, \\ \text{Re tr}_{\check{\rho}_{i,l}}(\hat{D}_v \tilde{X}_u \hat{X}_u \hat{D}_v) &\approx_{\epsilon_5} 1,\end{aligned} \quad (49)$$

As in the proof of Theorem 17, define isometry V as in Eq. (19) and quantum channels \mathfrak{T}_u as in Eq. (21) and operators

$$\check{D}_u = \mathfrak{T}_1 \circ \mathfrak{T}_2 \circ \dots \circ \mathfrak{T}_u(\tilde{D}_u).$$

Define operators R and R' as

$$\begin{aligned}R &= \mathfrak{T}_2 \circ \dots \circ \mathfrak{T}_u(\check{D}_u), \\ R' &= \frac{R + \tilde{Z}_1 R \tilde{Z}_1}{2}.\end{aligned}$$

For $l \in [L]$, we have

$$\begin{aligned}
\operatorname{Re tr}_{\hat{\rho}_{0,l}}(\check{D}_u \hat{D}_u) &= \frac{1}{2} \operatorname{Re tr}_{\check{\rho}_{0,l}}(R' \hat{D}_u + \tilde{X}_1 R' \tilde{X}_1 \hat{D}_u) \\
&\approx_{\epsilon_5^{1/2}} \frac{1}{2} \operatorname{Re tr}_{\check{\rho}_{0,l}}(R' \hat{D}_u + \hat{X}_1 R' \hat{X}_1 \hat{D}_u) \\
&\approx_{\epsilon_5^{1/2}} \frac{1}{2} \operatorname{Re tr}_{\check{\rho}_{0,l}}(R' \hat{D}_u + \hat{X}_1 R' \hat{D}_u \hat{X}_1) \\
&= \operatorname{Re tr}_{\check{\rho}_{1,l}}(R' \hat{D}_u),
\end{aligned}$$

where the first approximation is by Eqs. (48) and (49), and the second is by Eq. (47). Similarly, we have

$$\begin{aligned}
\operatorname{Re tr}_{\check{\rho}_{1,l}}(R' \hat{D}_u) &= \frac{1}{2} \operatorname{Re tr}_{\check{\rho}_{1,l}}(R \hat{D}_u + \tilde{Z}_1 R \tilde{Z}_1 \hat{D}_u) \\
&= \frac{1}{2} \operatorname{Re tr}_{\check{\rho}_{1,l}}(R \hat{D}_u + \hat{Z}_1 R \hat{Z}_1 \hat{D}_u) \\
&\approx_{\epsilon_5^{1/2}} \frac{1}{2} \operatorname{Re tr}_{\check{\rho}_{1,l}}(R \hat{D}_u + \hat{Z}_1 R \hat{D}_u \hat{Z}_1) \\
&= \operatorname{Re tr}_{\check{\rho}_{2,l}}(R \hat{D}_u),
\end{aligned}$$

Repeating the above procedure, we have for $l \in [L]$ and $\epsilon_6 = n\epsilon_5^{1/2}$,

$$\operatorname{Re tr}_{\hat{\rho}_{0,l}}(\check{D}_u \hat{D}_u) \approx_{\epsilon_6} 1. \quad (50)$$

This proves Eq. (38) in the theorem for the case $P \in \mathbb{P}_{n,1}$ with state $\hat{\rho}_0$ by taking $l = 0$.

Consider constraints of the form $P \prod_{u \in J} D_u = \mathbb{1}$ for $P = \prod_{u \in J} D_u$, we have by Eq. (46),

$$\operatorname{Re tr}_{\hat{\rho}_{0,l}}(\hat{P} \prod_{u \in J} \hat{D}_u) \approx_{\epsilon_4} 1.$$

By Eq. (50), we have

$$\operatorname{Re tr}_{\hat{\rho}_0}(\hat{P} \prod_{u \in J} \check{D}_u) \approx_{\epsilon_6^{1/2}} 1.$$

Finally, by Lemma 26 and Lemma 13, we have

$$\operatorname{Re tr}_{\hat{\rho}_0}(\hat{P} \check{P}) \approx_{\epsilon_6^{1/2}} 1.$$

By constraints of the form $(P|Q)P = \mathbb{1}$ in Eq. (46), we have

$$\operatorname{Re tr}_{\hat{\rho}_{0,l}}(\widehat{P|Q} \hat{P}) \approx_{\epsilon_4} 1,$$

and using Eq. (50) and Lemma 26, this implies that

$$\operatorname{Re tr}_{\hat{\rho}_{0,l'}}(\widehat{P|Q} \check{P}) \approx_{\epsilon_6^{1/2}} 1,$$

for all index l' that corresponds to sequences of the concatenations of empty primitive sequence and derived reflections. Lemma 27 then completes the proof for state $\hat{\rho}_0$.

Finally, using a similar argument as in the proof for Lemma 24, the statements in the theorem are proved for the state $\rho_0 = \operatorname{tr}_{\mathcal{X}} \rho_{0,0}$. \square

We mention that even though the definition of the (n, k) -constraint system game is an extended nonlocal game with a single player, it is straightforward to extend it the case of r players. For this, consider a copy of the (n, k) -constraint system for each player and take the union of all the constraints. The referee then does the same as in the one player case and direct questions of reflections of the i -th copy of the constraint system to the player (i) . Similar rigidity results can be established for this r -player version of the (n, k) -constraint propagation game.

The use of the (n, k) -constraint propagation game is to enforce that the players follow the measurement specifications. In order to check other properties, such as the correct propagation of some quantum computation, we need to first measure the clock register S and continue only when the result is 0. This is not very efficient and introduces another polynomial loss in efficiency. This is not important in our case. But this loss may be recovered by using a more intricate decoding of the players' answers when measurement outcome other than 0 appears in the (n, k) -constraint propagation game.

5 From Interactive Proofs to Nonlocal Games

5.1 Localization with Honest Players

In this section, we show how to transform an r -prover quantum interactive proof system to a game between a quantum referee and r honest players, in which the referee measures and asks each player to measure at most constant number of qubits. Therefore the questions in the game consist of at most logarithmic number of bits. We start with the following lemma proved in [34].

Lemma 28. *For any $r, m \in \text{poly}$, c' and s' satisfying $c' - s' \in \text{poly}^{-1}$, there exists an $s \in 1 - \text{poly}^{-1}$, such that $\text{QMIP}^*(r, m, c', s') \subseteq \text{QMIP}^*(r, 3, 1, s)$.*

It therefore suffices to start with r -prover, 3-message quantum interactive proof systems with perfect completeness. Recall that V is the private quantum register of the verifier V , P_i for $i \in [r]$ is the private quantum register of prover P_i , and M_i for $i \in [r]$ are the quantum registers for the message qubits. The registers V and M_i consist of $q_V, q_M \in \text{poly}$ qubits respectively, while there are no restrictions on the sizes of P_i 's. The associated Hilbert spaces of these registers are denoted as $\mathcal{V}, \mathcal{M}_i, \mathcal{P}_i$ respectively. Registers M, P refer to the collection of quantum registers $(M_i)_{i=1}^r$ and $(P_i)_{i=1}^r$ respectively, and $\mathcal{M} = \bigotimes_{i=1}^r \mathcal{M}_i$ and $\mathcal{P} = \bigotimes_{i=1}^r \mathcal{P}_i$ are their associated Hilbert spaces. Let (V^1, V^2) and (W^1, W^2, \dots, W^r) be the polynomial-time quantum verifier and the quantum provers' circuits for the 3-message interactive proof system. For simplicity, we assume that both V^1 and V^2 consist of L elementary gates from some universal gate set specified below. If they have different size, one can add extra elementary gates that act on auxiliary qubits. Define $T = 2L + 1$ be the total number of time steps including the provers' action as in Fig. 9.

In the r -prover, 3-message interactive proof system, the provers initialize a state $|\Psi\rangle \in \mathcal{M} \otimes \mathcal{P}$, send registers M_i to the verifier. The verifier then applies $V^1 \in U(\mathcal{V} \otimes \mathcal{M})$ and sends M_i to prover P_i . The provers apply $W^i \in U(\mathcal{M}_i \otimes \mathcal{P}_i)$ and sends M_i back to the verifier. Finally the verifier applies $V^2 \in U(\mathcal{V} \otimes \mathcal{M})$ and accepts if and only if the first qubit V measures to 1. Define projection $\Pi_{\text{acc}} = |1\rangle\langle 1|_{V,1}$. For any verifier described by (V^1, V^2) , The maximum acceptance probability of the provers is given by

$$\text{MAP}(V^1, V^2) = \sup \left\| \Pi_{\text{acc}} V^2 W V^1 (|0^{q_V}\rangle_V |\Psi\rangle_{M,P}) \right\|^2,$$

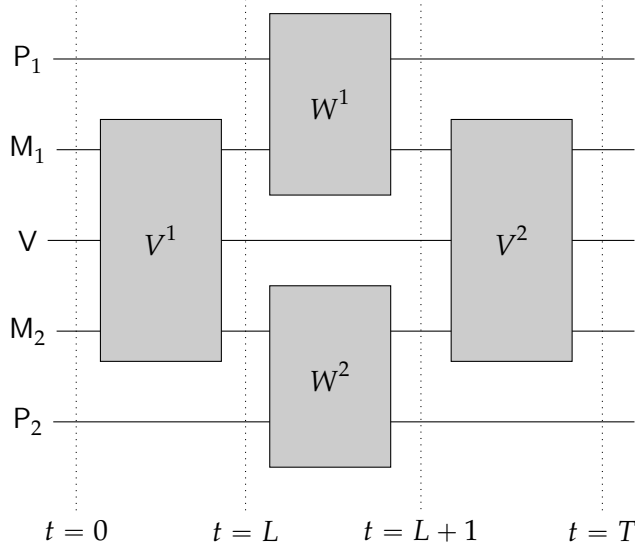


Figure 9: An illustration of two-prover, three-message quantum interactive proof system with verifier's circuits (V^1, V^2) and provers' circuits (W^1, W^2).

where $W = \bigotimes_{i=1}^r W^i$, and the supreme is taken over all possible Hilbert spaces \mathcal{P}_i , all quantum state $|\Psi\rangle$ and all quantum provers $W^i \in U(\mathcal{M}_i \otimes \mathcal{P}_i)$. For any language $A \in \text{QMIP}^*(r, 3, 1, s)$, we have $\text{MAP}(V^1, V^2) = 1$ if $x \in A$ and $\text{MAP}(V^1, V^2) \leq s$ if $x \notin A$ by the completeness and the soundness of the proof system.

Our transformation from this three-message interactive proof system to a one-round multi-player game with honest players can be regarded as a generalization of the circuit-to-Hamiltonian transformation of Kitaev to the interactive setting. The multi-player game consists of a referee and r players, playing the role of the verifier and provers respectively. The referee possesses a clock register C and a register V . For each $i \in [r]$, player (i) possesses register B_i , a copy of the $L + 1$ -th qubit in C , and two registers M_i, P_i . We use unary clock encoding for the clock register C consisting of T qubits. The legal states of the register are spanned by states of the form $|1^t 0^{T-t}\rangle$. Let T be the collection of the clock register C and all registers $(B_i)_{i=1}^r$. The legal clock states are spanned by

$$|\hat{t}\rangle_T = |1^t 0^{T-t}\rangle_C \otimes \left(\bigotimes_{i=1}^r |\delta_t\rangle_{B_i} \right),$$

where $\delta_t \in \{0, 1\}$ equals 0 if $t \leq L$ and equals 1 otherwise. We will use $\mathcal{C}, \mathcal{T}, \mathcal{B}_i$ to denote the corresponding Hilbert spaces of register C, T, B_i . Define Hilbert space \mathcal{H} to be $\mathcal{T} \otimes \mathcal{V} \otimes \mathcal{M} \otimes \mathcal{P}$.

In the game, there are two possible types of questions that the referee may ask. The first type is a measurement specification of either one or several commuting Pauli operators on qubits in registers B_i and M_i . The players are honest in the sense that they will perform the measurements corresponding to the received Pauli operators and reply with the measurement outcome. The second type consists of only one special question $*$, which asks player (i) to measure X on B_i after the application of the prover's circuit $(W^i)^*$ conditioned on the qubit in B_i . The player is however not required to follow this protocol exactly.

The game proceeds as follows. The players first prepare a state ρ in all the registers of the referee and the players. The players are not allowed to communicate after this initialization step. The referee sends questions to the players as in Fig. 12 and the players respond honestly in the sense

described above. Finally, the referee determines whether to accept or reject based on the questions, answers and his own measurement outcomes. The strategy of the players can be described by $\mathfrak{S} = (\rho, \hat{\ast}^{(i)})$, for state $\rho \in D(\mathcal{H})$ and reflection $\hat{\ast}^{(i)}$ the players applies for question \ast .

In order to use XZ-form Pauli measurements in the game, we will assume that circuit V^1, V^2 uses two elementary gates—the Toffoli gate and the Hadamard gate [55]. We further assume that the each Hadamard gate on a qubit in V (or M_i) appears in pair with another Hadamard gate on V (M_i respectively). This can be easily achieved by adding a dummy qubit to these registers and it is a technique first used in [8] to simplify the design of a zero-knowledge proof for QMA. With this convention of the verifier circuit, the referee will play the *Hadamard Check* and *Toffoli Check* given in Figs. 10 and 11 to check the propagation of the verifier's circuits.

Hadamard Check

Let u_1, u_2 be the two qubits that the two Hadamard gates act on in the Hadamard check at time t , the referee measures $X_{C,t}$ with outcome x . He samples $j \in \{0, 1\}$ uniformly at random and does the following:

1. If $j = 0$, measures or asks the players to measure $X_{u_1}X_{u_2}, Z_{u_1}Z_{u_2}$ (if both u_1, u_2 are qubits form register M) and let a_1, a_2 be the two outcome bits; rejects if $x \oplus a_1 = x \oplus a_2 = 1$ and accepts otherwise.
2. If $j = 1$, measures or asks the players to measure $X_{u_1}Z_{u_2}, Z_{u_1}X_{u_2}$ and let a_1, a_2 be the two outcome bits; rejects if $x \oplus a_1 = x \oplus a_2 = 1$ and accepts otherwise.

Figure 10: The protocol that checks the Hadamard gate propagation at time step t .

Lemma 29. *Let ρ be the shared state used in the Hadamard Check and U_t be the corresponding doubled Hadamard gate, the referee rejects with probability*

$$\frac{1}{4} \text{tr}_\rho [\mathbb{1} - X_{C,t} \otimes U_t].$$

Proof. The referee rejects with probability

$$\frac{1}{2} \text{tr}_\rho \left[\frac{\mathbb{1} - X_{C,t}}{2} \otimes \frac{[(\mathbb{1} + XX)(\mathbb{1} + ZZ)]_{u_1, u_2}}{4} + \frac{\mathbb{1} + X_{C,t}}{2} \otimes \frac{[(\mathbb{1} - XX)(\mathbb{1} - ZZ)]_{u_1, u_2}}{4} + \frac{\mathbb{1} - X_{C,t}}{2} \otimes \frac{[(\mathbb{1} + XZ)(\mathbb{1} + ZX)]_{u_1, u_2}}{4} + \frac{\mathbb{1} + X_{C,t}}{2} \otimes \frac{[(\mathbb{1} - XZ)(\mathbb{1} - ZX)]_{u_1, u_2}}{4} \right].$$

The above express simplifies to

$$\frac{1}{4} \text{tr}_\rho [\mathbb{1} - X_{C,t} \otimes U_t]$$

by a direct calculation. □

Lemma 30. *Let ρ be the shared state used in the Toffoli Check and U_t be the corresponding Toffoli gate, the referee rejects with probability*

$$\frac{1}{4} \text{tr}_\rho [\mathbb{1} - X_{C,t} \otimes U_t].$$

Toffoli Check

Let u_1, u_2, u_3 be the three qubits the Toffoli gate acts on with u_3 the target qubit. In the Toffoli check at time t , the referee measures $X_{C,t}$ with outcome x . He samples $j \in \{0, 1\}$ uniformly at random, accepts if $j = 1$ and continues otherwise. He measures or asks the players to measure $Z_{u_1}, Z_{u_2}, X_{u_3}$ (if any of u_1, u_2, u_3 is a qubit from register M) and let a_1, a_2, a_3 be the three outcome bits; rejects if $a_1 = a_2 = 1, x \oplus a_3 = 1$ or $a_1 a_2 = 0, x = 1$ and accepts otherwise.

Figure 11: The protocol that checks the Toffoli gate propagation at time step t .

Proof. Let u_1, u_2, u_3 be the qubits that U_t acts on. The referee rejects with probability

$$\frac{1}{2} \operatorname{tr}_\rho \left[\frac{\mathbb{1} - X_{C,t}}{2} \otimes (\mathbb{1} - |11\rangle\langle 11|)_{u_1, u_2} + \frac{\mathbb{1} - X_{C,t} X_{u_3}}{2} \otimes |11\rangle\langle 11|_{u_1, u_2} \right] = \frac{1}{4} \operatorname{tr}_\rho [\mathbb{1} - X_{C,t} \otimes U_t].$$

□

Note that we have scaled down the rejection probability by a half using the random bit j on purpose so that both checks have the rejection probabilities of the same form.

We use the following lemma in the analysis.

Lemma 31. *Let h, p, s be positive real numbers such that $s \in 1 - \text{poly}^{-1}$, $h \in \text{poly}$, $p \in \text{poly}^{-1}$ and $p \in 1 - \text{poly}^{-1}$. Let $\kappa \geq 0$ be a constant. Then for function*

$$f(\epsilon) = (1 - p)(1 - \epsilon) + p \min(1, s + h\epsilon^{1/\kappa}),$$

we have $\max_\epsilon f(\epsilon) \in 1 - \text{poly}^{-1}$.

Proof. If $\epsilon \leq [(1 - s)/2h]^\kappa$, then

$$f(\epsilon) \leq 1 - p + p(1 + s)/2 = 1 - p(1 - s)/2 \in 1 - \text{poly}^{-1}.$$

Otherwise,

$$f(\epsilon) \leq (1 - p)(1 - \epsilon) + p = 1 - (1 - p)\epsilon \in 1 - \text{poly}^{-1}.$$

□

Theorem 32. *For $r \in \text{poly}$, $s \in 1 - \text{poly}^{-1}$, there exists a real number $s' \in 1 - \text{poly}^{-1}$ such that, for any language $A \in \text{QMIP}^*(r, 3, 1, s)$ and an instance x , the following properties hold for the game in Fig. 12:*

1. *If $x \in A$, the referee accepts with certainty;*
2. *If $x \notin A$, the referee accepts with probability at most s' .*

Proof of Theorem 32. We first prove the case $x \in A$. Let (V^1, V^2) and (W^1, W^2, \dots, W^r) be the polynomial-time quantum verifier and provers' circuits in the three-message quantum interactive proof system. Let U_1, U_2, \dots, U_L and $U_{L+2}, U_{L+3}, \dots, U_T$ be the L elementary gates in V^1 and V^2 respectively. Define $U_{L+1} = \bigotimes_{i=1}^r W^i$. Let $|\Psi\rangle$ be the state that the provers initialize in registers M, P. In the multi-player game, the honest players share state

$$\frac{1}{\sqrt{T+1}} \sum_{t=0}^T |\hat{t}\rangle_{\text{T}} \otimes U_t U_{t-1} \cdots U_1 (|0^{qv}\rangle_{\text{V}} |\Psi\rangle_{\text{M,P}}).$$

Multi-Player One-Round Game for QMIP* with Honest Players

The referee performs the following checks with equal probability:

1. *Clock Check*. The referee checks the validity of clock states. He does the following with equal probability:
 - (a) Randomly samples $t \in [T - 1]$; measures $Z_{C,t}, Z_{C,t+1}$ and rejects if the outcomes are 0, 1 respectively; accepts otherwise.
 - (b) Randomly samples $i \in [r]$; sends measurement specification Z_{B_i} to player (i); measures $Z_{C,L+1}$ and rejects if the outcome is different from the player's answer bit; accepts otherwise.
2. *Verifier Propagation Check*. The referee checks propagation of the verifier steps:
 - (a) Samples $t \in [T] \setminus \{L + 1\}$ uniformly at random;
 - (b) Measures $Z_{C,t-1}$ and $Z_{C,t+1}$ (if $t = 1$, assume that the first measurement always has outcome 1, and if $t = T$, assume that the second measurement always has outcome 0); accepts if the outcomes are not 1, 0 respectively and continues otherwise;
 - (c) If U_t is a Toffoli gate, does the *Toffoli Check* at time t ; accepts or rejects as in the *Toffoli Check*;
 - (d) If U_t consists of two Hadamard gates, does the *Hadamard Check* at time t ; accepts or rejects as in the *Hadamard Check*.
3. *Prover Propagation Check*. The referee checks the propagation of the provers' step:
 - (a) Measures $Z_{C,L}, Z_{C,L+2}$ and accepts if the outcomes are not 1, 0 respectively; continues otherwise;
 - (b) Sends $*$ to player (i) and receives $a^{(i)}$ back;
 - (c) Measures $X_{C,L+1}$ and accepts if the outcome $a = \bigoplus a^{(i)}$; rejects otherwise.
4. *Initialization Check*. The referee checks that the state is correctly initialized:
 - (a) Measures $Z_{C,1}$ and accepts if the outcome is 1; continues otherwise;
 - (b) Samples $j \in V$; measures $Z_{V,j}$ and accepts if the outcome is 0; rejects otherwise.
5. *Output Check*. The output qubit should indicate acceptance in the interactive proof:
 - (a) Measures $Z_{C,T}$ and accepts if the outcome is 0; continues otherwise;
 - (b) Measures $Z_{V,1}$ and accepts if outcome is 1; rejects otherwise.

Figure 12: The multi-player one-round game for QMIP* with honest players.

The referee in the game then accepts with certainty, a fact which can be verified directly but it will also become clear in the next part of the proof. This proves the first part of the theorem.

Now we prove the second part and suppose that $x \notin A$. Let $\mathfrak{S} = (\rho, \hat{*})$ be the strategy of the players. Define three games G_2 , G_3 and G_4 derived from the honest player game G as follows. In game G_4 , the referee performs the *Initialization Check* and *Output Check* with equal probability. In game G_3 , the referee does the *Prover Propagation Check* and G_4 with probability $1/3$ and $2/3$ respectively. In game G_2 , the referee does the *Verifier Propagation Check* and G_3 with probability $1/4$ and $3/4$. Finally, game G is equivalent to the game in which the referee does the *Clock Check* and G_2 with probability $1/5$ and $4/5$ respectively. We will analyze the five checks in the game sequentially

as follows.

Step 1. For the *Clock Check*, define a Hamiltonian

$$H_{\text{clock}} = \frac{1}{2(T-1)} \sum_{t=1}^{T-1} |0\rangle\langle 0|_{C,t} \otimes |1\rangle\langle 1|_{C,t+1} + \frac{1}{2r} \sum_{a \in \{0,1\}} \sum_{i=1}^r |a\rangle\langle a|_{C,L+1} \otimes [\mathbb{1} - |a\rangle\langle a|]_{B_i}.$$

In this proof, we assume that the Hamiltonians are operators in $\text{Herm}(\mathcal{H})$. The referee then rejects in the *Clock Check* subgame with probability $\epsilon = \text{tr}_\rho(H_{\text{clock}})$. The Hamiltonian H_{clock} has an eigenspace $S_{\text{legal}} \subseteq \mathcal{H}$ with eigenvalue 0 spanned by the legal clock states of the form $|\hat{t}\rangle_{\mathcal{T}} |\zeta\rangle_{V,M,P}$, and the nonzero eigenvalues are least $\Omega(1/h)$ for $h = \max(T, r)$. We then have, by Lemma 5,

$$D(\rho, \rho_{\text{legal}}) \leq O(\sqrt{h\epsilon}),$$

where $\rho_{\text{legal}} = \Pi_{\text{legal}} \rho \Pi_{\text{legal}} / \text{tr}_\rho(\Pi_{\text{legal}})$ for Π_{legal} is the projection onto S_{legal} . Hence, by the monotonicity of the trace distance, the referee accepts in game G with probability at most

$$\frac{1}{5}(1 - \epsilon) + \frac{4}{5} \min(1, s_2 + c\sqrt{h\epsilon}),$$

where s_2 is the maximum acceptance probability of the referee in game G_2 for the shared state ρ supported on S_{legal} , namely, $\rho = \Pi_{\text{legal}} \rho \Pi_{\text{legal}}$. Lemma 31 then reduces our problem to proving that $s_2 \in 1 - \text{poly}^{-1}$.

Step 2. Suppose that the state ρ satisfies $\rho = \Pi_{\text{legal}} \rho \Pi_{\text{legal}}$. We want to prove that the acceptance probability s_2 of game G_2 is at most $1 - \text{poly}^{-1}$ under this condition. Define a Hamiltonian

$$H_{\text{prop},v} = \frac{1}{4(T-1)} \left[|0\rangle\langle 0|_{C,2} \otimes (\mathbb{1} - X_{C,1} \otimes U_1) + \sum_{\substack{t:1 < t < T \\ t \neq L+1}} |10\rangle\langle 10|_{C,t-1,t+1} \otimes (\mathbb{1} - X_{C,t} \otimes U_t) + |1\rangle\langle 1|_{C,T-1} \otimes (\mathbb{1} - X_{C,T} \otimes U_T) \right].$$

It is easy to verify that

$$H_{\text{prop},v} \upharpoonright_{S_{\text{legal}}} = \frac{1}{4(T-1)} \sum_{\substack{t:1 \leq t \leq T \\ t \neq L+1}} \left[(|\widehat{t-1}\rangle\langle \widehat{t-1}| + |\widehat{t}\rangle\langle \widehat{t}|)_{\mathcal{T}} \otimes \mathbb{1} - (|\widehat{t}\rangle\langle \widehat{t-1}| + |\widehat{t-1}\rangle\langle \widehat{t}|)_{\mathcal{T}} \otimes U_t \right]$$

By Lemmas 29 and 30, the referee rejects in the *Verifier Propagation Check* with probability

$$\text{tr}_\rho H_{\text{prop},v} = \text{tr}_\rho (H_{\text{prop},v} \upharpoonright_{S_{\text{legal}}}).$$

Define unitary operator $U \in U(S_{\text{legal}})$, and states $|\omega^1\rangle, |\omega^2\rangle$ as

$$U = \sum_{t=0}^L |\widehat{t}\rangle\langle \widehat{t}|_{\mathcal{T}} \otimes U_{t+1}^* U_{t+2}^* \cdots U_L^* + \sum_{t=L+1}^T |\widehat{t}\rangle\langle \widehat{t}|_{\mathcal{T}} \otimes U_t U_{t-1} \cdots U_{L+2},$$

and

$$|\omega^1\rangle = \frac{1}{\sqrt{L+1}} \sum_{t=0}^L |\widehat{t}\rangle_{\mathcal{T}},$$

$$|\omega^2\rangle = \frac{1}{\sqrt{L+1}} \sum_{t=L+1}^T |\widehat{t}\rangle_{\mathcal{T}}.$$

Define two subspaces $S_{\text{prop},v}^1$ and $S_{\text{prop},v}^2$ of S_{legal} as the spans respectively of states of the form

$$\begin{aligned} |\psi^1\rangle &= U(|\omega^1\rangle \otimes |\psi_L^1\rangle), \\ |\psi^2\rangle &= U(|\omega^2\rangle \otimes |\psi_0^2\rangle), \end{aligned}$$

for states $|\psi_L^1\rangle, |\psi_0^2\rangle \in \mathcal{V} \otimes \mathcal{M} \otimes \mathcal{P}$. Let $S_{\text{prop},v}$ be the direct sum of $S_{\text{prop},v}^1$ and $S_{\text{prop},v}^2$. Let $\Pi_{\text{prop},v}^1$, $\Pi_{\text{prop},v}^2$ and $\Pi_{\text{prop},v}$ be the projections onto subspaces $S_{\text{prop},v}^1$, $S_{\text{prop},v}^2$ and $S_{\text{prop},v}$ respectively. It is easy to verify that $S_{\text{prop},v}$ is the 0-eigenspace of $H_{\text{prop},v} \upharpoonright_{S_{\text{legal}}}$.

The Hamiltonian $H_{\text{prop},v} \upharpoonright_{S_{\text{legal}}}$ is the sum of two propagation checking Hamiltonians as in the circuit to Hamiltonian construction. It follows that the nonzero eigenvalues of $H_{\text{prop},v} \upharpoonright_{S_{\text{legal}}}$ is at least $\Omega(1/T^3)$. Suppose that the referee rejects in the *Verifier Propagation Check* part of G_2 with probability ϵ . Define state

$$\rho_{\text{prop},v} = \Pi_{\text{prop},v} \rho \Pi_{\text{prop},v} / \text{tr}_\rho(\Pi_{\text{prop},v}).$$

It then follows similarly that

$$\text{tr}_\rho \Pi_{\text{prop},v} \geq 1 - O(T^3 \epsilon),$$

and

$$D(\rho, \rho_{\text{prop},v}) \leq O(T^{3/2} \epsilon^{1/2}).$$

The referee then accepts strategy \mathfrak{S} in game G_2 with probability at most

$$\frac{1}{4}(1 - \epsilon) + \frac{3}{4} \min(1, s_3 + cT^{3/2} \epsilon^{1/2}),$$

where s_3 is the maximum acceptance probability of the referee in game G_3 for the shared state ρ supported on $S_{\text{prop},v}$. Lemma 31 then reduces the problem to showing that $s_3 \in 1 - \text{poly}^{-1}$.

Step 3. Suppose that the state ρ is supported on $S_{\text{prop},v}$. The aim is to show that the referee will reject with at least with inverse polynomial probability in game G_3 .

Define an operator

$$H_{\text{prop},p} = \frac{1}{2} |10\rangle\langle 10|_{C,L,L+2} \otimes \left[\mathbb{1} - X_{C,L+1} \otimes \left(\bigotimes_{i=1}^r \hat{\ast}^{(i)} \right) \right].$$

The restriction $H_{\text{prop},p} \upharpoonright_{S_{\text{prop},v}}$ can be computed as

$$\begin{aligned} H_{\text{prop},p} \upharpoonright_{S_{\text{prop},v}} &= (\Pi_{\text{prop},v}^1 + \Pi_{\text{prop},v}^2) H_{\text{prop},p} (\Pi_{\text{prop},v}^1 + \Pi_{\text{prop},v}^2) \\ &= \frac{1}{L+1} \left[U(|\omega^1\rangle\langle \widehat{L}| + |\omega^2\rangle\langle \widehat{L+1}|) R (|\widehat{L}\rangle\langle \omega^1| + |\widehat{L+1}\rangle\langle \omega^2|) U^* \right], \end{aligned}$$

where

$$R = \frac{1}{2} \left[\mathbb{1} - X_{C,L+1} \otimes \left(\bigotimes_{i=1}^r \hat{\ast}^{(i)} \right) \right].$$

Define operator B as

$$B = |\omega^1\rangle\langle \widehat{L}| + |\omega^2\rangle\langle \widehat{L+1}|.$$

The referee rejects in the *Prover Propagation Check* part of G_3 with probability

$$\text{tr}_\rho H_{\text{prop},p} = \text{tr}_\rho (H_{\text{prop},p} \upharpoonright_{S_{\text{prop},v}}) = \frac{1}{L+1} \text{tr}_{\rho'} R,$$

for $\rho' = B^* U^* \rho U B$. If the rejection probability is ϵ , it then follows that

$$\mathrm{tr}_{\rho'} \left[X_{C,L+1} \otimes \left(\bigotimes_{i=1}^r \hat{\star}^{(i)} \right) \right] = 1 - 2(L+1)\epsilon. \quad (51)$$

On the other hand,

$$\mathrm{tr}_{\rho'} [Z_{C,L+1} \otimes Z_{B_i}] = \mathrm{tr}_{\rho} [UB(Z_{C,L+1} \otimes Z_{B_i})B^*U^*] = \mathrm{tr}_{\rho} \Pi_{\mathrm{prop},v} = 1, \quad (52)$$

for all $i \in [r]$. These two conditions and Lemma 11 then imply that

$$\mathrm{tr}_{\rho'} [\hat{\star}^{(i)} Z_{B_i} \hat{\star}^{(i)} Z_{B_i}] \approx_{L\epsilon} -1,$$

for $i \in [r]$.

By Lemma 14, there exist unitary operators $W_i \in U(\mathcal{B}_i \otimes \mathcal{M}_i \otimes \mathcal{P}_i)$, such that for all $i \in [r]$

$$Z_{B_i} = W_i^* (Z \otimes \mathbb{1}) W_i, \quad (53)$$

and

$$d_{\rho'}(\hat{\star}^{(i)}, W_i^* (X \otimes \mathbb{1}) W_i) \leq O(\sqrt{L\epsilon}). \quad (54)$$

Define operator

$$\tilde{W} = \left(\bigotimes_{i=1}^r W_i \right).$$

These two conditions and Eqs. (51) and (52) then implies that the state

$$\tilde{\rho} = \tilde{W} \rho' \tilde{W}^*$$

satisfies that

$$\mathrm{tr}_{\tilde{\rho}} \left[X_{C,L+1} \otimes \left(\bigotimes_{i=1}^r X_{B_i} \right) \right] \geq 1 - O(r\sqrt{L\epsilon}),$$

and

$$\mathrm{tr}_{\tilde{\rho}} (Z_{C,L+1} \otimes Z_{B_i}) = 1.$$

State $\tilde{\rho}$ is therefore approximately stabilized by $X_{C,L+1} \otimes \left(\bigotimes_{i=1}^r X_{B_i} \right)$ and $Z_{C,L+1} \otimes Z_{B_i}$. These operators generate the stabilizer for the GHZ state

$$|\Phi_{\mathrm{GHZ}}\rangle = \frac{|0\rangle_{C,L+1} |0^r\rangle_{\mathrm{B}} + |1\rangle_{C,L+1} |1^r\rangle_{\mathrm{B}}}{\sqrt{2}}.$$

Let Π_{GHZ} be the projection to state $|\Phi_{\mathrm{GHZ}}\rangle$ and h be $r\sqrt{L}$. We have

$$\mathrm{tr}_{\tilde{\rho}} \Pi_{\mathrm{GHZ}} \geq 1 - O(h\sqrt{\epsilon}).$$

Define state

$$\tilde{\rho}_{\mathrm{GHZ}} = \frac{\Pi_{\mathrm{GHZ}} \tilde{\rho} \Pi_{\mathrm{GHZ}}}{\mathrm{tr}_{\tilde{\rho}} \Pi_{\mathrm{GHZ}}}.$$

By Lemma 4,

$$D(\tilde{\rho}, \tilde{\rho}_{\mathrm{GHZ}}) \leq O(h^{1/2} \epsilon^{1/4}). \quad (55)$$

By the condition in (53), it follows that each unitary W_i has the block form

$$W_i = |0\rangle\langle 0|_{B_i} \otimes W_i^0 + |1\rangle\langle 1|_{B_i} \otimes W_i^1.$$

Define

$$\tilde{W}^0 = \bigotimes_{i=1}^r W_i^0, \tilde{W}^1 = \bigotimes_{i=1}^4 W_i^1.$$

As ρ is supported on $S_{\text{prop},V}$, it follows by the definition of the state $\tilde{\rho}$ that the state $\tilde{\rho}$ is supported on states spanned by states of the form

$$|\widehat{L}\rangle|\zeta\rangle, |\widehat{L+1}\rangle|\zeta'\rangle.$$

Without loss of generality, we may assume that the state ρ in the strategy is a pure state. Then ρ' , $\tilde{\rho}$ and $\tilde{\rho}_{\text{GHZ}}$ are all pure states. It follows that the pure state corresponding to $\tilde{\rho}_{\text{GHZ}}$ can be written as

$$|\tilde{\Psi}_{\text{GHZ}}\rangle = \frac{|\widehat{L}\rangle + |\widehat{L+1}\rangle}{\sqrt{2}} \otimes |\psi\rangle.$$

By Eq. (55), it follows that state

$$|\Psi\rangle = UBW^*|\tilde{\Psi}_{\text{GHZ}}\rangle = \frac{U[|\omega^1\rangle(\tilde{W}^0)^*|\psi\rangle + |\omega^2\rangle(\tilde{W}^1)^*|\psi\rangle]}{\sqrt{2}}$$

is a good approximation of the state ρ .

Define state

$$|\psi_0\rangle = U_1^* U_2^* \cdots U_L^* (\tilde{W}^0)^* |\psi\rangle,$$

and unitary

$$W^i = (\tilde{W}_i^1)^* \tilde{W}_i^0.$$

Let $\hat{\mathcal{G}}$ be the strategy that uses the state and reflection in the following equation

$$\frac{1}{\sqrt{T+1}} \sum_{t=0}^T |\hat{t}\rangle \otimes U_t U_{t-1} \cdots U_1 |\psi_0\rangle, \quad \Lambda(W^i)(X \otimes \mathbf{1})\Lambda(W^i)^*. \quad (56)$$

They form a strategy that is accepted with probability $1 - O(h^{1/2}\epsilon^{1/4})$.

Assuming the condition that the state is supported on $S_{\text{prop},V}$, the strategy is accepted in game G_3 with probability at most

$$\frac{1}{3}(1 - \epsilon) + \frac{2}{3} \min(1, s_4 + ch^{1/2}\epsilon^{1/4}),$$

where s_4 is the maximum acceptance probability of the referee in game G_4 if the players use a strategy of the form in Eq. (56). By Lemma 31, the problem then reduces to proving $s_4 \in 1 - \text{poly}^{-1}$.

Step 4. Define Hamiltonian

$$H_{\text{in}} = \frac{1}{q_V} |0\rangle\langle 0|_{C,1} \otimes \sum_{j=1}^{q_V} |1\rangle\langle 1|_{V,j}.$$

For state ρ of the form in (56), the referee rejects with probability

$$\text{tr}_\rho H_{\text{in}} = \frac{1}{q_V(T+1)} \langle \psi_0 | \sum_{j=1}^{q_V} |1\rangle\langle 1|_{V,j} | \psi_0 \rangle.$$

Let Π_{in} be the projection

$$\Pi_{\text{in}} = |0^{q_V}\rangle\langle 0^{q_V}|_V.$$

Suppose the referee rejects with probability ϵ in *Initialization Check*, then

$$\langle \psi_0 | \Pi_{\text{in}} | \psi_0 \rangle \geq 1 - O(h\epsilon),$$

where $h = q_V(T+1)$.

Define ρ_{in} be density matrix of the pure state

$$\Pi_{\text{in}} | \psi_0 \rangle / \|\Pi_{\text{in}} | \psi_0 \rangle\|$$

Then

$$D(\rho_{\text{in}}, | \psi_0 \rangle\langle \psi_0 |) \leq O(\sqrt{h\epsilon}).$$

The referee rejects with probability

$$\frac{1}{2}(1 - \epsilon) + \frac{1}{2} \min(1, s + c\sqrt{h\epsilon}).$$

By Lemma 31, it is at most $1 - \text{poly}^{-1}$ as $s \in 1 - \text{poly}^{-1}$. This completes the proof. \square

5.2 Extended Nonlocal Game for QMIP

In this section, we transform the honest player game in the previous subsection to an extended nonlocal game.

The referee possesses registers C and V as in the honest player game and additional registers S and X. The register S will be used as the clock register for the constraint propagation subgame and its size q_S will be determined correspondingly. The player (i) possess registers B_i , M_i and P_i as in the honest player game. The questions have the same form as in the honest player game, which can be either a measurement specification or the special question \ast . But the players are not required to play honestly anymore. The game is specified in Fig. 13.

For later convenience, we now use unary clock encoding in register S and, to accommodate this change, the measurement Π_e used in the (n, k) -constraint propagation game will be updated accordingly as follows.

For $t \in [q_S]$ and edge $e = (t-1, t)$, the measurement Π_e is

$$\begin{aligned} \Pi_e^0 &= |10\rangle\langle 10|_{S,t-1,t+1} \otimes \frac{\mathbb{1} + X_{S,t}}{2} \\ \Pi_e^1 &= |10\rangle\langle 10|_{S,t-1,t+1} \otimes \frac{\mathbb{1} - X_{S,t}}{2} \\ \Pi_e^2 &= (\mathbb{1} - |10\rangle\langle 10|)_{S,t-1,t+1}. \end{aligned}$$

The measurement can be easily implemented by X, Z measurements on the $t-1$, t and $t+1$ -th qubit of register S.

For edge $e = (t_1, t_2)$ with $t_2 - t_1 = k$, the measurement Π_e is

$$\begin{aligned}\Pi_e^0 &= \frac{1}{2} |10\rangle\langle 10|_{S, t_1, t_2+1} \otimes \sum_{a, b \in \{0,1\}} |a^k\rangle\langle b^k|_{S, t_1+1, \dots, t_2} \\ \Pi_e^1 &= \frac{1}{2} |10\rangle\langle 10|_{S, t_1, t_2+1} \otimes \sum_{a, b \in \{0,1\}} (-1)^{a \oplus b} |a^k\rangle\langle b^k|_{S, t_1+1, \dots, t_2} \\ \Pi_e^2 &= \mathbb{1} - \Pi_e^0 - \Pi_e^1.\end{aligned}$$

For any constant k , the measurement can be implemented using collective X, Z measurements on constant number of qubits.

Extended Nonlocal Game for QMIP*

The referee does the following with equal probability:

1. *Clock Check*. Randomly samples $t \in [q_S - 1]$; measures $Z_{S,t}, Z_{S,t+1}$ and rejects if the outcomes are 0, 1 respectively; accepts otherwise.
2. *Constraint Propagation*. Plays the (n, k) -constraint propagation game with the r -players using registers S and X and accepts or rejects accordingly.
3. *Output Check*. Measures $Z_{S,1}$ with outcome a ; plays the honest player game as in Fig 12 using registers C and V ; rejects if $a = 0$ and the honest player game rejects; accepts otherwise.

Figure 13: The extended nonlocal game for QMIP*.

Theorem 33. For any $r \in \text{poly}$, $s \in 1 - \text{poly}^{-1}$, there is a $s' \in 1 - \text{poly}^{-1}$ such that for any language $A \in \text{QMIP}^*(r, 3, 1, s)$ and instance x , the extended game in Fig 13 has the property that

1. If $x \in A$, the referee accepts with certainty;
2. If $x \notin A$, the referee accepts with probability at most s' .

Proof. If $x \in A$, it is easy to see that the players can win the game with certainty.

Consider the case for $x \notin A$. Define game G_2 to be the game where the referee plays the *Constraint Propagation* and *Output Check* parts with equal probability. Let the strategy be $\mathfrak{S} = (\rho, \{\hat{P}\}, \{\hat{Q}\})$. Suppose that the referee rejects with probability ϵ in the *Clock Check*. It then follows that

$$D(\rho, \rho_{\text{legal}}) \leq O(\sqrt{q_S \epsilon}).$$

where $\rho_{\text{legal}} = \Pi_{\text{legal}} \rho \Pi_{\text{legal}} / \text{tr}_\rho \Pi_{\text{legal}}$ and Π_{legal} be the projection to the legal clock subspace as in the proof in Theorem 32. Let s_2 be the maximum acceptance probability in game G_2 for players who share a state supported on the legal clock subspace. Then the referee accepts with probability

$$\frac{1}{3}(1 - \epsilon) + \frac{2}{3} \min(1, s_2 + c\sqrt{q_S \epsilon}).$$

Lemma 31 then reduced the problem to proving that $s_2 \in 1 - \text{poly}^{-1}$.

We now analyze game G_2 with states supported on the legal clock subspace. Suppose the referee rejects with probability ϵ in the *Constraint Propagation*. Theorem 25 then implies that the

players must play approximately honestly for the measurement specification type of questions. That is, there is a constant κ , such that there exists an isometry V_i

$$\begin{aligned} d_{\rho_0}(\hat{P}, \check{P}) &\leq O(n^\kappa \epsilon^{1/\kappa}), \\ d_{\rho_0}(\hat{Q}, \check{Q}) &\leq O(n^\kappa \epsilon^{1/\kappa}), \end{aligned}$$

where $\check{P} = V_i^*(P \otimes \mathbb{1})V_i$ and \check{Q} is the measurement that measures Pauli operators in Q after application of V_i . Furthermore, the probability p_0 that outcome 0 occurs in the *Output Check* satisfies

$$p_0 \approx_{n^\kappa \epsilon^{1/\kappa}} \frac{1}{q_S}.$$

Consider the strategy \mathfrak{S}' with all measurements \hat{P} and \hat{Q} replaced by \check{P} and \check{Q} for all players. Strategy \mathfrak{S}' is accepted with probability at most $s_1 \in 1 - \text{poly}^{-1}$ conditioned on the event that the measurement $Z_{S,1}$ in *Output Check* has outcome 0, as promised by Theorem 32.

Therefore the referee accepts with probability at most

$$\frac{1}{2}(1 - \epsilon) + \frac{1}{2} \min\left(1, 1 - p_0 + p_0(s_1 + cn^\kappa \epsilon^{1/\kappa})\right). \quad (57)$$

The proof follows from the above equation and Lemma 31. \square

5.3 Nonlocal Games for QMIP

In this section, further transform the extended nonlocal game to a nonlocal game. The idea is to introduce eight extra players $(1'), (2'), \dots, (8')$ and let them to share an encoding of the referee's state and measure the local X, Z measurements the referee does in the extended nonlocal game. As in the extended nonlocal game, the referee's measurements on his registers correspond to at most k Pauli operators of weight at most k , the referee in the following nonlocal game can delegate the measurement to the eight extra players and the rigidity theorem for the (n, k) -stabilizer game guarantees that the players have to measure honestly.

Nonlocal Game for QMIP*

The referee does the following with equal probability:

1. Plays the (n, k) -stabilizer game in Fig. 4 with players $(1'), (2'), \dots, (8')$ and rejects or accepts accordingly.
2. Simulates the extended nonlocal game using logical X, Z measurements.

Figure 14: The nonlocal game for QMIP*.

Theorem 34. For $r \in \text{poly}$, $s \in 1 - \text{poly}^{-1}$ there is an $s' \in 1 - \text{poly}^{-1}$. For any language $A \in \text{QMIP}^*(r, 3, 1, s)$ and an instance x , the nonlocal game in Fig 14 satisfies that

1. If $x \in A$, the referee accepts with certainty;
2. If $x \notin A$, the referee accepts with probability at most s' .

Proof. Suppose that the referee rejects with probability ϵ in the first part. Theorem 17 then implies that the players must play approximately honestly for the measurement specification type of questions. Therefore the referee accepts with probability at most

$$\frac{1}{2}(1 - \epsilon) + \frac{1}{2} \min(1, s_1 + cn^\kappa \epsilon^{1/\kappa}), \quad (58)$$

where s_1 is the maximum acceptance probability for the second part where all the eight players $(1'), (2'), \dots, (8')$ measures honestly. By Theorem 33, it follows that $s_1 \in 1 - \text{poly}^{-1}$. The proof then follows from Lemma 31. \square

Our main theorem (Theorem 1) follows by observing that the questions of the nonlocal game in Fig. 14 are measurement specification of at most k Pauli operators of weight at most k and can be encoded with logarithmically number of bits.

References

- [1] AHARONOV, D., AND NAVEH, T. Quantum NP - A Survey. arXiv:quant-ph/0210077, 2002.
- [2] ARORA, S., LUND, C., MOTWANI, R., SUDAN, M., AND SZEGEDY, M. Proof verification and the hardness of approximation problems. *J. ACM* 45, 3 (1998), 501–555.
- [3] ARORA, S., AND SAFRA, S. Probabilistic Checking of Proofs: A New Characterization of NP. *J. ACM* 45, 1 (1998), 70–122.
- [4] BABAI, L. Trading group theory for randomness. In *Proceedings of the Seventeenth Annual ACM Symposium on Theory of Computing* (1985), STOC '85, pp. 421–429.
- [5] BABAI, L., FORTNOW, L., AND LUND, C. Nondeterministic exponential time has two-prover interactive protocols. In *Proceedings of the 31st Annual Symposium on Foundations of Computer Science* (1990), SFCS '90, pp. 16–25.
- [6] BELL, J. On the Einstein Podolsky Rosen Paradox. *Physics* 1, 3 (1964), 195–200.
- [7] BEN-OR, M., GOLDWASSER, S., KILIAN, J., AND WIGDERSON, A. Multi-prover Interactive Proofs: How to Remove Intractability Assumptions. In *Proceedings of the Twentieth Annual ACM Symposium on Theory of Computing* (1988), STOC '88, pp. 113–131.
- [8] BROADBENT, A., JI, Z., SONG, F., AND WATROUS, J. Zero-knowledge proof systems for QMA. arXiv:1604.02804, 2016.
- [9] CAMERON, P. J., MONTANARO, A., NEWMAN, M. W., SEVERINI, S., AND WINTER, A. On the quantum chromatic number of a graph. *Electronic Journal of Combinatorics* 14 (2007), R81.
- [10] CHUNG, F. R. *Spectral Graph Theory*. No. 92 in CBMS Regional Conference Series. Conference Board of the Mathematical Sciences, 1996.
- [11] CLAUSER, J. F., HORNE, M. A., SHIMONY, A., AND HOLT, R. A. Proposed experiment to test local hidden-variable theories. *Phys. Rev. Lett.* 23, 15 (1969), 880–884.
- [12] CLEVE, R., HOYER, P., TONER, B., AND WATROUS, J. Consequences and limits of nonlocal strategies. In *Proceedings of the 19th Annual IEEE Conference on Computational Complexity* (2004), CCC '04, pp. 236–249.

- [13] CLEVE, R., AND MITTAL, R. Characterization of binary constraint system games. In *Automata, Languages, and Programming*, J. Esparza, P. Fraigniaud, T. Husfeldt, and E. Koutsoupias, Eds., vol. 8572 of LNCS. 2014, pp. 320–331.
- [14] COOK, S. A. The complexity of theorem-proving procedures. In *Proceedings of the third annual ACM symposium on Theory of computing (1971)*, STOC '71, pp. 151–158.
- [15] COOK, S. A. A hierarchy for nondeterministic time complexity. *Journal of Computer and System Sciences* 7 (1973), 343–353.
- [16] DINUR, I. The PCP Theorem by Gap Amplification. *J. ACM* 54, 3 (2007).
- [17] EASTIN, B., AND KNILL, E. Restrictions on transversal encoded quantum gate sets. *Phys. Rev. Lett.* 102 (2009), 110502.
- [18] FITZSIMONS, J., AND VIDICK, T. A Multiprover Interactive Proof System for the Local Hamiltonian Problem. In *Proceedings of the 2015 Conference on Innovations in Theoretical Computer Science (2015)*, ITCS '15, pp. 103–112.
- [19] GOLDREICH, O., MICALI, S., AND WIGDERSON, A. Proofs That Yield Nothing but Their Validity or All Languages in NP Have Zero-knowledge Proof Systems. *J. ACM* 38, 3 (1991), 690–728.
- [20] GOLDWASSER, S., MICALI, S., AND RACKOFF, C. The knowledge complexity of interactive proof-systems. In *Proceedings of the Seventeenth Annual ACM Symposium on Theory of Computing (1985)*, STOC '85, pp. 291–304.
- [21] GOLDWASSER, S., MICALI, S., AND RACKOFF, C. The knowledge complexity of interactive proof systems. *SIAM Journal on Computing* 18, 1 (1989), 186–208.
- [22] GOTTESMAN, D. *Stabilizer Codes and Quantum Error Correction*. PhD thesis, California Institute of Technology, 1997.
- [23] GOTTESMAN, D., AND IRANI, S. The Quantum and Classical Complexity of Translationally Invariant Tiling and Hamiltonian Problems. *Theory of Computing* 9, 2 (2013), 31–116.
- [24] ITO, T., KOBAYASHI, H., AND MATSUMOTO, K. Oracularization and two-prover one-round interactive proofs against nonlocal strategies. In *Proceedings of the 2009 24th Annual IEEE Conference on Computational Complexity (2009)*, CCC '09, pp. 217–228.
- [25] ITO, T., AND VIDICK, T. A Multi-prover Interactive Proof for NEXP Sound Against Entangled Provers. In *Proceedings of the 2012 IEEE 53rd Annual Symposium on Foundations of Computer Science (2012)*, FOCS '12, pp. 243–252.
- [26] JAIN, R., JI, Z., UPADHYAY, S., AND WATROUS, J. QIP = PSPACE. *J. ACM* 58, 6 (2011), 30.
- [27] JI, Z. Binary Constraint System Games and Locally Commutative Reductions. arXiv:1310.3794, 2013.
- [28] JI, Z. Classical verification of quantum proofs. In *Proceedings of the 48th Annual ACM SIGACT Symposium on Theory of Computing (New York, NY, USA, 2016)*, STOC 2016, ACM, pp. 885–898.
- [29] JOHNSTON, N., MITTAL, R., V., R., AND WATROUS, J. Extended nonlocal games and monogamy-of-entanglement games. *Proceedings of the Royal Society A* 472 (2016), 20160003.

- [30] JORDAN, C. Essai sur la géométrie à n dimensions. *Bulletin de la Société Mathématique de France* 3 (1875), 103–174.
- [31] KARP, R. M. Reducibility among combinatorial problems. In *Complexity of Computer Computations* (1972), R. E. Miller and J. W. Thatcher, Eds., pp. 85–103.
- [32] KEMPE, J., KITAEV, A., AND REGEV, O. The Complexity of the Local Hamiltonian Problem. *SIAM J. Comput.* 35, 5 (2006), 1070–1097.
- [33] KEMPE, J., KOBAYASHI, H., MATSUMOTO, K., TONER, B., AND VIDICK, T. Entangled games are hard to approximate. In *Proceedings of the 2008 49th Annual IEEE Symposium on Foundations of Computer Science* (2008), FOCS '08, pp. 447–456.
- [34] KEMPE, J., KOBAYASHI, H., MATSUMOTO, K., AND VIDICK, T. Using entanglement in quantum multi-prover interactive proofs. In *Proceedings of the 2008 IEEE 23rd Annual Conference on Computational Complexity* (2008), CCC '08, pp. 211–222.
- [35] KITAEV, A., AND WATROUS, J. Parallelization, amplification, and exponential time simulation of quantum interactive proof systems. In *Proceedings of the Thirty-second Annual ACM Symposium on Theory of Computing* (2000), STOC '00, pp. 608–617.
- [36] KITAEV, A. Y. Lecture given in Hebrew University, Jerusalem, Israel, 1999.
- [37] KITAEV, A. Y., SHEN, A. H., AND VYALYI, M. N. *Classical and Quantum Computation*. American Mathematical Society, 2002.
- [38] KOBAYASHI, H., AND MATSUMOTO, K. Quantum multi-prover interactive proof systems with limited prior entanglement. *Journal of Computer and System Sciences* 66, 3 (2003), 429–450.
- [39] KOCHEN, S. B., AND SPECKER, E. The problem of hidden variables in quantum mechanics. *Journal of Mathematics and Mechanics* 17 (1967), 59–87.
- [40] LANCE FORTNOW AND JOHN ROMPEL AND MICHAEL SIPSER. On the power of multi-prover interactive protocols. *Theoretical Computer Science* 134, 2 (1994), 545–557.
- [41] LEVIN, L. Universal search problems. *Problems of Information Transmission* 9, 3 (1973), 115–116.
- [42] LUND, C., FORTNOW, L., KARLOFF, H., AND NISAN, N. Algebraic methods for interactive proof systems. *J. ACM* 39, 4 (1992), 859–868.
- [43] MARRIOTT, C., AND WATROUS, J. Quantum Arthur-Merlin games. *Computational Complexity* 14, 2 (2005), 122–152.
- [44] MAYERS, D., AND YAO, A. Quantum cryptography with imperfect apparatus. In *Proceedings of the 39th Annual Symposium on Foundations of Computer Science* (1998), FOCS '98, p. 503.
- [45] MCKAGUE, M. Self-testing graph states. In *Theory of Quantum Computation, Communication, and Cryptography*, D. Bacon, M. Martin-Delgado, and M. Roetteler, Eds., vol. 6745 of LNCS. 2014, pp. 104–120.
- [46] MCKAGUE, M., YANG, T. H., AND SCARANI, V. Robust self-testing of the singlet. *Journal of Physics A: Mathematical and Theoretical* 45, 45 (2012), 455304.

- [47] MERMIN, N. D. Simple unified form for the major no-hidden-variables theorems. *Phys. Rev. Lett.* 65, 27 (1990), 3373–3376.
- [48] MILLER, C. A., AND SHI, Y. Optimal robust quantum self-testing by binary nonlocal XOR games. arXiv:1207.1819, 2012.
- [49] NATARAJAN, A., AND VIDICK, T. Constant-soundness interactive proofs for local Hamiltonians. arXiv:1512.02090, 2015.
- [50] OLIVEIRA, R., AND TERHAL, B. M. The complexity of quantum spin systems on a two-dimensional square lattice. *Quant. Inf. Comp.* 8, 10 (2008), 0900–0924.
- [51] PERES, A. Incompatible results of quantum measurements. *Phys. Lett. A* 151, 3–4 (1990), 107–108.
- [52] REICHARDT, B. W., UNGER, F., AND VAZIRANI, U. Classical command of quantum systems. *Nature* 496, 7446 (2013), 456–460.
- [53] ROBERSON, D. E., AND MANČINSKA, L. Graph homomorphisms for quantum players. arXiv:1212.1724, 2012.
- [54] SHAMIR, A. $IP = PSPACE$. *J. ACM* 39, 4 (1992), 869–877.
- [55] SHI, Y. Both Toffoli and Controlled-NOT need little help to do universal quantum computation. arXiv:quant-ph/0205115, 2002.
- [56] SINCLAIR, A. *Algorithms for Random Generation and Counting: A Markov Chain Approach*. Birkhauser Verlag, Basel, Switzerland, Switzerland, 1993.
- [57] SLOFSTRA, W. Tsirelson’s problem and an embedding theorem for groups arising from non-local games. arXiv:1606.03140, 2016.
- [58] TSIRELSON, B. Quantum generalizations of Bell’s inequality. *Letters in Mathematical Physics* 4 (1980), 93–100.
- [59] VAN DAM, W., MAGNIEZ, F., MOSCA, M., AND SANTHA, M. Self-testing of universal and fault-tolerant sets of quantum gates. In *Proceedings of the Thirty-second Annual ACM Symposium on Theory of Computing* (2000), STOC ’00, pp. 688–696.
- [60] VIDICK, T. Three-Player Entangled XOR Games Are NP-Hard to Approximate. *2013 IEEE 54th Annual Symposium on Foundations of Computer Science* (2013), 766–775.
- [61] VIDICK, T., AND WATROUS, J. Quantum proofs. *Foundations and Trends in Theoretical Computer Science* 11, 1-2 (2016), 1–215.
- [62] WATROUS, J. PSPACE has constant-round quantum interactive proof systems. In *Foundations of Computer Science, 1999. 40th Annual Symposium on* (1999), pp. 112–119.
- [63] WATROUS, J. Zero-knowledge against quantum attacks. *SIAM Journal on Computing* 39, 1 (2009), 25–58.
- [64] WERNER, R. F. Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model. *Phys. Rev. A* 40, 8 (1989), 4277–4281.

- [65] WINTER, A. Coding theorem and strong converse for quantum channels. *IEEE Transactions on Information Theory* 45, 7 (1999), 2481–2485.
- [66] ZENG, B., CROSS, A., AND CHUANG, I. L. Transversality versus universality for additive quantum codes. *IEEE Transactions on Information Theory* 57, 9 (2011), 6272–6284.