

Exponential Separation of Quantum Communication and Classical Information

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Abstract

We exhibit a Boolean function for which the *quantum communication complexity* is exponentially larger than the *classical information complexity*. An exponential separation in the other direction was already known from the work of Kerenidis et. al. [SICOMP 44, pp. 1550–1572], hence our work implies that these two complexity measures are incomparable. As classical information complexity is an upper bound on *quantum information complexity*, which in turn is equal to *amortized quantum communication complexity*, our work implies that a tight direct sum result for distributional quantum communication complexity cannot hold. The function we use to present such a separation is the **Symmetric k -ary Pointer Jumping** function introduced by Rao and Sinha [ECCC TR15-057], whose classical communication complexity is exponentially larger than its classical information complexity. In this paper, we show that the quantum communication complexity of this function is polynomially equivalent to its classical communication complexity. The high-level idea behind our proof is arguably the simplest so far for such an exponential separation between information and communication, driven by a sequence of round-elimination arguments, allowing us to simplify further the approach of Rao and Sinha.

As another application of the techniques that we develop, we give a simple proof for an optimal trade-off between Alice’s and Bob’s communication while computing the related **Greater-Than** function on n bits: say Bob communicates at most b bits, then Alice must send $\frac{n}{2^{O(b)}}$ bits to Bob. This holds even when allowing pre-shared entanglement. We also present a *classical* protocol achieving this bound.

1 Introduction

Communication complexity is a core topic of computational complexity which studies the number of bits that the participants in a communication protocol need to exchange in order to accomplish a distributed task. Designing generic lower bound methods for communication complexity has been a central endeavor since the birth of this subject, see [KN96, LS07] as excellent surveys. One of the most powerful lower bound methods for *randomized communication complexity* (RCC) is *information complexity* (IC) introduced in [CSWY01, BJKS02, BBCR10], which studies the amount of information about the inputs that the players need to reveal in order to accomplish a communication task. Investigations of information complexity have led to numerous elegant compression protocols, which in turn have led to direct sum and direct product results [JRS03a, BBCR10, BR11, JPY12, JY12, Bra12, Jai15, BRWY13a, BRWY13b, BW15] (and many other works).

The notion of information complexity appears in two flavors. The first is termed *external information complexity*, introduced by Chakrabarti, Shi, Wirth and Yao [CSWY01], which measures the amount of information about the inputs that the players reveal to an external observer in the protocol. Formally, it is defined as $I(XY : MR)$, the mutual information between XY and MR , where XY is the joint input to the players (with respect to an implicit prior distribution μ); M is the set of messages exchanged in the protocol and R is the public coins shared between the players. The second notion is that of (*internal*) *information complexity*, formally introduced by Barak, Braverman, Chen and Rao in [BBCR10] (building on a related notion introduced by Bar-Yossef, Jayram, Kumar and Sivakumar [BJKS02]) and defined as $I(X : MR|Y) + I(Y : MR|X)$.

Following is a central question in the field of communication complexity. Given a communication protocol with external information complexity I^{ext} , information complexity I and communication complexity C , where $I^{\text{ext}}, I \ll C$, can this protocol be simulated by another communication protocol (or *compressed*) with a much smaller amount of communication? After a decade's efforts, it is now known that any such protocol can be compressed to one with communication complexity $2^{\mathcal{O}(I)}$ [Bra12]; $\mathcal{O}(\sqrt{IC} \log C)$ [BBCR10]; $\mathcal{O}(I^{\text{ext}} \log^2 C)$ [BBCR10]. For r -round protocols, it can be compressed to $I + \mathcal{O}(\sqrt{rI} + r)$ [BR11]. If the distribution of the input is product, then recent results show that the protocol can be simulated by another protocol with communication complexity $\mathcal{O}(I^2 \text{polylog} I)$ due to Kol [Kol16], and to $\mathcal{O}(I \log^2 I)$ in a later improvement by Sherstov [She16].

An immediate question towards this line of research is whether compressing to $\mathcal{O}(I)$, or even $\text{poly}(I)$, is possible in general. This question is tightly connected to the direct sum question, and unfortunately, the answer is negative. In a sequence of breakthrough works, Ganor, Kol and Raz [GKR14, GKR15] exhibited a function with information complexity I that requires $2^{\Omega(I)}$ communication to solve with constant error, say, $1/3$. A significantly simpler proof was later given by Rao and Sinha [RS15b]. These works imply that Braverman's exponential simulation theorem [Bra12] is tight in some cases. Moreover, since information complexity is equal to *amortized communication complexity* [BR11], this also proves that a tight direct sum result for distributional communication complexity is not possible in general, resolving a longstanding open problem in communication complexity.

Much work has been devoted to seeking the quantum analog of information complexity, inspired by the numerous successful applications of information complexity in classical communication complexity. A major obstacle towards extending the notion of information complexity to the quantum setting is that the messages exchanged between the players in different rounds in general

do not exist at the same time due to the no-cloning theorem [WZ82, Die82]. In spite of this, Jain, Radhakrishnan and Sen [JRS08] defined an information theoretic notion of *privacy loss* and presented several elegant compression schemes for quantum protocols. The same set of authors [JRS03b] also proposed a different measure called *information loss* which extended the work [BJKS02], lower bounding the communication complexity of the **Set-Disjointness** function, to the quantum setting. Recently, Touchette [Tou15] has extended (internal) information complexity to the quantum setting by defining quantum information complexity (QIC), inspired by the *quantum state-redistribution protocols* [DY08, YD09]. QIC has been shown to satisfy many of the natural properties possessed by IC, and in particular, it is equal to amortized quantum communication complexity. Meanwhile, Touchette has also shown a direct sum result for *bounded-round quantum communication complexity*. To add to these developments, Braverman, Garg, Ko, Mao and Touchette [BGK⁺15] have used QIC in a crucial way to give a nearly tight bound on the bound-round quantum communication complexity of **Set-Disjointness**. More recently, Nayak and Touchette [NT16] used QIC to extend the work of Jain and Nayak [JN14], using Augmented Index to lower bound the space complexity of streaming algorithms for **DYCK(2)**.

We study the gap between quantum communication complexity (QCC) and IC. It is known that $\text{QCC}(f, 1/3) \leq 2^{\mathcal{O}(\text{QIC}(f, 1/3))} \leq 2^{\mathcal{O}(\text{IC}(f, 1/3))}$ [BGK⁺15] for any Boolean function f , where $\text{QCC}(f, 1/3)$, $\text{IC}(f, 1/3)$ and $\text{QIC}(f, 1/3)$ represent the minimum QCC, the minimum IC and the minimum IC of a protocol that computes f with error at most $1/3$, respectively. However, in contrast to the classical analog of this result, their proof does not proceed via a direct compression argument and much remains to be added in our understanding of interactive quantum compression.

1.1 Results and Contributions

In this paper we show that there exists a Boolean function with an exponential gap between its QCC and IC. This gap is as large as possible [Bra12].

Theorem 1.1. *There exists a (family of) Boolean function f and a distribution μ on its input such that $\text{QCC}(f, \mu, 1/3) \geq 2^{\Omega(\text{IC}(f, \mu, 1/3))} \geq 2^{\Omega(\text{QIC}(f, \mu, 1/3))}$.*

Combining with the fact that QIC is equal to the amortized quantum communication complexity, this shows that a tight direct sum result for distributional QCC is not possible. In fact, our results show that for the task we consider, the amortized *classical* communication is exponentially smaller than the *quantum* communication complexity. Notice that for the **Vector-in-Subspace Problem**, Kerenidis, Laplante, Lerays, Roland and Xiao [KLL⁺15] proved that its *quantum* communication complexity is exponentially smaller than its amortized *classical* communication. Our results thus imply that these two notions, QCC and IC, are incomparable.

In [GKR14, GKR15], Ganor et.al. introduced the **Bursting-Noise** function and proved that the RCC of this function is exponentially larger than its IC. To this end, they introduced a new lower bound method for RCC, namely the *relative discrepancy bound*, and showed that the relative discrepancy bound of the bursting noise function is exponentially larger than the IC. An immediate question, which would directly imply Theorem 1.1, is whether the relative discrepancy bound is also a lower bound on QCC or they are polynomially equivalent. The answer is negative. In [RK11], Klartag and Regev essentially showed that the relative discrepancy bound of **Vector-in-Subspace** problem is $\Omega(n^{1/3})$, while the QCC is $\mathcal{O}(\log n)$. Later, Rao and Sinha [RS15b] simplified Ganor et.al's result by defining a similar but relatively simpler function called **Symmetric k -ary Pointer Jumping** function, a symmetrized variant of the **Iterated Index** function [KNTZ01]. They introduced

and used the *fooling distribution method* to prove the lower bound on the RCC of this function. However, in the same paper, they also showed that fooling distribution method subsumes the relative discrepancy bound, so that we cannot directly rely on their fooling distribution method to prove our desired separation. Currently, other than QIC, the strongest method to prove QCC lower bounds is γ_2 /*generalized discrepancy* [Kla07, She08]. However, at least in the prior-free setting, the generalized discrepancy is known to be upper bounded by QIC due to [BGK⁺15]. Moreover, in the distributional setting, the generalized discrepancy is known to lower bound IC [KLL⁺15], which we know is low for the task we consider. In particular, our result imply that for some specific functions, like the one we consider here, the generalized discrepancy bound can be exponentially smaller than the QCC. Hence, to prove Theorem 1.1, we need new techniques to prove the lower bound on QCC.

The function we use to exhibit the exponential separation is the *Symmetric k -ary Pointer Jumping* function, the same function used by Rao and Sinha [RS15b] to show the exponential gap between RCC and IC. To reach our goal of showing that QCC is also large, we adopt the same framework as developed in [RS15b], and essentially show that for their task, the fooling distribution they defined is also a *quantum fooling distribution*. However, the proof technique is significantly different from theirs. As explained above, a distribution fooling classical protocols with low communication does not necessary fools quantum protocols with low communication. Moreover, the proof in [RS15b] heavily relies on two ideas that have no clear quantum counterparts: first, that a protocol with low communication induces large monochromatic rectangles, and, second, that given a protocol with input XY drawn from a product distribution and a transcript M , $X - M - Y$ forms a Markov chain.

In order to avoid these obstacles, our proof is based on the *round elimination* technique [MNSW98, KNTZ01, JRS03b]. Even though we handle various technical difficulties surrounding quantum messages, we believe that, conceptually, the high-level outline of our proof, as described in section 3, is the simplest among aforementioned exponential separation results, simplifying further the ideas developed in [RS15b].

In particular, it is a simple consequence of our proof techniques that the *Greater-Than* function on n bits satisfies a communication trade-off similar to that of the *Index* function

Theorem 1.2. *In any (quantum) protocol computing Greater-Than on n bits with error $1/3$, if Bob communicates b bits to Alice, then Alice must communicate $\frac{n}{2^{O(b)}}$ bits to Bob.*

We provide a simple matching upper bound. To the best of our knowledge, this trade-off was not known before, even for classical communication [BW12, Vio13, RS15a]. This trade-off is the same as the one of *Index* function [MNSW98, JRS09], where Alice and Bob are given $x \in \{0, 1\}^n$ and $i \in [n]$, respectively, and $\text{Index}(x, i) \stackrel{\text{def}}{=} x_i$. In contrast to *Index* for which the upper bound can be achieved with only 2-messages (if Bob sends the first message), the protocol we give here to achieve the trade-off requires $\Omega(b)$ rounds of interaction if Bob sends fewer bits to Alice than Alice sends to Bob. Interaction is necessary here, since for any constant number of rounds r , the r -round communication complexity of *Greater-Than* on n bits is $\Omega(n^{1/r})$ [MNSW98].

We point out that the first communication task to be presented as a candidate separating information complexity from communication complexity [Bra13] was motivated by the *Greater-Than* function, and all tasks achieving such a separation have a hard distribution bearing some resemblance to the hard distribution for *Greater-Than*. We build on [RS15a], who gave a simple proof of the optimal symmetric $\Omega(\log n)$ lower bound, and apply our strengthening of a lemma, variants of which have appeared in all previous works on exponential separation between IC and RCC.

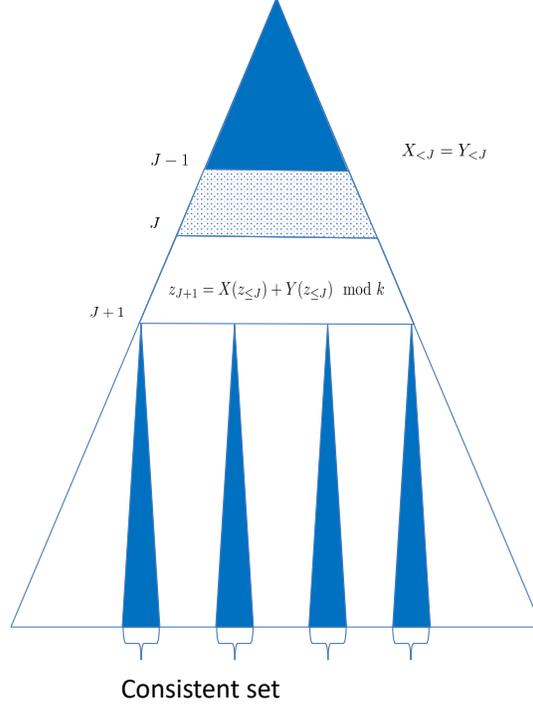


Figure 1: Depiction of the k -ary pointer jumping function. X and Y are defined for all internal nodes in a complete k -ary tree of depth n , and F and G are defined for all leaves. Given an hidden layer J , it holds that $X_{<J} = Y_{<J}$, and the set of consistent strings is defined through $X_J + Y_J \bmod k$. Under μ_b , $X_{>J} = Y_{>J}$ for all consistent internal nodes, and $F \oplus G = b$ for all consistent leaves.

2 The Function: Symmetric k -ary Pointer Jumping

To exhibit an exponential separation between QCC and IC, we consider the **Symmetric k -ary Pointer Jumping** function introduced in [RS15b], which in turn is based upon the ideas introduced in [Bra13, GKR14, GKR15]; see Figure 1.

We work with the set $[k] = \{0, 1, \dots, k-1\}$, endowed with addition (modulo k), and strings of elements from this set. For any integer j , the set of all strings of length less than j will be represented by $[k]^{<j}$. Another parameter characterizing this function is n . The functions $x, y : [k]^{<n} \rightarrow [k]$ map strings of length less than n to elements of $[k]$. The functions $f, g : [k]^n \rightarrow \{0, 1\}$ map strings of length n to binary values $\{0, 1\}$. Given an integer j and a string z with $|z| \geq j$, let $z_{\leq j}$ represent the string formed by taking the first j characters of z . Similarly, we define z_j as the j -th character of z . We use similar notation for the functions x, y , with $x_{\leq j}$ the restriction of x to strings z satisfying $|z| \leq j$, etc.

For an integer $j < n$ and functions x, y , we say that a string z is *consistent* with x, y, j if $|z| > j$ and it holds that $x(z_{\leq j}) + y(z_{\leq j}) = z_{j+1} \bmod k$. We follow [RS15b] and define a *quantum fooling distribution* p from which we derive a *hard distribution* μ by further conditioning p on an event \mathcal{E} .

We later show that low communication protocols cannot distinguish between 0-inputs to the hard distribution and inputs to the fooling distribution, and similarly for 1-inputs.

Definition 2.1. Fooling Distribution $p(x, y, f, g, j)$: Let J be a random variable taking value uniformly at random in $\{0, 1 \dots n-1\}$. We define $p(x, y, f, g, j) \stackrel{\text{def}}{=} \Pr_J(j) \cdot p(x, y, f, g|j)$, where the conditional distribution $p(x, y, f, g|j)$ is defined as follows: x, y, f, g are chosen uniformly at random, subject to the constraint that for all $z \in [k]^{<j}$, $x(z) = y(z)$.

Definition 2.2. Hard Distribution $\mu(x, y, f, g, j)$: Let \mathcal{E}_0 be the event that for every x, y, f, g, j and every z consistent with this choice of x, y, j , $x(z) = y(z)$ (when $|z| < n$) and $f(z) = g(z)$ (when $|z| = n$). Let \mathcal{E}_1 be the event that for every x, y, f, g, j and every z consistent with this choice of x, y, j , $x(z) = y(z)$ (when $|z| < n$) and $f(z) \neq g(z)$ (when $|z| = n$). Let $\mathcal{E} \stackrel{\text{def}}{=} \mathcal{E}_0 \vee \mathcal{E}_1$, then $\mu(x, y, f, g, j) \stackrel{\text{def}}{=} p(x, y, f, g, j|\mathcal{E})$. We further denote $\mu_0 = \mu|\mathcal{E}_0 = p|\mathcal{E}_0$, and $\mu_1 = \mu|\mathcal{E}_1 = p|\mathcal{E}_1$, so that $\mu = \frac{1}{2}\mu_0 + \frac{1}{2}\mu_1$.

This allows us to define the inputs to Alice and Bob and the required task.

Definition 2.3. The Communication Task.

- A referee draws x, y, f, g, j from the distribution $\mu(x, y, f, g, j)$. Alice is given input (x, f) , and Bob input (y, g) . The index j is kept hidden from both parties.
- Let $\hat{z} \in [k]^n$ be the unique string that satisfies, for all $r > 0$ (and $r < n$), $x(\hat{z}_{\leq r}) + y(\hat{z}_{\leq r}) = \hat{z}_{r+1}$, and $x(\epsilon) + y(\epsilon) = \hat{z}_1$ for ϵ the empty string. Alice and Bob must output $f(\hat{z}) + g(\hat{z}) \bmod 2$.

An important property of the distribution $\mu(x, y, f, g, j)$ is that the output $f(\hat{z}) + g(\hat{z}) \bmod 2$ is the same on all consistent strings, simply because $f(z) = g(z)$ (or $f(z) \neq g(z)$) on all consistent strings z , and the unique string \hat{z} on which $f(\hat{z}) + g(\hat{z}) \bmod 2$ must be evaluated is also a consistent string. Thus, we define S to be the set of all consistent strings for a given tuple x, y, j . This allows us to extend the definition of distributions p and μ to include S , as $p(x, y, f, g, s, j)$ and $\mu(x, y, f, g, s, j)$.

The proof of our main theorem, Theorem 1.1, follows from the following two theorems.

Theorem 2.4. *There exists a quantum protocol that accomplishes the communication task from Definition 2.3 with error $\varepsilon \leq \frac{1}{\log n}$ and with QIC upper bounded by IC, which in turn is upper bounded by $\mathcal{O}(\log(k \log n)2^{\frac{2 \log n}{k}})$.*

Theorem 2.5. *Any protocol which accomplishes the communication task from Definition 2.3 with constant error $\varepsilon \in (0, \frac{1}{2})$ requires a quantum communication cost lower bounded by $\min \left\{ \Omega \left(k^{1/5} \right), \Omega(\log n) \right\}$.*

If we choose $k = \log n$, then the IC is $\mathcal{O}(\log k)$ while the QCC is $\Omega \left(k^{1/5} \right)$.

Our technical contributions go into proving the lower bound on QCC stated in Theorem 2.5. The upper bound of QIC in Theorem 2.4 follows by combining the two theorems below, proven in [RS15b] and [LT17], respectively.

Theorem 2.6. [RS15b] *There exists a classical protocol that accomplishes the communication task from Definition 2.3 with constant error $\varepsilon > 0$ and with IC upper bounded by $\mathcal{O}(\log(k \log n)2^{\frac{2 \log n}{k}})$.*

Theorem 2.7. [LT17] *For any classical protocol Π , there exists a quantum protocol Π' exactly simulating the input-output behavior of Π while maintaining the same communication pattern as (the padded version of) Π , and also satisfying $\text{QIC}(\Pi', \mu) = \text{IC}(\Pi, \mu)$ for all μ .*

In [LT17], the bulk of the effort for showing the theorem about the quantum simulation of classical protocols goes into arguing how to quantumly simulate private randomness without affecting the information cost. Note that we could alternatively use the fact that IC is equal to amortized communication complexity to argue that the IC is also an upper bound on the QIC for any communication task in the distributional setting: $\text{QIC}(f, \mu, \epsilon) = \text{AQCC}(f, \mu, \epsilon) \leq \text{ACC}(f, \mu, \epsilon) = \text{IC}(f, \mu, \epsilon)$.

3 High-Level Proof Sketch for the Communication Lower Bound

In this section, we give a high-level proof sketch of Theorem 2.5. We also formally state the main technical lemmata that go into the proof. Formal proofs are given in Section 6. Our strategy for proving the lower bound is divided into two main steps.

- We first consider the fooling distribution $p(x, y, f, g, j)$ and show that in any quantum protocol Π with small communication, the state of the registers with Bob is almost independent of $X_S F_S$, conditioned on $x_{\leq j} y_{\leq j} j$, and similarly the state of the registers with Alice is almost independent of $Y_S G_S$, conditioned on $x_{\leq j} y_{\leq j} j$. For this, we argue by performing two different reductions to one-round protocols.
- Using the observation that, conditioned on $x_{\leq j} y_{\leq j} j$, $p(x, y, f, g, j)$ and $\mu(x, y, f, g, j)$ have the same marginals on (x, f) , and also the same marginals on (y, g) , we show that the ‘approximate independence’ concluded above for $p(x, y, f, g, j)$ implies that the final state on Alice’s or Bob’s registers is approximately the same for inputs according to either of $\mu_0(x, y, f, g) \stackrel{\text{def}}{=} p(x, y, f, g | \mathcal{E}_0)$, $\mu_1(x, y, f, g) \stackrel{\text{def}}{=} p(x, y, f, g | \mathcal{E}_1)$ or $p(x, y, f, g)$. For this, we argue by performing a round-by-round elimination.

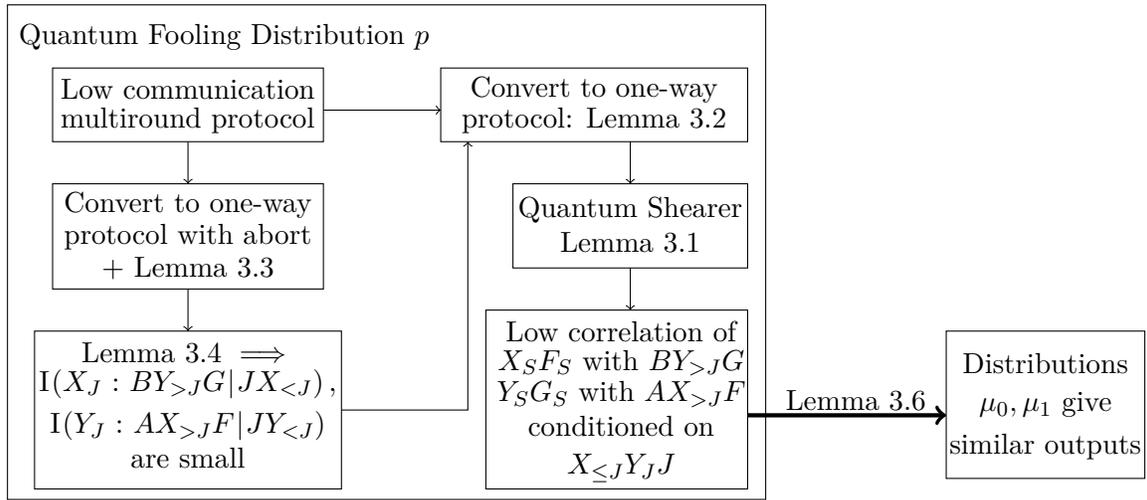


Figure 2: Structure of our proof. We have ignored purification of input registers for simplicity of presentation.

A sketch of our proof strategy appears in Figure 2. In more details, let us first consider the simpler case of a single-message protocol from Alice to Bob, under distribution p , with some fixed value of $y_{\leq j}$. As discussed above, we show that the output under p and the output under μ_0 are close, given that the message is short. A similar argument holds for μ_1 , leading to a contradiction. Denote by M_1 the register holding the first message (and possibly some pre-shared entanglement). Notice that for a single message, since the marginal on (x, f) is the same in p and μ_0 , the state on registers $XF M_1$ is also the same under these two distributions. But the correlations with Bob's input (y, g) are different: since XF is independent of YG under p (conditioned on the fixed value of $y_{\leq j}$), M_1 is also independent of YG ; whereas under μ_0 (and similarly μ_1), $X_S F_S = Y_S G_S$ which means that M_1 is highly correlated with $Y_S G_S$ (more precisely $Y_S G_S M_1 = X_S F_S M_1$). Notice that on restricting to the complement of S , $Y_{> j} G$ is independent of $X_S F_S M_1$ and distributed in the same way under both p and μ_0 . Now, the distance between the final output under p and under μ_0 can be upper bounded, using monotonicity, by the distance between $Y_S G_S \otimes M_1$ (under p) and $Y_S G_S M_1$ (under μ_0). By the above argument, this is same as the distance between $X_S F_S \otimes M_1$ (under p) and $X_S F_S M_1$ under μ_0 (which is distributed as $X_S F_S M_1$ under p). This is in turn upper bounded by the mutual information between $X_S F_S$ and M_1 under the distribution p . To complete the argument, we use the following lemma, which can be thought of as a quantum version of Shearer's Lemma [CGFS86, Rad03] for mutual information.

Lemma 3.1. *Consider registers U_1, U_2, \dots, U_m, V and define $U \stackrel{\text{def}}{=} U_1 U_2 \dots U_m$. Consider a quantum state Ψ_{UV} such that $\Psi_{U_1, U_2, \dots, U_m} = \Psi_{U_1} \otimes \Psi_{U_2} \otimes \dots \otimes \Psi_{U_m}$. Let $S = \{i_1, \dots, i_{|S|}\} \subseteq [m]$ be a random set independent of Ψ_{UV} satisfying $\Pr[i \in S] \leq \frac{1}{k}$ for all i and $U_S \stackrel{\text{def}}{=} U_{i_1} U_{i_2} \dots U_{i_{|S|}}$. Then it holds that*

$$I(U_S : V | S)_\Psi \leq \frac{I(U : V)_\Psi}{k},$$

Now, to extend the above argument to multi-round protocols, we want to ensure that even if Alice knows some information about S , the argument still goes through, as long as her information about S is small. We do so by specially crafting an input to the protocol and then reducing it to an essentially equivalent one-round protocol. For this, we use an asymmetric round-compression argument from [JRS05] to generate the state in each round of the protocol, up to a small error, by a one-way protocol with communication cost close to that in the original protocol. We also require a similar argument on Bob's side. Formally, we prove the following result, with some extra care needed since we wish, for technical reasons, to maintain correlations with the reference registers.

Lemma 3.2. *Consider a quantum state $|\Psi\rangle = \sum_{xy} \sqrt{\mu(x, y)} |xxy\rangle_{R_X X R_Y Y} \otimes |\psi^{xy}\rangle_{AB}$ satisfying $I(Y : R_X X A)_\Psi \leq \epsilon$, where $\mu = \mu_X \otimes \mu_Y$ is a product distribution, and the register X and register Y are held by Alice and Bob, respectively. Given $\delta > 0$, there exists a one-way quantum protocol where Alice sends $\mathcal{O}((I(X : Y R_Y B)_\Psi + 1)/\delta^2)$ qubits to Bob. Let $\tilde{\Psi}$ be the global state in the end of this protocol. It holds that ¹*

$$h^2(\tilde{\Psi}_{X A B Y R_Y}, \Psi_{X A B Y R_Y}) \leq 4\delta^2 + 6\epsilon,$$

To prove that the information about S is small, first notice that for fixed $x_{\leq j}$, S is determined by y_j , and vice-versa. Hence, we wish to bound the amount of information about Y_j that Alice has in

¹ $h(\cdot, \cdot)$ denotes the Hellinger distance which will be defined in section 5.

any round, conditioned on some fixed values of $x_{\leq j}$. In all previous works [GKR16, GKR15, RS15b] on exponential separation between information and communication, the proof relied on a statement of the form “the information Alice has about the j -th part of Bob’s input is upper bounded by $\frac{2^{O(\ell)}}{n}$ ”. This holds even when conditioning on some j playing a role similar to the hidden index j here, and also on some part of Alice’s input corresponding to j . ℓ is the total number of bits of communication in the protocol, and n is the number of parts of Alice’s input (usually related to the depth of some underlying communication tree), of size exponentially larger than the desired communication bound. This is usually proved via involved information-theoretic arguments that make use of the rectangular nature of classical protocols, hence such proof cannot be generalized to the quantum setting at all. We give a very simple two-step argument to achieve similar bounds. First, we once again use a reduction to a one-way protocol. Second, for such one-way protocols, we can use a simple direct sum argument and avoid the exponential blow-up. Formally, we have the following lemma for one-way protocols, variants of which have appeared in [KNTZ01, SV01].

Lemma 3.3. *Let Π be a quantum one-way protocol with correlated inputs XY , in which Alice sends ℓ qubits to Bob. Let $X = X_1 \cdots X_n$, and for a uniformly random index $J \in_R [n]$, decompose $Y = Y_1^J Y_2^J$ such that Y_2^J is a function of $X_{<J}$ and $(X_{\geq J} Y_1^J | JX_{<J} = jx_{<j}) = (X_{\geq J} \otimes Y_1^J | JX_{<J} = jx_{<j})$ for any $jx_{<j}$, that is, conditioned on J and $X_{<J}$, $X_{\geq J}$ and Y_1^J are independent. Let $\rho_{X R_X Y R_Y A B C}$ be the global state in the end of the protocol, where A is the register with Alice; C is the register of the message Alice sends to Bob; B is the register with Bob before receiving the message and $R_X R_Y$ are the canonical purification of the input XY . Then it holds that*

$$I(X_J : C B Y_1^J R_{Y_1^J} | JX_{<J})_\rho \leq \frac{2\ell}{n}. \quad (1)$$

Second, to extend the lemma to multiple-round protocols, we still have “enough room” to perform a one-way simulation of any interactive protocol, with at most an exponential blow-up in the communication and still achieve similar bounds as in the classical setting. Formally, we prove the following result by appealing to both compression arguments and to the notion of protocols with abort [KLL⁺15, LLR12], with some extra care needed since we again wish, for technical reasons, to maintain correlations with the reference registers.

Lemma 3.4. *Let Π be a quantum protocol with correlated input XY . Let $X = X_1 \cdots X_n$, and for a uniformly random index $J \in_R [n]$, decompose $Y = Y_1^J Y_2^J$ such that Y_2^J is a function of $X_{<J}$ and $(X_{\geq J} Y_1^J | JX_{<J}) = (X_{\geq J} \otimes Y_1^J | JX_{<J})$, that is, conditional on $JX_{<J}$, $X_{\geq J}$ and Y_1^J are independent. Then, for any r , it holds that*

$$I(X_J : C_r B_r Y_1^J R_{Y_1^J} | JX_{<J}) \leq \frac{\ell_{A,r} 2^{2\ell_{B,r}+2}}{n}, \quad (2)$$

where $\ell_{A,r}$ and $\ell_{B,r}$ are the number of qubits Alice and Bob send in the first r rounds, respectively.

Finally, in order to go from the distribution p to the distribution μ_0 , we have the following *distributional cut-and-paste* lemma. Intuitively, it states the following. Assume that in each round and on a product input distribution, the local states are almost independent of the other party’s input. Then, up to local isometries, the overall state stays independent of the joint input. Importantly, this holds even after conditioning the input distribution on an arbitrary joint event. Hence, if the input is replaced by another one with the same marginal distributions on both sides,

then the marginals of the global state in the final round on both sides are almost unchanged. Note that p and μ_0 have the same marginal distributions on the both sides and p is a product distribution conditioned on $x_{\leq j}y_{\leq j}$. Thus the following lemma enables us to show that neither Alice nor Bob is able to distinguish p from μ_0 and equivalently p from μ_1 . The lemma could be interesting on its own and we believe it should have other applications in quantum communication complexity. The proof is inspired from quantum versions of the cut-and-paste lemma [JRS03b, JN14, NT16], with extra care needed to go from one distribution to the other. Let us set some notation before stating the lemma.

Definition 3.5. Consider a protocol Π , and states $|\rho\rangle_{XR_XYR_Y} = |\rho\rangle_{XR_X} \otimes |\rho\rangle_{YR_Y}$ and $|\sigma\rangle_{XYR_XR_Y}$ such that $\sigma_X = \rho_X$, $\sigma_Y = \rho_Y$, and $\rho_{XY} = \rho_X \otimes \rho_Y$ and σ_{XY} are classical input distributions for Π with canonical purifications $|\rho\rangle_{XR_XYR_Y}$ and $|\sigma\rangle_{XYR_XR_Y}$, respectively. We denote by $|\rho^i\rangle_{XR_XYR_YA_iB_iC_i}$ and $|\sigma^i\rangle_{XR_XYR_YA_iB_iC_i}$ the state in round i when Π is run on input distributions ρ_{XY} and σ_{XY} , respectively. For any register L , we use \tilde{L} to represent a new register with the same dimension as L . For $i > 0$ odd, let

$$\epsilon_i \stackrel{\text{def}}{=} h(\rho_{R_XYR_YB_iC_i}^i, \rho_{R_X}^i \otimes \rho_{YR_YB_iC_i}^i), \quad (3)$$

and for $i > 0$ even,

$$\epsilon_i \stackrel{\text{def}}{=} h(\rho_{R_YXR_XA_iC_i}^i, \rho_{R_Y}^i \otimes \rho_{XR_XA_iC_i}^i). \quad (4)$$

For $i = 0$, let $C_0 = 1$ be a trivial register, let $\epsilon_0 = 0$ and let

$$V^0 \stackrel{\text{def}}{=} I_Y \otimes I_{B_0 \rightarrow \tilde{B}_0} \otimes V_{1 \rightarrow \tilde{Y}_0 \tilde{R}_{Y_0}}^Y, \quad (5)$$

in which $V_{1 \rightarrow \tilde{Y}_0 \tilde{R}_{Y_0}}^Y$ creates $|\rho^Y\rangle_{\tilde{Y}_0 \tilde{R}_{Y_0}}$ from nothing.

Also let, for odd $i > 0$, $V^i = V_{XA_i \rightarrow X\tilde{A}_i \tilde{X}_i \tilde{R}_{X_i}}^i$, satisfying

$$\epsilon_i = h(V^i(\rho_{XR_XYR_YA_iB_iC_i}^i), \rho_{XR_X} \otimes \rho_{\tilde{X}_i \tilde{R}_{X_i} YR_Y \tilde{A}_i B_i C_i}^i), \quad (6)$$

$$(7)$$

(note that $B_i = B_{i-1}$ for odd $i > 0$, and $A_i = A_{i-1}$ for even $i > 0$) and for $i > 0$ even, $V^i = V_{YB_i \rightarrow Y\tilde{B}_i \tilde{Y}_i \tilde{R}_{Y_i}}^i$ satisfying

$$\epsilon_i = h(V^i(\rho_{XR_XYR_YA_iB_iC_i}^i), \rho_{YR_Y} \otimes \rho_{XR_X \tilde{Y}_i \tilde{R}_{Y_i} A_i \tilde{B}_i C_i}^i). \quad (8)$$

$$(9)$$

The existence of V^i 's is guaranteed by Fact 5.7.

Lemma 3.6. *With the notation from Definition 3.5, let, for odd $i > 0$,*

$$\gamma_i = h(V^i V^{i-1}(\rho_{XR_XYR_YA_iB_iC_i}^i), \rho_{XR_XYR_Y} \otimes \rho_{\tilde{X}_i \tilde{R}_{X_i} \tilde{Y}_{i-1} \tilde{R}_{Y_{i-1}} \tilde{A}_i \tilde{B}_i C_i}^i), \quad (10)$$

and

$$\delta_i = h(V^i V^{i-1} (\sigma_{X R_X Y R_Y A_i B_i C_i}^i), \sigma_{X Y R_X R_Y} \otimes \rho_{\tilde{X}_i \tilde{R}_{X_i} \tilde{Y}_{i-1} \tilde{R}_{Y_{i-1}} \tilde{A}_i \tilde{B}_i C_i}^i), \quad (11)$$

and for $i > 0$ even, let

$$\gamma_i = h(V^i V^{i-1} (\rho_{X R_X Y R_Y A_i B_i C_i}^i), \rho_{X R_X Y R_Y} \otimes \rho_{\tilde{X}_{i-1} \tilde{R}_{X_{i-1}} \tilde{Y}_i \tilde{R}_{Y_i} \tilde{A}_i \tilde{B}_i C_i}^i), \quad (12)$$

and

$$\delta_i = h(V^i V^{i-1} (\sigma_{X R_X Y R_Y A_i B_i C_i}^i), \sigma_{X Y R_X R_Y} \otimes \rho_{\tilde{X}_{i-1} \tilde{R}_{X_{i-1}} \tilde{Y}_i \tilde{R}_{Y_i} \tilde{A}_i \tilde{B}_i C_i}^i). \quad (13)$$

Then it holds that for $i \geq 1$,

$$\gamma_i \leq \epsilon_i + \epsilon_{i-1} + 2 \sum_{j=1}^{i-2} \epsilon_j, \quad \delta_i \leq \epsilon_i + \epsilon_{i-1} + 2 \sum_{j=1}^{i-2} \epsilon_j.$$

The theorem follows by blending all of these ingredients together, using a concavity argument, and also optimizing over the number of rounds t .

Also, note that the polynomial rather than linear dependence on k is due to the last round-elimination argument, in Lemma 3.6, which works in a round-by-round fashion and from which a factor of t , the number of rounds, comes out and over which we must optimize. The other lemmata do not incur such blow-up, and if we take the corresponding lemmata in the classical setting, we could further use the Markov property of classical protocol run on product distributions along with the specific “ $x = y$ ” event, as done in Lemma 5 in [RS15b] in order to obtain a tight $\Omega(k)$ lower bound. Obtaining tight round elimination arguments in the quantum setting remains an important open question, and another interesting open question is whether one can avoid such a round-by-round argument, and the extra factor of t coming out of it, to complete the proof in the quantum setting as well.

4 Warm-up: Trade-off for Greater-Than

In this section, we investigate the trade-off between the communication from Alice to Bob and the one from Bob to Alice for Greater-Than function. For $x, y \in \{0, 1\}^n$, we define $x \geq y$ if the integer with binary representation x is at least as large as the integer with binary representation y . The Greater-Than function is defined as

$$\text{Greater-Than}(x, y) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } x \geq y, \\ 0 & \text{otherwise.} \end{cases}$$

Let us restate Theorem 1.2 more formally.

Theorem 4.1. *Given any constant $0 < \epsilon < \frac{1}{2}$ and a quantum protocol that computes Greater-Than: $\{0, 1\}^n \times \{0, 1\}^n \rightarrow \{0, 1\}$ with error at most ϵ , if Bob communicates b qubits to Alice, then Alice must communicate at least $\frac{n}{2^{\Omega(b+1)}}$ qubits to Bob. Moreover, this trade-off is tight.*

Proof. By a standard repetition argument, we may assume without loss of generality that ϵ is a sufficiently small constant ; this can at most increase Alice’s and Bob’s respective communication by a constant multiplicative factor. Suppose Alice communicates $a \geq 1$ qubits. Then by the proof of Lemma 3.4, there exists a one-way quantum protocol that computes **Greater-Than** with communication $a \cdot 2^{\mathcal{O}(b)}$ and error at most 2ϵ . Thus it suffices to show that the quantum one-way communication complexity of **Greater-Than** is $\Omega(n)$. Our proof is close to the one in [RS15a], where Ramamoorthy and Sinha provided a tight lower bound on the RCC of **Greater-Than**, $\Omega(\log n)$. We adopt the hard distribution of the inputs given in [RS15a] (slightly adapted from [BW12, Vio13]) and show that the distributional quantum one-way communication complexity of **Greater-Than** under this distribution is $\Omega(n)$. Then we further apply Yao’s minimax theorem [Yao79] to get the desired lower bound.

Let $J \in [\frac{n}{2}]$ be uniformly random. $X, Y \in \{0, 1\}^n$ are sampled uniformly conditioned on the event that $X_{<J} = Y_{<J}$, where $X_{<J} \stackrel{\text{def}}{=} X_1 \dots X_{J-1}$. Let Π be a quantum one-way protocol that computes **Greater-Than** with communication at most c and error at most 2ϵ . We use ACB to represent the state shared between Alice and Bob after Alice sends the message, where A is the remaining register with Alice; C is the register sent to Bob and B is the register owned by Bob in the beginning of the protocol (B is independent of the inputs). C contains at most c qubits. Consider

$$\begin{aligned} I(CBY : X_J | X_{<J} J) &= \mathbb{E}_{j \leftarrow J} [I(CB : X_j | X_{<j} j)] = \mathbb{E}_{j \leftarrow J} [I(CB : X_j | X_{<j})] = \frac{2}{n} I(CB : X_{\leq \frac{n}{2}}) \\ &= \frac{2}{n} I(C : X_{\leq \frac{n}{2}} | B) \leq \frac{4c}{n}; \end{aligned} \tag{14}$$

where the second equality is from the fact that J is independent of CBX_J given $X_{<J}$; the third equality is by the chain rule; the fourth equality is from the fact that B is independent of the inputs; the inequality is from Fact 5.17. Let O be the output of the protocol. The following claim is proved in [RS15a].

Claim 4.2. [RS15a] Suppose $n > 20$, it holds that

$$I(\text{Greater-Than}(X, Y) : O | X_{<J} Y_{<J} J) \geq 1 - \mathcal{O}\left(\sqrt{\epsilon} \log \frac{1}{\epsilon}\right), \tag{15}$$

and

$$I(\text{Greater-Than}(X, Y) : O | X_{\leq J} Y_{<J} J) < 0.84. \tag{16}$$

Hence,

$$\begin{aligned} &I(\text{Greater-Than}(X, Y) : O | X_{<J} Y_{<J} J) \\ &\leq I(X_J \text{Greater-Than}(X, Y) : O | X_{<J} Y_{<J} J) \\ &\leq I(X_J : O | X_{<J} Y_{<J} J) + I(\text{Greater-Than}(X, Y) : O | X_{\leq J} Y_{<J} J) \\ &\leq I(X_J : CBY_{\geq J} | X_{<J} Y_{<J} J) + I(\text{Greater-Than}(X, Y) : O | X_{\leq J} Y_{<J} J) \\ &\leq \frac{4c}{n} + 0.84; \end{aligned}$$

where the third inequality is from Fact 5.14 and the last inequality is from Eqs. (14) and (16) . Combining with Eq. (15), the result follows.

To prove the tightness, let's assume without loss of generality that Alice sends more qubits to Bob than Bob sends to Alice. It is well-known that the RCC of Greater-Than with bounded error is $\mathcal{O}(\log n)$ due to Nisan [Nis94]. Thus it suffices to consider the case that $\frac{n}{2^b} = n^{\Omega(1)}$. To achieve such a bound, Alice and Bob first check whether $x = y$ using shared hashing function with $\mathcal{O}(1)$ bits. Then, they equally divide the inputs into $2^{\Omega(b)}$ intervals of $\frac{n}{2^{\Omega(b)}}$ bits before running the protocol in Fact 4.3 below, in order to find the interval containing the most significant bit for which x and y differ. Alice further sends the part of her input in that interval to Bob, which requires $\frac{n}{2^{\Omega(b)}}$ bits, larger than b . Hence the total communication from Alice to Bob is $\frac{n}{2^{\Omega(b+1)}}$. \square

Fact 4.3. [FRPU94] There exists a randomized public-coin protocol with communication complexity $\mathcal{O}(\log k/\epsilon)$ such that on input two strings $x, y \in \mathcal{X}^k$, where \mathcal{X} is a finite set, it outputs the smallest index $i \in [k]$ such that $x_i \neq y_i$ with probability at least $1 - \epsilon$, if such i exists.

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5 Preliminaries

5.1 Information Theory

For an integer $n \geq 1$, let $[n]$ represent the set $\{1, 2, \dots, n\}$. Let \mathcal{X} and \mathcal{Y} be finite sets and k be a natural number. Let \mathcal{X}^k be the set $\mathcal{X} \times \dots \times \mathcal{X}$, the Cartesian product of \mathcal{X} , k times. Given $a = a_1, \dots, a_k$, we write $a_{\leq i}$ to denote a_1, \dots, a_i . We define $a_{< i}, a_{\geq i}, a_{> i}$ similarly. We write a_S to represent the projection of a to the coordinates specified in the set $S \subseteq [k]$. Let μ be a probability distribution on \mathcal{X} . Let $\mu(x)$ represent the probability of $x \in \mathcal{X}$ according to μ . Let X be a random variable distributed according to μ . We use the same symbol to represent a random variable and its distribution whenever it is clear from the context. The expectation value of function f on \mathcal{X} is defined as $\mathbb{E}_{x \leftarrow X}[f(x)] \stackrel{\text{def}}{=} \sum_{x \in \mathcal{X}} \Pr[X = x] \cdot f(x)$, where $x \leftarrow X$ means that x is drawn according to the distribution of X .

A quantum state (or just a state) ρ is a positive semi-definite matrix with unit trace. It is called pure if its rank is 1. For unit vector $|\psi\rangle$, with slight abuse of notation, we use ψ to represent the state and also the density matrix $|\psi\rangle\langle\psi|$, associated with $|\psi\rangle$. A classical distribution μ can be viewed as a diagonal quantum state with entries $\mu(x)$. For two quantum states ρ and σ , $\rho \otimes \sigma$ represents the tensor product (Kronecker product) of ρ and σ . A quantum super-operator $\mathcal{E}(\cdot)$ is

a completely positive and trace preserving (CPTP) linear map from states to states. Readers can refer to [CT91, NC00, Wat11, Wil13] for more details.

Definition 5.1. For quantum states ρ and σ , the ℓ_1 -distance between them is given by $\|\rho - \sigma\|_1$, where $\|X\|_1 \stackrel{\text{def}}{=} \text{Tr}\sqrt{X^\dagger X}$ is the sum of the singular values of X . We say that ρ is ε -close to σ if $\|\rho - \sigma\|_1 \leq \varepsilon$.

Definition 5.2. For quantum states ρ and σ , the *fidelity* between them is given by $F(\rho, \sigma) \stackrel{\text{def}}{=} \|\sqrt{\rho}\sqrt{\sigma}\|_1$. The *Hellinger distance* between them is defined as $h(\rho, \sigma) = \sqrt{1 - F(\rho, \sigma)}$. We also use $h\left(\frac{\rho}{\sigma}\right)$ for overlong expressions.

The following fact relates the ℓ_1 -distance and the fidelity between two states.

Fact 5.3 (Fuchs-van de Graaf inequalities [FVDG99]). For quantum states ρ and σ , it holds that

$$2(1 - F(\rho, \sigma)) = 2h^2(\rho, \sigma) \leq \|\rho - \sigma\|_1 \leq 2\sqrt{1 - F(\rho, \sigma)^2}.$$

For pure states $|\phi\rangle$ and $|\psi\rangle$, we have

$$\begin{aligned} \||\phi\rangle\langle\phi| - |\psi\rangle\langle\psi|\|_1 &= \sqrt{1 - F(|\phi\rangle\langle\phi|, |\psi\rangle\langle\psi|)^2} \\ &= \sqrt{1 - |\langle\phi|\psi\rangle|^2}. \end{aligned}$$

We use capital letters A, B, \dots to represent the registers; $\mathcal{H}_A, \mathcal{H}_B, \dots$ to represent the Hilbert spaces associated to them and $\mathcal{D}_A, \mathcal{D}_B, \dots$ to represent the set of all quantum states in $\mathcal{H}_A, \mathcal{H}_B, \dots$. For any register A , $|A|$ represents the number of qubits it contains, or equivalently, $\log \dim \mathcal{H}_A$. For bipartite ρ_{AB} , we define

$$\rho_B \stackrel{\text{def}}{=} \text{Tr}_A(\rho_{AB}) \stackrel{\text{def}}{=} \sum_i (\langle i| \otimes \mathbb{1}_B) \rho_{AB} (|i\rangle \otimes \mathbb{1}_B)$$

where $\{|i\rangle\}_i$ is a basis for the Hilbert space \mathcal{H}_A and $\mathbb{1}_B$ is the identity matrix in space \mathcal{H}_B . Tr_A is called the partial trace operation. The state ρ_B is referred to as the marginal state of ρ_{AB} in register B . The following fact states that the distance between two states can't be increased by quantum operations.

Fact 5.4. For states ρ, σ , and quantum operation $\mathcal{E}(\cdot)$, it holds that

$$\|\mathcal{E}(\rho) - \mathcal{E}(\sigma)\|_1 \leq \|\rho - \sigma\|_1$$

and

$$F(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \geq F(\rho, \sigma).$$

In particular, for any bipartite states ρ_{AB} and σ_{AB} , it holds that

$$\|\rho_{AB} - \sigma_{AB}\|_1 \geq \|\rho_A - \sigma_A\|_1, F(\rho_{AB}, \sigma_{AB}) \leq F(\rho_A, \sigma_A) \text{ and } h(\rho_{AB}, \sigma_{AB}) \geq h(\rho_A, \sigma_A).$$

Fact 5.5. Given bipartite states $\rho_{AB} = \sum_i p_i |i\rangle\langle i| \otimes \rho_i$ and $\sigma_{AB} = \sum_i q_i |i\rangle\langle i| \otimes \sigma_i$, where $\{p_i\}_i$ and $\{q_i\}_i$ are distributions, it holds that

$$F(\rho_{AB}, \sigma_{AB}) = \sum_i \sqrt{p_i q_i} F(\rho_i, \sigma_i).$$

Definition 5.6. We say that a pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ is a purification of some state ρ if $\text{Tr}_A(|\psi\rangle\langle\psi|) = \rho$. If $\rho = \sum_i p(i) |i\rangle\langle i|$ is a classical state, we say the *canonical purification* of ρ is $\sum_i \sqrt{p(i)} |i\rangle |i\rangle$.

Fact 5.7 (Uhlmann's theorem). Given quantum states ρ, σ , and a purification $|\psi\rangle$ of ρ , it holds that $F(\rho, \sigma) = \max_{|\phi\rangle} |\langle\phi|\psi\rangle|$, where the maximum is taken over all purifications of σ . Let $\rho = \sum_i \alpha_i |u_i\rangle\langle u_i|$ and $\sigma = \sum_i \beta_i |v_i\rangle\langle v_i|$ be spectral decompositions of ρ and σ , respectively; $|\phi\rangle_{AB}$ and $|\psi\rangle_{AB}$ be purifications of ρ and σ , respectively, with Schmidt decomposition $|\phi\rangle = \sum_i \sqrt{\alpha_i} |u_i\rangle_A |u'_i\rangle_B$ and $|\psi\rangle = \sum_i \sqrt{\beta_i} |v_i\rangle_A |v'_i\rangle_B$. Let $\tilde{\rho}, \tilde{\sigma}$ be marginals of $|\phi\rangle, |\psi\rangle$ on register B respectively. Let U be the unitary such that $\sqrt{\tilde{\rho}}\sqrt{\tilde{\sigma}}U$ is positive semidefinite (guaranteed by the polar decomposition). Then $\langle\phi|(\mathbf{1}_A \otimes U)|\psi\rangle = F(\rho, \sigma)$. In particular, if $\tilde{\rho}, \tilde{\sigma}$ are classical-quantum states, then U can be assumed to be a controlled isometry on classical register.

Definition 5.8. The *entropy* of a quantum state ρ (in register X) is defined as $S(\rho) \stackrel{\text{def}}{=} -\text{Tr}\rho \log \rho$. We also let $S(X)_\rho$ represent $S(\rho)$.

Definition 5.9. The *relative entropy* between quantum states ρ and σ is defined as $D(\rho||\sigma) \stackrel{\text{def}}{=} \text{Tr}\rho \log \rho - \text{Tr}\rho \log \sigma$.

Definition 5.10. Let ρ_{XY} be a quantum state in space $\mathcal{H}_X \otimes \mathcal{H}_Y$. The *mutual information* between registers X and Y is defined to be

$$I(X : Y)_\rho \stackrel{\text{def}}{=} S(X)_\rho + S(Y)_\rho - S(XY)_\rho.$$

It holds that $I(X : Y)_\rho = D(\rho_{XY} || \rho_X \otimes \rho_Y)$.

If X is a classical register, namely $\rho_{XY} = \sum_x \mu(x) |x\rangle\langle x| \otimes \rho_Y^x$, where μ is a probability distribution over X , then

$$\begin{aligned} I(X : Y)_\rho &= S(Y)_\rho - S(Y|X)_\rho \\ &= S\left(\sum_x \mu(x) \rho_Y^x\right) - \sum_x \mu(x) S(\rho_Y^x) \end{aligned}$$

where the *conditional entropy* is defined as

$$S(Y|X)_\rho \stackrel{\text{def}}{=} \mathbb{E}_{x \leftarrow \mu} [S(\rho_Y^x)].$$

For bipartite quantum state ρ_{XY} , $S(XY)_\rho - S(X)_\rho$ is not always nonnegative. For instance, $S(XY)_\rho - S(X)_\rho = -|X|$ if ρ_{XY} is an EPR-state.

Fact 5.11. [AL70] Given a bipartite state ρ_{AB} , it holds that

$$\left|S(A)_\rho - S(B)_\rho\right| \leq S(AB)_\rho \leq S(A)_\rho + S(B)_\rho.$$

Let ρ_{XYZ} be a quantum state with Y being a classical register. The mutual information between X and Z , conditioned on Y , is defined as

$$\begin{aligned} I(X : Z|Y)_\rho &\stackrel{\text{def}}{=} \mathbb{E}_{y \leftarrow Y} \left[I(X : Z|Y = y)_\rho \right] \\ &= S(X|Y)_\rho + S(Z|Y)_\rho - S(XZ|Y)_\rho. \end{aligned}$$

The following *chain rule* for mutual information follows easily from the definitions, when Y is a classical register.

$$I(X : YZ)_\rho = I(X : Y)_\rho + I(X : Z|Y)_\rho.$$

We will need the following basic facts.

Fact 5.12 ([Wat11, JRS03b]). For quantum states ρ and σ , it holds that

$$\|\rho - \sigma\|_1 \leq \sqrt{D(\rho\|\sigma)} \quad \text{and} \quad 1 - F(\rho, \sigma) = h^2(\rho, \sigma) \leq D(\rho\|\sigma).$$

Fact 5.13. For quantum states ρ_{XY} , σ_X , and τ_Y , it holds that

$$D(\rho_{XY}\|\sigma_X \otimes \tau_Y) \geq D(\rho_{XY}\|\rho_X \otimes \rho_Y) = I(X : Y)_\rho.$$

Combing with Fact 5.12, it holds that

$$h(\rho_{XY}, \rho_X \otimes \rho_Y) \leq \sqrt{I(X : Y)_\rho}.$$

Fact 5.14. Let ρ and σ be quantum states and $\mathcal{E}(\cdot)$ be a quantum channel. Then it holds that

$$D(\rho\|\sigma) \geq D(\mathcal{E}(\rho)\|\mathcal{E}(\sigma)).$$

Moreover, given a bipartite quantum state ρ_{XY} , let $\mathcal{E}^{Y \rightarrow Z}(\cdot)$ be a quantum operation on Y . Combining with Fact 5.13, we find that

$$I(X : Y)_\rho \geq I(X : Z)_{\mathcal{E}(\rho)}.$$

If $\mathcal{E}(\cdot)$ is an isometry, then

$$I(X : Y)_\rho = I(X : Z)_{\mathcal{E}(\rho)}.$$

Fact 5.15. (Data-processing inequality). Given a tripartite quantum state ρ_{XYB} , where XY are classical registers, with the property that X is determined by Y , that is, $S(X|Y)_\rho = 0$. Then

$$I(X : B)_\rho \leq I(Y : B)_\rho.$$

Fact 5.16. [Lie73, LR73](**Strong subadditivity theorem**) For any tripartite quantum state ρ_{ABC} , it holds that $I(A : C|B)_\rho \geq 0$.

Fact 5.17. Given a tripartite state ρ_{ABC} , it holds that $I(A : B|C)_\rho \leq 2|B|$.

Lemma 5.18. Consider a tripartite pure state $|\psi\rangle_{ABC}$ which satisfies $I(A : C) \leq \epsilon$. Then for any purifications $|\psi_1\rangle_{AB_1}$ and $|\psi_2\rangle_{B_2C}$ of ψ_A and ψ_C , respectively, there exists an isometry U mapping \mathcal{H}_B to $\mathcal{H}_{B_1} \otimes \mathcal{H}_{B_2}$ such that

$$|\langle \psi_1 | \langle \psi_2 | (\mathbf{1}_{AC} \otimes U_B) |\psi\rangle| \geq 1 - \epsilon.$$

Combining with Fact 5.3 we have

$$h\left((\mathbf{1}_{AC} \otimes U_B) \psi \left(\mathbf{1}_{AC} \otimes U_B^\dagger\right), \psi_1 \otimes \psi_2\right) \leq \sqrt{\epsilon}.$$

Proof. From Fact 5.12 and Fact 5.13, we have

$$F(\psi_{AC}, \psi_A \otimes \psi_C) \geq 1 - \epsilon.$$

The conclusion now follows from Uhlmann's theorem and the Fuchs-van de Graaf inequalities (Facts 5.3 and 5.7). \square

We need the following fact for state distribution.

Fact 5.19. [JRS05] Given a target quantum state $\rho_{XAB} = \sum_x p(x) |x\rangle\langle x|_X \otimes \rho_{AB}^x$, where the input register X is held by Alice. There exists a one-way quantum protocol where Alice sends $\mathcal{O}\left(\left(\mathbb{I}(X : B)_\rho + 1\right)/\delta^2\right)$ qubits to Bob such that $\mathbb{E}_{x \leftarrow p}[h^2(\rho_{AB}^x, \tilde{\rho}_{AB}^x)] \leq \delta^2$, where $\tilde{\rho}_{AB}^x$ is the state shared between Alice and Bob at the end of the protocol when the input is x .²

5.2 Models of Quantum Communication Complexity

Quantum communication complexity was introduced by Yao in [Yao93]. It studies the advantages and limitations of the players who are allowed to exchange quantum messages to accomplish a communication task. Here we describe two models of quantum communication complexity as follows.

Yao's Model

The model we use here is slightly different from the original one defined in Yao [Yao93]. It is closer to the one of Cleve and Buhrman [CB97], with pre-shared entanglement, but we allow the players to communicate with quantum messages. In this model, an r -round protocol Π for a given classical task from input registers $A_{in} = X$, $B_{in} = Y$ to output registers A_{out} , B_{out} is defined by a sequence of isometries U_1, \dots, U_{r+1} along with a pure state $\psi \in \mathcal{D}(T_A^{in} T_B^{in})$ shared between Alice and Bob, for arbitrary finite dimensional registers T_A^{in}, T_B^{in} : the pre-shared entanglement. We need $r + 1$ isometries in order to have r messages since a first isometry is applied before the first message is sent and a last one after the final message is received. In the case of even r , for appropriate finite dimensional quantum memory registers $A_1, A_3, \dots, A_{r-1}, A'$ held by Alice, $B_2, B_4, \dots, B_{r-2}, B'$ held by Bob, and quantum communication registers $C_1, C_2, C_3, \dots, C_r$ exchanged by Alice and Bob, we have $U_1 \in \mathcal{U}(A_{in} T_A^{in}, A_1 C_1)$, $U_2 \in \mathcal{U}(B_{in} T_B^{in} C_1, B_2 C_2)$, $U_3 \in \mathcal{U}(A_1 C_2, A_3 C_3)$, $U_4 \in \mathcal{U}(B_2 C_3, B_4 C_4)$, \dots , $U_r \in \mathcal{U}(B_{r-2} C_{r-1}, B_{out} B' C_r)$, $U_{r+1} \in \mathcal{U}(A_{r-1} C_r, A_{out} A')$, where $\mathcal{U}(A, B)$ is the set of unitary channels from \mathcal{H}_A to \mathcal{H}_B : see Figure 3. We adopt the convention that, at the outset, $A_0 = A_{in} T_A^{in}$, $B_0 = B_{in} T_B^{in}$, for odd i with $1 \leq i < r$, $B_i = B_{i-1}$, for even i with $1 < i \leq r$, $A_i = A_{i-1}$ and also $B_r = B_{r+1} = B_{out} B'$, and $A_{r+1} = A_{out} A'$. In this way, after application of U_i , Alice holds register A_i , Bob holds register B_i and the communication register is C_i . In the case of an odd number of messages r , the registers corresponding to U_r, U_{r+1} are changed accordingly. We slightly abuse notation and also write Π to denote the channel from registers $A_{in} B_{in}$ to $A_{out} B_{out}$

²In [JRS05], the theorem is stated in terms of ℓ_1 distance and the proof uses the quantum substate theorem [JRS02]. Later, Jain and Nayak [JN12] provided a simpler proof for the quantum substate theorem with better dependence on the parameters. With the strengthened quantum substate theorem, it is easy to verify that the compression in [JRS05] also has better dependence on the parameters in terms of Hellinger distance as stated in Fact 5.19.

implemented by the protocol, i.e. for any input distribution μ on XY and ρ_μ encoding μ on input registers $A_{in}B_{in}$,

$$\Pi(\rho_\mu) = \text{Tr}_{A'B'} U_{r+1} U_r \cdots U_2 U_1 (\rho_\mu \otimes \psi). \quad (17)$$

Note that the A' and B' registers are the final memory registers that are being discarded at the end of the protocol by Alice and Bob, respectively.

Recall that for a given state, all purifications are related by isometries on the purification registers. For classical input registers XY distributed according to μ , we consider a canonical purification $|\rho_\mu\rangle^{XR_XYR_Y}$ of $\rho_\mu^{A_{in}B_{in}}$, with

$$|\rho_\mu\rangle^{XR_XYR_Y} = \sum_{x,y} \sqrt{\mu(x,y)} |xxyy\rangle^{XR_XYR_Y}. \quad (18)$$

We then say that the purifying registers $R_X R_Y$ contain *quantum copies* of XY . We define the *global state* at round i to be the state on $X R_X Y R_Y A_i B_i C_i$, which is a pure state. Then the global state at round i is

$$\rho_i^{XR_XYR_Y A_i B_i C_i} = U_i \cdots U_1 (\rho^{XR_XYR_Y} \otimes \psi^{T_A^{in} T_B^{in}}) \quad (19)$$

Also, we require that the final marginal state $\Pi(\rho^{A_{in}B_{in}R_X R_Y})$ on $R_X R_Y A_{out} B_{out}$ is classical. We say that a protocol Π solves a function f with error ϵ with respect to input distribution μ if $\Pr_\mu[\Pi(x,y) \neq f(x,y)] \leq \epsilon$, and we say Π solves f with error ϵ if $\max_{(x,y)} \Pr[\Pi(x,y) \neq f(x,y)] \leq \epsilon$.

We also make use of the notion of a *control-isometry*: it is an isometry acting on a classical-quantum state that leaves the content of the classical register unchanged. Such a classical register is called a control-register. In Yao's model, we assume that all the isometries U_1, \dots, U_{r+1} are control-isometries with control-register being the inputs.

Cleve-Buhrman model

In 1997, Cleve and Buhrman [CB97] defined an alternative model for communication complexity in a quantum setting, in which the players are allowed to pre-share an arbitrary entangled state but transmit classical rather than quantum bits. This model is equivalent to Yao's model (with entanglement, up to a factor of 2), since entanglement can be used to teleport [BBC⁺93] the qubits with twice as many classical bits.

Quantum Communication Complexity and Quantum Information Complexity

Since Yao's model (augmented with entanglement) and coherent Cleve-Buhrman model are equivalent up to factor 2, in this paper, we do not differentiate between these two models unless particularly specified.

Definition 5.20. For a protocol Π and an input distribution μ , we define the *quantum communication cost* (QCC) and *quantum information cost* (QIC) of Π on input μ as

$$QCC(\Pi, \mu) \stackrel{\text{def}}{=} \sum_i |C_i|,$$

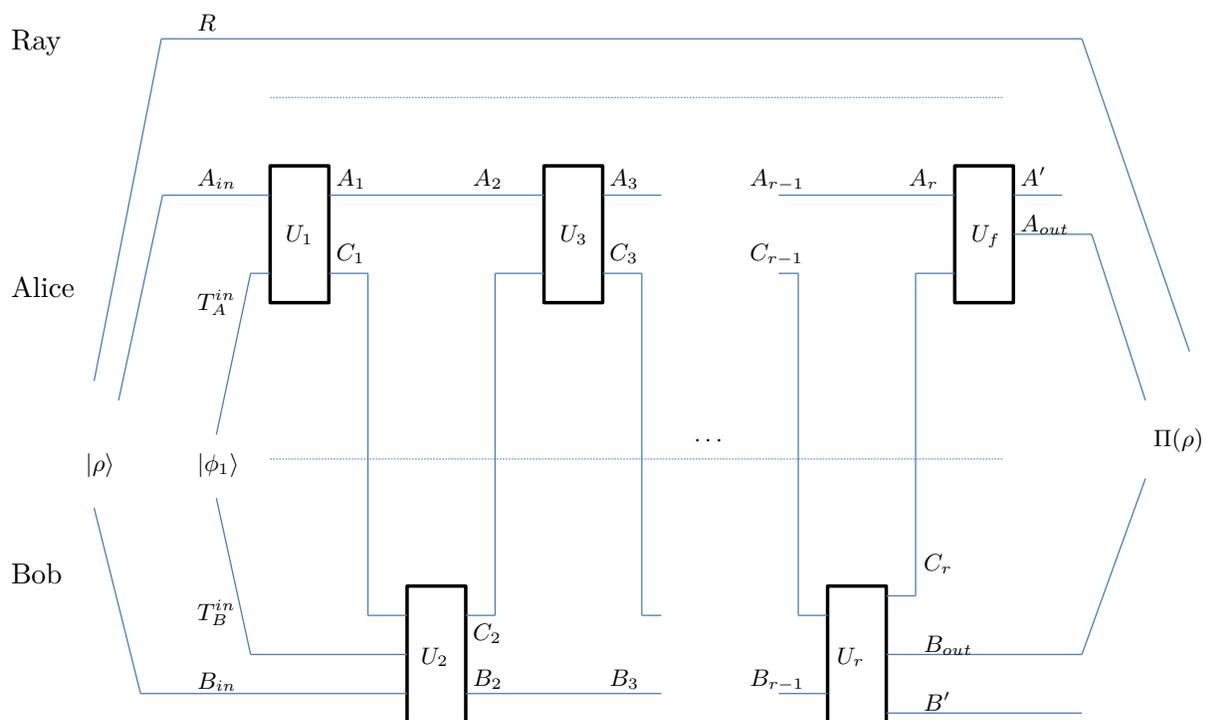


Figure 3: Depiction of a quantum protocol in the interactive model, adapted from the long version of [Tou15, Figure 1].

and

$$QIC(\Pi, \rho) \stackrel{\text{def}}{=} \sum_{i \geq 1, \text{ odd}} I(C_i : R_X R_Y | B_i) + \sum_{i \geq 1, \text{ even}} I(C_i : R_X R_Y | A_i),$$

respectively. For any function f , any input distribution μ , and any $\epsilon > 0$,

$$QCC(f, \mu, \epsilon) \stackrel{\text{def}}{=} \inf_{\Pi} QCC(\Pi, \mu), \quad (20)$$

and

$$QIC(f, \mu, \epsilon) \stackrel{\text{def}}{=} \inf_{\Pi} QIC(\Pi, \mu), \quad (21)$$

where the infimum is over the protocols Π computing f with error ϵ w.r.t μ .

6 Lower bound on quantum communication complexity

In this section, we prove Theorem 2.5 by following the high-level proof sketch given in Section 3. We assume throughout this section that the protocol runs for T rounds. We first prove the Theorem assuming the results of the Lemmata in Section 3, before proving these Lemmata.

6.1 Proof of main Theorem

Proof of Theorem 2.5. Let Π be a T -round quantum protocol with communication cost c . We assume without loss of generality that t is odd and in the end of the protocol, Bob outputs the correct answer with probability at least $1 - \epsilon > \frac{1}{2}$.

We first consider running protocol Π on inputs given according to p . We assume that the protocol is well-defined even outside the support of μ , otherwise, adding an error flag as a potential output can only increase the distance of the output depending on whether Π is run on p or on μ . Let inputs to Alice and Bob be given in registers XF and YG in the state

$$\sum_{x,y} p(x, y, f, g) |x\rangle\langle x|_X \otimes |y\rangle\langle y|_Y \otimes |f\rangle\langle f|_F \otimes |g\rangle\langle g|_G.$$

Let these registers be purified by $R_X R_F$ and $R_Y R_G$ respectively, which are not accessible to either players. Let Alice and Bob initially hold registers A_0, B_0 with shared entanglement $\Theta_{T_A T_B}^0$. Then the initial state is

$$|\Psi^0\rangle_{XYFGR_X R_Y R_F R_G T_A T_B} \stackrel{\text{def}}{=} \sum_{x,y,f,g} \sqrt{p(x, y, f, g)} |xyyffgg\rangle_{X R_X Y R_Y F R_F G R_G} |\Theta^0\rangle_{T_A T_B}.$$

Alice applies a control unitary $U^1 : XFT_A \rightarrow XFA_1 C_1$ such that the unitary acts on T_A controlled by XF , then sends C_1 to Bob. Let $B_1 \equiv T_B$ be a relabelling of Bob's register B_0 . He applies $U^2 : YGC_1 B_1 \rightarrow YGC_2 B_2$ such that the unitary acts on $C_1 B_1$ conditioned on YG . He sends C_2 to Alice. Players proceed in this fashion until the end of the protocol. At round r , let the registers be $A_r C_r B_r$, where C_r is the message register, A_r is with Alice and B_r is with Bob. If r is odd, then $B_r \equiv B_{r-1}$ and if r is even, then $A_r \equiv A_{r-1}$. Then the global state at round r is

$$|\Psi^r\rangle_{XYFGR_X R_Y R_F R_G A_r C_r B_r} \stackrel{\text{def}}{=} \sum_{x,y,f,g} \sqrt{p(x,y,f,g)} |xxyyffgg\rangle_{X R_X Y R_Y F R_F G R_G} \left| \Theta^{r,xyfg} \right\rangle_{A_r C_r B_r}.$$

$$\text{Set } c_i \stackrel{\text{def}}{=} |C_i|; \ell_{A,r} \stackrel{\text{def}}{=} \sum_{i \leq r, i \text{ odd}} c_i; \ell_{B,r} \stackrel{\text{def}}{=} \sum_{i \leq r, i \text{ even}} c_i.$$

$$\begin{aligned} \epsilon_{r,x \leq j y_j j} &\stackrel{\text{def}}{=} h \left(\Psi_{X_S F_S B_r Y_{\geq j}(R_Y)_{\geq j} G R_G}^{r,x \leq j j y_j}, \Psi_{X_S F_S}^{r,x \leq j j y_j} \otimes \Psi_{B_r Y_{\geq j}(R_Y)_{\geq j} G R_G}^{r,x \leq j j y_j} \right) \\ &= \sqrt{\mathbb{E}_{x_s f_s \leftarrow X_S F_S} \left[h^2 \left(\Psi_{B_r Y_{\geq j}(R_Y)_{\geq j} G R_G}^{r,x \leq j j y_j x_s f_s}, \Psi_{B_r Y_{\geq j}(R_Y)_{\geq j} G R_G}^{r,x \leq j j y_j} \right) \right]} \end{aligned}$$

when r is odd,

$$\begin{aligned} \epsilon_{r,x \leq j y_j j} &\stackrel{\text{def}}{=} h \left(\Psi_{Y_S G_S A_r X_{\geq j}(R_X)_{\geq j} F R_F}^{r,x \leq j j y_j}, \Psi_{Y_S G_S}^{r,x \leq j j y_j} \otimes \Psi_{A_r X_{\geq j}(R_X)_{\geq j} F R_F}^{r,x \leq j j y_j} \right) \\ &= \sqrt{\mathbb{E}_{y_s g_s \leftarrow Y_S G_S} \left[h^2 \left(\Psi_{A_r X_{\geq j}(R_X)_{\geq j} F R_F}^{r,x \leq j j y_j y_s g_s}, \Psi_{A_r X_{\geq j}(R_X)_{\geq j} F R_F}^{r,x \leq j j y_j} \right) \right]}, \end{aligned}$$

when r is even, where the equalities are from Fact 5.5. By taking appropriate choices of input into protocol Π , we can combine Lemmata 3.1, 3.2, 3.4, and prove that on average under p , Bob's state is almost independent of $x_s f_s$, and Alice's state is almost independent of $y_s g_s$. We get the following claim.

Claim 6.1. It holds that for all $r \leq t, 0 < \delta < 1$,

$$\mathbb{E}_{x \leq j y_j j} \left[\epsilon_{r,x \leq j y_j j}^2 \right] \leq \mathcal{O} \left(\frac{\ell_{A,r}}{k\delta^2} \right) + 12\delta^2 + 18 \frac{\ell_{B,r} 2^{2\ell_{A,r}+4}}{n}, \quad (22)$$

when r is odd, and

$$\mathbb{E}_{x \leq j y_j j} \left[\epsilon_{r,x \leq j y_j j}^2 \right] \leq \mathcal{O} \left(\frac{\ell_{B,r}}{k\delta^2} \right) + 12\delta^2 + 18 \frac{\ell_{A,r} 2^{2\ell_{B,r}+4}}{n}, \quad (23)$$

when r is even.

To go from distribution p to distributions μ_0 and μ_1 , we make yet another appropriate choice of the input into protocol Π , so that Lemma 3.6 can be used. Let

$$\begin{aligned} &\left(\Phi_b^{0,x \leq j y_j j} \right)_{X_S R_X S F_S R_{F_S} Y_S R_{Y_S} G_S R_{G_S}} \\ &= \sum_{x_s f_s y_s g_s} \sqrt{\mu_b(x_s, f_s, y_s, g_s \mid x \leq j, y_j, j)} |x_s x_s f_s f_s y_s y_s g_s g_s\rangle_{X_S R_X S F_S R_{F_S} Y_S R_{Y_S} G_S R_{G_S}} \end{aligned}$$

be canonical purifications, for $b \in \{0, 1\}$, of the inputs $(x_s f_s, y_s g_s)$ restricted to S and drawn under distribution μ_0 and μ_1 , respectively. Also let $\Phi_b^{T,x \leq j y_j j}$ be the final states after running protocol Π on inputs distributed according to μ_0 and μ_1 , respectively. According to our assumption in the beginning of the proof, T is odd and Bob outputs the answer. We get the following claim.

Claim 6.2. There exist registers \hat{A}, \hat{B} , control isometries

$$\begin{aligned} V_{x \leq j y_j j}^T &\in \mathcal{U} \left(X_S F_S X_{S^c} F_{S^c} R_{X_{S^c}} R_{F_{S^c}} A_T, X_S F_S \hat{A} \right), \\ V_{x \leq j y_j j}^{T-1} &\in \mathcal{U} \left(Y_S G_S Y_{S^c} G_{S^c} R_{Y_{S^c}} R_{G_{S^c}} B_T, Y_S G_S \hat{B} \right) \end{aligned}$$

controlled by $X_S F_S$ and $Y_S G_S$, respectively, and a quantum state $\hat{\Psi} \in \mathcal{D}_{\hat{A}\hat{B}}$ satisfying that

$$\begin{aligned} &h \left(V_{x \leq j y_j j}^T V_{x \leq j y_j j}^{T-1} \left(\Phi_0^{T, x \leq j y_j j} \right), \left(\Phi_0^{0, x \leq j y_j j} \right)_{X_S R_{X_S} F_S R_{F_S} Y_S R_{Y_S} G_S R_{G_S}} \otimes \hat{\Psi}_{\hat{A}\hat{B}} \right) \\ &\leq \epsilon_{T, x \leq j y_j j} + \epsilon_{T-1, x \leq j y_j j} + 2 \sum_{r=1}^{T-2} \epsilon_{r, x \leq j y_j j}, \end{aligned}$$

and

$$\begin{aligned} &h \left(V_{x \leq j y_j j}^T V_{x \leq j y_j j}^{T-1} \left(\Phi_1^{T, x \leq j y_j j} \right), \left(\Phi_1^{0, x \leq j y_j j} \right)_{X_S R_{X_S} F_S R_{F_S} Y_S R_{Y_S} G_S R_{G_S}} \otimes \hat{\Psi}_{\hat{A}\hat{B}} \right) \\ &\leq \epsilon_{T, x \leq j y_j j} + \epsilon_{T-1, x \leq j y_j j} + 2 \sum_{r=1}^{T-2} \epsilon_{r, x \leq j y_j j}. \end{aligned}$$

Using this claim, we proceed as follows. Note that $\left(\Phi_0^{0, x \leq j y_j j} \right)_{Y_S G_S} = \left(\Phi_1^{0, x \leq j y_j j} \right)_{Y_S G_S}$, so

$$\begin{aligned} &\text{Tr}_{X_S R_{X_S} F_S R_{F_S} R_{Y_S} R_{G_S}} \left(\left(\Phi_0^{0, x \leq j y_j j} \right)_{X_S R_{X_S} F_S R_{F_S} Y_S R_{Y_S} G_S R_{G_S}} \otimes \hat{\Psi}_{\hat{A}\hat{B}} \right) \\ &= \text{Tr}_{X_S R_{X_S} F_S R_{F_S} R_{Y_S} R_{G_S}} \left(\left(\Phi_1^{0, x \leq j y_j j} \right)_{X_S R_{X_S} F_S R_{F_S} Y_S R_{Y_S} G_S R_{G_S}} \otimes \hat{\Psi}_{\hat{A}\hat{B}} \right). \end{aligned}$$

By triangle inequality and Fact 5.4, we have

$$\begin{aligned} &h \left(\text{Tr}_{X_S R_{X_S} F_S R_{F_S} R_{Y_S} R_{G_S}} \left(V_{x \leq j y_j j}^T V_{x \leq j y_j j}^{T-1} \left(\Phi_0^{T, x \leq j y_j j} \right) \right), \right. \\ &\quad \left. \text{Tr}_{X_S R_{X_S} F_S R_{F_S} R_{Y_S} R_{G_S}} \left(V_{x \leq j y_j j}^T V_{x \leq j y_j j}^{T-1} \left(\Phi_1^{T, x \leq j y_j j} \right) \right) \right) \\ &\leq 2 \left(\epsilon_{T, x \leq j y_j j} + \epsilon_{T-1, x \leq j y_j j} + 2 \sum_{r=1}^{T-2} \epsilon_{r, x \leq j y_j j} \right) \end{aligned}$$

Further taking expectation over $x \leq j y_j j$, we have

$$\begin{aligned} &\mathbb{E}_{x \leq j y_j j} \left[h \left(\text{Tr}_{X_S R_{X_S} F_S R_{F_S} R_{Y_S} R_{G_S}} \left(V_{x \leq j y_j j}^T V_{x \leq j y_j j}^{T-1} \left(\Phi_0^{T, x \leq j y_j j} \right) \right), \right. \right. \\ &\quad \left. \left. \text{Tr}_{X_S R_{X_S} F_S R_{F_S} R_{Y_S} R_{G_S}} \left(V_{x \leq j y_j j}^T V_{x \leq j y_j j}^{T-1} \left(\Phi_1^{T, x \leq j y_j j} \right) \right) \right) \right] \\ &\leq \mathbb{E}_{x \leq j y_j j} \left[2 \left(\epsilon_{T, x \leq j y_j j} + \epsilon_{T-1, x \leq j y_j j} + 2 \sum_{r=1}^{T-2} \epsilon_{r, x \leq j y_j j} \right) \right] \\ &\leq 4 \sum_{r=1}^T \mathbb{E}_{x \leq j y_j j} \left[\epsilon_{r, x \leq j y_j j} \right] \end{aligned}$$

$$\begin{aligned}
&\leq 4 \mathbb{E}_{x_{\leq j} y_{jj}} \left[\sqrt{T \sum_{r=1}^T \epsilon_{r, x_{\leq j} y_{jj}}^2} \right] && \text{(Cauchy-Schwarz inequality)} \\
&\leq 4 \sqrt{T \sum_{r=1}^T \mathbb{E}_{x_{\leq j} y_{jj}} [\epsilon_{r, x_{\leq j} y_{jj}}^2]} && \text{(Concavity of } \sqrt{x} \text{)} \\
&\leq 4 \sqrt{T \sum_{r=1}^T \left(\left(\mathcal{O} \left(\frac{\ell_{A,r}}{k\delta^2} \right) + 12\delta^2 + 18 \frac{\ell_{B,r} 2^{2\ell_{A,r}+4}}{n} \right) + \left(\mathcal{O} \left(\frac{\ell_{B,r}}{k\delta^2} \right) + 12\delta^2 + 18 \frac{\ell_{A,r} 2^{2\ell_{B,r}+4}}{n} \right) \right)} \\
&\text{(by Eq. (22).)} \\
&\leq 4 \sqrt{\mathcal{O} \left(\frac{T^2 c}{k\delta^2} \right) + 24\delta^2 T^2 + \frac{c 2^{2c+10}}{n}} \\
&\leq 4 \sqrt{\mathcal{O} \left(\frac{c^3}{k\delta^2} \right) + 24\delta^2 c^2 + \frac{c 2^{2c+10}}{n}} && \text{(because } T \leq c \text{),}
\end{aligned}$$

If $\frac{c}{24k} \geq 1$, then $c \geq \Omega(k)$. Otherwise, choose $\delta \stackrel{\text{def}}{=} (\frac{c}{24k})^{1/4}$. Then we have

$$\begin{aligned}
&\mathbb{E}_{x_{\leq j} y_{jj}} \left[h \left(\begin{array}{c} \text{Tr}_{X_S R_{X_S} F_S R_{F_S} R_{Y_S} R_{G_S}} \left(V_{x_{\leq j} y_{jj}}^T V_{x_{\leq j} y_{jj}}^{T-1} \left(\Phi_0^{T, x_{\leq j} y_{jj}} \right) \right), \\ \text{Tr}_{X_S R_{X_S} F_S R_{F_S} R_{Y_S} R_{G_S}} \left(V_{x_{\leq j} y_{jj}}^T V_{x_{\leq j} y_{jj}}^{T-1} \left(\Phi_1^{T, x_{\leq j} y_{jj}} \right) \right) \end{array} \right) \right] \\
&\leq 4 \left(\mathcal{O} \left(\frac{c^5/2}{\sqrt{k}} \right) + \frac{c 2^{2c+10}}{n} \right). \tag{24}
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\mathbb{E}_{x_{\leq j} y_{jj}} \left[h \left(\begin{array}{c} \text{Tr}_{X_S R_{X_S} F_S R_{F_S} R_{Y_S} R_{G_S}} \left(V_{x_{\leq j} y_{jj}}^T V_{x_{\leq j} y_{jj}}^{T-1} \left(\Phi_0^{T, x_{\leq j} y_{jj}} \right) \right), \\ \text{Tr}_{X_S R_{X_S} F_S R_{F_S} R_{Y_S} R_{G_S}} \left(V_{x_{\leq j} y_{jj}}^T V_{x_{\leq j} y_{jj}}^{T-1} \left(\Phi_1^{T, x_{\leq j} y_{jj}} \right) \right) \end{array} \right) \right] \\
&\geq \mathbb{E}_{x_{\leq j} y_{jj}} \left[h \left(\begin{array}{c} \text{Tr}_{X_S R_{X_S} F_S R_{F_S} R_{Y_S} R_{G_S} \hat{A}} \left(V_{x_{\leq j} y_{jj}}^T V_{x_{\leq j} y_{jj}}^{T-1} \left(\Phi_0^{T, x_{\leq j} y_{jj}} \right) \right), \\ \text{Tr}_{X_S R_{X_S} F_S R_{F_S} R_{Y_S} R_{G_S} \hat{A}} \left(V_{x_{\leq j} y_{jj}}^T V_{x_{\leq j} y_{jj}}^{T-1} \left(\Phi_1^{T, x_{\leq j} y_{jj}} \right) \right) \end{array} \right) \right] \\
&= \mathbb{E}_{x_{\leq j} y_{jj}} \left[h \left(\text{Tr}_{F_S R_{F_S} R_{G_S} A_T} \left(\left(\Phi_0^{T, x_{\leq j} y_{jj}} \right) \right), \text{Tr}_{F_S R_{F_S} R_{G_S} A_T} \left(\left(\Phi_1^{T, x_{\leq j} y_{jj}} \right) \right) \right) \right] \geq \Omega(1).
\end{aligned}$$

The equality is because $V_{x_{\leq j} y_{jj}}^t$ and $V_{x_{\leq j} y_{jj}}^{t-1}$ are all control isometries controlled by $X_S F_S$ and $Y_S G_S$, respectively. The last inequality is because we assume that Bob outputs incorrect answers with probability at most constant $\varepsilon < \frac{1}{2}$.

Combining with (24), we have

$$4 \left(\mathcal{O} \left(\frac{c^5/2}{\sqrt{k}} \right) + \frac{c 2^{2c+10}}{n} \right) \geq \Omega(1).$$

Therefore, quantum communication complexity for the communication task from Definition 2.3 is at least $\min \left\{ \Omega \left(k^{1/5} \right), \Omega(\log n) \right\}$. \square

6.2 Proofs of the Claims

In order to prove Claims 6.1, 6.2, we need the following two additional claims.

Claim 6.3. For any $r \leq t$, it holds that

$$\begin{aligned} \mathbb{I}(XF : C_r B_r Y R_Y G R_G | JX_{<J}) &\leq 2\ell_{A,r} \\ \mathbb{I}(YG : C_r A_r X R_X F R_F | JY_{<J}) &\leq 2\ell_{B,r} \end{aligned}$$

Proof. Let's prove the first inequality. The second one follows by symmetry. We prove it by induction on r . When $r = 1$, any local operation on Bob's side does not increase $\mathbb{I}(XF : C_1 B_1 Y R_Y G R_G | JX_{<J})$. We consider the state when Bob has received the first message and does not perform any operation. Then the left hand side is

$$\mathbb{I}(XF : C_1 T_B Y R_Y G R_G | JX_{<J}) = \mathbb{I}(XF : C_1 T_B | JX_{<J}) = \mathbb{I}(XF : C_1 | T_B JX_{<J}) \leq 2|C_1| = 2\ell_{A,1}.$$

The first equality is because XF and YG are independent conditioning on $JX_{<J}$. The second inequality is because T_B is the part of the pre-shared entanglement, which is independent of the input. The last inequality is by Lemma 5.17. If r is odd, we have

$$\begin{aligned} &\mathbb{I}(XF : C_r B_r Y R_Y G R_G | JX_{<J}) \\ &\leq \mathbb{I}(XF : B_r Y R_Y G R_G | JX_{<J}) + \mathbb{I}(XF : C_r | JX_{<J} B_r Y R_Y G R_G) \\ &= \mathbb{I}(XF : B_{r-1} Y R_Y G R_G | JX_{<J}) + 2c_r \\ &\leq 2\ell_{A,r}, \end{aligned}$$

where the first inequality is Lemma 5.17 and the second inequality is from the induction. If r is even, we have

$$\mathbb{I}(XF : C_r B_r Y R_Y G R_G | JX_{<J}) = \mathbb{I}(XF : C_{r-1} B_{r-1} Y R_Y G R_G | JX_{<J}),$$

by Fact 5.14. □

Claim 6.4. It holds that for all $r \leq t$,

$$\mathbb{I}(X_J : C_r B_r Y_{>J} (R_Y)_{>J} G R_G | Y_{\leq J} J) \leq \frac{\ell_{A,r} 2^{2\ell_{B,r}+2}}{n}, \quad (25)$$

$$\mathbb{I}(Y_J : C_r A_r X_{>J} (R_X)_{>J} F R_F | X_{\leq J} J) \leq \frac{\ell_{B,r} 2^{2\ell_{A,r}+2}}{n}. \quad (26)$$

Proof. With the notation from Lemma 3.4, taking $X \leftarrow X, J \leftarrow J, C \leftarrow C_r, B \leftarrow B_r, Y_1^J \leftarrow Y_{\geq J} G, Y_2^J \leftarrow Y_{<J} = X_{<J}$, we have

$$\mathbb{I}\left(X_J : C_r B_r Y_{\geq J} R_{Y_{\geq J}} G R_G \middle| JX_{<J}\right) \leq \frac{\ell_{A,r} 2^{2\ell_{B,r}+2}}{n}. \quad (27)$$

Then

$$\begin{aligned} &\mathbb{I}(X_J : C_r B_r Y_{>J} (R_Y)_{>J} G R_G | Y_{\leq J} J) \\ &= \mathbb{I}(X_J : C_r B_r Y_{>J} (R_Y)_{>J} G R_G | Y_J X_{<J} J) \quad (\text{because } X_{<J} = Y_{<J}) \\ &= \mathbb{I}\left(X_J : C_r B_r Y_{\geq J} (R_Y)_{\geq J} G R_G \middle| X_{<J} J\right) \quad (Y_J \text{ is independent of } X \text{ given } X_{<J} J) \end{aligned}$$

Together with Eq. (27), we have Eq. (25). Eq. (26). follows by the symmetric argument. □

Proof of Claim 6.1. Let r be odd, the case of even r is proved similarly. We consider a new protocol Π' , where we have fixed $x_{\leq j}j$, and it is known to both Alice and Bob. The input to Alice is XF ($X_{\leq j}$ is fixed) and the input to Bob is Y_j . Note that Y_j and S determine each other given $x_{\leq j}j$. Bob locally generates the registers $Y_{>j}(R_Y)_{>j}GR_G$, which are independent of XF whenever $X_{\leq j}j$ is fixed. After that, Alice and Bob together simulate the original protocol Π till round r . The global joint state is $\Psi^{r,x_{\leq j}j}$, which is the state Ψ^r conditioned on fixing $x_{\leq j}j$.

$I(X_{>j}F : B_r Y_{>j}(R_Y)_{>j}GR_G)_{\Psi^{r,x_{\leq j}j}} \leq 2\ell_{A,r}$ by Claim 6.3. As $\ell_{A,r} \geq 1$, from Lemma 3.2, there exists a one-way entanglement assisted protocol Π'' with communication cost $\mathcal{O}(\ell_{A,r}/\delta^2)$ and the final global state $\tilde{\Psi}_{X_{>j}(R_X)_{>j}FR_F A_r B_r Y_{>j}(R_Y)_{>j}GR_G}^{x_{\leq j}j}$ such that

$$h^2 \left(\tilde{\Psi}_{X_{>j}F A_r B_r Y_{>j}(R_Y)_{>j}GR_G}^{x_{\leq j}j}, \Psi_{X_{>j}F A_r B_r Y_{>j}(R_Y)_{>j}GR_G}^{r,x_{\leq j}j} \right) \leq 4\delta^2 + 6I(Y_J : A_r X_{>J}(R_X)_{>J}GR_G)_{\Psi^{r,x_{\leq j}j}}. \quad (28)$$

As Y_j and S determine each other and XF and YG are independent for fixed $x_{\leq j}y_{\leq j}j$, we can further apply Lemma 3.1 to obtain

$$I(X_S F_S : B_r Y_{\geq j}(R_Y)_{\geq j}GR_G | S)_{\tilde{\Psi}^{x_{\leq j}y_{\leq j}j}} \leq \mathcal{O}\left(\frac{\ell_{A,r}}{k\delta^2}\right). \quad (29)$$

The reason is that $\tilde{\Psi}^{x_{\leq j}y_{\leq j}j}$ is obtained from the protocol Π'' with communication cost $\mathcal{O}(\ell_{A,r}/\delta^2)$. Combined with Claim 6.3, (29) follows. By Fact 5.13, we have

$$\mathbb{E}_{s \leftarrow S} \left[h^2 \left(\tilde{\Psi}_{B_r Y_{\geq j}(R_Y)_{\geq j}GR_G X_S F_S}^{x_{\leq j}j y_{\leq j}j s}, \tilde{\Psi}_{B_r Y_{\geq j}(R_Y)_{\geq j}GR_G}^{x_{\leq j}j y_{\leq j}j s} \otimes \tilde{\Psi}_{X_S F_S}^{x_{\leq j}j y_{\leq j}j s} \right) \right] \leq \mathcal{O}\left(\frac{\ell_{A,r}}{k\delta^2}\right).$$

It implies

$$\mathbb{E}_{y_j s_x s_f s \leftarrow Y_j S X_S F_S} \left[h^2 \left(\tilde{\Psi}_{B_r Y_{\geq j}(R_Y)_{\geq j}GR_G}^{x_{\leq j}j y_j s_x s_f s}, \tilde{\Psi}_{B_r Y_{\geq j}(R_Y)_{\geq j}GR_G}^{x_{\leq j}j y_j s} \right) \right] \leq \mathcal{O}\left(\frac{\ell_{A,r}}{k\delta^2}\right). \quad (30)$$

Combining (28) (30) and triangle inequality, we have

$$\mathbb{E}_{y_j s_x s_f s \leftarrow Y_j S X_S F_S} \left[h^2 \left(\Psi_{B_r Y_{\geq j}(R_Y)_{\geq j}GR_G}^{r,x_{\leq j}j y_j s_x s_f s}, \Psi_{B_r Y_{\geq j}(R_Y)_{\geq j}GR_G}^{r,x_{\leq j}j y_j s} \right) \right] \leq \mathcal{O}\left(\frac{\ell_{A,r}}{k\delta^2}\right) + 12\delta^2 + 18I(Y_J : A_r X_{>J}(R_X)_{>J}GR_G)_{\Psi^{r,x_{\leq j}j}}.$$

Taking expectation over $x_{\leq j}j$, we have

$$\begin{aligned} & \mathbb{E}_{j x_{\leq j} y_j s_x s_f s \leftarrow J X_{\leq j} Y_j S X_S F_S} \left[h^2 \left(\Psi_{B_r Y_{\geq j}(R_Y)_{\geq j}GR_G}^{r,x_{\leq j}j y_j s_x s_f s} - \Psi_{B_r Y_{\geq j}(R_Y)_{\geq j}GR_G}^{r,x_{\leq j}j y_j s} \right) \right] \\ & \leq \mathcal{O}\left(\frac{\ell_{A,r}}{k\delta^2}\right) + 12\delta^2 + 18 \mathbb{E}_{x_{\leq j}j \leftarrow X_{\leq j}J} \left[I(Y_J : A_r X_{>J}(R_X)_{>J}GR_G)_{\Psi^{r,x_{\leq j}j}} \right] \\ & = \mathcal{O}\left(\frac{\ell_{A,r}}{k\delta^2}\right) + 12\delta^2 + 18I(Y_J : A_r X_{>J}(R_X)_{>J}GR_G | X_{\leq j}J)_{\Psi^r} \\ & \leq \mathcal{O}\left(\frac{\ell_{A,r}}{k\delta^2}\right) + 12\delta^2 + 18 \frac{\ell_{B,r} 2^{2\ell_{A,r}+4}}{n}, \end{aligned}$$

where the first equality is from the definition of $h(\cdot)$ and Fact 5.5 and the last inequality is from Claim 6.4. \square

Proof of Claim 6.2. Consider the following communication task. Alice and Bob share $x_{\leq j}y_jj \leftarrow X_{\leq j}Y_jJ$. They are given $(x_s f_s, y_s g_s)$ as input. Note that S is determined by $x_{\leq j}y_jj$. Moreover, since we are considering for now the fooling distribution p , XF and YG are independent when $x_{\leq j}y_jj$ is fixed. Alice and Bob locally sample the missing part of XF and YG , respectively, and execute the protocol Π . Applying Lemma 3.6, we get the claim. \square

6.3 Proofs of the Lemmata

We now provide the proof of our lemmas.

Proof of Lemma 3.1. The result follows from the following chain of inequalities:

$$\begin{aligned}
& \mathbb{I}(U_S : V | S) \\
&= \mathbb{E}_s \left[\sum_{i \in s} \mathbb{I}(U_i : V | U_{[<i] \cap s}, S = s) \right] \\
&= \mathbb{E}_s \left[\sum_{i \in s} \mathbb{I}(U_i : V U_{[<i] \setminus s} | U_{[<i] \cap s}, S = s) - \mathbb{I}(U_i : U_{[<i] \setminus s} | V U_{[<i] \cap s}, S = s) \right] \quad (\text{Chain rule}) \\
&\leq \mathbb{E}_s \left[\sum_{i \in s} \mathbb{I}(U_i : V U_{[<i] \setminus s} | U_{[<i] \cap s}, S = s) \right] \quad (\text{Fact 5.16}) \\
&= \mathbb{E}_s \left[\sum_{i \in s} \mathbb{I}(U_i : V | U_{<i}, S = s) \right] \quad (\text{Chain rule and independence of } U = U_1 \otimes \dots \otimes U_m \text{ and } S) \\
&= \mathbb{E}_s \left[\sum_{i \in s} \mathbb{I}(U_i : V | U_{<i}) \right] \quad (UV \text{ is independent of } S) \\
&= \sum_i \Pr[i \in S] \mathbb{I}(U_i : V | U_{<i}) \\
&\leq \frac{1}{k} \mathbb{I}(U : V). \quad (\text{Chain rule}).
\end{aligned}$$

\square

Proof of Lemma 3.2. Note that $\mathbb{I}(R_Y : R_X X A)_{\Psi} = \mathbb{I}(Y : R_X X A)_{\Psi}$. By Lemma 5.18, there exists a register B' and an isometry U_{YB} mapping $\mathcal{H}_Y \otimes \mathcal{H}_B$ to $\mathcal{H}_Y \otimes \mathcal{H}_{B'}$ such that

$$h^2 \left(U_{YB} \Psi (U_{YB})^\dagger, \left(\sum_y \sqrt{\mu_Y(y)} |yy\rangle \right) \otimes \left(\sum_y \sqrt{\mu_Y(y)} \langle yy| \right)_{Y R_Y} \otimes |\Phi\rangle\langle\Phi|_{X R_X A B'} \right) \leq \epsilon, \quad (31)$$

where $|\Phi\rangle$ is a purification of $\Psi_{X R_X Y A}$.

Note that

$$\mathbb{I}(X : B')_{U_{YB} \Psi (U_{YB})^\dagger} \leq \mathbb{I}(X : Y R_Y B')_{U_{YB} \Psi (U_{YB})^\dagger} = \mathbb{I}(X : Y R_Y B)_{\Psi},$$

where the inequality is from Fact 5.16. By Fact 5.19, there exists a one-way quantum protocol, where Alice is given $x \sim \mu_X(x)$ and she sends $\mathcal{O}((\mathbb{I}(X : Y R_Y B)_{\Psi} + 1) / \delta^2)$ qubits to Bob such that

$$h^2 \left(\sum_x \mu_X(x) |x\rangle\langle x|_X \otimes \tilde{\psi}_{AB'}^x, \left(U_{YB} \Psi (U_{YB})^\dagger \right)_{XAB'} \right) \leq \delta^2,$$

where $\tilde{\psi}_x$ is the shared state between Alice and Bob in the end of the protocol given input x . Combining with the previous inequality and Fact 5.4, we have

$$h^2 \left(\sum_x \mu_X(x) |x\rangle\langle x|_X \otimes \tilde{\psi}_{AB'}^x, \Phi_{XAB'} \right) \leq 2\delta^2 + 2\epsilon.$$

Hence

$$h^2 \left(\begin{array}{c} \sum_x \mu_X(x) |x\rangle\langle x|_X \otimes \left(\sum_y \sqrt{\mu_Y(y)} |yy\rangle \right) \otimes \left(\sum_y \sqrt{\mu_Y(y)} \langle yy| \right)_{YR_Y} \otimes \tilde{\psi}_{AB'}^x, \\ \left(\sum_y \sqrt{\mu_Y(y)} |yy\rangle \right) \otimes \left(\sum_y \sqrt{\mu_Y(y)} \langle yy| \right)_{YR_Y} \otimes \Phi_{XAB'} \end{array} \right) \leq 2\delta^2 + 2\epsilon. \quad (32)$$

Combining (31) (32), Fact 5.4 and triangle inequality, we have

$$h^2 \left(\begin{array}{c} \sum_x \mu_X(x) |x\rangle\langle x|_X \otimes \left(\sum_y \sqrt{\mu_Y(y)} |yy\rangle \right) \otimes \left(\sum_y \sqrt{\mu_Y(y)} \langle yy| \right)_{YR_Y} \otimes \tilde{\psi}_{AB'}^x, \\ \left(U_{YB} \Psi (U_{YB})^\dagger \right)_{XYR_Y AB'} \end{array} \right) \leq 4\delta^2 + 6\epsilon.$$

Bob further applies $(U_{YB})^{-1}$ on the registers YB' . He may need to extend the space \mathcal{H}_B by adding the ancilla and trace out after applying $(U_{YB})^{-1}$. By Fact 5.4, the Hellinger distance does not increase. We reach the desired conclusion. \square

Proof of Lemma 3.3. Note that any quantum operation Bob performs does not increase $I(X_J : CBY_1^J R_{Y_1^J} | JX_{<J})$ because of Fact 5.14 and the assumption that Y_2^J is a function of $X_{<J}$. It suffices to consider the global state when Bob receives C and does not perform any operation. Denote the global state by $\hat{\rho}$. It follows that

$$\begin{aligned} & I(X_J : C_1 B_1 Y_1^J R_{Y_1^J} | JX_{<J})_{\hat{\rho}} \\ & \leq I(X_J : C_1 T_B Y_1^J R_{Y_1^J} | JX_{<J})_{\hat{\rho}} \quad (T_B \text{ is the marginal of the pre-shared states on Bob's side}) \\ & = \mathbb{E}_{j, x_{<j}} \left[I(X_J; C_1 T_B Y_1^J R_{Y_1^J} | J = j, X_{<J} = x_{<j}) \right]_{\hat{\rho}} \\ & = \mathbb{E}_{j, x_{<j}} \left[I(X_J; C_1 T_B | J = j, X_{<J} = x_{<j}) \right]_{\hat{\rho}} \quad (Y_1^J R_{Y_1^J} \text{ is independent of } X_J C_1 T_B \text{ given } jx_{<j}.) \\ & = \mathbb{E}_{j, x_{<j}} \left[I(X_j : C_1 T_B | X_{<j} = x_{<j}) \right]_{\hat{\rho}} \quad (XC_1 T_B \text{ is independent of } J.) \\ & = \frac{1}{n} \sum_j I(X_j : C_1 T_B | X_{<j})_{\hat{\rho}} \\ & = \frac{1}{n} I(X : C_1 T_B)_{\hat{\rho}} \quad (\text{The chain rule of mutual information.}) \\ & = \frac{1}{n} I(X : C_1 | T_B)_{\hat{\rho}} \quad (T_B \text{ is part of the pre-shared entangled state, independent of the input.}) \\ & \leq \frac{2|C_1|}{n} \leq \frac{2\ell}{n}. \quad (\text{Lemma 5.17.}) \end{aligned}$$

\square

Proof of Lemma 3.4. We assume the protocol Π is in the Yao model defined in Section 5.2. Note that any local operation on Bob's side does not increase $I(X_J : C_r B_r Y_1^J R_{Y_1^J} | JX_{<J})$. It suffices to consider the case that r is odd. We first convert Π to a new protocol Π' in a Cleve-Buhrman model using quantum teleportation, where the communication cost doubles. We construct a one-way protocol Π'' simulating the first r rounds of Π' as follows. For this, we use an additional register P on Bob's side. It informs whether Bob aborts the protocol or not. In Π'' , Alice and Bob share the entanglement state as in Π and $c_2 + c_4 + \dots + c_{r-1}$ copies of EPR states additionally.

For each odd round t , Alice measures $(\frac{c_{t-1}}{2} + 1)$ -th, \dots , $\frac{c_{t+1}}{2}$ -th copies of the EPR pairs on her side in computational basis and treats the outcome as the message Bob sent in round $t - 1$. Alice performs exactly same as in Π . For each even round t , Bob measures the register P . If it is 1, he does not perform any further operation. If it is 0, Bob performs and prepares the message same as he is supposed to send in round t of Π' , denoted by M_t . Meanwhile, he also measures $(\frac{c_{t-1}}{2} + 1)$ -th, \dots , $\frac{c_t}{2}$ -th copies of the EPR pairs in computational basis. If the outcome is not same as M_t , he flips the bit in P to 1. Otherwise, he proceeds to the next round directly. For protocol Π'' , we define $\Gamma_{XYR_X R_Y A C B}$ to be the global state when Bob has received message C and does not perform any quantum operation; and $\Theta_{XYR_X R_Y \tilde{A} \tilde{B} P}$ to be the global state after Bob performs his quantum operation. Here we drop the superscript r to simplify the notations. It is easy to see that $\Pr[P = 0] = 2^{-2\ell_{B,r}}$. Set

$$\Theta_{XYR_X R_Y \tilde{A} \tilde{B}} \stackrel{\text{def}}{=} \tilde{\Theta}_{XYR_X R_Y \tilde{A} \tilde{B}}^{P=0},$$

$\Psi_{XYR_X R_Y A_r B_r C_r}^r$ to be the global state of protocol Π in round r . We have

$$\begin{aligned} & I(X_J : B_r C_r Y_1^J R_{Y_1^J} | JX_{<J})_{\Psi^r} \\ & \leq I(X_J : \tilde{B} Y_1^J R_{Y_1^J} | JX_{<J})_{\Theta} \\ & \leq 2^{2\ell_{B,r}} I(X_J : \tilde{B} Y_1^J R_{Y_1^J} | JX_{<J} P)_{\tilde{\Theta}}, \end{aligned} \quad (33)$$

where the first inequality is from the fact that the Cleve-Buhrman model obtained by quantum teleportation can be converted back to Yao's model via local quantum operations and Fact 5.4. As Π'' is a one-way protocol, we have

$$\begin{aligned} & I(X_J : \tilde{B} Y_1^J R_{Y_1^J} | JX_{<J} P)_{\tilde{\Theta}} \\ & \leq I(X_J : \tilde{B} Y_1^J R_{Y_1^J} P | JX_{<J})_{\tilde{\Theta}} \quad (\text{Chain rule of the mutual information}) \\ & \leq I(X_J : C B Y_1^J R_{Y_1^J} | JX_{<J})_{\Gamma} \quad (\text{Fact 5.14 and the fact that } Y_2^J \text{ is a function of } X_{<J}) \\ & = I(X_J : C T_B | JX_{<J})_{\Gamma} \quad (\text{From the assumption } Y_1^J \text{ is independent of } X_{\geq J} \text{ given } JX_{<J}) \end{aligned}$$

By Lemma 3.3, we have

$$I(X_J : C T_B | JX_{<J})_{\Gamma} \leq \frac{4\ell_{A,r}}{n}$$

Combining with (33), we obtain that for $r = \text{odd}$,

$$I(X_J : B_r C_r Y_1^J R_{Y_1^J} | JX_{<J})_{\Psi^r} \leq \frac{\ell_{A,r} 2^{2\ell_{B,r}+2}}{n}.$$

□

Proof of Lemma 3.6. We will show that for $i = 1$, $\gamma_1 = \delta_1 = \epsilon_1$, and for $i > 1$,

$$\begin{aligned}\gamma_i &\leq \gamma_{i-1} + \epsilon_{i-2} + \epsilon_i, \\ \delta &\leq \delta_{i-1} + \epsilon_{i-2} + \epsilon_i,\end{aligned}$$

from which the result follows.

First, notice that $\rho_{R_X Y R_Y B_i C_i}^i$, $\rho_{R_X}^i \otimes \rho_{Y R_Y B_i C_i}^i$, $\rho_{R_Y X R_X A_i C_i}^i$, $\rho_{R_Y}^i \otimes \rho_{X R_X A_i C_i}^i$ are all classical-quantum states. By Fact 5.7, we can assume that the V^i 's are control isometries controlled by X for odd i and controlled by Y for even i (note that $X = R_X$ and $Y = R_Y$). For $i = 1$, we can rewrite

$$\begin{aligned}\epsilon_1 &= h\left(\rho_{R_X Y R_Y B_1 C_1}^1, \rho_{R_X}^1 \otimes \rho_{Y R_Y B_1 C_1}^1\right) \\ &= h\left(V^1\left(\rho_{X R_X Y R_Y A_1 B_1 C_1}^1\right), \rho_{X R_X} \otimes \rho_{\tilde{X}_1 \tilde{R}_{X_1} Y R_Y \tilde{A}_1 B_1 C_1}^1\right) \\ &= h\left(V^1\left(\rho_{X R_X A_1 B_1 C_1}^1\right) \otimes \rho_{Y R_Y}, \rho_{X R_X} \otimes \rho_{Y R_Y} \otimes \rho_{\tilde{X}_1 \tilde{R}_{X_1} \tilde{A}_1 B_1 C_1}^1\right).\end{aligned}$$

For γ_1 , we can further apply V_0 on both sides. Note that V^0 and V^1 commute, we have

$$\begin{aligned}\epsilon_1 &= h\left(V^0\left(V^1\left(\rho_{X R_X A_1 B_1 C_1}^1\right) \otimes \rho_{Y R_Y}\right), V^0\left(\rho_{X R_X} \otimes \rho_{Y R_Y} \otimes \rho_{\tilde{X}_1 \tilde{R}_{X_1} \tilde{A}_1 B_1 C_1}^1\right)\right) \\ &= h\left(V^1 V^0\left(\rho_{X R_X A_1 B_1 C_1}^1 \otimes \rho_{Y R_Y}\right), \rho_{X R_X} \otimes \rho_{Y R_Y} \otimes \rho_{\tilde{X}_1 \tilde{R}_{X_1} \tilde{Y}_0 \tilde{R}_{Y_0} \tilde{A}_1 \tilde{B}_1 C_1}^1\right) \\ &= \gamma_1,\end{aligned}$$

where the second equality follows from the fact that $\rho_{\tilde{X}_1 \tilde{R}_{X_1} \tilde{Y}_0 \tilde{R}_{Y_0} \tilde{A}_1 \tilde{B}_1 C_1}^1 = \rho_{\tilde{Y}_0 \tilde{R}_{Y_0}}^1 \otimes \rho_{\tilde{X}_1 \tilde{R}_{X_1} \tilde{A}_1 \tilde{B}_1 C_1}^1$.

For δ_1 , we instead get rid of the uncorrelated state ρ_Y before applying $V^{Y|X} = V_{X \rightarrow XY R_Y}^{Y|X}$ acting as a control unitary on X and such that $V^{Y|X}(\rho_{X R_X}) = \sigma_{X Y R_X R_Y}$, as well as $V^{Y|X}(\rho_{X R_X A_1 B_1 C_1}^1) = \sigma_{X R_X Y R_Y A_1 B_1 C_1}^1$, and get by then applying V_0 ,

$$\begin{aligned}\epsilon_1 &= h\left(V^1\left(\rho_{X R_X A_1 B_1 C_1}^1\right), \rho_{X R_X} \otimes \rho_{\tilde{X}_1 \tilde{R}_{X_1} \tilde{A}_1 B_1 C_1}^1\right) \\ &= h\left(V^{Y|X} V^1\left(\rho_{X R_X A_1 B_1 C_1}^1\right), V^{Y|X}\left(\rho_{X R_X}\right) \otimes \rho_{\tilde{X}_1 \tilde{R}_{X_1} \tilde{A}_1 B_1 C_1}^1\right) \\ &= h\left(V^1\left(\sigma_{X R_X Y R_Y A_1 B_1 C_1}^1\right), \sigma_{X R_X Y R_Y} \otimes \rho_{\tilde{X}_1 \tilde{R}_{X_1} \tilde{A}_1 B_1 C_1}^1\right) \\ &= h\left(V^0\left(V^1\left(\sigma_{X R_X Y R_Y A_1 B_1 C_1}^1\right)\right), V^0\left(\sigma_{X R_X Y R_Y} \otimes \rho_{\tilde{X}_1 \tilde{R}_{X_1} \tilde{A}_1 B_1 C_1}^1\right)\right) \\ &= h\left(V^1 V^0\left(\sigma_{X R_X Y R_Y A_1 B_1 C_1}^1\right), \sigma_{X R_X Y R_Y} \otimes \rho_{\tilde{X}_1 \tilde{R}_{X_1} \tilde{Y}_0 \tilde{R}_{Y_0} \tilde{A}_1 \tilde{B}_1 C_1}^1\right) \\ &= \delta_1,\end{aligned}$$

where the third equality is from the fact that V^1 and $V^{Y|X}$ commute.

For $i > 1$, we focus on even i ; the case odd i is proven similarly. Denote

$$\begin{aligned}U^i &= U_{Y B_{i-1} C_{i-1} \rightarrow Y B_i C_i}^i, \\ U^{i-1} &= U_{X A_{i-2} C_{i-2} \rightarrow X A_{i-1} C_{i-1}}^{i-1},\end{aligned}$$

the protocol unitaries. Then

$$U^i \left(\rho_{X\bar{R}_X Y R_Y A_{i-1} B_{i-1} C_{i-1}}^{i-1} \right) = \rho_{X\bar{R}_X Y R_Y A_i B_i C_i}^i, \quad (34)$$

$$U^{i-1} \left(\rho_{X\bar{R}_X Y R_Y A_{i-2} B_{i-2} C_{i-2}}^{i-2} \right) = \rho_{X\bar{R}_X Y R_Y A_{i-1} B_{i-1} C_{i-1}}^{i-1}, \quad (35)$$

$$U^i \left(\sigma_{X\bar{R}_X Y R_Y A_{i-1} B_{i-1} C_{i-1}}^{i-1} \right) = \sigma_{X\bar{R}_X Y R_Y A_i B_i C_i}^i, \quad (36)$$

$$U^{i-1} \left(\sigma_{X\bar{R}_X Y \bar{R}_Y A_{i-2} B_{i-2} C_{i-2}}^{i-2} \right) = \sigma_{X\bar{R}_X Y R_Y A_{i-1} B_{i-1} C_{i-1}}^{i-1}. \quad (37)$$

For γ_i , we first reduce to γ_{i-1} using the triangle inequality:

$$\begin{aligned} \gamma_i &= h \left(V^i V^{i-1} \left(\rho_{X\bar{R}_X Y R_Y A_i B_i C_i}^i \right), \rho_{X\bar{R}_X} \otimes \rho_{Y R_Y} \otimes \rho_{\tilde{X}_{i-1} \tilde{R}_{X_{i-1}} \tilde{Y}_i \tilde{R}_{Y_i} \tilde{A}_i \tilde{B}_i C_i} \right) \\ &\leq h \left(\begin{array}{c} V^i V^{i-1} \left(\rho_{X\bar{R}_X Y R_Y A_i B_i C_i}^i \right), \\ V^i U^i (V^{i-2})^\dagger \left(\rho_{X\bar{R}_X} \otimes \rho_{Y R_Y} \otimes \rho_{\tilde{X}_{i-1} \tilde{R}_{X_{i-1}} \tilde{Y}_{i-2} \tilde{R}_{Y_{i-2}} \tilde{A}_{i-1} \tilde{B}_{i-1} C_{i-1}}^{i-1} \right) \end{array} \right) \\ &+ h \left(\begin{array}{c} V^i U^i (V^{i-2})^\dagger \left(\rho_{X\bar{R}_X} \otimes \rho_{Y R_Y} \otimes \rho_{\tilde{X}_{i-1} \tilde{R}_{X_{i-1}} \tilde{Y}_{i-2} \tilde{R}_{Y_{i-2}} \tilde{A}_{i-1} \tilde{B}_{i-1} C_{i-1}}^{i-1} \right), \\ \rho_{X\bar{R}_X} \otimes \rho_{Y R_Y} \otimes \rho_{\tilde{X}_{i-1} \tilde{R}_{X_{i-1}} \tilde{Y}_i \tilde{R}_{Y_i} \tilde{A}_i \tilde{B}_i C_i} \end{array} \right). \end{aligned}$$

Indeed, by rearranging and using that $(U^i)^\dagger$, acting on Bob's side, and V_{i-1} , acting on Alice's side, commute, we get that the first term is equal to γ_{i-1} :

$$\begin{aligned} &h \left(\begin{array}{c} V^i V^{i-1} \left(\rho_{X\bar{R}_X Y R_Y A_i B_i C_i}^i \right), \\ V^i U^i (V^{i-2})^\dagger \left(\rho_{X\bar{R}_X}^X \otimes \rho_{Y R_Y}^Y \otimes \rho_{\tilde{X}_{i-1} \tilde{R}_{X_{i-1}} \tilde{Y}_{i-2} \tilde{R}_{Y_{i-2}} \tilde{A}_{i-1} \tilde{B}_{i-1} C_{i-1}}^{i-1} \right) \end{array} \right) \\ &= h \left(\begin{array}{c} V^{i-2} (U^i)^\dagger V^{i-1} \left(\rho_{X\bar{R}_X Y R_Y A_i B_i C_i}^i \right), \\ \rho_{X\bar{R}_X}^X \otimes \rho_{Y R_Y}^Y \otimes \rho_{\tilde{X}_{i-1} \tilde{R}_{X_{i-1}} \tilde{Y}_{i-2} \tilde{R}_{Y_{i-2}} \tilde{A}_{i-1} \tilde{B}_{i-1} C_{i-1}}^{i-1} \end{array} \right) \\ &= h \left(\begin{array}{c} V^{i-2} V^{i-1} \left(\rho_{X\bar{R}_X Y R_Y A_i B_i C_i}^i \right), \\ \rho_{X\bar{R}_X}^X \otimes \rho_{Y R_Y}^Y \otimes \rho_{\tilde{X}_{i-1} \tilde{R}_{X_{i-1}} \tilde{Y}_{i-2} \tilde{R}_{Y_{i-2}} \tilde{A}_{i-1} \tilde{B}_{i-1} C_{i-1}}^{i-1} \end{array} \right) \\ &= h \left(\begin{array}{c} V^{i-1} V^{i-2} (U^i)^\dagger \left(\rho_{X\bar{R}_X Y R_Y A_{i-1} B_{i-1} C_{i-1}}^{i-1} \right), \\ \rho_{X\bar{R}_X}^X \otimes \rho_{Y R_Y}^Y \otimes \rho_{\tilde{X}_{i-1} \tilde{R}_{X_{i-1}} \tilde{Y}_{i-2} \tilde{R}_{Y_{i-2}} \tilde{A}_{i-1} \tilde{B}_{i-1} C_{i-1}}^{i-1} \end{array} \right) = \gamma_{i-1}, \end{aligned}$$

where the last equality is from Eq. (34) and the commutativity of V^{i-1} and V^{i-2} . For the second term, we again use the triangle inequality to reduce to ϵ_i :

$$\begin{aligned} &h \left(\begin{array}{c} V^i U^i (V^{i-2})^\dagger \left(\rho_{X\bar{R}_X} \otimes \rho_{Y R_Y} \otimes \rho_{\tilde{X}_{i-1} \tilde{R}_{X_{i-1}} \tilde{Y}_{i-2} \tilde{R}_{Y_{i-2}} \tilde{A}_{i-1} \tilde{B}_{i-1} C_{i-1}}^{i-1} \right), \\ \rho_{X\bar{R}_X} \otimes \rho_{Y R_Y} \otimes \rho_{\tilde{X}_{i-1} \tilde{R}_{X_{i-1}} \tilde{Y}_i \tilde{R}_{Y_i} \tilde{A}_i \tilde{B}_i C_i} \end{array} \right) \\ &\leq h \left(\begin{array}{c} V^i U^i (V^{i-2})^\dagger \left(\rho_{Y R_Y} \otimes \rho_{\tilde{X}_{i-1} \tilde{R}_{X_{i-1}} \tilde{Y}_{i-2} \tilde{R}_{Y_{i-2}} \tilde{A}_{i-1} \tilde{B}_{i-1} C_{i-1}}^{i-1} \right), \\ V^i \left(\rho_{\tilde{X}_{i-1} \tilde{R}_{X_{i-1}} Y R_Y \tilde{A}_i B_i C_i}^i \right) \end{array} \right) \end{aligned}$$

$$\begin{aligned}
& + h \left(V^i \left(\rho_{\tilde{X}_{i-1}\tilde{R}_{X_{i-1}}YR_Y\tilde{A}_i\tilde{B}_iC_i}^i \right), \rho_{YR_Y} \otimes \rho_{\tilde{X}_{i-1}\tilde{R}_{X_{i-1}}\tilde{Y}_i\tilde{R}_{Y_i}\tilde{A}_i\tilde{B}_iC_i}^i \right) \\
& = h \left(V^i U^i (V^{i-2})^\dagger \left(\rho_{YR_Y} \otimes \rho_{\tilde{X}_{i-1}\tilde{R}_{X_{i-1}}\tilde{Y}_{i-2}\tilde{R}_{Y_{i-2}}\tilde{A}_{i-1}\tilde{B}_{i-1}C_{i-1}}^{i-1} \right), \right. \\
& \quad \left. V^i \left(\rho_{\tilde{X}_{i-1}\tilde{R}_{X_{i-1}}YR_Y\tilde{A}_i\tilde{B}_iC_i}^i \right) \right) + \epsilon_i, \tag{38}
\end{aligned}$$

in which we also use the fact that U^i , V^i and V^{i-2} all act on Bob's side to get rid of the uncorrelated state ρ_{XR_X} . Notice that $\tilde{X}_{i-1}\tilde{R}_{X_{i-1}} = \tilde{X}_i\tilde{R}_{X_i}$ as i is even. For the first term, we use the fact that V^{i-2} , acting on Bob's side, and U^{i-1} , acting on Alice's side, commute to go from ρ^{i-1} to ρ^{i-2} , and find that it equals ϵ_{i-2} :

$$\begin{aligned}
& h \left(V^i U^i (V^{i-2})^\dagger \left(\rho_{YR_Y} \otimes \rho_{\tilde{X}_{i-1}\tilde{R}_{X_{i-1}}\tilde{Y}_{i-2}\tilde{R}_{Y_{i-2}}\tilde{A}_{i-1}\tilde{B}_{i-1}C_{i-1}}^{i-1} \right), \right. \\
& \quad \left. V^i \left(\rho_{\tilde{X}_{i-1}\tilde{R}_{X_{i-1}}YR_Y\tilde{A}_i\tilde{B}_iC_i}^i \right) \right) \\
& = h \left(\rho_{YR_Y} \otimes \rho_{\tilde{X}_{i-1}\tilde{R}_{X_{i-1}}\tilde{Y}_{i-2}\tilde{R}_{Y_{i-2}}\tilde{A}_{i-1}\tilde{B}_{i-1}C_{i-1}}^{i-1}, \right. \\
& \quad \left. V^{i-2} (U^i)^\dagger \left(\rho_{\tilde{X}_{i-1}\tilde{R}_{X_{i-1}}YR_Y\tilde{A}_i\tilde{B}_iC_i}^i \right) \right) \\
& = h \left(\rho_{YR_Y} \otimes \rho_{\tilde{X}_{i-1}\tilde{R}_{X_{i-1}}\tilde{Y}_{i-2}\tilde{R}_{Y_{i-2}}\tilde{A}_{i-1}\tilde{B}_{i-1}C_{i-1}}^{i-1}, \right. \\
& \quad \left. V^{i-2} \left(\rho_{\tilde{X}_{i-1}\tilde{R}_{X_{i-1}}YR_Y\tilde{A}_{i-1}\tilde{B}_{i-1}C_{i-1}}^{i-1} \right) \right) \\
& = h \left(U^{i-1} \left(\rho_{YR_Y} \otimes \rho_{\tilde{X}_{i-2}\tilde{R}_{X_{i-2}}\tilde{Y}_{i-2}\tilde{R}_{Y_{i-2}}\tilde{A}_{i-2}\tilde{B}_{i-2}C_{i-2}}^{i-2} \right), \right. \\
& \quad \left. V^{i-2} U^{i-1} \left(\rho_{\tilde{X}_{i-2}\tilde{R}_{X_{i-2}}YR_Y\tilde{A}_{i-2}\tilde{B}_{i-2}C_{i-2}}^{i-2} \right) \right) \\
& = h \left(\rho_{YR_Y} \otimes \rho_{\tilde{X}_{i-2}\tilde{R}_{X_{i-2}}\tilde{Y}_{i-2}\tilde{R}_{Y_{i-2}}\tilde{A}_{i-2}\tilde{B}_{i-2}C_{i-2}}^{i-2}, \right. \\
& \quad \left. V^{i-2} \left(\rho_{\tilde{X}_{i-2}\tilde{R}_{X_{i-2}}YR_Y\tilde{A}_{i-2}\tilde{B}_{i-2}C_{i-2}}^{i-2} \right) \right) = \epsilon_{i-2}. \tag{39}
\end{aligned}$$

The bound on γ_i follows by combining these.

To handle δ_i , similarly to $V^{Y|X}$, we define $V^{X|Y} = V_{Y \rightarrow YXR_X}^{X|Y}$ acting as a control unitary on Y and such that $V^{X|Y}(\rho_{YR_Y}) = \sigma_{XR_XYR_Y}$. We first reduce δ_i to δ_{i-1} using the triangle inequality:

$$\begin{aligned}
\delta_i & = h \left(V^i V^{i-1} \left(\sigma_{XR_XYR_Y\tilde{A}_i\tilde{B}_iC_i}^i \right), \sigma_{XR_XYR_Y} \otimes \rho_{\tilde{X}_{i-1}\tilde{R}_{X_{i-1}}\tilde{Y}_i\tilde{R}_{Y_i}\tilde{A}_i\tilde{B}_iC_i}^i \right) \\
& \leq h \left(V^i V^{i-1} \left(\sigma_{XR_XYR_Y\tilde{A}_i\tilde{B}_iC_i}^i \right), \right. \\
& \quad \left. V^i U^i (V^{i-2})^\dagger \left(\sigma_{XR_XYR_Y} \otimes \rho_{\tilde{X}_{i-1}\tilde{R}_{X_{i-1}}\tilde{Y}_{i-2}\tilde{R}_{Y_{i-2}}\tilde{A}_{i-1}\tilde{B}_{i-1}C_{i-1}}^{i-1} \right) \right) \\
& + h \left(V^i U^i (V^{i-2})^\dagger \left(\sigma_{XR_XYR_Y} \otimes \rho_{\tilde{X}_{i-1}\tilde{R}_{X_{i-1}}\tilde{Y}_{i-2}\tilde{R}_{Y_{i-2}}\tilde{A}_{i-1}\tilde{B}_{i-1}C_{i-1}}^{i-1} \right), \right. \\
& \quad \left. \sigma_{XR_XYR_Y} \otimes \rho_{\tilde{X}_{i-1}\tilde{R}_{X_{i-1}}\tilde{Y}_i\tilde{R}_{Y_i}\tilde{A}_i\tilde{B}_iC_i}^i \right).
\end{aligned}$$

Similarly to γ_i , we get that the first term is equal to δ_{i-1} :

$$\delta_{i-1} = h \left(V^i V^{i-1} \left(\sigma_{XR_X Y R_Y A_i B_i C_i}^i \right), \left(\sigma_{XR_X Y R_Y} \otimes \rho_{\tilde{X}_{i-1} \tilde{R}_{X_{i-1}} \tilde{Y}_{i-2} \tilde{R}_{Y_{i-2}} \tilde{A}_{i-1} \tilde{B}_{i-1} C_{i-1}}^{i-1} \right) \right).$$

For the second term, since U_i , V_i and V_{i-2} all act on Bob's side, we apply $(V^{X|Y})^\dagger$ on both side to get the same term as for γ_i , which was proved to be at most $\epsilon_i + \epsilon_{i-2}$:

$$\begin{aligned} & h \left(V^i U^i (V^{i-2})^\dagger \left(\sigma_{XR_X Y R_Y} \otimes \rho_{\tilde{X}_{i-1} \tilde{R}_{X_{i-1}} \tilde{Y}_{i-2} \tilde{R}_{Y_{i-2}} \tilde{A}_{i-1} \tilde{B}_{i-1} C_{i-1}}^{i-1} \right), \right. \\ & \quad \left. \sigma_{XR_X Y R_Y} \otimes \rho_{\tilde{X}_{i-1} \tilde{R}_{X_{i-1}} \tilde{Y}_i \tilde{R}_{Y_i} \tilde{A}_i \tilde{B}_i C_i}^i \right) \\ &= h \left((V^{Y|X})^\dagger V^i U^i (V^{i-2})^\dagger \left(\sigma_{XR_X Y R_Y} \otimes \rho_{\tilde{X}_{i-1} \tilde{R}_{X_{i-1}} \tilde{Y}_{i-2} \tilde{R}_{Y_{i-2}} \tilde{A}_{i-1} \tilde{B}_{i-1} C_{i-1}}^{i-1} \right), \right. \\ & \quad \left. (V^{Y|X})^\dagger \sigma_{XR_X Y R_Y} \otimes \rho_{\tilde{X}_{i-1} \tilde{R}_{X_{i-1}} \tilde{Y}_i \tilde{R}_{Y_i} \tilde{A}_i \tilde{B}_i C_i}^i \right) \\ &= h \left(V^i U^i (V^{i-2})^\dagger \left(\rho_{Y R_Y} \otimes \rho_{\tilde{X}_{i-1} \tilde{R}_{X_{i-1}} \tilde{Y}_{i-2} \tilde{R}_{Y_{i-2}} \tilde{A}_{i-1} \tilde{B}_{i-1} C_{i-1}}^{i-1} \right), \rho_{Y R_Y} \otimes \rho_{\tilde{X}_{i-1} \tilde{R}_{X_{i-1}} \tilde{Y}_i \tilde{R}_{Y_i} \tilde{A}_i \tilde{B}_i C_i}^i \right) \\ &= h \left(V^i U^i (V^{i-2})^\dagger \left(\rho_{X R_X} \otimes \rho_{Y R_Y} \otimes \rho_{\tilde{X}_{i-1} \tilde{R}_{X_{i-1}} \tilde{Y}_{i-2} \tilde{R}_{Y_{i-2}} \tilde{A}_{i-1} \tilde{B}_{i-1} C_{i-1}}^{i-1} \right), \right. \\ & \quad \left. \rho_{X R_X} \otimes \rho_{Y R_Y} \otimes \rho_{\tilde{X}_{i-1} \tilde{R}_{X_{i-1}} \tilde{Y}_i \tilde{R}_{Y_i} \tilde{A}_i \tilde{B}_i C_i}^i \right) \\ &\leq \epsilon_i + \epsilon_{i-2} \quad (\text{Eqs. (38)(39)}) \end{aligned}$$

The bound on δ_i follows by combining these. □

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