Abstracts

Is F automatic?

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Let G be a group with finite symmetric generating set $X = X^{-1}$. An *automatic structure* for (G, X) is the following collection of finite state automata (FSA):

- an FSA M accepting $L \subseteq X^*$ in bijection[1] with G
- for each $x \in X \cup \{\epsilon\}$ an FSA M_x accepting $\{u \otimes v \mid u, v \in L, v =_G ux\}$

where the notation $u \otimes v$ means words of the form

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \cdots \begin{pmatrix} u_s \\ v_s \end{pmatrix} \begin{pmatrix} \$ \\ v_{s+1} \end{pmatrix} \cdots \begin{pmatrix} \$ \\ v_t \end{pmatrix}$$

if $u = u_1 \dots u_s, v = v_1 \dots v_t$ with $t \geqslant s$,

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \cdots \begin{pmatrix} u_t \\ v_t \end{pmatrix} \begin{pmatrix} u_{t+1} \\ \$ \end{pmatrix} \cdots \begin{pmatrix} u_s \\ \$ \end{pmatrix}$$

if $s > t$, and \$ is a padding symbol[2]. If such a structure exists then (G, X) is *automatic*.

An equivalent, more geometric definition is (G, X) is automatic if there is:

- a regular language $L \subseteq X^*$ in bijection with G
- a constant $k \in \mathbb{N}$ such that for each $u, v \in L$ with $v =_G ux$ for some $x \in X \cup \{\epsilon\}$

$$d_X(u(t), v(t)) \leqslant k.$$

That is, in the Cayley graph for (G, X) L-words which start at the identity and end distance at most 1 apart must *synchronously* k*-fellow travel*.

Example 1. $\mathbb{Z}^2 = \langle a, b \mid ab = ba \rangle$, $L = \{a^i b^j \mid i, j \in \mathbb{Z}\}$. Figure 1 shows the automaton M_a.

Example 2. If G is any δ-hyperbolic group with finite generating set $X = X^{-1}$, the set of all shortlex geodesics is regular and satisfies the synchronous fellow travelling condition for a constant depending on δ. In fact, the set of all geodesics also gives an automatic structure (replacing bijection by surjection in the definition), as does the set of all (λ, μ)-quasigeodesics provided $\lambda \in \mathbb{Q}$ and some mild extra conditions [23].

Here are some facts [18]:

- being automatic is independent of the choice of finite generating set

[1] Equivalently, L surjects to G.

[2] Equivalently, (u, v) are accepted by a *synchronous 2-tape automaton*.

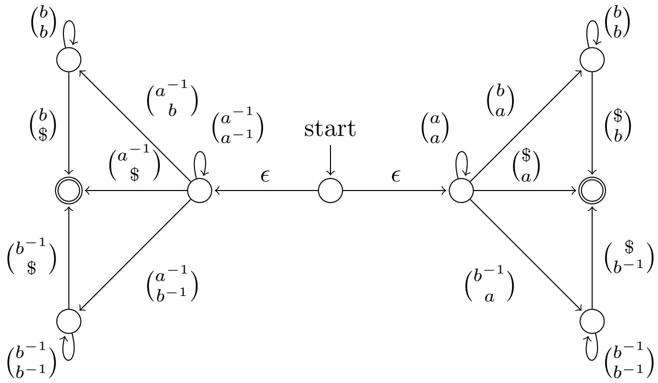

FIGURE 1. The FSA M_a for $\mathbb{Z}^2$.

- L-words are quasi-geodesics; this follows easily from the pumping lemma for regular languages as follows. Let $u \in L$ be the L-word for the identity, $|u| = c$, m the maximum number of states in any M_x, and consider a geodesic $v = a_1 \ldots a_n \in X^*$. Define a sequence of L-words recursively by $v_0 = u, v_i =_G v_{i-1}a_i$ Then $|v_i| \leqslant |v_{i-1}|+m$ since otherwise one could pump the suffix containing $\binom{\$}{x}$ symbols and obtain infinitely many L-words for v. Then $|v_n| \leqslant mn + c$.
- the word problem for automatic groups can be solved in at most quadratic time and linear space (use the previous argument to compute the L-words v_i for a given input word $v = a_1 \ldots a_n$)
- automatic implies G has a Dehn function that is at most quadratic
- automatic implies G is type FP_∞ [20, 1].

So, is F automatic? Recall that Thompson's group F has the finite presentation

$$\langle x_0, x_1 \mid [x_0x_1^{-1}, x_0^{-1}x_1x_0], [x_0x_1^{-1}, x_0^{-2}x_1x_0^2]\rangle.$$

It is known that F has quadratic Dehn function [21], is type FP_∞ [10], has a quasi-linear ($n \log n$) time word problem (algorithm: draw the tree pair diagram). So none of the obvious properties rule F out from being automatic.

Guba and Sapir give the following regular normal form for elements of F: $L =$ all freely reduced words which avoid factors ($i > 0$):

- $x_1^{\pm 1}x_0^ix_1$
- $x_1^{\pm 1}x_0^{i+1}x_1^{-1}$.

The comparison automaton M_{x_0} is easy to construct, since multiplying a word in L on the right by x_0 changes the suffix by at most one letter. However multiplication by x_1 can cause word length to explode: consider $w_i = x_1x_0^i$ with $i > 0$. Then

$$x_1x_0^ix_1 \to x_0^ix_1x_0^{-i-1}x_1x_0^{i+1}.$$

Then the L-words for $w_i, w_i x_1$ have length difference $2i+3$ so when i is greater than then number of states of M_{x_1} we can apply the pumping lemma to obtain infinitely many words u with $w_i \otimes u$ accepted, which is a contradiction.

Note that a weaker version of automatic is to allow words that end at most an edge apart to *asynchronously* fellow travel, or equivalently the comparator automata M_x to read words asynchronously. Consider $w_{m,i} = x_1^m x_0^i$ with $m, i > 0$. The L-word for $w_{m,i}x_1$ is

$$x_0^i x_1 x_0^{-i-1} x_1^m x_0^{i+1}$$

and a careful pumping lemma argument also leads to a contradiction showing that the language also fails to give an asynchronous automatic structure for F.

Non-automatic groups with quadratic Dehn function. Stallings' group

$$\left\langle a,b,c,d,s \;\middle|\; \begin{array}{l} [a,c]=[a,d]=[b,c]=[b,d]=1, \\ (a^{-1}b)^s = a^{-1}b, (a^{-1}c)^s = a^{-1}c, (a^{-1}d)^s = a^{-1}d \end{array} \right\rangle$$

is not type FP_3 [25] and has quadratic Dehn function [15]. It can be seen as the kernel of the map $F_2 \times F_2 \times F_2 \to \mathbb{Z}$ which sends words to their exponent sum; taking n copies of F_2 gives the n-th Bieri-Stallings group which is type FP_{n-1} but not type FP_n [5], and these (for $n > 3$) were also shown to have quadratic Dehn function [12].

Another interesting example is

$$\langle a,b,s,t \mid ab = ba, a^s = ab, a^t = ab^{-1} \rangle$$

which is type FP_∞, not CAT(0) [19], has a quadratic Dehn function [6], has an asynchronously automatic structure [16], but does not admit an automatic structure [7]. The proof of non-automatic relies on a direct argument that, if it were, the set of *slopes* you would expect to see in the embedded $\mathbb{Z}^2$ planes in the Cayley graph should be finite, which leads to a contradiction. It is possible that some similar direct argument can be constructed to rule out the possibility that F is automatic.

Why should F not be automatic? None of the following facts prove that F cannot have an automatic structure, but they do not bode well.

- F has many "bad" subgroups such as $\mathbb{Z}^d$ for any $d \in \mathbb{N} \cup \{\infty\}$, and arbitrary iterated wreath products of $\mathbb{Z}$.
- Cleary, the author and Taback [13] showed that for the standard generating set, any set of words that contains at least one geodesic for each element cannot be regular, so $(F, \{x_0, x_1\})$ has no geodesic automatic structure.
- Jeremy Hauze [22] strengthened this to: languages that have at least one representative of each element of F of word length that is within a *fixed constant* of the geodesic length cannot be part of an automatic structure.

Is F graph automatic? Weakening the notion of automatic further we arrive at the following. A *graph automatic structure* [24] for (G, X) is:

- a finite *symbol alphabet* S (not necessarily corresponding to group elements)

– an FSA M accepting $L \subseteq S^*$ in bijection[3] with G
– for each $x \in X \cup \{\epsilon\}$ an FSA M_x accepting $\{u \otimes v \mid u, v \in L, v =_G ux\}$.

Example 3. The 3-dimensional Heisenberg group consisting of matrices

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

which correspond to triples (a, b, c) of integers. Writing a, b, c in binary we can use an alphabet $S =$ consisting of symbols (i, j, k) with $i, j, k \in \{0, 1, +, -\}$. For example

$$\begin{pmatrix} 1 & -3 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

is represented as $(-, +, +)(1, 0, 0)(1, 0, 1)(0, 1, 0)$. It is easy to check that multiplication by generators $(1, 0, 0), (0, 1, 0)$ simply adds 1 in one position. Berdinsky and Trakuldit [4] attribute this observation to Sénizergues.

Other examples of graph automatic groups include include all Baumslag-Solitar groups, various wreath products, all finitely generated nilpotent groups of nilpotency class at most two [24, 3, 2]. As for automatic groups we have [24]:

– L-words (over symbols) have quasi-geodesic length
– at most a quadratic time word problem
– being graph automatic is invariant under change of finite generating set
– can assume without loss of generality that S is a subset of the generating set. However, paths in the Cayley graph labeled by S-edges do not necessarily end anywhere near the group element represented by the label of the path. See [4].

Thompson's group F seems like a natural candidate for graph automaticity, since we have many nice ways to represent elements, for example as tree pair diagrams. However, any encoding of a tree pair using a finite alphabet will require some memory. This leads to the notion of a $\mathscr{C}$-graph automatic structure where we replace regular languages by languages in the class $\mathscr{C}$ in the definition. This even weaker notion still implies some nice properties: for counter-graph automatic with a quasigeodesic normal form we still have a polynomial time algorithm to compute L-words, which means a polynomial time word problem [17]. In [26] Taback and Younes constructs a (3-counter)-graph automatic structure based on tree pair diagrams for F.

Encoding the infinite normal form in a certain way, the author and Taback were able to lower the complexity to (1-counter)-graph automatic. We write words

$$x_0^{i_0} x_1^{i_1} \dots x_r^{i_r} x_s^{-j_s} \dots x_0^{-j_0}$$

as strings over an alphabet $\{\#, a, b\}$ in such a way that the conditions required to have unique representatives are regular to check. The single counter is needed to

[3]Equivalently, L surjects to G

check multiplication by x_1. Specifically we represent $x_0^{i_0} \dots x_r^{i_r} x_s^{-j_s} \dots x_0^{-j_0}$ as

$$a^{i_0} b^{j_0} \# \dots \# a^{i_m} b^{j_m}$$

where $m = \max\{r, s\}$. The words obtained are quasigeodesic [11].

Final remarks. Another extension of the notion of automatic which I did not discuss in the talk is *autostackable* [9] and the weaker notion of *algorithmically stackable* [8]. Brittenham, Hermiller and Holt introduced these notions, showing that they also imply some nice computation properties. Cleary, Hermiller, Stein and Taback prove that F is algorithmically stackable with respect to a deterministic context-free language of normal forms [14, 8].

Whether F is another example of a group with quadratic Dehn function that is not automatic, or if in fact it admits some nice automatic or graph automatic structure remains open. Once again F proves itself to be an enigma.

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