Less-Expensive Pricing and Hedging of Extreme-Maturity Interest Rate Derivatives and Equity Index Options under the Real-World Measure

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Certificate of Original Authorship

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Notation

The following list provides the meaning of symbols and notation used throughout this thesis.

Symbol	Meaning
$\mathcal{A}_t$	Information available at time t written as a σ-algebra
$\underline{\mathcal{A}}$	Filtration or evolution of the flow of information over time
$A^+_{\bar{T},K}(t,U)$	Price of an asset binary call option on an underlying asset having price process U with expiry time $\bar{T}$ and strike price K
$A^-_{\bar{T},K}(t,U)$	Price of an asset binary put option on an underlying asset having price process U with expiry time $\bar{T}$ and strike price K
AIC	Akaike Information Criterion
$\bar{\alpha}_0$	Initial drift of the discounted GOP
$\bar{\alpha}_t$	Drift of the discounted GOP
$B^+_{\bar{T},K}(t,U)$	Price of a bond binary call option on an underlying asset having price process U with expiry time $\bar{T}$ and strike price K
$B^-_{\bar{T},K}(t,U)$	Price of a bond binary put option on an underlying asset having price process U with expiry time $\bar{T}$ and strike price K
B_t	Value of savings account at time t
BA	Benchmark approach
BS	Black-Scholes
$c_{\bar{T},K}(t,U)$	Price of a call option on an underlying asset having price process U with expiry time $\bar{T}$ and strike price K
$\mathbf{cap}_{\mathcal{T},\mathcal{K}}(t)$	Price of a cap in respect of a start date and subsequent payment dates in the set $\mathcal{T} = \{T_0, T_1, \ldots, T_n\}$
$\mathbf{caplet}_{\bar{T},T,K}(t)$	Price of a caplet with start time $\bar{T}$ and end time T

Symbol	Meaning
CEV	Constant elasticity of variance
CIR	Cox-Ingersoll-Ross
δ_*	Strategy associated with the growth optimal portfolio
$\mathrm{E}(U)$	Expectation of a random variable U
$\mathrm{E}(U\|\mathcal{A}_t)$	Expectation of a random variable U given information available at time t
$f_T(t)$	Instantaneous T-forward rate at time t
$f_\infty(t)$	Asymptotic forward rate as at time t
$F_{\bar{T},T}(t)$	Discrete $[\bar{T},T]$-forward rate at time t
$\mathbf{floor}_{\mathcal{T},\mathcal{K}}(t)$	Price of a floor in respect of a start date and subsequent payment dates in the set $\mathcal{T} = \{T_0, T_1, \ldots, T_n\}$
$\mathbf{floorlet}_{\bar{T},T,K}(t)$	Price of a floorlet with start time $\bar{T}$ and end time T
$F_T(t,U)$	T-Forward price of an asset having price process U
FRA	Forward rate agreement
GOP	Growth optimal portfolio
$g_T(t)$	Contribution of the short rate to the instantaneous T-forward rate at time t
$g_\infty(t)$	Contribution of the short rate to the asymptotic instantaneous forward rate as at time t
$G_T(t)$	Contribution of the short rate to the T-maturity zero-coupon bond price
$G(\alpha,\gamma)$	Gamma distribution with shape parameter α and scale parameter γ
$G(x;\alpha,\gamma)$	Cumulative distribution function of the gamma distribution
GBM	Geometric Brownian motion
GIG	Generalised inverse Gaussian
GMMM	Generalised minimal market model
$\Gamma(x)$	Gamma function of x given by $\int_0^\infty u^{x-1}\mathrm{e}^{-u}\,du$
$h_T(t)$	Contribution of the short rate to the T-maturity zero-coupon bond yield at time t
$h_\infty(t)$	Contribution of the short rate to the long zero-coupon bond yield as at time t
$\mathbf{1}_X$	Indicator function, equalling 1 if the statement X is true and 0 otherwise
$\mathcal{I}$	Fisher's information matrix
$I_\nu(x)$	Modified Bessel function of the first kind with index ν
η	Net market growth rate of the GOP
κ	Speed of mean reversion associated with a short rate model
K	Strike price of an option
$K_\lambda(\omega)$	Modified Bessel function of the third kind with index λ

Symbol	Meaning
$\ell(\Theta)$	Logarithm of likelihood function of model parameters Θ
$L(\Theta)$	Likelihood function of model parameters Θ
LHS	Left hand side
$LN(\mu, \sigma)$	Lognormal distribution with location parameter μ and scale parameter σ
$LN(x; \mu, \sigma)$	Cumulative distribution function of the lognormal distribution
$m_T(t)$	Contribution of the discounted GOP to the T-forward rate at time t
$M_T(t)$	Contribution of the discounted GOP to the T-maturity zero-coupon bond price
$M(\alpha, \gamma, z)$	Confluent hypergeometric function
MGF	Moment generating function
$MGF_X(t)$	Moment generating function of the random variable X
MLE	Maximum likelihood estimate
MMM	Minimal market model
MSCI	Morgan Stanley Capital International
$n(x)$	Probability density function for the standard normal distribution
$N(x)$	Cumulative distribution function for the standard normal distribution
$N(\mu, \sigma^2)$	Normal distribution having mean μ and variance σ^2
$n_T(t)$	Contribution of the discounted GOP to the T-maturity zero-coupon bond yield at time t
$NCG(\alpha, \gamma, \lambda)$	Non-central gamma distribution having scale parameter γ, shape parameter α and non-centrality parameter λ
$NCG(x; \alpha, \gamma, \lambda)$	Cumulative distribution function of the non-central gamma distribution
OTC	Over the counter
$p_{\bar{T},K}(t, U)$	Price of a put option on an underlying asset having price process U with expiry time $\bar{T}$ and strike price K
$P(t, T)$	Price of T-maturity zero-coupon bond at time t
$P_{\mathcal{T},c}(t)$	Price at time t of a coupon bond having unit notional, coupon rate c and most recent coupon payment date and subsequent coupon payment dates in the set $\mathcal{T} = \{T_0, T_1, \ldots, T_n\}$
$\textbf{payerswaption}_{\mathcal{T},K,N}(t)$	Price at time t of a payer swaption having strike rate K and underlying swap with notional N and start date (same as expiry date of swaption) and subsequent payment dates in the set $\mathcal{T} = \{T_0, T_1, \ldots, T_n\}$
PDE	Partial differential equation
$Poi(\lambda)$	Poisson distribution having rate parameter λ
QV	Quadratic variation
$[X]_t$	Quadratic variation of a stochastic process X

Symbol	Meaning
r	Short rate
$\bar{r}$	Level of mean reversion associated with a short rate model
$\mathbf{receiverswaption}_{\mathcal{T},K,N}(t)$	Price at time t of a receiver swaption having strike rate K and underlying swap with notional N and start date (same as expiry date of swaption) and subsequent payment dates in the set $\mathcal{T} = \{T_0, T_1, \ldots, T_n\}$
$\mathbb{R}$	Real numbers
RHS	Right hand side
$\mathbf{swaprate}_{\mathcal{T}}(t)$	Swap rate at time t in respect of a start date and subsequent payment dates in the set $\mathcal{T} = \{T_0, T_1, \ldots, T_n\}$
s, t	Time
σ	Diffusion parameter of a short rate process
$S_t^{\delta_*}$	Value of the growth optimal portfolio at time t
$\bar{S}_t^{\delta_*}$	Discounted value of the growth optimal portfolio at time t
S&P 500	Standard and Poor's 500 equity index
SDE	Stochastic differential equation
SGH	Symmetric generalised hyperbolic
Skew(U)	Skew of a random variable U
SE$(\hat{p})$	Standard error of an estimate of the parameter p
T, $\bar{T}$	Time of option expiry or bond maturity
θ	Volatility of the discounted GOP
USD	United States Dollar
Var(U)	Variance of a random variable U
Var$(U\|\mathcal{A}_t)$	Variance of a random variable U given information available at time t
VaR	Value at risk
W	Wiener process driving the discounted GOP
WSI	Diversified world stock index
χ^2_ν	Chi-squared distribution with ν degrees of freedom
$\chi^2_\nu(x)$	Cumulative distribution function for the chi-squared distribution with ν degrees of freedom
$\chi^2_{\nu,\lambda}$	Non-central chi-squared distribution with ν degrees of freedom and non-centrality parameter λ
$\chi^2_{\nu,\lambda}(x)$	Cumulative distribution function for the non-central chi-squared distribution with ν degrees of freedom and non-centrality parameter λ
$y_T(t)$	Continuously compounded T-maturity zero-coupon bond yield at time t
$y_\infty(t)$	Continuously compounded long zero-coupon bond yield as at time t

Symbol	Meaning
Z	Wiener process driving the short rate
ZCB	Zero-coupon bond
$\mathbf{zcbcall}_{\bar{T},T,K}$	Price of $\bar{T}$-expiry call option on a T-maturity zero-coupon bond with strike price K
$\mathbf{zcbput}_{\bar{T},T,K}$	Price of $\bar{T}$-expiry put option on a T-maturity zero-coupon bond with strike price K
$(x)_n$	Pockhammer function which is shorthand for the product $x\,(x+1)\,(x+2)\,\ldots\,(x+n-1)$, where n is a non-negative integer and we use the convention that the empty product is one

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Abstract

This thesis is practically oriented towards the pricing and hedging of long-dated interest rate derivatives and equity index options under Platen's benchmark approach. It aims to be self-contained for convenience of the reader, including all proofs. Among leading banks and insurance companies there does not appear to exist a generally accepted methodology of accurately pricing and hedging such over-the-counter derivatives. This remains the case, despite significant efforts by academics and market practitioners since the early 1990s. This thesis revisits this problem in the light of empirical evidence in a much wider modelling framework than that provided by the classical risk neutral approach.

The models considered in this thesis are specified by stochastic differential equations that describe the real-world dynamics of two market variables, namely the short rate and the volatility of the growth optimal portfolio (GOP). The latter is essentially a diversified equity index.

This thesis assesses for these models their ability to generate reasonably accurate prices and hedges of typical interest rate term structure derivatives and equity index options. When the discounted GOP is modelled as a time-transformed squared Bessel process, fair prices differ from classical risk neutral prices, resulting in lower prices and lower values-at-risk of long-dated derivatives. Also, such models reflect well empirical market features, such as leptokurtic returns, the leverage effect and a stochastic, yet stationary, volatility structure of the equity index.

The results of this analysis, which are contained in this thesis, have been supplemented by the publications of Fergusson and Platen [2006], Fergusson and Platen [2014a], Fergusson and Platen [2015b], Fergusson [2017a] and Fergusson [2017b] and the research reports of Fergusson and Platen [2013], Fergusson and Platen [2014b] and Fergusson and Platen [2015a]. In addition, as by-products of the work done in this thesis, the following papers have been published: Thompson et al. [2017], Calderin et al. [2017] and the following have been submitted to journals for publication: Fergusson and Platen [2014c], Fergusson and Platen [2017]. Finally, the following working papers are to be submitted to journals shortly: Fergusson [2017c], Fergusson [2017d], Fergusson [2017e].

Chapter 1

Brief Outline of the Thesis

The aim of this thesis is to assess the performance of hedging interest rate derivatives and equity index options entirely under the real-world probability measure for a variety of models in a much wider modelling framework than that provided by classical theories. The thesis aims also to derive all formulae employed, using new or alternative proofs.

The pricing of interest rate derivatives is performed under the benchmark approach, where the growth optimal portfolio (GOP) plays a central role. The GOP is the strictly positive portfolio that maximises expected logarithmic utility of terminal wealth. It is approximated in this thesis by large, well-diversified portfolios such as a world stock index (WSI).

The short rate is the continuously compounded annualised rate of interest earned on cash deposited for an infinitesimally small period of time. The cash account is the accumulation of cash continuously reinvested at the short rate of interest. The discounted GOP is the ratio of the GOP to the cash account.

The models considered in this thesis are described via stochastic differential equations (SDEs) for the discounted GOP and the short rate.

The benchmark approach (BA) to pricing derivative securities provides a methodology more general than that of risk neutral pricing in complete markets, used by most participants in the market today. A challenging problem is the pricing and hedging of extreme-maturity fixed income derivatives. In this regard, this thesis will document that prices and replicating portfolios can be established under the BA that are considerably less expensive than those currently suggested under the prevailing theories and practices.

For instance, over the last ten years there has been increasing demand for fifty year bonds by pension funds. This thesis points the way to hedging longer dated derivatives in practice. It shows that the existing market prices for extreme-maturity derivatives are potentially overpriced. Based on our assumptions we infer prices for extreme-maturity derivatives in a manner which we aim to make transparent to both academics and market practitioners.

The key to our less-expensive valuation is that a diversified equity market index, interpreted as a proxy for the GOP, is growing faster on average in the very long term than the savings account. This is a stylised empirical fact, well documented, for instance, in the equity risk premium literature. The hedging of extreme-maturity derivatives exploits as much as possible the higher growth of the index rather than the lesser growth of the savings account. The latter is, in principle, currently used as the reference asset for the market.

Our main question that we would like to answer is the following: When using historical data for the underlying assets, is the hedging of long term fixed income market instruments aligned more to the risk neutral pricing paradigm than to the general paradigm offered under the benchmark approach?

Part of the above purpose is to marry historical estimation of model parameters to realistic theoretical prices and hedges of important fixed income derivatives. This is to say, in this thesis, we use retrospective statistical estimates to fit interest rate term structure models under the real-world probability measure with the aim that the resulting derivative prices can be reasonably hedged in reality.

In this thesis we deliberately avoid assuming that extreme-maturity and other derivative prices are correctly priced in the market already, and instead rely on real-world pricing within the benchmark paradigm to assess their possibly low values.

Chapter 2 of this thesis introduces in detail Platen's benchmark approach (BA) and how it is applied to the pricing and hedging of European interest rate derivatives. The central building block of the BA is the growth optimal portfolio (GOP), which is the portfolio having maximal expected logarithmic utility from terminal wealth. The GOP satisfies in the given continuous market a very particular SDE, where the risk premium is the square of the volatility.

The GOP will be used as the numéraire portfolio (see Long [1990]) and benchmark. Prices expressed in units of the GOP are called benchmarked prices. All benchmarked nonnegative portfolios are supermartingales, which means that they trend, in the long run, downwards or have no (upward) trend.

Using the martingale property of benchmarked fair prices of derivatives, we compute general formulae for forward rate agreements, interest rate swaps, interest rate caps, interest rate swaptions, bond options and options on the GOP under various models. The resulting interest rate derivatives are then discussed in the light of historical market data. One way that we exploit is the use of hedge simulations of interest rate derivatives. A given market model, which consists of an interest rate model and an associated model of the discounted GOP, induces a pricing formula for an interest rate derivative and also induces a related dynamic hedging strategy. A hedging strategy is assessed by looking at the cost of hedging the derivative: the lower the cost of hedging, and the greater the accuracy, the better the hedging strategy. We can, therefore, compare various financial market models with regard to the performance of their induced prices and hedging strategies.

In Chapter 3, three models of the short rate are described, ranging in simplicity from the Vasicek model, to the more complicated Cox-Ingersoll-Ross and 3/2 interest rate models. In Chapter 4, two models of the discounted GOP are described, geometric Brownian motion and a time transformed squared Bessel process.

In Chapter 5 we investigate several market models which combine the short rate models of Chapter 3 with the models of the discounted GOP in Chapter 4. The particular combinations of models for the discounted GOP and the short rate are the following:

For the discounted GOP we consider the Black-Scholes (BS) model and the minimal market model (MMM). For the short rate we study the deterministic short rate model, Vasicek model, CIR model and 3/2 model. The parameters of these short rate models are deliberately chosen without any time dependency. The parameter estimation methods for the short rate models have been published in Fergusson and Platen [2015b]. For each model we compute pricing formulae of major interest rate derivatives and equity index options. The pricing formulae of interest rate swaptions under the two-factor market model are original, published in Fergusson and Platen [2014a]. The pricing formulae of equity index options when interest rates are stochastic are original and the corresponding approximate pricing formulae are developed which allow rapid calculation of the prices of equity index options, reported in Fergusson and Platen [2015a].

In Chapter 6 we describe stylised features of each model with regard to yield curve shapes, interest rate volatility term structure and GOP volatility. These findings have been published in Fergusson and Platen [2006] and reported in Fergusson and Platen [2014b].

In Chapter 7 we compare each model's performance in hedging zero-coupon bonds of various terms to maturity. Chapter 8 presents a comparison of each model's performance in hedging swaptions. These comparisons have been published in Fergusson and Platen [2014a]. Chapter 9 involves a comparison of each model's performance in hedging options on the GOP, reported in Fergusson and Platen [2015a].

Chapter 10 concludes with the narrowing down of our search for the best market model, this being a model which exhibits many of the stylised features observed in the market. In particular, it is the model under which all the regimes of the yield curve, namely steepening, inverse and humped shapes, are attainable, the volatility term structure of forward rates exhibits stylised features of the cap and swaption markets and the leverage effect is realistically reflected. We find that extreme-maturity fixed income derivatives are less expensive under MMM discounted GOP models than under BS discounted GOP models. Also, hedge simulations of long-dated and extreme-maturity derivatives within the paradigm of MMM discounted GOP models demonstrate significantly lower Values-at-Risk than those within the paradigm of BS discounted GOP models. Finally, we indicate a way of generalising the market models in this thesis to multi-currency models, which can be used to price and hedge more exotic long-dated interest rate and FX derivatives.

Chapter 2

Derivatives Pricing

2.1 Introduction

We describe in this chapter the pricing of derivatives using the GOP as numéraire and the real-world probability measure as pricing measure, as done in the earlier work of Platen [2002b], Platen [2004], Platen and Heath [2006] and Miller [2007]. We commence describing primary security accounts as the building blocks of our portfolios. Of all possible non-negative portfolios that can be constructed, we explain the idea of a growth optimal portfolio (GOP). We define a benchmarked portfolio as a non-negative portfolio denominated in terms of the GOP.

In Platen [2002b] it is proven that in a continuous time jump diffusion market a benchmarked self-financing portfolio process is a local martingale, and when the portfolio is non-negative it is also a supermartingale. The supermartingale property of portfolios ensures that the portfolios do not permit strong arbitrage in the sense described in Platen and Heath [2006], see Definition 2.5.1. The minimal supermartingale that replicates a given benchmarked payoff is the corresponding martingale. This is why we will use martingales to determine derivative prices choosing the GOP as numéraire. Now the pricing measure is the probability measure associated with the real-world dynamics of the market and its benchmarked portfolios. In our setting a contingent claim is a payoff at a given time whose value depends on the values of the underlying assets. By computing the expectation of a benchmarked contingent claim with respect to the real-world probability measure, we obtain the fair (or real-world) price for the benchmarked contingent claim, and hence a fair price process for the contingent claim. This price process is minimal in the set of possible replicating portfolio processes. The minimal prices when valuing contingent claims are those that follow via real-world pricing under the benchmark approach.

Of course, when given a model we can calculate associated sensitivities of the price of the contingent claim to changes in prices of underlying assets. In this manner we can develop a dynamic trading strategy involving the underlying assets, which hedges the contingent claim to its expiry date.

The benchmark approach to hedging a contingent claim is described later in this chapter including several results on derivative contracts, which differ from classical risk neutral prices.

2.2 Financial Market

Our market consists of $d+1$ *primary assets*, which are continuously traded and whose randomness is modelled by d independent standard Wiener processes $W^k = \{W^k_t,\ t \in [0,T]\},\ k \in \{1,2,\ldots,d\}$. These processes are defined on a filtered probability space $(\Omega, \mathcal{A}_T, \underline{\mathcal{A}}, P)$ with finite time horizon $T \in (0,\infty)$, fulfilling the usual conditions, namely that the probability space $(\Omega, \mathcal{A}_T, P)$ is complete, the σ-algebras $\mathcal{A}_t$ contain all the sets in $\mathcal{A}_T$ of zero probability and the filtration $\underline{\mathcal{A}} = \{\mathcal{A} : t \in [0,T]\}$ is right-continuous, as given in Protter [2004]. Here the filtration $\underline{\mathcal{A}} = \{\mathcal{A} : t \in [0,T]\}$ models the evolution of market information over time, while $\mathcal{A}_t$ describes the information available at time $t \in [0,T]$. P is the real-world probability measure. For unexplained notions and definitions, we refer to Platen and Heath [2006].

2.2.1 Primary Security Accounts

We introduce $d+1$ primary assets, $d \in \{1,2,\ldots\}$, where the 0-th primary asset is the domestic currency and where the other primary assets are, for example, stocks, currencies, bonds and commodities. Associated to each primary asset is an income, or carrying cost, derived from holding the asset. For example, the income earned from holding US dollars is interest calculated at the US deposit rate and the income earned from holding stocks is the paid dividends.

The j-th primary asset, $j \in \{0,1,\ldots,d\}$, when measured in units of the domestic currency, is modelled via the *primary security account process* $S^j = \{S^j_t,\ t \in [0,T]\}$. Each primary security account represents the accumulation of all income, carrying costs plus capital gains or losses achieved while holding the underlying primary asset. For example, the IBM stock has its primary security account consisting of the initial share held at time $t = 0$ plus all dividends reinvested into the stock since the initial time $t = 0$, with the total value of these holdings expressed in units of the domestic currency, which is in this thesis usually US dollars (USD). We also assume that the units of primary assets are infinitely divisible such that it is possible for trading to be both continuous and frictionless, that is to say, without transaction costs.

We assume that S^j_t is the unique solution of the stochastic differential equation

(SDE)

$$dS_t^j = a_t^j\, S_t^j\, dt + S_t^j \sum_{k=1}^{d} b_t^{j,k}\, dW_t^k \tag{2.2.1}$$

for $t \in [0,T]$ and $j \in \{0,1,\dots,d\}$ with finite initial value $S_0^j > 0$. The j-th *appreciation rate* $a^j = \{a_t^j,\, t \in [0,T]\}$ is the expected return of holding the j-th primary security in units of the domestic currency. The (j,k)-th *volatility* $b^{j,k} = \{b_t^{j,k},\, t \in [0,T]\}$ expresses the fluctuations generated by the k-th Wiener process W^k, $k \in \{1,2,\dots,d\}$ of the return of the j-th primary security when denominated in the domestic currency. The appreciation rates and the volatilities are assumed to be finite predictable processes.

Assumption 2.2.1 *We assume that the* volatility matrix $b_t = [b_t^{j,k}]_{j,k=1}^d$ *exists and is for Lebesgue-almost-every* t*,* invertible*, hence* $(b_t)^{-1} < \infty$ *for* $t \in [0,T]$*.*

This ensures that market prices of risk for the given currency denomination are uniquely determined. In this study we only consider the case where there exists a unique market price of risk for every source of uncertainty that is generated by the Wiener processes W^k, $k \in \{1,2,\dots,d\}$, in the domestic currency denomination.

2.2.2 Savings Account

We consider the 0-th primary asset and form an accumulation account in units of this security. We refer to such an account as the domestic savings account process $B = \{B_t,\, t \in [0,T]\}$ for the 0-th primary security. It is locally riskless in the sense as for domestic cash holdings in a bank account denominated in the domestic currency. The growth rate of the domestic savings account at time t is referred to as the domestic short rate r_t. We assume that the domestic savings account B_t satisfies the equation

$$B_t = B_0 \exp\left(\int_0^t r_s\, ds\right) \tag{2.2.2}$$

for $t \in [0,T]$, where we set $B_0 = 1$ without loss of generality. Comparing (2.2.1) and (2.2.2) we observe that $a_t^0 = r_t$ and $b_t^{0,k} = 0$ for $k \in \{1,\dots,d\}$ and $t \in [0,T]$.

Also note that the domestic savings account process $B = \{B_t,\, t \in [0,T]\}$ and the domestic short rate process $r = \{r_t,\, t \in [0,T]\}$ provide equivalent characterisations of the time value of the domestic currency. We will later see that the savings account can be interpreted as the limit of a sequence of roll-over short-term bond accounts.

If we introduce the appreciation rate vector $a_t = (a_t^1,\dots,a_t^d)^\top$ in the domestic currency denomination and the unit vector $\mathbf{1} = (1,\dots,1)^\top$, then by Assumption 2.2.1 we obtain the *market price of risk* vector for the domestic currency denomination as

$$\theta_t = (\theta_t^1, \theta_t^2, \dots, \theta_t^d)^\top = (b_t)^{-1}\big(a_t - r_t \times \mathbf{1}\big) \tag{2.2.3}$$

for $t \in [0, T]$. Using (2.2.3), we can rewrite the SDE (2.2.1) in the form

$$dS_t^j = S_t^j \left(r_t + \sum_{k=1}^{d} b_t^{j,k}\, \theta_t^k \right) dt + S_t^j \sum_{k=1}^{d} b_t^{j,k}\, dW_t^k \tag{2.2.4}$$

for $t \in [0, T]$ and $j \in \{0, 1, \dots, d\}$. Let $S = \{S_t = (S_t^0, S_t^1, \dots, S_t^d)^\top, t \in [0, T]\}$ denote the vector process of the primary security accounts. Also note that, thus far, no major restrictions have been placed on the dynamics of the primary security accounts.

2.2.3 Trading Strategies and Portfolios

We can now construct a portfolio of $d + 1$ primary security accounts. We say that a stochastic process $\delta = \{\delta_t = (\delta_t^0, \dots, \delta_t^d)^\top,\ t \in [0, T]\}$ is a *strategy*, if δ is $\underline{\mathcal{A}}$-predictable and S-integrable, as defined in Protter [2004]. The j-th component of the strategy at time t is denoted $\delta_t^j \in (-\infty, \infty)$ and represents the number of units of the j-th primary security account that are held in the portfolio at time $t \in [0, T]$ for all $j \in \{0, 1, \dots, d\}$.

For a given strategy δ we introduce the corresponding *wealth process* as $S^\delta = \{S_t^\delta,\ t \in [0, T]\}$, which is determined as

$$S_t^\delta = \sum_{j=0}^{d} \delta_t^j\, S_t^j \tag{2.2.5}$$

for $t \in [0, T]$.

Definition 2.2.2 *A strategy δ and the corresponding wealth process S^δ are called* self-financing *if*

$$dS_t^\delta = \sum_{j=0}^{d} \delta_t^j\, dS_t^j \tag{2.2.6}$$

for $t \in [0, T]$.

In economic terms, self-financing strategies infer that there are no inflows or outflows of funds to the corresponding wealth process S^δ and that all changes in value are due to gains from trade. We only consider self-financing strategies and wealth processes in the remainder of the thesis.

Definition 2.2.3 *A strategy δ is called* admissible *with respect to the domestic currency if it is self-financing and its corresponding wealth process S^δ is non-negative.*

For a given strategy δ and portfolio value $S_t^\delta > 0$, let $\pi_t^{j,\delta}$ equal the *proportion* of the value of the portfolio, that is invested in the j-th primary security account at time t, which is calculated as

$$\pi_t^{j,\delta} = \frac{\delta_t^j \, S_t^j}{S_t^\delta} \tag{2.2.7}$$

for $t \in [0, T]$ and $j \in \{0, 1, \ldots, d\}$. Note from (2.2.5) and (2.2.7) that these proportions must sum to unity, that is

$$\sum_{j=0}^{d} \pi_t^{j,\delta} = 1 \tag{2.2.8}$$

for all $t \in [0, T]$. For the strategy δ the corresponding wealth process S_t^δ satisfies the SDE

$$dS_t^\delta = S_t^\delta \left(r_t + \sum_{k=1}^{d} \sum_{j=0}^{d} \pi_t^{j,\delta} \, b_t^{j,k} \, \theta_t^k \right) dt + S_t^\delta \sum_{k=1}^{d} \sum_{j=0}^{d} \pi_t^{j,\delta} \, b_t^{j,k} \, dW_t^k \tag{2.2.9}$$

for all $t \in [0, T]$.

2.3 Growth Optimal Portfolio

It is our aim to construct a *benchmark*, that can be used as a numéraire (or reference unit). The benchmark approach uses the growth optimal portfolio (GOP) as benchmark, see Platen and Heath [2006]. Therefore, we seek the uniquely determined GOP of the above described market. The GOP achieves the maximum possible expected growth rate at any time, and also the maximum growth in the long run, as shown in Platen [2004]. Furthermore, when used as benchmark, each benchmarked non-negative portfolio is a supermartingale. As such, the GOP is the *best performing portfolio* in this sense. It has been studied previously, for example in Kelly [1956], Long [1990], Karatzas and Shreve [1998], Platen [2002b] and by many other authors.

We can conveniently determine the GOP by calculating the maximum of the drift of the logarithm of all strictly positive portfolios. Hence from (2.2.9) and the Itô formula we find

$$d\log(S_t^\delta) = g_t^\delta \, dt + \sum_{k=1}^{d} \sum_{j=0}^{d} \pi_t^{j,\delta} \, b_t^{j,k} \, dW_t^k \tag{2.3.1}$$

for $t \in [0,T]$ and $j \in \{0,1,\ldots,d\}$, with the resulting *portfolio growth rate* g_t^δ as at time t. The optimal strategy $\delta_* = \{\delta_*(t) = (\delta_*^0(t),\ldots,\delta_*^d(t))^\top,\, t \in [0,T]\}$ follows from solving the first order conditions of the corresponding quadratic maximisation problem for g_t^δ, as shown in Platen [2002b]. This optimal strategy translates into the optimal proportions

$$\pi_t^{\delta_*} = (\pi_t^{1,\delta_*},\ldots,\pi_t^{d,\delta_*})^\top = [(b_t)^{-1}]^\top \theta_t \tag{2.3.2}$$

for $t \in [0,T]$. Therefore, substitution of the optimal proportions (2.3.2) into (2.2.9) uniquely forms the GOP $S_t^{\delta_*}$ at time t, satisfying the SDE

$$dS_t^{\delta_*} = S_t^{\delta_*}\left(r_t + \sum_{k=1}^d (\theta_t^k)^2\right) dt + S_t^{\delta_*} \sum_{k=1}^d \theta_t^k \, dW_t^k \tag{2.3.3}$$

for $t \in [0,T]$ with a given strictly positive initial value $S_0^{\delta_*} > 0$. By (2.2.9), (2.3.1) and (2.3.3) we obtain the optimal growth rate $g_t^{\delta_*}$ of the GOP at time t in the form

$$g_t^{\delta_*} = r_t + \frac{1}{2}\sum_{k=1}^d (\theta_t^k)^2 \tag{2.3.4}$$

for $t \in [0,T]$. The two terms of (2.3.4), these being the short rate and half the sum of squares of the GOP volatilities, are also the key ingredients that need to be specified in an interest rate term structure model, as we will see later and also as employed in Long [1990] and Karatzas and Shreve [1998].

The *squared total market price of risk*, which is the risk premium of the GOP in (2.3.3), equates to the expression

$$|\theta_t|^2 = (\theta_t)^\top \theta_t = \sum_{k=1}^d (\theta_t^k)^2 \tag{2.3.5}$$

at time $t \in [0,T]$. It is also observed as the square of the GOP volatility. Hence we can rewrite (2.3.3) as the SDE

$$dS_t^{\delta_*} = S_t^{\delta_*}\left(r_t + |\theta_t|^2\right) dt + S_t^{\delta_*}\, |\theta_t| \, d\hat{W}_t \tag{2.3.6}$$

for $t \in [0,T]$. Here the Wiener process $\hat{W} = \{\hat{W}_t,\, t \in [0,T]\}$ has the stochastic differential

$$d\hat{W}_t = \frac{1}{|\theta_t|}\sum_{k=1}^d \theta_t^k \, dW_t^k \tag{2.3.7}$$

for $t \in [0,T]$.

The following remark highlights one of the most important practical features of the benchmark approach, namely the possibility to use a diversified market index to approximate the GOP.

Remark 2.3.1 *Platen [2005b] and Platen and Rendek [2012a] prove that appropriately defined* diversified portfolios *also represent approximate GOPs. This result implies that appropriately defined diversified portfolios exhibit similar behaviour. Therefore, a number of commonly used, well-diversified stock market indices can be used to approximate the GOP, including but not limited to the following: the Standard and Poor's* 500 *Index (S&P* 500*) and the Russell* 2000 *Index for the US market and the MSCI Growth World Stock Index (MSCI) for global modelling.*

Furthermore, inputs used in models discussed in later chapters can be based on market observable indices. As an example, the volatility of the GOP can be reasonably approximated by the volatility of the S&P 500 or the MSCI. By using this approach, model inputs will be independent of investor preferences. Additionally, parameters, probabilities and expectations are estimated and taken with respect to the real-world probability measure, as discussed in the following subsection.

An example of a well-diversified index that approximates the GOP is the *World Stock Index* (WSI) provided by Global Financial Data. This long-term time series is a careful reconstruction of an accumulated world stock index weighted by market capitalisation beginning in 1920. A similar World Stock Index is studied by Dimson et al. [2002], where the USD WSI discounted by the savings account achieves an average net growth rate of 4.9% per annum over the previous century. We have sourced data for the S&P Composite Index, which also approximates the GOP, from Shiller [1989]. In Figure 2.1 we plot the logarithm of the S&P Composite Index denominated in United States dollars (USD) for the period from January 1871 to September 2014. By assuming that the GOP is approximated by the market portfolio, Figure 2.1 can be interpreted as the logarithm of a historical sample path for the GOP over the same period. We will later use the S&P Composite Index covering the period from 1871 until 2014 to highlight the effects on extreme-maturity term structure derivatives.

Note that the GOP dynamics are completely characterized by the short rate r_t and the market prices of risk θ_t^k for $k \in \{1, \ldots, d\}$ and $t \in [0, T]$. We can separate these two effects by considering the *discounted GOP* process $\bar{S}^{\delta_*} = \{\bar{S}_t^{\delta_*},\, t \in [0, T]\}$, given by

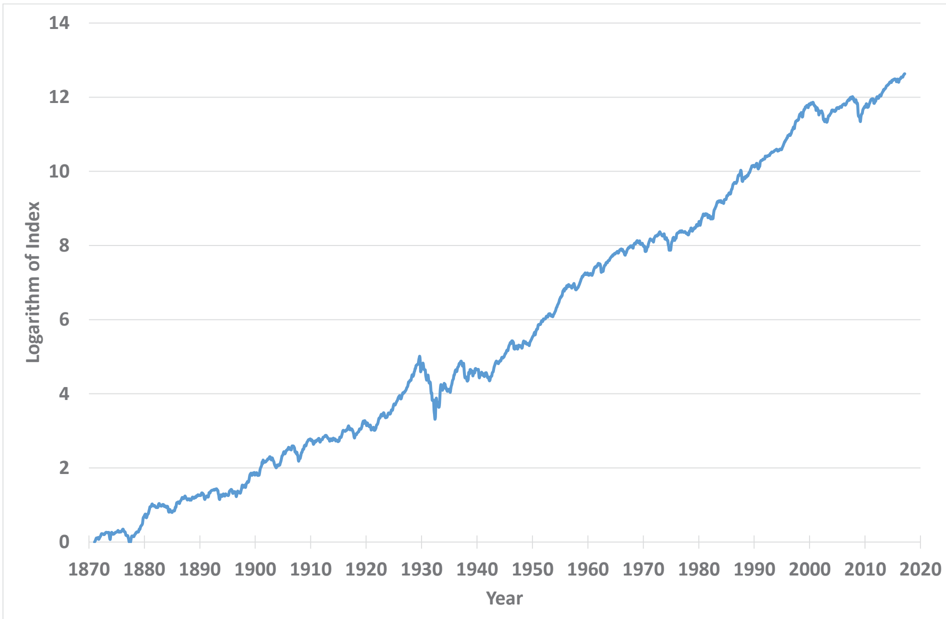
$$\bar{S}_t^{\delta_*} = \frac{S_t^{\delta_*}}{B_t} \tag{2.3.8}$$

satisfying the SDE

$$d\bar{S}_t^{\delta_*} = \bar{S}_t^{\delta_*} \, |\theta_t|^2 \, dt + \bar{S}_t^{\delta_*} \, |\theta_t| \, d\hat{W}_t \tag{2.3.9}$$

for $t \in [0, T]$. Thus, discounting provides a natural way to separate the corresponding short rate and the market price of risk component of the GOP.

Figure 2.1: Logarithm of the WSI in USD from January 1871 to September 2014.



2.4 Benchmarked Prices

Investment managers *benchmark* their performance against that of the market index. Hence, the market index acts as a benchmark. According to Remark 2.3.1 a WSI, such as the MSCI, is a reasonable proxy for the GOP. Therefore, we can infer that investors who use market indices to benchmark their performance are actually comparing their performance against a proxy of the GOP. As we will see below, in pricing derivatives we should simply compare the performance of the derivative price against the best performing portfolio, which is in many ways the GOP.

As explained in Geman et al. [1995], many strictly positive numéraires can be selected as a reference unit. Traditionally in derivative pricing, an equivalent pricing probability measure is assumed to exist based on the choice of the savings account as numéraire. In the following, we select the GOP as the numéraire and refer to prices expressed in units of the GOP as *benchmarked prices*. Long [1990] showed under classical assumptions that when pricing securities in a complete market with the GOP as numéraire, it is not necessary to perform a measure transformation and it suffices to have the real-world probability as the pricing measure. Therefore, the GOP as numéraire under real-world pricing plays a similar role to a savings account for risk neutral pricing or a zero-coupon bond under forward-adjusted pricing. We will see that under certain conditions the resulting derivative prices coincide.

For any non-negative portfolio S^δ, the corresponding *benchmarked portfolio process* $\hat{S}^\delta = \{\hat{S}^\delta_t,\ t \in [0,T]\}$ is defined as

$$\hat{S}^\delta_t = \frac{S^\delta_t}{S^{\delta_*}_t} \tag{2.4.1}$$

for any $t \in [0,T]$.

Application of the Itô formula together with (2.2.9), (2.3.3) and (2.4.1) results in the value of the benchmarked portfolio $\hat{S}^\delta_t$ at time t satisfying the SDE

$$d\hat{S}^\delta_t = -\sum_{k=1}^{d}\sum_{j=0}^{d} \delta^j_t\, \hat{S}^j_t \left(\theta^k_t - b^{j,k}_t\right) dW^k_t = -\sum_{k=1}^{d}\sum_{j=0}^{d} \delta^j_t\, \hat{S}^j_t\, \sigma^{j,k}_t\, dW^k_t \tag{2.4.2}$$

for $t \in [0,T]$. Note that $\sigma^{j,k}_t = \theta^k_t - b^{j,k}_t$ for all $t \in [0,T]$, $j \in \{0,1,\ldots,d\}$ and $k \in \{1,\ldots,d\}$, as shown in Platen [2001]. This leads to the following corollary found in Platen [2002b].

Corollary 2.4.1 *Any self-financing benchmarked wealth process $\hat{S}^\delta$ is an $(\underline{\mathcal{A}}, P)$-local martingale. If the corresponding portfolio is non-negative and its strategy δ is predictable, then its benchmarked wealth process is also an $(\underline{\mathcal{A}}, P)$-supermartingale.*

Consider the j-th *benchmarked primary security account process* $\hat{S}^j = \{\hat{S}^j_t,\ t \in [0,T]\}$, which is obtained by

$$\hat{S}^j_t = \frac{S^j_t}{S^{\delta_*}_t} \tag{2.4.3}$$

for $t \in [0,T]$ and $j \in \{0,1,\ldots,d\}$. Using (2.2.4), (2.3.3) and (2.4.3) and with the Itô formula we find that $\hat{S}^j_t$ satisfies the SDE

$$d\hat{S}^j_t = -\hat{S}^j_t \sum_{k=1}^{d} \left(\theta^k_t - b^{j,k}_t\right) dW^k_t = -\hat{S}^j_t \sum_{k=1}^{d} \sigma^{j,k}_t\, dW^k_t \tag{2.4.4}$$

for $t \in [0,T]$ and $j \in \{0,1,\ldots,d\}$. We observe that the right-hand-side of (2.4.4) is driftless and, therefore, the benchmarked primary security account process $\hat{S}^j$ is an $(\underline{\mathcal{A}}, P)$-local martingale. Additionally, since $\hat{S}^j$ is non-negative, it is also an $(\underline{\mathcal{A}}, P)$-supermartingale (see Karatzas and Shreve [1991]).

2.5 Strong Arbitrage

As we will show, some of the models studied in this thesis do not fit under the standard risk neutral framework. Therefore, we use the benchmark approach with its associated concept of *real-world* or *fair pricing*, outlined in Platen [2002b] and Platen and Heath [2006]. As mentioned above, the choice of the GOP as numéraire leads us to price benchmarked securities by using conditional expectations with respect to the real-world probability measure P. In contrast, under the widely known risk neutral approach, a change to an assumed equivalent risk neutral probability measure is necessary for pricing. This restricts the range of stochastic dynamics available for modelling financial quantities to processes that exist under an equivalent risk neutral probability measure transformation. Actually, in a complete market the corresponding Radon-Nikodym derivative process needs to be a martingale under the real-world probability measure, as discussed in Karatzas and Shreve [1998] and Platen and Heath [2006]. For example, simple models such as the Black-Scholes model for the GOP, driven by geometric Brownian motion, satisfies the necessary restrictions. However, more complicated models, which reflect empirical features found in financial data, may not. For instance, a number of difficulties are encountered for the well-known constant elasticity of variance (CEV) model, as discussed in Delbaen and Shirakawa [2002], Heath and Platen [2003] and Heston et al. [2007]. More generally, complications exist for several classes of stochastic volatility models, discussed in Sin [1998] and Lewis [2000]. However, since these models exist under the real-world probability measure, and potentially provide a good fit to market data, the wider selection of stochastic processes that becomes available for modelling under the benchmark approach has significant benefits. In each model, a benchmarked portfolio satisfies a driftless SDE and, therefore, will form an $(\underline{\mathcal{A}}, P)$-local martingale. From Karatzas and Shreve [1991] we know that all continuous non-negative local martingales are $(\underline{\mathcal{A}}, P)$-supermartingales, and, thus, from Corollary 2.4.1 all non-negative benchmarked price processes are $(\underline{\mathcal{A}}, P)$-supermartingales. This supermartingale property is used in Platen [2002b] to show that the resulting price system of securities does not permit a form of strong arbitrage. In words, the definition of strong arbitrage used is that strictly positive profits cannot be generated under limited liability with strictly positive probability from zero initial wealth. The precise mathematical definition of strong arbitrage underlying this thesis is that described within Platen and Heath [2006].

Definition 2.5.1 *A* strong arbitrage *for an admissible trading strategy δ is the corresponding associated non-negative wealth process S^δ beginning with zero initial wealth, hence $S^\delta_t = 0$ almost-surely, and satisfying the relationships*

$$P(S^\delta_T \geq 0 \,|\, \mathcal{A}_t) = 1 \tag{2.5.1}$$

and

$$P(S^\delta_T > 0 \,|\, \mathcal{A}_t) > 0 \tag{2.5.2}$$

for all $t \in [0, \bar{T}]$ *and* $i \in \{0, 1, \ldots, d\}$*, where* $\bar{T}$ *is a stopping time taking values in* $[0, T]$.

This means that one has strong arbitrage if one can generate under limited liability strictly positive wealth from zero initial capital. We have already noted in Corollary 2.4.1 that non-negative benchmarked wealth processes are $(\underline{\mathcal{A}}, P)$-supermartingales. Platen [2002b] proves that if the initial value of a non-negative wealth process is zero, then it must remain zero indefinitely. Therefore, strong arbitrage, as per Definition 2.5.1, is excluded in the given continuous market. This also means that pricing by excluding strong arbitrage does not make much sense. We will price later by the reasoning used to obtain the minimal possible price.

It is also interesting to distinguish between the type of strong arbitrage we have defined here and the kind of classical arbitrage discussed in the existing classical literature. Usually the absence of classical arbitrage opportunities is defined in terms of the existence of an equivalent risk neutral martingale probability measure, starting with Harrison and Kreps [1979] and Harrison and Pliska [1981] and culminating with the *no-free-lunch-with-vanishing-risk* (NFLVR) condition in Delbaen and Schachermayer [1994], in their fundamental theorem of asset pricing. This means that in our framework there exist some models that exclude the strong arbitrage of Definition 2.5.1, yet may not satisfy the NFLVR condition. We argue in this thesis that the NFLVR condition is most likely too restrictive since the existence of an equivalent risk neutral probability measure is not necessary to capture the essence of no-arbitrage in the real market. There is typically a need to show collateral when fully aiming to exploit a classical arbitrage that is not a strong arbitrage.

2.6 Real-World Pricing

Next we introduce an important notion from Platen [2002b] that is required to explain the mechanism of real-world pricing.

Definition 2.6.1 *A price process* $U = \{U_t,\, t \in [0, T]\}$*, denominated in the domestic currency, with* $\mathrm{E}\left(\frac{|U_t|}{S_t^{\delta_*}}\right) < \infty$ *for* $t \in [0, T]$*, is called* fair *if the corresponding benchmarked price process* $\hat{U} = \{\hat{U}_t = U_t / S_t^{\delta_*},\, t \in [0, T]\}$ *forms an* $(\underline{\mathcal{A}}, P)$*-martingale, that is*

$$\hat{U}_t = \mathrm{E}\left(\hat{U}_{\bar{T}} \,\middle|\, \mathcal{A}_t\right) \tag{2.6.1}$$

for all $0 \leq t \leq \bar{T} \leq T$.

The $(\underline{\mathcal{A}}, P)$-martingale property of a fair price process means that its last observed benchmarked value is the best forecast for any of its future benchmarked values.

Various definitions can be found in the literature for contingent claims and related derivative securities, usually in the form given in Baxter and Rennie [1996] or Hull [1997]. The latter provides the definition of a derivative as "a financial instrument whose value depends on the value of other, more basic underlying variables". We provide in this thesis the following definition.

Definition 2.6.2 *We define a* contingent claim $H_{\bar{T}}$*, denominated in the domestic currency, that matures at a stopping time* $\bar{T} \in [0, T]$ *as an* $\mathcal{A}_{\bar{T}}$*-measurable, non-negative payoff with*

$$\mathrm{E}\left(\frac{H_{\bar{T}}}{S^{\delta_*}_{\bar{T}}} \,\middle|\, \mathcal{A}_t \right) < \infty \tag{2.6.2}$$

for all $t \in [0, \bar{T}]$.

Henceforth, we refer to a *derivative* as a contract that has a contingent claim as its payoff. In Platen and Heath [2006] it is shown that the minimal portfolio price process that hedges a replicable contingent claim is the fair price process that matches the payoff at maturity. This comes from the fact that the minimal non-negative supermartingale that matches a given payoff is the corresponding martingale. Under this rationale we will see that we generalise the classical risk neutral no-arbitrage pricing by fair pricing, which provides the minimal possible replicating hedge portfolios. Furthermore, Platen and Heath [2006] also show that the price process resulting from the utility indifference pricing approach of Davis [1997], is again a fair price process, even for the case of an incomplete market when contingent claims are priced that are not replicable. It is, therefore, reasonable in this thesis to make the following assumption.

Assumption 2.6.3 *All* derivative price processes *are* fair*.*

We remark that in Du and Platen [2016] the concept of benchmarked risk minimisation has been introduced, which yields also for not fully replicable contingent claims fair derivative price processes for a competitive liquid market. To calculate the *fair value* $U^{H_{\bar{T}}}_t$ at time $t \in [0, \bar{T}]$ for the contingent claim $H_{\bar{T}}$, in the domestic currency, the corresponding price process $U^{H_{\bar{T}}} = \{U^{H_{\bar{T}}}_t,\, t \in [0, \bar{T}]\}$ is assumed to satisfy the replication condition

$$U^{H_{\bar{T}}}_{\bar{T}} = H_{\bar{T}} \tag{2.6.3}$$

at the stopping time $\bar{T}$, P-almost surely. By Assumption 2.6.3 the derivative price process $U^{H_{\bar{T}}}$ must be fair and by Definition 2.6.1 the benchmarked price process $\hat{U}^{\hat{H}_{\bar{T}}}$ must then be an $(\underline{\mathcal{A}}, P)$-martingale. Thus $\hat{U}^{\hat{H}_{\bar{T}}}_t = E(\hat{H}_{\bar{T}} \,|\, \mathcal{A}_t)$, see (2.6.1). This yields the following result, which can be found in Platen [2002b].

Theorem 2.6.4 (Platen) *The fair value of the derivative price* $U_t^{H_{\bar{T}}}$ *in the domestic currency can be obtained at time* t *by the* real-world pricing formula

$$U_t^{H_{\bar{T}}} = \mathrm{E}\left(\frac{S_t^{\delta_*}}{S_{\bar{T}}^{\delta_*}} H_{\bar{T}} \,\middle|\, \mathcal{A}_t \right) \tag{2.6.4}$$

for $t \in [0, \bar{T}]$.

We emphasise that while it is possible that other self-financing price processes may replicate the contingent claim $H_{\bar{T}}$, the fair price process is the minimal replicating price process. This follows since a martingale is the unique minimal replicating non-negative supermartingale, see for example Du and Platen [2016].

2.7 Link to Risk Neutral Pricing

We now show that the real-world pricing methodology generalises the classical risk neutral pricing framework in a complete market.

Recall from Karatzas and Shreve [1998] that in a complete market the candidate Radon-Nikodym derivative process $\Lambda^\theta = \{\Lambda_t^\theta,\, t \in [0, T]\}$ for the putative risk neutral measure P_θ with

$$\Lambda_t^\theta = \frac{dP_\theta}{dP}\bigg|_{\mathcal{A}_t} = \frac{B_t}{S_t^{\delta_*}} \frac{S_0^{\delta_*}}{B_0} = \frac{\hat{B}_t}{\hat{B}_0} = \frac{\bar{S}_0^{\delta_*}}{\bar{S}_t^{\delta_*}} \tag{2.7.1}$$

equals, up to a constant normalisation factor, the benchmarked savings account or inverse of the discounted GOP, with initial value $\Lambda_0^\theta = 1$. Recall that the candidate Radon-Nikodym derivative can be interpreted as the inverse of the discounted stock market index.

Under the assumption that Λ^θ is an $(\underline{\mathcal{A}}, P)$-martingale, the application of Girsanov's Theorem (see for example Heath and Platen [2006]) to the GOP (2.3.6) allows us to obtain the SDE

$$dS_t^{\delta_*} = r_t\, S_t^{\delta_*}\, dt + |\theta_t|\, S_t^{\delta_*}\, d\hat{W}_t^\theta, \tag{2.7.2}$$

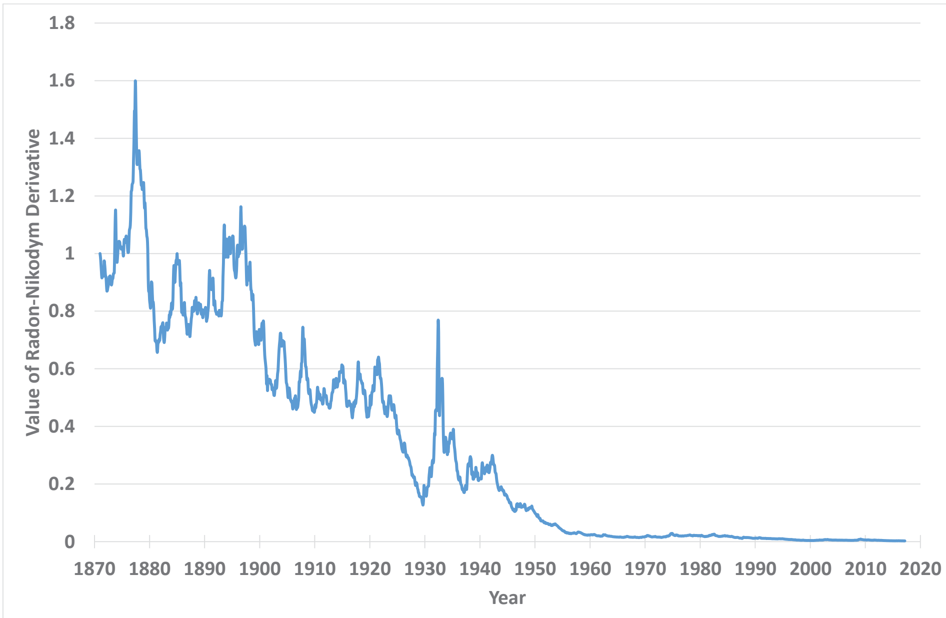
where

$$d\hat{W}_t^\theta = |\theta_t|\, dt + d\hat{W}_t \tag{2.7.3}$$

for $t \in [0, T]$. Here $\hat{W}^\theta$ is a Wiener process under the assumed risk neutral probability measure P_θ. The existence of P_θ follows when Λ^θ is a martingale. In the case when Λ^θ is not a martingale the risk neutral pricing methodology is not applicable.

Let us now try to understand whether this martingale assumption is a realistic assumption. Using the data underlying the WSI in Figure 2.1 and corresponding

Figure 2.2: Radon-Nikodym derivative for USD from 1871 to 2014.



USD short rate data, we can construct a plot of the candidate Radon-Nikodym derivative (2.7.1) for USD from 1871 to 2014, as illustrated in Figure 2.2. Notice that the candidate Radon-Nikodym derivative process declines systematically on average. Its long-term downward trend makes it unlikely to represent the trajectory of a true martingale. The consistent average decline of the candidate Radon-Nikodym derivative, displayed in Figure 2.2, is not surprising from an economic perspective, given the evidence that average returns on long-term investments in the stock market are consistently greater than the short rate, as demonstrated in Dimson et al. [2002]. An important implication of this observation is that the key assumption of risk neutral pricing, which postulates that the candidate Radon-Nikodym derivative process is an $(\underline{\mathcal{A}}, P)$-martingale, is questionable. This assumption is crucially required for the application of Girsanov's Theorem and should, therefore, be avoided, as first suggested in Platen [2002b]. In particular, one has to be careful when aiming to apply the risk neutral approach for long-term derivatives.

The candidate Radon-Nikodym derivative is found from (2.3.9) and (2.7.1) to equal

$$\Lambda_t^\theta = \exp\left(-\frac{1}{2}\int_0^t |\theta_s|^2\, ds - \int_0^t |\theta_s|\, d\hat{W}_s\right) \tag{2.7.4}$$

for $t \in [0, T]$. This representation for the candidate Radon-Nikodym derivative process Λ^θ in (2.7.4) is sometimes described as state-price density or deflator, see for instance Duffie [2001] or Karatzas and Shreve [1998]. Similarly, the inverse of the GOP can be used as the stochastic discount factor in the stochastic discount factor approach, see for example Cochrane [2001]. In the context of the work of Loewenstein and Willard [2000] or in Platen [2002b], Λ^θ need not necessarily be a martingale, and thus may not correspond to an equivalent change of probability measure. For the special case, when Λ^θ is strictly positive on $[0,T]$ and $\mathrm{E}\big(\Lambda^\theta_T \,\big|\, \mathcal{A}_0\big) = \Lambda^\theta_0 = 1$, then Λ^θ is indeed a state-price density or deflator. However, as we have just observed in Figure 2.2 the empirical data indicate that the candidate Radon-Nikodym derivative process is realistically described by a strict $(\underline{\mathcal{A}}, P)$-supermartingale. This translates into the relation $\mathrm{E}\big(\Lambda^\theta_T \,\big|\, \mathcal{A}_0\big) < 1$, for $T \in (0,\infty)$. We emphasise that real-world pricing, as outlined in Section 2.6 and suggested in Platen [2002b], makes no assumptions for the process Λ^θ, because it does not form part of the real-world pricing methodology.

One can obtain the SDE for the candidate Radon-Nikodym derivative from either (2.7.4) or by using (2.3.9) and (2.7.1) with the Itô formula as

$$d\Lambda^\theta_t = -\Lambda^\theta_t \,|\theta_t|\, d\hat{W}_t \tag{2.7.5}$$

for $t \in [0,T]$, which is an $(\underline{\mathcal{A}}, P)$-local martingale. Whether the SDE (2.7.5) describes a martingale will depend upon the nature of the volatility $|\theta_t|$ of the GOP. The SDE will be that of a true martingale if the Novikov condition or other similar sufficient conditions are met which can be found in the literature, see for instance, Revuz and Yor [1999], Heath and Platen [2006] or Hulley and Platen [2012].

It is possible to write the real-world pricing formula (2.6.4) with the aid of (2.7.1) as

$$U^{H_{\bar{T}}}_t = \mathrm{E}\left(\frac{\Lambda^\theta_{\bar{T}}}{\Lambda^\theta_t}\, \frac{B_t}{B_{\bar{T}}}\, H_{\bar{T}} \,\middle|\, \mathcal{A}_t \right) \tag{2.7.6}$$

for $t \in [0,\bar{T}]$. For models where the candidate risk neutral measure P_θ and the real-world measure P are equivalent, see Heath and Platen [2006], and the Radon-Nikodym derivative process Λ^θ is an $(\underline{\mathcal{A}}, P)$-martingale, the relation (2.7.6) reverts via the Bayes rule to the standard *risk neutral pricing formula*

$$U^{H_{\bar{T}}}_t = \mathrm{E}_\theta\left(\frac{B_t}{B_{\bar{T}}}\, H_{\bar{T}} \,\middle|\, \mathcal{A}_t \right) \tag{2.7.7}$$

for $t \in [0,\bar{T}]$, see Heath and Platen [2006]. Here E_θ denotes conditional expectation with respect to the risk neutral probability measure P_θ. However, it should be noted that there are realistic models, such as some of those discussed in later chapters, where the real-world and candidate risk neutral probability measures are not equivalent because the Radon-Nikodym derivative process Λ^θ is not an $(\underline{\mathcal{A}}, P)$-martingale. In these cases, the assumptions underlying the risk neutral pricing formula are not satisfied and, thus, (2.7.7) cannot be used. In contrast, the real-world pricing formula (2.6.4) still remains applicable under these circumstances.

2.8 Forward Adjusted Pricing

Zero-coupon bonds are fundamental securities in financial markets. With the aid of the real-world pricing formula (2.6.4) we can formally define zero-coupon bonds under the benchmark framework.

Definition 2.8.1 *The* real-world *price of a* zero-coupon bond $P(t,\bar{T})$ *at time* t *with fixed maturity* $\bar{T} \in [0,T]$ *is defined as the fair value at time* t *of a payoff of one unit of the domestic currency, and is given by*

$$P(t,\bar{T}) = \mathrm{E}\left(\frac{S_t^{\delta_*}}{S_{\bar{T}}^{\delta_*}} \,\middle|\, \mathcal{A}_t\right) \tag{2.8.1}$$

for $t \in [0,\bar{T}]$.

Note that $P(\bar{T},\bar{T}) = 1$. Throughout this thesis we will find it convenient to make the following assumption.

Assumption 2.8.2 *We assume that the driving processes of the short rate process* r *and the discounted GOP* $\bar{S}^{\delta_*}$ *are independent.*

This assumption can be relaxed if requested. It allows us to characterise the real-world zero-coupon bond price $P(t,\bar{T})$ in the following multiplicative way:

Theorem 2.8.3 *A zero-coupon bond with price* $P(t,\bar{T})$ *as at time* t *and maturing at time* $\bar{T}$ *satisfies the formula*

$$P(t,\bar{T}) = \mathrm{E}\left(\frac{\bar{S}_t^{\delta_*}}{\bar{S}_{\bar{T}}^{\delta_*}}\,\frac{B_t}{B_{\bar{T}}} \,\middle|\, \mathcal{A}_t\right) = M_{\bar{T}}(t)\, G_{\bar{T}}(t), \tag{2.8.2}$$

where the discounted GOP contribution *to the zero-coupon bond price is defined by*

$$M_{\bar{T}}(t) = \mathrm{E}\left(\frac{\bar{S}_t^{\delta_*}}{\bar{S}_{\bar{T}}^{\delta_*}} \,\middle|\, \mathcal{A}_t\right) = \mathrm{E}\left(\frac{\Lambda_{\bar{T}}^{\theta}}{\Lambda_t^{\theta}} \,\middle|\, \mathcal{A}_t\right) \tag{2.8.3}$$

and the short rate contribution *to the bond price is given as*

$$G_{\bar{T}}(t) = \mathrm{E}\left(\frac{B_t}{B_{\bar{T}}} \,\middle|\, \mathcal{A}_t\right) \tag{2.8.4}$$

for $t \in [0,\bar{T}]$.

Proof. The result follows if we introduce the sigma-algebra $\mathcal{A}_t^r$ generated by $\mathcal{A}_t$ and the entire path of the short rate r until time $\bar{T}$. That is $\mathcal{A}_t^r = \sigma\{r_s,\ s \in [0,\bar{T}]\} \cup \mathcal{A}_t$, noting that $\mathcal{A}_t \subseteq \mathcal{A}_t^r$. Then using the independence of the driving processes for the short rate r and the discounted GOP $\bar{S}^{\delta_*}$, given in (2.8.1) and (2.3.9) and assumed in Assumption 2.8.2, the real-world price of a zero-coupon bond is found to equal

$$\begin{aligned}
P(t,\bar{T}) &= \mathrm{E}\left(\frac{B_t}{B_{\bar{T}}}\,\frac{\bar{S}_t^{\delta_*}}{\bar{S}_{\bar{T}}^{\delta_*}}\,\middle|\,\mathcal{A}_t\right)\\
&= \mathrm{E}\left(\frac{B_t}{B_{\bar{T}}}\,\mathrm{E}\left(\frac{\bar{S}_t^{\delta_*}}{\bar{S}_{\bar{T}}^{\delta_*}}\,\middle|\,\mathcal{A}_t^r\right)\,\middle|\,\mathcal{A}_t\right)\\
&= \mathrm{E}\left(\frac{B_t}{B_{\bar{T}}}\,M_{\bar{T}}(t)\,\middle|\,\mathcal{A}_t\right)\\
&= \mathrm{E}\left(\frac{B_t}{B_{\bar{T}}}\,\middle|\,\mathcal{A}_t\right)\,M_{\bar{T}}(t)
\end{aligned} \tag{2.8.5}$$

for $t \in [0,\bar{T}]$, which proves (2.8.2). **Q.E.D.**

An example, where the driving processes of the discounted GOP and the short rate are assumed to be independent, is studied in Miller and Platen [2005]. Another example is when the short rate is assumed to be deterministic, as appears regularly in the literature on options in equity markets.

Remark 2.8.4 *The expression in* (2.8.3)*, referred to as the discounted GOP contribution to the zero-coupon bond price, is one of the most important quantities we encounter in this thesis. The second equality in* (2.8.3) *reminds us that it can also be interpreted as the expected value of either the Radon-Nikodym derivative for the candidate risk neutral measure, or equivalently, the potential state-price density. An equivalent risk neutral probability measure will exist if and only if* Λ^θ *is an* $(\underline{\mathcal{A}}, P)$*-martingale and hence* $M_{\bar{T}}(t) = 1$ *in* (2.8.3) *for all* $t \in [0,\bar{T}]$*. This is the case for the Black-Scholes model that we consider in the next chapter. Mathematically, this means that the underlying assumptions of Girsanov's Theorem, see Heath and Platen [2006], are satisfied for this specific model. However, for the models of MMM type that will be discussed later on, the Radon-Nikodym derivative* Λ^θ *forms an* $(\underline{\mathcal{A}}, P)$ *strict supermartingale, meaning that* $M_{\bar{T}}(t) < 1$ *for all* $t \in [0,\bar{T})$*. In this latter case, Girsanov's Theorem is not applicable since its assumptions are not satisfied.*

Given zero-coupon bond prices, we can calculate the *continuously compounded yield to maturity* $y_{\bar{T}}(t)$ at time t for the maturity date $\bar{T} \in [0,T]$, expressed as

$$y_{\bar{T}}(t) = -\frac{1}{\bar{T}-t}\,\log\big(P(t,\bar{T})\big) \tag{2.8.6}$$

for all $t \in [0, \bar{T}]$. Therefore, in the case when the zero-coupon bond price can be written in the form of (2.8.2), then the yield to maturity (2.8.6) takes the form

$$y_{\bar{T}}(t) = n_{\bar{T}}(t) + h_{\bar{T}}(t) \tag{2.8.7}$$

where the *discounted GOP contribution* to the yield to maturity is

$$n_{\bar{T}}(t) = -\frac{1}{\bar{T} - t} \log \big(M_{\bar{T}}(t) \big) \tag{2.8.8}$$

and the *short rate contribution* to the yield to maturity is

$$h_{\bar{T}}(t) = -\frac{1}{\bar{T} - t} \log \big(G_{\bar{T}}(t) \big) \tag{2.8.9}$$

for $t \in [0, \bar{T}]$.

Given zero-coupon bond prices, we can also calculate the *instantaneous forward rate* $f_{\bar{T}}(t)$ at time t for the maturity date $\bar{T} \in [0, T]$, expressed as

$$f_{\bar{T}}(t) = -\frac{\partial}{\partial \bar{T}} \log \big(P(t, \bar{T}) \big) \tag{2.8.10}$$

for all $t \in [0, \bar{T}]$. Therefore, in the case when the zero-coupon bond price can be written in the form of (2.8.2), then the forward rate (2.8.10) takes the form

$$f_{\bar{T}}(t) = m_{\bar{T}}(t) + g_{\bar{T}}(t) \tag{2.8.11}$$

where the *discounted GOP contribution* to the forward rate is

$$m_{\bar{T}}(t) = -\frac{\partial}{\partial \bar{T}} \log \big(M_{\bar{T}}(t) \big) \tag{2.8.12}$$

and the *short rate contribution* to the forward rate is

$$g_{\bar{T}}(t) = -\frac{\partial}{\partial \bar{T}} \log \big(G_{\bar{T}}(t) \big) \tag{2.8.13}$$

for $t \in [0, \bar{T}]$.

In *forward-adjusted pricing* a zero-coupon bond process $P(\cdot, \bar{T})$ with fixed maturity $\bar{T}$ is used as numéraire. The candidate measure for this numéraire is referred to as the *$\bar{T}$-forward risk-adjusted measure* and sometimes more briefly as *$\bar{T}$-forward measure*. For the domestic currency denomination, the $\bar{T}$-forward measure is denoted by $P_{\bar{T}}$ and its related expectation is denoted $E_{\bar{T}}$.

In the case of the $\bar{T}$-forward measure, the corresponding candidate Radon-Nikodym derivative process $\Lambda^{\bar{T}} = \{\Lambda^{\bar{T}}_t,\, t \in [0, T]\}$ equals

$$\Lambda^{\bar{T}}_t = \left.\frac{dP_{\bar{T}}}{dP}\right|_{\mathcal{A}_t} = \frac{\hat{P}(t, \bar{T})}{\hat{P}(0, \bar{T})} = \frac{P(t, \bar{T})\, S_0^{\delta_*}}{P(0, \bar{T})\, S_t^{\delta_*}} \tag{2.8.14}$$

with initial value $\Lambda_0^{\bar{T}} = 1$. To derive the corresponding pricing formula, we can rewrite the real-world pricing formula (2.6.4) using (2.8.14) to find

$$U_t^{H_{\bar{T}}} = \mathrm{E}\left(\frac{\Lambda_{\bar{T}}^{\bar{T}}}{\Lambda_t^{\bar{T}}} \frac{P(t,\bar{T})}{P(\bar{T},\bar{T})} H_{\bar{T}} \,\middle|\, \mathcal{A}_t \right) \tag{2.8.15}$$

for $t \in [0,\bar{T}]$. The candidate Radon-Nikodym derivative process $\Lambda^{\bar{T}}$ is an $(\underline{\mathcal{A}}, P)$-martingale, because according to our Assumption 2.6.3 the zero-coupon bond price process is fair. This ensures the following result:

Corollary 2.8.5 *Applying Bayes' Rule (see Heath and Platen [2006]), the formula* (2.8.15) *reduces to the* forward-adjusted pricing formula

$$U_t^{H_{\bar{T}}} = P(t,\bar{T})\, \mathrm{E}_{\bar{T}}\left(H_{\bar{T}} \,\middle|\, \mathcal{A}_t \right) \tag{2.8.16}$$

for $t \in [0,\bar{T}]$.

Applications of (2.8.16) are discussed in detail, for instance, within Musiela and Rutkowski [1997]. Hence, we have shown that real-world pricing generalises forward-adjusted pricing. We emphasise that under real-world pricing there is no need to assume that the candidate Radon-Nikodym derivative process $\Lambda^{\bar{T}}$ in (2.8.14) is an $(\underline{\mathcal{A}}, P)$-martingale, nor that the probability measures $P_{\bar{T}}$ and P are equivalent. This is guaranteed by Assumption 2.6.3, according to which benchmarked zero-coupon bonds are martingales. We do emphasise that there may exist significant computational advantages when using the forward-adjusted pricing formula, as will become clear later.

2.9 Forward Prices and Contract Valuation

In this section we examine forward prices and the valuation of forward contracts. First, consider a forward contract on an underlying nonnegative price process $U = \{U_t,\, t \in [0,T]\}$ that pays $U_{\bar{T}}$ at maturity date $\bar{T}$ for the fixed *delivery price* K_0, set at time 0.

2.9.1 Long Position in a Forward Contract

For a long position in a forward contract the net payoff at the maturity date is given by

$$H_{\bar{T}} = U_{\bar{T}} - K_0. \tag{2.9.1}$$

For this payoff, we calculate the time $t = 0$ fair value $V_{\bar{T},K_0}(0,U)$ of such a contract using the real-world pricing formula (2.6.4) and (2.9.1) to obtain

$$\begin{aligned} V_{\bar{T},K_0}(0,U) &= \mathrm{E}\left(\frac{S_0^{\delta_*}}{S_{\bar{T}}^{\delta_*}}\,(U_{\bar{T}} - K_0)\,\bigg|\,\mathcal{A}_0\right) \\ &= S_0^{\delta_*}\,\mathrm{E}\left(\frac{U_{\bar{T}}}{S_{\bar{T}}^{\delta_*}}\,\bigg|\,\mathcal{A}_0\right) - K_0\,\mathrm{E}\left(\frac{S_0^{\delta_*}}{S_T^{\delta_*}}\,\bigg|\,\mathcal{A}_0\right) \\ &= S_0^{\delta_*}\,\hat{U}_0 - K_0\,P(0,\bar{T}), \end{aligned} \tag{2.9.2}$$

where

$$\hat{U}_0 = \mathrm{E}\left(\frac{U_{\bar{T}}}{S_{\bar{T}}^{\delta_*}}\,\bigg|\,\mathcal{A}_0\right) \tag{2.9.3}$$

for $t \in [0,\bar{T}]$. We also make use of the real-world zero-coupon bond pricing formula (2.8.1). Note that if the process U is fair, as per Definition 2.6.1, then (2.9.2) simplifies to

$$V_{\bar{T},K_0}(0,U) = U_0 - K_0\,P(0,\bar{T}) \tag{2.9.4}$$

for $t \in [0,\bar{T}]$.

It is well known that the forward contract must have zero value at inception. Therefore, at the initiation of the forward contract, the delivery price is set equal to the currently observed forward price. This means that the *forward price* $F_{\bar{T}}(0,U)$ associated with the forward contract written at time $t = 0$, is obtained as a substitution for the delivery price K_0 when the forward contract value (2.9.2) is set to zero. That is, $F_{\bar{T}}(0,U) = K_0$ when $V_{\bar{T},K_0}(0,U) = 0$. A rearrangement of (2.9.2) results in the formula

$$F_{\bar{T}}(0,U) = \frac{S_0^{\delta_*}\,\hat{U}_0}{P(0,\bar{T})} \tag{2.9.5}$$

for maturity time $\bar{T} \in [0,T]$.

2.9.2 Long Position in a Forward Contract after Inception

We can generalise the relationship in (2.9.2) to the value of a forward contract $V_{\bar{T},K_t}(t,U)$ on the price process U paying the identical cashflow $U_{\bar{T}}$ at maturity time $\bar{T}$ for the time t delivery price K_t. The fair value of such a contract at time t is found as

$$V_{\bar{T},K_t}(t,U) = \mathrm{E}\left(\frac{S_t^{\delta_*}}{S_{\bar{T}}^{\delta_*}}\,(U_{\bar{T}} - K_t)\,\bigg|\,\mathcal{A}_t\right) = S_t^{\delta_*}\,\hat{U}_t - K_t\,P(t,\bar{T}), \tag{2.9.6}$$

where

$$\hat{U}_t = \mathrm{E}\left(\frac{U_{\bar{T}}}{S_{\bar{T}}^{\delta_*}}\,\bigg|\,\mathcal{A}_t\right) \tag{2.9.7}$$

for all $t \in [0, \bar{T}]$. The time t forward price $F_{\bar{T}}(t, U)$ corresponding to (2.9.5) is found once more as a substitution for the time t delivery price K_t when the value of the forward contract is set to zero. Therefore, using (2.9.6) and (2.9.7), we obtain

$$F_{\bar{T}}(t, U) = K_t = \frac{S_t^{\delta_*} \hat{U}_t}{P(t, \bar{T})} \tag{2.9.8}$$

for $t \in [0, \bar{T}]$. In the case when the underlying price process U is *fair*, then the forward price at time t in (2.9.8) reduces to

$$F_{\bar{T}}(t, U) = \frac{U_t}{P(t, \bar{T})} \tag{2.9.9}$$

for $t \in [0, \bar{T}]$, which recovers the classical forward price formula, as we will discuss in Subsection 2.9.4.

2.9.3 Offsetting Short Position in a Forward Contract

Next consider a long position in a forward contract on a price process U that pays $U_{\bar{T}}$ at maturity time $\bar{T}$, originally transacted at time $t = 0$ with the delivery price K_0. In order to offset this original long position at the intermediate time t, one needs to sell a new forward contract with the same payout $U_{\bar{T}}$ at maturity time $\bar{T}$, but at a new time t fixed delivery price K_t. The net payoff for such a portfolio of forward contracts at the maturity time $\bar{T}$ is

$$H_{\bar{T}} = (U_{\bar{T}} - K_0) - (U_{\bar{T}} - K_t) = K_t - K_0 \tag{2.9.10}$$

for $t \in [0, \bar{T}]$. Thus, the fair value at time t of this portfolio of forward contracts is determined by the real-world pricing formula (2.6.4) for the payoff (2.9.10), and is found to be

$$\begin{aligned} V_{\bar{T}, K_0}(t, U) - V_{\bar{T}, K_t}(t, U) &= \mathrm{E}\left(\frac{S_t^{\delta_*}}{S_{\bar{T}}^{\delta_*}} \, (K_t - K_0) \,\Big|\, \mathcal{A}_t \right) \\ &= (K_t - K_0)\, P(t, \bar{T}) \end{aligned} \tag{2.9.11}$$

for $t \in [0, \bar{T}]$. The above analysis shows that for a forward contract entered at time $t = 0$, its subsequent value at time t prior to maturity $\bar{T}$, will always be known with certainty. To see this, first recall that the delivery price at inception K_0 must equal the original forward price $F_{\bar{T}}(0, U)$. Next, note that the value of the time t forward contract $V_{\bar{T}, K_t}(t, U)$ is zero when its delivery price K_t equals the time t forward price. That is, $V_{\bar{T}, K_t}(t, U) = 0$ when $K_t = F_{\bar{T}}(t, U)$. Substituting this information into (2.9.11) we observe that

$$V_{\bar{T}, K_0}(t, U) = (F_{\bar{T}}(t, U) - F_{\bar{T}}(0, U))\; P(t, \bar{T}) \tag{2.9.12}$$

for $t \in [0, \bar{T}]$. Intuitively, expressions (2.9.11) and (2.9.12), state that at time t, the value of a forward contract originally written at time $t = 0$, is simply the change in the market forward price, discounted by the appropriate fair zero-coupon bond. This result applies to all types of forward contracts.

2.9.4 Comparison with Classical Pricing of Forwards

For comparison, we discuss the classical approach to the pricing of forward contracts, described for instance in Cox et al. [1981]. These authors give, in Proposition 1 of their paper, the *forward price* $F_{\bar{T}}(t, U)$, written at time t, as the value of a contract on an underlying price process $U = \{U_t,\, t \in [0, T]\}$ that has a payoff at the maturity time $\bar{T}$ of

$$H_{\bar{T}} = \frac{U_{\bar{T}}}{P(t, \bar{T})} \tag{2.9.13}$$

for $t \in [0, \bar{T}]$. Application of the real-world pricing formula (2.6.4) to the payoff (2.9.13) results in the relation

$$F_{\bar{T}}(t, U) = \mathrm{E}\left(\frac{S_t^{\delta_*}}{S_{\bar{T}}^{\delta_*}} \frac{U_{\bar{T}}}{P(t, \bar{T})} \,\middle|\, \mathcal{A}_t \right) = \frac{S_t^{\delta_*}\, \hat{U}_t}{P(t, \bar{T})} \tag{2.9.14}$$

for $t \in [0, \bar{T}]$, which by (2.9.7) shows that (2.9.14) is equivalent to (2.9.8). The only difference between (2.9.8) and (2.9.14) is the path taken to derive the result. In the former derivation (2.9.6)−(2.9.8), we examined the payoff of a *forward contract* net of the underlying price process at maturity. In the latter derivation (2.9.13)−(2.9.14), we calculated the fair value of the payoff for the corresponding *forward price*.

An important distinction is the case when the underlying price process U is *fair*, as defined in Definition 2.6.1. Then the forward price (2.9.14) simplifies to

$$F_{\bar{T}}(t, U) = \frac{U_t}{P(t, \bar{T})} \tag{2.9.15}$$

for $t \in [0, \bar{T}]$. Otherwise, if the price process U is not fair, then we deduce from (2.9.7) and (2.9.14) that

$$F_{\bar{T}}(t, U) \leq \frac{U_t}{P(t, \bar{T})} \tag{2.9.16}$$

for $t \in [0, \bar{T}]$. The equality in (2.9.15) is the typical representation given in the standard finance literature. However, it is important to recognise that (2.9.15) contains the fair zero-coupon bond price and not the classical risk neutral zero-coupon bond price, as used, for instance, in Hull [1997] or Ritchken [1996]. The inequality in (2.9.16) illustrates the difference between underlying price processes that are fair and those that are not. It stems from the fact that any admissible benchmarked non-negative wealth process is an $(\mathcal{A}, P)$-supermartingale, as stated in Corollary 2.4.1. Therefore, when the price process is *not* fair, the forward price should be calculated according to formula (2.9.14), and not by the classical formula of the form given in (2.9.15), which by (2.9.16) may lead to a more expensive price.

2.9.5 Forward Contracts on Primary Security Accounts

We now calculate forward prices using (2.9.14) for each of the primary security account processes discussed thus far and for zero-coupon bonds. To begin, we consider the calculation of a forward price $F_{\bar{T}}(t, S^j)$ at time t for a primary security account value $S^j_{\bar{T}}$ with maturity time $\bar{T}$, which is found using (2.8.3) and (2.9.14) to be

$$\begin{aligned} F_{\bar{T}}(t, S^j) &= \frac{S^{\delta_*}_t}{P(t,\bar{T})} \, \mathrm{E}\left(\frac{S^j_{\bar{T}}}{S^{\delta_*}_{\bar{T}}} \,\bigg|\, \mathcal{A}_t \right) \\ &= \frac{S^j_t}{P(t,\bar{T})} \, \mathrm{E}\left(\frac{S^{\delta_*}_t}{S^{\delta_*}_{\bar{T}}} \, \frac{S^j_{\bar{T}}}{S^j_t} \,\bigg|\, \mathcal{A}_t \right) \\ &= \frac{S^j_t}{P(t,\bar{T})} \, \mathrm{E}\left(\frac{\hat{S}^j_{\bar{T}}}{\hat{S}^j_t} \,\bigg|\, \mathcal{A}_t \right) \\ &= S^j_t \, \frac{M^j_{\bar{T}}(t)}{P(t,\bar{T})}, \end{aligned} \tag{2.9.17}$$

where

$$M^j_{\bar{T}}(t) = \mathrm{E}\left(\frac{\hat{S}^j_{\bar{T}}}{\hat{S}^j_t} \,\bigg|\, \mathcal{A}_t \right) \tag{2.9.18}$$

for $t \in [0, \bar{T}]$ and $j \in \{0, 1, \ldots, d\}$. The expression $M^j_{\bar{T}}(t)$ in (2.9.17) and (2.9.18) equals 1 if S^j is fair. As can be seen from (2.9.17), $M^j_{\bar{T}}(t)$ arises naturally in the context of calculating the forward price on a primary security account. In contrast, the discounted GOP contribution $M_{\bar{T}}(t) = M^0_{\bar{T}}(t)$ only appears in fair zero-coupon bond pricing under the assumption of independence between the driving processes of the discounted GOP and the short rate.

Furthermore, the appearance of the expression $M^j_{\bar{T}}(t)$ within (2.9.17) is quite general in the sense that we have not specified a model so far. In the case when S^j is an $(\underline{\mathcal{A}}, P)$-martingale we have $M^j_{\bar{T}}(t) = 1$ for all $t \in [0, \bar{T}]$. From another perspective, when the term $M^j_{\bar{T}}(t)$ does not equal unity, this will represent an explicit difference between the fair forward price of a primary security account calculated using real-world pricing and that obtained using formally classical risk neutral pricing. Hence these forward prices can be materially different depending on pricing methodology. Examples of models for which this happens will be provided later within this thesis.

2.9.6 Forward Contracts on a Portfolio

A more complex example is the forward contract on an arbitrary nonnegative portfolio process S^{δ}. Denoting the time t forward price of a contract that pays at maturity time $\bar{T}$ the portfolio value $S^{\delta}_{\bar{T}}$ as $F_{\bar{T}}(t, S^{\delta})$, then we obtain via (2.9.14) the result

$$F_{\bar{T}}(t, S^{\delta}) = \frac{S^{\delta_*}_t}{P(t,\bar{T})} \, \mathrm{E}\left(\frac{S^{\delta}_{\bar{T}}}{S^{\delta_*}_{\bar{T}}} \,\Big|\, \mathcal{A}_t \right) \leq \frac{S^{\delta}_t}{P(t,\bar{T})} \tag{2.9.19}$$

for $t \in [0, \bar{T}]$. The inequality in (2.9.19) is a consequence of S^{δ} not necessarily being a fair price process as given by Definition 2.6.1. An example where the portfolio price process may not be fair is the case of the savings account, evident from the graph of the savings account denominated by the GOP in Figure 2.2. However, when the portfolio price process *is* fair, as is the case for any derivative under Assumption 2.6.3, then equality holds for the second relation in (2.9.19).

Now we consider a forward contract on the GOP itself. The time t forward price $F_{\bar{T}}(t, S^{\delta_*})$ with maturity $\bar{T}$ is calculated using (2.9.14) to give

$$F_{\bar{T}}(t, S^{\delta_*}) = \frac{S^{\delta_*}_t}{P(t,\bar{T})} \, \mathrm{E}\left(\frac{S^{\delta_*}_{\bar{T}}}{S^{\delta_*}_{\bar{T}}} \,\Big|\, \mathcal{A}_t \right) = \frac{S^{\delta_*}_t}{P(t,\bar{T})} \tag{2.9.20}$$

for $t \in [0, \bar{T}]$. This result is consistent with the classical result of (2.9.15) because the benchmarked GOP is an $(\underline{\mathcal{A}}, P)$-martingale, since $\hat{S}^{\delta_*}_t = 1$ for all $t \in [0, T]$. By Definition 2.6.1 the GOP is a fair price process.

2.9.7 Forward Contract on a Zero-Coupon Bond

The last forward price we consider is denoted as $F_{\bar{T}}(t, P(\cdot, T))$. It represents the time t price associated with a forward contract, where the underlying security is the zero-coupon bond $P(\bar{T}, T)$. Hence the expiry of the forward contract is $\bar{T}$ while the maturity of the zero-coupon bond, paying one unit of the domestic currency, is the time $T > \bar{T}$. We determine by (2.8.1), (2.9.14) and the law of

Table 2.1: Notation and payoffs of various European options

Option Type	Payoff at Expiry	Notation
First order asset binary call option	$U_{\bar{T}}\mathbf{1}_{U_{\bar{T}}>K}$	$A^{+}_{\bar{T},K}(t,U)$
First order asset binary put option	$U_{\bar{T}}\mathbf{1}_{U_{\bar{T}}\leq K}$	$A^{-}_{\bar{T},K}(t,U)$
First order bond binary call option	$\mathbf{1}_{U_{\bar{T}}>K}$	$B^{+}_{\bar{T},K}(t,U)$
First order bond binary put option	$\mathbf{1}_{U_{\bar{T}}\leq K}$	$B^{-}_{\bar{T},K}(t,U)$
European call option	$(U_{\bar{T}}-K)^{+}$	$c_{\bar{T},K}(t,U)$
European put option	$(K-U_{\bar{T}})^{+}$	$p_{\bar{T},K}(t,U)$

iterated expectations that

$$\begin{aligned} F_{\bar{T}}(t,P(\cdot,T)) &= \frac{S^{\delta_*}_t}{P(t,\bar{T})}\,\mathrm{E}\left(\frac{P(\bar{T},T)}{S^{\delta_*}_{\bar{T}}}\,\bigg|\,\mathcal{A}_t\right) \\ &= \frac{1}{P(t,\bar{T})}\,\mathrm{E}\left(\frac{S^{\delta_*}_t}{S^{\delta_*}_{\bar{T}}}\,\mathrm{E}\left(\frac{S^{\delta_*}_{\bar{T}}}{S^{\delta_*}_T}\,\bigg|\,\mathcal{A}_{\bar{T}}\right)\,\bigg|\,\mathcal{A}_t\right) \\ &= \frac{1}{P(t,\bar{T})}\,\mathrm{E}\left(\frac{S^{\delta_*}_t}{S^{\delta_*}_T}\,\bigg|\,\mathcal{A}_t\right) \\ &= \frac{P(t,T)}{P(t,\bar{T})} \end{aligned} \tag{2.9.21}$$

for $0 \leq t \leq \bar{T} \leq T$. In Section 2.10.3 we discuss a transformation of the above result, known as a forward rate agreement (FRA), which is traded in interest rate markets.

2.10 European Option Pricing and Hedging

In this section we will consider a variety of European style derivatives including call and put options, binaries and other related derivatives. We use the notation in Table 2.1, similar to that of Buchen [2001] and Buchen [2004]. The options are on an underlying asset U and each option has strike price K and expiry date $\bar{T}$.

In particular, binary options on the underlying asset are referred to as first order asset binary options. Rubinstein and Reiner [1991] used the term “asset-or-nothing” options. Binary options on a cash payout are referred to as first order bond binaries. These were termed “cash-or-nothing” options by Rubinstein and Reiner [1991].

2.10.1 First Order Binary Options

Let us denote by $U = \{U_t,\, t \in [0, \bar{T}]\}$ the price process of an underlying security. Also, let us denote by $s \in \{+, -\}$ the sign indicator corresponding to the direction of the underlying security price relative to the strike price in which the binary option has a nonzero payoff. The price $V^s_{\bar{T},K}(t, U)$ of a *general first order binary option* on U is a derivative contract with non-negative strike price K and fixed expiry date $\bar{T} \geq t$ with the payoff

$$H_{\bar{T}} = f(U_{\bar{T}})\, \mathbf{1}_{\{s\, U_{\bar{T}} \geq s\, K\}}. \tag{2.10.1}$$

Here $f(\cdot)$ is an appropriately defined function such that the conditional expectations below are defined, and $s = +$ and $s = -$ are the sign indicators for call and put binary options, respectively.

We can price first order binary options using the real-world pricing formula (2.6.4) as

$$V^s_{\bar{T},K}(t, U) = \mathrm{E}\left(\frac{S^{\delta_*}_t}{S^{\delta_*}_{\bar{T}}}\, f(U_{\bar{T}})\, \mathbf{1}_{\{s\, U_{\bar{T}} \geq s\, K\}} \,\bigg|\, \mathcal{A}_t \right) \tag{2.10.2}$$

for $t \in [0, \bar{T}]$.

A first order asset binary is obtained by the selection $f(U_{\bar{T}}) = U_{\bar{T}}$ in the payoff (2.10.1). Thus, it represents an option on one unit of the underlying asset at expiry. The price $A^s_{\bar{T},K}(t, U)$ of a *first order asset binary option* simplifies from (2.10.2) to

$$A^s_{\bar{T},K}(t, U) = \mathrm{E}\left(\frac{S^{\delta_*}_t}{S^{\delta_*}_{\bar{T}}}\, U_{\bar{T}}\, \mathbf{1}_{\{s\, U_{\bar{T}} \geq s\, K\}} \,\bigg|\, \mathcal{A}_t \right) \tag{2.10.3}$$

for $t \in [0, \bar{T}]$.

On the other hand, a *first order bond binary option* is given by the selection of $f(U_{\bar{T}}) = 1$ in (2.10.1), representing an option on one unit of the domestic currency at expiry. Hence the price $B^s_{\bar{T},K}(t, U)$ of a first order bond binary option becomes

$$B^s_{\bar{T},K}(t, U) = \mathrm{E}\left(\frac{S^{\delta_*}_t}{S^{\delta_*}_{\bar{T}}}\, \mathbf{1}_{\{s\, U_{\bar{T}} \geq s\, K\}} \,\bigg|\, \mathcal{A}_t \right) \tag{2.10.4}$$

for $t \in [0, \bar{T}]$.

We can also derive *parity relationships* for first order binary options. Note that for a given strike price K we can express any arbitrary payoff function $f(U_{\bar{T}})$ as

$$f(U_{\bar{T}}) = f(U_{\bar{T}})\, \mathbf{1}_{\{U_{\bar{T}} > K\}} + f(U_{\bar{T}})\, \mathbf{1}_{\{U_{\bar{T}} \leq K\}} \tag{2.10.5}$$

for $t \in [0, \bar{T}]$. If we express the price of a European option that pays $f(U_{\bar{T}})$ at expiry time $\bar{T}$ by $V_{\bar{T}}(t, U)$, then it follows from (2.10.5) that

$$V_{\bar{T}}(t, U) = V^+_{\bar{T},K}(t, U) + V^-_{\bar{T},K}(t, U) \tag{2.10.6}$$

for $t \in [0, \bar{T}]$. For simplicity, we assume that the event $U_{\bar{T}} = K$ has zero probability. Irrespective of the dynamics for the underlying price process U, it is straightforward to verify that the following parity relationships are satisfied by first order binary options

$$A^{+}_{\bar{T},K}(t,U) + A^{-}_{\bar{T},K}(t,U) = S^{\delta_*}_t \, \hat{U}_t \qquad (2.10.7)$$

$$B^{+}_{\bar{T},K}(t,U) + B^{-}_{\bar{T},K}(t,U) = P(t,\bar{T}) \qquad (2.10.8)$$

for $t \in [0, \bar{T}]$ with (2.9.7). Note that the asset binary parity relationship (2.10.7) simplifies to

$$A^{+}_{\bar{T},K}(t,U) + A^{-}_{\bar{T},K}(t,U) = U_t \qquad (2.10.9)$$

when U is a *fair* price process as per Definition 2.6.1.

2.10.2 Call and Put Options

First order binary options can be thought of as basic building blocks for European derivatives. If we let $c_{\bar{T},K}(t,U)$ denote the price of a standard *European call option* with strike price K and expiry date $\bar{T} \geq t$, then using the real-world pricing formula (2.6.4), (2.10.3) and (2.10.4), one obtains

$$\begin{aligned} c_{\bar{T},K}(t,U) &= \mathrm{E}\left(\frac{S^{\delta_*}_t}{S^{\delta_*}_{\bar{T}}} \, (U_{\bar{T}} - K)^+ \,\middle|\, \mathcal{A}_t \right) \\ &= \mathrm{E}\left(\frac{S^{\delta_*}_t}{S^{\delta_*}_{\bar{T}}} \, U_{\bar{T}} \, \mathbf{1}_{\{U_{\bar{T}} > K\}} \,\middle|\, \mathcal{A}_t \right) - K \, \mathrm{E}\left(\frac{S^{\delta_*}_t}{S^{\delta_*}_{\bar{T}}} \, \mathbf{1}_{\{U_{\bar{T}} > K\}} \,\middle|\, \mathcal{A}_t \right) \\ &= A^{+}_{\bar{T},K}(t,U) - K \, B^{+}_{\bar{T},K}(t,U) \end{aligned} \qquad (2.10.10)$$

for $t \in [0, \bar{T}]$. Similarly, if we let $p_{\bar{T},K}(t,U)$ denote the price of a standard *European put option* with strike price K and expiry date $\bar{T} \geq t$, then we obtain

$$p_{\bar{T},K}(t,U) = -A^{-}_{\bar{T},K}(t,U) + K \, B^{-}_{\bar{T},K}(t,U) \qquad (2.10.11)$$

for $t \in [0, \bar{T}]$.

The well-known form of the put-call parity relationship is recovered from (2.10.7), (2.10.8), (2.10.10), (2.10.11) as

$$c_{\bar{T},K}(t,U) + K \, P(t,\bar{T}) = p_{\bar{T},K}(t,U) + S^{\delta_*}_t \, \hat{U}_t \qquad (2.10.12)$$

for $t \in [0, \bar{T}]$, where $\hat{U}_t$ is defined in (2.9.7). This put-call parity relationship is general. We emphasise in (2.10.12) the use of the fair zero-coupon bond price and not the ratio of savings account values, even in the case of a deterministic short rate. This reflects the underlying economic spirit of put-call parity, whereby a portfolio comprising a long call option plus the discounted value of the strike price, equals a portfolio of a long position in a put option and the discounted value of the underlying from the option expiry back to the transaction date. Hence there are two important differences between the put-call parity relationship derived under the benchmark approach above and that which results from traditional risk neutral pricing rules: The first is that under the benchmark approach the discounted value of the underlying cannot be simplified beyond the representation given by the last term in relation (2.10.12) above. This is because, in general, we do not know whether the underlying price process is fair. The second difference is that it is essential that put-call parity be expressed using the fair zero-coupon bond price (2.8.1), as it was for generic forward pricing in (2.9.14)−(2.9.16). We emphasise that the ratio of savings accounts, even for a deterministic short rate, could lead to results indicating that put-call parity does not hold, as has been argued in Cox and Hobson [2005], Li [2005] and Heston et al. [2007].

Now we consider the application of the put-call parity relationship to the financial quantities discussed previously. Starting with the GOP itself, by using (2.9.7) and (2.10.12) it is elementary to show that

$$c_{\bar{T},K}(t, S^{\delta_*}) + K\,P(t,\bar{T}) = p_{\bar{T},K}(t, S^{\delta_*}) + S_t^{\delta_*} \tag{2.10.13}$$

for $t \in [0, \bar{T}]$, as expected.

Finally, for now, we provide the result for put-call parity relating to primary security accounts, which itself requires (2.2.2), (2.9.18), (2.9.7) and (2.10.12) to compute

$$c_{\bar{T},K}(t, S^{j}) + K\,P(t,\bar{T}) = p_{\bar{T},K}(t, S^{j}) + S_t^{j}\, M_{\bar{T}}^{j}(t) \tag{2.10.14}$$

for $t \in [0, \bar{T}]$ and $j \in \{0, 1, \ldots, d\}$, where $M_{\bar{T}}^{j}(t)$ is as in (2.9.18). Once more, we have obtained a result for a primary security account that includes the quantity $M_{\bar{T}}^{j}(t)$, defined in (2.9.18). Thus, similar considerations to the primary security account forward price (2.9.17) apply here.

2.10.3 Interest Rate Derivatives

Let us also introduce the necessary relationships for pricing basic *over-the-counter* (OTC) interest rate derivatives, including interest rate caps and floors and swaptions.

Under the benchmark approach, all interest rate sensitive financial quantities are dependent on both the discounted GOP $\bar{S}_t^{\delta_*}$ and the short rate r_t for $t \in [0, T]$. Following the zero-coupon bond decomposition in (2.8.2), it is usually possible to obtain certain simplifications that make fair pricing of interest rate derivatives feasible. However, in order to define these instruments, we must first provide definitions for the more basic quantities of discrete forward rates and forward rate agreements (FRAs).

Discrete forward rates, as opposed to instantaneous forward rates defined by (2.8.10), are one of the fundamental financial quantities in interest rate markets. We introduce a simply-compounded discrete forward rate $\tilde{F}_{\bar{T},T}(t)$ representing the interest rate that can be obtained at time t for the forward period defined from the expiry time $\bar{T} > t$ to the maturity time $T > \bar{T}$. The most commonly used discrete forward rate is the LIBOR rate. The simple-compounding property of discrete forward rates means that the forward bond price $F_{\bar{T}}(t, P(\cdot, T))$, the corresponding traded instrument, previously calculated in (2.9.21), is priced in the market as a discount instrument, and hence

$$F_{\bar{T}}(t, P(\cdot, T)) = \frac{1}{1 + \tilde{F}_{\bar{T},T}(t)\,(T - \bar{T})} \tag{2.10.15}$$

for $0 \leq t \leq \bar{T} < T$. By re-arranging (2.10.15) in terms of the discrete forward rate $\tilde{F}_{\bar{T},T}(t)$ with the aid of (2.9.21) we obtain

$$\tilde{F}_{\bar{T},T}(t) = \frac{1}{T - \bar{T}} \left(\frac{P(t, \bar{T})}{P(t, T)} - 1 \right) \tag{2.10.16}$$

for $0 \leq t \leq \bar{T} < T$. This discrete forward rate (2.10.16) is the fixed interest rate in a *forward rate agreement* (FRA) that sets the FRA contract value equal to zero at inception.

Also note that by using smoothness properties of zero-coupon bonds, the instantaneous forward rate (2.8.10) can be shown to be the limit of the discrete forward rate (2.10.16) as the maturity approaches the expiry date, that is

$$f_{\bar{T}}(t) = \lim_{T \to \bar{T}^+} \tilde{F}_{\bar{T},T}(t) \tag{2.10.17}$$

almost-surely.

We continue with the following definition of options on zero-coupon bonds.

Definition 2.10.1 *The* fair *prices of* call *and* put *options on a zero-coupon bond at time t with expiry $\bar{T}$, bond maturity $T \geq \bar{T}$ and strike price K are defined as*

$$\mathbf{zcbcall}_{\bar{T},\,T,\,K}(t) = \mathrm{E}\left(\frac{S_t^{\delta_*}}{S_{\bar{T}}^{\delta_*}} \left(P(\bar{T}, T) - K \right)^+ \,\middle|\, \mathcal{A}_t \right) \tag{2.10.18}$$

$$\mathbf{zcbput}_{\bar{T},\,T,\,K}(t) = \mathrm{E}\left(\frac{S_t^{\delta_*}}{S_{\bar{T}}^{\delta_*}} \left(K - P(\bar{T}, T) \right)^+ \,\middle|\, \mathcal{A}_t \right) \tag{2.10.19}$$

for $0 \leq t \leq \bar{T} \leq T$.

Also, we can determine the put-call relationship for options on zero-coupon bonds using (2.10.12) and Definition 2.10.1 as

$$\mathbf{zcbcall}_{\bar{T},T,K}(t) + K\,P(t,\bar{T}) = \mathbf{zcbput}_{\bar{T},T,K}(t) + P(t,T) \tag{2.10.20}$$

for $0 \leq t \leq \bar{T} \leq T$.

We note here the price of options on zero-coupon bonds under the assumptions that an equivalent risk neutral probability measure exists and that the short rate is constant, hence $r_t = r$, for $t \in [0,T]$. In this case, we trivially obtain

$$\mathbf{zcbcall}_{\bar{T},T,K}(t) = \big(+\exp\{-r\,(T-t)\} - K\,\exp\{-r\,(\bar{T}-t)\}\big)^+ \tag{2.10.21}$$
$$\mathbf{zcbput}_{\bar{T},T,K}(t) = \big(-\exp\{-r\,(T-t)\} + K\,\exp\{-r\,(\bar{T}-t)\}\big)^+ \tag{2.10.22}$$

for $0 \leq t \leq \bar{T} \leq T$. Of course, both call and put options will be zero when $K = G_T(\bar{T}) = \exp\{-r\,(T-\bar{T})\}$. This special case is included here, for comparison with the corresponding results we will derive for each model considered in subsequent chapters of this thesis.

We also note here the price of options on zero-coupon bonds under Assumption 2.8.2, namely when there is independence of the noise processes driving the short rate and the discounted GOP. In this case we obtain the formulae in the following lemma.

Theorem 2.10.2 *Suppose the short rate r_t and the discounted GOP $\bar{S}^{\delta_*}_t$ satisfy Assumption 2.8.2. Then the fair prices of a call option and a put option on a zero-coupon bond are given respectively by*

$$\mathbf{zcbcall}_{\bar{T},T,K}(t) = E\left(\frac{\bar{S}^{\delta_*}_t}{\bar{S}^{\delta_*}_{\bar{T}}} M_T(\bar{T})\, c_{\bar{T},K/M_T(\bar{T})}(t, G_T(\cdot), \bar{S}^{\delta_*}_{\bar{T}}) \,\middle|\, \mathcal{A}_t\right) \tag{2.10.23}$$
$$\mathbf{zcbput}_{\bar{T},T,K}(t) = E\left(\frac{\bar{S}^{\delta_*}_t}{\bar{S}^{\delta_*}_{\bar{T}}} M_T(\bar{T})\, p_{\bar{T},K/M_T(\bar{T})}(t, G_T(\cdot), \bar{S}^{\delta_*}_{\bar{T}}) \,\middle|\, \mathcal{A}_t\right), \tag{2.10.24}$$

where $0 \leq t \leq \bar{T} \leq T < \infty$,

$$c_{\bar{T},K/M_T(\bar{T})}(t, G_T, \bar{S}^{\delta_*}_{\bar{T}}) = \mathrm{E}\left(\frac{B_t}{B_{\bar{T}}}(G_T(\bar{T}) - K/M_T(\bar{T}))^+ \,\middle|\, \bar{S}^{\delta_*}_{\bar{T}}, \mathcal{A}_t\right) \tag{2.10.25}$$
$$p_{\bar{T},K/M_T(\bar{T})}(t, G_T, \bar{S}^{\delta_*}_{\bar{T}}) = \mathrm{E}\left(\frac{B_t}{B_{\bar{T}}}(K/M_T(\bar{T}) - G_T(\bar{T}))^+ \,\middle|\, \bar{S}^{\delta_*}_{\bar{T}}, \mathcal{A}_t\right)$$

and $G_T(\cdot)$ is the process $(G_T(s))_{t\leq s\leq T}$, with $G_T(s)$ defined in (2.8.4).

Proof. We only prove the formula for the call option (2.10.23) on the zero-coupon bond. The proof for the put option formula (2.10.24) is similar: We have

$$\begin{aligned}
\mathbf{zcbcall}_{\bar{T},T,K}(t) &= E\left(\frac{S_t^{\delta_*}}{S_{\bar{T}}^{\delta_*}}(P(\bar{T},T)-K)^+ \Big| \mathcal{A}_t\right) \qquad (2.10.26)\\
&= E\left(\frac{\bar{S}_t^{\delta_*}}{\bar{S}_{\bar{T}}^{\delta_*}} \mathrm{E}\left(\frac{B_t}{B_{\bar{T}}}(P(\bar{T},T)-K)^+ \Big| \bar{S}_{\bar{T}}^{\delta_*}\right) \Big| \mathcal{A}_t\right)\\
&= E\left(\frac{\bar{S}_t^{\delta_*}}{\bar{S}_{\bar{T}}^{\delta_*}} \mathrm{E}\left(\frac{B_t}{B_{\bar{T}}}(M_T(\bar{T})G_T(\bar{T})-K)^+ \Big| \bar{S}_{\bar{T}}^{\delta_*}\right) \Big| \mathcal{A}_t\right)\\
&= E\left(\frac{\bar{S}_t^{\delta_*}}{\bar{S}_{\bar{T}}^{\delta_*}} M_T(\bar{T}) \mathrm{E}\left(\frac{B_t}{B_{\bar{T}}}(G_T(\bar{T})-K/M_T(\bar{T}))^+ \Big| \bar{S}_{\bar{T}}^{\delta_*}\right) \Big| \mathcal{A}_t\right)\\
&= E\left(\frac{\bar{S}_t^{\delta_*}}{\bar{S}_{\bar{T}}^{\delta_*}} M_T(\bar{T})\, c_{\bar{T},K/M_T(\bar{T})}(t, G_T(\cdot), \bar{S}_{\bar{T}}^{\delta_*}) \Big| \mathcal{A}_t\right),
\end{aligned}$$

which is the call option pricing formula (2.10.23). **Q.E.D.**

This lemma is useful for pricing options on ZCBs when the expectations in (2.10.25) are closed form expressions. This is because the expectations in (2.10.23) and (2.10.24) can be readily evaluated when the probability density function of $\bar{S}_{\bar{T}}^{\delta_*}$, given the starting value $\bar{S}_t^{\delta_*}$, is known.

In the particular case, where the discounted GOP obeys a Black-Scholes model, we have $M_T(t) = 1$ and the pricing formulae (2.10.23) and (2.10.24) simplify to

$$\begin{aligned}
c_{\bar{T},K}(t, G_T(\cdot)) &= \mathrm{E}\left(\frac{B_t}{B_{\bar{T}}}(G_T(\bar{T})-K)^+ \Big| \mathcal{A}_t\right) \qquad (2.10.27)\\
p_{\bar{T},K}(t, G_T(\cdot)) &= \mathrm{E}\left(\frac{B_t}{B_{\bar{T}}}(K-G_T(\bar{T}))^+ \Big| \mathcal{A}_t\right).
\end{aligned}$$

2.10.4 Caps and Floors

As mentioned in standard texts such as in Baxter and Rennie [1996], Hull [1997], or Brigo and Mercurio [2006], interest rate caps and floors are introduced as simple transformations of put and call options on zero-coupon bonds, respectively. An *interest rate cap* can be decomposed into a portfolio of individual *caplets*. Each caplet itself is equivalent to a put option on a zero-coupon bond with adjustments to the strike rate and notional principal. Analogously, an *interest rate floor* comprises a portfolio of *floorlets*, and each floorlet is equivalent to a call option on a zero-coupon bond with adjustments to the strike rate and notional principal.

Let us denote an *interest rate cap* contract by $\mathbf{cap}_{\mathcal{T},K,N}(t)$ at time $t \leq T_0$ with strike rate K, notional principal N and the set of dates $\mathcal{T} = \{T_0, T_1, \ldots, T_n\}$. The ℓ-th individual *caplet* $\mathbf{caplet}_{T_{\ell-1},T_\ell,K}(t)$ relates to the ℓ-th zero-coupon bond put option for $\ell \in \{1, \ldots, n\}$ as in the following theorem.

Theorem 2.10.3 *Suppose the short rate r_t and the discounted GOP $\bar{S}_t^{\delta_*}$ satisfy Assumption 2.8.2. Then the fair price of the caplet with start time $\bar{T}$, end time $T > \bar{T}$ and strike rate K is*

$$\mathbf{caplet}_{\bar{T},T,K}(t) = (1 + K(T - \bar{T}))\mathbf{zcbput}_{\bar{T},T,1/(1+(T-\bar{T})K)}(t) \tag{2.10.28}$$

and the fair price of the floorlet with the same parameters is

$$\mathbf{floorlet}_{\bar{T},T,K}(t) = (1 + K(T - \bar{T}))\mathbf{zcbcall}_{\bar{T},T,1/(1+(T-\bar{T})K)}(t). \tag{2.10.29}$$

Proof. The payoff of the caplet with start time $\bar{T}$, end time $T \geq \bar{T}$ and strike rate K is

$$H_{\bar{T}} = (T - \bar{T})(\tilde{F}_{\bar{T},T}(\bar{T}) - K)^+ \tag{2.10.30}$$

and using Theorem 2.6.4 we have

$$\begin{aligned}
\mathbf{caplet}_{\bar{T},T,K}(t) &= E\left(\frac{S_t^{\delta_*}}{S_T^{\delta_*}}(T - \bar{T})(\tilde{F}_{\bar{T},T}(\bar{T}) - K)^+ \middle| \mathcal{A}_t\right) \\
&= E\left(\frac{S_t^{\delta_*}}{S_T^{\delta_*}}(1 + (T - \bar{T})\tilde{F}_{\bar{T},T}(\bar{T}) - 1 - (T - \bar{T})K)^+ \middle| \mathcal{A}_t\right) \\
&= E\left(\frac{S_t^{\delta_*}}{S_{\bar{T}}^{\delta_*}}(1 - (1 + (T - \bar{T})K)P(\bar{T}, T))^+ \middle| \mathcal{A}_t\right) \\
&= (1 + (T - \bar{T})K)E\left(\frac{S_t^{\delta_*}}{S_{\bar{T}}^{\delta_*}}(1/(1 + (T - \bar{T})K) - P(\bar{T}, T))^+ \middle| \mathcal{A}_t\right) \\
&= (1 + (T - \bar{T})K)\mathbf{zcbput}_{\bar{T},T,1/(1+(T-\bar{T})K)}(t),
\end{aligned} \tag{2.10.31}$$

which verifies the caplet formula. The floorlet formula is derived in a similar manner. **Q.E.D.**

Consequently, we can write the price of the cap as

$$\begin{aligned}
\mathbf{cap}_{\mathcal{T},K,N}(t) &= \sum_{\ell=1}^{n} \mathbf{caplet}_{T_{\ell-1},T_\ell,K,N}(t) \\
&= N \sum_{\ell=1}^{n} (T_\ell - T_{\ell-1})\,\mathrm{E}\left(\frac{S_t^{\delta_*}}{S_{T_\ell}^{\delta_*}}\ \left(F_{T_{\ell-1},T_\ell}(T_{\ell-1}) - K\right)^+ \middle| \mathcal{A}_t\right) \\
&= \sum_{\ell=1}^{n} N'_\ell\,\mathbf{zcbput}_{T_{\ell-1},T_\ell,K'_\ell}(t),
\end{aligned} \tag{2.10.32}$$

where

$$N'_\ell = N\ (1 + K\,(T_\ell - T_{\ell-1})) \quad \text{and} \quad K'_\ell = \frac{1}{1 + K\,(T_\ell - T_{\ell-1})} \tag{2.10.33}$$

for $0 \leq t \leq T_0 < \ldots < T_n$.

We also provide the analogous result for an *interest rate floor* contract $\mathbf{floor}_{\mathcal{T},K,N}(t)$ valued at time $t \leq T_0$. The ℓ-th individual floorlet $\mathbf{floorlet}_{T_{\ell-1},T_\ell,K}(t)$ relates to the ℓ-th zero-coupon bond call option for $\ell \in \{1,\ldots,n\}$ as in Theorem 2.10.3. Therefore, we can write the price of a floor as

$$\begin{aligned}
\mathbf{floor}_{\mathcal{T},K,N}(t) &= \sum_{\ell=1}^{n} \mathbf{floorlet}_{T_{\ell-1},T_\ell,K,N}(t) \\
&= N \sum_{\ell=1}^{n} (T_\ell - T_{\ell-1})\, \mathrm{E}\left(\frac{S_t^{\delta_*}}{S_{T_\ell}^{\delta_*}} \left(K - F_{T_{\ell-1},T_\ell}(T_{\ell-1})\right)^+ \,\middle|\, \mathcal{A}_t \right) \\
&= \sum_{\ell=1}^{n} N'_\ell\, \mathbf{zcbcall}_{T_{\ell-1},T_\ell,K'_\ell}(t) \qquad (2.10.34)
\end{aligned}$$

for $0 \leq t \leq T_0 < \ldots < T_n$ with adjusted strikes and notional principal values as per (2.10.33).

2.10.5 Options on Coupon Bonds

A coupon bond is a security which pays a series of coupons at regular times over the life of the bond in addition to a return of principal, otherwise termed the face value of the bond, at the maturity date. The coupon payment is typically calculated as the product of the principal amount, the annualised coupon rate and the fraction of the year elapsed since the previous coupon payment, mathematically written as

$$COUPON(T_k) = N \times c \times (T_k - T_{k-1}), \qquad (2.10.35)$$

where T_k is the coupon payment date, T_{k-1} is the previous coupon payment date, c is the annualised coupon rate and N is the principal value of the bond.

Let $\mathcal{T}$ be the set consisting of the accrual start time T_0 along with the prescribed coupon payment dates T_1, ..., T_n, which satisfy the inequalities

$$T_0 < T_1 < \ldots < T_n. \qquad (2.10.36)$$

Here, T_0 is the date from which the first coupon payment accrues and where T_n is the maturity date of the bond.

Then the price at time $t < T_1$ of the coupon bond having face value N and coupon rate c is given by the formula

$$\begin{aligned}
P_{\mathcal{T},c,N}(t) &= E\left(c \times N \times \sum_{k=1}^{n} (T_k - T_{k-1}) \frac{S_t^{\delta_*}}{S_{T_k}^{\delta_*}} + N \times \frac{S_t^{\delta_*}}{S_{T_n}^{\delta_*}} \,\middle|\, \mathcal{A}_t \right) \qquad (2.10.37) \\
&= c \times N \times \sum_{k=1}^{n} (T_k - T_{k-1}) P(t,T_k) + N \times P(t,T_n).
\end{aligned}$$

When the face value of the coupon bond is one domestic currency unit, that is $N = 1$, we write

$$P_{\mathcal{T},c}(t) = P_{\mathcal{T},c,1}(t) \tag{2.10.38}$$

and, therefore, for any value of N we have

$$P_{\mathcal{T},c,N}(t) = N \times P_{\mathcal{T},c}(t). \tag{2.10.39}$$

Pricing formulae and efficient pricing algorithms for options on coupon bonds under a variety of short rate models have been supplied by many authors, including Jamshidian [1989], Longstaff [1993], Singleton and Umantsev [2002] and Schrager and Pelsser [2005].

Singleton and Umantsev [2002] provide a numerically accurate and computationally fast approximation to the prices of European options on coupon bonds under all affine term structure models, namely those models of the short rate r_t describable by the system of SDEs

$$\begin{aligned} r_t &= a + b^\top Y_t \\ Y_t &= \mathcal{K}(\theta - Y_t)dt + \Sigma\sqrt{S_t}dW_t \\ S_t &= \text{diag}(c_i + d_i^\top Y_t), \end{aligned} \tag{2.10.40}$$

where W_t is n-dimensional Brownian motion, Y_t is an n-dimensional stochastic process, $\mathcal{K}$ is an $n \times n$ matrix, Σ is an $n \times n$ covariance matrix, S_t is an $n \times n$ diagonal matrix, b is an n-dimensional column vector and θ is an n-dimensional column vector.

For affine term structure models, Schrager and Pelsser [2005] approximate the price of options on coupon bonds by deriving approximate dynamics in which the relevant swap rate obeys a square root process.

Jamshidian [1989] provides a closed form pricing formula for a European option on a portfolio of zero-coupon bonds under the assumption of a mean reverting Gaussian interest rate model as in Vasicek [1977]. The derivation of the formula relies on the observation that the monotonicity of the zero-coupon bond price as a function of the short rate implies that the exercise short rate of the portfolio of ZCBs is the same as each of the exercise short rates of the options on the component ZCBs.

Following the work of Jamshidian, there is a semi-closed-form formula for European options on a portfolio of $G_{\bar{T}_i}(t)$, $i = 1, 2, \ldots, n$, when there is monotonicity in the value of $G_{\bar{T}}(t)$ as a function of the short rate r_t at time t. We state without proof Jamshidian's proposition that the price of a call option on a portfolio of $G_{\bar{T}_i}(t)$ is equal to the sum of the prices of call options on constituent $G_{\bar{T}_i}(t)$ with specific strike price K_i, under the assumption of a mean-reverting Gaussian interest rate model. Denote by $P(r, t, T)$ the price at time t of a ZCB maturing at time T for $r_t = r$, and by $C(r, t, T, s, K)$ the price at time t of a call option on the s-maturity ZCB with exercise price K and expiry time T, where $T < s$ and $r_t = r$.

Proposition 2.10.4 *(Jamshidian) The price C_a at time t and interest rate $r_t = r$ of a European call option with exercise price K and expiration T on a portfolio of n ZCBs, where the i-th ZCB has face value a_i and maturity s_i for $i \in \{1, 2, \ldots, n\}$, is given by*

$$C_a = P(r, t, T)\mathrm{E}\left(\max\{0, \tilde{P}_a - K\}\right), \tag{2.10.41}$$

where

$$\tilde{P}_a = \sum_{j:T<s_j} a_j P(R_{r,t,T}, T, s_j) \tag{2.10.42}$$

and $R_{r,t,T}$ denotes the random interest rate at time T conditional on the interest rate at time t being r. We also have the decomposition

$$\max\{0, \tilde{P}_a - K\} = \sum a_j \max\{0, P(R_{r,t,T}, T, s_j) - K_j\}, \tag{2.10.43}$$

where $K_j = P(r^*, t, s_j)$ and r^* is the solution to

$$\sum_{j:T<s_j} a_j P(r^*, t, s_j) = K. \tag{2.10.44}$$

Hence,

$$C_a = \sum a_j C(r, t, T, s_j, K_j), \tag{2.10.45}$$

where $C(r, t, T, s, K)$ is the price at time t, given that $r_t = r$, of a call option on the s-maturity pure discount bond with exercise price K and expiration $T < s$.

Employing this idea of Jamshidian we can calculate the price of an option on a portfolio of contributions $G_{\bar{T}_i}(t)$, $i = 1, 2, \ldots, n$ as the portfolio sum of options on a single contributions $G_{\bar{T}_i}(t)$, $i = 1, 2, \ldots, n$.

Using the same observation, Longstaff [1993] derives simple closed form expressions for European options on coupon bonds under the CIR short rate model, see Cox et al. [1985].

In deriving the pricing formulae used in this thesis we make use of the assumed independence of the short rate process and the discounted GOP process, and the closed form expressions for coupon bond options given by Jamshidian [1989] and Longstaff [1993] that emerge when the coupon bond option price is conditioned on the state of the discounted GOP process at the option expiry date.

Now, suppose we have a European call option on a coupon bond with unit face value, having strike price K and expiring at time $\bar{T}$, where $\bar{T} < T_1$. Then the payoff at expiry is

$$H_{\bar{T}} = \left(P_{T,c}(\bar{T}) - K\right)^+ \tag{2.10.46}$$

We have the following theorem, which supplies the pricing formulae of European options on a coupon bond when the short rate process is such that $G_T(t)$ can be written in the form $G_T(t) = A(t,T)\exp(-r_t B(t,T))$.

Theorem 2.10.5 *Suppose the short rate r_t and the discounted GOP $\bar{S}^{\delta_*}_t$ satisfy Assumption 2.8.2 and that $G_T(t)$ has the form $A(t,T)\exp(-r_t B(t,T))$ for deterministic functions A and B. Then the fair prices of a call option and put option on a coupon bond are given by*

$$c_{\bar{T},K}(t, P_{\mathcal{T},c}) = E\left(\frac{\bar{S}^{\delta_*}_t}{\bar{S}^{\delta_*}_{\bar{T}}} \sum_{i=1}^{n} \big((T_i - T_{i-1})c + \mathbf{1}_{i=n}\big) M_{T_i}(\bar{T}) c_{\bar{T},K'_i}(t, G_{T_i}, \bar{S}^{\delta_*}_{\bar{T}}) \Big| \mathcal{A}_t\right) \tag{2.10.47}$$

$$p_{\bar{T},K}(t, P_{\mathcal{T},c}) = E\left(\frac{\bar{S}^{\delta_*}_t}{\bar{S}^{\delta_*}_{\bar{T}}} \sum_{i=1}^{n} \big((T_i - T_{i-1})c + \mathbf{1}_{i=n}\big) M_{T_i}(\bar{T}) p_{\bar{T},K'_i}(t, G_{T_i}, \bar{S}^{\delta_*}_{\bar{T}}) \Big| \mathcal{A}_t\right), \tag{2.10.48}$$

respectively, where for $i = 1, 2, \dots, n$ the modified strike price $K'_i(\bar{S}^{\delta_*}_{\bar{T}})$ is given by

$$K'_i(\bar{S}^{\delta_*}_{\bar{T}}) = A(\bar{T}, T_i)\exp(-r^*(\bar{S}^{\delta_*}_{\bar{T}}) B(\bar{T}, T_i)) \tag{2.10.49}$$

and where $r^*(\bar{S}^{\delta_*}_{\bar{T}})$ satisfies the equation

$$\sum_{i=1}^{n} \big((T_i - T_{i-1})c + \mathbf{1}_{i=n}\big) M_{T_i}(\bar{T}) A(\bar{T}, T_i)\exp(-r^*(\bar{S}^{\delta_*}_{\bar{T}}) B(\bar{T}, T_i)) = K. \tag{2.10.50}$$

Proof. We have that with $H_{\bar{T}}$ as in (2.10.46) the fair price of the call option is

$$\begin{aligned} c_{\bar{T},K}(t, P_{\mathcal{T},c}(\cdot)) &= E\left(\frac{S^{\delta_*}_t}{S^{\delta_*}_{\bar{T}}} H_{\bar{T}} \Big| \mathcal{A}_t\right) \qquad (2.10.51)\\ &= E\left(\frac{\bar{S}^{\delta_*}_t}{\bar{S}^{\delta_*}_{\bar{T}}} \frac{B_t}{B_{\bar{T}}} H_{\bar{T}} \Big| \mathcal{A}_t\right) \\ &= E\left(\frac{\bar{S}^{\delta_*}_t}{\bar{S}^{\delta_*}_{\bar{T}}} \mathrm{E}\left(\frac{B_t}{B_{\bar{T}}} H_{\bar{T}} \Big| \bar{S}^{\delta_*}_{\bar{T}}\right) \Big| \mathcal{A}_t\right) \\ &= E\left(\frac{\bar{S}^{\delta_*}_t}{\bar{S}^{\delta_*}_{\bar{T}}} \mathrm{E}\left(\frac{B_t}{B_{\bar{T}}} \sum_{i=1}^{n} H^{(i)}_{\bar{T}}(\bar{S}^{\delta_*}_{\bar{T}}) \Big| \bar{S}^{\delta_*}_{\bar{T}}\right) \Big| \mathcal{A}_t\right), \end{aligned}$$

where

$$\begin{aligned} H^{(i)}_{\bar{T}}(\bar{S}^{\delta_*}_{\bar{T}}) &= \Big(\big((T_i - T_{i-1})c + \mathbf{1}_{i=n}\big) P(\bar{T}, T_i) - K_i(\bar{S}^{\delta_*}_{\bar{T}})\Big)^+ \qquad (2.10.52)\\ &= \Big(\big((T_i - T_{i-1})c + \mathbf{1}_{i=n}\big) M_{T_i}(\bar{T}) G_{T_i}(\bar{T}) - K_i(\bar{S}^{\delta_*}_{\bar{T}})\Big)^+, \end{aligned}$$

and where $K_i(\bar{S}^{\delta_*}_{\bar{T}})$ is given by

$$K_i(\bar{S}^{\delta_*}_{\bar{T}}) = \big((T_i - T_{i-1})c + \mathbf{1}_{i=n}\big) M_{T_i}(\bar{T}) A(\bar{T}, T_i) \exp(-r^*(\bar{S}^{\delta_*}_{\bar{T}}) B(\bar{T}, T_i)), \quad (2.10.53)$$

with $r^*(\bar{S}^{\delta_*}_{\bar{T}})$ satisfying the equation

$$\sum_{i=1}^{n} \big((T_i - T_{i-1})c + \mathbf{1}_{i=n}\big) M_{T_i}(\bar{T}) A(\bar{T}, T_i) \exp(-r^*(\bar{S}^{\delta_*}_{\bar{T}}) B(\bar{T}, T_i)) = K. \quad (2.10.54)$$

It follows from (2.10.51) that

$$\begin{aligned} & c_{\bar{T},K}(t, P_{\mathcal{T},c}(\cdot)) \qquad (2.10.55) \\ &= E\left(\frac{\bar{S}^{\delta_*}_t}{\bar{S}^{\delta_*}_{\bar{T}}} \sum_{i=1}^{n} \mathrm{E}\left(\frac{B_t}{B_{\bar{T}}} H^{(i)}_{\bar{T}}(\bar{S}^{\delta_*}_{\bar{T}}) \middle| \bar{S}^{\delta_*}_{\bar{T}}\right) \middle| \mathcal{A}_t\right) \\ &= E\left(\frac{\bar{S}^{\delta_*}_t}{\bar{S}^{\delta_*}_{\bar{T}}} \sum_{i=1}^{n} \big((T_i - T_{i-1})c + \mathbf{1}_{i=n}\big) M_{T_i}(\bar{T}) c_{\bar{T},K'_i}(t, G_{T_i}(\cdot), \bar{S}^{\delta_*}_{\bar{T}}) \middle| \mathcal{A}_t\right), \end{aligned}$$

where for $i = 1, 2, \ldots, n$ the modified strike price $K'_i(\bar{S}^{\delta_*}_{\bar{T}})$ is given by

$$K'_i(\bar{S}^{\delta_*}_{\bar{T}}) = A(\bar{T}, T_i) \exp(-r^*(\bar{S}^{\delta_*}_{\bar{T}}) B(\bar{T}, T_i)) \quad (2.10.56)$$

and $r^*(\bar{S}^{\delta_*}_{\bar{T}})$ as in (2.10.50). The proof of the put option formula follows a similar line of reasoning. **Q.E.D.**

Thus Theorem 2.10.5 permits the computation of the price of a coupon bond option by integrating a closed form expression over the probability density of the value of the discounted GOP at option expiry. This leads to an efficient means of computing coupon bond option prices and swaption prices as we will see in Chapter 5.

2.10.6 Swaps

The holder of an interest rate payer swap receives variable interest payments and pays fixed interest payments on a set of prescribed payment dates. The fixed interest payment is called the swap rate, denoted **swaprate**, and the prescribed set of payment dates T_1, ..., T_n satisfy

$$T_0 < T_1 < \ldots < T_n, \quad (2.10.57)$$

where T_0 is the swap's start date from which the first interest payment accrues.

The T_0-forward swap rate as at time t, in respect of a swap commencing at time T_0 with payment dates $T_1, \ldots, T_n$, is denoted by $\mathbf{swaprate}_{\mathcal{T}}(t)$, where $\mathcal{T}$ is the set of dates $\{T_0, T_1, \ldots, T_n\}$.

The T_0-forward swap rate $\mathbf{swaprate}_{\mathcal{T}}(t)$ is determined as that rate at which the value of the stream of fixed payments equates to the stream of floating payments.

Theorem 2.10.6 *The swap rate can be calculated using the formula*

$$\mathbf{swaprate}_{\mathcal{T}}(t) = \frac{P(t,T_0) - P(t,T_n)}{\sum_{k=1}^{n}(T_k - T_{k-1})P(t,T_k)}, \tag{2.10.58}$$

where $P(t,T)$ is again the price of a T-maturity zero-coupon bond as at time t.

Proof. The equation defining the swap rate is

$$\begin{aligned} &E\left(\sum_{k=1}^{n} \frac{S_t^{(\delta_*)}(T_k - T_{k-1})\mathbf{swaprate}_{\mathcal{T}}(t)}{S_{T_k}^{(\delta_*)}} \middle| \mathcal{A}_t\right) \\ &= E\left(\sum_{k=1}^{n} \frac{S_t^{(\delta_*)}(T_k - T_{k-1})F_{T_{k-1},T_k}(T_{k-1})}{S_{T_k}^{(\delta_*)}} \middle| \mathcal{A}_t\right), \end{aligned} \tag{2.10.59}$$

where $F_{T',T}(t)$ is the T'-starting, T-maturity interest rate (forward rate) as at time t and is given by the formula

$$F_{T',T}(t) = \frac{1}{T - T'}\left(\frac{P(t,T')}{P(t,T)} - 1\right). \tag{2.10.60}$$

The left hand side (LHS) of (2.10.59) can be rewritten as

$$\sum_{k=1}^{n} P(t,T_k)(T_k - T_{k-1})\mathbf{swaprate}_{\mathcal{T}}(t) \tag{2.10.61}$$

and the right hand side (RHS) of (2.10.59) can be rewritten as

$$\begin{aligned} &\sum_{k=1}^{n} E\left(\frac{S_t^{(\delta_*)}}{S_{T_k}^{(\delta_*)}}\left(\frac{1}{P(T_{k-1},T_k)} - 1\right) \middle| \mathcal{A}_t\right) \\ &= \sum_{k=1}^{n}\left(E\left(\frac{S_t^{(\delta_*)}}{S_{T_k}^{(\delta_*)}}\frac{1}{P(T_{k-1},T_k)} \middle| \mathcal{A}_t\right) - P(t,T_k)\right). \end{aligned} \tag{2.10.62}$$

Now we note that

$$
\begin{aligned}
&E\left(\frac{S_t^{(\delta_*)}}{S_{T_k}^{(\delta_*)}}\frac{1}{P(T_{k-1},T_k)}\middle|\mathcal{A}_t\right)\\
&= E\left(E\left(\frac{S_t^{(\delta_*)}}{S_{T_k}^{(\delta_*)}}\frac{1}{P(T_{k-1},T_k)}\middle|\mathcal{A}_{T_{k-1}}\right)\middle|\mathcal{A}_t\right)\\
&= E\left(E\left(\frac{S_t^{(\delta_*)}}{S_{T_{k-1}}^{(\delta_*)}}\frac{S_{T_{k-1}}^{(\delta_*)}}{S_{T_k}^{(\delta_*)}}\frac{1}{P(T_{k-1},T_k)}\middle|\mathcal{A}_{T_{k-1}}\right)\middle|\mathcal{A}_t\right)\\
&= E\left(\frac{S_t^{(\delta_*)}}{S_{T_{k-1}}^{(\delta_*)}}\frac{1}{P(T_{k-1},T_k)}E\left(\frac{S_{T_{k-1}}^{(\delta_*)}}{S_{T_k}^{(\delta_*)}}\middle|\mathcal{A}_{T_{k-1}}\right)\middle|\mathcal{A}_t\right)\\
&= E\left(\frac{S_t^{(\delta_*)}}{S_{T_{k-1}}^{(\delta_*)}}\frac{1}{P(T_{k-1},T_k)}P(T_{k-1},T_k)\middle|\mathcal{A}_t\right)\\
&= E\left(\frac{S_t^{(\delta_*)}}{S_{T_{k-1}}^{(\delta_*)}}\middle|\mathcal{A}_t\right)\\
&= P(t,T_{k-1})
\end{aligned}
\tag{2.10.63}
$$

and so (2.10.62) simplifies to

$$\sum_{k=1}^{n}\left(P(t,T_{k-1})-P(t,T_k)\right)=P(t,T_0)-P(t,T_n). \tag{2.10.64}$$

Equating (2.10.61) to (2.10.64) gives the result. **Q.E.D.**

2.10.7 Swaptions

In practice swaptions are rather common. As mentioned in standard texts such as Baxter and Rennie [1996], Hull [1997] and Brigo and Mercurio [2006], a payer interest rate swaption entitles the buyer to enter into a payer interest rate swap at a specified strike rate. By a payer swap, we mean that the exerciser of the payer swaption enters into a swap, where fixed rate payments at the strike rate are paid to the counterparty and floating rate payments at the prevailing floating rates are received by the exerciser.

Let us denote the price of a *payer interest rate swaption* contract by

$$\mathbf{payerswaption}_{\mathcal{T},K,N}(t) \tag{2.10.65}$$

at time $t \leq T_0$ with strike rate K, notional principal N and the set of dates $\mathcal{T}=\{T_0,T_1,\dots,T_n\}$. The swaption will be exercised if the prevailing swap rate at time T_0 exceeds the strike rate K. At the ℓth coupon date T_ℓ the payoff is

$$N\times(T_\ell-T_{\ell-1})\times\left(\frac{S_t^{\delta_*}}{S_{T_\ell}^{\delta_*}}\left(\mathbf{swaprate}_{\mathcal{T}}(T_0)-K\right)^+\right) \tag{2.10.66}$$

for $\ell \in \{1, \ldots, n\}$.

The formula for the payer swaption price is

$$\begin{aligned} &\mathbf{payerswaption}_{\mathcal{T},K,N}(t) \\ &= N \sum_{\ell=1}^{n} (T_\ell - T_{\ell-1}) \, \mathrm{E}\left(\frac{S_t^{\delta_*}}{S_{T_\ell}^{\delta_*}} \left(\mathbf{swaprate}_{\mathcal{T}}(T_0) - K\right)^+ \,\middle|\, \mathcal{A}_t \right), \end{aligned} \tag{2.10.67}$$

where $\mathbf{swaprate}_{\mathcal{T}}$ is as in (2.10.58).

Similarly a receiver interest rate swaption entitles the buyer to enter into a swap receiving fixed rate payments at the strike rate and paying floating rate payments at the prevailing floating rate.

The formula for the receiver swaption price is

$$\begin{aligned} &\mathbf{receiverswaption}_{\mathcal{T},K,N}(t) \\ &= N \sum_{\ell=1}^{n} (T_\ell - T_{\ell-1}) \, \mathrm{E}\left(\frac{S_t^{\delta_*}}{S_{T_\ell}^{\delta_*}} \left(K - \mathbf{swaprate}_{\mathcal{T}}(T_0)\right)^+ \,\middle|\, \mathcal{A}_t \right), \end{aligned} \tag{2.10.68}$$

where $\mathbf{swaprate}_{\mathcal{T}}$ is as in (2.10.58).

Employing the identity $(x-K)^+ - (K-x)^+ = x - K$ we deduce from (2.10.67) and (2.10.68) the put-call parity formula for swaptions as follows:

$$\begin{aligned} &\mathbf{payerswaption}_{\mathcal{T},K,N}(t) - \mathbf{receiverswaption}_{\mathcal{T},K,N}(t) \\ &= (\mathbf{swaprate}_{\mathcal{T}}(T_0) - K) \times N \times \sum_{\ell=1}^{n} (T_\ell - T_{\ell-1}) \, P(t, T_\ell). \end{aligned} \tag{2.10.69}$$

To price a swaption we use the pricing formula for a coupon bond option. We have the following corollary to Theorem 2.10.5, which shows that the payer swaption price is equal to a put option on a coupon bond.

Corollary 2.10.7 *For a payer swaption with strike rate K, notional amount N dollars and payment dates $\mathcal{T}$ the pricing formula is*

$$\boldsymbol{payerswaption}_{\mathcal{T},K,N}(t) = N \times p_{T_0,1}(t, P_{\mathcal{T},K}(\cdot)), \tag{2.10.70}$$

where $P_{\mathcal{T},K}(t)$ is the price of a coupon bond as in (2.10.38) *and $p_{T_0,1}(t, P_{\mathcal{T},K}(\cdot))$ is as in* (2.10.48). *Also, for a receiver swaption with the same strike rate, notional amount and payment dates the pricing formula is*

$$\boldsymbol{receiverswaption}_{\mathcal{T},K,N}(t) = N \times c_{T_0,1}(t, P_{\mathcal{T},K}(\cdot)), \tag{2.10.71}$$

where $c_{T_0,1}(t, P_{\mathcal{T},K}(\cdot))$ is as in (2.10.47).

Proof. The right hand side of (2.10.67) can be simplified as follows:

$$
\begin{aligned}
&= N \sum_{\ell=1}^{n} (T_\ell - T_{\ell-1})\, \mathrm{E}\left(\frac{S_t^{\delta_*}}{S_{T_\ell}^{\delta_*}} \, (\mathbf{swaprate}_{\mathcal{T}}(T_0) - K)^+ \,\Big|\, \mathcal{A}_t \right) \qquad (2.10.72)\\
&= N\, \mathrm{E}\left((\mathbf{swaprate}_{\mathcal{T}}(T_0) - K)^+ \sum_{\ell=1}^{n} (T_\ell - T_{\ell-1}) \frac{S_t^{\delta_*}}{S_{T_\ell}^{\delta_*}} \,\Big|\, \mathcal{A}_t \right)\\
&= N\, \mathrm{E}\left((\mathbf{swaprate}_{\mathcal{T}}(T_0) - K)^+ \sum_{\ell=1}^{n} (T_\ell - T_{\ell-1}) \frac{S_t^{\delta_*}}{S_{T_0}^{\delta_*}} \frac{S_{T_0}^{\delta_*}}{S_{T_\ell}^{\delta_*}} \,\Big|\, \mathcal{A}_t \right)\\
&= N\, \mathrm{E}\left(\frac{S_t^{\delta_*}}{S_{T_0}^{\delta_*}} \mathrm{E}\left((\mathbf{swaprate}_{\mathcal{T}}(T_0) - K)^+ \sum_{\ell=1}^{n} (T_\ell - T_{\ell-1}) \frac{S_{T_0}^{\delta_*}}{S_{T_\ell}^{\delta_*}} \,\Big|\, \mathcal{A}_{T_0} \right) \,\Big|\, \mathcal{A}_t \right)\\
&= N\, \mathrm{E}\left(\frac{S_t^{\delta_*}}{S_{T_0}^{\delta_*}} \mathrm{E}\left((\mathbf{swaprate}_{\mathcal{T}}(T_0) - K)^+ \sum_{\ell=1}^{n} (T_\ell - T_{\ell-1}) \frac{S_{T_0}^{\delta_*}}{S_{T_\ell}^{\delta_*}} \,\Big|\, \mathcal{A}_{T_0} \right) \,\Big|\, \mathcal{A}_t \right)\\
&= N\, \mathrm{E}\left(\frac{S_t^{\delta_*}}{S_{T_0}^{\delta_*}} \, (\mathbf{swaprate}_{\mathcal{T}}(T_0) - K)^+ \sum_{\ell=1}^{n} (T_\ell - T_{\ell-1})\, \mathrm{E}\left(\frac{S_{T_0}^{\delta_*}}{S_{T_\ell}^{\delta_*}} \,\Big|\, \mathcal{A}_{T_0} \right) \,\Big|\, \mathcal{A}_t \right)\\
&= N\, \mathrm{E}\left(\frac{S_t^{\delta_*}}{S_{T_0}^{\delta_*}} \, (\mathbf{swaprate}_{\mathcal{T}}(T_0) - K)^+ \sum_{\ell=1}^{n} (T_\ell - T_{\ell-1})\, P(T_0, T_\ell) \,\Big|\, \mathcal{A}_t \right).
\end{aligned}
$$

The formula for the swap rate at time $t = T_0$ is computed from (2.10.58) to be

$$
\mathbf{swaprate}_{\mathcal{T}}(T_0) = \frac{1 - P(T_0, T_n)}{\sum_{k=1}^{n} (T_k - T_{k-1}) P(T_0, T_k)} \qquad (2.10.73)
$$

and inserting into the last line of (2.10.72) gives

$$
\begin{aligned}
&N\, \mathrm{E}\left(\frac{S_t^{\delta_*}}{S_{T_0}^{\delta_*}} \left(1 - P(T_0, T_n) - K \sum_{\ell=1}^{n} (T_\ell - T_{\ell-1})\, P(T_0, T_\ell) \right)^+ \,\Big|\, \mathcal{A}_t \right) \qquad (2.10.74)\\
&= N\, p_{T_0,1}(t, P_{\mathcal{T},K}(\cdot)),
\end{aligned}
$$

as required. The formula for the receiver swaption is similarly proven. **Q.E.D.**

Corollary 2.10.7 equips us with an efficient swaption pricing formula.

2.10.8 Hedging

Generally speaking, hedging a derivative security over a specified time period involves buying a hedge portfolio of securities at the outset and then periodically rebalancing its constituents according to a hedging strategy. The hedge universe is the set of tradeable and investible securities permitted to be used in the hedge portfolio. The hedge universe is chosen in a way that allows the hedge portfolio to replicate a desired exposure. For example, the hedge universe can be trivially chosen to be the set of primary securities. Alternatively, the hedge universe can be chosen as the cash account plus near dated futures contracts on each of the non-cash primary securities.

Under the Benchmark Approach (BA) and within a market comprised of $d+1$ primary securities the hedge portfolio at the outset is constructed such that its value equates to the fair price of the derivative being hedged. The hedge strategy that is adopted is typically delta hedging, whereby the dollar sensitivities of the hedge portfolio to the non-cash primary security accounts at each rebalancing time match the respective dollar sensitivities of the derivative to the non-cash primary security accounts.

The market models considered in this thesis derive jointly from a model of the short interest rate r_t and a model of the discounted GOP $\bar{S}_t^{\delta_*}$ each of whose driving processes satisfy Assumption 2.8.2. Thus the market models considered in this thesis have $d=1$ or $d=2$ according to whether the short rate is deterministic or stochastic, respectively. For the case $d=1$ the primary assets are the cash account and the GOP account and for the case $d=2$ a zero-coupon bond forms the additional primary security account. Therefore, when $d=1$ our only hedgeable risk is that pertaining to the GOP account, whereas when $d=2$ our only hedgeable risks are those pertaining to the GOP account and the zero-coupon bond account. The risk pertaining to the zero-coupon bond account can be captured by the short rate risk, and for this reason when $d=2$ we can delta hedge our derivative by maintaining an appropriate exposure in our hedge portfolio to the GOP uncertainty and the short rate uncertainty.

Let our given derivative security V that we wish to hedge be European with payoff H at expiry time $\bar{T} \in [0,T]$.

Our hedging strategy over a hedge period $[0,\bar{T}]$ involves buying a hedge portfolio consisting of a quantity of units of the domestic savings account, a quantity of units of the GOP account (and, for the case $d=2$, a quantity of units of the zero-coupon bond account) such that the risk sensitivities of the derivative to the GOP (and, for the case $d=2$, the short rate) equate to those of the hedge portfolio. This is classical delta hedging, as discussed in Hull [1997] or in Section 8.6 of Heath and Platen [2006]. Recall, B_t is the value of the savings account at time t, $S_t^{\delta_*}$ is the value of the GOP account at time t, $P(t,T)$ is the value of the T-maturity ZCB at time t and V_t is the value of derivative as at time t. The cost of the hedging strategy is computed as the value of the derivative less gains from trading the hedge portfolio, namely

$$C_t = V_t - \int_0^t (\delta_s^{(0)} dB_s + \delta_s^{(1)} dS_s^{\delta_*} + \delta_s^{(2)} dP(s,T)) \tag{2.10.75}$$

for $t \in [0,T]$ with $C_0 = V_0$. Here $\delta_t^{(0)}$ is the number of units of the domestic savings account, $\delta_t^{(1)}$ is the number of units of the GOP account and, for the case $d = 2$, $\delta_t^{(2)}$ is the number of units of the ZCB account. Equivalently the cost series can be regarded as being equal to the initial value of the hedge portfolio plus the amount required to purchase the derivative in the market in excess of the prevailing value of the hedge portfolio.

The discounted cost of hedging is given by the SDE

$$\bar{C}_t = \frac{C_t}{B_t} = \frac{V_t}{B_t} - \int_0^t (\delta_s^{(1)} d\bar{S}_s^{\delta_*} + \delta_s^{(2)} d\bar{P}(s,T)) \tag{2.10.76}$$

for $t \in [0, \bar{T}]$.

For a perfect hedge we have complete replication of the derivative payoff at expiry by the hedge portfolio, that is, $V_{\bar{T}} = V_{\bar{T}}^{(\pi)}$ under all market scenarios. Equivalently stated, for a perfect hedge we have $\bar{C}_{\bar{T}} = \bar{V}_0^{(\pi)}$ under all market scenarios.

We emphasise, under the BA there may be several self-financing portfolios π, which replicate a given payoff. We are interested in the minimum replicating portfolio whose benchmarked value $\hat{V}_t^{(\pi)}$ equates to the benchmarked value of the derivative, which is obtained from the formula

$$\hat{V}_t^{(\pi)} = \hat{V}_t = E(\hat{V}_{\bar{T}} | \mathcal{A}_t). \tag{2.10.77}$$

For this minimum replicating portfolio the cost of hedging is $C_{\bar{T}} = V_0^{(\pi)} = V_0$ under all market scenarios.

In the case when the underlying processes are Markovian, which we will always assume, the conditional expectation in (2.10.77) can be calculated by using the Feynman-Kac formula, see Heath and Platen [2006]. The resulting benchmarked pricing function is then a function of time and a number of Markovian factor processes solving a related partial differential equation (PDE). For the models we consider, this is possible.

2.10.9 Backtesting the Hedge Strategy

Given a fully specified model with known parameters, we can backtest hedging of the derivative over the time interval $[0, \bar{T}]$ by setting the $n-1$ rebalancing times

$$t_1 < t_2 < ... < t_{n-1}$$

satisfying $0 = t_0 < t_1$ and $t_{n-1} < t_n = \bar{T}$.

The hedge portfolio $V^{(\pi)}$ is adjusted at the rebalancing times and is computed iteratively using the formula

$$V_{t_i}^{(\pi)} = \delta_{t_{i-1}}^{(0)} B_{t_i} + \delta_{t_{i-1}}^{(1)} S_{t_i}^{\delta_*} + \delta_{t_{i-1}}^{(2)} P(t_i, T) \tag{2.10.78}$$

for $i = 1, 2, \ldots, n$ with initial condition

$$V_0^{(\pi)} = V_0, \tag{2.10.79}$$

where, for $i = 1, 2, \ldots n-1$, the numbers of units held in the GOP account and ZCB account at time t_i are computed as

$$\begin{aligned} \delta_{t_i}^{(1)} &= \frac{\partial}{\partial S_s^{\delta_*}} V(r_s, S_s^{\delta_*})\big|_{s=t_i} - \delta_{t_i}^{(2)} \frac{\partial}{\partial S_s^{\delta_*}} P(s,T)\big|_{s=t_i} \\ \delta_{t_i}^{(2)} &= \frac{\partial}{\partial r_s} V_s(r_s, S_s^{\delta_*})\big|_{s=t_i} \Big/ \frac{\partial}{\partial r_s} P(s,T)\big|_{s=t_i} \end{aligned} \tag{2.10.80}$$

and the number of units held in the cash account at time t_i is computed as

$$\delta_{t_i}^{(0)} = \left(V_{t_i}^{(\pi)} - \delta_{t_i}^{(1)} S_{t_i}^{\delta_*} - \delta_{t_i}^{(2)} P(t_i, T) \right) / B_{t_i}. \tag{2.10.81}$$

Since trading is not continuous hedge errors arise, which are small for small time step sizes and are neglected in the first instance.

Within this thesis we assess the performance of hedging strategies under particular market models in the following way:

We calculate the total benchmarked cost $\hat{C}_{\bar{T}}$ from start date $t = 0$ to payoff date $t = \bar{T}$. The smaller the value of $\hat{C}_{\bar{T}}$, the cheaper the dynamic hedge strategy replicates the payoff. Because the fair price of the derivative has minimal initial cost, the initial cost of hedging C_0 will be the minimal one in cost among all self-financing replicating portfolios. Also if the particular model is correct then we expect that the cost of hedging will be minimal among competing models. In this thesis we compare the costs of hedging zero-coupon bonds, equity index options and swaptions under several market models using historical data which forms the basis of our comparison of market models studied in this thesis. As a result we expect to obtain a reasonable view about which model provides on average the least expensive hedge portfolios, and whether such model goes beyond the classical risk neutral setting. We emphasise that we do not study historical fixed income data, which would be another task and is beyond the scope of the thesis. In case the market uses risk neutral pricing for long-dated contracts and one may be able to price and hedge those contracts less expensively under the BA, one first has to verify this using the available historical data.

2.11 Conclusion

We have described the pricing of derivatives using the GOP as numéraire and the real-world probability measure as pricing measure. We have seen how the supermartingale property of portfolios ensures that the portfolios do not permit strong arbitrage. Thus, by computing the expectation of a benchmarked contingent claim with respect to the real-world probability measure, we obtain the fair or real-world price for the benchmarked contingent claim which gives a price process that is minimal in the set of possible replicating portfolio processes. We have shown how to develop a dynamic trading strategy involving the underlying assets, which hedges the contingent claim to its expiry date. It is clear that the benchmark approach to hedging contingent claims differs from classical risk neutral approaches. In Chapters 7, 8 and 9 we illustrate the benefits of the benchmark approach in hedging long-dated zero-coupon bonds, swaptions and index options.

Chapter 3

Three Short Rate Models

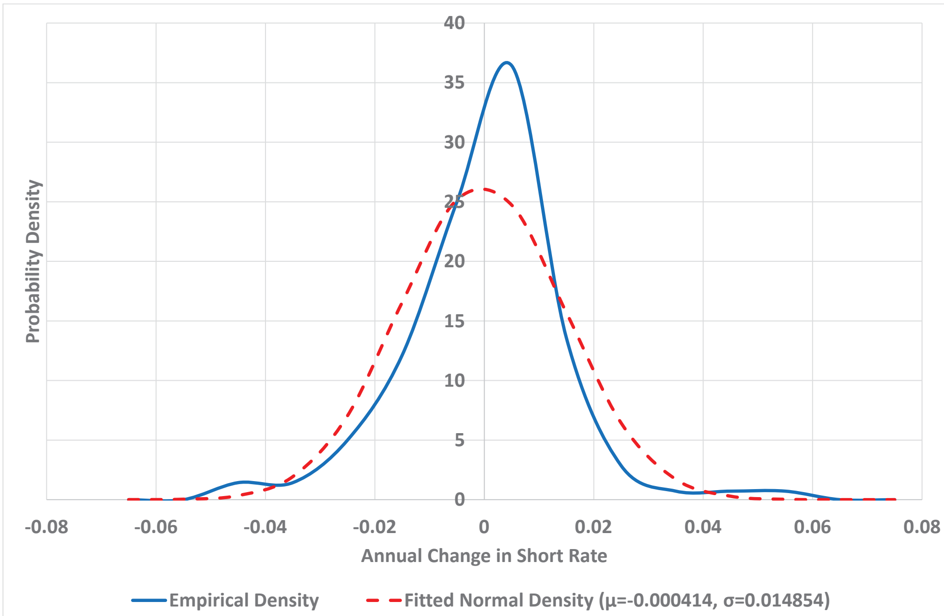
3.1 Introduction

In actuarial science the short-term interest rate plays a central role in valuations of future cashflows, particularly those pertaining to short-tail insurance policies. A short rate model is a mathematical model of the instantaneous, continuously compounded deposit rate for a specific currency. The most realistic proxy for the short rate among investible securities is probably the overnight cash deposit rate, expressed as a continuously compounded rate. Short rates are typically modelled as stochastic processes and coverages of short rate models can be found, for example, in Rebonato [1998] and Brigo and Mercurio [2006].

The short rate models considered in this chapter are specified by stochastic differential equations (SDEs) with a single noise source and with time-dependent coefficients. From an actuarial pricing perspective the availability of explicit pricing formulae for calculations involving the short-term interest rate is of extreme importance. The class of short rate dynamics where one can probably expect the widest range of explicit valuation formulae is probably the Gaussian class. They are convenient and also reasonably realistic for pricing future cash flows and contingent claims. They have explicit closed-form formulae for their transition density functions and also allow negative values. Of particular importance for actuaries is the requirement that the long-term bond yield implied by the model be a finite constant, which is guaranteed for the Vasicek model but not necessarily for extended Vasicek models. Figure 3.1 illustrates the asymmetry of the distribution of annual changes in the short rate for US cash rates, which corresponds to the leverage effect in bond markets. While this effect is not captured by extended Vasicek models, it is a short-term effect which is less pronounced when analysing the asymptotic behaviour of bond yields and volatilities.

A particular example of a Gaussian short rate model is the well-known Vasicek model, which is a linear mean reverting stochastic model, see Vasicek [1977]. This

Figure 3.1: Comparison of empirical probability density function of annual change in short rate with that of the fitted normal distribution (US 1Y cash rates 1871 - 2010).



ensures that interest rates adhere to a long run reference level.

Working on a filtered probability space $(\Omega, \underline{\mathcal{A}}, (\mathcal{A}_t)_{t\geq 0}, P)$, the SDE for the extended Vasicek short rate model or Hull-White extension is given as

$$dr_t = \kappa_t(\bar{r}_t - r_t)dt + \sigma_t dZ_t, \tag{3.1.1}$$

where r_t is the short rate at time $t \geq 0$, Z_t is a Wiener process adapted to the filtration $(\mathcal{A}_t)_{t\geq 0}$ and $\bar{r}$, κ and σ are positive deterministic functions of time. One goal in this chapter is to provide for this type of model a wide range of valuation formulae that are useful in actuarial valuations and, for suitable conditions on the model parameters, to show that the implied long-term bond yield is finite[1].

Another single-factor short rate model considered is the CIR model is a linear mean reverting stochastic model, which avoids the possibility of negative interest rates experienced in the Vasicek model.

Finally we examine the 3/2 model which also prohibits negative interest rates but is not linear mean reverting. As we will see, its inverse is linear mean reverting and, thus, adheres to a long run reference level.

[1]This work has been published in Fergusson [2017a].

We examine each of the three short rate models and, in doing so, calculate the transition density function, relevant contribution G to the ZCB price, relevant contribution to the instantaneous forward rate and the option on the ZCB contribution G. We go deliberately through all steps of the derivations, even though some of these may be well-known under risk neutral assumptions, since our pricing will be done under the real-world probability measure and not under some assumed risk neutral measure.

The availability of explicit formulae for the transition density functions makes possible the fitting of each model using maximum likelihood estimation. For estimating the drift parameters the length of the observation window is crucial. Therefore, we fit each model to Shiller's monthly data set comprised of US one-year rates from 1871 to 2012. We retain the use of this particular data set throughout the thesis because the fitted model parameters are used for backtesting hedge strategies on this same data set. The use of the one-year deposit rate as a proxy for the short rate is an assumption that is made here. The magnitude of the biases of a short-term deposit rate in lieu of the unobservable short rate was investigated in Chapman et al. [1999], where it was found not to be economically significant.

Aside from fitting three short rate models to US cash rate data over more than a century, this chapter provides convenient formulae, which are essential for pricing zero-coupon bonds, options on zero-coupon bonds and options on the GOP. An explicit formula for the fair price of a zero-coupon bond demands an explicit formula for the short rate contribution G and this will be supplied in respect of each of the three short rate models. Having explicit formulae for prices of options on the short rate contribution G allows for a single dimensional integral formula for the price of options on a zero-coupon bond. Finally, having formulae for the moment generating function of $\int_t^{\bar{T}} r_s ds$ leads to pricing formulae for options on the GOP.

3.2 Vasicek Short Rate Model and Extensions

The Vasicek model was proposed in Vasicek [1977], and extended in Hull and White [1990] to the Hull-White model whose drift and diffusion parameters are made time-dependent, which also became known as the extended Vasicek model.

This SDE (3.1.1) is the Ornstein-Uhlenbeck SDE whose explicit solution is obtained by solving the SDE of $q_t = r_t e_t$ with

$$e_t = \exp\left\{ \int_0^t \kappa_s \, ds \right\}, \tag{3.2.1}$$

where

$$dq_t = d(r_t \, e_t) = \kappa_t e_t \bar{r}_t dt + e_t \sigma_t dZ_t. \tag{3.2.2}$$

Vasicek's model, which is a special case of (3.1.1) with κ_t, $\bar{r}_t$, σ_t constant, and whose SDE is

$$dr_t = \kappa(\bar{r} - r_t)\,dt + \sigma\,dZ_t, \tag{3.2.3}$$

was probably the first interest rate model to capture mean reversion, an essential characteristic of the interest rate that sets it apart from simpler models. Thus, under the real-world probability measure, as opposed to stock prices, for instance, interest rates are not expected to rise indefinitely. This is because at very high levels they would hamper economic activity, prompting a decrease in interest rates. Similarly, interest rates are unlikely to decrease indefinitely. As a result, interest rates move mainly in a range, showing a tendency to revert to a long run value.

The drift factor $\kappa(\bar{r} - r_t)$ represents the expected instantaneous change in the interest rate at time t. The parameter $\bar{r}$ represents the long-run reference value towards which the interest rate reverts. Indeed, in the absence of uncertainty, the interest rate would remain constant when it has reached $r_t = \bar{r}$. The parameter κ, governing the speed of adjustment, needs to be positive to ensure stability around the long-term value. For example, when r_t is below $\bar{r}$, the drift term $\kappa(\bar{r} - r_t)$ becomes positive for positive κ, generating a tendency for the interest rate to move upwards.

The main disadvantage seemed that, under Vasicek's model, it is theoretically possible for the interest rate to become negative. In the previous academic literature this has been interpreted as an undesirable feature. However, on several occasions the market generated in recent years negative interest rates, for example in Switzerland and in Europe. The possibility of negative interest rates is excluded in the Cox-Ingersoll-Ross model (see Cox et al. [1985]), the exponential Vasicek model (see Brigo and Mercurio [2001]), the model of Black et al. [1990] and the model of Black and Karasinski [1991], among many others. See Brigo and Mercurio [2006] for further discussions.

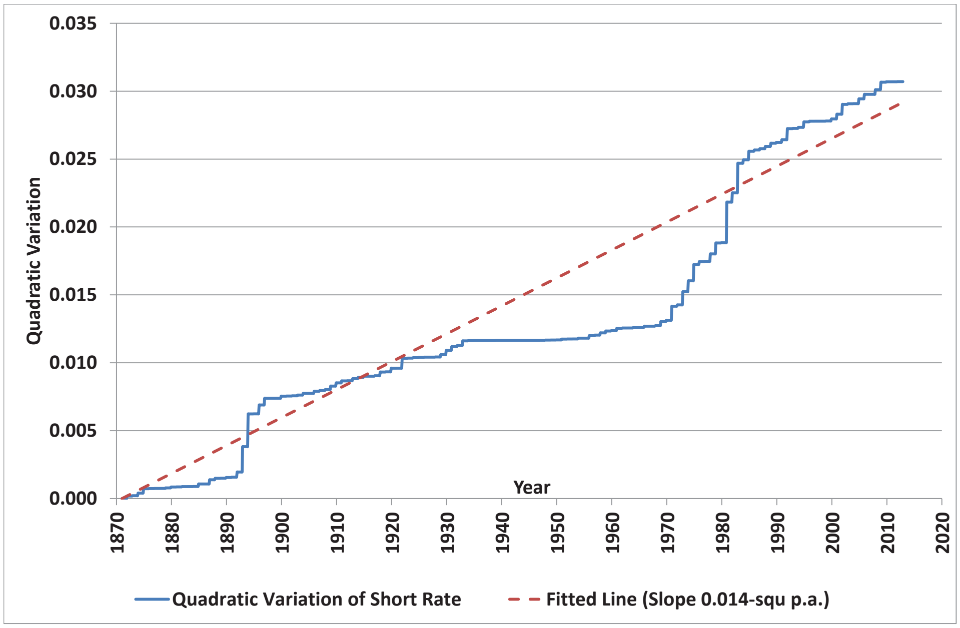
Another disadvantage is that the Vasicek model does not capture stochastic volatility, evident in the graph of the quadratic variation of the short rate in Figure 3.2. Therefore, a serious consideration of real-world dynamics would require models whose stochastic differential equations of the short rate have stochastic volatility, such as the Cox-Ingersoll-Ross model and the 3/2 model (see Platen [1999]). However, owing to the mean reverting nature of stochastic volatility, this will have less impact on the asymptotic behaviour of bond volatilities.

The Vasicek model was further extended in the Hull-White model (see Hull and White [1990]), by allowing time dependence in the drift parameters. The Hull-White model is specified by the SDE

$$dr_t = \{\theta(t) + a(t)(b - r_t)\}dt + \sigma(t)dZ_t, \tag{3.2.4}$$

where $\theta(t)$, $a(t)$ and $\sigma(t)$ are deterministic functions of t, satisfying $a(t) > 0$ and $\sigma(t) > 0$ and b is a constant. When setting $\kappa_t = a(t)$ and $\bar{r}_t = b + \theta(t)/a(t)$ in

Figure 3.2: Quadratic variation of the short rate (monthly series of US 1Y cash rates 1871 - 2012, see Data Set B in Section L.2 of Appendix).



(3.1.1) we obtain (3.2.4). Further, in (3.2.4) when setting $a(t) = 0$ and $\sigma(t)$ equal to a positive constant σ we obtain

$$dr_t = \theta(t)\,dt + \sigma dZ_t, \tag{3.2.5}$$

which is implicitly what is employed in Ho and Lee [1986].

We now provide an explicit solution to each of the SDE (3.2.3) and the SDE (3.2.4) from which we determine the associated transition density function.

3.2.1 Explicit Formula for the Short Rate

An explicit solution to the Ornstein-Uhlenbeck process is straightforwardly obtained in the following theorem.

Proposition 3.2.1 *The short rate r_t satisfying the Vasicek SDE* (3.2.3) *has solution*

$$r_t = r_s \exp(-\kappa(t-s)) + \bar{r}(1 - \exp(-\kappa(t-s))) + \sigma \int_s^t \exp(-\kappa(t-u))dZ_u \tag{3.2.6}$$

for times s and t with $0 \leq s < t$ and for positive constants $\bar{r}$, κ and σ. Here Z is the Wiener process in (3.2.3).

Proof. Integrating both sides of (3.2.2) between times s and t gives

$$r_t \exp(\kappa t) - r_s \exp(\kappa s) = \kappa \bar{r} \int_s^t \exp(\kappa u) du + \sigma \int_s^t \exp(\kappa u) dZ_u. \tag{3.2.7}$$

Multiplying both sides by $\exp(-\kappa t)$ and simplifying gives (3.2.6). **Q.E.D.**

The proof is similar for the solution to the Hull-White SDE in (3.2.4).

Proposition 3.2.2 *The short rate r_t satisfying the Hull-White SDE* (3.2.4) *has solution*

$$\begin{aligned} r_t = r_s \exp\left\{ - \int_s^t a(\tau)\, d\tau \right\} + \int_s^t \exp\left\{ - \int_u^t a(\tau)\, d\tau \right\} \{\theta(u) + a(u) b\}\, du \\ + \int_s^t \exp\left\{ - \int_u^t a(\tau)\, d\tau \right\} \sigma(u) dZ_u \end{aligned} \tag{3.2.8}$$

for times s and t with $0 \leq s < t$, for positive functions $\theta(u)$, $a(u)$ and $\sigma(u)$ and for a constant b. Here Z is the Wiener process in (3.2.4).

3.2.2 Transition Density of the Short Rate

As is the case for the Ho-Lee model in (3.2.5) and the Hull-White model in (3.2.4), the transition density function of the Vasicek short rate is that of a normal distribution.

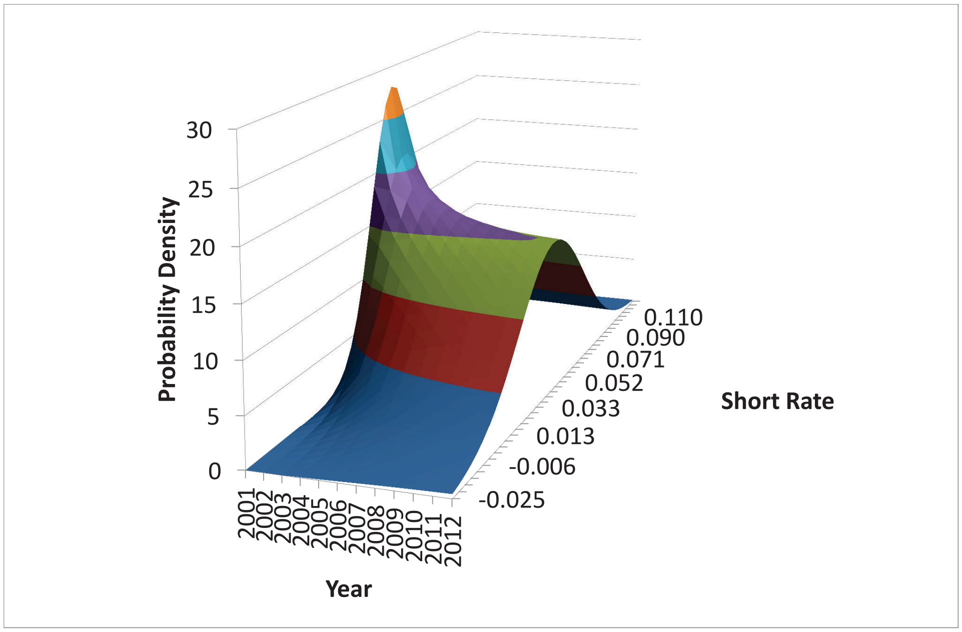
Corollary 3.2.3 *For times s and t with $0 \leq s < t \leq T$ the transition density of the short rate r_t in* (3.2.3) *is given by*

$$\begin{aligned} p_r(s, r_s, t, r_t) = & \frac{1}{\sqrt{2\pi\sigma^2 \frac{1-\exp(-2\kappa(t-s))}{2\kappa}}} \\ & \times \exp\left(-\frac{1}{2} \left(\frac{r_t - r_s \exp(-\kappa(t-s)) - \bar{r}(1 - \exp(-\kappa(t-s)))}{\sqrt{\sigma^2 \frac{1-\exp(-2\kappa(t-s))}{2\kappa}}} \right)^2 \right). \end{aligned} \tag{3.2.9}$$

Proof. From (3.2.6) we see that r_t conditioned upon r_s is normally distributed and has expected value

$$\mathrm{E}(r_t | \mathcal{A}_s) = r_s \exp(-\kappa(t-s)) + \bar{r}(1 - \exp(-\kappa(t-s))) \tag{3.2.10}$$

Figure 3.3: Vasicek transition density function of US cash rates based at year 2000 and short rate 0.064.



and variance

$$\mathrm{Var}(r_t|\mathcal{A}_s) = \sigma^2 \int_s^t \exp(-2\kappa(t-u))du = \sigma^2(1-\exp(-2\kappa(t-s)))/(2\kappa). \quad (3.2.11)$$

The transition density function must, therefore, be given by (3.2.9). **Q.E.D.**

A graph of the transition density function is shown in Figure 3.3 for parameters shown in (3.2.25).

As for other Gaussian short rate models such as the Ho-Lee model and the Hull-White model, a potential disadvantage of the Vasicek model is the possibility of negative interest rates.

A lemma, which can be deduced from Corollary 3.2.3 and which will be used later, is as follows.

Lemma 3.2.4 *For the Vasicek process in* (3.2.3) *and times* s, t *with* $s \leq t$ *let the mean and variance of* r_t *given* r_s *be defined as*

$$m_s(t) = \mathrm{E}(r_t|\mathcal{A}_s) \quad (3.2.12)$$
$$v_s(t) = \mathrm{Var}(r_t|\mathcal{A}_s) = \mathrm{E}((r_t - m_s(t))^2|\mathcal{A}_s).$$

Then we have the explicit formulae

$$m_s(t) = \bar{r}\kappa B(s,t) + r_s(1 - \kappa B(s,t)) \tag{3.2.13}$$
$$v_s(t) = \sigma^2\left(B(s,t) - \frac{1}{2}\kappa B(s,t)^2\right),$$

where

$$B(s,t) = (1 - \exp(-\kappa(t-s)))/\kappa. \tag{3.2.14}$$

Proof. See Appendix A.

For the Hull-White model we have the following corollary.

Corollary 3.2.5 *For times s and t with $0 \le s < t \le T$ the transition density of the short rate r_t in* (3.2.4) *is given by*

$$p_r(s, r_s, t, r_t) = \frac{1}{\sqrt{2\pi v_s(t)}} \times \exp\left(-\frac{1}{2}\left(\frac{r_t - m_s(t)}{\sqrt{v_s(t)}}\right)^2\right), \tag{3.2.15}$$

where

$$m_s(t) = r_s \exp\left\{-\int_s^t a(\tau)\,d\tau\right\} + \int_s^t \exp\left\{-\int_u^t a(\tau)\,d\tau\right\}\left\{\theta(u) + a(u)b\right\}du \tag{3.2.16}$$
$$v_s(t) = \int_s^t \exp\left\{-2\int_u^t a(\tau)\,d\tau\right\}\sigma(u)^2 du.$$

3.2.3 Fitting the Vasicek Model

Estimating the parameters of the Vasicek model is achieved using maximum likelihood estimation. To avoid any potential confusion we derive the estimators showing all steps. Because the transition density function of the Vasicek short rate is normal it suffices to have formulae for the conditional mean and variance, which are given in Lemma 3.2.4, and therefore our log-likelihood function under the Vasicek model for the set of observed short rates r_{t_i}, for $i = 0, 2, \ldots, n$ is

$$\ell(\bar{r}, \kappa, \sigma) = -\frac{1}{2}\sum_{i=1}^{n}\left(\log(2\pi v_{t_{i-1}}(t_i)) + \frac{(r_{t_i} - m_{t_{i-1}}(t_i))^2}{v_{t_{i-1}}(t_i)}\right) \tag{3.2.17}$$

where

$$
\begin{aligned}
m_{t_{i-1}}(t_i) &= \bar{r}\kappa B(t_{i-1}, t_i) + r_{t_{i-1}}(1 - \kappa B(t_{i-1}, t_i)) \\
v_{t_{i-1}}(t_i) &= \sigma^2 \left(B(t_{i-1}, t_i) - \frac{1}{2}\kappa B(t_{i-1}, t_i)^2 \right)
\end{aligned}
\tag{3.2.18}
$$

and $B(s, t)$ is as in (3.2.14).

The following theorem provides explicit maximum likelihood estimates (MLEs) of $\bar{r}$, κ and σ for a fixed value of $\bar{r}$.

Theorem 3.2.6 *Assume that the times $t_0 < t_1 < \ldots < t_n$ are equidistant with spacing Δ. Then the MLEs of $\bar{r}$, κ and σ are given by*

$$
\begin{aligned}
\bar{r} &= \frac{S_1 S_{00} - S_0 S_{01}}{S_0 S_1 - S_0^2 - S_{01} + S_{00}} \\
\kappa &= \frac{1}{\Delta} \log \frac{S_0 - \bar{r}}{S_1 - \bar{r}} \\
\sigma^2 &= \frac{1}{n\beta(1 - \frac{1}{2}\kappa\beta)} \sum_{i=1}^{n} (r_{t_i} - m_{t_{i-1}}(t_i))^2
\end{aligned}
\tag{3.2.19}
$$

where

$$
\begin{aligned}
S_0 &= \frac{1}{n} \sum_{i=1}^{n} r_{t_{i-1}} \\
S_1 &= \frac{1}{n} \sum_{i=1}^{n} r_{t_i} \\
S_{00} &= \frac{1}{n} \sum_{i=1}^{n} r_{t_{i-1}} r_{t_{i-1}} \\
S_{01} &= \frac{1}{n} \sum_{i=1}^{n} r_{t_{i-1}} r_{t_i}
\end{aligned}
\tag{3.2.20}
$$

and $\beta = \frac{1}{\kappa}(1 - \exp(-\kappa\Delta))$.

Proof. See Appendix A.

As a result, Theorem 3.2.6 supplies the explicit MLEs $(\hat{\bar{r}}, \hat{\kappa}, \hat{\sigma})$.

To provide standard errors of these MLEs, we note that their variances satisfy the Cramér-Rao inequality

$$
\mathrm{Var}((\hat{\bar{r}}, \hat{\kappa}, \hat{\sigma})) \geq \frac{1}{\mathcal{I}(\bar{r}, \kappa, \sigma)}, \tag{3.2.21}
$$

where $\mathcal{I}(\bar{r}, \kappa, \sigma)$ is the Fisher information matrix. As the number of observations approaches infinity the variance is asymptotic to the lower bound. Also the Fisher information matrix is approximated by the observed Fisher information matrix

$$\mathcal{I}(\bar{r}, \kappa, \sigma) \approx \mathcal{I}(\hat{\bar{r}}, \hat{\kappa}, \hat{\sigma}) = -\nabla^2 \ell(\hat{\bar{r}}, \hat{\kappa}, \hat{\sigma}). \tag{3.2.22}$$

The following theorem supplies the observed Fisher information matrix in respect of MLEs of the Vasicek model.

Theorem 3.2.7 *The observed Fisher information matrix in respect of the MLEs in Theorem 3.2.6 is given by*

$$\begin{pmatrix} \frac{n\kappa^2\beta}{\sigma^2(1-\frac{1}{2}\kappa\beta)} & \frac{m_{\bar{r}}}{v}\Delta\, \exp(-\kappa\Delta)(\bar{r}-S_0) & 0 \\ \frac{m_{\bar{r}}}{v}\Delta\, \exp(-\kappa\Delta)(\bar{r}-S_0) & \frac{n}{v}\left(\Delta\, \exp(-\kappa\Delta)\right)^2(\bar{r}^2 - 2\bar{r}S_0 + S_{00}) - \frac{n}{2v^2}v_\kappa^2 & \frac{1}{2}v_\sigma v_\kappa \frac{n}{v^2} \\ 0 & \frac{1}{2}v_\sigma v_\kappa \frac{n}{v^2} & \frac{2n}{\sigma^2} \end{pmatrix} \tag{3.2.23}$$

where

$$\begin{aligned} \beta &= \frac{1}{\kappa}(1 - \exp(-\kappa\,\Delta)) \\ v &= \sigma^2\beta(1 - \frac{1}{2}\kappa\beta) \\ m_{\bar{r}} &= 1 - \exp(-\kappa\,\Delta) \\ v_\sigma &= \frac{2v}{\sigma} \\ v_\kappa &= -\frac{1}{\kappa}v + 2\,\Delta\, v + \frac{\sigma^2}{\kappa}\Delta \end{aligned} \tag{3.2.24}$$

and we have assumed that the times $t_0 < t_1 < \ldots < t_n$ are equidistant with spacing Δ.

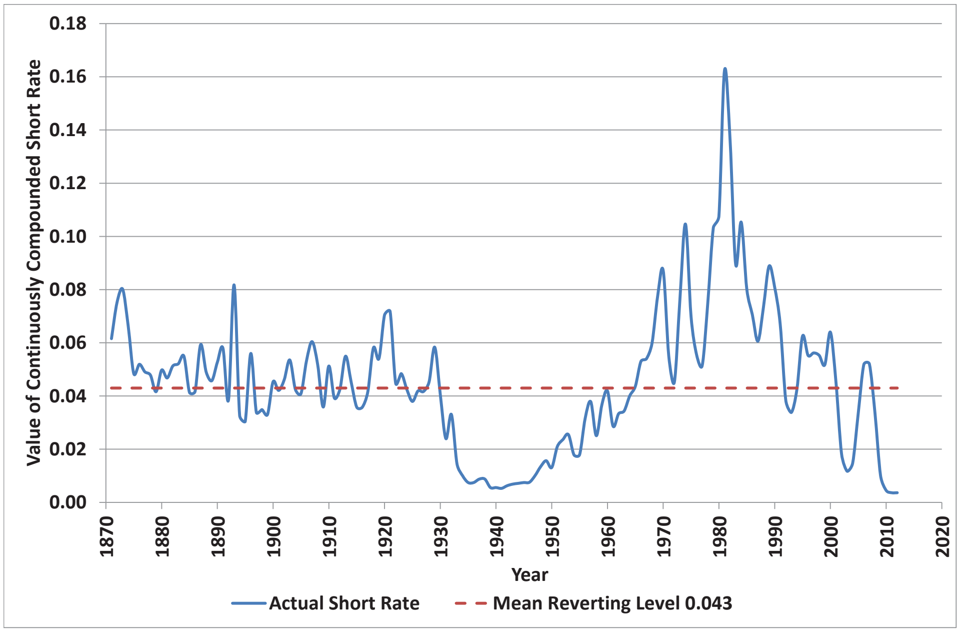
Proof. See Appendix A.

We fit the Vasicek model to the annual series of one-year deposit rates from 1871 to 2012, referred to as Data Set A in Section L.1 of Appendix L. We obtain the MLEs, with standard errors shown in brackets,

$$\begin{aligned} \bar{r} &= 0.042994\,(0.0080023) \\ \kappa &= 0.162953\,(0.053703) \\ \sigma &= 0.015384\,(0.00099592). \end{aligned} \tag{3.2.25}$$

Remark 3.2.8 *Fitting the Vasicek model to the monthly series of one-year deposit rates from 1871 to 2017, referred to as Data Set C in Section L.3 of Appendix L, we obtain the MLEs, with standard errors shown in brackets,*

Figure 3.4: Actual short rate and fitted Vasicek mean reverting level for US cash rates.



$$
\begin{aligned}
\bar{r} &= 0.043285\,(0.008696) \\
\kappa &= 0.148315\,(0.045877) \\
\sigma &= 0.015537\,(0.000234).
\end{aligned}
\tag{3.2.26}
$$

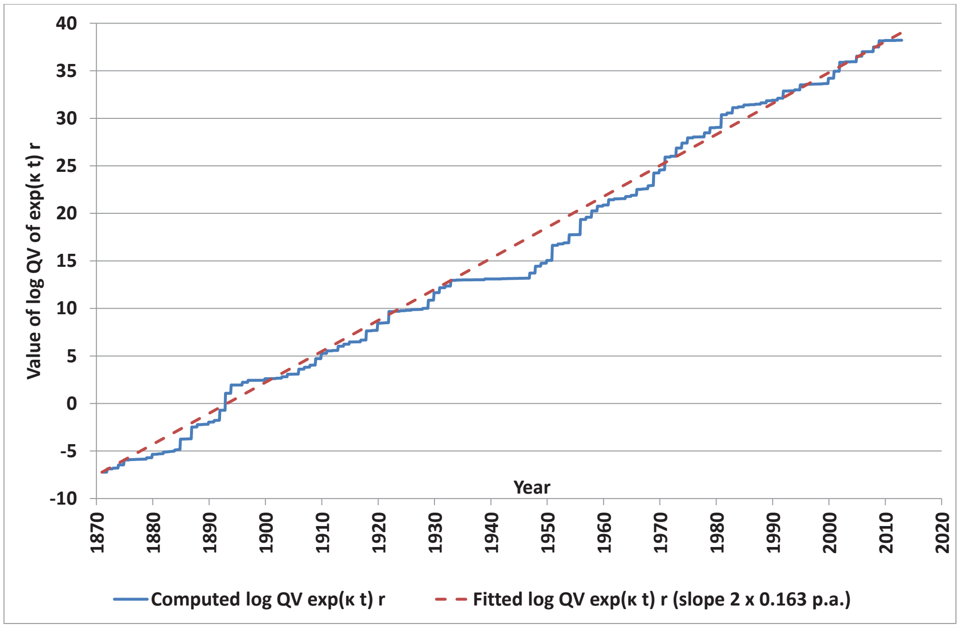
Thus, we see good agreement between the respective estimates based on annual and monthly data in (3.2.25) *and* (3.2.26).

We show the parameter estimate for the mean reverting level $\bar{r}$ alongside the historical short rates in Figure 3.4. We note that for the periods after 1930 a time-dependent reference level may be appropriate but we deliberately keep constant parameters in this thesis to clarify firstly this case for the different models[2].

One way of assessing the goodness of fit of the parameter estimates for σ and κ in a graphical fashion is to compare the theoretical quadratic variation of $q_t = r_t \exp(\kappa t)$ with the observed quadratic variation.

[2]Time dependency in the reference level has been addressed in Fergusson [2017a] and Fergusson [2018] for the Vasicek and CIR models respectively.

Figure 3.5: Logarithm of quadratic variation of $\exp(\kappa t) r_t$.



From the SDE (3.2.2) the theoretical quadratic variation of $q_t = r_t \exp(\kappa t)$ is

$$[q]_t = \sigma^2 \int_0^t \exp(2\kappa s) ds = \sigma^2 \frac{\exp(2\kappa t) - 1}{2\kappa} \tag{3.2.27}$$

and the observed quadratic variation of q is computed using the formula

$$[q]_t \approx \sum_{j:t_j \leq t} (r_{t_j} \exp(\kappa t_j) - r_{t_{j-1}} \exp(\kappa t_{j-1}))^2. \tag{3.2.28}$$

The logarithm of the observed quadratic variation of $q_t = r_t \exp(\kappa t)$ in (3.2.28) is shown alongside the logarithm of the fitted quadratic variation function (3.2.27) in Figure 3.5. We note that we have visually a good fit.

3.2.4 The Savings Account and its Transition Density

The savings account consists of the dollar wealth accumulated continuously at the short rate, given an initial deposit of one dollar at time zero. The value of the savings account at time t is given in (2.2.2). The following lemma leads to the formula for the savings account value under the Vasicek model.

Lemma 3.2.9 *Let r_t satisfy the Vasicek SDE* (3.2.3). *Then*

$$\int_t^{\bar{T}} r_s ds = r_t B(t,\bar{T}) + \bar{r}(\bar{T} - t - B(t,\bar{T})) + \sigma \int_t^{\bar{T}} B(u,\bar{T}) dZ_u, \tag{3.2.29}$$

where

$$B(t,\bar{T}) = \frac{1}{\kappa}(1 - \exp(-\kappa(\bar{T} - t))). \tag{3.2.30}$$

Proof. From (3.2.6) we have for $s \in [t,\bar{T}]$

$$r_s = r_t \exp(-\kappa(s-t)) + \bar{r}(1 - \exp(-\kappa(s-t))) + \sigma \int_t^s \exp(-\kappa(s-u)) dZ_u. \tag{3.2.31}$$

Integrating both sides with respect to s between t and $\bar{T}$ gives

$$\begin{aligned}
&\int_t^{\bar{T}} r_s ds \qquad (3.2.32)\\
&= \int_t^{\bar{T}} \Big(r_t \exp(-\kappa(s-t)) + \bar{r}(1 - \exp(-\kappa(s-t))) \\
&+ \sigma \int_t^s \exp(-\kappa(s-u)) dZ_u \Big) ds \\
&= \int_t^{\bar{T}} r_t \exp(-\kappa(s-t)) ds + \bar{r}(\bar{T} - t) + \bar{r}\frac{1}{\kappa}(\exp(-\kappa(\bar{T} - t)) - 1) \\
&+ \sigma \int_t^{\bar{T}} \int_t^s \exp(-\kappa(s-u)) dZ_u \, ds \\
&= r_t \frac{1}{\kappa}(1 - \exp(-\kappa(\bar{T} - t))) + \bar{r}(\bar{T} - t - \frac{1}{\kappa}(1 - \exp(-\kappa(\bar{T} - t))) \\
&+ \sigma \frac{1}{\kappa} \int_t^{\bar{T}} (1 - \exp(-\kappa(\bar{T} - u))) dZ_u
\end{aligned}$$

which completes the proof. **Q.E.D.**

A similar lemma applies to the Hull-White model.

Lemma 3.2.10 *Let r_t satisfy the Hull-White SDE* (3.2.4). *Then*

$$\int_t^{\bar{T}} r_s ds = r_t B(t,\bar{T}) + \int_t^{\bar{T}} B(u,\bar{T}) \big\{ \theta(u) + a(u) b \big\} \, du \int_t^{\bar{T}} B(u,\bar{T}) \sigma(u) dZ_u, \tag{3.2.33}$$

where

$$B(t,\bar{T}) = \int_t^{\bar{T}} \exp \Big\{ - \int_t^s a(\tau) \, d\tau \Big\} ds. \tag{3.2.34}$$

The following proposition provides the formula for the savings account under the Vasicek short rate model.

Proposition 3.2.11 *Let r_t satisfy the Vasicek SDE* (3.2.3). *Then the SDE*

$$dB_t = r_t B_t dt \tag{3.2.35}$$

of the savings account B_t has the solution

$$B_{\bar{T}} = B_t \exp\left(r_t B(t,\bar{T}) + \bar{r}(\bar{T} - t - B(t,\bar{T})) + \sigma \int_t^{\bar{T}} B(u,\bar{T}) dZ_u \right) \tag{3.2.36}$$

where $B(t,\bar{T})$ is as in (3.2.30).

Proof. Combining (3.2.29) and (2.2.2) gives the formula for the savings account as

$$\begin{aligned} B_{\bar{T}} &= B_t \exp\left(\int_t^{\bar{T}} r_s ds \right) &(3.2.37)\\ &= B_t \exp\left(r_t B(t,\bar{T}) + \bar{r}(\bar{T} - t - B(t,\bar{T})) + \sigma \int_t^{\bar{T}} B(u,\bar{T}) dZ_u \right), \end{aligned}$$

which completes the proof. **Q.E.D.**

From (3.2.36) we immediately see that the transition density function

$$p_B(t, B_t, \bar{T}, B_{\bar{T}}) \tag{3.2.38}$$

of the savings account value is a lognormal density function.

Proposition 3.2.12 *Let r_t satisfy the Vasicek SDE* (3.2.3). *Then the transition density function of the savings account value $B_{\bar{T}}$ is*

$$p_B(t, B_t, \bar{T}, B_{\bar{T}}) = \frac{1}{B_{\bar{T}} \sqrt{2\pi v(t,\bar{T})}} \exp\left(-\frac{1}{2} \left(\log(B_{\bar{T}}/B_t) - m(t,\bar{T}) \right)^2 / v(t,\bar{T}) \right), \tag{3.2.39}$$

where

$$\begin{aligned} m(t,\bar{T}) &= r_t B(t,\bar{T}) + \bar{r}(\bar{T} - t - B(t,\bar{T})) &(3.2.40)\\ v(t,\bar{T}) &= \frac{\sigma^2}{\kappa^2} \left(\bar{T} - t - B(t,\bar{T}) - \frac{1}{2} \kappa B(t,\bar{T})^2 \right). \end{aligned}$$

Proof. From (3.2.36) we can write

$$B_{\bar{T}} = B_t \exp\left(m(t,\bar{T}) + \sqrt{v(t,\bar{T})}Z\right), \tag{3.2.41}$$

where

$$\begin{aligned} m(t,\bar{T}) &= r_t B(t,\bar{T}) + \bar{r}(\bar{T} - t - B(t,\bar{T})) \\ v(t,\bar{T}) &= \sigma^2 \int_t^{\bar{T}} B(u,\bar{T})^2 du \end{aligned} \tag{3.2.42}$$

and Z is a standard normal random variable. We can simplify the squared volatility $v(t,\bar{T})$ as follows

$$\begin{aligned} v(t,\bar{T}) &= \sigma^2 \int_t^{\bar{T}} \frac{1}{\kappa^2}(1 - \exp(-\kappa(\bar{T}-u)))^2 du \\ &= \frac{\sigma^2}{\kappa^2}\left((\bar{T}-t) - 2\frac{1-\exp(-\kappa(\bar{T}-t))}{\kappa} + \frac{1-\exp(-2\kappa(\bar{T}-t))}{2\kappa}\right) \\ &= \frac{\sigma^2}{\kappa^2}\left((\bar{T}-t) - 2B(t,\bar{T}) + \frac{1-(1-\kappa B(t,\bar{T}))^2}{2\kappa}\right) \\ &= \frac{\sigma^2}{\kappa^2}\left(\bar{T} - t - B(t,\bar{T}) - \frac{1}{2}\kappa B(t,\bar{T})^2\right) \end{aligned} \tag{3.2.43}$$

and we have the result. **Q.E.D.**

Therefore, we can write the conditional distribution of the savings account value as

$$\log B_{\bar{T}} \sim N\left(\log B_t + m(t,\bar{T}), v(t,\bar{T})\right) \tag{3.2.44}$$

given B_t for $m(t,\bar{T})$ and $v(t,\bar{T})$ as in (3.2.40), where $N(m,v)$ denotes the Gaussian distribution with mean m and variance v.

Analogously, for the Hull-White model we can write the conditional distribution of the savings account value as

$$\log B_{\bar{T}} \sim N\left(\log B_t + m(t,\bar{T}), v(t,\bar{T})\right) \tag{3.2.45}$$

given B_t, where $m(t,\bar{T})$ and $v(t,\bar{T})$ are given by

$$\begin{aligned} m(t,\bar{T}) &= r_t \int_t^{\bar{T}} \exp\left\{-\int_t^s a(\tau)\,d\tau\right\} ds \\ &\quad + \int_t^{\bar{T}} \left[\int_t^s \exp\left\{-\int_u^s a(\tau)\,d\tau\right\}\big(\theta(u) + a(u)b\big)\,du\right] ds \\ v(t,\bar{T}) &= \int_t^{\bar{T}} \left[\int_u^{\bar{T}} \exp\left\{-\int_u^s a(\tau)\,d\tau\right\} ds\right]^2 \sigma(u)^2\,du. \end{aligned} \tag{3.2.46}$$

The transition density of the savings account allows us to calculate the contribution of the short rate to the zero-coupon bond price in the following section.

3.2.5 Short Rate Contribution to ZCB Price

In the following lemma we calculate the contribution $G_{\bar{T}}(t)$ to the zero-coupon bond price which is due to the short rate.

Lemma 3.2.13 *For time $t \in [0, \bar{T}]$ the short rate contribution to the ZCB price is*

$$G_{\bar{T}}(t) = A(t, \bar{T}) \exp(-r_t B(t, \bar{T})), \tag{3.2.47}$$

where

$$B(t, \bar{T}) = \frac{1 - \exp(-\kappa(\bar{T} - t))}{\kappa} \tag{3.2.48}$$

and

$$A(t, \bar{T}) = \exp\left((\bar{r} - \frac{\sigma^2}{2\kappa^2})(B(t, \bar{T}) - \bar{T} + t) - \frac{\sigma^2}{4\kappa} B(t, \bar{T})^2 \right). \tag{3.2.49}$$

Proof. From (3.2.44)

$$\log B_{\bar{T}} \sim N(\log B_t + m(t, \bar{T}), v(t, \bar{T})) \tag{3.2.50}$$

given B_t and using (2.8.4) we have

$$\begin{aligned}
G_{\bar{T}}(t) &= E\left(\frac{B_t}{B_{\bar{T}}} \middle| \mathcal{A}_t \right) \qquad (3.2.51)\\
&= B_t E\left(\exp(-\log B_{\bar{T}}) \middle| \mathcal{A}_t \right)\\
&= B_t \exp(-\mathrm{E}(\log B_{\bar{T}} | \mathcal{A}_t) + \frac{1}{2}\mathrm{Var}(B_{\bar{T}} | \mathcal{A}_t))\\
&= B_t \exp(-\log B_t - m(t, \bar{T}) + \frac{1}{2} v(t, \bar{T}))\\
&= \exp(-m(t, \bar{T}) + \frac{1}{2} v(t, \bar{T}))\\
&= \exp\Bigg(- r_t B(t, \bar{T}) - \bar{r}(\bar{T} - t - B(t, \bar{T}))\\
&\quad + \frac{\sigma^2}{2\kappa^2}\left(\bar{T} - t - B(t, \bar{T}) - \frac{1}{2}\kappa B(t, \bar{T})^2 \right)\Bigg)\\
&= \exp(-r_t B(t, \bar{T})) \exp\left((-\bar{r} + \frac{\sigma^2}{2\kappa^2})(\bar{T} - t - B(t, \bar{T})) - \frac{\sigma^2}{4\kappa} B(t, \bar{T})^2 \right)
\end{aligned}$$

which is the result. **Q.E.D.**

A similar result can be proven for the Hull-White short rate model.

Lemma 3.2.14 *Let r_t satisfy the Hull-White SDE* (3.2.4). *Then for time $t \in [0, \bar{T}]$ the $\bar{T}$-maturity ZCB price is*

$$G_{\bar{T}}(t) = A(t, \bar{T}) \exp(-r_t B(t, \bar{T})), \tag{3.2.52}$$

where

$$B(t, \bar{T}) = \int_t^{\bar{T}} \exp\Big\{ - \int_t^s a(\tau)\, d\tau \Big\}\, ds \tag{3.2.53}$$

and

$$A(t, \bar{T}) = \exp\bigg(- \int_t^{\bar{T}} \Big[\int_t^s \exp\Big\{ - \int_u^s a(\tau)\, d\tau \Big\} \big(\theta(u) + a(u) b\big)\, du \Big]\, ds \tag{3.2.54}$$
$$+ \frac{1}{2} \int_t^{\bar{T}} B(u, \bar{T})^2 \sigma(u)^2\, du \bigg).$$

3.2.6 Short Rate Contribution to Bond Yields and Forward Rates

We investigate the asymptotic level of the yield curve under the Vasicek model. As a corollary of Lemma 3.2.13 we calculate the $\bar{T}$-maturity ZCB yield $h_{\bar{T}}(t)$, as given in

$$h_{\bar{T}}(t) = - \frac{1}{\bar{T} - t} \log G_{\bar{T}}(t), \tag{3.2.55}$$

as $\bar{T} \to \infty$, which we call the long ZCB yield.

Corollary 3.2.15 *Let r_t satisfy the Vasicek SDE* (3.2.3). *Then the long ZCB yield is*

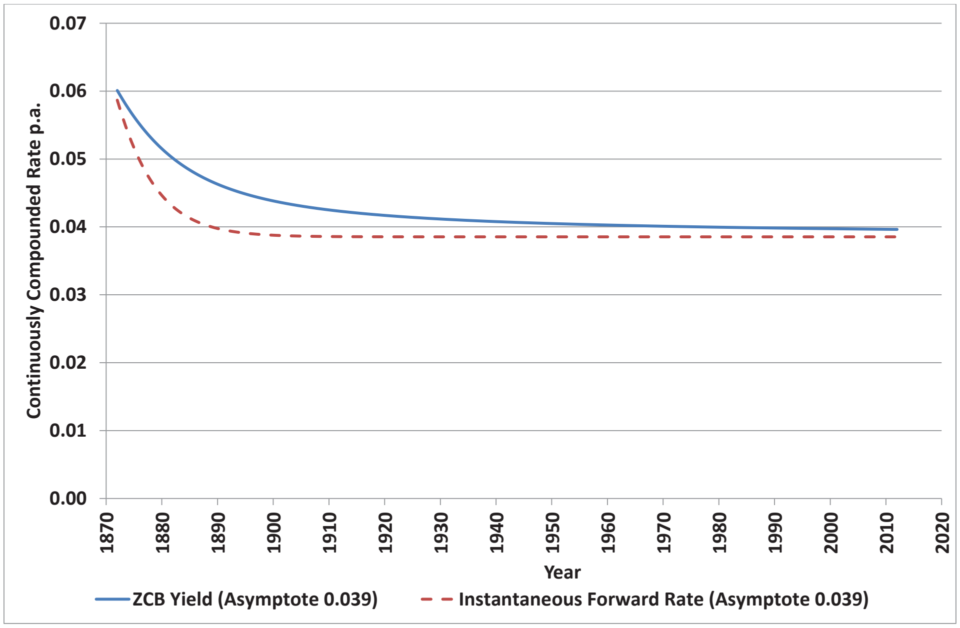
$$h_\infty(t) = \bar{r} - \frac{\sigma^2}{2\kappa^2}. \tag{3.2.56}$$

Proof. From (3.2.55), the ZCB yield is given by the formula

$$\begin{aligned} h_\infty(t) &= - \lim_{\bar{T} \to \infty} \frac{1}{\bar{T} - t} \log G_{\bar{T}}(t) \\ &= \lim_{\bar{T} \to \infty} \frac{1}{\bar{T} - t} (- \log A(t, \bar{T}) + r_t B(t, \bar{T})) \\ &= \lim_{\bar{T} \to \infty} r_t \frac{B(t, \bar{T})}{\bar{T} - t} - (\bar{r} - \frac{\sigma^2}{2\kappa^2}) \frac{B(t, \bar{T}) - \bar{T} + t}{\bar{T} - t} + \frac{\sigma^2}{4\kappa(\bar{T} - t)} B(t, \bar{T})^2. \end{aligned} \tag{3.2.57}$$

But $\lim_{\bar{T} \to \infty} B(t, \bar{T}) = \frac{1}{\kappa}$ and, therefore, the long ZCB yield simplifies to $\bar{r} - \frac{\sigma^2}{2\kappa^2}$. **Q.E.D.**

Figure 3.6: Zero coupon yield curve under the Vasicek model based at 1871.



In Figure 3.6 the continuously compounded yield curve is plotted as at the time of 1871. We have an inverted yield curve and this portends an economic recession because decreasing forward rates indicate expectations of low inflation and low economic growth, as discussed in Harvey [1991].

We calculate the forward rate $g_{\bar{T}}(t)$, given by

$$g_{\bar{T}}(t) = -\frac{\partial}{\partial \bar{T}} \log G_{\bar{T}}(t). \qquad (3.2.58)$$

Lemma 3.2.16 *For time $t \in [0, \bar{T}]$ the forward rate is computed to be*

$$g_{\bar{T}}(t) = (r_t - \bar{r}) \exp(-\kappa(\bar{T} - t)) + \bar{r} - \frac{\sigma^2}{2\kappa^2}\left(1 - \exp(-\kappa(\bar{T} - t))\right)^2. \qquad (3.2.59)$$

Proof. Using (3.2.58) and (3.2.47) we have

$$
\begin{aligned}
g_{\bar{T}}(t) &= -\frac{\partial}{\partial \bar{T}} \log G_{\bar{T}}(t) \\
&= -\frac{\partial}{\partial \bar{T}}\Bigg\{ -(r_t-\bar{r})\frac{(1-\exp(-\kappa(\bar{T}-t)))}{\kappa} - \bar{r}(\bar{T}-t) \\
&+ \frac{\sigma^2}{2\kappa^2}\bigg((\bar{T}-t) - 2\frac{1-\exp(-\kappa(\bar{T}-t))}{\kappa} + \frac{1-\exp(-2\kappa(\bar{T}-t))}{2\kappa}\bigg)\Bigg\} \\
&= (r_t-\bar{r})\exp(-\kappa(\bar{T}-t)) + \bar{r} \\
&- \frac{\sigma^2}{2\kappa^2}\Big(1 - 2\exp(-\kappa(\bar{T}-t)) + \exp(-2\kappa(\bar{T}-t))\Big)
\end{aligned}
\tag{3.2.60}
$$

and simplifying gives the result. **Q.E.D.**

As a corollary of this lemma we calculate directly the asymptotic instantaneous forward rate.

Corollary 3.2.17 *For the Vasicek short rate model, the asymptotic instantaneous forward rate is*

$$
g_\infty(t) = \bar{r} - \frac{\sigma^2}{2\kappa^2}. \tag{3.2.61}
$$

In Figure 3.6 the instantaneous forward rate $g_{\bar{T}}$ is plotted and can be seen to be asymptotic to $g_\infty(t) = 0.0385$ based upon the parameters in (3.2.25).

3.2.7 Expectations Involving $G_{\bar{T}}(t)$

Motivated by pricing call and put options on zero-coupon bonds in Chapter 5 we seek formulae for the following expectations

$$
\begin{aligned}
f_1(t,T,K,\bar{T}) &= \mathrm{E}\bigg(\exp\Big(-\int_t^T r_s ds\Big) G_{\bar{T}}(T)\mathbf{1}_{G_{\bar{T}}(T)>K} \bigg| \mathcal{A}_t\bigg) \\
f_2(t,T,K,\bar{T}) &= \mathrm{E}\bigg(\exp\Big(-\int_t^T r_s ds\Big) G_{\bar{T}}(T)\mathbf{1}_{G_{\bar{T}}(T)\le K} \bigg| \mathcal{A}_t\bigg) \\
f_3(t,T,K,\bar{T}) &= \mathrm{E}\bigg(\exp\Big(-\int_t^T r_s ds\Big) \mathbf{1}_{G_{\bar{T}}(T)>K} \bigg| \mathcal{A}_t\bigg) \\
f_4(t,T,K,\bar{T}) &= \mathrm{E}\bigg(\exp\Big(-\int_t^T r_s ds\Big) \mathbf{1}_{G_{\bar{T}}(T)\le K} \bigg| \mathcal{A}_t\bigg) \\
f_5(t,T,K,\bar{T}) &= \mathrm{E}\bigg(\exp\Big(-\int_t^T r_s ds\Big) \big(G_{\bar{T}}(T)-K\big)^+ \bigg| \mathcal{A}_t\bigg) \\
f_6(t,T,K,\bar{T}) &= \mathrm{E}\bigg(\exp\Big(-\int_t^T r_s ds\Big) \big(K-G_{\bar{T}}(T)\big)^+ \bigg| \mathcal{A}_t\bigg),
\end{aligned}
\tag{3.2.62}
$$

where $0 \leq t < T < \bar{T}$ and $K > 0$. These expectations correspond to prices of various call and put options on zero-coupon bonds under the Vasicek short rate model.

It is well known that the Vasicek short rate model is an example of a Gaussian interest rate model and that for such models the prices of call options on zero-coupon bonds employ the Black-Scholes option pricing formula. It is this formula that we establish here, employing the following three lemmas and subsequent corollary.

Lemma 3.2.18 *Let Y be a normally distributed random variable. Then for any real number y we have*

$$\mathrm{E}(\exp(Y)\mathbf{1}_{Y\leq y}) = \mathrm{E}(\exp(Y)) \times \mathrm{E}(\mathbf{1}_{Y\leq y-\mathrm{Var}(Y)}) \tag{3.2.63}$$

and

$$\mathrm{E}(\exp(Y)\mathbf{1}_{Y>y}) = \mathrm{E}(\exp(Y)) \times \mathrm{E}(\mathbf{1}_{Y>y-\mathrm{Var}(Y)}). \tag{3.2.64}$$

Proof. See Appendix B.

In the following lemma we state an extension of the above lemma which we prove later in the thesis.

Lemma 3.2.19 *Let Y_1 and Y_2 be normally distributed random variables. Then for any real number y,*

$$\mathrm{E}(\exp(Y_1)\mathbf{1}_{Y_2\leq y}) = \mathrm{E}(\exp(Y_1)) \times \mathrm{E}(\mathbf{1}_{Y_2\leq y-\mathrm{Cov}(Y_1,Y_2)}). \tag{3.2.65}$$

Also we have

$$\mathrm{E}(\exp(Y_1)\mathbf{1}_{Y_2>y}) = \mathrm{E}(\exp(Y_1)) \times \mathrm{E}(\mathbf{1}_{Y_2>y-\mathrm{Cov}(Y_1,Y_2)}). \tag{3.2.66}$$

Proof. See Appendix B.

We can readily prove the pricing formulae for ZCB call and put options using Lemma 3.2.19 when the integral of the short rate is a normally distributed random variable whose variance parameter is a deterministic function, that is when the following condition holds:

Condition 3.2.20 *The integral $\int_t^T r_s ds$ is normally distributed, that is,*

$$\int_t^T r_s ds \sim N\big(m, v\big), \tag{3.2.67}$$

where the parameter v is a deterministic function involving the parameters $\bar{r}$, κ, σ, t and T.

This condition is satisfied by the Ho-Lee short rate model, the Hull-White short rate model and various extended versions of these. Therefore, our lemmas apply to these models, which result in a proof of the Black-Scholes formula for options on zero-coupon bonds under each of these models.

Lemma 3.2.21 *Let r_t be a process for the short rate which satisfies Condition 3.2.20 and let*

$$G_{\bar{T}}(T) = \mathrm{E}\left(\exp\left(-\int_T^{\bar{T}} r_s ds\right)\Big|\mathcal{A}_T\right). \tag{3.2.68}$$

Then the random variable L conditional on information up to time t, given by

$$L = \log G_{\bar{T}}(T), \tag{3.2.69}$$

is normally distributed whose expected value satisfies

$$\begin{aligned} \mathrm{E}(L|\mathcal{A}_t) &= \log G_{\bar{T}}(t)/G_T(t) \\ &\quad - \frac{1}{2}\mathrm{Var}(L|\mathcal{A}_t) + \mathrm{Cov}\left(L, \int_t^T r_s ds\Big|\mathcal{A}_t\right) \end{aligned} \tag{3.2.70}$$

and whose variance $\mathrm{Var}(L|\mathcal{A}_t)$ *satisfies*

$$\mathrm{Var}(L|\mathcal{A}_t) = \mathrm{Var}\left(\mathrm{E}\left(\int_T^{\bar{T}} r_s ds\Big|\mathcal{A}_T\right)\Big|\mathcal{A}_t\right). \tag{3.2.71}$$

Proof. See Appendix B.

Theorem 3.2.22 *Let r_t be a process for the short rate which satisfies Condition 3.2.20. Then the formulae for the expectations f_1 and f_2 in* (3.2.62) *are given by*

$$\begin{aligned} f_1(t, T, K, \bar{T}) &= G_{\bar{T}}(t) N(d_1) \\ f_2(t, T, K, \bar{T}) &= G_{\bar{T}}(t)(1 - N(d_1)), \end{aligned} \tag{3.2.72}$$

where

$$\begin{aligned} d_1 &= \frac{1}{2}\sigma_G + \frac{1}{\sigma_G}\log\frac{G_{\bar{T}}(t)}{G_T(t)K} \\ \sigma_G^2 &= \mathrm{Var}\left(\mathrm{E}\left(\int_T^{\bar{T}} r_s ds\Big|\mathcal{A}_T\right)\Big|\mathcal{A}_t\right). \end{aligned} \tag{3.2.73}$$

Proof. See Appendix B.

Theorem 3.2.23 *Let r_t be a process for the short rate which satisfies Condition 3.2.20. Then the formulae for the expectations f_3 and f_4 in* (3.2.62) *are given by*

$$\begin{aligned} f_3(t,T,K,\bar{T}) &= G_T(t)N(d_2) \\ f_4(t,T,K,\bar{T}) &= G_T(t)(1-N(d_2)), \end{aligned} \tag{3.2.74}$$

where

$$\begin{aligned} d_2 &= -\frac{1}{2}\sigma_G + \frac{1}{\sigma_G}\log\frac{G_{\bar{T}}(t)}{G_T(t)K} \\ \sigma_G^2 &= \mathrm{Var}\left(\mathrm{E}\left(\int_T^{\bar{T}} r_s ds \middle| \mathcal{A}_T\right) \middle| \mathcal{A}_t\right). \end{aligned} \tag{3.2.75}$$

Proof. See Appendix B.

Theorem 3.2.24 *Let r_t be a process for the short rate which satisfies Condition 3.2.20. Then the formulae for the expectations f_5 and f_6 in* (3.2.62) *are given by*

$$\begin{aligned} f_5(t,T,K,\bar{T}) &= G_{\bar{T}}(t)N(d_1) - KG_T(t)N(d_2) \\ f_6(t,T,K,\bar{T}) &= -G_{\bar{T}}(t)(1-N(d_1)) + KG_T(t)(1-N(d_2)), \end{aligned} \tag{3.2.76}$$

where

$$\begin{aligned} d_1 &= \frac{1}{2}\sigma_G + \frac{1}{\sigma_G}\log\frac{G_{\bar{T}}(t)}{G_T(t)K} \\ d_2 &= -\frac{1}{2}\sigma_G + \frac{1}{\sigma_G}\log\frac{G_{\bar{T}}(t)}{G_T(t)K} \\ \sigma_G^2 &= \mathrm{Var}\left(\mathrm{E}\left(\int_T^{\bar{T}} r_s ds \middle| \mathcal{A}_T\right) \middle| \mathcal{A}_t\right). \end{aligned} \tag{3.2.77}$$

Proof. See Appendix B.

When the short rate obeys a Vasicek process, the pricing formulae for call and put options on the ZCB contribution $G_{\bar{T}}$ follow as a corollary of the above theorem.

Corollary 3.2.25 *Under the Vasicek model, for a strike price K and valuation time t, the price of a T-expiry call option on a $\bar{T}$-maturity ZCB is*

$$c_{T,K,G_{\bar{T}}}(t) = G_{\bar{T}}(t)N(h) - KG_T(t)N(h-\sigma_G) \tag{3.2.78}$$

and the price of a T-expiry put option on a $\bar{T}$-maturity ZCB is

$$p_{T,K,G_{\bar{T}}}(t) = -G_{\bar{T}}(t)N(-h) + KG_T(t)N(-h+\sigma_G), \tag{3.2.79}$$

where

$$h = \frac{1}{\sigma_G} \log \frac{G_{\bar{T}}(t)}{G_T(t)K} + \frac{1}{2}\sigma_G \tag{3.2.80}$$

$$\sigma_G = \sigma B(T, \bar{T}) \sqrt{\frac{1}{2\kappa}(1 - \exp(-2\kappa(T-t)))} \tag{3.2.81}$$

and

$$B(t,T) = \begin{cases} \frac{1}{\kappa}(1 - \exp(-\kappa(T-t))), & \text{if } \kappa > 0 \\ T - t, & \text{if } \kappa = 0 \end{cases}. \tag{3.2.82}$$

Equation (3.2.78) agrees with the formula for the price of a call option on a zero-coupon bond given in Jamshidian [1989]. However, Jamshidian has made an assumption of risk neutral dynamics and has calculated expectations involving lognormal ZCB prices, omitting many details of the proof, to arrive at the result, whereas we have calculated expectations under the real-world measure, giving all details of the proof.

When the short rate obeys a Hull-White process, as in (3.2.4), the pricing formulae for call and put options on the $\bar{T}$-maturity ZCB follow as a corollary of Theorem 3.2.24.

Theorem 3.2.26 *Under the Hull-White model, for a strike price K and valuation time t, the price of a T-expiry call option on a $\bar{T}$-maturity ZCB is*

$$c_{T,K,G_{\bar{T}}}(t) = G_{\bar{T}}(t)N(h) - KG_T(t)N(h - \sigma_G) \tag{3.2.83}$$

and the price of a T-expiry put option on a $\bar{T}$-maturity ZCB is

$$p_{T,K,G_{\bar{T}}}(t) = -G_{\bar{T}}(t)N(-h) + KG_T(t)N(-h + \sigma_G), \tag{3.2.84}$$

where

$$h = \frac{1}{\sigma_G} \log \frac{G_{\bar{T}}(t)}{G_T(t)K} + \frac{1}{2}\sigma_G \tag{3.2.85}$$

$$\sigma_G = B(T, \bar{T}) \sqrt{\int_t^T \exp\left\{ -2 \int_u^T a(\tau)\, d\tau \right\} \sigma(u)^2\, du} \tag{3.2.86}$$

and

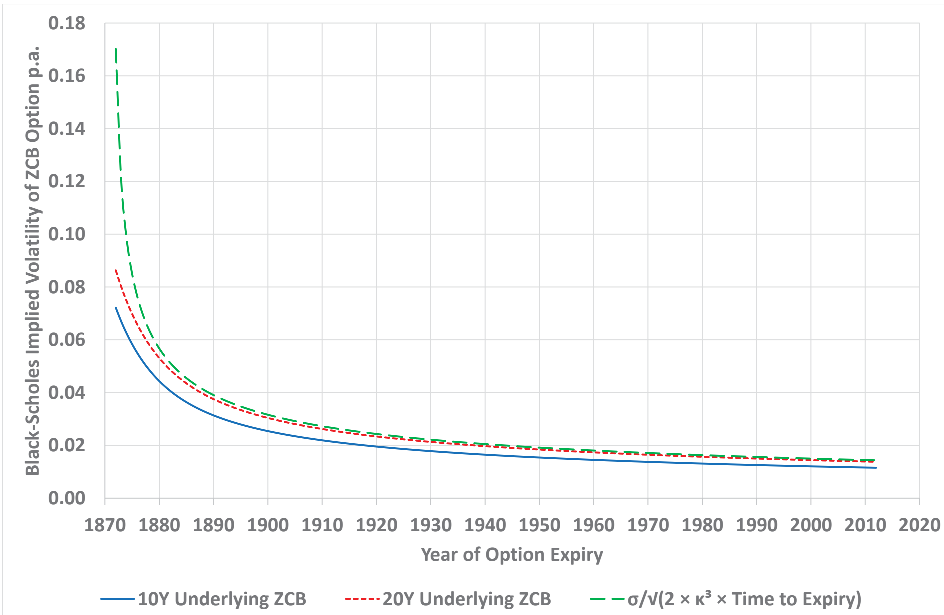
$$B(t,T) = \int_t^T \exp\left\{ - \int_t^s a(\tau)\, d\tau \right\} ds. \tag{3.2.87}$$

In Figure 3.7 the Black-Scholes implied volatilities of options on ten-year and twenty-year ZCBs are shown, based upon the parameters in (3.2.25). There is good agreement with the theoretical asymptotic formula of the Black-Scholes implied volatility, obtained as

$$\sigma_{BS} = \frac{\sigma_G}{\sqrt{T-t}} \to \frac{\bar{\sigma}}{\sqrt{2\bar{a}^3\,(T-t)}}, \tag{3.2.88}$$

as $\bar{T} \to \infty$.

Figure 3.7: Comparison of asymptotic formula with Black-Scholes implied option volatilities of options on 10Y and 20Y zero-coupon bonds based at year 1871.



3.3 Cox-Ingersoll-Ross Short Rate Model

The Cox-Ingersoll-Ross (CIR) model was introduced in 1985 by Cox et al. [1985] as an alternative to the Vasicek model. A good explanation of the model is given in Hull [1997]. The short rate is described by the SDE

$$dr_t = \kappa(\bar{r} - r_t)dt + \sigma\sqrt{r_t}dZ_t \tag{3.3.1}$$

for positively valued constants $\bar{r}$, σ and κ. A zero valued short rate is avoided because we set $\kappa\bar{r} > \frac{1}{2}\sigma^2$. The parameter κ denotes the speed of reversion of the short rate r_t to the mean reverting level $\bar{r}$. As in the Ho-Lee, Hull-White and Vasicek models, $\bar{r}$ can be thought of as a smoothed average short rate which is targeted by the central bank.

The case where mean reverting level $\bar{r}_t$ and variation parameter σ_t are time varying deterministic functions is called the extended CIR model for which explicit pricing formulae have been supplied by Maghsoodi [1996].

The CIR model has the property, different from the Ho-Lee, Hull-White and Vasicek models, that for the above conditions on the parameters the interest rates can never be negative.

We can remove the occurrence of r_t in the drift term of (3.3.1) by means of the integrating factor $\exp(\kappa t)$, giving the SDE for $q_t = r_t \exp(\kappa t)$ in the form

$$dq_t = \kappa \exp(\kappa t)\bar{r}dt + \exp(\kappa t/2)\sigma\sqrt{q_t}dZ_t. \tag{3.3.2}$$

We now remove the occurrence of q_t in the diffusion coefficient by making the transformation $\sqrt{q_t}$ for which we have the SDE

$$\begin{aligned} d\sqrt{q_t} &= \frac{1}{2\sqrt{q_t}}dq_t - \frac{1}{8\sqrt{q_t^3}}d[q]_t \\ &= \frac{\sigma^2\exp(\kappa t)}{8\sqrt{q_t}}\left(\frac{4\kappa\bar{r}}{\sigma^2} - 1\right)dt + \frac{1}{2}\exp(\kappa t/2)\sigma dZ_t. \end{aligned} \tag{3.3.3}$$

We now provide an explicit solution to (3.3.1) which gives rise to a transition density function of the short rate, later used for fitting the CIR model to data.

3.3.1 Explicit Formula for CIR Short Rate

We first state some lemmas which lead to a solution to the SDE (3.3.1), the proofs of which can be found in Platen and Heath [2006].

Lemma 3.3.1 *Let ν be a fixed integer, $\nu > 2$ and for each $j \in \{1, 2, \ldots, \nu\}$ let $Z^{(j)} = (Z_t^{(j)})_{t\geq 0}$ be a Wiener process, independent of the other Wiener processes $Z^{(k)}$, $k \in \{1, 2, \ldots, \nu\}$ $\{k\}$. Let the stochastic process X be defined by*

$$X_t = \sum_{j=1}^{\nu}(\lambda^{(j)} + Z_t^{(j)})^2 \tag{3.3.4}$$

for $t \geq 0$ and $\lambda^{(j)} \in R$. Then X has the SDE

$$dX_t = \nu dt + 2\sqrt{X_t}\,dZ_t \tag{3.3.5}$$

with initial condition $X_0 = \sum_{j=1}^{\nu}(\lambda^{(j)})^2$ and where

$$dZ_t = \sum_{k=1}^{\nu}\frac{(\lambda^{(k)} + Z_t^{(k)})\,dZ_t^{(k)}}{\sqrt{\sum_{j=1}^{\nu}(\lambda^{(j)} + Z_t^{(j)})^2}} \tag{3.3.6}$$

is a Brownian motion.

Definition 3.3.2 *A stochastic process X satisfying the SDE* (3.3.5) *is a squared Bessel process of dimension ν.*

Lemma 3.3.3 *Let Y be the process defined by*

$$Y_t = z_t X_{\varphi_t} \tag{3.3.7}$$

for $t \geq 0$, where X is as in (3.3.5), $z_t = z_0 \exp(\int_0^t b_u du)$ and $\varphi_t = \varphi_0 + \frac{1}{4}\int_0^t c_u^2/z_u \, du$, for deterministic functions b and c. Then Y has the SDE

$$dY_t = \left(\frac{\nu c_t^2}{4} + b_t Y_t\right) dt + c_t \sqrt{Y_t}\, dZ_t. \tag{3.3.8}$$

Proof. See Appendix C.

In the following theorem we apply this lemma to solving (3.3.1).

Theorem 3.3.4 *For the integer $\nu = \frac{4\kappa\bar{r}}{\sigma^2} > 2$ a solution to the SDE (3.3.1) is*

$$r_t = \exp(-\kappa t) \sum_{i=1}^{\nu} (\lambda^{(i)} + Z_{\varphi_t}^{(i)})^2 \tag{3.3.9}$$

where $\lambda^{(1)}, \ldots, \lambda^{(\nu)}$ are chosen such that $r_0 = \sum_{i=1}^{\nu} (\lambda^{(i)})^2$,where $\varphi_t = \varphi_0 + \frac{1}{4}\sigma^2(\exp(\kappa t) - 1)/\kappa$, where $Z^{(1)}, \ldots, Z^{(\nu)}$ are independent Brownian motions and $Z = (Z_t)_{t\geq 0}$ is given in (3.3.6).

Proof. See Appendix C.

3.3.2 Transition Density of CIR Short Rate

The transition density function of the CIR short rate model is that of the non-central chi-square distribution. We demonstrate that this is the case by commencing with the following definition.

Definition 3.3.5 *The* non-central chi-squared distribution *with non-centrality parameter λ and integer degrees of freedom parameter $\nu > 2$ is the distribution of a random variable Y equalling the sum of the squares of ν normally and independently distributed random variables each having variance one and which have their sum of squared means equalling λ. So we say that if $X^{(1)} \sim N(\mu^{(1)}, 1), \ldots, X^{(\nu)} \sim N(\mu^{(\nu)}, 1)$ with*

$$\lambda = \sum_{i=0}^{\nu} (\mu^{(i)})^2 \tag{3.3.10}$$

then the random variable

$$Y = \sum_{i=0}^{\nu} (X^{(i)})^2 \tag{3.3.11}$$

is distributed as $\chi^2_{\nu,\lambda}$.

Lemma 3.3.6 *The moment generating function of the random variable Y in* (3.3.11) *is*

$$MGF_Y(t) = (1-2t)^{-\nu/2}\exp(-\frac{1}{2}\lambda)\exp(\frac{1}{2}\lambda(1-2t)^{-1}). \tag{3.3.12}$$

Proof. See Appendix D.

We state the definition of the chi-squared distribution, which will be of help in identifying the probability density function of the non-central chi-squared distribution.

Definition 3.3.7 *The* chi-squared distribution *with integer degrees of freedom parameter $\nu \geq 0$ is the distribution of a random variable X equalling the sum of the squares of ν normally and independently distributed random variables each having mean zero and variance one. So we say that if*

$$X^{(1)} \sim N(0,1), \ldots, X^{(\nu)} \sim N(0,1) \tag{3.3.13}$$

then the random variable

$$X = \begin{cases} \sum_{i=1}^{\nu}(X^{(i)})^2 & \text{if } \nu > 0 \\ 1 & \text{if } \nu = 0 \end{cases} \tag{3.3.14}$$

is distributed as χ^2_ν. The probability density function of the chi-squared distribution is

$$f_{\chi^2_\nu}(x) = \frac{1}{2^{\nu/2}\Gamma(\nu/2)}x^{\nu/2-1}\exp(-x/2) \tag{3.3.15}$$

for $\nu > 0$ and is equal to the Dirac delta function $\delta_0(x)$ when $\nu = 0$, where $\Gamma(\cdot)$ is the gamma function.

Lemma 3.3.8 *The probability density function of the non-central chi-squared distribution is equal to a linear combination of probability density functions of chi-squared distributions as prescribed by the formula*

$$f_{\chi^2_{\nu,\lambda}}(x) = \exp(-\lambda/2)\sum_{i=0}^{\infty}\frac{(\lambda/2)^i}{i!}f_{\chi^2_{\nu+2i}}(x) \tag{3.3.16}$$

for non-centrality parameter λ and degrees of freedom parameter ν.

Proof. See Appendix D.

Remark 3.3.9 *We can see from* (3.3.16) *that if a random variable Y is defined as*

$$Y \sim \chi^2_{\nu+2Z}$$

where $Z \sim \text{Poisson}(\lambda/2)$, then Y is a $\chi^2_{\nu,\lambda}$-distributed random variable.

Corollary 3.3.10 *The probability density function of the non-central chi-squared distribution has the form*

$$f_{\chi^2_{\nu,\lambda}}(x) = \exp(-\lambda/2)\exp(-x/2)x^{\nu/2-1}\frac{1}{2^{\nu/2}}\sum_{i=0}^{\infty}\frac{(\lambda x/4)^i}{i!}\frac{1}{\Gamma(i+\nu/2)} \tag{3.3.17}$$

for non-centrality parameter λ and integer degrees of freedom parameter ν.

Proof. See Appendix D.

We now concern ourselves with characterising the distribution of the squared Bessel process X defined in (3.3.5).

Lemma 3.3.11 *For $t > s$ and the process X defined in (3.3.5), the conditional random variable*

$$\frac{1}{t-s}X_t \tag{3.3.18}$$

given X_s has a non-central chi-squared distribution, namely

$$\frac{1}{t-s}X_t \sim \chi^2_{\nu,X_s/(t-s)}. \tag{3.3.19}$$

The preceding lemma allows us to write down the formula for the transition density function of the squared Bessel process X defined in (3.3.5).

Lemma 3.3.12 *The transition density function for the squared Bessel process X in (3.3.5) is*

$$p_X(s,X_s,t,X_t) = \frac{1}{2(t-s)}\left(\frac{X_t}{X_s}\right)^{\frac{1}{2}(\frac{\nu}{2}-1)}\exp\left(-\frac{1}{2}\frac{X_s+X_t}{t-s}\right)I_{\frac{\nu}{2}-1}\left(\frac{\sqrt{X_sX_t}}{t-s}\right) \tag{3.3.20}$$

where

$$I_\nu(x) = \sum_{i=0}^{\infty}\frac{1}{i!\Gamma(i+\nu+1)}\left(\frac{x}{2}\right)^{2i+\nu} \tag{3.3.21}$$

is the power series expansion of the modified Bessel function of the first kind.

Lemma 3.3.13 *Let Y be the process defined by*

$$Y_t = z_t X_{\varphi_t} \tag{3.3.22}$$

for $t \geq 0$, where X is as in (3.3.5), $z_t = z_0\exp(\int_0^t b_u du)$ and $\varphi_t = \varphi_0 + \frac{1}{4}\int_0^t c_u^2/z_u\,du$, for deterministic functions b and c. Then Y has the transition density function

$$p_Y(s,Y_s,t,Y_t) = \frac{1}{2(\varphi_t-\varphi_s)z_t}\left(\frac{Y_t/z_t}{Y_s/z_s}\right)^{\frac{1}{2}(\frac{\nu}{2}-1)} \times \exp\left(-\frac{1}{2}\frac{Y_s/z_s+Y_t/z_t}{\varphi_t-\varphi_s}\right)I_{\frac{\nu}{2}-1}\left(\frac{\sqrt{Y_sY_t/(z_sz_t)}}{\varphi_t-\varphi_s}\right) \tag{3.3.23}$$

Proof. From the definition of Y in (3.3.22) we know that the transition density function of Y is

$$p_Y(s, Y_s, t, Y_t) = \frac{1}{z_t} p_X(\varphi_s, Y_s/z_s, \varphi_t, Y_t/z_t) \tag{3.3.24}$$

and combining with (3.3.20) gives (3.3.23). **Q.E.D.**

This provides the following result.

Theorem 3.3.14 *The transition density function of the short rate process in* (3.3.1) *is*

$$\begin{aligned} p_r(s, r_s, t, r_t) = & \frac{1}{2(\varphi_t - \varphi_s)\exp(-\kappa t)} \left(\frac{r_t \exp(\kappa t)}{r_s \exp(\kappa s)} \right)^{\frac{1}{2}(\frac{\nu}{2}-1)} \\ & \times \exp\left(-\frac{1}{2} \frac{r_s \exp(\kappa s) + r_t \exp(\kappa t)}{(\varphi_t - \varphi_s)} \right) I_{\frac{\nu}{2}-1}\left(\frac{\sqrt{r_s r_t \exp(\kappa(s+t))}}{(\varphi_t - \varphi_s)} \right), \end{aligned} \tag{3.3.25}$$

where $\varphi_t = \varphi_0 + \frac{1}{4}\sigma^2(\exp(\kappa t) - 1)/\kappa$ *and* $\nu = \frac{4\kappa\bar{r}}{\sigma^2}$ *and where*

$$I_\nu(x) = \sum_{i=0}^{\infty} \frac{1}{i!\Gamma(i+\nu+1)} \left(\frac{x}{2}\right)^{2i+\nu} \tag{3.3.26}$$

is the power series expansion of the modified Bessel function of the first kind.

Proof. Making the substitutions (C.7) in (3.3.23) provides the transition density function for r_t. **Q.E.D.**

Now, we can state directly the following result:

Corollary 3.3.15 *For* $t > s$ *and for the short rate process* r *defined in* (3.3.1), *the conditional random variable*

$$\frac{\exp(\kappa t)}{\varphi_t - \varphi_s} r_t \tag{3.3.27}$$

given r_s *has a non-central chi-squared distribution with* $\nu = 4\kappa\bar{r}/\sigma^2$ *degrees of freedom and non-centrality parameter* $\lambda = r_s \exp(\kappa s)/(\varphi_t - \varphi_s)$, *namely*

$$\frac{\exp(\kappa t)}{\varphi_t - \varphi_s} r_t \sim \chi^2_{\nu, r_s \exp(\kappa s)/(\varphi_t - \varphi_s)}, \tag{3.3.28}$$

where $\varphi_t = \varphi_0 + \frac{1}{4}\sigma^2(\exp(\kappa t) - 1)/\kappa$.

3.3.3 Fitting the CIR Model

Estimating the parameters of the CIR model is achieved using the maximum likelihood method. The log-likelihood function is given by

$$\ell(\bar{r}, \kappa, \sigma) = \sum_{i=1}^{n} \log p_r(t_{i-1}, r_{t_{i-1}}, t_i, r_{t_i}), \tag{3.3.29}$$

where the transition density function p_r is as in Theorem 3.3.14.

We fit the CIR model to the annual series of one-year deposit rates from 1871 to 2012, referred to as Data Set A in Section L.1 of Appendix L, obtaining the maximum likelihood estimates

$$\begin{aligned} \bar{r} &= 0.041078\,(0.011421) \\ \kappa &= 0.092540\,(0.038668) \\ \sigma &= 0.064670\,(0.0040761), \end{aligned} \tag{3.3.30}$$

where the standard errors are shown in brackets.

In Figure 3.8 we plot the actual short rate versus the fitted mean reversion level. We obtain a similar reference level as for the Vasicek model, see (3.2.25).

The logarithm of the empirically calculated quadratic variation of $\sqrt{r_t \exp(\kappa t)}$ is shown alongside the logarithm of the theoretically computed quadratic variation function, $[\sqrt{q}]_t = \frac{1}{4}\sigma^2(\exp(\kappa t) - 1)/\kappa$, in Figure 3.9. We note visually a reasonable long-term fit.

Remark 3.3.16 *From Theorem 3.3.4 we have $\nu = \frac{4\kappa\bar{r}}{\sigma^2}$ and substituting the values for $\bar{r}$, κ and σ in* (3.3.30) *gives $\nu = 4 \times 0.041078 \times 0.092540/(0.064670)^2 = 3.6357$, which could be approximated by $\nu \approx 4$.*

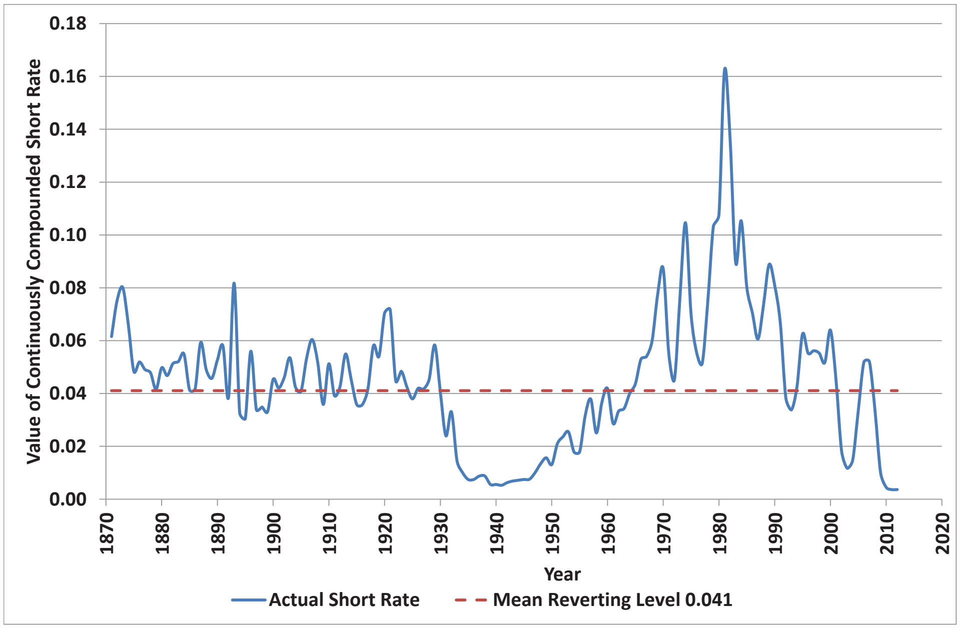
A graph of the transition density function is shown in Figure 3.10 in respect of the fitted parameters in (3.3.30) and commencing at $s = 2000$.

Unlike the Vasicek model, under the CIR model there is no possibility of negative interest rates, as demonstrated by (3.3.9) and as illustrated by comparing the CIR density in Figure 3.10 with the Vasicek density in Figure 3.3. It is also evident from Figures 3.3 and 3.10 that the CIR density is more peaked than the Vasicek density.

3.3.4 Fitting the CIR Model using Log Normal Approximation

Poulsen [1999] shows that good estimates can be obtained by approximating the transition density of the CIR process with a Gaussian distribution having the

Figure 3.8: Actual short rate and fitted CIR mean reversion level for US cash rates.



same mean and variance as the transition density function. However, our fitting of the CIR model to the data suggests that the Gaussian distribution is inadequate as an approximation and further, we find that almost exact estimates can be obtained with the lognormal approximation. This is not surprising given that the lognormal distribution is positively skewed and is defined for positive values of the random variable, as is the case for the non-central chi-squared distribution. We have the following lemma which gives the mean and variance of the CIR transition density function, using moment equations which are described in Section 7.3 of Platen and Heath [2006].

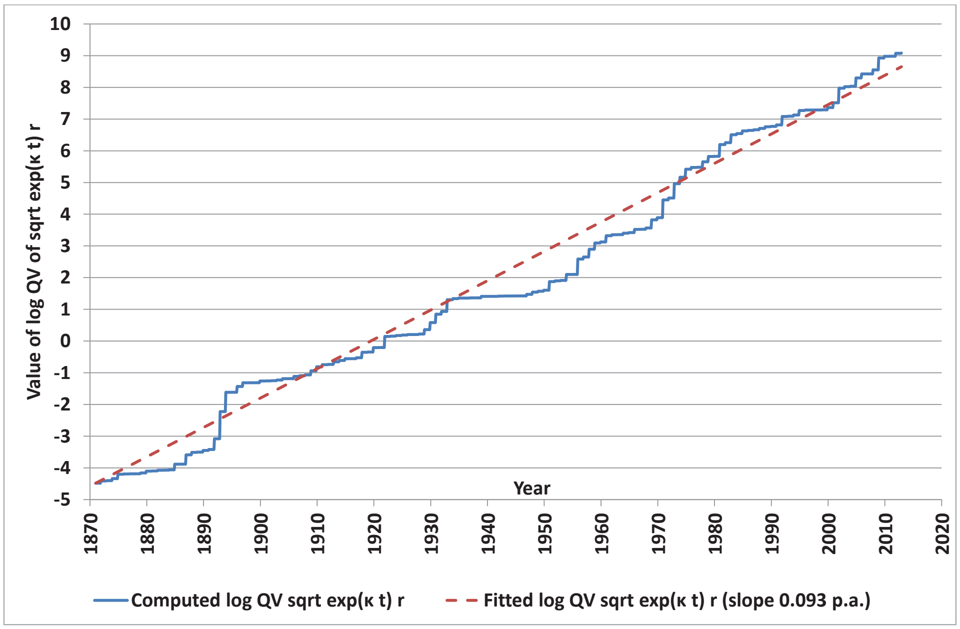
Lemma 3.3.17 *For the CIR process in* (3.3.1) *and times s, t with $s \leq t$ let the mean and variance of r_t given r_s be defined as*

$$\begin{aligned} m_s(t) &= \mathrm{E}(r_t|\mathcal{A}_s) \\ v_s(t) &= \mathrm{Var}(r_t|\mathcal{A}_s). \end{aligned} \tag{3.3.31}$$

Then we have the explicit formulae

$$\begin{aligned} m_s(t) &= \bar{r}\kappa B(s,t) + r_s(1 - \kappa B(s,t)) \\ v_s(t) &= \sigma^2 \left(\frac{1}{2}\kappa B(s,t)^2 \bar{r} + (B(s,t) - \kappa B(s,t)^2) r_s \right), \end{aligned} \tag{3.3.32}$$

Figure 3.9: Logarithm of quadratic variation of $\sqrt{\exp(\kappa t) r_t}$ for US cash rates.



where

$$B(s,t) = (1 - \exp(-\kappa(t-s)))/\kappa. \tag{3.3.33}$$

Proof. See Appendix C.

We approximate the transition density of the CIR process from time s to time t by a lognormal distribution that matches the mean and variance. It is straightforward to show that the approximating lognormal distribution has parameters $\mu_s(t)^{(LN)}$, $\sigma_s(t)^{(LN)}$ given by

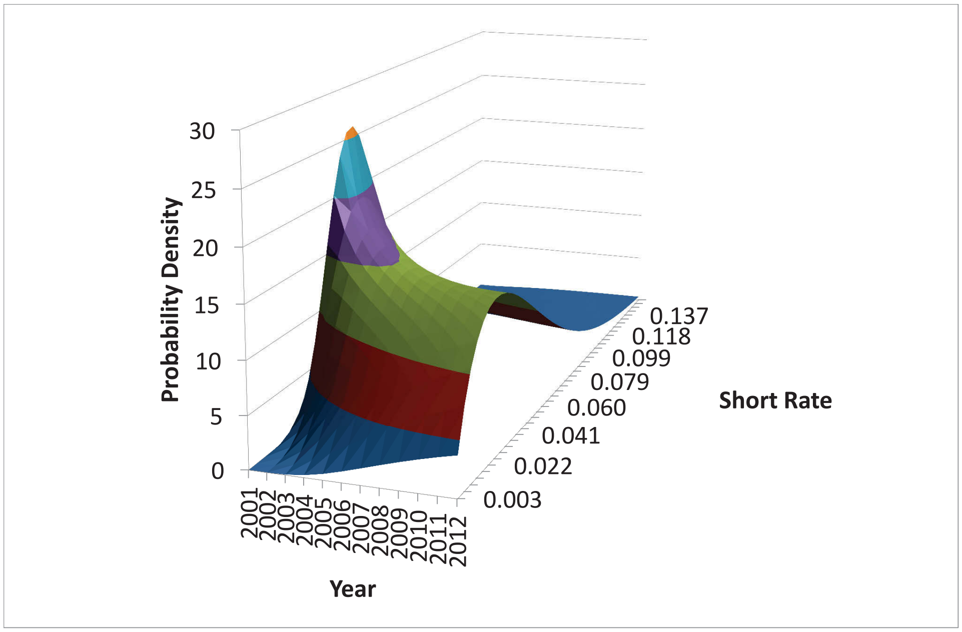
$$\begin{aligned} \mu_s(t)^{(LN)} &= \log m_s(t) - \frac{1}{2}(\sigma_s(t)^{(LN)})^2 \\ (\sigma_s(t)^{(LN)})^2 &= \log\left(1 + \frac{v_s(t)}{m_s(t)^2}\right) \end{aligned} \tag{3.3.34}$$

where

$$\begin{aligned} m_{t_{i-1}}(t_i) &= \bar{r}\kappa B(t_{i-1}, t_i) + r_{t_{i-1}}(1 - \kappa B(t_{i-1}, t_i)) \\ v_{t_{i-1}}(t_i) &= \sigma^2\left(\frac{1}{2}\kappa B(t_{i-1}, t_i)^2 \bar{r} + (B(t_{i-1}, t_i) - \kappa B(t_{i-1}, t_i)^2) r_{t_{i-1}}\right) \end{aligned} \tag{3.3.35}$$

and $B(s,t)$ is as in (3.3.33).

Figure 3.10: CIR transition density function of US cash rates based at year 2000 and short rate 0.064.



Therefore, our approximating log-likelihood function on the set of observed short rates r_{t_i}, for $i = 0, 2, \ldots, n$ is

$$\ell(\bar{r}, \kappa, \sigma) = -\frac{1}{2} \sum_{i=1}^{n} \left(\log(2\pi v_{t_{i-1}}(t_i)) + \log r_{t_i} + \frac{(\log r_{t_i} - \mu_{t_{i-1}}(t_i)^{(LN)})^2}{(\sigma_{t_{i-1}}(t_i)^{(LN)})^2} \right). \tag{3.3.36}$$

We fit the CIR model to the annual series of one-year deposit rates from 1871 to 2012, Data Set A in Section L.1 of Appendix L, obtaining the maximum likelihood estimates

$$\begin{aligned} \bar{r} &= 0.042470\,(0.010935) \\ \kappa &= 0.102677\,(0.041105) \\ \sigma &= 0.070948\,(0.0041198). \end{aligned} \tag{3.3.37}$$

We note that these estimates are close to those in (3.3.30).

3.3.5 CIR Savings Account and Transition Density

The SDE of the savings account B_t is given by

$$dB_t = r_t B_t dt, \tag{3.3.38}$$

for $t \geq 0$ with $B_0 = 1$, and we aim now to determine the transition density function of the savings account B_t.

When the characteristic function of the random variable is known, many practitioners, as mentioned in Carr and Madan [1999], employ inverse Fourier transforms to retrieve the probability density function.

A related approach is to firstly determine the moment generating function of the logarithm of appreciation factor of the savings account value, $L = \log B_T/B_t$, and secondly compute the inverse Laplace transform. By virtue of the relationship

$$\exp(sL) = (B_T/B_t)^s \tag{3.3.39}$$

calculating the MGF is equivalent to determining the expectation $\mathrm{E}((B_T/B_t)^s|\mathcal{A}_t)$, which demands some preliminary results involving the expectation of a stochastic exponential, given in Section 5.5 of Baldeaux and Platen [2013].

We now compute the expectation $\mathrm{E}((B_T/B_t)^k|\mathcal{A}_t)$ in the following lemma. We note that letting $L = \log(B_T/B_t)$, this lemma provides a formula for the moment generating function of the random variable L.

Lemma 3.3.18 *Let the short rate r obey the SDE* (3.3.1) *and let the savings account B obey the SDE* (3.3.38)*. Then for any number k and time T such that $T > t$,*

$$\begin{aligned}\mathrm{E}\left(\left(\frac{B_T}{B_t}\right)^k \middle| \mathcal{A}_t\right) &= \left(\frac{h\exp(\frac{1}{2}\kappa(T-t))}{\kappa\sinh\frac{1}{2}(T-t)h + h\cosh\frac{1}{2}(T-t)h}\right)^{2\kappa\bar{r}/\sigma^2} \\ &\quad\times\exp\left(k\frac{2\sinh\frac{1}{2}(T-t)h}{\kappa\sinh\frac{1}{2}(T-t)h + h\cosh\frac{1}{2}(T-t)h}r_t\right),\end{aligned} \tag{3.3.40}$$

where $h = \sqrt{\kappa^2 - 2k\sigma^2}$.

Proof. See Appendix D.

We can apply the inverse Laplace transform formula, as given in Section 8.2 of Marsden and Hoffman [1999], to give a formula for the transition density function of the savings account.

Theorem 3.3.19 *For a CIR short rate the transition density function of the savings account is given by*

$$p_B(t, B_t, T, B_T) = \frac{1}{2\pi}\frac{B_t}{B_T}\int_{-\infty}^{\infty}\exp\big((c+iu)\log(B_T/B_t)\big)F_L(c+iu)du \tag{3.3.41}$$

where $F_L = MGF_L(-s)$ and c is a constant greater than the real part of any singularities of F_L in the complex plane.

Proof. Let f be the probability density function of the random variable $L = \log B_T/B_t$. Because r_t is always non-negative, B_T/B_t is always at least one in value and, therefore, L assumes values in the interval $[0, \infty)$. Thus the Laplace transform of f can be written as

$$F(s) = \int_0^\infty \exp(-sx) f(x) dx = \mathrm{E}(\exp(-sL)) \tag{3.3.42}$$

and the inverse Laplace transform of F is

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \exp(sx) F(s) ds, \tag{3.3.43}$$

where the integration is performed along the line $\Re(z) = c$ in the complex plane, with c being a constant such that $F(s)$ has no singularities with the real part exceeding the constant c. Making the change of variables $s = c + iu$ in the integral and the change of variables $B_T/B_t = \exp(L)$ in the probability density gives the result. **Q.E.D.**

The transition density function of the logarithm of the savings account is shown in Figure 3.11. As will be shown in Chapter 5, the transition density function of the logarithm of the savings account is useful in computing the price of an equity index option.

3.3.6 Short Rate Contribution to ZCB Price

We calculate the contribution $G_{\bar{T}}(t)$ to the zero-coupon bond price that is due to the short rate.

The following theorem is easily proven in the light of the results of the preceding subsection. This result is also supplied in Cox et al. [1985], and the details of that proof are omitted.

Theorem 3.3.20 *For time $t \in [0, \bar{T}]$ the short rate contribution to the ZCB price is computed to be*

$$G_{\bar{T}}(t) = A(t, \bar{T}) \exp(-B(t, \bar{T}) r_t), \tag{3.3.44}$$

where

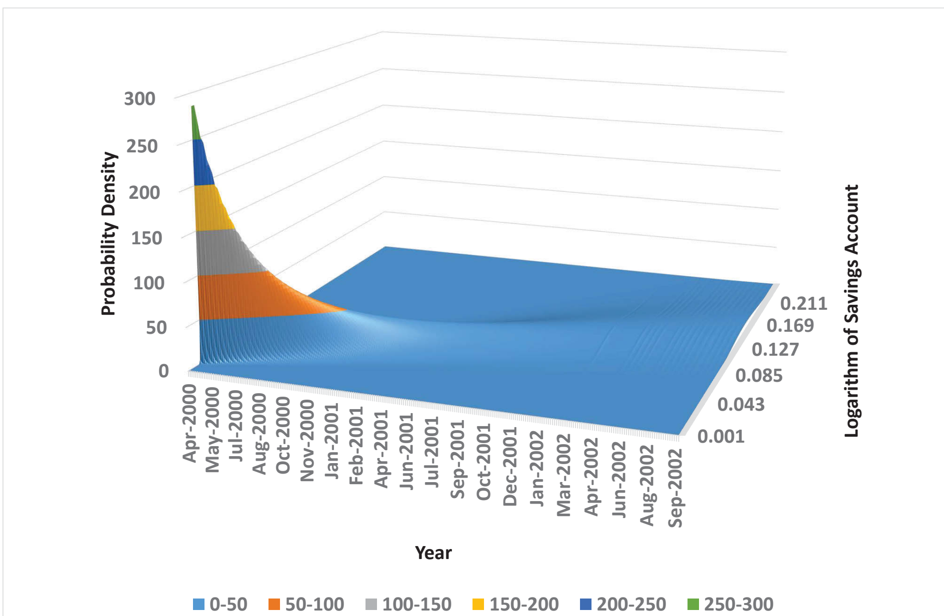
$$A(t, \bar{T}) = \left(\frac{h \exp(\frac{1}{2}\kappa(\bar{T} - t))}{\kappa \sinh \frac{1}{2} h(\bar{T} - t) + h \cosh \frac{1}{2} h(\bar{T} - t)} \right)^{2\kappa\bar{r}/\sigma^2} \tag{3.3.45}$$

$$B(t, \bar{T}) = \frac{2 \sinh \frac{1}{2} h(\bar{T} - t)}{\kappa \sinh \frac{1}{2} h(\bar{T} - t) + h \cosh \frac{1}{2} h(\bar{T} - t)} \tag{3.3.46}$$

and

$$h = \sqrt{\kappa^2 + 2\sigma^2}. \tag{3.3.47}$$

Figure 3.11: CIR transition density of the logarithm of savings account based at year 2000.



Proof. Using Lemma 3.3.18 gives the result. **Q.E.D.**

The following corollary is a result which agrees with the formula for the long maturity yield $R(r_t, t, \infty) = 2\kappa\bar{r}/(\gamma + \kappa + \lambda)$ given in Cox et al. [1985], when $\lambda = 0$, where γ is given by

$$\gamma = \sqrt{(\kappa + \lambda)^2 + 2\sigma^2} \tag{3.3.48}$$

and λ is the market price of risk.

Corollary 3.3.21 *For the CIR short rate, the contribution of the short rate to the long ZCB yield is*

$$h_\infty(t) = \frac{\kappa\bar{r}}{\sigma^2}(h - \kappa). \tag{3.3.49}$$

Proof. See Appendix F.

In Figure 3.12 the short rate contribution to the continuously compounded yield curve is plotted, viewed from 1871. It is an inverted yield curve and occurs when the future economic growth is expected to be subdued. This is similar to the yield curve shown in Figure 3.6.

3.3.7 Short Rate Contribution to the Forward Rate

We calculate the contribution $g_{\bar{T}}(t)$ to the forward rate that is due to the short rate, which is a new explicit formula.

Lemma 3.3.22 *For time $t \in [0, \bar{T}]$ the short rate contribution to the $\bar{T}$-forward rate is computed to be*

$$g_{\bar{T}}(t) = \frac{2\kappa\bar{r}}{\sigma^2}\left(\frac{1}{2}h\coth(\bar{T} - t)h - \frac{1}{2}\kappa - \frac{h^2}{2\sinh\frac{1}{2}h(\bar{T} - t)C(t, \bar{T})}\right) + r_t\frac{h^2}{C(t, \bar{T})^2}, \tag{3.3.50}$$

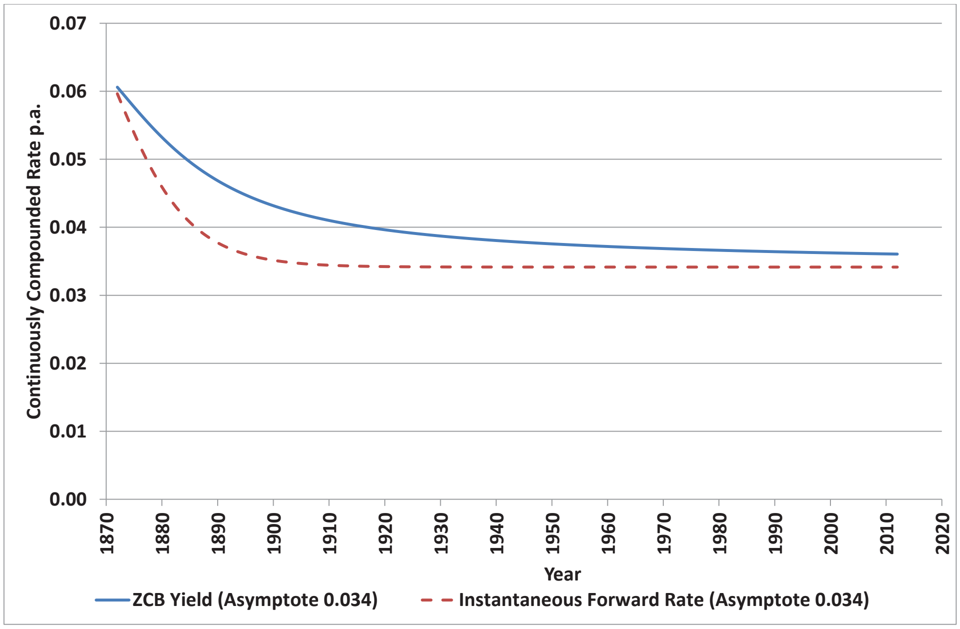
where $C(t, \bar{T})$ is given by

$$C(t, \bar{T}) = \kappa\sinh\frac{1}{2}h(\bar{T} - t) + h\cosh\frac{1}{2}h(\bar{T} - t) \tag{3.3.51}$$

and h is as in (3.3.47).

Proof. See Appendix F.

Figure 3.12: Short rate contribution to the zero coupon yield curve under the Cox-Ingersoll-Ross model based at 1871.



Corollary 3.3.23 *For the CIR short rate, the short rate contribution to the instantaneous forward rate has the asymptotic value*

$$g_\infty(t) = \frac{\kappa\bar{r}}{\sigma^2}(h - \kappa). \tag{3.3.52}$$

Proof. From (3.3.50) it follows

$$g_\infty(t) = \lim_{\bar{T}\to\infty} g_{\bar{T}}(t) = \frac{\kappa\bar{r}}{\sigma^2}(h - \kappa) \tag{3.3.53}$$

as required. **Q.E.D.**

In Figure 3.12 the instantaneous forward rate is plotted and the asymptotic value is seen to be $g_\infty(t) = 0.03415$.

3.3.8 Expectations Involving $G_{\bar{T}}(t)$

In Cox et al. [1985] the formula for the price of a ZCB call option is supplied, but there are no details of the proof. We supply here a proof of this result, formulated in the following theorem.

Theorem 3.3.24 *Under a CIR short rate model, for a strike price K and valuation time t, the price of a T-expiry call option on a $\bar{T}$-maturity ZCB is*

$$c_{T,K,G_{\bar{T}}}(t) = G_{\bar{T}}(t)\chi^2_{\nu,\omega}\Big(2r'(\rho+\psi+B(T,\bar{T}))\Big) - KG_T(t)\chi^2_{\nu,\omega'}\Big(2r'(\rho+\psi)\Big) \tag{3.3.54}$$

and the price of a T-expiry put option on a $\bar{T}$-maturity ZCB is

$$\begin{aligned} p_{T,K,G_{\bar{T}}}(t) = -G_{\bar{T}}(t)\Big(1-\chi^2_{\nu,\omega}\Big(2r'(\rho+\psi+B(T,\bar{T}))\Big)\Big) \\ + KG_T(t)\Big(1-\chi^2_{\nu,\omega'}\Big(2r'(\rho+\psi)\Big)\Big) \end{aligned} \tag{3.3.55}$$

where

$$\nu = \frac{4\kappa\bar{r}}{\sigma^2} \tag{3.3.56}$$

$$\omega = \frac{2\rho^2 r_t \exp(h(T-t))}{\rho+\psi+B(T,\bar{T})} \tag{3.3.57}$$

$$\omega' = \frac{2\rho^2 r_t \exp(h(T-t))}{\rho+\psi} \tag{3.3.58}$$

$$\rho = \frac{2h}{\sigma^2(\exp(h(T-t))-1)} \tag{3.3.59}$$

$$\psi = \frac{\kappa+h}{\sigma^2} \tag{3.3.60}$$

$$r' = \frac{\log(A(T,\bar{T})/K)}{B(T,\bar{T})} \tag{3.3.61}$$

$$h = \sqrt{\kappa^2+2\sigma^2} \tag{3.3.62}$$

$$B(t,T) = \frac{2\sinh\frac{1}{2}h(T-t)}{\kappa\sinh\frac{1}{2}h(T-t)+h\cosh\frac{1}{2}h(T-t)} \tag{3.3.63}$$

$$A(t,T) = \left(\frac{h\exp(\frac{1}{2}\kappa(T-t))}{\kappa\sinh\frac{1}{2}h(T-t)+h\cosh\frac{1}{2}h(T-t)}\right)^{2\kappa\bar{r}/\sigma^2} \tag{3.3.64}$$

We prove these call and put option formulae in this section by pricing the asset binary call option on $G_{\bar{T}}(t)$ and the bond binary call option on $G_{\bar{T}}(t)$.

Firstly, we formulate a lemma which will be of use in determining the moment generating function of the short rate under the T-forward measure.

Lemma 3.3.25 *Let the short rate r obey the SDE* (3.3.1). *Then for any real* *number u and time T such that $T > t$,*

$$\begin{aligned}
&\mathrm{E}\left(\exp\left(r_T u - \int_t^T r_s ds \right) \middle| \mathcal{A}_t \right) \qquad (3.3.65)\\
&= \left(\frac{h \exp(\frac{1}{2}\kappa(T-t))}{(\kappa - \sigma^2 u)\sinh\frac{1}{2}h(T-t) + h\cosh\frac{1}{2}h(T-t)} \right)^{2\kappa\bar{r}/\sigma^2}\\
&\times \exp\left(\frac{hu\cosh\frac{1}{2}h(T-t) - (\kappa u + 2)\sinh\frac{1}{2}h(T-t)}{(\kappa - \sigma^2 u)\sinh\frac{1}{2}h(T-t) + h\cosh\frac{1}{2}h(T-t)} r_t \right).
\end{aligned}$$

The proof of this result can be found in Section 5.4 in Baldeaux and Platen [2013].

We can now prove the following theorem concerning asset binary options on a zero-coupon bond.

Theorem 3.3.26 *Given an expiry time T and a strike price K the price of the asset binary call option on $G_{\bar{T}}(t)$ is*

$$A^+_{T,K,G_{\bar{T}}}(t) = G_{\bar{T}}(t)\chi^2_{\nu,\omega}(R/\gamma), \qquad (3.3.66)$$

where

$$\begin{aligned}
\nu &= 4\kappa\bar{r}/\sigma^2 \qquad (3.3.67)\\
\gamma &= \frac{1}{4}\sigma^2 B(t,T)(1 + \frac{1}{2}\sigma^2 B(t,T)B(T,\bar{T}))^{-1}\\
\omega &= r_t \frac{4}{\sigma^2}\left(\frac{1}{2}h\coth\frac{1}{2}h(T-t) - \frac{1}{2}\kappa \right)\left(1 - 2B(T,\bar{T})\gamma \right) - r_t\frac{8}{\sigma^2}\gamma\\
R &= \frac{1}{B(T,\bar{T})}\log\frac{A(T,\bar{T})}{K}.
\end{aligned}$$

Also, the price of the asset binary put option on $G_{\bar{T}}(t)$ is

$$A^-_{T,K,G_{\bar{T}}}(t) = G_{\bar{T}}(t)\left(1 - \chi^2_{\nu,\omega}(R/\gamma) \right). \qquad (3.3.68)$$

Proof. See Appendix E.

The formulae for the prices of the bond binary call and put options on a zero-coupon bond are less complicated and appear in the following theorem.

Theorem 3.3.27 *Given an expiry time T and a strike price K the price of the bond binary call option on $G_{\bar{T}}(t)$ is*

$$B^+_{T,K,G_{\bar{T}}}(t) = G_T(t)\chi^2_{\nu,\omega'}(R/\gamma'), \qquad (3.3.69)$$

where

$$\begin{aligned}
\nu &= 4\kappa\bar{r}/\sigma^2 \qquad (3.3.70)\\
\gamma' &= \frac{1}{4}\sigma^2 B(t,T)\\
\omega' &= r_t\frac{4}{\sigma^2}\left(\frac{1}{2}h\coth\frac{1}{2}h(T-t)-\frac{1}{2}\kappa\right)-r_t\frac{8}{\sigma^2}\gamma'\\
R &= \frac{1}{B(T,\bar{T})}\log\frac{A(T,\bar{T})}{K}.
\end{aligned}$$

Also, the price of the bond binary put option on $G_{\bar{T}}(t)$ is

$$B^-_{T,K,G_{\bar{T}}}(t) = G_T(t)\Big(1-\chi^2_{\nu,\omega'}(R/\gamma')\Big). \qquad (3.3.71)$$

We are now poised to supply formulae for the call and put options on zero-coupon bonds under the CIR model, which follow from Theorems 3.3.26 and 3.3.27.

Corollary 3.3.28 *Given an expiry time T and a strike price K the price of the call option on $G_{\bar{T}}(t)$ is*

$$c_{T,K,G_{\bar{T}}}(t) = G_{\bar{T}}(t)\chi^2_{\nu,\omega}(R/\gamma) - KG_T(t)\chi^2_{\nu,\omega'}(R/\gamma'), \qquad (3.3.72)$$

where

$$\begin{aligned}
\nu &= 4\kappa\bar{r}/\sigma^2 \qquad (3.3.73)\\
\gamma' &= \frac{1}{4}\sigma^2 B(t,T)\\
\omega' &= r_t\frac{4}{\sigma^2}\left(\frac{1}{2}h\coth\frac{1}{2}h(T-t)-\frac{1}{2}\kappa\right)-r_t\frac{8}{\sigma^2}\gamma'\\
\gamma &= \frac{1}{4}\sigma^2 B(t,T)(1+\frac{1}{2}\sigma^2 B(t,T)B(T,\bar{T}))^{-1}\\
\omega &= r_t\frac{4}{\sigma^2}\left(\frac{1}{2}h\coth\frac{1}{2}h(T-t)-\frac{1}{2}\kappa\right)\Big(1-2B(T,\bar{T})\gamma\Big)-r_t\frac{8}{\sigma^2}\gamma\\
R &= \frac{1}{B(T,\bar{T})}\log\frac{A(T,\bar{T})}{K}.
\end{aligned}$$

Also, the price of the put option on $G_{\bar{T}}(t)$ is

$$p_{T,K,G_{\bar{T}}}(t) = KG_T(t)\Big(1-\chi^2_{\nu,\omega'}(R/\gamma')\Big) - G_{\bar{T}}(t)\Big(1-\chi^2_{\nu,\omega}(R/\gamma)\Big). \qquad (3.3.74)$$

This completes our coverage of the CIR short rate model, which is sufficient for the calculation of zero-coupon bond prices and equity and interest rate derivatives in Chapter 5. We now proceed to our discussion of the 3/2 short rate model.

3.4 The 3/2 Short Rate Model

The 3/2 power law for the diffusion coefficient in a short rate model was shown in Chan et al. [1992] to be the best fitting power law. Also the nonlinear drift term of this model could not be rejected in Aït-Sahalia [1996]. The 3/2 short rate model was derived in Platen [1999] and studied by Ahn and Gao [1999], the SDE of which is

$$dr_t = (pr_t + qr_t^2)dt + \sigma r_t^{3/2} dZ_t, \tag{3.4.1}$$

where $q < \sigma^2/2$ and $\sigma > 0$ so as to avoid explosive values of r_t.

Setting $R_t = 1/r_t$ we obtain the SDE

$$dR_t = (\sigma^2 - q - pR_t)dt - \sigma\sqrt{R_t}dZ_t, \tag{3.4.2}$$

which shows that $R_t = 1/r_t$ follows a square root process. This fact makes the derivation of the transition density function of the 3/2 model straightforward.

3.4.1 Explicit Formula for 3/2 Short Rate

The following theorem provides the solution to the SDE of the 3/2 model.

Theorem 3.4.1 *A solution to the SDE* (3.4.1) *is*

$$r_t = \exp(pt) / \sum_{i=1}^{\nu} (\lambda^{(i)} + Z_{\varphi_t}^{(i)})^2 \tag{3.4.3}$$

where ν is an integer such that $\nu = \frac{4(\sigma^2 - q)}{\sigma^2}$, $\lambda^{(1)}, \ldots, \lambda^{(\nu)} \in \mathbb{R}$ are chosen such that $r_0 = 1/\sum_{i=1}^{\nu}(\lambda^{(i)})^2$, where $\varphi_t = \varphi_0 + \frac{1}{4}\sigma^2(\exp(pt) - 1)/p$ and $Z^{(1)}, \ldots, Z^{(\nu)}$ are independent Brownian motions.

Proof. From (3.4.2) $R_t = 1/r_t$ follows a square root process and, therefore, from Theorem 3.3.4 the explicit formula for R_t is

$$R_t = \exp(-pt) \sum_{i=1}^{\nu} (\lambda^{(i)} + Z_{\varphi_t}^{(i)})^2, \tag{3.4.4}$$

where $\lambda^{(1)}, \ldots, \lambda^{(\nu)}$ are chosen such that $R_0 = \sum_{i=1}^{\nu}(\lambda^{(i)})^2$ and where $\varphi_t = \varphi_0 + \frac{1}{4}\sigma^2(\exp(pt) - 1)/p$ and $\nu = \frac{4(\sigma^2 - q)}{\sigma^2}$. The formula for r_t is simply the reciprocal of (3.4.4). **Q.E.D.**

3.4.2 Transition Density of 3/2 Short Rate

The transition density function of the 3/2 short rate model is easily deduced from the transition density of the CIR short rate model as specified in Theorem 3.3.14 because the SDE of the reciprocal of a 3/2-process short rate obeys the CIR SDE (3.3.1), as shown in (3.4.2).

Lemma 3.4.2 *The transition density function of the short rate process in* (3.4.1) *is*

$$\begin{aligned} p_r(s, r_s, t, r_t) = & \frac{r_t^{-2}}{2(\varphi_t - \varphi_s)\exp(-pt)} \left(\frac{r_t^{-1}\exp(pt)}{r_s^{-1}\exp(ps)} \right)^{\frac{1}{2}(\frac{\nu}{2}-1)} \\ & \times \exp\left(-\frac{1}{2} \frac{r_s^{-1}\exp(ps) + r_t^{-1}\exp(pt)}{(\varphi_t - \varphi_s)} \right) \\ & \times I_{\frac{\nu}{2}-1}\left(\frac{\sqrt{r_s^{-1} r_t^{-1}\exp(p(s+t))}}{(\varphi_t - \varphi_s)} \right) \end{aligned} \tag{3.4.5}$$

where $\varphi_t = \varphi_0 + \frac{1}{4}\sigma^2(\exp(pt) - 1)/p$ *and* $\nu = 4(1 - q/\sigma^2)$ *and where*

$$I_\nu(x) = \sum_{i=0}^{\infty} \frac{1}{i!\Gamma(i + \nu + 1)} \left(\frac{x}{2}\right)^{2i+\nu} \tag{3.4.6}$$

is the power series expansion of the modified Bessel function of the first kind as in (3.3.26).

Proof. From Theorem 3.3.14 we have that the transition density function of the process R_t in (3.4.2) is

$$\begin{aligned} p_R(s, R_s, t, R_t) = & \frac{1}{2(\varphi_t - \varphi_s)\exp(-\kappa t)} \left(\frac{R_t\exp(\kappa t)}{R_s\exp(\kappa s)} \right)^{\frac{1}{2}(\frac{\nu}{2}-1)} \\ & \times \exp\left(-\frac{1}{2} \frac{R_s\exp(\kappa s) + R_t\exp(\kappa t)}{(\varphi_t - \varphi_s)} \right) I_{\frac{\nu}{2}-1}\left(\frac{\sqrt{R_s R_t\exp(\kappa(s+t))}}{(\varphi_t - \varphi_s)} \right) \end{aligned} \tag{3.4.7}$$

where $\varphi_t = \varphi_0 + \frac{1}{4}\sigma^2(\exp(\kappa t) - 1)/\kappa$ and $\nu = \frac{4\kappa\bar{r}}{\sigma^2}$ and where

$$I_\nu(x) = \sum_{i=0}^{\infty} \frac{1}{i!\Gamma(i + \nu + 1)} \left(\frac{x}{2}\right)^{2i+\nu} \tag{3.4.8}$$

is the power series expansion of the modified Bessel function of the first kind as in (3.3.26). Now comparing coefficients of (3.4.1) with those in (3.3.1) gives the correspondences

$$\begin{aligned} \bar{r} &\leftrightarrow \frac{\sigma^2 - q}{p} \\ \kappa &\leftrightarrow p \\ \sigma &\leftrightarrow \sigma \end{aligned} \tag{3.4.9}$$

and making these substitutions in (3.4.7) gives

$$
\begin{aligned}
p_R(s, R_s, t, R_t) = & \frac{1}{2(\varphi_t - \varphi_s)\exp(-pt)} \left(\frac{R_t \exp(pt)}{R_s \exp(ps)} \right)^{\frac{1}{2}(\frac{\nu}{2}-1)} \\
& \times \exp\left(-\frac{1}{2} \frac{R_s \exp(ps) + R_t \exp(pt)}{(\varphi_t - \varphi_s)} \right) \\
& \times I_{\frac{\nu}{2}-1}\left(\frac{\sqrt{R_s R_t \exp(p(s+t))}}{(\varphi_t - \varphi_s)} \right).
\end{aligned} \tag{3.4.10}
$$

Finally, the transition density function of $r_t = 1/R_t$ is

$$
\begin{aligned}
p_r(s, r_s, t, r_t) &= \frac{d}{dr_t} \int_0^{r_t} p_r(s, r_s, t, w)\,dw \\
&= \frac{d}{dr_t} \int_{R_t}^{\infty} p_R(s, R_s, t, x)\,dx \\
&= -\frac{dR_t}{dr_t} \times p_R(s, R_s, t, R_t) \\
&= r_t^{-2} p_R(s, r_s^{-1}, t, r_t^{-1})
\end{aligned} \tag{3.4.11}
$$

and using (3.4.10) gives the result. **Q.E.D.**

The following corollary explicitly gives the distribution of the reciprocal of the 3/2 short rate.

Corollary 3.4.3 *For $t > s$ and for the short rate process r defined in* (3.4.1), *the conditional random variable*

$$
\frac{\exp(pt)}{r_t(\varphi_t - \varphi_s)} \tag{3.4.12}
$$

given r_s has a non-central chi-squared distribution with $\nu = 4(\sigma^2 - q)/\sigma^2$ degrees of freedom and non-centrality parameter $\lambda = \exp(ps)/(r_s(\varphi_t - \varphi_s))$, namely

$$
\frac{\exp(pt)}{r_t(\varphi_t - \varphi_s)} \sim \chi^2_{\nu, \exp(ps)/(r_s(\varphi_t - \varphi_s))}, \tag{3.4.13}
$$

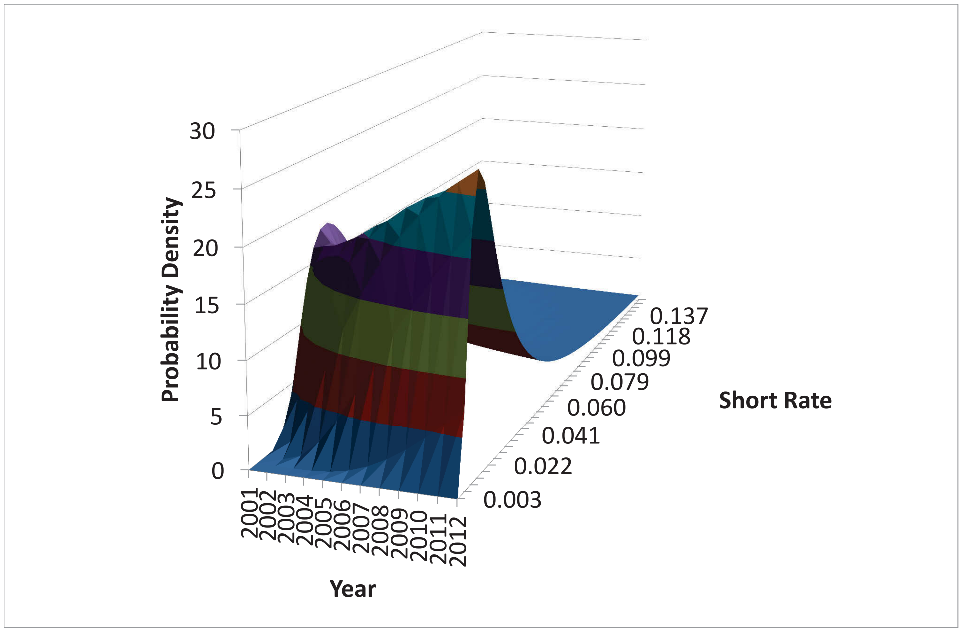
given r_s, where $\varphi_t = \varphi_0 + \frac{1}{4}\sigma^2(\exp(pt) - 1)/p$.

3.4.3 Fitting the 3/2 Model

For the annual series of one-year deposit rates from 1871 to 2012, Data Set A in Section L.1 of Appendix L, the fitted parameters of the 3/2 model, using the maximum likelihood method, are given by

$$
\begin{aligned}
p &= 0.038506\,(0.042284) \\
q &= 0.877908\,(1.177853) \\
\sigma &= 2.0681\,(0.13241),
\end{aligned} \tag{3.4.14}
$$

Figure 3.13: 3/2 transition density of US cash rates.



where the standard errors are shown in brackets.

In Figure 3.13 the transition density of the 3/2 model is plotted.

The mean reverting level of $R_t = 1/r_t$ in (3.4.2) is given by

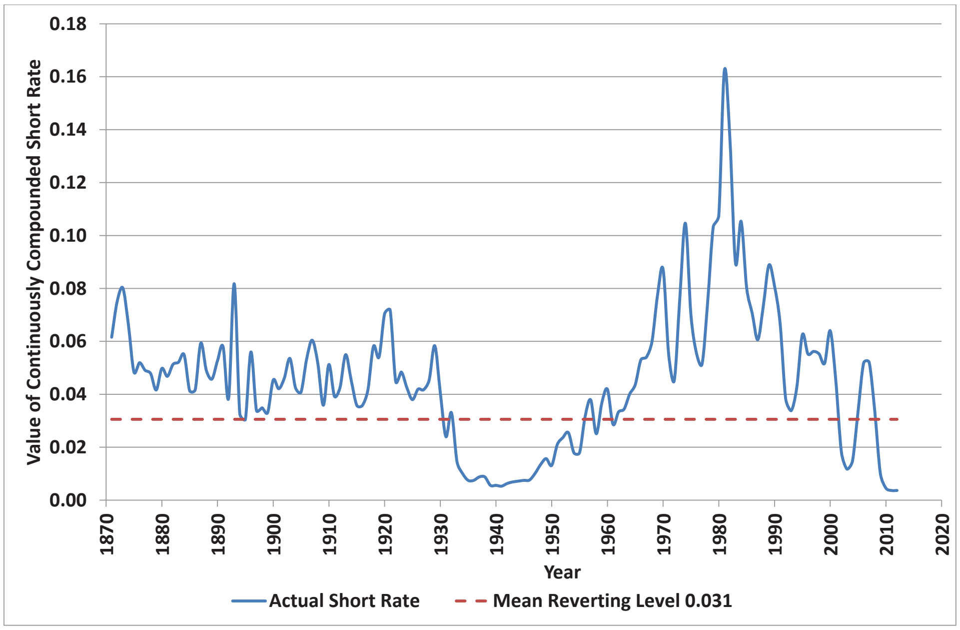
$$\frac{\sigma^2 - q}{p}. \tag{3.4.15}$$

The inverse of this level is not the mean reverting level of r_t. The limiting distribution of $\frac{1}{4}\sigma^2 r_t/p$ as $t \to \infty$ is an inverse chi-squared distribution with ν degrees of freedom and the mean of r_t as $t \to \infty$ is deduced to be

$$\frac{4p/\sigma^2}{\nu - 2} = \frac{4p}{2\sigma^2 - 4q}. \tag{3.4.16}$$

In Figure 3.14 the graph of the short rate is shown along with the implied reverting level, the mean, of 0.03054 of the 3/2 model. The dimension of the square root process $1/r_t$ is here estimated as $\nu = 4(\sigma^2 - q)/\sigma^2 \approx 3.1790$, which is reasonably close to three.

Figure 3.14: The fitted reverting level under the 3/2 model.



3.4.4 Short Rate Contribution to ZCB Price

We calculate the contribution $G_{\bar{T}}(t)$ to the zero-coupon bond price which is due to the short rate. The following lemma gives a formula for the Laplace transform of the random variable $\int_t^T r_s ds$.

Lemma 3.4.4 *The conditional Laplace transform of* $\int_t^T r_s ds$ *satisfies*

$$\mathrm{E}\left(\exp(-u \int_t^T r_s ds) \Big| \mathcal{A}_t \right) = \frac{\Gamma(\gamma_u - \alpha_u)}{\Gamma(\gamma_u)} \left(\frac{2}{\sigma^2 y(t, r_t)} \right)^{\alpha_u} M(\alpha_u, \gamma_u, \frac{-2}{\sigma^2 y(t, r_t)}) \tag{3.4.17}$$

where

$$y(t, r_t) = r_t \int_t^T \exp((w - t)p) dw = \frac{r_t}{p} (\exp((T - t)p) - 1) \tag{3.4.18}$$

$$\alpha_u = -\left(\frac{1}{2} - \frac{q}{\sigma^2} \right) + \sqrt{\left(\frac{1}{2} - \frac{q}{\sigma^2} \right)^2 + \frac{2u}{\sigma^2}}$$

$$\gamma_u = 2\left(\alpha_u + 1 - \frac{q}{\sigma^2} \right)$$

and $M(\alpha, \gamma, z)$ is the confluent hypergeometric function, given by

$$M(\alpha, \gamma, z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma - \alpha)} \int_0^1 \exp(zu) u^{\alpha-1}(1-u)^{\gamma-\alpha-1} du \tag{3.4.19}$$

or the alternative expression

$$M(\alpha, \gamma, z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\gamma)_n} \frac{z^n}{n!}. \tag{3.4.20}$$

Proof. A slight generalisation of this result is proven in Theorem 3 in Carr and Sun [2007] and an adapted proof specifically for the 3/2 short rate model is supplied in Appendix G. For proofs involving Lie symmetries of partial differential equations, see Craddock and Lennox [2009] and Baldeaux and Platen [2013]. **Q.E.D.**

Equivalently, we have that the moment generating function of the random variable $L = \int_t^T r_s ds$ is

$$MGF_L(-u) = \frac{\Gamma(\gamma - \alpha)}{\Gamma(\gamma)} \left(\frac{2}{\sigma^2 y(t, r_t)} \right)^{\alpha} M(\alpha, \gamma, \frac{-2}{\sigma^2 y(t, r_t)}), \tag{3.4.21}$$

where the variables have their meaning in the above lemma.

The following corollary follows straightforwardly from the above lemma.

Corollary 3.4.5 *For time $t \in [0, \bar{T}]$ the short rate contribution to the ZCB price is*

$$G_{\bar{T}}(t) = \frac{\Gamma(\gamma_1 - \alpha_1)}{\Gamma(\gamma_1)} \left(\frac{2}{\sigma^2 y(t, r_t)} \right)^{\alpha_1} M(\alpha_1, \gamma_1, \frac{-2}{\sigma^2 y(t, r_t)}), \tag{3.4.22}$$

where

$$y(t, r_t) = \frac{r_t}{p} (\exp((T - t)p) - 1) \tag{3.4.23}$$

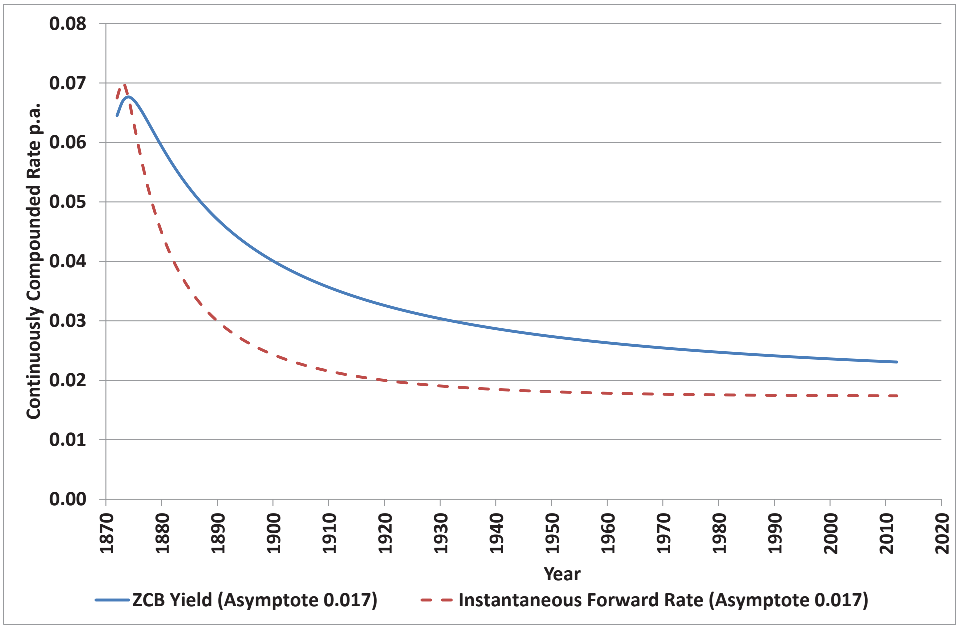
$$\alpha_1 = -\left(\frac{1}{2} - \frac{q}{\sigma^2} \right) + \sqrt{\left(\frac{1}{2} - \frac{q}{\sigma^2} \right)^2 + \frac{2}{\sigma^2}}$$

$$\gamma_1 = 2\left(\alpha_1 + 1 - \frac{q}{\sigma^2} \right).$$

Proof. Substituting $u = 1$ in Lemma 3.4.4 gives the result. **Q.E.D.**

The asymptotic value of the short rate contribution to the ZCB yield is given in the following corollary.

Figure 3.15: Short rate contribution to the zero coupon yield curve under the 3/2 model as at 1871.



Corollary 3.4.6 *For the 3/2 short rate, the contribution of the short rate to the long ZCB yield is*

$$h_\infty(t) = \alpha_1 p, \tag{3.4.24}$$

where

$$\alpha_1 = -\left(\frac{1}{2} - \frac{q}{\sigma^2}\right) + \sqrt{\left(\frac{1}{2} - \frac{q}{\sigma^2}\right)^2 + \frac{2}{\sigma^2}}. \tag{3.4.25}$$

Proof. See Appendix H.

In Figure 3.15 the continuously compounded yield curve as of January 1871 is plotted corresponding to the base time in Figure 3.13. We have an inverted yield curve, which portends an economic recession because decreasing forward rates indicate expectations of low inflation and low economic growth, see for example Harvey [1991].

3.4.5 Short Rate Contribution to the Forward Rate

We calculate the contribution $g_{\bar{T}}(t)$ to the forward rate that is due to the short rate. This is a new result which has to date not appeared in the literature.

Lemma 3.4.7 *For time $t \in [0, \bar{T}]$ the short rate contribution to the forward rate is computed to be*

$$g_{\bar{T}}(t) = \alpha_1 p \left(1 + \frac{1}{\exp(p(\bar{T}-t)) - 1}\right)\left(1 - \frac{z}{\gamma_1}\frac{M(\alpha_1+1, \gamma_1+1, -z)}{M(\alpha_1, \gamma_1, -z)}\right) \quad (3.4.26)$$

where $z = \frac{2}{\sigma^2 y(t, r_t)}$.

Proof. See Appendix H.

The asymptotic value of the forward rate is given by the following lemma.

Corollary 3.4.8 *For the 3/2 short rate model, the asymptotic instantaneous forward rate is*

$$g_\infty(t) = \alpha_1 p \quad (3.4.27)$$

where

$$\alpha_1 = -\left(\frac{1}{2} - \frac{q}{\sigma^2}\right) + \sqrt{\left(\frac{1}{2} - \frac{q}{\sigma^2}\right)^2 + \frac{2}{\sigma^2}}. \quad (3.4.28)$$

Proof. From (3.4.26) as $\bar{T} \to \infty$

$$g_{\bar{T}}(t) = \alpha_1 p \Big(1 + o(1)\Big)\Big(1 + O(z)\Big) \quad (3.4.29)$$

and because $z \to 0$ as $\bar{T} \to \infty$ we have the result. **Q.E.D.**

In Figure 3.15 the instantaneous forward rate $g_{\bar{T}}$ is plotted and can be seen to be asymptotic to $g_\infty(t) = 0.01732$ based upon the parameters in (3.4.14).

This concludes our treatment of the 3/2 short rate model and we move on to a comparison of the short rate models discussed in this chapter.

3.5 Comparison of Models

The three models considered in this chapter have explicit formulae for their transition density functions and this has allowed the fitting of parameters using maximum likelihood estimation. The Vasicek model is most easily fitted to the data because it has closed form expressions for the parameter estimates. In contrast, the CIR and 3/2 models each require three-dimensional grid searches to find the best fitting parameters.

Table 3.1: Values of the AIC in respect of various short rate models (US cash rates, see Data Set A in Section L.1 of the Appendix).

Model	Parameters	Log Likelihood	AIC
Vasicek	3	399.7019	-793.4038
CIR	3	427.8116	-849.6232
3/2	3	406.2713	-806.5426

In fitting the three models to the US cash rates we can identify which model provides the best fit to the data by looking at the Akaike Information Criterion, see Burnham and Anderson [2004] for example,

$$AIC^{(model)} = 2p - 2\log L^{(model)}(\theta_1, \dots, \theta_p) \tag{3.5.1}$$

where $L^{(model)}$ is the likelihood function and θ_j is the j-th parameter estimate among a total of p parameters of the model. The AIC measures the loss of information and, therefore, the best model is that which has the least loss of information. The AIC value of each model is shown in Table 3.1, where the CIR model appears to be the best fitting model.

To establish further whether the CIR model is a good fitting model we consider Pearson's goodness-of-fit chi-squared statistic, described in Kendall and Stuart [1961], given by

$$S = \sum_{i=1}^{k} (O_i - E_i)^2 / E_i, \tag{3.5.2}$$

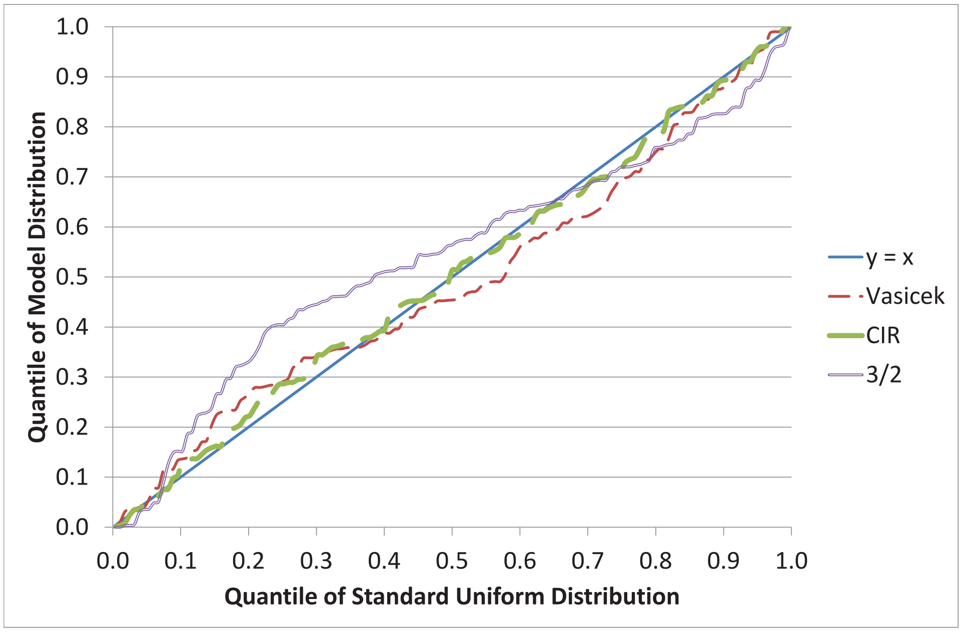
where O_i is the number of observations in category i and E_i is the corresponding expected number of observations according to the hypothesised model. The test statistic S is asymptotically distributed as χ^2_ν, where ν equals the number of categories k less the number of constraints and estimated parameters of the model.

Given a time series of interest rates $\{r_{t_j} : j = 1, 2, \dots, n\}$ and given a hypothesised transition density function with corresponding cumulative distribution function F we compute the $n-1$ quantiles $q_j = F(t_{j-1}, r_{t_{j-1}}, t_j, r_{t_j})$ for $j = 2, 3, \dots, n$. Under the hypothesised model the quantiles q_j are independent and uniformly distributed. These quantiles are graphed against those of the uniform distribution in Figure 3.16. One notes that the CIR model remains in some sense visually closest over the $[0, 1]$ interval.

A similar comparison is shown in Figure 3.17 for the monthly data series of one-year US Treasury bond yields from January 1962 to June 2014, sourced from the US Federal Reserve Bank website and referred to as Data Set D in Section L.4 of the Appendix, where a similar conclusion follows.

For a fixed integer k satisfying $2 \leq k \leq (n-1)/5$ we partition the unit interval into k equally sized subintervals. Hence we compute the number of observations

Figure 3.16: Comparison of quantile-quantile plots of short rate models (Shiller annual US data, see Data Set A in Section L.1 of the Appendix).



O_i in the i-th subinterval $((i-1)/k, i/k]$ for $i = 1, 2, \ldots, k$. The corresponding expected number of observations E_i in the i-th subinterval is $(n-1)/k$. Our test statistic is thus computed as

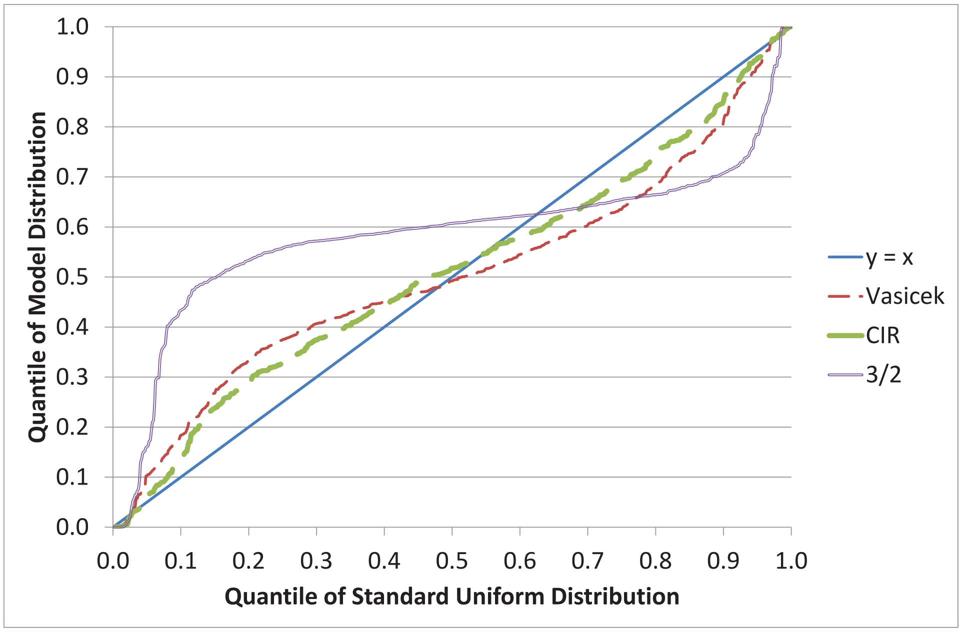
$$S = k \sum_{i=1}^{k} (O_i - (n-1)/k)^2/(n-1), \tag{3.5.3}$$

which is approximately chi-squared distributed with $\nu = k - 1 - n_{parameters}$ degrees of freedom.

The value of the Pearson's chi-squared statistic and corresponding p-value for each model and for a range of partition sizes is shown in Table 3.2. It is evident that the 3/2 model and, for some partition sizes, the Vasicek model can be rejected at the 1% level of significance and that the CIR model cannot be rejected at this level of significance. We conclude that the CIR model cannot be rejected as a valid model whereas we can reject the validity of the Vasicek model and the 3/2 model.

Another test of goodness-of-fit is the Kolmogorov-Smirnov test, as described by Stephens [1974]. Under the null hypothesis that the set of n observations $u_1, u_2, \ldots, u_n$ emanate from a uniform distribution, the Kolmogorov test statistic

Figure 3.17: Comparison of quantile-quantile plots of short rate models (US Federal Reserve monthly data, see Data Set D in Section L.4 of the Appendix).



is

$$D_n = \sup_{x \in \{u_1, u_2, \ldots, u_n\}} \max\left(F^{(n)}(x) - x, x - F^{(n)}(x) - \frac{1}{n}\right) \tag{3.5.4}$$

and the modified test statistic $K_n = \sqrt{n} D_n$ has the limiting distribution function, as $n \to \infty$,

$$F(x) = \frac{\sqrt{2\pi}}{x} \sum_{k=1}^{\infty} \exp(-(2k-1)^2 \pi^2 / (8x^2)), \tag{3.5.5}$$

where

$$F^{(n)}(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{u_i \leq x} \tag{3.5.6}$$

is the empirical cumulative distribution function. We compute the test statistics in Table 3.3 where we see that only the 3/2 short rate model can be rejected at the 1% level of significance.

Finally, another test of goodness-of-fit is the Anderson-Darling test, as described in Stephens [1974]. Under the null hypothesis that the set of n observations $u_1, u_2, \ldots, u_n$ emanate from a uniform distribution, the test statistic A is given by

$$A = \sqrt{-n - S}, \tag{3.5.7}$$

Table 3.2: Pearson's chi-squared statistic with p-values shown in brackets in respect of various short rate models (US cash rates, see Data Set A in Section L.1 of the Appendix).

k	ν	Vasicek	CIR	3/2
5	1	8.7143 (0.3157%)	0.5714 (44.9703%)	30.6429 (0.0000%)
10	6	13.7143 (3.2996%)	1.8571 (93.2361%)	35.5714 (0.0003%)
15	11	21.7857 (2.6087%)	10.2143 (51.1220%)	37.6429 (0.0090%)
20	16	34.0000 (0.5433%)	11.1429 (80.0580%)	42.5714 (0.0324%)
25	21	30.3571 (8.5044%)	18.9286 (58.9720%)	41.7857 (0.4476%)

Table 3.3: Kolmogorov-Smirnov test statistics in respect of various short rate models (US cash rates, see Data Set A in Section L.1 of the Appendix).

	Vasicek	CIR	3/2
D_n	0.08468	0.04526	0.17092
n	141	141	141
K_n	1.00556	0.53749	2.02958
$F(K_n)$	0.73592	0.06518	0.99947
p-value	0.26408	0.93482	0.00053

where

$$S = \sum_{i=1}^{n} \frac{2i-1}{n} \big(\log u_i + \log(1 - u_{n+1-i}) \big). \tag{3.5.8}$$

We compute the test statistics in Table 3.4 where, as for the Kolmogorov-Smirnov test, we see that only the 3/2 short rate model can be rejected at the 1% level of significance. The p-values of the test statistic A in Table 3.4 were estimated using sample Anderson-Darling statistics of 1000 simulations of sets of 141 uniformly distributed observations.

Table 3.4: Anderson-Darling test statistics in respect of various short rate models (US cash rates, see Data Set A in Section L.1 of the Appendix).

	Vasicek	CIR	3/2
S	-142.50533	-141.28374	-146.21838
n	141	141	141
$A^2 = -n - S$	1.50533	0.28374	5.21838
A	1.22692	0.53267	2.28438
p-value	0.1898	0.9546	0.0015

3.6 Conclusions

In this chapter we have provided explicit formulae for transition densities, contributions to the ZCB prices, contributions to the forward rates and contributions to the prices of options on ZCBs under the three short rate models. The proofs of formulae for expectations involving the short rate contribution G, which lead to explicit formulae for Vasicek and CIR bond option prices, are original. Furthermore, we have adjudicated on the goodness of fits of the short rate models to US data, where it is evident that the CIR short rate model provides the best fit.

Chapter 4

Two Discounted GOP Models

4.1 Introduction

A discounted equity index is computed as the ratio of an equity index to the accumulated savings account denominated in the same currency. In this way, discounting provides a natural way of separating the modelling of the short rate from the market price of risk component of the equity index. In this vein, we investigate the applicability of maximum likelihood estimation to stochastic models of a discounted equity index, providing explicit formulae for parameter estimates[1]. We restrict our consideration to two important index models, namely the Black-Scholes model and the minimal market model of Platen, each having an explicit formula for the transition density function. The first model is the standard continuous time market model under the classical risk neutral assumption, whereas the second model is the standard model under the benchmark approach, discussed in Platen and Heath [2006]. Explicit formulae for the estimates of model parameters and their standard errors are derived which then are used in fitting the two models to US data. Further, we demonstrate the effect of the model choice on the classical no-arbitrage assumption employed in risk neutral pricing.

The application of maximum likelihood estimation is well studied for stochastic models of equity indices, starting from Mandelbrot [1963] and Fama [1963] and summarised more recently in Behr and Pötter [2009]. However, in the current article we are interested in the application to models of discounted equity indices, also examined in Baldeaux et al. [2015]. Our motivation stems from the benchmark approach, introduced in Platen [2004], whereby a *benchmark* is constructed as the "best" (in several ways) performing portfolio for use as a numéraire or reference unit. The benchmark approach uses the growth optimal portfolio (GOP) as benchmark, as discussed in Platen and Heath [2006]. The GOP achieves the maximum possible expected growth rate at any time, and also the almost

[1]This work has been published in Fergusson [2017b].

surely maximum long-term growth rate, as shown in Platen [2004]. When used as benchmark, each benchmarked non-negative portfolio can be shown to be a supermartingale with its current benchmarked value greater than or equal to the expected future benchmarked values. As such, the GOP is the *"best" performing portfolio* in this sense. It has been studied and employed previously, for example in Kelly [1956], Long [1990], Merton [1971], Karatzas and Shreve [1998], Platen [2002b] and by many other authors.

We work on a filtered probability space $(\Omega, \underline{\mathcal{A}}, (\mathcal{A}_t)_{t\geq 0}, P)$, where Ω is the sample space, $\underline{\mathcal{A}}$ is the set of events, $(\mathcal{A}_t)_{t\geq 0}$ is the filtration of events modelling the evolution of information and P is the real-world probability measure. It is shown in Platen [2004] that the GOP, denoted by $S_t^{\delta_*}$ at time t, for $t \in [0,T]$, satisfies the SDE

$$dS_t^{\delta_*} = S_t^{\delta_*}\left(r_t + \theta_t^2\right) dt + S_t^{\delta_*}\,\theta_t\, d\hat{W}_t, \tag{4.1.1}$$

where r_t is the short rate, θ_t is the market price of risk and $\hat{W}$ is a Wiener process. In Platen [2005b] and Platen and Rendek [2012a] it is shown that appropriately *diversified portfolios* represent approximate GOPs. Therefore, a number of common, well-diversified stock market indices can be used to approximate the GOP, including but not limited to the following: the Standard and Poor's 500 Index (S&P 500) and the Russell 2000 Index for the US market and the MSCI Growth World Stock Index (MSCI) for global modelling.

In Figure 4.1 we plot the logarithm of the S&P 500 denominated in United States dollars (USD), and normalised to one at the start, over the period from January 1871 to March 2017, using data sourced from the website of Shiller [1989] (see Data Set C in Section L.3 of Appendix L). By assuming that the GOP for the US equity market is approximated by the S&P 500, Figure 4.1 can be interpreted as the logarithm of a historical sample path for the GOP.

Note from (4.1.1) that the GOP dynamics are completely characterized by the short rate r_t and the market price of risk θ_t. Letting B_t denote the accumulated value of the savings account, satisfying the SDE $dB_t = r_t B_t dt$, with $B_0 = 1$, we can separate these two effects by considering the *discounted GOP* process $\bar{S}^{\delta_*} = \{\bar{S}_t^{\delta_*},\, t \in [0,T]\}$, given by

$$\bar{S}_t^{\delta_*} = \frac{S_t^{\delta_*}}{B_t}, \tag{4.1.2}$$

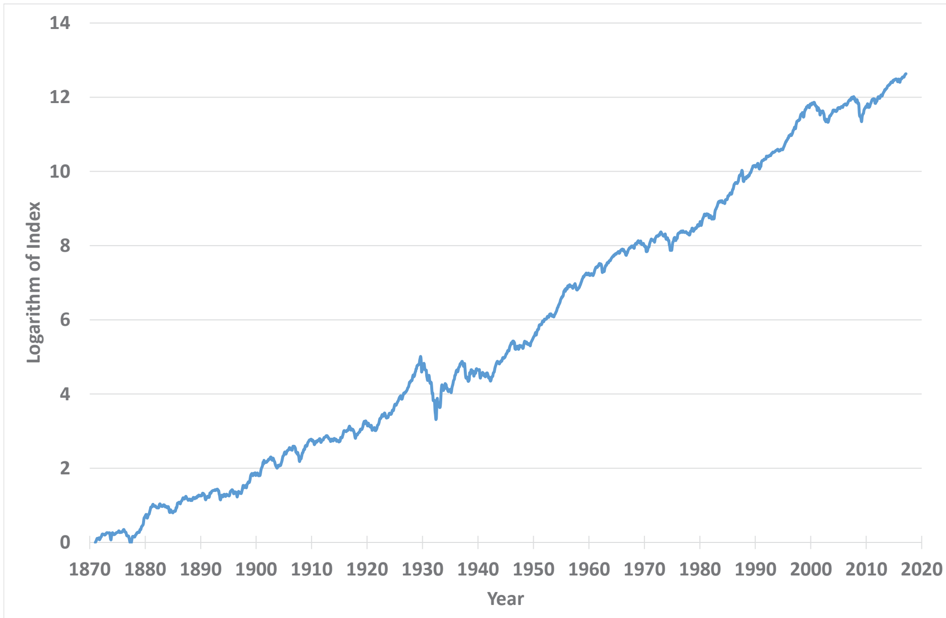
satisfying the SDE

$$d\bar{S}_t^{\delta_*} = \bar{S}_t^{\delta_*}\,\theta_t^2\, dt + \bar{S}_t^{\delta_*}\,\theta_t\, d\hat{W}_t, \tag{4.1.3}$$

for $t \in [0,T]$.

From, for example, Karatzas and Shreve [1998], one notes that in a complete market, the candidate Radon-Nikodym derivative process $\Lambda^Q = \{\Lambda_t^Q : t \in [0,T]\}$ for the putative risk neutral measure Q is equal to the reciprocal of the discounted

Figure 4.1: Logarithm of the S&P 500 in USD from January 1871 to March 2017.



GOP. That is,

$$\Lambda_t^Q = \left.\frac{dQ}{dP}\right|_{\mathcal{A}_t} = \frac{B_t}{S_t^{\delta_*}}\frac{S_0^{\delta_*}}{B_0} = \frac{\bar{S}_0^{\delta_*}}{\bar{S}_t^{\delta_*}}. \tag{4.1.4}$$

A necessary condition for Q to be a probability measure is that

$$Q(\Omega) = 1. \tag{4.1.5}$$

Violation of Condition 4.1.5 implies that Q is not equivalent to P and, therefore, an equivalent risk neutral probability measure does not exist in this case. The Fundamental Theorem of Asset Pricing, as given in Delbaen and Schachermayer [2006], states the classical no-arbitrage condition which is equivalent to the existence of an equivalent risk neutral probability measure. Therefore, violation of Condition 4.1.5 means that the corresponding dynamics permit some form of classical arbitrage. It is important to understand whether in reality one can support the classical no-arbitrage assumption or whether there is significant evidence from historical data that favour a more general modelling framework where this assumption is not imposed.

In this article we consider a complete market. Two models of the discounted GOP $\bar{S}^{\delta_*}$ are considered, these being the Black-Scholes (BS) model,

$$d\bar{S}_t^{\delta_*} = \bar{S}_t^{\delta_*}\theta_t^2 dt + \bar{S}_t^{\delta_*}\theta_t dW_t, \tag{4.1.6}$$

where $\theta_t = \theta > 0$, and the minimal market model (MMM),

$$d\bar{S}_t^{\delta_*} = \bar{\alpha}_t dt + \sqrt{\bar{S}_t^{\delta_*} \bar{\alpha}_t} dW_t, \tag{4.1.7}$$

where $\theta_t = \sqrt{\bar{\alpha}_t / \bar{S}_t^{\delta_*}}$ with $\bar{\alpha}_t = \bar{\alpha}_0 \exp(\eta t)$, $\eta > 0$.

The Black-Scholes model is the standard market model under classical no-arbitrage assumptions. It derives its name from the geometric Brownian model assumed in Black and Scholes [1973].

The MMM includes the discounted GOP process in (4.1.7) with the requirement that the net GOP drift term $\bar{\alpha}_t = \bar{S}_t^{\delta_*} |\theta_t|^2$ grows exponentially as $\bar{\alpha}_t = \bar{\alpha}_0 \exp(\eta t)$ with the net growth rate $\eta > 0$ reflecting the long-term average growth rate of $\bar{S}^{\delta_*}$ and, thus, the economy. Each of these models has one key parameter, the volatility θ for the BS model and the net growth rate η for the MMM.

We examine each of the two discounted GOP models and, in doing so, fit each to historical data, calculate the relevant contribution M to the ZCB price, the relevant contribution m to the instantaneous forward rate and the price of the option on the ZCB contribution M.

4.2 Black-Scholes Model of Discounted GOP

From (4.1.3) we have the SDE that is assumed to be satisfied by the discounted GOP under the BS model. By insisting that the market price of risk θ_t be a constant θ, the SDE for the discounted GOP $\bar{S}_t^{\delta_*}$ becomes

$$d\bar{S}_t^{\delta_*} = \bar{S}_t^{\delta_*} (\theta^2 dt + \theta dW_t). \tag{4.2.1}$$

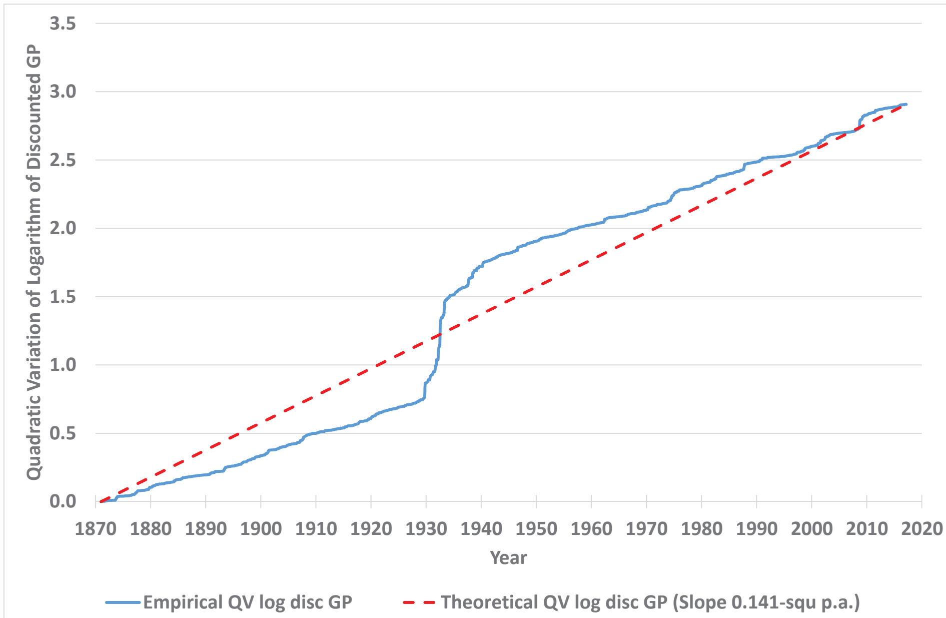
From (4.2.4) the theoretical quadratic variation of $\log\left(\bar{S}_t^{\delta_*}\right)$ is

$$[\log\left(\bar{S}^{\delta_*}\right)]_t = \int_0^t \theta^2 ds = \theta^2 t, \tag{4.2.2}$$

which we show in Figure 4.2 alongside the actual empirical quadratic variation of the logarithm of the discounted GOP, computed from Shiller's monthly series of discounted values of the S&P 500 over the period from 1871 to 2017, referred to as Data Set C in Section L.3 of Appendix L.

Remark 4.2.1 *It is evident from Figure 4.2 that the Black-Scholes model of the discounted GOP fails to capture the stochastic nature of the market price of risk θ_t since the slope of the quadratic variation of the logarithm of the discounted stock index moves significantly over time.*

Figure 4.2: Quadratic variation of logarithm of discounted stock index.



From (4.1.4) the Radon-Nikodym derivative of the risk neutral probability measure is equal to the reciprocal of the discounted GOP which, under the BS model, is given by

$$\Lambda_t^Q = \exp(-\frac{1}{2}\theta^2 t - \theta W_t). \tag{4.2.3}$$

Remark 4.2.2 *Risk neutral pricing of derivative contracts relies on the Radon-Nikodym derivative Λ^Q being a martingale with respect to the real-world probability measure P. This condition is indeed satisfied under the model because $\mathrm{E}(\Lambda_t^Q|\mathcal{A}_s) = \Lambda_s^Q$ for $0 \leq s \leq t < \infty$.*

We show in Figure 4.3 $\Lambda_t^Q = 1/\bar{S}_t^{\delta_*}$ for the S&P 500, where we note that this trajectory seems unlikely to be that of a true martingale.

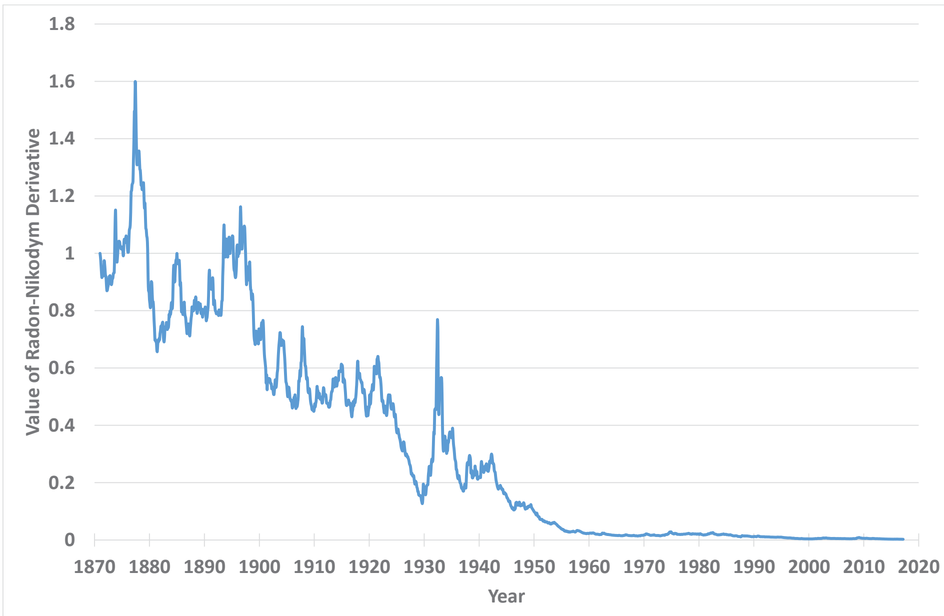
In the following section we derive the transition density function of the discounted GOP, used to fit the parameters to data.

4.2.1 Transition Density of Discounted GOP

The logarithm of the discounted GOP obeys the SDE

$$d\log\left(\bar{S}_t^{\delta_*}\right) = \frac{1}{2}\theta^2 dt + \theta dW_t \tag{4.2.4}$$

Figure 4.3: Radon-Nikodym derivative $\Lambda_t^Q = 1/\bar{S}_t^{\delta_*}$ of the putative risk neutral probability measure.



whose solution is immediately found to be $\log\left(\bar{S}_t^{\delta_*}\right) = \log\left(\bar{S}_0^{\delta_*}\right) + \frac{1}{2}\theta^2 t + \theta(W_t - W_0)$. This gives rise to the following lemma.

Lemma 4.2.3 *The transition density function of the logarithm of the discounted GOP in* (4.2.1) *is*

$$p_{\log\left(\bar{S}^{\delta_*}\right)}\left(t, \log(x_t), \bar{T}, \log(x_{\bar{T}})\right) \tag{4.2.5}$$
$$= \frac{1}{\sqrt{2\pi}\sqrt{(\bar{T}-t)\theta^2}} \exp\left(-\frac{1}{2}\left(\frac{\log\left(\frac{x_{\bar{T}}}{x_t}\right) - \frac{1}{2}(\bar{T}-t)\theta^2}{\sqrt{(\bar{T}-t)\theta^2}}\right)^2\right)$$

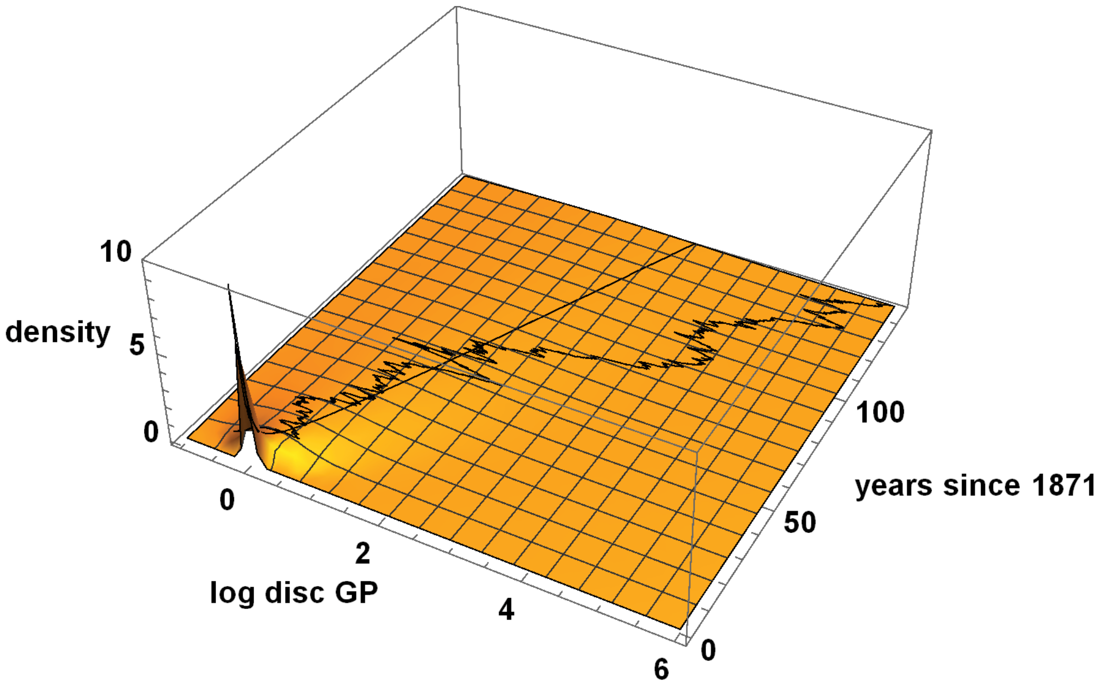
and the transition density function of the discounted GOP is

$$p_{\bar{S}^{\delta_*}}(t, x_t, \bar{T}, x_{\bar{T}}) = \frac{1}{x_{\bar{T}}\sqrt{2\pi}\sqrt{(\bar{T}-t)\theta^2}} \exp\left(-\frac{1}{2}\left(\frac{\log\left(\frac{x_{\bar{T}}}{x_t}\right) - \frac{1}{2}(\bar{T}-t)\theta^2}{\sqrt{(\bar{T}-t)\theta^2}}\right)^2\right). \tag{4.2.6}$$

By virtue of this lemma we can write the conditional distribution of $\bar{S}_{\bar{T}}^{\delta_*}$ given $\bar{S}_t^{\delta_*}$ as

$$\log\left(\bar{S}_{\bar{T}}^{\delta_*}\right) \sim N\left(\log \bar{S}_t^{\delta_*} + \frac{1}{2}(\bar{T}-t)\theta^2, (\bar{T}-t)\theta^2\right) \tag{4.2.7}$$

Figure 4.4: Transition density function of logarithm of Black-Scholes discounted GOP based at 1871 with trajectory of log discounted S&P 500 and theoretical mean.



Below, we plot the transition density function of the logarithm of the discounted GOP in Figure 4.4 for the fitted parameter θ when starting in January 1871 with $\bar{S}_0^{\delta_*} = 1$. We also include in the graph the theoretical mean of the logarithm of the index and the logarithm of the index.

4.2.2 Fitting the Black-Scholes Model of Discounted GOP

We use maximum likelihood estimation to fit the model (4.2.1) to Shiller's monthly series of discounted values of the S&P 500 over the period from 1871 to 2017, referred to as Data Set C in Section L.3 of Appendix L.

For a time discretisation $t_i = i\Delta$, $\Delta > 0$, our log-likelihood function is

$$\ell_\Delta(\theta) = -\frac{1}{2}\sum_{i=1}^{n}\left\{\log(2\pi\theta^2\Delta) + \frac{\left(\log\left(\bar{S}_{t_i}^{\delta_*}\right) - \log\left(\bar{S}_{t_{i-1}}^{\delta_*}\right) - \frac{1}{2}\theta^2\Delta\right)^2}{\theta^2\Delta}\right\}, \quad (4.2.8)$$

and, although widely known (for example, see Chapter 8 of Rice [2007]), an explicit formula for the maximum likelihood estimate of the parameter θ and its standard error is supplied in the following theorem.

Theorem 4.2.4 *The square of the maximum likelihood estimate $\hat{\theta}_\Delta$ of θ in the*

SDE (4.2.1), *with time step size* Δ, *is given by*

$$\hat{\theta}^2_\Delta = \frac{-2 + 2\sqrt{1 + \frac{1}{n}\sum_{i=1}^n (\log \bar{S}^{\delta_*}_{t_i}/\bar{S}^{\delta_*}_{t_{i-1}})^2}}{\Delta} \tag{4.2.9}$$

where there are $n+1$ *observations of the discounted GOP* $\bar{S}^{\delta_*}_{t_i}$, *for* $i = 0, 1, 2, \ldots, n$. *Further, the standard error of the parameter estimate is given by*

$$\mathrm{SE}(\hat{\theta}_\Delta) = \frac{1}{\sqrt{n\Delta + \frac{2n}{\hat{\theta}^2_\Delta}}}. \tag{4.2.10}$$

Proof. We rewrite (4.2.8) as

$$\ell(\theta) = -\frac{1}{2}\sum_{i=1}^n \left\{ \log(2\pi\Delta) + 2\log(\theta) + \frac{\big(\log(\bar{S}^{\delta_*}_{t_i}/\bar{S}^{\delta_*}_{t_{i-1}})\big)^2}{\theta^2\Delta} - \log(\bar{S}^{\delta_*}_{t_i}/\bar{S}^{\delta_*}_{t_{i-1}}) + \frac{1}{4}\theta^2\Delta \right\} \tag{4.2.11}$$

and differentiating with respect to θ gives

$$\ell'(\theta) = -\frac{1}{2}\sum_{i=1}^n \left\{ \frac{2}{\theta} - \frac{2\big(\log(\bar{S}^{\delta_*}_{t_i}/\bar{S}^{\delta_*}_{t_{i-1}})\big)^2}{\theta^3\Delta} + \frac{1}{2}\theta\Delta \right\}. \tag{4.2.12}$$

Rearranging (4.2.12) gives

$$\begin{aligned} \ell'(\theta) &= -\frac{1}{4\theta^3}\sum_{i=1}^n \left\{ 4\theta^2 - \frac{4\big(\log(\bar{S}^{\delta_*}_{t_i}/\bar{S}^{\delta_*}_{t_{i-1}})\big)^2}{\Delta} + \theta^4\Delta \right\} \qquad (4.2.13) \\ &= -\frac{1}{4\theta^3}\left(n\Delta\,\theta^4 + 4n\theta^2 - \sum_{i=1}^n \frac{4\big(\log(\bar{S}^{\delta_*}_{t_i}/\bar{S}^{\delta_*}_{t_{i-1}})\big)^2}{\Delta} \right). \end{aligned}$$

The solution to the equation $\ell'(\theta) = 0$ is given by the formula (4.2.9). To determine the standard error of the estimate of θ we take the second derivative of ℓ, giving

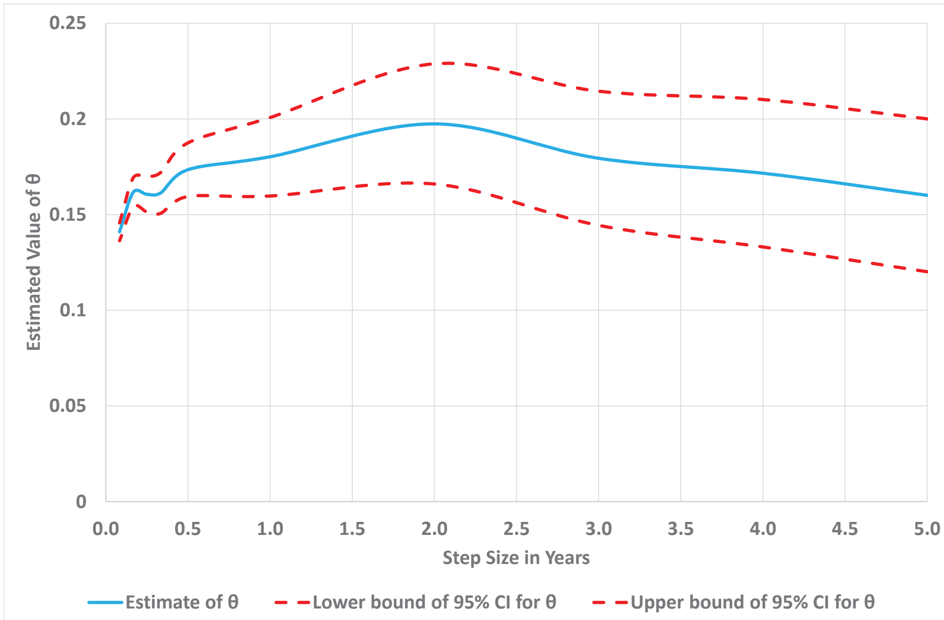
$$\ell''(\theta) = -\frac{1}{4\theta^4}\left(n\Delta\,\theta^4 - 4n\theta^2 + 3\sum_{i=1}^n \frac{4\big(\log(\bar{S}^{\delta_*}_{t_i}/\bar{S}^{\delta_*}_{t_{i-1}})\big)^2}{\Delta} \right). \tag{4.2.14}$$

When $\theta = \hat{\theta}_\Delta$ the second derivative can be simplified to

$$\begin{aligned} \ell''(\theta) &= -\frac{1}{4\theta^4}\left(4n\Delta\,\theta^4 + 8n\theta^2 \right) \qquad (4.2.15) \\ &= -\left(n\Delta + \frac{2n}{\theta^2} \right). \end{aligned}$$

The standard error is given by the reciprocal of the square root of the negative of the second derivative, giving the result (4.2.10). **Q.E.D.**

Figure 4.5: Graph showing how the estimates of θ vary with step size.



From this theorem we compute the maximum likelihood estimate for various time step sizes. In Table 4.3 we show the estimate $\hat{\theta} = 0.141002\,(0.002380)$ with the standard error shown in brackets.

In Figure 4.5 we plot the estimate with its 95% confidence interval in dependence on the time step size. We note that for some larger time step sizes the confidence intervals become unrealistic as the normality assumption of the standard error is invalid.

4.2.3 Discounted GOP Contribution to the ZCB Price

The following lemma gives the contribution of the discounted GOP to the ZCB price. We remark that the same contributions emerge for any model which has a risk neutral probability measure and where the savings account and the discounted GOP are independent.

Lemma 4.2.5 *For $\bar{S}^{\delta_*}$ satisfying* (4.2.1) *the discounted GOP contribution to the ZCB price is*

$$M_{\bar{T}}(t) = 1 \tag{4.2.16}$$

Proof. From (2.8.3) we have

$$
\begin{aligned}
M_{\bar{T}}(t) &= \mathrm{E}\left(\frac{\bar{S}_t^{\delta_*}}{\bar{S}_{\bar{T}}^{\delta_*}} \middle| \mathcal{A}_t \right) \\
&= \int_0^\infty \frac{\bar{S}_t^{\delta_*}}{\bar{S}_{\bar{T}}^{\delta_*}} p_{\bar{S}^{\delta_*}}(t, \bar{S}_t^{\delta_*}, \bar{T}, \bar{S}_{\bar{T}}^{\delta_*})\, d\bar{S}_{\bar{T}}^{\delta_*} \\
&= 1,
\end{aligned}
\tag{4.2.17}
$$

which is the result. **Q.E.D.**

The following corollary is trivial.

Corollary 4.2.6 *The discounted GOP contribution to the long ZCB yield is zero, that is*

$$n_\infty(t) = 0. \tag{4.2.18}$$

We now consider the contribution of the discounted GOP to the forward rate.

4.2.4 Discounted GOP Contribution to the Forward Rate

The following lemma gives the contribution of the discounted GOP to the instantaneous forward rate. As mentioned in the previous section, we remark that the same contributions emerge for any model which has a risk neutral probability measure and where the savings account and the discounted GOP are independent.

Lemma 4.2.7 *For $\bar{S}^{\delta_*}$ satisfying* (4.2.1) *the discounted GOP contribution to the forward rate is*

$$m_{\bar{T}}(t) = 0 \tag{4.2.19}$$

Proof. From (2.8.12) and (4.2.16) we have

$$m_{\bar{T}}(t) = \frac{\partial}{\partial \bar{T}} \log M_{\bar{T}}(t) = \frac{\partial}{\partial \bar{T}} \log 1 = 0 \tag{4.2.20}$$

which is the result. **Q.E.D.**

The following section contains some integrals, which will be useful in calculating option prices in Chapter 5.

Table 4.1: Formulae for expectations involving a standard normal random variable Z.

Expectation	Integral	Formula
$E(\exp(\alpha Z)\mathbf{1}_{Z>z})$	$\int_z^\infty e^{\alpha u} n(u)du$	$\exp(\frac{1}{2}\alpha^2)\big(1-N(z-\alpha)\big)$
$E(\exp(\alpha Z)\mathbf{1}_{Z\leq z})$	$\int_{-\infty}^z e^{\alpha u} n(u)du$	$\exp(\frac{1}{2}\alpha^2)N(z-\alpha)$
$E(\exp(\alpha Z))$	$\int_{-\infty}^\infty e^{\alpha u} n(u)du$	$\exp(\frac{1}{2}\alpha^2)$
$E(N(\alpha Z+\beta))$	$\int_{-\infty}^\infty N(\alpha u+\beta)n(u)du$	$N\left(\frac{\beta}{\sqrt{1+\alpha^2}}\right)$
$E\big(\exp(\gamma Z)N(\alpha Z+\beta)\big)$	$\int_{-\infty}^\infty e^{\gamma u}N(\alpha u+\beta)n(u)du$	$\exp(\frac{1}{2}\gamma^2)N\left(\frac{\alpha\gamma+\beta}{\sqrt{1+\alpha^2}}\right)$

4.2.5 Expectations Involving a Standard Normal Random Variable

We provide formulae for expectations involving a standard normal random variable Z in Table 4.1 which are used in pricing options on the discounted GOP under the real-world measure. We employ the notation n and N where $n(u) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}u^2)$ is the probability density function of the standard normal distribution and $N(z) = \int_{-\infty}^z n(u)du$ is the cumulative probability function of the standard normal distribution.

The following lemma proves the first three expectations in Table 4.1.

Lemma 4.2.8 *We have*

$$E(\exp(\alpha Z)) = \exp(\frac{1}{2}\alpha^2) \tag{4.2.21}$$

$$E(\exp(\alpha Z)\mathbf{1}_{Z>z}) = \exp(\frac{1}{2}\alpha^2)\big(1-N(z-\alpha)\big)$$

$$E(\exp(\alpha Z)\mathbf{1}_{Z\leq z}) = \exp(\frac{1}{2}\alpha^2)N(z-\alpha)$$

Proof. This is a particular case of Lemma 3.2.18 with $Y = \alpha Z$. See Appendix J. **Q.E.D.**

The fourth expectation in Table 4.1 is proven in the following lemma.

Lemma 4.2.9

$$E(N(\alpha Z+\beta)) = N\left(\frac{\beta}{\sqrt{1+\alpha^2}}\right) \tag{4.2.22}$$

Proof. See Appendix J.

The following lemma validates the fifth expectation in Table 4.1.

Lemma 4.2.10

$$E(\exp(\gamma Z)N(\alpha Z+\beta)) = \exp(\frac{1}{2}\gamma^2)N\left(\frac{\alpha\gamma+\beta}{\sqrt{1+\alpha^2}}\right) \tag{4.2.23}$$

Proof. See Appendix J.

We have sufficient machinery developed for the Black-Scholes discounted GOP model to price derivatives in Chapter 5 and so we now proceed with another model of the discounted GOP.

4.3 Minimal Market Model of Discounted GOP

In this section we consider the *minimal market model* (MMM) originally proposed in Platen [2001]. By making the assumption that the drift $\hat{\alpha}_t = \theta_t^2 \bar{S}_t^{\delta_*}$ of the discounted GOP in (4.1.3) behaves exponentially we arrive at the SDE (4.1.7) for the discounted GOP, where

$$\bar{\alpha}_t = \bar{\alpha}_0 \exp(\eta t) \tag{4.3.1}$$

and η is the *net growth rate*. Here the net growth rate η can be viewed as the growth rate of the GOP in excess of the short rate.

The square root of the discounted GOP has SDE

$$d\big(\sqrt{\bar{S}_t^{\delta_*}}\big) = \frac{3}{8\sqrt{\bar{S}_t^{\delta_*}}}\bar{\alpha}_t\,dt + \frac{1}{2}\sqrt{\bar{\alpha}_t}\,dW_t. \tag{4.3.2}$$

It follows that the discounted GOP drift can be observed as four times the first order time derivative of the quadratic variation of the square root of the discounted GOP, that is,

$$\bar{\alpha}_t = 4\frac{d}{dt}[\sqrt{\bar{S}^{\delta_*}}]_t. \tag{4.3.3}$$

In Figure 4.8 the quadratic variation of the square root of the discounted stock index is shown. It is clear that the exponential function $\bar{\alpha}_t = \bar{\alpha}_0 \exp(\eta t)$ provides a good fit and thus confirms the assumptions for the net market trend in (4.3.1) and constant growth rate η as being reasonable.

Under the MMM, Platen [2002b] showed that the discounted GOP obeys a time-transformed squared Bessel process of dimension four.

We define the normalised discounted GOP process Y_t as the ratio of the discounted GOP to its net market trend. The SDE satisfied by Y_t is

$$\begin{aligned} dY_t &= \bar{\alpha}_t^{-1} d\bar{S}_t^{\delta_*} - \bar{\alpha}_t^{-2} \bar{S}_t^{\delta_*} d\bar{\alpha}_t \\ &= dt + \sqrt{\bar{\alpha}_t^{-1} \bar{S}_t^{\delta_*}}\, dW_t - \bar{\alpha}_t^{-1} Y_t \times \eta \bar{\alpha}_t \\ &= (1 - \eta Y_t) dt + \sqrt{Y_t} dW_t. \end{aligned} \tag{4.3.4}$$

Making the substitutions $c_t = 1$, $\nu = 4$, $b_t = -\eta$, $z_0 = \bar{\alpha}_0$ in Lemma 3.3.3 in Chapter 3 we see that $Y_t = \bar{\alpha}_0 \exp(-\eta t) X_{\varphi_t}$, where X is a squared Bessel process of dimension four with $X_0 = \bar{S}_0^{\delta_*}$ and $\varphi_t = (\exp(\eta t) - 1)/(4\eta)$.

We observe that the market price of risk $\theta_t = \sqrt{\bar{\alpha}_t / \bar{S}_t^{\delta_*}} = 1/\sqrt{Y_t}$ is given as the reciprocal of the square root of the normalised discounted GOP. Therefore, the squared market price of risk $v_t = \theta_t^2 = 1/Y_t$ obeys the SDE

$$dv_t = d(Y_t^{-1}) \tag{4.3.5}$$

$$= -Y_t^{-2} dY_t + \frac{1}{2} \times 2 Y_t^{-3} d[Y]_t \tag{4.3.6}$$

$$= -(Y_t^{-2} - \eta Y_t^{-1}) dt - Y_t^{-3/2} dW_t + Y_t^{-2} dt \tag{4.3.7}$$

$$= \eta v_t dt - v_t^{3/2} dW_t. \tag{4.3.8}$$

This SDE is known as the 3/2 volatility model, proposed in Platen [1997]. The *leverage effect*, as explained in Black [1976], pertains to the multiplied losses associated with severely adverse movements in the stock index. The negative correlation between the variable v_t and the discounted GOP $\bar{S}_t^{\delta_*}$ shows how the leverage effect is naturally incorporated into the MMM.

4.3.1 Transition Density of Discounted GOP

Before we give the transition density function of the discounted GOP we prove that the time transformed discounted GOP is a squared Bessel process of dimension four.

Lemma 4.3.1 *The discounted GOP process $\bar{S}^{\delta_*} = \{\bar{S}_t^{\delta_*},\, t \in [0, T]\}$ given by the SDE* (4.1.7) *is a time-transformed squared Bessel process of dimension four.*

Proof. Define the $(\underline{\mathcal{A}}, P)$-local martingale $\bar{U} = \{\bar{U}_t,\, t \in [0, T]\}$ as a solution to the SDE

$$d\bar{U}_t = \sqrt{\frac{\bar{\alpha}_t}{4}}\, d\hat{W}_t \tag{4.3.9}$$

for $t \in [0, T]$. Applying the Dambis, Dubins-Schwarz (DDS) Theorem, for example as given in Klebaner [1998], the DDS Wiener process is $\bar{U}_t = W_{[\bar{U}]_t} = W_{\varphi_t}$, where $W = \{W_\varphi,\, \varphi \in [\varphi_0, \varphi_T]\}$ is a standard Wiener process in φ-time. The time change itself $\varphi = \{\varphi_t,\, t \in [\varphi_0, \varphi_T]\}$ is defined as

$$\varphi_t = \varphi_0 + [\bar{U}]_t = \varphi_0 + \frac{1}{4}\int_0^t \bar{\alpha}_s\, ds, \tag{4.3.10}$$

or equivalently

$$d\varphi_t = \frac{1}{4}\,\bar{\alpha}_t\, dt \tag{4.3.11}$$

for $t \in [0, T]$. Therefore, the discounted GOP process $X = \{X_{\varphi_t},\, \varphi_t \in [\varphi_0, \varphi_T]\}$ with time transform of (4.3.10) is found by setting

$$X_{\varphi_t} = \bar{S}_t^{\delta_*} \tag{4.3.12}$$

for $t \in [0, T]$ with the initial condition $X_{\varphi_0} = \bar{S}_0^{\delta_*}$. Hence the SDE (4.1.7) can now be written as

$$dX_\varphi = 4\, d\varphi + \sqrt{4\, X_\varphi}\, dW_\varphi \tag{4.3.13}$$

for $\varphi \in [\varphi_0, \varphi_T]$. It follows from Revuz and Yor [1999] that X is, in the transformed φ-time (4.3.10), a squared Bessel process of dimension four. **Q.E.D.**

This lemma gives rise to the following lemma concerning the conditional distribution of the discounted GOP.

Lemma 4.3.2 *For the discounted GOP process $\bar{S}^{\delta_*}$ satisfying (4.1.7) and φ_t given by (4.3.10) we have for times $\bar{T} > t$ and given $\bar{S}_t^{\delta_*}$ that*

$$\frac{\bar{S}_{\bar{T}}^{\delta_*}}{\varphi_{\bar{T}} - \varphi_t} \tag{4.3.14}$$

is non-central chi-squared distributed with four degrees of freedom and with non-centrality parameter $\lambda = \bar{S}_t^{\delta_*}/(\varphi_{\bar{T}} - \varphi_t)$, written as

$$\frac{\bar{S}_{\bar{T}}^{\delta_*}}{\varphi_{\bar{T}} - \varphi_t} \sim \chi^2_{4, \bar{S}_t^{\delta_*}/(\varphi_{\bar{T}} - \varphi_t)}. \tag{4.3.15}$$

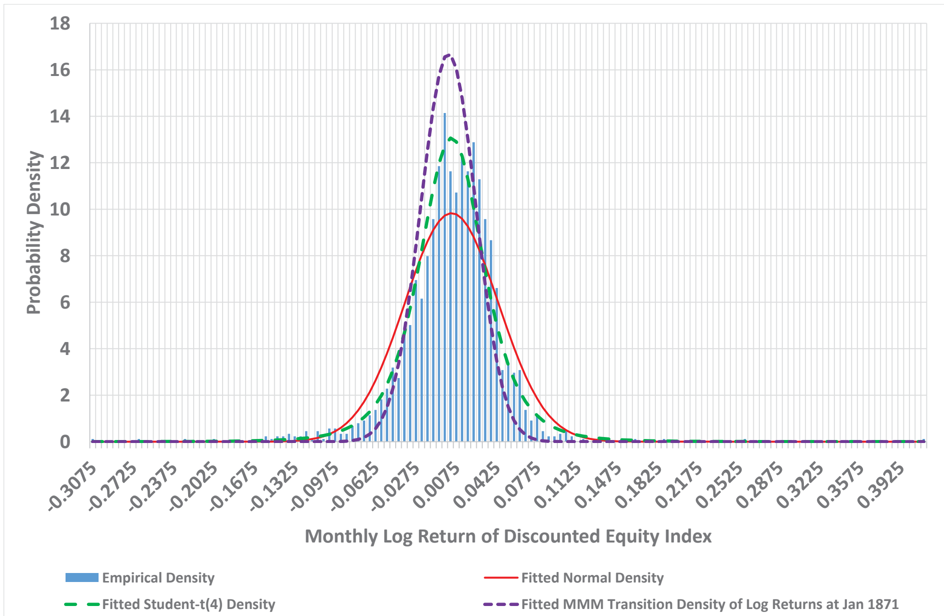
The transition density function of the discounted GOP is given explicitly in the following lemma.

Lemma 4.3.3 *Let t and $\bar{T}$ be such that $\bar{T} > t \geq 0$. The transition density function of the discounted GOP in (4.1.7) is*

$$p_{\bar{S}^{\delta_*}}(t, x_t, \bar{T}, x_{\bar{T}}) = \frac{1}{2(\varphi_{\bar{T}} - \varphi_t)}\sqrt{\frac{x_{\bar{T}}}{x_t}}\exp\left(-\frac{x_t + x_{\bar{T}}}{2(\varphi_{\bar{T}} - \varphi_t)}\right) I_1\left(\frac{\sqrt{x_t x_{\bar{T}}}}{\varphi_{\bar{T}} - \varphi_t}\right), \tag{4.3.16}$$

where $I_\nu(z) = \sum_{m=0}^{\infty} \frac{(\frac{1}{2}z)^{\nu+2m}}{m!\Gamma(\nu+m+1)}$ is the modified Bessel function of the first kind with index ν and φ_t is the quadratic variation of $\sqrt{\bar{S}_t^{\delta_*}}$ as given by (4.3.10).

Figure 4.6: Graphs of empirical, normal and Student-t stationary density functions as well as the transition density of log returns of MMM discounted GOP based at January 1871.



Remark 4.3.4 *From* (4.1.7) *the logarithm of the GOP has the SDE*

$$d\log\left(\bar{S}_t^{\delta_*}\right) = \frac{1}{2}Y_t dt + \sqrt{\frac{1}{Y_t}}dW_t, \tag{4.3.17}$$

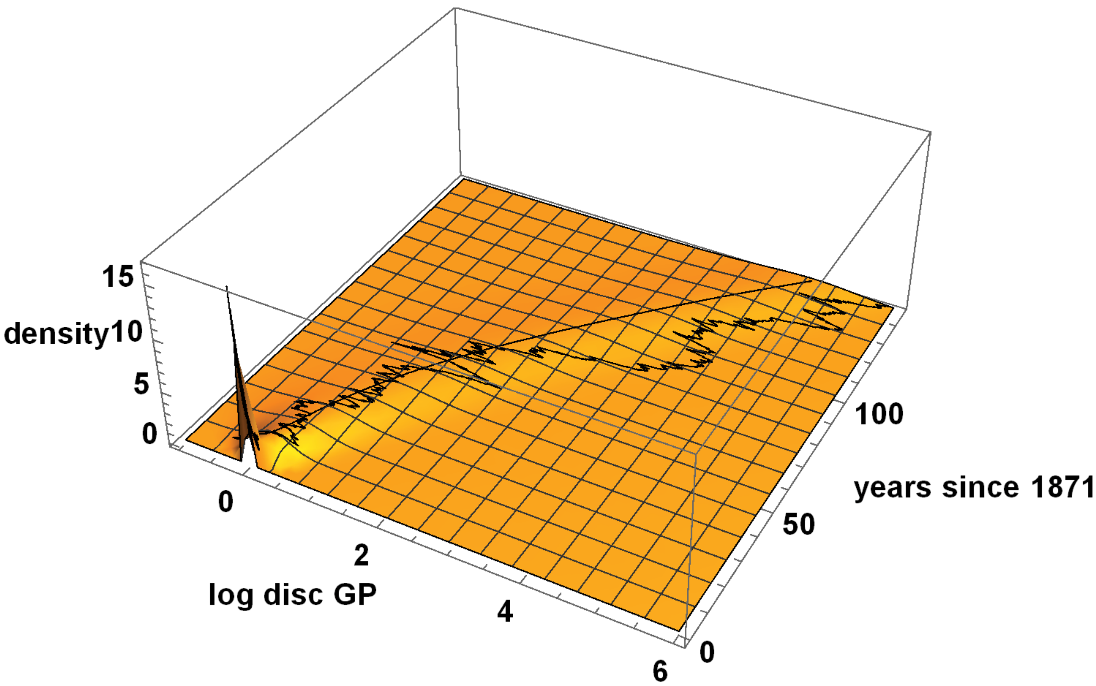
where Y_t *obeys SDE* (4.3.4). *The stationary density* $\bar{p}$ *of* Y_t *is that of a scaled chi-square distribution and is given by*

$$\bar{p}(y) = \frac{(2\eta)^2}{\Gamma(2)} y \exp(-2\eta y), \tag{4.3.18}$$

(see, for example, Chapter 4 of Platen and Heath [2006]). Therefore, under the MMM, the distribution of log returns of the discounted GOP is Student-t with four degrees of freedom. This contrasts with the corresponding distribution under the BS model being Gaussian. These stationary densities are shown in Figure 4.6. Also shown, is the transition density of log returns of MMM discounted GOP based at January 1871 which demonstrates the ability of the MMM to capture asymmetry in log returns.

The graph of the transition density function of the discounted GOP with parameter values $\bar{\alpha}_0 = 0.0068370$ and $\eta = 0.045486$ is shown in Figure 4.7.

Figure 4.7: Transition density function of MMM discounted GOP based at January 1871, with S&P 500 trajectory and theoretical mean.



4.3.2 Fitting the MMM Discounted GOP

Estimation of the parameters of the MMM discounted GOP is achieved using maximum likelihood estimation.

The log-likelihood function in respect of the observed values of the discounted GOP $\bar{S}^{\delta_*}_{t_i}$, $i = 0, 1, 2, \ldots, n$, is

$$\ell(\bar{\alpha}_0, \eta) = \sum_{i=1}^{n} \log \Bigg[\frac{1}{2(\varphi_{t_i} - \varphi_{t_{i-1}})} \sqrt{\frac{\bar{S}^{\delta_*}_{t_i}}{\bar{S}^{\delta_*}_{t_{i-1}}}} \times \exp\left\{ -\frac{1}{2} \frac{\bar{S}^{\delta_*}_{t_i} + \bar{S}^{\delta_*}_{t_{i-1}}}{\varphi_{t_i} - \varphi_{t_{i-1}}} \right\} I_1\Big(\sqrt{\bar{S}^{\delta_*}_{t_i} \bar{S}^{\delta_*}_{t_{i-1}}} / (\varphi_{t_i} - \varphi_{t_{i-1}}) \Big) \Bigg], \tag{4.3.19}$$

where $\varphi_t = \frac{1}{4}\bar{\alpha}_0(\exp(\eta t) - 1)/\eta$.

The following theorem supplies an accurate approximation to the maximum likelihood estimates as well as an explicit formula for the standard errors of the parameter estimates.

Theorem 4.3.5 *Assume that the series*

$$\left(\log \left[\left\{ \sqrt{\bar{S}^{\delta_*}_{t_i}} - \sqrt{\bar{S}^{\delta_*}_{t_{i-1}}} \right\}^2 \right] \right)_{i=1}^{n} \tag{4.3.20}$$

is approximately linear in t_i, for $i = 1, 2, \dots, n$, and that the resulting residuals ϵ_i of the linear fit have sufficiently small moments, that is

$$\frac{1}{n}\sum_{i=1}^{n}\big(\exp(\epsilon_i) - \epsilon_i - 1\big) \le \frac{K_3}{n}, \tag{4.3.21}$$

where

$$\epsilon_i = \log\left[\left\{\sqrt{\bar{S}^{\delta_*}_{t_i}} - \sqrt{\bar{S}^{\delta_*}_{t_{i-1}}}\right\}^2\right] - K_1 - K_2 t_i \tag{4.3.22}$$

and where K_1, K_2 and K_3 are positive constants independent of n. The maximum likelihood estimates $\hat{\bar{\alpha}}_{0,\Delta}$ and $\hat{\eta}_\Delta$ of $\bar{\alpha}_0$ and η respectively in the SDE (4.1.7) with time step size Δ are given by

$$\begin{aligned}\hat{\bar{\alpha}}_{0,\Delta} &\approx \frac{1}{n}\sum_{i=1}^{n}\frac{\left(\sqrt{\bar{S}^{\delta_*}_{t_i}} - \sqrt{\bar{S}^{\delta_*}_{t_{i-1}}}\right)^2}{\frac{1}{4\eta}\big(\exp(\eta\Delta) - 1\big)\exp(\eta(i-1)\Delta)} \\ \hat{\eta}_\Delta &\approx \frac{6}{\Delta(n+1)}\left[\frac{1}{\binom{n}{2}}\sum_{i\in K}(i-1)\log\left\{\left(\sqrt{\bar{S}^{\delta_*}_{t_i}} - \sqrt{\bar{S}^{\delta_*}_{t_{i-1}}}\right)^2\right\}\right. \\ &\quad \left. - \frac{1}{n}\sum_{i\in K}\log\left\{\left(\sqrt{\bar{S}^{\delta_*}_{t_i}} - \sqrt{\bar{S}^{\delta_*}_{t_{i-1}}}\right)^2\right\}\right],\end{aligned} \tag{4.3.23}$$

where summation is over the set $K = \{i = 1, 2, \dots, n : \bar{S}^{\delta_*}_{t_i} \neq \bar{S}^{\delta_*}_{t_{i-1}}\}$ and where there are $n+1$ observations of the discounted GOP $\bar{S}^{\delta_*}_t$, for $i = 0, 1, 2, \dots, n$. Further, the standard errors of the parameter estimates are approximately given by

$$\begin{aligned}\mathrm{SE}(\hat{\bar{\alpha}}_{0,\Delta}) &\approx 2\hat{\bar{\alpha}}_{0,\Delta}\sqrt{\frac{2n-1}{n(n+1)}} \\ \mathrm{SE}(\hat{\eta}_\Delta) &\approx \sqrt{\frac{24}{n(n^2-1)\Delta^2}}.\end{aligned} \tag{4.3.24}$$

Proof. Firstly, we rewrite the log likelihood function in (4.3.19) in terms of η and a new parameter $a = \frac{1}{4\eta}\bar{\alpha}_0(\exp(\eta\Delta) - 1)$, giving

$$\begin{aligned}\ell(a,\eta) &= -n\log 2 - n\log a - \eta\binom{n}{2} + \frac{1}{2}\log\left\{\frac{\bar{S}^{\delta_*}_{t_n}}{\bar{S}^{\delta_*}_{t_0}}\right\} \\ &\quad - \frac{1}{2}\sum_{i=1}^{n} y_i + \sum_{i=1}^{n} f(z_i),\end{aligned} \tag{4.3.25}$$

where we have used the notation

$$y_i = \frac{\left\{\sqrt{\bar{S}^{\delta_*}_{t_i}} - \sqrt{\bar{S}^{\delta_*}_{t_{i-1}}}\right\}^2}{\varphi_{t_i} - \varphi_{t_{i-1}}} = \frac{\left\{\sqrt{\bar{S}^{\delta_*}_{t_i}} - \sqrt{\bar{S}^{\delta_*}_{t_{i-1}}}\right\}^2}{a\exp(\eta(i-1)\Delta)} \tag{4.3.26}$$

$$z_i = \frac{\sqrt{\bar{S}^{\delta_*}_{t_i}\bar{S}^{\delta_*}_{t_{i-1}}}}{\varphi_{t_i} - \varphi_{t_{i-1}}} = \frac{\sqrt{\bar{S}^{\delta_*}_{t_i}\bar{S}^{\delta_*}_{t_{i-1}}}}{a\exp(\eta(i-1)\Delta)}$$

and where the function f is defined as

$$f(x) = \log\{\exp(-x)I_1(x)\}, \tag{4.3.27}$$

which has convenient asymptotic properties as $x \to \infty$, demonstrated in Appendix N. We straightforwardly obtain the following first order partial derivatives with respect to the parameters a and η, that is,

$$\frac{\partial \ell}{\partial a} = -\frac{n}{a} + \frac{1}{2a}\sum_{i=1}^{n} y_i - \frac{1}{a}\sum_{i=1}^{n} f'(z_i)\, z_i \tag{4.3.28}$$

$$\frac{\partial \ell}{\partial \eta} = -\binom{n}{2}\Delta + \frac{\Delta}{2}\sum_{i=1}^{n} y_i\,(i-1) - \Delta\sum_{i=1}^{n} f'(z_i)\, z_i\,(i-1)$$

from which we obtain the second order partial derivatives

$$\frac{\partial^2 \ell}{\partial a^2} = \frac{n}{a^2} - \frac{1}{a^2}\sum_{i=1}^{n} y_i + \frac{1}{a^2}\sum_{i=1}^{n}\left\{2f'(z_i)\, z_i + f''(z_i)\, z_i^2\right\} \tag{4.3.29}$$

$$\frac{\partial^2 \ell}{\partial \eta \partial a} = -\frac{\Delta}{2a}\sum_{i=1}^{n} y_i\,(i-1) + \frac{\Delta}{a}\sum_{i=1}^{n}\left\{f'(z_i)\, z_i\,(i-1) + f''(z_i)\, z_i^2\,(i-1)\right\}$$

$$\frac{\partial^2 \ell}{\partial \eta^2} = -\frac{\Delta^2}{2}\sum_{i=1}^{n} y_i\,(i-1)^2 + \Delta^2\sum_{i=1}^{n}\left\{f'(z_i)\, z_i\,(i-1)^2 + f''(z_i)\, z_i^2\,(i-1)^2\right\}.$$

Note that, from Appendix N, we have as $y \to \infty$

$$yf'(y) = -\frac{1}{2} + \frac{3}{8y} + O\left(\frac{1}{y^2}\right) \tag{4.3.30}$$

$$y^2 f''(y) = \frac{1}{2} - \frac{3}{4y} + O\left(\frac{1}{y^2}\right).$$

Inserting these asymptotic estimates into (4.3.28) we obtain

$$\begin{aligned}
\frac{a}{n}\frac{\partial \ell}{\partial a} &= -1 + \frac{1}{2n}\sum_{i=1}^{n} y_i - \frac{1}{n}\sum_{i=1}^{n}\left\{-\frac{1}{2} + O\left(\frac{1}{z_i}\right)\right\} \\
&= -\frac{1}{2} + \frac{1}{2n}\sum_{i=1}^{n} y_i + O(\Delta) \\
\frac{1}{\binom{n}{2}\Delta}\frac{\partial \ell}{\partial \eta} &= -1 + \frac{1}{2\binom{n}{2}}\sum_{i=1}^{n} y_i\,(i-1) - \frac{1}{\binom{n}{2}}\sum_{i=1}^{n}\left\{-\frac{1}{2} + O\left(\frac{1}{z_i}\right)\right\}(i-1) \\
&= -\frac{1}{2} + \frac{1}{2\binom{n}{2}}\sum_{i=1}^{n} y_i\,(i-1) + O(\Delta).
\end{aligned} \tag{4.3.31}$$

Here we have employed the two approximations $\bar{S}_{t_i}^{\delta_*} \approx \bar{\alpha}_0 \exp(\eta i \Delta)$ and

$$\begin{aligned}
z_i &\approx \frac{\bar{\alpha}_0 \exp(\eta(i-1/2)\Delta)}{\frac{1}{4\eta}\bar{\alpha}_0(\exp(\eta\Delta)-1)\exp(\eta(i-1)\Delta)} \\
&\approx \frac{4\eta\exp(\eta\Delta/2)}{\eta\Delta + O((\eta\Delta)^2)} \\
&\approx \frac{4}{\Delta}(1+O(\Delta))
\end{aligned} \tag{4.3.32}$$

to arrive at $1/z_i = O(\Delta)$. Our maximum likelihood equations become

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n} y_i &= 1 + O(\Delta) \\
\frac{1}{\binom{n}{2}}\sum_{i=1}^{n} y_i\,(i-1) &= 1 + O(\Delta).
\end{aligned} \tag{4.3.33}$$

The solutions to these equations are close to the exact ones by virtue of the smallness of Δ for large values of n. If the series $(y_i)_{i=1}^{n}$ is nearly constant, as per our assumption, Jensen's inequality $E[\log X] \leq \log E[X]$ is nearly strict so that we have the approximations

$$\begin{aligned}
\frac{1}{n}\sum_{i\in K} \log(y_i) &\approx \log(1+O(\Delta)) \\
\frac{1}{\binom{n}{2}}\sum_{i\in K} \log(y_i)\,(i-1) &\approx \log(1+O(\Delta)),
\end{aligned} \tag{4.3.34}$$

where summation is over those i in the set $K = \{i : y_i > 0\}$. The solutions to these equations are straightforwardly shown to be those given in (4.3.23). Using the approximations (4.3.31) in (4.3.29) we obtain the simplified expressions

$$\begin{aligned} \frac{a^2}{n}\frac{\partial^2 \ell}{\partial a^2} &\approx -\frac{1}{2} + O(\Delta) \qquad (4.3.35)\\ \frac{a}{\binom{n}{2}\Delta}\frac{\partial^2 \ell}{\partial \eta \partial a} &\approx -\frac{1}{2} + O(\Delta)\\ \frac{1}{\frac{1}{6}n(n-1)(2n-1)\Delta^2}\frac{\partial^2 \ell}{\partial \eta^2} &\approx -\frac{1}{2} + O(\Delta). \end{aligned}$$

Fisher's information matrix is computed to be

$$-\begin{pmatrix} \frac{\partial^2 \ell}{\partial a^2} & \frac{\partial^2 \ell}{\partial \eta \partial a} \\ \frac{\partial^2 \ell}{\partial \eta \partial a} & \frac{\partial^2 \ell}{\partial \eta^2} \end{pmatrix}^{-1} = \frac{24a^2}{n^2(n^2-1)\Delta^2}\begin{pmatrix} \frac{n(n-1)(2n-1)\Delta}{24} & -\binom{n}{2}\frac{\Delta}{a} \\ -\binom{n}{2}\frac{\Delta}{a} & \frac{n}{a^2} \end{pmatrix} \qquad (4.3.36)$$

and the standard errors of $\hat{a}$ and $\hat{\eta}$ are

$$\begin{aligned} \mathrm{SE}(\hat{a}) &\approx 2\hat{a}\sqrt{\frac{2n-1}{n(n+1)}} \qquad (4.3.37)\\ \mathrm{SE}(\hat{\eta}_\Delta) &\approx \sqrt{\frac{24}{n(n^2-1)\Delta^2}}. \end{aligned}$$

Because $a = \frac{1}{4\eta}\bar{\alpha}_0(\exp(\eta\Delta) - 1) \approx \frac{\Delta}{4}\bar{\alpha}_0(1 + O(\Delta))$ we have

$$\mathrm{SE}(\hat{\bar{\alpha}}_{0,\Delta}) \approx \frac{4}{\Delta}\mathrm{SE}(\hat{a}) \approx \frac{8\hat{a}}{\Delta}\sqrt{\frac{2n-1}{n(n+1)}} \approx 2\hat{\bar{\alpha}}_{0,\Delta}\sqrt{\frac{2n-1}{n(n+1)}}. \qquad (4.3.38)$$

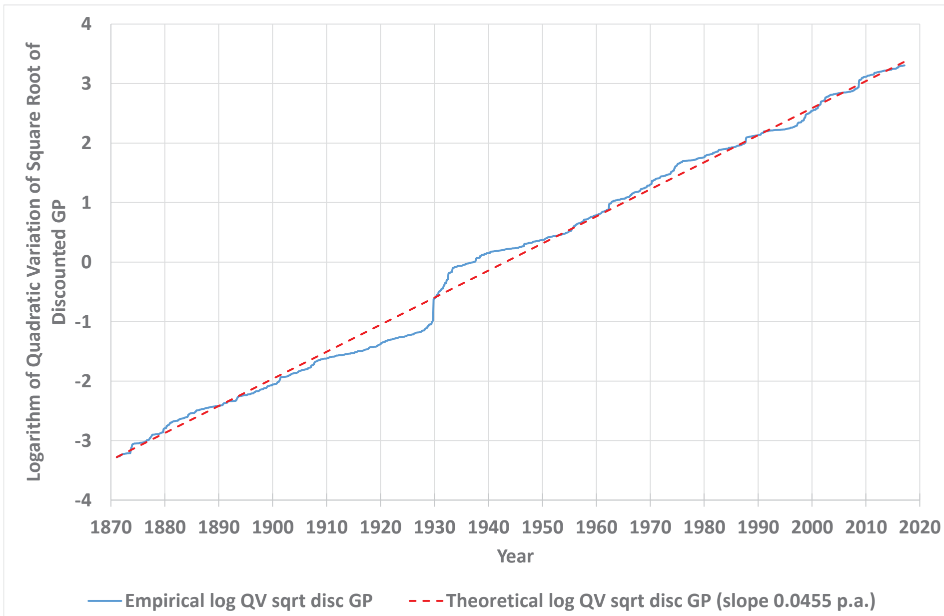
Q.E.D.

We remark that the assumptions (4.3.21) and (4.3.22) underlying the maximum likelihood estimation in Theorem 4.3.5 are likely to be satisfied in empirical applications where the quadratic variation of the square root of elements of the time series has approximate logarithmic growth, as posited for the MMM discounted GOP, and where this approximation is sufficiently good, as specified in (4.3.21). For an arbitrary series, a suitable transformation may need to be applied so that the transformed series satisfies the assumptions (4.3.21) and (4.3.22).

For Shiller's monthly data set of US one-year deposit rates and stock index values from 1871 to 2017, referred to as Data Set C in Section L.3 of Appendix L, the following estimates are obtained by applying the Newton-Raphson root-finding method to the first-order partial derivatives of the log-likelihood function,

$$\begin{aligned} \bar{\alpha}_0 &= 0.006837\,(0.000462), \qquad (4.3.39)\\ \eta &= 0.045486\,(0.000800), \end{aligned}$$

Figure 4.8: Logarithm of quadratic variation of square root of discounted GOP (monthly US data, see Data Set B in Section L.2 of the Appendix).



where the standard errors are shown in brackets.

We note that the estimate of the net growth rate η is close to the empirical finding of $\eta \approx 0.06$ derived from the estimates 10.1% and 4.1% of annualised returns for the last century of equities and short-dated treasury bills respectively, given in Chapter 16 of Dimson et al. [2002].

The quadratic variation of $\sqrt{\bar{S}_t^{\delta_*}}$ is

$$\varphi_t - \varphi_0 = [\sqrt{\bar{S}^{\delta_*}_{\cdot}}]_t = \frac{1}{4}\int_0^t \bar{\alpha}_s ds = \frac{1}{4\eta}\bar{\alpha}_0(\exp(\eta t) - 1), \tag{4.3.40}$$

where $\varphi_0 = \frac{1}{4\eta}\bar{\alpha}_0$. We can visually assess the accuracy of the two parameters $\bar{\alpha}_0$ and η by comparing the theoretical quadratic variation of the square root of the discounted GOP, namely $\frac{1}{4\eta}\bar{\alpha}_0(\exp(\eta t) - 1)$, with the quadratic variation of $\sqrt{\bar{S}_t^{\delta_*}}$. The graphs are shown in Figure 4.8.

Remark 4.3.6 *Similar to the parameter estimation method employed in Section 4 of Hulley and Platen [2012], a crude estimate of* $\bar{\alpha}_0$ *and* η *can be obtained by solving the two equations* $\varphi_{t_n} - \varphi_{t_0} = \bar{\alpha}_0(\exp(\eta t_n) - 1)/(4\eta) = QV_{t_n}$ *and* $\varphi_{t_{n/2}} - \varphi_{t_0} = \bar{\alpha}_0(\exp(\eta t_{n/2}) - 1)/(4\eta) = QV_{t_{n/2}}$ *for* η*, giving* $\exp(\eta \frac{1}{2} n\Delta) + 1 =$

$QV_{t_n}/QV_{t_{n/2}}$ and hence $\hat{\eta} = 0.041917$, with log-likelihood -957.907584. The estimate of η provided by Theorem 4.3.5 is $\hat{\eta} = 0.046020$, with log-likelihood -943.885294, which is very close to the optimum in (4.3.39).

4.3.3 Fitting the MMM Discounted GOP using a Lognormal Approximation

We will see that good estimates of the parameters of the MMM discounted GOP can be obtained by approximating the non-central chi-squared distribution by a lognormal distribution having the same mean and variance. The following lemma supplies formulae for the conditional mean and variance of the discounted GOP.

Lemma 4.3.7 *For the discounted GOP obeying the SDE* (4.1.7) *and for times s, t such that $s < t$ we have*

$$m_s(t) = \mathrm{E}(\bar{S}_t^{\delta_*}|\mathcal{A}_s) = \bar{S}_s^{\delta_*} + \frac{\bar{\alpha}_0}{\eta}\big(\exp(\eta t) - \exp(\eta s)\big) \tag{4.3.41}$$

$$\begin{aligned} v_s(t) = \mathrm{Var}(\bar{S}_t^{\delta_*}|\mathcal{A}_s) &= \bar{S}_s^{\delta_*}\frac{\bar{\alpha}_0}{\eta}\big(\exp(\eta t) - \exp(\eta s)\big) \\ &\quad + \frac{\bar{\alpha}_0^2}{2\eta^2}\big(\exp(\eta t) - \exp(\eta s)\big)^2. \end{aligned}$$

Proof. Integrating the SDE (4.1.7) from time s to time t and taking expectations conditional on $\bar{S}_s^{\delta_*}$ gives

$$\begin{aligned} \mathrm{E}(\bar{S}_t^{\delta_*}|\mathcal{A}_s) &= \bar{S}_s^{\delta_*} + \int_s^t \bar{\alpha}_u du \\ &= \bar{S}_s^{\delta_*} + \bar{\alpha}_0 \int_s^t \exp(\eta u) du \\ &= \bar{S}_s^{\delta_*} + \frac{\bar{\alpha}_0}{\eta}\big(\exp(\eta t) - \exp(\eta s)\big), \end{aligned} \tag{4.3.42}$$

which is the first equation. The SDE for $(\bar{S}_u^{\delta_*})^2$ is, by Ito's Lemma,

$$\begin{aligned} d(\bar{S}_u^{\delta_*})^2 &= 2\bar{S}_u^{\delta_*} d\bar{S}_u^{\delta_*} + \frac{1}{2} \times 2 \times d[\bar{S}^{\delta_*}]_u \\ &= 2\bar{S}_u^{\delta_*}\bar{\alpha}_u du + 2\sqrt{\bar{\alpha}_u(\bar{S}_u^{\delta_*})^3} dW_u + \bar{\alpha}_u \bar{S}_u^{\delta_*} du \\ &= 3\bar{S}_u^{\delta_*}\bar{\alpha}_u du + 2\sqrt{\bar{\alpha}_u(\bar{S}_u^{\delta_*})^3} dW_u. \end{aligned} \tag{4.3.43}$$

As done for $\bar{S}_u^{\delta_*}$, we integrate this SDE for $(\bar{S}_u^{\delta_*})^2$ from time s to time t and take expectations conditional on $\bar{S}_s^{\delta_*}$. By noticing that the integral with respect to the

Wiener process is a martingale, we obtain

$$
\begin{aligned}
\mathrm{E}((\bar{S}_t^{\delta_*})^2|\mathcal{A}_s) &= (\bar{S}_s^{\delta_*})^2 + \int_s^t 3\bar{\alpha}_u \mathrm{E}(\bar{S}_u^{\delta_*}|\mathcal{A}_s)du \\
&= (\bar{S}_s^{\delta_*})^2 + \int_s^t 3\bar{\alpha}_u \bar{S}_s^{\delta_*} du + \int_s^t 3\bar{\alpha}_u \frac{\bar{\alpha}_0}{\eta}\big(\exp(\eta u) - \exp(\eta s)\big)du \\
&= (\bar{S}_s^{\delta_*})^2 + (3\bar{S}_s^{\delta_*} - 3\frac{\bar{\alpha}_0}{\eta}\exp(\eta s))\int_s^t \bar{\alpha}_u du + 3\frac{\bar{\alpha}_0}{\eta}\int_s^t \bar{\alpha}_u \exp(\eta u)du \\
&= (\bar{S}_s^{\delta_*})^2 + (3\bar{S}_s^{\delta_*} - 3\frac{\bar{\alpha}_0}{\eta}\exp(\eta s))\frac{\bar{\alpha}_0}{\eta}\big(\exp(\eta t) - \exp(\eta s)\big) \\
&+ 3\frac{\bar{\alpha}_0^2}{2\eta^2}\big(\exp(2\eta t) - \exp(2\eta s)\big) \\
&= (\bar{S}_s^{\delta_*})^2 + 3\bar{S}_s^{\delta_*}\frac{\bar{\alpha}_0}{\eta}\big(\exp(\eta t) - \exp(\eta s)\big) \\
&+ 3\frac{\bar{\alpha}_0^2}{2\eta^2}\big(\exp(\eta t) - \exp(\eta s)\big)^2.
\end{aligned} \tag{4.3.44}
$$

The formula for the variance is straightforwardly given by

$$
\begin{aligned}
\mathrm{Var}(\bar{S}_t^{\delta_*}|\mathcal{A}_s) &= \mathrm{E}((\bar{S}_t^{\delta_*})^2|\mathcal{A}_s) - (\mathrm{E}(\bar{S}_t^{\delta_*}|\mathcal{A}_s))^2 \\
&= \bar{S}_s^{\delta_*}\frac{\bar{\alpha}_0}{\eta}\big(\exp(\eta t) - \exp(\eta s)\big) \\
&+ \frac{\bar{\alpha}_0^2}{2\eta^2}\big(\exp(\eta t) - \exp(\eta s)\big)^2,
\end{aligned} \tag{4.3.45}
$$

which is the second equation. **Q.E.D.**

We approximate the transition density of the square root process from time t_{i-1} to time t_i by a lognormal distribution that matches the mean and variance. The lognormal distribution for the exponential of a Gaussian random walk with mean μ and variance σ^2 has mean

$$
m = \exp(\mu + \frac{1}{2}\sigma^2) \tag{4.3.46}
$$

and variance

$$
v = \exp(2\mu + \sigma^2)\left(\exp(\sigma^2) - 1\right). \tag{4.3.47}
$$

It is straightforward to show that the approximating lognormal distribution has, according to (4.3.46) and (4.3.47), parameters $\mu_{t_i}^{(LN)}$, $\sigma_{t_i}^{(LN)}$ given by

$$
\begin{aligned}
\mu_{t_i}^{(LN)} &= \log\left(m_{t_{i-1}}(t_i)\right) - \frac{1}{2}(\sigma_{t_i}^{(LN)})^2 \\
(\sigma_{t_i}^{(LN)})^2 &= \log\left(1 + \frac{v_{t_{i-1}}(t_i)}{m_{t_{i-1}}(t_i)^2}\right),
\end{aligned} \tag{4.3.48}
$$

respectively, where

$$m_{t_{i-1}}(t_i) = \bar{S}^{\delta_*}_{t_{i-1}} + \frac{\bar{\alpha}_0}{\eta}\big(\exp(\eta t_i) - \exp(\eta t_{i-1})\big) \tag{4.3.49}$$

$$\begin{aligned} v_{t_{i-1}}(t_i) &= \bar{S}^{\delta_*}_{t_{i-1}} \frac{\bar{\alpha}_0}{\eta}\big(\exp(\eta t_i) - \exp(\eta t_{i-1})\big) \\ &\quad + \frac{\bar{\alpha}_0^2}{2\eta^2}\big(\exp(\eta t_i) - \exp(\eta t_{i-1})\big)^2. \end{aligned}$$

Therefore, our approximating log-likelihood function on the set of observed values of the discounted GOP $\bar{S}^{\delta_*}_{t_i}$, for $i = 0, 2, \ldots, n$ is

$$\ell(\bar{\alpha}_0, \eta) = -\frac{1}{2}\sum_{i=1}^{n}\left\{\log\left(2\pi(\sigma^{(LN)}_{t_i})^2\right) + 2\log\left(\bar{S}^{\delta_*}_{t_i}\right) + \frac{\left(\log(\bar{S}^{\delta_*}_{t_i}) - \mu^{(LN)}_{t_i}\right)^2}{(\sigma^{(LN)}_{t_i})^2}\right\}. \tag{4.3.50}$$

Remark 4.3.8 *Because the skew of the lognormal distribution is*

$$\begin{aligned} &m_{t_{i-1}}(t_i)^3\big(\exp((\sigma^{(LN)}_{t_i})^2) - 1\big)^2\big(\exp((\sigma^{(LN)}_{t_i})^2) + 2\big) \\ &\approx 16(\varphi_{t_i} - \varphi_{t_{i-1}})^2 m_{t_{i-1}}(t_i)\big(1 + O((\varphi_{t_i} - \varphi_{t_{i-1}})/m_{t_{i-1}}(t_i))\big) \end{aligned} \tag{4.3.51}$$

and the skew of the noncentral chi-squared distribution is

$$\begin{aligned} &-64(\varphi_{t_i} - \varphi_{t_{i-1}})^3 + 24(\varphi_{t_i} - \varphi_{t_{i-1}})^2 m_{t_{i-1}}(t_i) \\ &\approx 24(\varphi_{t_i} - \varphi_{t_{i-1}})^2 m_{t_{i-1}}(t_i)\big(1 + O((\varphi_{t_i} - \varphi_{t_{i-1}})/m_{t_{i-1}}(t_i))\big) \end{aligned} \tag{4.3.52}$$

we see that the lognormal approximation accounts for roughly two-thirds of the skew present in the noncentral chi-squared distribution which is not captured by the normal approximation, for example.

We fit the MMM to the monthly discounted GOP series derived from Shiller's data set, referred to as Data Set C in Section L.3 of Appendix L, obtaining the maximum likelihood estimates

$$\begin{aligned} \bar{\alpha}_0 &= 0.006904\,(0.000466) \\ \eta &= 0.045555\,(0.000800). \end{aligned} \tag{4.3.53}$$

We note that the estimates for $\bar{\alpha}_0$ and η are in close agreement with those in (4.3.39).

4.3.4 Discounted GOP Contribution to the ZCB Price

We state and prove two lemmas concerning expectations of Poisson and Gamma random variables before stating and proving a lemma to be used later. The motivation for this approach stems from the representation of the non-central chi-squared distribution as a chi-squared distribution having a random number of degrees of freedom. As far as can be established from the literature, this is a new approach to proving the expectation of the reciprocal of the MMM discounted GOP and expectations of discounted payoffs of options for a MMM discounted GOP.

Lemma 4.3.9 *Let Z be a* Poisson(λ) *distributed random variable. Then*

$$E\left(\frac{1}{Z+1}\right) = \frac{1-\exp(-\lambda)}{\lambda}. \tag{4.3.54}$$

Proof. We have straightforwardly

$$\begin{aligned} E\left(\frac{1}{Z+1}\right) &= \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} \exp(-\lambda) \frac{1}{i+1} \qquad (4.3.55) \\ &= \sum_{i=0}^{\infty} \frac{\lambda^i}{(i+1)!} \exp(-\lambda) \\ &= \frac{1}{\lambda} \sum_{i=0}^{\infty} \frac{\lambda^{i+1}}{(i+1)!} \exp(-\lambda) \\ &= \frac{1}{\lambda} \left(-\exp(-\lambda) + \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} \exp(-\lambda) \right) \\ &= \frac{1}{\lambda} \left(-\exp(-\lambda) + 1 \right), \end{aligned}$$

which is the result. **Q.E.D.**

The following lemma is useful in determining the expectation of the inverse of the square root process, over the long term, in (3.4.16).

Lemma 4.3.10 *Let X be a* Gamma(α, λ)*-distributed random variable. Then*

$$\mathrm{E}(X^{-1}) = \lambda/(\alpha - 1). \tag{4.3.56}$$

Proof. The probability density function of the Gamma(α, λ) distribution is

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-x/\lambda) \tag{4.3.57}$$

and so

$$
\begin{aligned}
E(X^{-1}) &= \int_0^\infty x^{-1} f(x)dx \qquad (4.3.58)\\
&= \int_0^\infty x^{-1}\frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1}\exp(-x/\lambda)dx\\
&= \int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-2}\exp(-x/\lambda)dx\\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)}\frac{\Gamma(\alpha-1)}{\lambda^{\alpha-1}}\int_0^\infty \frac{\lambda^{\alpha-1}}{\Gamma(\alpha-1)} x^{\alpha-2}\exp(-x/\lambda)dx\\
&= \frac{\lambda}{\alpha-1}\int_0^\infty \frac{\lambda^{\alpha-1}}{\Gamma(\alpha-1)} x^{\alpha-2}\exp(-x/\lambda)dx\\
&= \frac{\lambda}{\alpha-1},
\end{aligned}
$$

which is the result. **Q.E.D.**

Corollary 4.3.11 *Let X be a χ^2_ν-distributed random variable for $\nu > 2$. Then*

$$E(X^{-1}) = 1/(\nu - 2). \qquad (4.3.59)$$

Proof. Because a χ^2_ν-distributed random variable is also a Gamma$(\nu/2, 1/2)$-distributed random variable, using Lemma 4.3.10 gives the result. **Q.E.D.**

The following lemma concerns the expected value of the reciprocal of a $\chi^2_{\nu,\lambda}$-distributed random variable when $\nu = 4$.

Lemma 4.3.12 *Suppose the random variable Y is distributed as $\chi^2_{\nu,\lambda}$ for $\nu = 4$ and $\lambda > 0$. Then*

$$E(Y^{-1}) = \frac{1-\exp(-\lambda/2)}{\lambda}. \qquad (4.3.60)$$

Proof. From Remark 3.3.9 we can write, for a Poisson$(\lambda/2)$-distributed random variable Z,

$$
\begin{aligned}
E(Y^{-1}) &= E\left(\frac{1}{\chi^2_{\nu+2Z}}\right) \qquad (4.3.61)\\
&= E\left(E\left(\frac{1}{\chi^2_{\nu+2Z}}\middle| Z\right)\right).
\end{aligned}
$$

Now the expectation of the conditional random variable

$$\frac{1}{\chi^2_{\nu+2Z}}$$

given Z is computed from Corollary 4.3.11 to be

$$E\left(\frac{1}{\chi^2_{\nu+2Z}}\middle| Z\right) = \frac{1}{\nu+2Z-2} \tag{4.3.62}$$

and so

$$\mathrm{E}(Y^{-1}) = E\left(\frac{1}{\nu+2Z-2}\right). \tag{4.3.63}$$

When $\nu = 4$ this simplifies to

$$\mathrm{E}(Y^{-1}) = E\left(\frac{1}{2Z+2}\right). \tag{4.3.64}$$

By virtue of Lemma 4.3.9 this can be simplified as follows

$$\begin{aligned}\mathrm{E}(Y^{-1}) &= \frac{1}{2}E\left(\frac{1}{Z+1}\right) \\ &= \frac{1}{2}\frac{1-\exp(-\lambda/2)}{\lambda/2} \\ &= \frac{1-\exp(-\lambda/2)}{\lambda},\end{aligned} \tag{4.3.65}$$

which is the result. **Q.E.D.**

Lemma 4.3.13 *Let t and $\bar{T}$ be times such that $\bar{T} > t \geq 0$. For $\bar{S}^{\delta_*}_t$ satisfying* (4.1.7) *and φ_t as in* (4.3.10) *the discounted GOP contribution to the ZCB price is*

$$M_{\bar{T}}(t) = 1 - \exp\left(-\frac{\bar{S}^{\delta_*}_t}{2(\varphi_{\bar{T}} - \varphi_t)}\right). \tag{4.3.66}$$

Proof. This is proven in Chapter 13 Section 3 of Platen and Heath [2006]. From (2.8.3) and (4.3.16) we have

$$\begin{aligned}M_{\bar{T}}(t) &= \mathrm{E}\left(\frac{\bar{S}^{\delta_*}_t}{\bar{S}^{\delta_*}_{\bar{T}}}\middle|\mathcal{A}_t\right) \\ &= \int_0^\infty \frac{\bar{S}^{\delta_*}_t}{\bar{S}^{\delta_*}_{\bar{T}}} p_{\bar{S}^{\delta_*}}(t, \bar{S}^{\delta_*}_t, \bar{T}, \bar{S}^{\delta_*}_{\bar{T}})\, d\bar{S}^{\delta_*}_{\bar{T}} \\ &= \int_0^\infty \frac{\lambda}{u}\frac{1}{2}\sqrt{\frac{u}{\lambda}}\exp(-\frac{1}{2}(u+\lambda))I_1(\sqrt{u\lambda})\, du,\end{aligned} \tag{4.3.67}$$

where we have used the transformation of variables

$$\begin{aligned}u &= \bar{S}^{\delta_*}_{\bar{T}}/(\varphi_{\bar{T}} - \varphi_t) \\ \lambda &= \bar{S}^{\delta_*}_t/(\varphi_{\bar{T}} - \varphi_t)\end{aligned} \tag{4.3.68}$$

By virtue of Lemma 4.3.2 we can interpret the above integral as the expectation of the reciprocal of a $\chi^2_{4,\lambda}$-random variable U. Using Lemma 4.3.12 we now have

$$M_{\bar{T}}(t) = \lambda \mathrm{E}(U^{-1}) = 1 - \exp\left(-\frac{1}{2}\lambda\right) = 1 - \exp\left(-\frac{1}{2}\bar{S}_t^{\delta_*}/(\varphi_{\bar{T}} - \varphi_t)\right) \tag{4.3.69}$$

which is the result. **Q.E.D.**

We have the following corollary.

Corollary 4.3.14 *The discounted GOP contribution to the long ZCB yield is η, that is*

$$n_\infty(t) = \eta. \tag{4.3.70}$$

4.3.5 Discounted GOP Contribution to the Forward Rate

Under the MMM we compute the discounted GOP contribution to the forward rate as in the following lemma.

Lemma 4.3.15 *Let t and $\bar{T}$ be times such that $\bar{T} > t \geq 0$. For $\bar{S}_t^{\delta_*}$ satisfying* (4.1.7) *and φ_t as in* (4.3.10) *the discounted GOP contribution to the forward rate is*

$$m_{\bar{T}}(t) = \frac{1}{M_{\bar{T}}(t)} \exp\left(-\frac{\bar{S}_t^{\delta_*}}{2(\varphi_{\bar{T}} - \varphi_t)}\right) \times \frac{\bar{S}_t^{\delta_*}}{2(\varphi_{\bar{T}} - \varphi_t)^2} \times \varphi'_{\bar{T}}. \tag{4.3.71}$$

Proof. From (2.8.12) and (4.3.66) we have

$$m_{\bar{T}}(t) = \frac{\partial}{\partial \bar{T}} \log M_{\bar{T}}(t) = \frac{\partial}{\partial \bar{T}} \log\left(1 - \exp(-\frac{1}{2}\bar{S}_t^{\delta_*}/(\varphi_{\bar{T}} - \varphi_t))\right) \tag{4.3.72}$$

which leads to the result. **Q.E.D.**

The following corollary provides an estimate of this contribution to the forward rate at the long end of the yield curve.

Corollary 4.3.16 *As $\bar{T} \to \infty$ the discounted GOP contribution to the forward rate is η.*

Proof. The intrinsic time has first derivative

$$\varphi'_{\bar{T}} = \frac{1}{4}\bar{\alpha}_0 \exp(\eta \bar{T}) = \eta \varphi_{\bar{T}} + \frac{1}{4}\bar{\alpha}_0. \tag{4.3.73}$$

Define the variable $x_{\bar{T}}$ as

$$x_{\bar{T}} = \frac{\bar{S}_t^{\delta_*}}{2(\varphi_{\bar{T}} - \varphi_t)} \tag{4.3.74}$$

so that we can write

$$\varphi_{\bar{T}} = \varphi_t + \frac{\bar{S}_t^{\delta_*}}{2x_{\bar{T}}} \tag{4.3.75}$$

and

$$\varphi'_{\bar{T}} = \eta \frac{\bar{S}_t^{\delta_*}}{2x_{\bar{T}}} + \eta \varphi_t + \frac{1}{4}\bar{\alpha}_0. \tag{4.3.76}$$

We have from (4.3.66) that

$$M_{\bar{T}}(t) = 1 - \exp(-x_{\bar{T}}) \tag{4.3.77}$$

and we can rewrite (4.3.71) in terms of $x_{\bar{T}}$ as

$$\begin{aligned} m_{\bar{T}}(t) &= \frac{1}{1 - \exp(-x_{\bar{T}})} \exp(-x_{\bar{T}}) \times \frac{2}{\bar{S}_t^{\delta_*}} x_{\bar{T}}^2 \times \big(\eta \frac{\bar{S}_t^{\delta_*}}{2x_{\bar{T}}} + \eta \varphi_t + \frac{1}{4}\bar{\alpha}_0\big) \\ &= \frac{x_{\bar{T}}}{\exp(x_{\bar{T}}) - 1} \times \big(\eta + x_{\bar{T}} \frac{2}{\bar{S}_t^{\delta_*}} (\eta \varphi_t + \frac{1}{4}\bar{\alpha}_0)\big). \end{aligned} \tag{4.3.78}$$

L'Hospital's Rule gives

$$\lim_{x_{\bar{T}} \to 0} \frac{x_{\bar{T}}}{\exp(x_{\bar{T}}) - 1} = \lim_{x_{\bar{T}} \to 0} \frac{1}{\exp(x_{\bar{T}})} = 1 \tag{4.3.79}$$

and therefore

$$\begin{aligned} \lim_{\bar{T} \to \infty} m_{\bar{T}}(t) &= \lim_{x_{\bar{T}} \to 0} \frac{x_{\bar{T}}}{\exp(x_{\bar{T}}) - 1} \times \big(\eta + x_{\bar{T}} \frac{2}{\bar{S}_t^{\delta_*}} (\eta \varphi_t + \frac{1}{4}\bar{\alpha}_0)\big) \\ &= \eta, \end{aligned} \tag{4.3.80}$$

which is the result. **Q.E.D.**

4.3.6 Expectations Involving a Non-Central Chi-Squared Random Variable

We provide the formulae of the expectations in Table 4.2, where U is distributed as non-central chi-squared with four degrees of freedom and non-centrality parameter λ and where x is a positive real number. The proofs are supplied in Appendix K.

We first state two lemmas which provide straightforward proofs of the above formulae.

Table 4.2: Table of formulae of expectations involving a non-central chi-squared random variable U having four degrees of freedom and non-centrality parameter λ.

Expectation	Formula
$E(\frac{\lambda}{U}\mathbf{1}_{U\le x})$	$\chi^2_{0,\lambda}(x) - \exp(-\lambda/2)$
$E(\frac{\lambda}{U}\mathbf{1}_{U>x})$	$1 - \chi^2_{0,\lambda}(x)$
$E(\frac{\lambda}{U})$	$1 - \exp(-\lambda/2)$
$E(\frac{\lambda}{U}\exp(-\tau U)\mathbf{1}_{U\le x})$	$\mathrm{e}^{-\frac{\tau}{2\tau+1}\lambda}\chi^2_{0,\lambda/(2\tau+1)}((2\tau+1)x) - \mathrm{e}^{-\frac{1}{2}\lambda}$
$E(\frac{\lambda}{U}\exp(-\tau U)\mathbf{1}_{U>x})$	$\mathrm{e}^{-\frac{\tau}{2\tau+1}\lambda}\{1 - \chi^2_{0,\lambda/(2\tau+1)}((2\tau+1)x)\}$
$E(\frac{\lambda}{U}\exp(-\tau U))$	$\mathrm{e}^{-\frac{\tau}{2\tau+1}\lambda} - \mathrm{e}^{-\frac{1}{2}\lambda}$

Lemma 4.3.17 *Let* $Z \sim \text{Poisson}(\mu)$ *be a Poisson random variable. Then*

$$\mathrm{E}\left(\frac{1}{Z+1}f(Z)\right) = \frac{1}{\mu}\mathrm{E}\left(f(Z-1)\right) - \frac{1}{\mu}\exp(-\mu)f(-1) \tag{4.3.81}$$

for a real valued function f *on the non-negative integers.*

Proof. See Appendix K.

Lemma 4.3.18 *Let* $X \sim \chi^2_\nu$ *be a chi-squared random variable. Then*

$$\mathrm{E}\left(\frac{1}{X}\mathbf{1}_{X\le x}\right) = \frac{1}{\nu-2}\mathrm{E}\left(\mathbf{1}_{Y\le x}\right) \tag{4.3.82}$$

for any non-negative real number x*, where* $Y \sim \chi^2_{\nu-2}$ *is also chi-squared distributed with* $\nu - 2$ *degrees of freedom.*

Proof. See Appendix K.

We are ready to formulate the first, second and third expectations in Table 4.2 in the following lemma.

Lemma 4.3.19 *Let* $U \sim \chi^2_{4,\lambda}$ *be a non-central chi-squared random variable. Then*

$$\mathrm{E}\left(\frac{\lambda}{U}\mathbf{1}_{U\le x}\right) = \chi^2_{0,\lambda}(x) - \exp(-\lambda/2) \tag{4.3.83}$$

$$\mathrm{E}\left(\frac{\lambda}{U}\mathbf{1}_{U>x}\right) = 1 - \chi^2_{0,\lambda}(x)$$

$$\mathrm{E}\left(\frac{\lambda}{U}\right) = 1 - \exp(-\lambda/2)$$

for a non-negative real number λ.

Proof. See Appendix K.

To prove the fourth, fifth and sixth expectations in Table 4.2 we require the following two lemmas.

Lemma 4.3.20 *Let* $Z \sim \text{Poisson}(\mu)$ *be a Poisson random variable. Then for a real valued function* f *on the non-negative integers*

$$\mathrm{E}\bigg(\frac{1}{Z+1}\gamma^Z f(Z)\bigg) = \frac{1}{\mu\gamma}\exp(-\mu(1-\gamma))\mathrm{E}\bigg(f(W-1)\bigg) - \frac{1}{\mu\gamma}\exp(-\mu)f(-1) \tag{4.3.84}$$

where $W \sim \text{Poisson}(\mu\gamma)$ *is a Poisson random variable.*

Proof. See Appendix K.

Lemma 4.3.21 *Let* $X \sim \chi^2_\nu$ *be a chi-squared random variable. Then*

$$\begin{aligned}\mathrm{E}\bigg(\frac{1}{X}\exp(-\tau X)\mathbf{1}_{X\le x}\bigg) &= \frac{1}{\nu-2}\frac{1}{(2\tau+1)^{\nu/2-1}}\mathrm{E}\bigg(\mathbf{1}_{Y\le(2\tau+1)x}\bigg) \qquad (4.3.85)\\ \mathrm{E}\bigg(\frac{1}{X}\exp(-\tau X)\mathbf{1}_{X> x}\bigg) &= \frac{1}{\nu-2}\frac{1}{(2\tau+1)^{\nu/2-1}}\mathrm{E}\bigg(\mathbf{1}_{Y>(2\tau+1)x}\bigg)\end{aligned}$$

for any non-negative real number x*, where* $Y \sim \chi^2_{\nu-2}$ *is also chi-squared distributed but with* $\nu-2$ *degrees of freedom.*

Proof. See Appendix K.

We now proceed with the formulation of the remaining expectations in Table 4.2.

Lemma 4.3.22 *Let* $U \sim \chi^2_{4,\lambda}$ *be a non-central chi-squared random variable. Then*

$$\begin{aligned}\mathrm{E}\bigg(\frac{\lambda}{U}\exp(-\tau U)\mathbf{1}_{U\le x}\bigg) &= \mathrm{e}^{-\frac{\tau}{2\tau+1}\lambda}\chi^2_{0,\lambda/(2\tau+1)}((2\tau+1)x) - \exp(-\lambda/2) \qquad (4.3.86)\\ \mathrm{E}\bigg(\frac{\lambda}{U}\exp(-\tau U)\mathbf{1}_{U> x}\bigg) &= \mathrm{e}^{-\frac{\tau}{2\tau+1}\lambda}(1-\chi^2_{0,\lambda/(2\tau+1)}((2\tau+1)x))\\ \mathrm{E}\bigg(\frac{\lambda}{U}\exp(-\tau U)\bigg) &= \mathrm{e}^{-\frac{\tau}{2\tau+1}\lambda} - \exp(-\lambda/2)\end{aligned}$$

for a non-negative real number λ*.*

Proof. See Appendix K.

Table 4.3: Values of the AIC in respect of the discounted GOP models (Shiller's monthly US data set, see Data Set C in Section L.3 of the Appendix).

Model	Parameters	Log Likelihood	AIC
Black-Scholes	$\theta = 0.141002$ $\mathrm{SE}(\theta) = 0.002380$	-1060.320193	2122.640385
MMM	$\bar{\alpha}_0 = 0.006837$ $\mathrm{SE}(\bar{\alpha}_0) = 0.000462$ $\eta = 0.045486$ $\mathrm{SE}(\eta) = 0.000800$	-943.670030	1891.340059

4.4 Comparison of Models

The two models considered in this chapter have explicit formulae for their transition density functions and this has allowed the fitting of parameters using maximum likelihood estimation. The Black-Scholes model is most easily fitted to the data because it has a closed form expression for its parameter estimate. In contrast, the MMM requires two-dimensional grid searches to find the best fitting parameters.

In fitting the two models to the US discounted GOP data we can identify which model provides the best fit to the data by looking at the Akaike Information Criterion, shown in Table 4.3, where the MMM appears to be the best fitting model[2].

To establish whether the MMM is a good fitting model we consider Pearson's goodness-of-fit chi-squared statistic, described in Kendall and Stuart [1961].

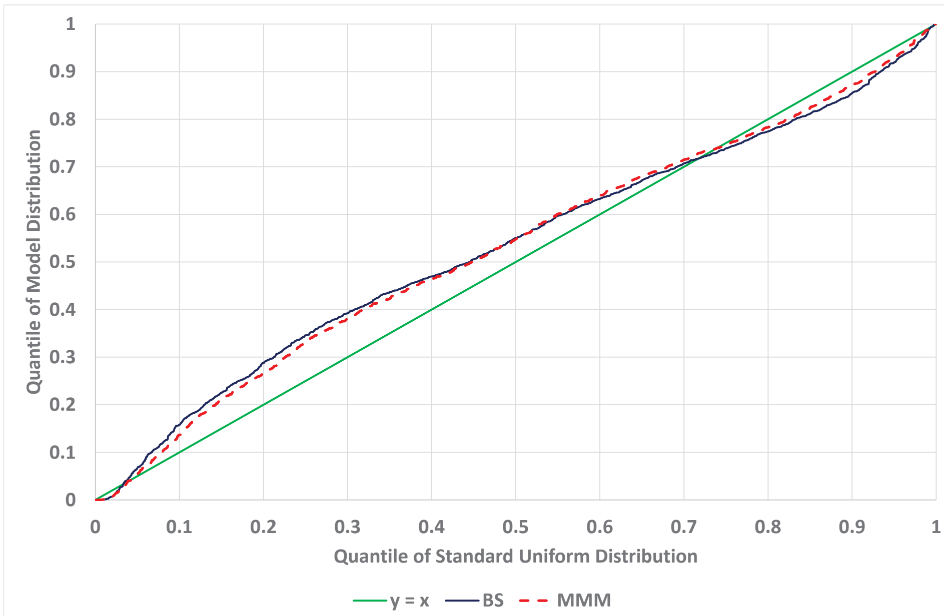
Given a time series of discounted GOP values $\{\bar{S}^{\delta_*}_{t_j} : j = 1, 2, \ldots, n\}$ and given a hypothesised transition density function with corresponding cumulative distribution function F we compute the $n-1$ quantiles $q_j = F(t_{j-1}, \bar{S}^{\delta_*}_{t_{j-1}}, t_j, \bar{S}^{\delta_*}_{t_j})$ for $j = 2, 3, \ldots, n$. Under the hypothesised model the quantiles q_j are independent and uniformly distributed. These quantiles are graphed against those of the uniform distribution in Figure 4.9. One notes that the MMM model remains in some sense visually closest over the $[0, 1]$ interval.

A similar comparison is shown in Figure 4.10 for the daily data series of the S&P 500 total return index values and Federal Funds Rates from January 1970 to May 2017, sourced from the Bloomberg data services and referred to as Data Set E in Section L.5 of the Appendix, where a similar conclusion follows.

For a fixed integer k satisfying $2 \leq k \leq (n-1)/5$ we partition the unit interval into k equally sized subintervals. Hence we compute the number of observations

[2]The Bayesian Information Criterion (BIC) values for the BS and MMM models are 2128.110040 and 1902.279368 respectively and, therefore, model selection based on the BIC concurs with that based on the AIC.

Figure 4.9: Comparison of quantile-quantile plots of discounted GOP models (Shiller's monthly US data set, see Data Set C in Section L.3 of the Appendix).



O_i in the i-th subinterval $((i-1)/k, i/k]$ for $i = 1, 2, \ldots, k$. The corresponding expected number of observations E_i in the i-th subinterval is $(n-1)/k$. Our test statistic is thus computed as

$$S = k \sum_{i=1}^{k} (O_i - (n-1)/k)^2/(n-1) \tag{4.4.1}$$

which is approximately chi-squared distributed with $\nu = k - 1 - n_{parameters}$ degrees of freedom.

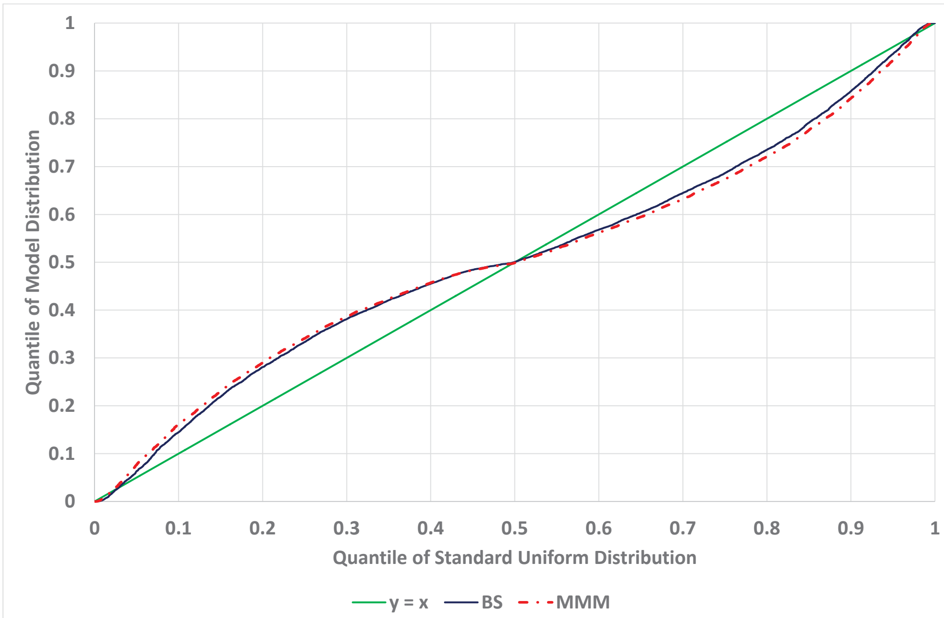
The value of Pearson's chi-squared statistic and corresponding p-value for each model and for a range of partition sizes is shown in Table 4.4. It is evident that both the BS model and MMM can be rejected at the 1% level of significance.

Another test of goodness-of-fit is the Kolmogorov-Smirnov test, as described by Stephens [1974]. Under the null hypothesis that the set of n observations $u_1, u_2, \ldots, u_n$ emanate from a uniform distribution, the Kolmogorov test statistic is

$$D_n = \sup_{x \in \{u_1, u_2, \ldots, u_n\}} \max \left(F^{(n)}(x) - x, x - F^{(n)}(x) - \frac{1}{n} \right) \tag{4.4.2}$$

and the modified test statistic $K_n = \sqrt{n} D_n$ has the limiting distribution function,

Figure 4.10: Comparison of quantile-quantile plots of discounted GOP models (US daily data from Bloomberg, see Data Set E in Section L.5 in the Appendix).



as $n \to \infty$,

$$F(x) = \frac{\sqrt{2\pi}}{x} \sum_{k=1}^{\infty} \exp(-(2k-1)^2\pi^2/(8x^2)), \tag{4.4.3}$$

where

$$F^{(n)}(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{u_i \le x} \tag{4.4.4}$$

is the empirical cumulative distribution function. We compute the test statistics in Table 4.5 where we see both models can be rejected at the 1% level of significance.

Finally, another test of goodness-of-fit is the Anderson-Darling test, as described in Stephens [1974]. Under the null hypothesis that the set of n observations $u_1 \le u_2 \le \ldots \le u_n$ emanate from a uniform distribution, the test statistic A is given by

$$A = \sqrt{-n - S}, \tag{4.4.5}$$

where

$$S = \sum_{i=1}^{n} \frac{2i-1}{n} \big(\log(u_i) + \log(1 - u_{n+1-i}) \big). \tag{4.4.6}$$

Table 4.4: Pearson's chi-squared statistics and p-values in respect of various discounted GOP models (Shiller's monthly US data set, see Data Set C in Section L.3 of the Appendix).

k	BS	ν	p-value	MMM	ν	p-value
5	128.9133	3	1.1768E-14	93.7822	2	1.5767E-10
10	147.8016	8	0.0000E0	113.7537	7	0.0000E0
15	165.1619	13	0.0000E0	116.1425	12	0.0000E0
20	179.3181	18	0.0000E0	130.9715	17	0.0000E0
25	180.7206	23	0.0000E0	130.4356	22	0.0000E0

Table 4.5: Kolmogorov-Smirnov test statistics in respect of various discounted GOP models (Shiller's monthly US data set, see Data Set C in Section L.3 of the Appendix).

	BS	MMM
D_n	0.096819	0.084347
n	1754	1754
K_n	4.054863	3.532536
$F(K_n)$	1.000000	1.000000
p-value	1.0436E-14	2.8979E-11

We compute the test statistics in Table 4.6 where, as for the Kolmogorov-Smirnov test, we see that both of the discounted GOP models can be rejected at the 1% level of significance. The p-values of the test statistic A in Table 4.6 were estimated using sample Anderson-Darling statistics of 1,000,000 simulations of sets of 1754 uniformly distributed observations.

4.5 Conclusions

In this chapter we have demonstrated the applicability of maximum likelihood estimation of parameters of discounted equity index models, giving explicit formulae

Table 4.6: Anderson-Darling test statistics in respect of various discounted GOP models (Shiller's monthly US data set, see Data Set C in Section L.3 of the Appendix).

	BS	MMM
S	-1783.440143	-1774.793996
n	1754	1754
$A^2 = -n - S$	29.440143	20.793996
A	5.425877	4.560043
p-value	0.0000E0	5.0092E-10

for maximum likelihood estimates of model parameters. The model parameters fitted to the US data set have values consistent with estimates obtained by others, for example Dimson et al. [2002]. Also, we have demonstrated several ways of assessing the goodness-of-fits of the models.

As mentioned in Remark 4.2.2, the log-returns of the discounted GOP under the Black-Scholes model are Gaussian with constant variance, whereas under the MMM they are Student-t distributed. Empirical evidence in Chapter 6, Fergusson and Platen [2006] and in Platen and Rendek [2008] indicates that the log returns of discounted equity indices are Student-t distributed with four degrees of freedom. This also shows that the MMM is a more realistic model that appears to capture better the stochastic volatility and leptokurtic log-returns observed in the market than the classical Black-Scholes model.

As mentioned in Remark 4.3.4, for the Black-Scholes model, the Radon-Nikodym derivative is a martingale and therefore risk neutral pricing of contingent claims is possible. Importantly, however, the Radon-Nikodym derivative pertaining to the MMM is a strict supermartingale and therefore the MMM does not admit an equivalent risk neutral probability measure. Thus, for the MMM we must resort to another valuation framework, such as the Benchmark Approach, where some arbitrage is permitted.

In addition to fitting the two models to data, this chapter has provided convenient formulae which are essential for pricing zero-coupon bonds, options on zero-coupon bonds and options on the GOP. Explicit formulae for the fair price of a zero-coupon bond demands an explicit formula short rate contribution M and this has been supplied in respect of each of the discounted GOP models. Having explicit formulae for prices of options on the discounted GOP contribution M allows for a single dimensional integral formula for the price of options on a zero-coupon bond.

Chapter 5

Derivatives Pricing Formulae

5.1 Introduction

In this chapter we supply the derivatives pricing formulae under several market models.

From (2.6.4) the fair price $V_t^{\delta_{H_T}}$ of a derivative security with payoff H_T at maturity T satisfies the real-world pricing formula

$$V_t^{\delta_{H_T}} = \mathrm{E}\left(\frac{S_t^{\delta_*}}{S_T^{\delta_*}} H_T \,\middle|\, \mathcal{A}_t\right), \tag{5.1.1}$$

where $\mathrm{E}(.|\mathcal{A}_t)$ denotes the real-world conditional expectation under the real-world probability measure given the information available at time t.

Applying the benchmark approach, we provide explicit pricing formulae for derivatives under several market models each of which is composed of a previously examined short rate model and discounted GOP model.

Taking all possible combinations of short rate and discounted GOP models gives six market models and adding in the deterministic short rate model gives an extra two market models.

Pricing formulae are provided in respect of the following derivative securities: zero-coupon bond, option on GOP, option on ZCB, option on coupon bond, caplet, floorlet, cap, floor and swaption. Most of these formulae are original and have been published in Fergusson and Platen [2014a] and reported in Fergusson and Platen [2015a].

From equation (2.8.2) we see that the zero-coupon bond price has contributions from both the short rate and the discounted GOP. We combine the short rate contributions of each of the three short rate models with the discounted GOP contributions of each of the two discounted GOP models to calculate derivative prices in respect of each of the six models.

In Chapter 6 the behaviour of each derivative is examined under each market model. In particular, we examine the shapes of yield curves and implied volatility surfaces generated by each market model.

5.2 Fair Price of Zero-Coupon Bond

Under our considered market models the real-world pricing formula (5.1.1) and (2.8.2) gives the price of a ZCB as

$$P(t,T) = \mathrm{E}\left(\frac{B_t}{B_T}\bigg|\mathcal{A}_t\right)\mathrm{E}\left(\frac{\bar{S}_t^{\delta_*}}{\bar{S}_T^{\delta_*}}\bigg|\mathcal{A}_t\right). \tag{5.2.1}$$

For the deterministic model of the short rate we have

$$\mathrm{E}\left(\frac{B_t}{B_T}\bigg|\mathcal{A}_t\right) = \exp\left(-\int_t^T r(s)ds\right). \tag{5.2.2}$$

For the Vasicek model of the short rate we have from Vasicek [1977]

$$\mathrm{E}\left(\frac{B_t}{B_T}\bigg|\mathcal{A}_t\right) = A(t,T)\exp(-r_t B(t,T)), \tag{5.2.3}$$

where

$$B(t,T) = \frac{1-\exp(-\kappa(T-t))}{\kappa} \tag{5.2.4}$$

and

$$A(t,T) = \exp\left((\bar{r}-\frac{\sigma^2}{2\kappa^2})(B(t,T)-T+t) - \frac{\sigma^2}{4\kappa}B(t,T)^2\right). \tag{5.2.5}$$

For the CIR model of the short rate we have from Cox et al. [1985]

$$\mathrm{E}\left(\frac{B_t}{B_T}\bigg|\mathcal{A}_t\right) = A(t,T)\exp(-r_t B(t,T)), \tag{5.2.6}$$

where

$$A(t,T) = \left(\frac{h\exp(\frac{1}{2}\kappa(T-t))}{\kappa\sinh\frac{1}{2}h(T-t)+h\cosh\frac{1}{2}h(T-t)}\right)^{2\kappa\bar{r}/\sigma^2} \tag{5.2.7}$$

$$B(t,T) = \frac{2\sinh\frac{1}{2}h(T-t)}{\kappa\sinh\frac{1}{2}h(T-t)+h\cosh\frac{1}{2}h(T-t)} \tag{5.2.8}$$

and

$$h = \sqrt{\kappa^2+2\sigma^2}. \tag{5.2.9}$$

For the 3/2 model of the short rate we have from Ahn and Gao [1999]

$$\mathrm{E}\left(\frac{B_t}{B_T}\bigg|\mathcal{A}_t\right) = \frac{\Gamma(\gamma_1-\alpha_1)}{\Gamma(\gamma_1)}\left(\frac{2}{\sigma^2 y(t,r_t)}\right)^{\alpha_1} M(\alpha_1,\gamma_1,\frac{-2}{\sigma^2 y(t,r_t)}), \tag{5.2.10}$$

where

$$y(t, r_t) = \frac{r_t}{p}(\exp((T-t)p) - 1) \tag{5.2.11}$$

$$\alpha_u = -\left(\frac{1}{2} - \frac{q}{\sigma^2}\right) + \sqrt{\left(\frac{1}{2} - \frac{q}{\sigma^2}\right)^2 + \frac{2u}{\sigma^2}}$$

$$\gamma_u = 2\left(\alpha_u + 1 - \frac{q}{\sigma^2}\right).$$

Here M is the confluent hypergeometric function given by

$$M(\alpha, \gamma, z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\gamma)_n} \frac{z^n}{n!} \tag{5.2.12}$$

and $\Gamma(x) = \int_0^\infty u^{x-1} \exp(-u)\, du$ is the gamma function. For the Black-Scholes discounted GOP we have

$$\mathrm{E}\left(\frac{\bar{S}_t^{\delta_*}}{\bar{S}_T^{\delta_*}} \middle| \mathcal{A}_t\right) = 1, \tag{5.2.13}$$

whereas for the MMM discounted GOP we have from Platen and Heath [2006]

$$\mathrm{E}\left(\frac{\bar{S}_t^{\delta_*}}{\bar{S}_T^{\delta_*}} \middle| \mathcal{A}_t\right) = 1 - \exp\left(-\frac{1}{2}\bar{S}_t^{\delta_*}/(\varphi_T - \varphi_t)\right), \tag{5.2.14}$$

where $\varphi_t = \frac{1}{4}\bar{\alpha}_0(\exp(\eta t) - 1)/\eta$. Thus various combinations of (5.2.3), (5.2.6), (5.2.10), (5.2.13) and (5.2.14) inserted into (5.2.1) give explicit formulae for the real-world prices of ZCBs under each considered market model. We see that because of the multiplier $1 - \exp(-\frac{1}{2}\bar{S}_t^{\delta_*}/(\varphi_{\bar{T}} - \varphi_t))$ the prices of ZCBs under models with a MMM discounted GOP are lower than corresponding prices under models with a BS discounted GOP. This translates into higher bond yields and forward rates under models with a MMM discounted GOP.

5.3 Options on GOP

When the claim is $H_T = 1$ the real-world pricing formula (5.1.1) gives at time t the price $P(t,T)$ of a zero-coupon bond (ZCB). Further, when $H_T = (S_T^{\delta_*} - K)^+$ formula (5.1.1) gives at time t the price $c_{T,K}(t, S_t^{\delta_*})$ of an equity index call option having strike price K, and when $H_T = (K - S_T^{\delta_*})^+$ formula (5.1.1) gives the price $p_{T,K}(t, S_t^{\delta_*})$ of an equity index put option having strike price K. Because of the following relation between payoffs

$$(S_T^{\delta_*} - K)^+ = (K - S_T^{\delta_*})^+ + S_T^{\delta_*} - K \tag{5.3.1}$$

the put-call parity relation

$$c_{T,K}(t, S_t^{\delta_*}) = p_{T,K}(t, S_t^{\delta_*}) + S_t^{\delta_*} - KP(t,T) \tag{5.3.2}$$

holds. Additionally, when $H_T = S_T^{\delta_*}\mathbf{1}_{S_T^{\delta_*}>K}$ formula (5.1.1) gives at time t the price $A^+_{T,K}(t, S_t^{\delta_*})$ of an asset-or-nothing binary call option having strike price K and when $H_T = S_T^{\delta_*}\mathbf{1}_{S_T^{\delta_*}\leq K}$ formula (5.1.1) gives the price $A^-_{T,K}(t, S_t^{\delta_*})$ of an asset-or-nothing binary put option having strike price K. Finally, when $H_T = \mathbf{1}_{S_T^{\delta_*}>K}$ formula (5.1.1) gives the price $B^+_{T,K}(t, S_t^{\delta_*})$ of a cash-or-nothing binary call option having strike price K and when $H_T = \mathbf{1}_{S_T^{\delta_*}\leq K}$ formula (5.1.1) gives the price $B^-_{T,K}(t, S_t^{\delta_*})$ of a cash-or-nothing binary put option having strike price K.

Under our considered market models (5.1.1) gives the price of a call option as

$$c_{T,K}(t, S^{\delta_*}) = \mathrm{E}\left(\frac{S_t^{\delta_*}}{S_T^{\delta_*}}(S_T^{\delta_*} - K)^+ \,\middle|\, \mathcal{A}_t\right) \tag{5.3.3}$$

and the price of a put option as

$$p_{T,K}(t, S^{\delta_*}) = \mathrm{E}\left(\frac{S_t^{\delta_*}}{S_T^{\delta_*}}(K - S_T^{\delta_*})^+ \,\middle|\, \mathcal{A}_t\right). \tag{5.3.4}$$

The prices of the respective asset-or-nothing and cash-or-nothing call and put options are

$$\begin{aligned} A^+_{T,K}(t, S^{\delta_*}) &= S_t^{\delta_*}\mathrm{E}\left(\mathbf{1}_{S_T^{\delta_*}>K} \,\middle|\, \mathcal{A}_t\right) \\ A^-_{T,K}(t, S^{\delta_*}) &= S_t^{\delta_*}\mathrm{E}\left(\mathbf{1}_{S_T^{\delta_*}\leq K} \,\middle|\, \mathcal{A}_t\right) \\ B^+_{T,K}(t, S^{\delta_*}) &= \mathrm{E}\left(\frac{S_t^{\delta_*}}{S_T^{\delta_*}}\mathbf{1}_{S_T^{\delta_*}>K} \,\middle|\, \mathcal{A}_t\right) \\ B^-_{T,K}(t, S^{\delta_*}) &= \mathrm{E}\left(\frac{S_t^{\delta_*}}{S_T^{\delta_*}}\mathbf{1}_{S_T^{\delta_*}\leq K} \,\middle|\, \mathcal{A}_t\right). \end{aligned} \tag{5.3.5}$$

Let $f_{S_T^{\delta_*}}(x)$ denote the probability density function of the random variable $S_T^{\delta_*}$, and define the random variable $R_T^{\delta_*}$ as being related to $S_T^{\delta_*}$ having the probability density function

$$f_{R_T^{\delta_*}}(x) = \frac{S_t^{\delta_*}}{x} f_{S_T^{\delta_*}}(x) \Big/ \mathrm{E}\left(\frac{S_t^{\delta_*}}{S_T^{\delta_*}} \,\middle|\, \mathcal{A}_t\right). \tag{5.3.6}$$

The pricing formulae of the various call and put options can then be shown to be

$$\begin{aligned}
c_{T,K}(t,S^{\delta_*}) &= S_t^{\delta_*}\Big(1-F_{S_T^{\delta_*}}(K)\Big) - P(t,T)K\Big(1-F_{R_T^{\delta_*}}(K)\Big) \qquad (5.3.7)\\
p_{T,K}(t,S^{\delta_*}) &= -S_t^{\delta_*}F_{S_T^{\delta_*}}(K) + P(t,T)K\,F_{R_T^{\delta_*}}(K)\\
A^+_{T,K}(t,S^{\delta_*}) &= S_t^{\delta_*}\Big(1-F_{S_T^{\delta_*}}(K)\Big)\\
A^-_{T,K}(t,S^{\delta_*}) &= S_t^{\delta_*}\,F_{S_T^{\delta_*}}(K)\\
B^+_{T,K}(t,S^{\delta_*}) &= P(t,T)\Big(1-F_{R_T^{\delta_*}}(K)\Big)\\
B^-_{T,K}(t,S^{\delta_*}) &= P(t,T)\,F_{R_T^{\delta_*}}(K),
\end{aligned}$$

where $F_{S_T^{\delta_*}}(x)$ and $F_{R_T^{\delta_*}}(x)$ denote the cumulative distribution functions of the random variables $S_T^{\delta_*}$ and $R_T^{\delta_*}$, respectively.

The following two theorems give exact expressions for the cumulative distribution functions $F_{S_T^{\delta_*}}(K)$ and $F_{R_T^{\delta_*}}(K)$ in terms of the cumulative distribution functions of the lognormal distribution and the noncentral gamma distribution, a straightforward generalisation of the noncentral chi-squared distribution, these being

$$\begin{aligned}
LN(y;\mu,\sigma^2) &= \int_0^y \frac{1}{x\sqrt{2\pi\sigma^2}}\exp\left(-\frac{1}{2\sigma^2}(\log x-\mu)^2\right)dx \qquad (5.3.8)\\
NCG(y;\alpha,\gamma,\lambda) &= \int_0^y \gamma\left(\frac{2\gamma x}{\lambda}\right)^{\alpha/2-1/2}\exp\big(-\frac{1}{2}(\lambda+2\gamma x)\big)\,I_{\alpha-1}(\sqrt{2\lambda\gamma x})\,dx,
\end{aligned}$$

respectively, where $I_\nu(x)$ is the modified Bessel function of the first kind with index ν, given by

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{j=0}^\infty \frac{1}{j!\Gamma(j+\nu+1)}\left(\frac{z^2}{4}\right)^j. \qquad (5.3.9)$$

In the proof of Theorem 5.3.2 we make use of an equivalent expression for the cumulative distribution function of the noncentral gamma distribution as a Poisson mixture with a gamma distribution, analogous to the noncentral chi-squared distribution being a Poisson mixture with a chi-squared distribution, namely

$$NCG(y;\alpha,\gamma,\lambda) = \sum_{j=0}^\infty \frac{(\lambda/2)^j}{j!}\exp(-\lambda/2)G(y;\alpha+j,\gamma), \qquad (5.3.10)$$

where $G(y;\alpha,\gamma)$ is the cumulative distribution function of the gamma distribution

$$G(y;\alpha,\gamma) = \frac{\gamma^\alpha}{\Gamma(\alpha)}\int_0^y x^{\alpha-1}\exp(-\gamma x)\,dx. \qquad (5.3.11)$$

Because a lognormal random variable $X \sim LN(\mu, \sigma^2)$ is one for which its logarithm is normally distributed, that is $\log X \sim N(\mu, \sigma^2)$, we have that the cumulative distribution function of the lognormal distribution satisfies $LN(y; \mu, \sigma^2) = N\big((\log y - \mu)/\sigma\big)$ where $N(x)$ denotes the cumulative distribution function of the standard normal distribution. Also, because a noncentral gamma random variable $X \sim NCG(\alpha, \gamma, \lambda)$ is one for which the product with 2γ is noncentral chi-squared distributed, that is $2\gamma X \sim \chi^2_{2\alpha,\lambda}$, we have that the cumulative distribution function of the noncentral gamma distribution satisfies $NCG(y; \alpha, \gamma, \lambda) = \chi^2_{2\alpha,\lambda}(2\gamma y)$ where $\chi^2_{\nu,\lambda}(x)$ denotes the cumulative distribution function of the noncentral chi-squared distribution having ν degrees of freedom and noncentrality parameter λ.

Theorem 5.3.1 *For the Black-Scholes discounted GOP $\bar{S}^{\delta_*}_T$ and random variable $L = \log(B_T/B_t)$ we have*

$$F_{S^{\delta_*}_T}(K) = \mathrm{E}\big(LN(K; \log S^{\delta_*}_t + L + \frac{1}{2}\theta^2(T-t), \theta^2(T-t))\big) = \mathrm{E}\big(N(-d_1(L))\big) \tag{5.3.12}$$

$$\begin{aligned} F_{R^{\delta_*}_T}(K) &= \mathrm{E}\big(\exp(-L) LN(K; \log S^{\delta_*}_t + L - \frac{1}{2}\theta^2(T-t), \theta^2(T-t))\big)/\mathrm{E}\big(\exp(-L)\big) \\ &= \frac{\mathrm{E}\big(\exp(-L) N(-d_2(L))\big)}{\mathrm{E}\big(\exp(-L)\big)} \end{aligned}$$

where

$$\begin{aligned} d_1(L) &= \frac{L + \frac{1}{2}\theta^2(T-t) + \log\frac{S^{\delta_*}_t}{K}}{\sqrt{\theta^2(T-t)}} \\ d_2(L) &= d_1(L) - \sqrt{\theta^2(T-t)}. \end{aligned} \tag{5.3.13}$$

Proof. We know that under the Black-Scholes model of the discounted GOP $\bar{S}^{\delta_*}_T$ is lognormally distributed, that is

$$\bar{S}^{\delta_*}_T \sim LN(\log \bar{S}^{\delta_*}_t + \frac{1}{2}\theta^2(T-t), \theta^2(T-t)), \tag{5.3.14}$$

and, therefore, $S^{\delta_*}_T$ conditioned on the random variable $L = \log(B_T/B_t)$ is also lognormally distributed, that is

$$S^{\delta_*}_T \sim LN(\log B_T + \log \bar{S}^{\delta_*}_t + \frac{1}{2}\theta^2(T-t), \theta^2(T-t)), \tag{5.3.15}$$

which can be rewritten as

$$S^{\delta_*}_T \sim LN(L + \log S^{\delta_*}_t + \frac{1}{2}\theta^2(T-t), \theta^2(T-t)). \tag{5.3.16}$$

Hence

$$\begin{aligned} F_{S_T^{\delta_*}}(K) &= \mathrm{E}(\mathbf{1}_{S_T^{\delta_*}\le K}) \\ &= \mathrm{E}\big(\mathrm{E}(\mathbf{1}_{S_T^{\delta_*}\le K}|L)\big) \\ &= \mathrm{E}\big(LN(K; L+\log S_t^{\delta_*}+\frac{1}{2}\theta^2(T-t), \theta^2(T-t))\big) \\ &= \mathrm{E}(N(-d_1(L))). \end{aligned} \tag{5.3.17}$$

Also the cumulative distribution function of the random variable $R_T^{\delta_*}$ is computed to be

$$F_{R_T^{\delta_*}}(K) = \mathrm{E}\Big(\mathrm{E}\big(S_t^{\delta_*}(S_T^{\delta_*})^{-1}\mathbf{1}_{S_T^{\delta_*}\le K}|L\big)\Big)/\mathrm{E}\Big(\mathrm{E}\big(S_t^{\delta_*}(S_T^{\delta_*})^{-1}|L\big)\Big). \tag{5.3.18}$$

But

$$\begin{aligned} \mathrm{E}\big(S_t^{\delta_*}(S_T^{\delta_*})^{-1}|L\big) &= S_t^{\delta_*}\mathrm{E}\big((S_T^{\delta_*})^{-1}|L\big) \\ &= S_t^{\delta_*}\exp\big(-(L+\log S_t^{\delta_*}+\frac{1}{2}\theta^2(T-t))+\frac{1}{2}\theta^2(T-t)\big) \\ &= \exp(-L) \end{aligned} \tag{5.3.19}$$

and

$$\begin{aligned} &\mathrm{E}\big(S_t^{\delta_*}(S_T^{\delta_*})^{-1}\mathbf{1}_{S_T^{\delta_*}\le K}|L\big) \\ &= S_t^{\delta_*}\int_0^K \frac{1}{x} f_{S_T^{\delta_*}|L}(x)\,dx \\ &= S_t^{\delta_*}\int_0^K \exp(-\log x) f_{S_T^{\delta_*}|L}(x)\,dx. \end{aligned} \tag{5.3.20}$$

Inserting the explicit expression for the lognormal density function $f_{S_T^{\delta_*}|L}(x)$ gives

$$\begin{aligned} &\mathrm{E}\big(S_t^{\delta_*}(S_T^{\delta_*})^{-1}\mathbf{1}_{S_T^{\delta_*}\le K}|L\big) \\ &= S_t^{\delta_*}\int_0^K \frac{1}{x\sqrt{2\pi\theta^2(T-t)}} \\ &\times \exp\Big\{-\frac{1}{2\theta^2(T-t)}\big(\log x-(L+\log S_t^{\delta_*}+\frac{1}{2}\theta^2(T-t))\big)^2+2\theta^2(T-t)\log x\Big\}\,dx \end{aligned} \tag{5.3.21}$$

and completing the square in the exponential in the integrand above gives

$$S_t^{\delta_*} \int_0^K \frac{1}{x\sqrt{2\pi\theta^2(T-t)}} \times \exp\Big\{ -\frac{1}{2\theta^2(T-t)}\big(\log x \tag{5.3.22}$$
$$- (L + \log S_t^{\delta_*} - \frac{1}{2}\theta^2(T-t))\big)^2 + 2(L+\log S_t^{\delta_*})\theta^2(T-t)\Big\}\, dx$$
$$= S_t^{\delta_*} \exp(-(L+\log S_t^{\delta_*})) \int_0^K \frac{1}{x\sqrt{2\pi\theta^2(T-t)}}$$
$$\times \exp\Big\{ -\frac{1}{2\theta^2(T-t)}\big(\log x - (L+\log S_t^{\delta_*} - \frac{1}{2}\theta^2(T-t))\big)^2\Big\}\, dx$$
$$= \exp(-L)\, LN\big(K; L + \log S_t^{\delta_*} - \frac{1}{2}\theta^2(T-t), \theta^2(T-t)\big).$$

Therefore,

$$F_{R_T^{\delta_*}}(K) \tag{5.3.23}$$
$$= \mathrm{E}\Big(\exp(-L)\, LN\big(K; L+\log S_t^{\delta_*} - \frac{1}{2}\theta^2(T-t), \theta^2(T-t)\big)\Big)/\mathrm{E}\big(\exp(-L)\big)$$
$$= \mathrm{E}(\exp(-L)\, N(-d_2(L)))/\mathrm{E}\big(\exp(-L)\big),$$

as required. **Q.E.D.**

Theorem 5.3.2 *For the MMM discounted GOP $\bar{S}_T^{\delta_*}$ and the random variable $L = \log(B_T/B_t)$ we have*

$$F_{S_T^{\delta_*}}(K) = \mathrm{E}\big(NCG(K; 2, \exp(-L)/(2(\varphi_T - \varphi_t)B_t), \lambda)\big) = \mathrm{E}\big(\chi^2_{4,\lambda}(u(L))\big) \tag{5.3.24}$$

$$F_{R_T^{\delta_*}}(K) = \frac{\mathrm{E}\big(\exp(-L)(NCG(K; 0, \exp(-L)/(2(\varphi_T-\varphi_t)B_t), \lambda) - \exp(-\lambda/2))\big)}{(1-\exp(-\lambda/2))\mathrm{E}\big(\exp(-L)\big)}$$
$$= \frac{\mathrm{E}\big(\exp(-L)(\chi^2_{0,\lambda}(u(L)) - \exp(-\lambda/2))\big)}{(1-\exp(-\lambda/2))\mathrm{E}\big(\exp(-L)\big)},$$

where

$$u(L) = \frac{K}{B_t(\varphi_T - \varphi_t)\exp(L)}, \tag{5.3.25}$$

$$\varphi_t = \frac{1}{4}\bar{\alpha}_0(\exp(\eta t) - 1)/\eta \tag{5.3.26}$$

and

$$\lambda = \frac{\bar{S}_T^{\delta_*}}{\varphi_T - \varphi_t}. \tag{5.3.27}$$

Proof. We know that under the MMM, the discounted GOP $\bar{S}_T^{\delta_*}$ is noncentral gamma distributed, that is

$$\bar{S}_T^{\delta_*} \sim NCG(2, 1/(2(\varphi_T - \varphi_t)), \lambda), \tag{5.3.28}$$

and therefore $S_T^{\delta_*}$ conditioned on the random variable $L = \log(B_T/B_t)$ is also noncentral gamma distributed, that is

$$S_T^{\delta_*} \sim NCG(2, \exp(-L)/(2B_t(\varphi_T - \varphi_t)), \lambda), \tag{5.3.29}$$

which can be rewritten as

$$S_T^{\delta_*}/(\exp(L)B_t(\varphi_T - \varphi_t)) \sim \chi^2_{4,\lambda}. \tag{5.3.30}$$

Hence

$$\begin{aligned} F_{S_T^{\delta_*}}(K) &= \mathrm{E}(\mathbf{1}_{S_T^{\delta_*} \leq K}) \\ &= \mathrm{E}\big(\mathrm{E}(\mathbf{1}_{S_T^{\delta_*} \leq K}|L)\big) \\ &= \mathrm{E}\big(NCG(K; 2, \exp(-L)/(2B_t(\varphi_T - \varphi_t)), \lambda)\big) \\ &= \mathrm{E}(\chi^2_{4,\lambda}(u(L))). \end{aligned} \tag{5.3.31}$$

Also the cumulative distribution function of the random variable $R_T^{\delta_*}$ is computed to be

$$F_{R_T^{\delta_*}}(K) = \mathrm{E}\bigg(\mathrm{E}\big(S_t^{\delta_*}(S_T^{\delta_*})^{-1}\mathbf{1}_{S_T^{\delta_*} \leq K}|L\big)\bigg)/\mathrm{E}\bigg(\mathrm{E}\big(S_t^{\delta_*}(S_T^{\delta_*})^{-1}|L\big)\bigg). \tag{5.3.32}$$

But

$$\begin{aligned} \mathrm{E}\big(S_t^{\delta_*}(S_T^{\delta_*})^{-1}|L\big) &= \bar{S}_t^{\delta_*}\exp(-L)\mathrm{E}\big((\bar{S}_T^{\delta_*})^{-1}\big) \\ &= \exp(-L)(1 - \exp(-\lambda/2)) \end{aligned} \tag{5.3.33}$$

and

$$\begin{aligned} &\mathrm{E}\bigg(\frac{S_t^{\delta_*}}{S_T^{\delta_*}}\mathbf{1}_{S_T^{\delta_*} \leq K}|L\bigg) \\ &= \exp(-L)\mathrm{E}\bigg(\frac{\bar{S}_t^{\delta_*}}{\bar{S}_T^{\delta_*}}\mathbf{1}_{\bar{S}_T^{\delta_*} \leq K/B_T}\bigg) \\ &= \exp(-L)\mathrm{E}\bigg(\frac{\lambda}{\bar{S}_T^{\delta_*}/(\varphi_T - \varphi_t)}\mathbf{1}_{\bar{S}_T^{\delta_*}/(\varphi_T - \varphi_t) \leq K\exp(-L)/(B_t(\varphi_T - \varphi_t))}\bigg) \\ &= \exp(-L)\int_0^{K\exp(-L)/(B_t(\varphi_T - \varphi_t))} \frac{\lambda}{x} f_{\bar{S}_T^{\delta_*}/(\varphi_T - \varphi_t)}(x)\,dx. \end{aligned} \tag{5.3.34}$$

The random variable $\bar{S}_T^{\delta_*}/(\varphi_T - \varphi_t)$ is distributed as $\chi^2_{4,\lambda}$, which has probability density function

$$f(x) = \sum_{j=0}^{\infty} \exp(-\lambda/2)\frac{(\lambda/2)^j}{j!} f_{\chi^2_{4+2j}}(x), \tag{5.3.35}$$

where the probability density function of the chi-squared distribution having $4+2j$ degrees of freedom has the formula

$$f_{\chi^2_{4+2j}}(x) = \frac{(1/2)^{2+j}}{\Gamma(2+j)} x^{1+j} \exp(-x/2). \tag{5.3.36}$$

Therefore, the integrand in the RHS of (5.3.34) can be written as

$$\begin{aligned}
&\frac{\lambda}{x} f_{\bar{S}^{\delta_*}_T/(\varphi_T-\varphi_t)}(x) \qquad (5.3.37)\\
&= \frac{\lambda}{x} \sum_{j=0}^{\infty} \exp(-\lambda/2) \frac{(\lambda/2)^j}{j!} \frac{(1/2)^{2+j}}{\Gamma(2+j)} x^{1+j} \exp(-x/2)\\
&= \lambda \sum_{j=0}^{\infty} \exp(-\lambda/2) \frac{(\lambda/2)^j}{j!} \frac{(1/2)^{2+j}}{\Gamma(2+j)} x^{j} \exp(-x/2)\\
&= \lambda \sum_{j=0}^{\infty} \exp(-\lambda/2) \frac{(\lambda/2)^j}{j!} \frac{(1/2)^{2+j}}{\Gamma(2+j)} \frac{\Gamma(1+j)}{(1/2)^{1+j}} f_{\chi^2_{2+2j}}(x)\\
&= \lambda \sum_{j=0}^{\infty} \exp(-\lambda/2) \frac{(\lambda/2)^j}{j!} \frac{1/2}{1+j} f_{\chi^2_{2+2j}}(x)\\
&= \sum_{j=1}^{\infty} \exp(-\lambda/2) \frac{(\lambda/2)^j}{j!} f_{\chi^2_{2j}}(x)\\
&= f_{\chi^2_{0,\lambda}}(x) - \exp(-\lambda/2) \mathbf{1}_{x=0}.
\end{aligned}$$

Hence (5.3.34) becomes

$$\begin{aligned}
&\exp(-L) \int_0^{K\exp(-L)/(B_t(\varphi_T-\varphi_t))} \frac{\lambda}{x} f_{\bar{S}^{\delta_*}_T/(\varphi_T-\varphi_t)}(x)\, dx \qquad (5.3.38)\\
&= \exp(-L) \Bigg(\int_0^{K\exp(-L)/(B_t(\varphi_T-\varphi_t))} f_{\chi^2_{0,\lambda}}(x)\, dx - \exp(-\lambda/2) \Bigg)\\
&= \exp(-L) \Big(\chi^2_{0,\lambda}\big(K\exp(-L)/(B_t(\varphi_T-\varphi_t))\big) - \exp(-\lambda/2) \Big),
\end{aligned}$$

which leads to the result. **Q.E.D.**

For a deterministic short rate the cumulative distribution functions $F_{S^{\delta_*}_T}(x)$ and

$F_{R_T^{\delta_*}}(x)$ are readily computed to be, under the BS discounted GOP,

$$\begin{aligned} F_{S_T^{\delta_*}}(x) &= LN(x; \log S_t^{\delta_*} + \int_t^T r(s)ds + \frac{1}{2}\theta^2(T-t), \theta^2(T-t)) \\ &= N\Big(\big(\log \frac{x/B_T}{S_t^{\delta_*}/B_t} - \frac{1}{2}\theta^2(T-t)\big)/\sqrt{\theta^2(T-t)}\Big), \\ F_{R_T^{\delta_*}}(x) &= LN(x; \log S_t^{\delta_*} + \int_t^T r(s)ds - \frac{1}{2}\theta^2(T-t), \theta^2(T-t)) \\ &= N\Big(\big(\log \frac{x/B_T}{S_t^{\delta_*}/B_t} + \frac{1}{2}\theta^2(T-t)\big)/\sqrt{\theta^2(T-t)}\Big) \end{aligned} \tag{5.3.39}$$

and, under the MMM discounted GOP,

$$\begin{aligned} F_{S_T^{\delta_*}}(x) &= \big(NCG(x; 2, 1/(2(\varphi_T - \varphi_t)B_t \exp(\int_t^T r(s)ds)), \lambda)\big) \\ &= \chi^2_{4,\lambda}\big(x/((\varphi_T - \varphi_t)B_T)\big), \\ F_{R_T^{\delta_*}}(x) &= \frac{NCG(x; 0, 1/(2(\varphi_T - \varphi_t)B_t \exp(\int_t^T r(s)ds)), \lambda) - \exp(-\lambda/2)}{1 - \exp(-\lambda/2)} \\ &= \frac{\chi^2_{0,\lambda}\big(x/((\varphi_T - \varphi_t)B_T)\big) - \exp(-\lambda/2)}{1 - \exp(-\lambda/2)}. \end{aligned} \tag{5.3.40}$$

This leads to the following two corollaries to Theorems 5.3.1 and 5.3.2. The first corollary recovers the well-known Black-Scholes option pricing formulae in Black and Scholes [1973].

Corollary 5.3.3 *For a deterministic the short rate and BS discounted GOP, the various call and put options in* (5.3.7) *have the following explicit formulae*

$$\begin{aligned} c_{T,K}(t, S^{\delta_*}) &= S_t^{\delta_*} N(d_1) - \frac{B_t}{B_T} K\, N(d_1 - \sqrt{\theta^2(T-t)}) \\ p_{T,K}(t, S^{\delta_*}) &= -S_t^{\delta_*} N(-d_1) + \frac{B_t}{B_T} K\, N(-d_1 + \sqrt{\theta^2(T-t)}) \\ A^+_{T,K}(t, S^{\delta_*}) &= S_t^{\delta_*}\, N(d_1) \\ A^-_{T,K}(t, S^{\delta_*}) &= S_t^{\delta_*}\, N(-d_1) \\ B^+_{T,K}(t, S^{\delta_*}) &= \frac{B_t}{B_T}\, N(d_1 - \sqrt{\theta^2(T-t)}) \\ B^-_{T,K}(t, S^{\delta_*}) &= \frac{B_t}{B_T}\, N(-d_1 + \sqrt{\theta^2(T-t)}), \end{aligned} \tag{5.3.41}$$

where

$$d_1 = \big(\log \frac{S_t^{\delta_*}/B_t}{K/B_T} + \frac{1}{2}\theta^2(T-t)\big)/\sqrt{\theta^2(T-t)}. \tag{5.3.42}$$

The second corollary recovers the formulae supplied in Hulley et al. [2005] and in Section 13.3 of Platen and Heath [2006].

Corollary 5.3.4 *For a deterministic the short rate and MMM discounted GOP, the various call and put options in* (5.3.7) *have the following explicit formulae*

$$c_{T,K}(t, S^{\delta_*}) = S_t^{\delta_*}\left(1 - \chi^2_{4,\lambda}\left(\frac{K/B_T}{\varphi_T - \varphi_t}\right)\right) - \frac{B_t}{B_T}K\left(1 - \chi^2_{0,\lambda}\left(\frac{K/B_T}{\varphi_T - \varphi_t}\right)\right) \tag{5.3.43}$$

$$p_{T,K}(t, S^{\delta_*}) = -S_t^{\delta_*}\chi^2_{4,\lambda}\left(\frac{K/B_T}{\varphi_T - \varphi_t}\right) + \frac{B_t}{B_T}K\left(\chi^2_{0,\lambda}\left(\frac{K/B_T}{\varphi_T - \varphi_t}\right) - \exp(-\lambda/2)\right)$$

$$A^+_{T,K}(t, S^{\delta_*}) = S_t^{\delta_*}\left(1 - \chi^2_{4,\lambda}\left(\frac{K/B_T}{\varphi_T - \varphi_t}\right)\right)$$

$$A^-_{T,K}(t, S^{\delta_*}) = S_t^{\delta_*}\,\chi^2_{4,\lambda}\left(\frac{K/B_T}{\varphi_T - \varphi_t}\right)$$

$$B^+_{T,K}(t, S^{\delta_*}) = \frac{B_t}{B_T}\left(1 - \chi^2_{0,\lambda}\left(\frac{K/B_T}{\varphi_T - \varphi_t}\right)\right)$$

$$B^-_{T,K}(t, S^{\delta_*}) = \frac{B_t}{B_T}\left(\chi^2_{0,\lambda}\left(\frac{K/B_T}{\varphi_T - \varphi_t}\right) - \exp(-\lambda/2)\right).$$

For a Vasicek model of the short rate

$$L \sim N\big(m(t,T), v(t,T)\big) \tag{5.3.44}$$

where

$$m(t,T) = (r_t - \bar{r})B(t,T) + \bar{r}(T-t) \tag{5.3.45}$$

$$v(t,T) = \frac{\sigma^2}{\kappa^2}(T - t - B(t,T) - \frac{1}{2}\kappa B(t,T)^2)$$

and

$$B(t,T) = \frac{1 - \exp(-\kappa(T-t))}{\kappa}. \tag{5.3.46}$$

So for a Vasicek short rate and BS discounted GOP

$$S_T^{\delta_*} \sim LN\big(\log S_t + \frac{1}{2}\theta^2(T-t) + m(t,T), \theta^2(T-t) + v(t,T)\big) \tag{5.3.47}$$

$$R_T^{\delta_*} \sim LN\big(\log S_t - \frac{1}{2}\theta^2(T-t) + m(t,T) - v(t,T), \theta^2(T-t) + v(t,T)\big)$$

and the cumulative distribution functions $F_{S_T^{\delta_*}}(x)$ and $F_{R_T^{\delta_*}}(x)$ are readily computed to be

$$F_{S_T^{\delta_*}}(x) = LN\big(x; \log S_t + \frac{1}{2}\theta^2(T-t) + m(t,T), \theta^2(T-t) + v(t,T)\big) \tag{5.3.48}$$

$$F_{R_T^{\delta_*}}(x) = LN\big(x; \log S_t - \frac{1}{2}\theta^2(T-t) + m(t,T) - v(t,T), \theta^2(T-t) + v(t,T)\big).$$

Also for a Vasicek short rate and MMM discounted GOP

$$F_{S_T^{\delta_*}}(x) = \int_{-\infty}^{\infty} \chi^2_{4,\lambda}(u(z))n(z)dz \tag{5.3.49}$$

$$F_{R_T^{\delta_*}}(x) = \Big\{ \int_{-\infty}^{\infty} \exp(-m(t,T) - \sqrt{v(t,T)}z)\chi^2_{0,\lambda}(u(z))n(z)dz$$
$$- \exp(-\lambda/2)\exp(-m(t,T) + \frac{1}{2}v(t,T))\Big\}$$
$$\times \Big\{(1-\exp(-\lambda/2))\exp(-m(t,T)+\frac{1}{2}v(t,T))\Big\}^{-1},$$

where $m(t,T)$ and $v(t,T)$ are given in (5.3.45), $u(z)$ is given in (5.3.25) and $n(z)$ is the probability density function of the standard normal distribution.

For the CIR short rate model and the 3/2 short rate model the probability density function of L is computed as the inverse Fourier transform of the moment generating function (MGF), that is

$$f_L(x) = \int_{-\infty}^{\infty} \exp(2\pi i x s) MGF_L(-2\pi i s)\, ds. \tag{5.3.50}$$

Here the MGF of L under the CIR short rate model is

$$MGF_L(u) = \left(\frac{h_u \exp(\frac{1}{2}\kappa(T-t))}{\kappa \sinh\frac{1}{2}(T-t)h_u + h_u \cosh\frac{1}{2}(T-t)h_u}\right)^{2\kappa\bar{r}/\sigma^2} \tag{5.3.51}$$
$$\times \exp\left(u\frac{2\sinh\frac{1}{2}(T-t)h_u}{\kappa\sinh\frac{1}{2}(T-t)h_u + h_u\cosh\frac{1}{2}(T-t)h_u} r_t\right),$$

where $h_u = \sqrt{\kappa^2 - 2u\sigma^2}$. From Theorem 3 of Carr and Sun [2007] the MGF of L under the 3/2 short rate model is

$$MGF_L(-u) = \frac{\Gamma(\gamma_u - \alpha_u)}{\Gamma(\gamma_u)}\left(\frac{2}{\sigma^2 y(t,r_t)}\right)^{\alpha_u} M(\alpha_u, \gamma_u, \frac{-2}{\sigma^2 y(t,r_t)}), \tag{5.3.52}$$

where the variables are as in (5.2.11). The cumulative distribution functions $F_{S_T^{\delta_*}}(x)$ and $F_{R_T^{\delta_*}}(x)$ under the BS discounted GOP become

$$F_{S_T^{\delta_*}}(x) = \int_0^{\infty} N(-d_1(x)) f_L(x)\, dx \tag{5.3.53}$$
$$F_{R_T^{\delta_*}}(x) = \frac{\int_0^\infty \mathrm{e}^{-x} N(-d_2(x)) f_L(x)\, dx}{MGF_L(-1)}$$

and, under the MMM discounted GOP, become

$$F_{S_T^{\delta_*}}(x) = \int_0^{\infty} \chi^2_{4,\lambda}(u(x)) f_L(x)\, dx \tag{5.3.54}$$
$$F_{R_T^{\delta_*}}(x) = \frac{\int_0^\infty \mathrm{e}^{-x}(\chi^2_{0,\lambda}(u(x)) - \mathrm{e}^{-\lambda/2}) f_L(x)\, dx}{MGF_L(-1)},$$

where $u(x)$ is given by

$$u(x) = \frac{K}{B_t \exp(x)(\varphi_{\bar{T}} - \varphi_t)}, \tag{5.3.55}$$

and $\varphi_t = \varphi_0 + \frac{1}{4}\bar{\alpha}_0(\exp(\eta t) - 1)/\eta$.

Thus we have demonstrated how the various cumulative distribution functions can be computed and, combined with (5.3.7), how prices of various call, put, asset-or-nothing and cash-or-nothing options can be computed. In Appendix I we provide approximate formulae for the various cumulative distribution functions, which lead to simplified and rapid computations.

5.4 Options on ZCBs

Pricing formulae for options on zero-coupon bonds are of paramount importance for most interest rate options because they are the building blocks for these securities. This is exemplified by the relationship of zero-coupon bond options to interest rate caps and floors provided in (2.10.32)−(2.10.34) and we exploit this in the following section on caps and floors.

In this section we provide formulae for the options on zero-coupon bonds for market models involving deterministic, Vasicek and CIR short rates only. We omit the cases involving the 3/2 short rate because a ZCB option formula is not currently available in the literature, although a suggested approach is given by Lo [2013].

Option pricing formulae under any market model considered in this thesis can be calculated using Theorem 2.10.2.

When the discounted GOP obeys Black-Scholes dynamics the formulae for the options on ZCBs are those obtained under classical risk neutral assumptions, as the following lemma demonstrates.

Lemma 5.4.1 *For a stochastic short rate r_t and a BS discounted GOP as in* (4.2.1)*, the* real-world *price at time t of* call *and* put *options on a zero-coupon bond with expiry $\bar{T}$, bond maturity $T \geq \bar{T}$ and strike price K are*

$$\mathbf{zcbcall}_{\bar{T},T,K}(t) = \mathrm{E}\left(\frac{B_t}{B_{\bar{T}}} \left(G_T(\bar{T}) - K\right)^+ \,\middle|\, \mathcal{A}_t\right) \tag{5.4.1}$$

$$\mathbf{zcbput}_{\bar{T},T,K}(t) = \mathrm{E}\left(\frac{B_t}{B_{\bar{T}}} \left(K - G_T(\bar{T})\right)^+ \,\middle|\, \mathcal{A}_t\right) \tag{5.4.2}$$

for $0 \leq t \leq \bar{T} \leq T$. Here $G_T(\bar{T})$ is the short rate contribution to the zero-coupon bond price, as defined in (2.8.4).

Proof. As in the proof of Theorem 2.8.3 for the price of a zero-coupon bond we introduce the sigma-algebra $\mathcal{A}_t^r$ which is generated by $\mathcal{A}_t$ and the path of the short rate r until time T, that is, $\mathcal{A}_t^r = \sigma\{r_s,\, s \in [0,T]\} \cup \mathcal{A}_t$. Using (2.8.1) and Theorem 2.8.3 and noting that $\mathcal{A}_t \subseteq \mathcal{A}_t^r$, the price of a call option on a zero-coupon bond is

$$\begin{aligned}
\mathbf{zcbcall}_{\bar{T},T,K}(t) &= \mathrm{E}\left(\frac{S_t^{\delta_*}}{S_{\bar{T}}^{\delta_*}} \left(P(\bar{T},T) - K\right)^+ \,\Big|\, \mathcal{A}_t \right) \\
&= \mathrm{E}\left(\mathrm{E}\left(\frac{S_t^{\delta_*}}{S_{\bar{T}}^{\delta_*}} \left(P(\bar{T},T) - K\right)^+ \,\Big|\, \mathcal{A}_t^r \right) \Big|\, \mathcal{A}_t \right) \\
&= \mathrm{E}\left(\mathrm{E}\left(\frac{\bar{S}_t^{\delta_*}}{\bar{S}_{\bar{T}}^{\delta_*}} \frac{B_t}{B_{\bar{T}}} \,\Big|\, \mathcal{A}_t^r \right) \left(G_T(\bar{T}) - K\right)^+ \,\Big|\, \mathcal{A}_t \right) \\
&= \mathrm{E}\left(\frac{B_t}{B_{\bar{T}}} M_{\bar{T}}(t) \left(G_T(\bar{T}) - K\right)^+ \,\Big|\, \mathcal{A}_t \right) \\
&= \mathrm{E}\left(\frac{B_t}{B_{\bar{T}}} \left(G_T(\bar{T}) - K\right)^+ \,\Big|\, \mathcal{A}_t \right)
\end{aligned} \tag{5.4.3}$$

for $0 \leq t \leq \bar{T} \leq T$. The formula for a put option on a zero-coupon bond $\mathbf{zcbput}_{\bar{T},T,K}(t)$ is proven similarly. **Q.E.D.**

In respect of a deterministic short rate and a Black-Scholes discounted GOP we state the following theorem whose proof is straightforward.

Theorem 5.4.2 *Let our market model consist of a determinstic short rate and a Black-Scholes discounted GOP. The fair price at time t of a first order asset binary call (respectively put) option on a T-maturity zero-coupon bond with expiry date $\bar{T}$ and strike price K is*

$$A^+_{\bar{T},K}(t, P(\cdot,T)) = G_T(t)\mathbf{1}_{G_T(\bar{T})>K}, \tag{5.4.4}$$

$$A^-_{\bar{T},K}(t, P(\cdot,T)) = G_T(t)\mathbf{1}_{G_T(\bar{T})\leq K}. \tag{5.4.5}$$

The fair price at time t of a first order bond binary call (respectively put) option on the GOP with expiry date $\bar{T}$ and strike price K is

$$B^+_{\bar{T},K}(t, P(\cdot,T)) = G_{\bar{T}}(t)\mathbf{1}_{G_T(\bar{T})>K}, \tag{5.4.6}$$

$$B^-_{\bar{T},K}(t, P(\cdot,T)) = G_{\bar{T}}(t)\mathbf{1}_{G_T(\bar{T})\leq K}. \tag{5.4.7}$$

The fair price at time t of a call (respectively put) option on the GOP with expiry date $\bar{T}$ and strike price K is

$$c_{\bar{T},K}(t, P(\cdot,T)) = \Big(G_T(t) - G_{\bar{T}}(t)\,K\Big)\mathbf{1}_{G_T(\bar{T})>K}, \tag{5.4.8}$$

$$p_{\bar{T},K}(t, P(\cdot,T)) = \Big(-G_T(t) + G_{\bar{T}}(t)\,K\Big)\mathbf{1}_{G_T(\bar{T})\leq K}. \tag{5.4.9}$$

In respect of a deterministic short rate and an MMM discounted GOP we have the following theorem.

Theorem 5.4.3 *Let our market model consist of a determinstic short rate and a MMM discounted GOP. The fair price at time t of a first order asset binary call (respectively put) option on a T-maturity zero-coupon bond with expiry date $\bar{T}$ and strike price K is*

$$\begin{aligned} A^{+}_{\bar{T},K}(t, P(\cdot,T)) = &\frac{B_t}{B_T}\left(1-\chi^2_{0,\lambda}(u^*)\right) \\ &- \frac{B_t}{B_T}\exp\left(-\frac{\tau}{1+2\tau}\lambda\right)\left(1-\chi^2_{0,\lambda/(1+2\tau)}((1+2\tau)u^*)\right), \end{aligned} \tag{5.4.10}$$

$$A^{-}_{\bar{T},K}(t, P(\cdot,T)) = \frac{B_t}{B_T}\left(\chi^2_{0,\lambda}(u^*) - \exp\left(-\frac{\tau}{1+2\tau}\lambda\right)\chi^2_{0,\lambda/(1+2\tau)}\big((1+2\tau)u^*\big)\right). \tag{5.4.11}$$

The fair price at time t of a first order bond binary call (respectively put) option on the GOP with expiry date $\bar{T}$ and strike price K is

$$B^{+}_{\bar{T},K}(t, P(\cdot,T)) = \frac{B_t}{B_{\bar{T}}}\left(1-\chi^2_{0,\lambda}(u^*)\right), \tag{5.4.12}$$

$$B^{-}_{\bar{T},K}(t, P(\cdot,T)) = \frac{B_t}{B_{\bar{T}}}\left(\chi^2_{0,\lambda}(u^*) - \exp(-\lambda/2)\right). \tag{5.4.13}$$

The fair price at time t of a call (respectively put) option on the GOP with expiry date $\bar{T}$ and strike price K is

$$\begin{aligned} \mathbf{zcbcall}_{\bar{T},T,K}(t) &= c_{\bar{T},K}(t, P(\cdot,T)) \\ &= \frac{B_t}{B_T}\left(1-\chi^2_{0,\lambda}(u^*)\right) \\ &\quad - \frac{B_t}{B_T}\exp\left(-\frac{\tau}{1+2\tau}\lambda\right)\left(1-\chi^2_{0,\lambda/(1+2\tau)}((1+2\tau)u^*)\right) \\ &\quad - K\frac{B_t}{B_{\bar{T}}}\left(1-\chi^2_{0,\lambda}(u^*)\right), \end{aligned} \tag{5.4.14}$$

$$\begin{aligned} \mathbf{zcbput}_{\bar{T},T,K}(t)(t) &= p_{\bar{T},K}(t, P(\cdot,T)) \\ &= -\frac{B_t}{B_T}\left(\chi^2_{0,\lambda}(u^*) - \exp\left(-\frac{\tau}{1+2\tau}\lambda\right)\chi^2_{0,\lambda/(1+2\tau)}\big((1+2\tau)u^*\big)\right) \\ &\quad + K\frac{B_t}{B_{\bar{T}}}\left(\chi^2_{0,\lambda}(u^*) - \exp(-\lambda/2)\right). \end{aligned} \tag{5.4.15}$$

Here

$$u^* = \begin{cases} 2\frac{\varphi_T - \varphi_{\bar{T}}}{\varphi_{\bar{T}} - \varphi_t} \log \frac{1}{1 - KB_T/B_{\bar{T}}} \text{ if } 1 > KB_T/B_{\bar{T}}; \\ \infty \text{ otherwise} \end{cases} \tag{5.4.16}$$

$$\lambda = \frac{\bar{S}_t^{\delta_*}}{\varphi_{\bar{T}} - \varphi_t}$$

$$\tau = \frac{1}{2}\frac{\varphi_{\bar{T}} - \varphi_t}{\varphi_T - \varphi_{\bar{T}}}.$$

Proof. Using the real-world pricing formula (2.6.4) we have

$$\begin{aligned} A^+_{\bar{T},K}(t, P(\cdot, T)) &= E\left(\frac{S_t^{\delta_*}}{S_{\bar{T}}^{\delta_*}} P(\bar{T}, T)\mathbf{1}_{P(\bar{T},T)>K}\bigg|\mathcal{A}_t\right) \\ &= \frac{B_t}{B_T} E\left(\frac{\bar{S}_t^{\delta_*}}{\bar{S}_{\bar{T}}^{\delta_*}} M_T(\bar{T})\mathbf{1}_{M_T(\bar{T})>KB_T/B_{\bar{T}}}\bigg|\mathcal{A}_t\right). \end{aligned} \tag{5.4.17}$$

Denoting by U the random variable

$$\frac{\bar{S}_{\bar{T}}^{\delta_*}}{\varphi_T - \varphi_t}\bigg|\mathcal{A}_t \tag{5.4.18}$$

and by λ the value

$$\frac{\bar{S}_t^{\delta_*}}{\varphi_T - \varphi_t} \tag{5.4.19}$$

and noting that

$$M_T(\bar{T}) = 1 - \exp\left(-\frac{\bar{S}_{\bar{T}}^{\delta_*}}{2(\varphi_T - \varphi_{\bar{T}})}\right) = 1 - \exp(-\tau U) \tag{5.4.20}$$

and that $M_T(\bar{T}) \geq KB_T/B_{\bar{T}}$ is equivalent to

$$U \geq \frac{1}{\tau}\log\frac{1}{1 - K\,B_T/B_{\bar{T}}} = u^* \tag{5.4.21}$$

we have

$$\begin{aligned} A^+_{\bar{T},K}(t, P(\cdot, T)) &= \frac{B_t}{B_T} E\left(\frac{\lambda}{U}\Big(1 - \exp(-\tau U)\Big)\mathbf{1}_{U>u^*}\right) \\ &= \frac{B_t}{B_T} E\left(\frac{\lambda}{U}\mathbf{1}_{U>u^*}\right) \\ &- \frac{B_t}{B_T} E\left(\frac{\lambda}{U}\exp(-\tau U)\mathbf{1}_{U>u^*}\right) \\ &= \frac{B_t}{B_T}\left(1 - \chi^2_{0,\lambda}(u^*)\right) \\ &- \frac{B_t}{B_T}\exp\left(-\frac{\tau}{1+2\tau}\lambda\right)\left(1 - \chi^2_{0,\lambda/(1+2\tau)}((1+2\tau)u^*)\right), \end{aligned} \tag{5.4.22}$$

where we have made use of Lemma 4.3.19 and Lemma 4.3.22 and where we have the relations in (5.4.16).

To find the formula for the price of the asset binary put option on a ZCB we use the real-world pricing formula (2.6.4), giving

$$\begin{aligned} A^-_{\bar{T},K}(t, P(\cdot,T)) &= E\left(\frac{S^{\delta_*}_t}{S^{\delta_*}_{\bar{T}}} P(\bar{T},T)\mathbf{1}_{P(\bar{T},T)\le K}\Big|\mathcal{A}_t\right) \\ &= \frac{B_t}{B_T} E\left(\frac{\bar{S}^{\delta_*}_t}{\bar{S}^{\delta_*}_{\bar{T}}} M_T(\bar{T})\mathbf{1}_{M_T(\bar{T})\le KB_T/B_{\bar{T}}}\Big|\mathcal{A}_t\right). \end{aligned} \tag{5.4.23}$$

Using the same notation as for the asset binary call option on a ZCB and applying Lemma 4.3.19 and Lemma 4.3.22 we have

$$\begin{aligned} A^-_{\bar{T},K}(t, P(\cdot,T)) &= \frac{B_t}{B_T} E\left(\frac{\lambda}{U}(1-\exp(-\tau U))\mathbf{1}_{U\le u^*}\right) \\ &= \frac{B_t}{B_T} E\left(\frac{\lambda}{U}\mathbf{1}_{U\le u^*}\right) - \frac{B_t}{B_T} E\left(\frac{\lambda}{U}\exp(-\tau U)\mathbf{1}_{U\le u^*}\right) \\ &= \frac{B_t}{B_T}\left(\chi^2_{0,\lambda}(u^*) - \exp(-\lambda/2)\right) \\ &\quad - \frac{B_t}{B_T}\left(\exp\left(-\frac{\tau}{1+2\tau}\lambda\right)\chi^2_{0,\lambda/(1+2\tau)}((1+2\tau)u^*) - \exp(-\lambda/2)\right) \\ &= \frac{B_t}{B_T}\left(\chi^2_{0,\lambda}(u^*) - \exp\left(-\frac{\tau}{1+2\tau}\lambda\right)\chi^2_{0,\lambda/(1+2\tau)}\big((1+2\tau)u^*\big)\right), \end{aligned} \tag{5.4.24}$$

where the parameters are as in (5.4.16).

Using the real-world pricing formula (2.6.4) and applying Lemma 4.3.19 we have

$$\begin{aligned} B^+_{\bar{T},K}(t, P(\cdot,T)) &= E\left(\frac{S^{\delta_*}_t}{S^{\delta_*}_{\bar{T}}}\mathbf{1}_{P(\bar{T},T)>K}\Big|\mathcal{A}_t\right) \\ &= \frac{B_t}{B_{\bar{T}}} E\left(\frac{\bar{S}^{\delta_*}_t}{\bar{S}^{\delta_*}_{\bar{T}}}\mathbf{1}_{\bar{S}^{\delta_*}_{\bar{T}}>K^*}\Big|\mathcal{A}_t\right). \end{aligned} \tag{5.4.25}$$

Employing the same notation as above we have

$$\begin{aligned} B^+_{\bar{T},K}(t, P(\cdot,T)) &= \frac{B_t}{B_{\bar{T}}} E\left(\frac{\lambda}{U}\mathbf{1}_{U>u^*}\right) \\ &= \frac{B_t}{B_{\bar{T}}}(1-\chi^2_{0,\lambda}(u^*)) \end{aligned} \tag{5.4.26}$$

which is the result for the first order bond binary call option. The formula for the price of the bond binary put option on a ZCB follows analogously.

The formula (2.10.10) connects the price of a call option with the first order asset binary call option and the first order bond binary call option. Combining this with (5.4.10) and (5.4.12) gives the formula for the call option. Also combining (2.10.11), (5.4.11) and (5.4.13) gives the formula for the put option. **Q.E.D.**

From Lemma 5.4.1 and Corollary 3.2.25 we have the following theorem in respect of a model involving a Vasicek short rate and a Black-Scholes discounted GOP.

Theorem 5.4.4 *Let our market model consist of a Vasicek short rate and a Black-Scholes discounted GOP. The fair price at time t of a first order asset binary call (respectively put) option on a T-maturity zero-coupon bond with expiry date $\bar{T}$ and strike price K is*

$$A^{+}_{\bar{T},K}(t, P(\cdot,T)) = G_T(t) N(d_1), \tag{5.4.27}$$

$$A^{-}_{\bar{T},K}(t, P(\cdot,T)) = G_T(t) N(-d_1). \tag{5.4.28}$$

The fair price at time t of a first order bond binary call (respectively put) option on a T-maturity zero-coupon bond with expiry date $\bar{T}$ and strike price K is

$$B^{+}_{\bar{T},K}(t, P(\cdot,T)) = G_{\bar{T}}(t) N(d_1 - \sigma_G), \tag{5.4.29}$$

$$B^{-}_{\bar{T},K}(t, P(\cdot,T)) = G_{\bar{T}}(t) N(-d_1 + \sigma_G). \tag{5.4.30}$$

The fair price at time t of a call (respectively put) option on a T-maturity zero-coupon bond with expiry date $\bar{T}$ and strike price K is

$$c_{\bar{T},K}(t, P(\cdot,T)) = G_T(t) N(d_1) - G_{\bar{T}}(t)\, K\, N(d_1 - \sigma_G), \tag{5.4.31}$$

$$p_{\bar{T},K}(t, P(\cdot,T)) = -G_T(t) N(-d_1) + G_{\bar{T}}(t)\, K\, N(-d_1 + \sigma_G). \tag{5.4.32}$$

Here d_1 is given by

$$d_1 = \frac{\log(G_T(t)/(K\, G_{\bar{T}}(t))) + \frac{1}{2}\sigma_G^2}{\sigma_G}, \tag{5.4.33}$$

$\sigma_G = \sigma B(\bar{T}, T)$ *and* $B(\bar{T}, T)$ *is given in* (3.2.82).

From Theorem 2.10.2 and Corollary 3.2.25 we have the following theorem in respect of a model involving a Vasicek short rate and a MMM discounted GOP.

Theorem 5.4.5 *Let our market model consist of a Vasicek short rate and a MMM discounted GOP. The fair price at time t of a first order asset binary call (respectively put) option on a T-maturity zero-coupon bond with expiry date $\bar{T}$ and strike price K is*

$$A^{+}_{\bar{T},K}(t, P(\cdot,T)) = G_T(t) \int_0^\infty \frac{\lambda}{u}(1 - \mathrm{e}^{-\tau u}) N(d_1(u)) f_{\chi^2_{4,\lambda}}(u)\, du, \tag{5.4.34}$$

$$A^{-}_{\bar{T},K}(t, P(\cdot,T)) = G_T(t) \int_0^\infty \frac{\lambda}{u}(1 - \mathrm{e}^{-\tau u}) N(-d_1(u)) f_{\chi^2_{4,\lambda}}(u)\, du. \tag{5.4.35}$$

The fair price at time t of a first order bond binary call (respectively put) option on a T-maturity zero-coupon bond with expiry date $\bar{T}$ and strike price K is

$$B^{+}_{\bar{T},K}(t,P(\cdot,T)) = G_{\bar{T}}(t)\int_0^\infty \frac{\lambda}{u} N(d_1(u)-\sigma_G) f_{\chi^2_{4,\lambda}}(u)\,du, \tag{5.4.36}$$

$$B^{-}_{\bar{T},K}(t,P(\cdot,T)) = G_{\bar{T}}(t)\int_0^\infty \frac{\lambda}{u} N(-d_1(u)+\sigma_G) f_{\chi^2_{4,\lambda}}(u)\,du. \tag{5.4.37}$$

The fair price at time t of a call (respectively put) option on a T-maturity zero-coupon bond with expiry date $\bar{T}$ and strike price K is

$$\begin{aligned} c_{\bar{T},K}(t,P(\cdot,T)) &= G_T(t)\int_0^\infty \frac{\lambda}{u}(1-\mathrm{e}^{-\tau u}) N(d_1(u)) f_{\chi^2_{4,\lambda}}(u)\,du \\ &\quad - G_{\bar{T}}(t)\,K\int_0^\infty \frac{\lambda}{u} N(d_1(u)-\sigma_G) f_{\chi^2_{4,\lambda}}(u)\,du, \end{aligned} \tag{5.4.38}$$

$$\begin{aligned} p_{\bar{T},K}(t,P(\cdot,T)) &= -G_T(t)\int_0^\infty \frac{\lambda}{u}(1-\mathrm{e}^{-\tau u}) N(-d_1(u)) f_{\chi^2_{4,\lambda}}(u)\,du \\ &\quad + G_{\bar{T}}(t)\,K\int_0^\infty \frac{\lambda}{u} N(-d_1(u)+\sigma_G) f_{\chi^2_{4,\lambda}}(u)\,du. \end{aligned} \tag{5.4.39}$$

Here $d_1(u)$ is given by

$$d_1(u) = \frac{\log(G_T(t)/(K_u\,G_{\bar{T}}(t))) + \frac{1}{2}\sigma_G^2}{\sigma_G}, \tag{5.4.40}$$

K_u is given by

$$K_u = \frac{K}{1-\exp(-\tau u)}, \tag{5.4.41}$$

τ is given by

$$\tau = \frac{\varphi_{\bar{T}} - \varphi_t}{2(\varphi_T - \varphi_{\bar{T}})}, \tag{5.4.42}$$

$\sigma_G = \sigma B(\bar{T},T)$ and $B(\bar{T},T)$ is given in (3.2.82).

From Lemma 5.4.1 and Corollary 3.3.28 we have the following theorem in respect of a model involving a CIR short rate and a Black-Scholes discounted GOP.

Theorem 5.4.6 *Let our market model consist of a CIR short rate and a Black-Scholes discounted GOP. The fair price at time t of a first order asset binary call (respectively put) option on a T-maturity zero-coupon bond with expiry date $\bar{T}$ and strike price K is*

$$A^{+}_{\bar{T},K}(t,P(\cdot,T)) = G_T(t)\chi^2_{\nu,\omega}(R/\gamma), \tag{5.4.43}$$

$$A^{-}_{\bar{T},K}(t,P(\cdot,T)) = G_T(t)\Big(1-\chi^2_{\nu,\omega}(R/\gamma)\Big). \tag{5.4.44}$$

The fair price at time t of a first order bond binary call (respectively put) option on a T-maturity zero-coupon bond with expiry date $\bar{T}$ and strike price K is

$$B^+_{\bar{T},K}(t, P(\cdot, T)) = G_{\bar{T}}(t)\chi^2_{\nu,\omega'}(R/\gamma'), \tag{5.4.45}$$

$$B^-_{\bar{T},K}(t, P(\cdot, T)) = G_{\bar{T}}(t)\Big(1 - \chi^2_{\nu,\omega'}(R/\gamma')\Big). \tag{5.4.46}$$

The fair price at time t of a call (respectively put) option on a T-maturity zero-coupon bond with expiry date $\bar{T}$ and strike price K is

$$c_{\bar{T},K}(t, P(\cdot, T)) = G_T(t)\chi^2_{\nu,\omega}(R/\gamma) - G_{\bar{T}}(t)\, K\, \chi^2_{\nu,\omega'}(R/\gamma'), \tag{5.4.47}$$

$$p_{\bar{T},K}(t, P(\cdot, T)) = -G_T(t)\Big(1 - \chi^2_{\nu,\omega}(R/\gamma)\Big) + G_{\bar{T}}(t)\, K\, \Big(1 - \chi^2_{\nu,\omega'}(R/\gamma')\Big). \tag{5.4.48}$$

Here we have

$$\begin{aligned}
\nu &= 4\kappa\bar{r}/\sigma^2 \\
\gamma' &= \frac{1}{4}\sigma^2 B(t, T) \\
\omega' &= r_t\frac{4}{\sigma^2}\left(\frac{1}{2}h\coth\frac{1}{2}h(T-t) - \frac{1}{2}\kappa\right) - r_t\frac{8}{\sigma^2}\gamma' \\
\gamma &= \frac{1}{4}\sigma^2 B(t, T)(1 + \frac{1}{2}\sigma^2 B(t, T)B(T, \bar{T}))^{-1} \\
\omega &= r_t\frac{4}{\sigma^2}\left(\frac{1}{2}h\coth\frac{1}{2}h(T-t) - \frac{1}{2}\kappa\right)\Big(1 - 2B(T, \bar{T})\gamma\Big) - r_t\frac{8}{\sigma^2}\gamma \\
R &= \frac{1}{B(T, \bar{T})}\log\frac{A(T, \bar{T})}{K}.
\end{aligned} \tag{5.4.49}$$

From Theorem 2.10.2 and Corollary 3.3.28 we have the following theorem in respect of a model involving a Vasicek short rate and a MMM discounted GOP.

Theorem 5.4.7 *Let our market model consist of a CIR short rate and a MMM discounted GOP. The fair price at time t of a first order asset binary call (respectively put) option on a T-maturity zero-coupon bond with expiry date $\bar{T}$ and strike price K is*

$$A^+_{\bar{T},K}(t, P(\cdot, T)) = G_T(t)\int_0^\infty \frac{\lambda}{u}(1 - \mathrm{e}^{-\tau u})\chi^2_{\nu,\omega}(R_u/\gamma)f_{\chi^2_{4,\lambda}}(u)\, du, \tag{5.4.50}$$

$$A^-_{\bar{T},K}(t, P(\cdot, T)) = G_T(t)\left(1 - \mathrm{e}^{-\lambda\tau/(1+2\tau)} - \int_0^\infty \frac{\lambda}{u}(1 - \mathrm{e}^{-\tau u})\chi^2_{\nu,\omega}(R_u/\gamma)f_{\chi^2_{4,\lambda}}(u)\, du\right). \tag{5.4.51}$$

The fair price at time t of a first order bond binary call (respectively put) option on a T-maturity zero-coupon bond with expiry date $\bar{T}$ and strike price K is

$$B^{+}_{\bar{T},K}(t, P(\cdot, T)) = G_{\bar{T}}(t) \int_0^\infty \frac{\lambda}{u} \chi^2_{\nu,\omega'}(R_u/\gamma') f_{\chi^2_{4,\lambda}}(u)\, du, \tag{5.4.52}$$

$$B^{-}_{\bar{T},K}(t, P(\cdot, T)) = G_{\bar{T}}(t) \left(1 - \mathrm{e}^{-\lambda/2} - \int_0^\infty \frac{\lambda}{u} \chi^2_{\nu,\omega'}(R_u/\gamma') f_{\chi^2_{4,\lambda}}(u)\, du\right). \tag{5.4.53}$$

The fair price at time t of a call (respectively put) option on a T-maturity zero-coupon bond with expiry date $\bar{T}$ and strike price K is

$$\begin{aligned} c_{\bar{T},K}(t, P(\cdot, T)) &= G_T(t) \int_0^\infty \frac{\lambda}{u} (1 - \mathrm{e}^{-\tau u}) \chi^2_{\nu,\omega}(R_u/\gamma) f_{\chi^2_{4,\lambda}}(u)\, du \\ &\quad - G_{\bar{T}}(t)\, K \int_0^\infty \frac{\lambda}{u} \chi^2_{\nu,\omega'}(R_u/\gamma') f_{\chi^2_{4,\lambda}}(u)\, du, \end{aligned} \tag{5.4.54}$$

$$\begin{aligned} p_{\bar{T},K}(t, P(\cdot, T)) &= -G_T(t) \left(1 - \mathrm{e}^{-\lambda\tau/(1+2\tau)} - \int_0^\infty \frac{\lambda}{u} (1 - \mathrm{e}^{-\tau u}) \chi^2_{\nu,\omega}(R_u/\gamma) f_{\chi^2_{4,\lambda}}(u)\, du\right) \\ &\quad + G_{\bar{T}}(t)\, K \left(1 - \mathrm{e}^{-\lambda/2} - \int_0^\infty \frac{\lambda}{u} \chi^2_{\nu,\omega'}(R_u/\gamma') f_{\chi^2_{4,\lambda}}(u)\, du\right). \end{aligned} \tag{5.4.55}$$

Here we have

$$\begin{aligned} \nu &= 4\kappa\bar{r}/\sigma^2 \\ \gamma' &= \frac{1}{4}\sigma^2 B(t,T) \\ \omega' &= r_t \frac{4}{\sigma^2}\left(\frac{1}{2}h \coth \frac{1}{2}h(T-t) - \frac{1}{2}\kappa\right) - r_t \frac{8}{\sigma^2}\gamma' \\ \gamma &= \frac{1}{4}\sigma^2 B(t,T)(1 + \frac{1}{2}\sigma^2 B(t,T) B(T,\bar{T}))^{-1} \\ \omega &= r_t \frac{4}{\sigma^2}\left(\frac{1}{2}h \coth \frac{1}{2}h(T-t) - \frac{1}{2}\kappa\right)\left(1 - 2B(T,\bar{T})\gamma\right) - r_t \frac{8}{\sigma^2}\gamma \\ R_u &= \frac{1}{B(T,\bar{T})} \log \frac{A(T,\bar{T})}{K_u} \\ K_u &= K/(1 - \exp(-\tau u)) \\ \tau &= \frac{\varphi_{\bar{T}} - \varphi_t}{2(\varphi_T - \varphi_{\bar{T}})}. \end{aligned} \tag{5.4.56}$$

We have provided explicit pricing formulae for options on zero-coupon bonds and are now positioned to price caps, floors and swaptions in the following sections.

5.5 Caps and Floors

An interest rate cap provides compensation to the owner whenever the single period interest rate exceeds a specified cap rate over any among a set of consecutive time periods. Therefore for someone with a floating rate liability, owning a cap would in effect limit the interest rate liability to the level of the cap rate.

Similarly an interest rate floor provides compensation to the owner when the single period interest rate falls below a specified floor rate over any among a set of consecutive time periods. Therefore for some with a floating rate asset, owning a floor would in effect guarantee a minimum interest rate equal to the specified floor rate.

We restate the pricing formulae of caps and floors given in (2.10.32) and (2.10.34). For a cap and a floor having a strike rate K and based upon a notional amount N and a set of reset dates $\mathcal{T} = \{T_0, T_1, \ldots, T_n\}$ the formulae for the prices are

$$\begin{aligned}
\mathbf{cap}_{\mathcal{T},\,K,\,N}(t) &= \sum_{\ell=1}^{n} \mathbf{caplet}_{T_{\ell-1},\,T_\ell,\,K,\,N}(t) \\
&= \sum_{\ell=1}^{n} N'_\ell \, \mathbf{zcbput}_{T_{\ell-1},\,T_\ell,\,K'_\ell}(t), \qquad (5.5.1) \\
\mathbf{floor}_{\mathcal{T},\,K,\,N}(t) &= \sum_{\ell=1}^{n} \mathbf{floorlet}_{T_{\ell-1},\,T_\ell,\,K,\,N}(t) \\
&= \sum_{\ell=1}^{n} N'_\ell \, \mathbf{zcbcall}_{T_{\ell-1},\,T_\ell,\,K'_\ell}(t) \qquad (5.5.2)
\end{aligned}$$

where, for $\ell = 1, 2, \ldots, n$,

$$N'_\ell = N\ (1 + K\,(T_\ell - T_{\ell-1})) \quad \text{and} \quad K'_\ell = \frac{1}{1 + K\,(T_\ell - T_{\ell-1})} \qquad (5.5.3)$$

and $0 \le t \le T_0 < \ldots < T_n$.

5.6 Swaptions

Swaptions are interest rate derivatives used to manage interest rate risk. Swaptions can be classified into two types: Payer swaptions and receiver swaptions. Payer swaptions entitle the owner the right to enter into a swap in which he pays fixed and receives floating interest rate payments. Receiver swaptions entitle the owner the right to enter into a swap in which he receives fixed and pays floating interest rate payments.

We restate the following payer swaption and receiver swaption formulae from Corollary 2.10.7.

For a payer swaption with strike rate K, unit notional amount and payment dates $\mathcal{T}$ the pricing formula is

$$\mathbf{payerswaption}_{\mathcal{T},\,K,\,N=1}(t) = p_{T_0,1}(t, P_{\mathcal{T},K}(\cdot)) \qquad (5.6.1)$$

whereas the corresponding receiver swaption pricing formula is

$$\mathbf{receiverswaption}_{\mathcal{T},\,K,\,N=1}(t) = c_{T_0,1}(t, P_{\mathcal{T},K}(\cdot)) \tag{5.6.2}$$

where $P_{\mathcal{T},K}(t)$ is the price of a coupon bond as in (2.10.38), $c_{T_0,1}(t, P_{\mathcal{T},K}(\cdot))$ is the price of a call option as in (2.10.47) and $p_{T_0,1}(t, P_{\mathcal{T},K}(\cdot))$ is the price of a put option as in (2.10.48).

When the discounted GOP obeys Black-Scholes dynamics we can see from Theorem 2.10.5 that the payer swaption and receiver swaption pricing formulae are given in the following two corollaries.

Corollary 5.6.1 *For a deterministic short rate r_t and for a discounted GOP obeying Black-Scholes dynamics the fair prices of a payer swaption and receiver swaption are given by*

$$\begin{aligned}
&\mathbf{payerswaption}_{\mathcal{T},\,K,\,N=1}(t) \\
&= \left(\exp\left\{ -\int_t^{T_0} r_s ds \right\} - \sum_{i=1}^{n} (K\,(T_i - T_{i-1}) + \mathbf{1}_{i=n}) \exp\left\{ -\int_t^{T_i} r_s ds \right\} \right)^+ \\
&\mathbf{receiverswaption}_{\mathcal{T},\,K,\,N=1}(t) \\
&= \left(\sum_{i=1}^{n} (K\,(T_i - T_{i-1}) + \mathbf{1}_{i=n}) \exp\left\{ -\int_t^{T_i} r_s ds \right\} - \exp\left\{ -\int_t^{T_0} r_s ds \right\} \right)^+ .
\end{aligned} \tag{5.6.3}$$

Corollary 5.6.2 *For a short rate r_t obeying Vasicek or CIR dynamics and for a discounted GOP obeying Black-Scholes dynamics the fair prices of a payer swaption and receiver swaption, barring the particular case below, are given by*

$$\begin{aligned}
&\mathbf{payerswaption}_{\mathcal{T},\,K,\,N=1}(t) \\
&= \sum_{i-1}^{n} (K\,(T_i - T_{i-1}) + \mathbf{1}_{i=n}) \mathbf{zcbput}_{T_0,T_i,K_i}(t) \\
&\mathbf{receiverswaption}_{\mathcal{T},\,K,\,N=1}(t) \\
&= \sum_{i-1}^{n} (K\,(T_i - T_{i-1}) + \mathbf{1}_{i=n}) \mathbf{zcbcall}_{T_0,T_i,K_i}(t),
\end{aligned} \tag{5.6.4}$$

where

$$K_i = A(T_0, T_i) \exp(-r^* B(T_0, T_i)) \tag{5.6.5}$$

and where r^* solves the equation

$$\sum_{i=1}^{n} \big((T_i - T_{i-1})K + \mathbf{1}_{i=n}\big) A(T_0, T_i) \exp(-r^* B(T_0, T_i)) = 1. \tag{5.6.6}$$

In the particular case when r_t obeys CIR dynamics and

$$\sum_{i=1}^{n} \big((T_i - T_{i-1})K + \mathbf{1}_{i=n}\big) A(T_0, T_i) < 1 \tag{5.6.7}$$

we have

$$\mathbf{payerswaption}_{\mathcal{T},K,N=1}(t) = P(t,T_0) - \sum_{i-1}^{n}(K\,(T_i - T_{i-1}) + \mathbf{1}_{i=n})P(t,T_i) \tag{5.6.8}$$

$$\mathbf{receiverswaption}_{\mathcal{T},K,N=1}(t) = 0.$$

When the discounted GOP obeys MMM dynamics we can see from Theorem 2.10.5 that the payer swaption and receiver swaption pricing formulae are given in the following two corollaries.

Corollary 5.6.3 *For a deterministic short rate r_t and for a discounted GOP obeying MMM dynamics the fair prices of a payer swaption and receiver swaption are given by*

$$\begin{aligned}
&\mathbf{payerswaption}_{\mathcal{T},K,N=1}(t) \\
&= \sum_{i=1}^{n}(K\,(T_i - T_{i-1}) + \mathbf{1}_{i=n})\Bigg(-\frac{B_t}{B_{T_i}}\big(\chi^2_{0,\lambda}(x) - \exp(-\lambda/2)\big) \\
&+ \frac{B_t}{B_{T_i}}\left(\exp\left(-\frac{\tau_i}{1+2\tau_i}\lambda\right)\chi^2_{0,\lambda/(1+2\tau_i)}((1+2\tau_i)x) - \exp(-\lambda/2)\right) \\
&+ K_i\frac{B_t}{B_{T_0}}\big(\chi^2_{0,\lambda}(x) - \exp(-\lambda/2)\big)\Bigg),
\end{aligned} \tag{5.6.9}$$

$$\begin{aligned}
&\mathbf{receiverswaption}_{\mathcal{T},K,N=1}(t) \\
&= \sum_{i=1}^{n}(K\,(T_i - T_{i-1}) + \mathbf{1}_{i=n})\Bigg(\frac{B_t}{B_{T_i}}\big(1 - \chi^2_{0,\lambda}(x)\big) \\
&- \frac{B_t}{B_{T_i}}\left(\exp\left(-\frac{\tau_i}{1+2\tau_i}\lambda\right)\big(1 - \chi^2_{0,\lambda/(1+2\tau_i)}((1+2\tau_i)x)\big)\right) \\
&- K_i\frac{B_t}{B_{T_0}}\big(1 - \chi^2_{0,\lambda}(x)\big)\Bigg),
\end{aligned}$$

where, for $i = 1, 2, \ldots, n$, K_i is given by

$$K_i = \frac{B_{T_0}}{B_{T_i}}\big(1 - \exp(-x\tau_i)\big). \tag{5.6.10}$$

Here x is the solution to the equation

$$1 = \sum_{i=1}^{n}(K\,(T_i - T_{i-1}) + \mathbf{1}_{i=n})\frac{B_{T_0}}{B_{T_i}}\big(1 - \exp(-x\tau_i)\big) \tag{5.6.11}$$

when $1 < \sum_{i=1}^{n}(K\,(T_i - T_{i-1}) + \mathbf{1}_{i=n})\frac{B_{T_0}}{B_{T_i}}$ and otherwise x is set to ∞.

Corollary 5.6.4 *For a short rate r_t obeying Vasicek or CIR dynamics and for a discounted GOP obeying MMM dynamics the fair prices of a payer swaption and receiver swaption are given by*

$$\begin{aligned}
&\mathbf{payerswaption}_{\mathcal{T},K,N=1}(t) \qquad (5.6.12)\\
&= \int_0^\infty \frac{\lambda}{u} \sum_{i-1}^{n} \left(K\,(T_i - T_{i-1}) + \mathbf{1}_{i=n}\right)\left(1 - \exp(-\tau_i u)\right) V_{t,i}^{(PUT)}(u) f_{\chi^2_{4,\lambda}}(u)\, du\\
&\mathbf{receiverswaption}_{\mathcal{T},K,N=1}(t)\\
&= \int_0^\infty \frac{\lambda}{u} \sum_{i-1}^{n} \left(K\,(T_i - T_{i-1}) + \mathbf{1}_{i=n}\right)\left(1 - \exp(-\tau_i u)\right) V_{t,i}^{(CALL)}(u) f_{\chi^2_{4,\lambda}}(u)\, du,
\end{aligned}$$

where

$$\begin{aligned}
V_{t,i}^{(PUT)}(u) &= \mathrm{E}\left(\frac{B_t}{B_{T_0}}\left(A(T_0,T_i)\exp(-r^*(u)B(T_0,T_i)) - G_{T_i}(T_0)\right)^+ \middle| \mathcal{A}_t\right) \qquad (5.6.13)\\
V_{t,i}^{(CALL)}(u) &= \mathrm{E}\left(\frac{B_t}{B_{T_0}}\left(G_{T_i}(T_0) - A(T_0,T_i)\exp(-r^*(u)B(T_0,T_i))\right)^+ \middle| \mathcal{A}_t\right).
\end{aligned}$$

Here $r^*(u)$ solves the equation

$$\sum_{i=1}^{n} \left((T_i - T_{i-1})K + \mathbf{1}_{i=n}\right)(1 - \exp(-\tau_i u))A(T_0,T_i)\exp(-r^*(u)B(T_0,T_i)) = 1, \qquad (5.6.14)$$

with τ_i, for $i = 1, 2, \ldots, n$, given by

$$\tau_i = \frac{\varphi_{T_0} - \varphi_t}{2(\varphi_{T_i} - \varphi_{T_0})}. \qquad (5.6.15)$$

5.7 Conclusions

This chapter illustrates how the benchmark approach recovers results consistent with the standard risk neutral approach based on geometric Brownian motion and the initial work of Black and Scholes [1973] and Merton [1973]. Also this chapter illustrates how these results can be extended to incorporate leptokurtic stock market behaviour when the discounted GOP obeys a squared Bessel process and interest rates are stochastic.

For example, in Section 5.3 we recovered the Black-Scholes formulae for pricing European options on the GOP and in Sections 5.2 and 5.4 we recovered the standard risk neutral short rate pricing of zero-coupon bonds and bond options for stochastic short rates. This was made possible by the benchmark approach and real-world pricing which generalise the risk neutral framework.

Significantly, we have extended the pricing of options on the GOP, zero-coupon bonds and bond options to richer classes of market models where the discounted GOP obeys a squared Bessel process and the short rate is stochastic. Consequently, caps and swaptions can be consistently priced within these market models which, as will be shown in Chapter 6, exhibit stylised facts such as volatility humps and volatility skews.

The derivatives pricing formulae developed in this chapter permit us to backtest derivatives hedging strategies in Chapters 7, 8 and 9.

Chapter 6

Empirical Stylised Facts of Equity Indices and Interest Rate Term Structures

6.1 Introduction

In this chapter we describe the empirical stylised facts of equity indices[1] and interest rate term structures[2]. In summary we find that most likely equity indices are Student-t distributed, swap rates are Student-t distributed, forward rates are Student-t distributed, the volatility term structure of forward rates is often humped and the dependency structure of interest rates is typically that of a Student-t copula.

6.2 Discounted GOP Log-Returns

Commencing with papers Mandelbrot [1963] and Fama [1963] a vast number of empirical studies on the distributions of log-returns of financial security prices has ensued. Using a wide variety of statistical techniques the majority of authors in the recent literature concludes that the assumption of normality of log-returns of stocks and exchange rates has to be rejected. The most obvious empirical feature that contradicts normality of log-returns is the large excess kurtosis that is observed.

In the two papers Markowitz and Usmen [1996a] and Markowitz and Usmen [1996b], S&P 500 Index log-returns over the twenty year period from 1963 to

[1]These findings have been published in Fergusson and Platen [2006].

[2]These findings have been reported in Fergusson and Platen [2014b] and submitted for publication.

1983 were analysed statistically in a Bayesian framework. Within the family of Pearson distributions, see Stuart and Ord [1994], they identified the Student-t distribution with about 4.5 degrees of freedom as the best fit to observed daily log-returns. The Pearson family includes as special cases the normal, chi-square, gamma, beta, Student-t, uniform, Pareto and exponential distributions.

It was demonstrated in Fergusson and Platen [2006] that log returns L_t of a globally diversified world stock index, for thirty four different currency denominations, over the period 1970 to 2004 are leptokurtic, having a Student-t distribution with approximately four degrees of freedom. The study was conducted within the class of symmetric generalised hyperbolic (SGH) models as specified by the probability density function

$$f_L(x;\mu,\delta,\bar{\alpha},\lambda) = \frac{1}{\delta K_\lambda(\bar{\alpha})}\sqrt{\frac{\bar{\alpha}}{2\pi}}\left(1+\frac{(x-\mu)^2}{\delta^2}\right)^{\frac{1}{2}(\lambda-\frac{1}{2})} K_{\lambda-\frac{1}{2}}\left(\bar{\alpha}\sqrt{1+\frac{(x-\mu)^2}{\delta^2}}\right) \tag{6.2.1}$$

for $x \in (-\infty,\infty)$, location parameter $\mu \in (-\infty,\infty)$ and two shape parameters $\lambda \in (-\infty,\infty)$ and $\bar{\alpha} = \alpha\delta \in [0,\infty)$, defined so that they are invariant under scale transformations. The scale parameter is $\delta \in [0,\infty)$. The parameters α and δ are such that $\bar{\alpha} = \alpha\delta$ with $\alpha, \delta \in [0,\infty)$ and

$$\delta > 0,\ \alpha \geq 0, \qquad \text{if } \lambda < 0, \tag{6.2.2}$$
$$\delta > 0,\ \alpha > 0, \qquad \text{if } \lambda = 0, \tag{6.2.3}$$
$$\delta \geq 0,\ \alpha > 0, \qquad \text{if } \lambda > 0. \tag{6.2.4}$$

Also

$$K_\lambda(\omega) = \frac{1}{2}\int_0^\infty u^{\lambda-1}\exp(-\frac{1}{2}\omega(u+u^{-1}))du \tag{6.2.5}$$

is the modified Bessel function of the third kind with index λ. In the particular instance where $\lambda = n+\frac{1}{2}$ is a positive half-integer we have the closed formula

$$K_{n+\frac{1}{2}}(\omega) = \sqrt{\frac{\pi}{2\omega}}\exp(-\omega)\left\{1+\sum_{i=1}^{n}\frac{(n+i)!}{(n-i)!i!}(2\omega)^{-i}\right\}. \tag{6.2.6}$$

The mean of the SGH distribution is μ and the variance is

$$\sigma^2 = \delta^2 K_{\lambda+1}(\bar{\alpha})/(\bar{\alpha}K_\lambda(\bar{\alpha})). \tag{6.2.7}$$

.

Special cases of the SGH$(\mu,\sigma,\bar{\alpha},\lambda)$ distribution include the normal inverse Gaussian (NIG) with $\lambda = -1/2$, the hyperbolic distribution with $\lambda = 1$, the Student-t distribution with $\nu = -2\lambda$ degrees of freedom when $\bar{\alpha} = 0$ and $\delta > 0$ and the Variance Gamma distribution with $\delta = 0$, $\alpha > 0$ and $\lambda > 0$.

The Generalised Inverse Gaussian distribution, denoted by $\mathrm{GIG}(\alpha, \delta, \lambda)$, has density function

$$f_{\Delta T}(u; \alpha, \delta, \lambda) = \frac{(\alpha/\delta)^{\lambda}}{2K_{\lambda}(\alpha\delta)} u^{\lambda-1} \exp\left(-\frac{1}{2}\left(\frac{\delta^2}{u} + \alpha^2 u\right)\right), \; u > 0 \tag{6.2.8}$$

with parameter domain given by

$$\delta > 0, \; \alpha \geq 0, \qquad \text{if } \lambda < 0, \tag{6.2.9}$$
$$\delta > 0, \; \alpha > 0, \qquad \text{if } \lambda = 0, \tag{6.2.10}$$
$$\delta \geq 0, \; \alpha > 0, \qquad \text{if } \lambda > 0. \tag{6.2.11}$$

The SGH distribution is formed by mixing the GIG distribution with the normal distribution by writing

$$L_t = \mu + \Delta W_t \tag{6.2.12}$$

where $\Delta T = T(t + \delta t) - T(t) \sim GIG(\alpha, \delta, \lambda)$. Note that ΔT has mean given by

$$E(\Delta T) = \begin{cases} \frac{\delta^2 K_{\lambda+1}(\alpha\delta)}{\alpha\delta K_{\lambda}(\alpha\delta)} & \text{if } \bar{\alpha} > 0 \\ \frac{2\lambda}{\alpha^2} & \text{if } \bar{\alpha} = 0 \end{cases}. \tag{6.2.13}$$

Also, because the conditional distribution of ΔW_t is normal, i.e. $\Delta W_t \big| \Delta T \sim N(0, \Delta T)$, we can express the cumulative distribution function of L_t as

$$F_L(x) = \int_0^{\infty} N\left(\frac{x-\mu}{u}\right) f_{\Delta T}(u) du \tag{6.2.14}$$

and the probability density function as

$$f_L(x) = \int_0^{\infty} \frac{1}{u} n\left(\frac{x-\mu}{u}\right) f_{\Delta T}(u) du. \tag{6.2.15}$$

Employing the maximum likelihood estimation algorithm in Appendix M we estimated the SGH parameters of the daily log returns of the discounted S&P 500 Index over the period 1971 to 2010. As shown in Figure 6.1 the set of MLE parameters $\bar{\alpha} = 0$ and $\lambda = -1.8$ corresponds to the Student-t distribution with 3.6 degrees of freedom.

We note that for a squared Bessel process X of dimension ν described by the SDE

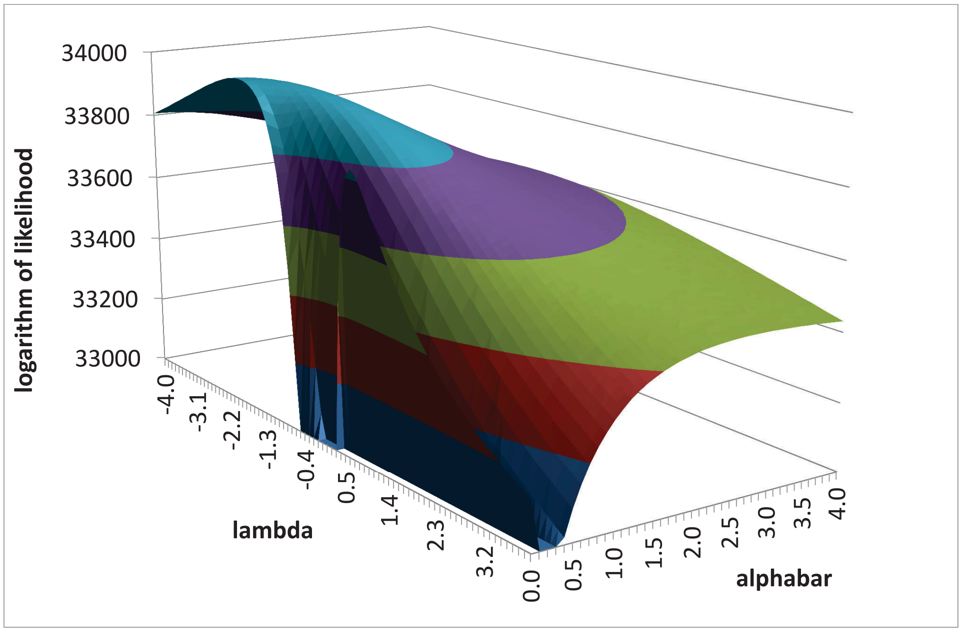
$$dX_t = \nu dt + 2\sqrt{X_t} dW_t \tag{6.2.16}$$

the log returns satisfy the SDE

$$d \log X_t = \frac{\nu - 2}{X_t} dt + \frac{2}{\sqrt{X_t}} dW_t \tag{6.2.17}$$

and it is apparent that the distribution of log returns is a mixture of a normal distribution with a variance that is inverse gamma distributed, yielding the

Figure 6.1: Contour plot of log likelihood function for daily log returns of the discounted S&P 500 Index over the period 1971 to 2010.



Student-t distribution with four degrees of freedom. This is precisely the SDE of the normalised discounted GOP, $Y_t = \bar{S}_t^{\delta_*}/\bar{\alpha}_t$, prescribed in the MMM, that is

$$dY_t = (1 - \eta Y_t)dt + \sqrt{Y_t}dW_t. \tag{6.2.18}$$

It has the stationary density function

$$\bar{p}(y) = \frac{(2\eta)^2}{\Gamma(2)} y \exp(-2\eta y), \tag{6.2.19}$$

which is the density function of a scaled chi-squared distribution with four degrees of freedom, scaled by the factor 4η. Also, the squared volatility V_t of the normalised discounted GOP is equal to its inverse, that is

$$V_t = \left(\frac{\sqrt{Y_t}}{Y_t}\right)^2 = \frac{1}{Y_t}, \tag{6.2.20}$$

and so Y_t has squared volatility or variance which is inverse gamma distributed with four degrees of freedom. Therefore, the Student-t distribution of the log returns of the discounted equity index is captured by the MMM as a model for the discounted GOP.

6.3 The Yield Curve

The yield curve shows the relationship between the interest rate or borrowing cost and the time to maturity of the debt for a given borrower in a given currency. Yield curves are usually upward sloping asymptotically, whereby the longer the maturity, the higher the yield, with diminishing marginal growth. In this situation the corresponding forward rate curve is decreasing. According to the standard liquidity premium theory of interest rates, this stylised upward sloping feature is explained by risk averse investors having higher demand for short-term debt instruments than long-term debt instruments, see for example Chapter 6 of Mishkin [2010]. The benchmark approach, see Platen [2002b], Platen [2006b] and Platen and Heath [2010], allows us to capture this stylised feature by incorporating the discounted stock index into the pricing of zero-coupon bonds (ZCBs) which affects forward rates. In this chapter we examine, under a general modelling framework, the interest rate term structure induced by various market models each of which is driven by two independent factors corresponding to the short rate and the discounted stock index.

Interest rate term structure models have sought to capture various stylised empirical properties such as, for example, mean reversion of interest rates, attaining normal yield curve and inverted yield curve shapes, humped volatility term structures of forward rates, imperfect correlations between adjacent forward rates, leptokurtic distributions for both forward and swap rates and tail dependencies between swap rates, see, for example, Rebonato [1998, 1999] and Brigo and Mercurio [2001].

Single-factor short rate models such as those of Vasicek [1977], Dothan [1978], Rendleman and Bartter [1980], Cox et al. [1985], Black and Karasinski [1991], Ahn and Gao [1999] are a starting point for modelling the term structure of interest rates. However, these only allow for parallel shifts in the yield curve and therefore attain a limited range of yield curve shapes. A related shortcoming of single-factor models is their inability to consistently price both caps and swaptions caused by the single-factor imposing perfect correlations between adjacent forward rates and, thereby, overpricing swaptions when model volatility parameters are fitted to observed cap prices. Several two-factor short rate models, such as those of Brennan and Schwartz [1979], Longstaff and Schwartz [1992] and Hull and White [1994], and three-factor models, such as that of Chen [1996], seek to improve the flexibility of attainable yield curve shapes and forward rate correlations and yet still fall short, but not by as much as single-factor models, in consistently pricing both caps and swaptions, as shown by Rebonato and Cooper [1994], for example.

Within the framework of Heath et al. [1990], the Libor Market Model of Brace et al. [1997] overcame several of these aforementioned deficiencies, modelling the forward rates as being lognormally distributed. Analogously, the Swap Market Model of Jamshidian [1997], modelling the forward swap rates as lognormally distributed, likewise overcame some of these deficiencies. However, these multifactor models still fail to capture the skew and smile features of caplets and floorlets. This led to the idea of the SABR Libor Market Model of Rebonato [2007], employing a stochastic volatility model of the forward rates.

The preceding approaches have priced derivatives under the assumption of the existence of an equivalent risk neutral probability measure. By removing this restrictive assumption the benchmark approach of Platen [2002b], described also by Platen and Heath [2006], involves calculating prices of interest rate derivatives as expected benchmarked payoffs under the real-world probability measure where the benchmark portfolio employed is the numéraire portfolio which is the growth optimal portfolio (GOP). The GOP is well approximated by a diversified stock index, see Platen and Heath [2010]. We emphasise, since under the benchmark approach there is no requirement that an equivalent risk neutral probability measure exists, we work in a much wider modelling world than that accessible under the classical risk neutral approach.

The interest rate term structure under the benchmark approach was first analysed in Platen [2002b], where he showed that the drift term of the SDE for the forward rate under the real-world probability measure depends on volatilities of forward rates, analogous to those discussed in Heath et al. [1992] under the risk neutral measure.

The first model for the interest rate term structure under the benchmark approach was provided in Platen [2005a] for which he showed that the discounted GOP contribution to the forward rate converges asymptotically to the net market growth rate. Also it was noted that medium and long-term maturities of the yield curve were impacted mainly by the discounted GOP dynamics, whereas the short-term maturities were impacted mainly by the short rate, thereby providing a natural segmentation of the yield curve. Intuitively, this is what is profferred by the liquidity premium theory of interest rates, namely that long-term bond yields incorporate a premium to compensate investors for the fluctuation in long-term bond prices, whereas short-term bond yields are rather close to the short rate.

Viewing the yield curve as being tethered to the short rate r_t at short-term maturities and to the asymptotic equilibrium forward rate at long-term maturities, the rates at intermediate maturities can vary according to supply and demand.

Also we would expect the forward rates, as well as the variance of the forward rates, to be hump-shaped, which Platen [2005a] demonstrates and which are supported by empirical research of, for example, Bouchaud et al. [1999], Bouchaud and Matacz [2000] and Brown and Schaefer [2000].

A theorem of Dybvig et al. [1996] states that if long-term continuously compounded rates are finite then they can never fall. Therefore, if one is using the long-term continuously compounded rate as a factor of the model then only under restricted dynamics of the long-term rate can a viable model result. This is borne out by many models having constant long-term forward rates, including those models considered in this chapter. More recently, Brody and Hughston [2013] developed a class of models for the purposes of discounting cash flows from long-term projects carried out for the benefit of society, where the state variable is the long-term simple rate and is assumed to be finite. A consequence of this assumption is that the long-term continuously compounded rate is zero, which differs to the resulting long-term continuously compounded rates of models considered henceforth.

Aside from deriving long-term forward rates under various market models, the current section illustrates the ability of the benchmark approach to explain the empirical level of long-term bond yields when the market model has the discounted GOP modelled by the minimal market model (MMM) of Platen and Heath [2006]. The market models we consider are described in Section 5.1. In Section 6.4 we describe the data employed in our empirical analysis and the methodology of fitting the models. In Section 6.5 we obtain estimates of long-term yields and forward rates. In Section 6.6 we describe the evolution of yield curves. In Section 6.7 we investigate the volatility of forward rates. In Section 6.8 we demonstrate the ability of the MMM in explaining long-term swap rates and examine the volatility of swap rates. In Section 6.9 we illustrate the leptokurtic distribution of swap rates. In Section 6.10 we examine the historical correlation of swap rates. In Section 6.11 we examine the dependency structure of swap rates.

6.4 Market Data and Fitting the Models

The US data set used for our empirical analysis is the annual series of US one-year deposit rates, ten-year treasury bond yields and S&P Composite Stock Index from 1871 to 2012, shown in Chapter 26 of Shiller [1989] and subsequently updated on http://aida.wss.yale.edu/ shiller/data/chapt26.xls (see Data Set A in Section L.1 of Appendix L). The 141 year length of this data series makes it a useful series for analysing the interest rate term structure and index dynamics because many different economic conditions have been experienced over that time. Also, because in this data set there are ten-year bond yields accompanying the one-year deposit rates and stock index values, we are able to compare actual with theoretical ten-year bond yields. The maximum likelihood estimates of the parameters for all models fitted to US data are shown in Table 6.1, where it is evident that the CIR model is the best fitting short rate model and the MMM is the best fitting discounted GOP model.

In Section 6.9 and subsequent sections the analyses of volatility and correlation demanded the use of daily and monthly data and therefore swap rate data and treasury bond data from the Federal Reserve Bank were used.

Table 6.1: Maximum likelihood estimates of model parameters fitted to US data 1871-2012 (see Data Set A in Section L.1 of Appendix L).

Model	Parameters	Standard Errors	Log Likelihood
Vasicek	$\bar{r} = 0.042994$	0.0080023	399.7019
	$\kappa = 0.162953$	0.053703	
	$\sigma = 0.015384$	0.00099592	
CIR	$\bar{r} = 0.041078$	0.011421	427.8116
	$\kappa = 0.092540$	0.038668	
	$\sigma = 0.064670$	0.0040761	
3/2	$p = 0.038506$	0.042284	406.2713
	$q = 0.877908$	1.177853	
	$\sigma = 2.0681$	0.13241	
Black-Scholes	$\theta = 0.177297$	0.059087	-267.4135
MMM	$\bar{\alpha}_0 = 0.010028$	0.0023389	-264.6433
	$\eta = 0.045486$	0.000800	

6.5 Asymptotic Long-Term Yields and Forward Rates

In this section we derive formulae for the asymptotic long-term yield and the asymptotic long-term forward rate of the yield curve under our market models. As in Section 2.8, denote by $y_T(t)$ the continuously compounded yield to maturity of a T-maturity ZCB as at time t, the formula of which is

$$y_T(t) = -\frac{1}{T-t} \log P(t,T). \tag{6.5.1}$$

From (2.8.7) we have

$$y_\infty(t) = h_\infty(t) + n_\infty(t). \tag{6.5.2}$$

The asymptotic long-term instantaneous forward rate $f_\infty(t)$ as at time t is computed using the formula

$$f_\infty(t) = \lim_{T\to\infty} -\frac{\partial}{\partial T} \log P(t,T), \tag{6.5.3}$$

and from (2.8.11) we have

$$f_\infty(t) = g_\infty(t) + m_\infty(t). \tag{6.5.4}$$

When r_t obeys the Vasicek short rate model one obtains from Corollary 3.2.15 and Corollary 3.2.17

$$g_\infty(t) = h_\infty(t) = \bar{r} - \frac{\sigma^2}{2\kappa^2}. \tag{6.5.5}$$

Table 6.2: Contributions by the short rate and the discounted GOP to the asymptotic equilibrium forward rate.

Model	Contribution to $f_\infty(t)$	Value
Vasicek	$\bar{r} - \frac{\sigma^2}{2\kappa^2}$	0.0385
CIR	$\frac{\kappa\bar{r}}{\sigma^2}(\sqrt{\kappa^2+2\sigma^2}-\kappa)$	0.03415
3/2	$\alpha_1 p$	0.01732
Black-Scholes	0	0
MMM	η	0.045486

When r_t obeys the CIR short rate model one obtains from Corollary 3.3.21 and Corollary 3.3.23

$$g_\infty(t) = h_\infty(t) = \frac{\kappa\bar{r}}{\sigma^2}(\sqrt{\kappa^2+2\sigma^2}-\kappa). \tag{6.5.6}$$

When r_t obeys the 3/2 short rate model one obtains from Corollary 3.4.6 and Corollary 3.4.8

$$g_\infty(t) = h_\infty(t) = \alpha_1 p, \tag{6.5.7}$$

where α_1 is defined in (3.4.23). Also, when $\bar{S}_t^{\delta_*}$ obeys the Black-Scholes dynamics one obtains from Corollary 4.2.6 and Lemma 4.2.7

$$m_\infty(t) = n_\infty(t) = 0 \tag{6.5.8}$$

and when $\bar{S}_t^{\delta_*}$ obeys the MMM dynamics one obtains from Corollary 4.3.14 and Corollary 4.3.16

$$m_\infty(t) = n_\infty(t) = \eta. \tag{6.5.9}$$

We see that under the MMM, the long-term equilibrium forward rate is, as shown in Platen [2005a], the sum of the short rate contribution and the discounted GOP contribution, as in (6.5.4). So from the parameter estimates given in Table 6.1 and the contributions to $f_\infty(t)$ in Table 6.2 we have the asymptotic long-term forward rates given in Table 6.3. This gives a highly sought answer to the question of what the long-term bond yield is, as asked by actuaries and accountants when performing market based valuations of long dated liabilities, see for example Mulquiney and Miller [2014] and Hibbert [2012].

6.6 Evolution of Yield Curves under the Models

The rates at intermediate maturities, as given by a market model, are determined by the mean reverting level of the short rate model, shown in Table 6.4. These levels correspond to the values of $\bar{r}$ given in Table 6.1 when the short rate obeys a Vasicek or CIR process and to the value of $\frac{4p}{2\sigma^2-4q}$, as given in (3.4.16), when the short rate obeys a 3/2 model.

Table 6.3: Asymptotic equilibrium forward rates for various models.

Model	$f_\infty(t)$
Vasicek SR & BS DGOP	0.0385
CIR & BS DGOP	0.03415
3/2 & BS DGOP	0.01732
Vasicek SR & MMM DGOP	0.085341
CIR & MMM DGOP	0.080991
3/2 & MMM DGOP	0.064161

Table 6.4: Mean reverting levels of the short rate.

Model	$\bar{r}$
Vasicek	0.042994
CIR	0.041078
3/2	0.03054

We illustrate the possible shapes of the yield curve as at the times 1871, 1930 and 1970 in Figures 6.2, 6.3 and 6.4 respectively.

In 1871 inverted (downward sloping) yield curves arise from some of the models. This is because the short rate r_{1871} is over 6% while the asymptotic equilibrium forward rate is less than 4% for market models involving the Black-Scholes discounted GOP.

In 1930 normal (upward sloping) yield curves arise from the models with an MMM discounted GOP, whereas near flat yield curves arise for models having a Black-Scholes discounted GOP. The 3/2 short rate model induces an inverted yield curve because its contribution to the asymptotic forward rate is small, being less than the level of short-term rates.

In 1970 inverted (downward sloping) yield curves arise from the models.

Clearly, flat yield curves are not attainable under any of the market models considered here because of the uncertainty or volatility of the short rate.

6.7 Forward Rate Volatilities under the Models

To illustrate the distributional properties of forward rates under the considered models the realised volatilities of three-month forward rates implied by each model are shown in Table 6.5. It is clear that the hump-shaped feature of forward rates is exhibited under models having a 3/2 short rate model whereas the other short rate models do not exhibit this stylised fact.

Figure 6.2: Zero-coupon yield curves for several models at 1871.

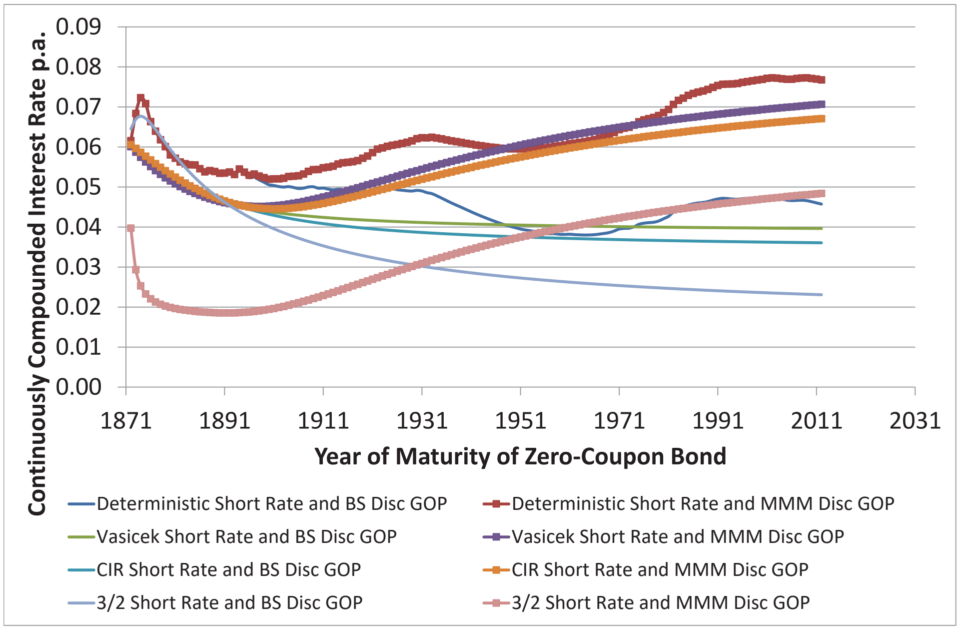


Table 6.5: Realised volatilities of model-implied three-month forward rates for various start times.

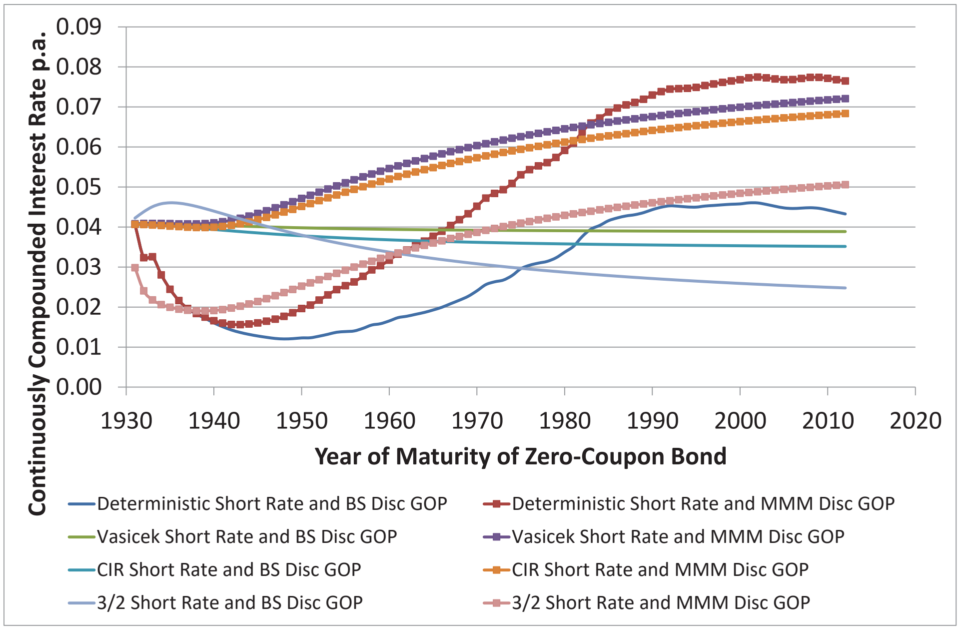
Short Rate	Disc GOP	3M	6M	9M	1Y	15M	18M	21M	2Y
Deterministic	BS	0.3152	0.3152	0.3152	0.3147	0.3147	0.3147	0.3147	0.3146
Deterministic	MMM	0.3152	0.3152	0.3152	0.3147	0.3147	0.3146	0.3146	0.3143
Vasicek	BS	0.2865	0.2712	0.2579	0.2459	0.2350	0.2249	0.2156	0.2069
Vasicek	MMM	0.2865	0.2712	0.2579	0.2459	0.2350	0.2249	0.2156	0.2068
CIR	BS	0.2984	0.2887	0.2798	0.2716	0.2639	0.2566	0.2498	0.2432
CIR	MMM	0.2984	0.2887	0.2798	0.2716	0.2639	0.2566	0.2497	0.2431
3/2	BS	0.3194	0.3217	0.3229	0.3225	0.3205	0.3170	0.3124	0.3070
3/2	MMM	0.3194	0.3217	0.3229	0.3225	0.3205	0.3170	0.3123	0.3068

To appreciate how the yield movements are affected by the short rate, recall from (6.5.1) that the T-maturity yield $y_T(t)$ is computed as

$$
\begin{aligned}
y_T(t) &= -\frac{1}{T-t}\log P(t,T) \qquad (6.7.1)\\
&= -\frac{1}{T-t}\log A(t,T) + \frac{B(t,T)}{T-t}r_t - \frac{1}{T-t}\log(1-\exp(\frac{1}{2}\bar{S}_t^{\delta_*}/(\varphi_T-\varphi_t))),
\end{aligned}
$$

which is an affine transformation of the short rate r_t. So if the daily movement of the short rate r_t is normally distributed, then so will the daily movement of the T-maturity yield. Further, if the daily movement of the short rate is Student-t distributed with ν degrees of freedom, then so will the daily movement of the T-maturity yield. Additionally, if the daily movement of the logarithm of the short rate is Student-t distributed with ν degrees of freedom, then so will the daily movement of the logarithm of the T-maturity yield. Of the considered models,

Figure 6.3: Zero-coupon yield curves for several models at 1930.



the 3/2-model, whose solution is the reciprocal of a squared-Bessel process, is such that the movement in the logarithm of the short rate has a diffusion term, which is nearly square-root inverse Gamma distributed and, therefore, mixing this diffusion term with a Wiener process gives rise to an estimated distribution which is nearly Student-t.

6.8 Ten-Year Swap Rates under the Models

Using the parameter estimates in Table 6.1 we graphically illustrate the evolution of the 10-year semi-annual swap rate under each market model in Figure 6.5 and compare with the empirical 10-year swap rates of the chosen data set.

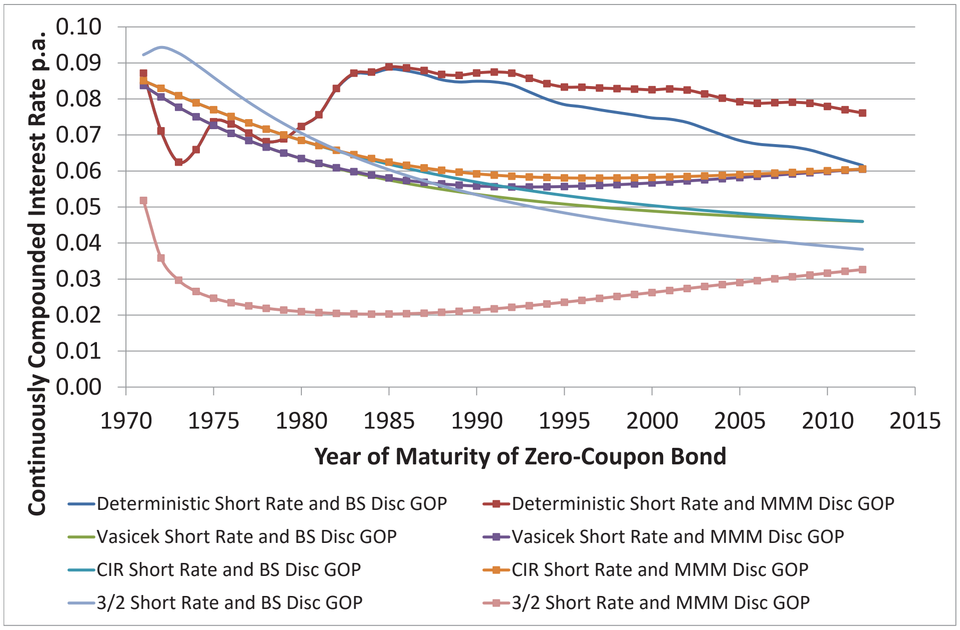
The swap rate is calculated using the swap rate formula

$$\mathbf{swap}_{T_0,T_1,\ldots,T_n}(t) = \frac{P(t,T_0) - P(t,T_n)}{\sum_{i=1}^{n}(T_i - T_{i-1})P(t,T_i)}, \tag{6.8.1}$$

where $P(t,T)$ is calculated from the relevant ZCB pricing formula given in Section 5.2.

The bias and root-mean-squared (RMS) error of each model-implied ten-year swap rate are shown in Table 6.6. It is evident that the lowest RMS market

Figure 6.4: Zero-coupon yield curves for several models at 1970.



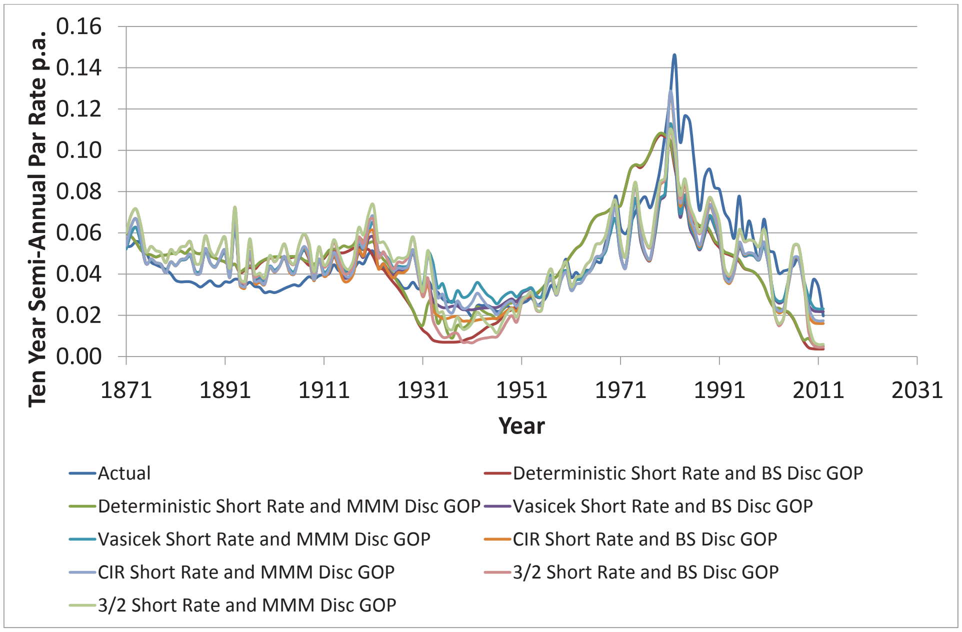
model is the one composed of the CIR short rate and MMM discounted GOP and a comparison is shown in Figure 6.6. In Section 6.4 the CIR short rate model was identified as having the best fit among candidate short rate models and the MMM discounted GOP was identified as having the best fit among the candidate discounted GOP models. In Table 6.6 we see that the best fitting component models combine to produce the best fitting model of swap rates. Therefore, the fair pricing of ZCBs appears to be vindicated by the results in Table 6.6.

We also calculated the model-implied realised volatilities of swap rates over the period 1871 to 2012 of the chosen data set using each model's fitted parameters, the results of which are shown in Table 6.7. None of the models exhibits a volatility hump, which appears as a stylised fact in swaption markets. In forthcoming work we will analyse a generalisation of the MMM, which has the potential to explain volatility humps.

6.9 Leptokurtic Distribution of Swap Rates

Within the class of symmetric generalised hyperbolic distributions we estimated the maximum likelihood parameters pertaining to the daily change in one, two, three, four, five, seven, ten and thirty year swap rates using data over the pe-

Figure 6.5: Comparison of actual ten-year swap rates with those implied by models.



riod from March 2001 to January 2013 obtained from the Federal Reserve Bank website. These swap rates are the mid-market par swap rates given by the International Swaps and Derivatives Association (ISDA). Rates are for a Fixed Rate Payer in return for receiving three month LIBOR, and are based on rates collected at 11:00 a.m. Eastern time by Garban Intercapital plc and published on Reuters Page ISDAFIX1.

The historical series of swap rates is graphically shown in Figure 6.7.

The summary statistics of the daily changes in swap rates is shown in Table 6.8.

We observe that the standard deviations of swap rates increases up to the five-year term to maturity, then decreases. This volatility hump of swap rates is a stylised feature.

The parameter estimates are shown in Table 6.9. It is evident that daily moves in swap rates are most likely distributed as a Student-t distribution with degrees of freedom ν ranging from 3.2 at the one-year term to 6 at the thirty-year term to maturity.

Table 6.6: Comparison of model with actual ten-year swap rates.

Short Rate Model	Discounted GOP Model	Bias	RMS Error
Deterministic	BS	-0.2099	1.7145
Deterministic	MMM	-0.0348	1.6275
Vasicek	BS	-0.2600	1.3370
Vasicek	MMM	-0.0904	1.3433
CIR	BS	-0.2979	1.2788
CIR	MMM	-0.1267	**1.2522**
3/2	BS	-0.0829	1.4833
3/2	MMM	0.0911	1.4270

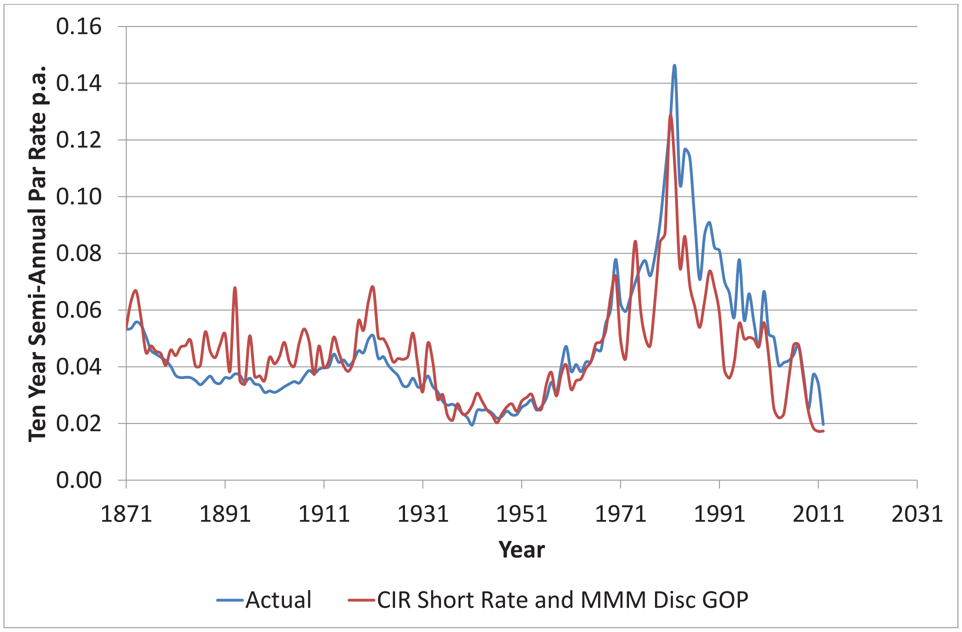
Table 6.7: Model-implied realised volatilities of swap rates.

Short Rate	Discounted GOP	1Y	2Y	3Y	4Y	5Y	7Y	10Y	30Y
Deterministic	BS	0.3184	0.2402	0.1921	0.1631	0.1416	0.1191	0.1040	0.0626
Deterministic	MMM	0.3184	0.2402	0.1917	0.1618	0.1412	0.1301	0.1315	0.0793
Vasicek	BS	0.2822	0.2567	0.2366	0.2202	0.2063	0.1840	0.1596	0.0974
Vasicek	MMM	0.2822	0.2567	0.2365	0.2198	0.2061	0.1853	0.1617	0.0888
CIR	BS	0.2968	0.2795	0.2650	0.2524	0.2413	0.2225	0.2004	0.1342
CIR	MMM	0.2968	0.2795	0.2648	0.2519	0.2409	0.2233	0.2007	0.1142
3/2	BS	0.3235	0.3223	0.3151	0.3057	0.2958	0.2774	0.2546	0.1790
3/2	MMM	0.3235	0.3223	0.3149	0.3048	0.2946	0.2756	0.2457	0.1274

Table 6.8: Summary statistics of daily moves in US swap rates.

Tenor	Min	Max	Mean	Std Dev	Skew	Excess Kurtosis
1Y	-0.41%	0.32%	-0.0025%	0.0479%	-0.7502	8.4153
2Y	-0.40%	0.34%	-0.0023%	0.0630%	-0.0987	3.7128
3Y	-0.35%	0.38%	-0.0021%	0.0678%	0.0436	2.8718
4Y	-0.40%	0.37%	-0.0019%	0.0700%	0.0528	2.6665
5Y	-0.44%	0.37%	-0.0018%	0.0722%	0.0477	2.6438
7Y	-0.48%	0.35%	-0.0015%	0.0707%	-0.0690	2.8103
10Y	-0.50%	0.32%	-0.0013%	0.0693%	-0.1506	3.2234
30Y	-0.49%	0.25%	-0.0011%	0.0617%	-0.1689	3.4677

Figure 6.6: Comparison of actual ten-year swap rates with those under CIR short rate and MMM discounted GOP.



6.10 Correlation of Swap Rates

The graph of historical US swap rates in Figure 6.7 illustrates the comovements in swap rates over the period from July 2000 to June 2014. Several features of the graphs are worthy of articulation. The dot-com bubble had burst in March 2000 and in January 2001 America Online merged with Time Warner, at the time the second biggest M&A deal in history. The period from 2006 to 2007 saw the threat of inflation with high oil and metal prices and this is mirrored in the figure by the consonant high levels of swap rates. Also we see that in September 2008 swap rates fall in concert and subsequent to the Lehman Brothers collapse there is dispersion of swap rates. Finally, in August 2010 we see a concerted fall in swap rates in response to the commencement of the second program of quantitative easing by the US Federal Reserve Bank.

Also, under our one factor short rate models, combined with the Black-Scholes model of the discounted GOP, the daily movements in yields of different maturities are fully correlated. This is not supported by the empirical stylised behaviour of yield correlations, typically decreasing with the difference in term to maturity. The correlation surface is shown in Figure 6.8.

To explicitly illustrate the relationships between yields under the considered mod-

Table 6.9: MLEs of SGH parameters of daily moves in US swap rates.

Tenor	$\bar{\alpha}$	$\nu = -\lambda/2$	Log Likelihood
1Y	0.0	3.2	16958
2Y	0.0	4.0	16037
3Y	0.0	5.0	15798
4Y	0.0	5.0	15697
5Y	0.0	5.0	15606
7Y	0.0	5.2	15657
10Y	0.0	5.4	15718
30Y	0.0	6.0	16028

Table 6.10: Correlation coefficients between one-year and ten-year swap rates under the market models.

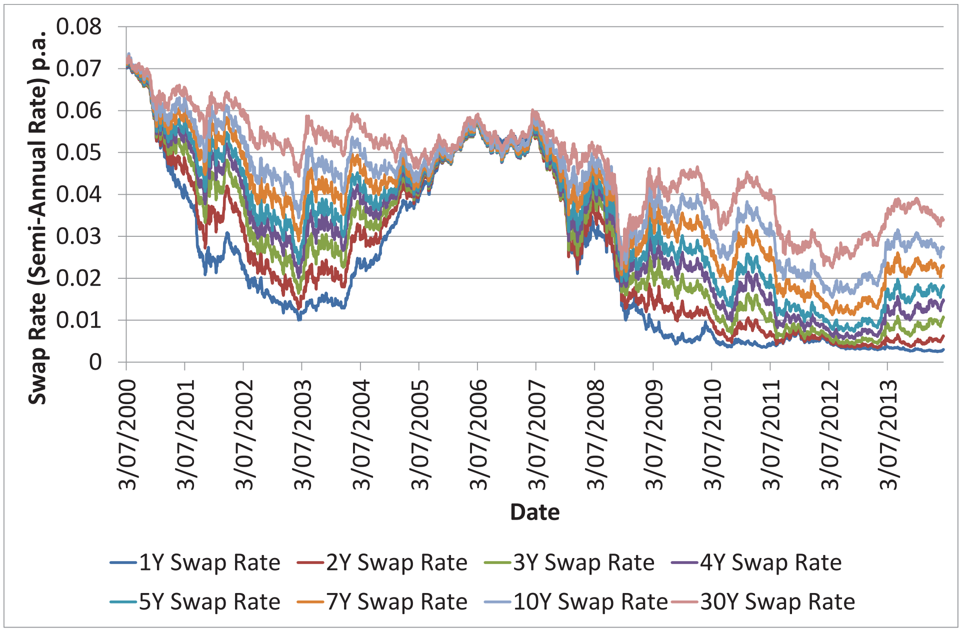
Short Rate	Disc GOP	1Y-10Y Correlation
Determinstic	BS	0.43605
Determinstic	MMM	0.35886
Vasicek	BS	0.9994
Vasicek	MMM	0.97355
CIR	BS	0.99955
CIR	MMM	0.98218
3/2	BS	0.96275
3/2	MMM	0.94643

els the correlation coefficients of annual movements in one-year and ten-year swap rates are shown in Table 6.10, where it is evident that the Black-Scholes discounted GOP ensures nearly full correlation of one and ten-year swap rates for the models having non-deterministic short rate models. The correlation between actual one-year and ten-year swap rates is 0.83133, which is lower than any correlation coefficient given in Table 6.10, aside from models involving deterministic short rates. This could potentially indicate that the stochastic interest rates are not fully independent.

6.11 Dependence Structure of Swap Rates

We examine the dependence structure of annual changes in swap rates implied by models and the dependence structure of monthly changes in actual swap rates.

Figure 6.7: Historical US swap rates (July 2000 to June 2014).



6.11.1 Annual Changes in Swap Rates Implied by Models

Tables 6.12 and 6.13 show the fitted parameters in respect of a variety of dependence structures for each of the market models. The Student-t copula has the best fit for each market model, except for the deterministic short rate and discounted GOP model where the Clayton copula has the best fit.

6.11.2 Changes in Actual Swap Rates

Various copulas are fitted to annual changes in actual 1Y, 5Y and 10Y treasury bond (par) rates in Table 6.11. Here the Gaussian copula gives the best fit according to the AIC.

Monthly swap rate data over the period from 2000 to 2014, sourced from the Federal Reserve Bank, was examined. Copulas fitted to the monthly changes in rates are shown in Table 6.14 where the Student-t copula gives the lowest AIC.

Furthermore, monthly treasury bond yield data over the period from 1962 to 2014, sourced from the Federal Reserve Bank, was examined. Copulas fitted to the monthly changes in yields are shown in Table 6.15 where it is evident that the Student-t copula has the lowest AIC.

Table 6.11: Trivariate copulas fitted to annual changes in actual 1Y, 5Y and 10Y par rates.

Copula	MLEs
Clayton	$\theta = 3.229$
	$loglkhd = 71.8772$
	$AIC = -141.7543$
Frank	$\theta = 9.8112$
	$loglkhd = 44.1719$
	$AIC = -86.3437$
Gumbel	$\theta = 2.891$
	$loglkhd = 71.592$
	$AIC = -141.184$
Gauss	$\theta_1 = 0.497$
	$\theta_2 = 0.1679$
	$\theta_3 = 0.6229$
	$loglkhd = 123.5103$
	$AIC = -241.0205$
Student's t	$\nu = 999$
	$\theta_1 = 0.4971$
	$\theta_2 = 0.168$
	$\theta_3 = 0.6229$
	$loglkhd = 123.4975$
	$AIC = -238.9951$

Table 6.12: Trivariate copulas fitted to annual changes in model 1Y, 5Y and 10Y par rates.

Copula	DSR BSDGOP	DSR MMMDGOP	VasicekSR BSDGOP	VasicekSR MMMDGOP
Clayton	$\theta = 1.0986$	$\theta = 0.8542$	$\theta = 47$	$\theta = 12.0243$
	$loglkhd = 65.3856$	$loglkhd = 48.3505$	$loglkhd = 854.4114$	$loglkhd = 454.0228$
	$AIC = -128.7712$	$AIC = -94.701$	$AIC = -1706.8228$	$AIC = -906.0455$
Frank	$\theta = 1.6557$	$\theta = 1.0091$	$\theta = 37$	$\theta = 37$
	$loglkhd = 5.6146$	$loglkhd = 2.5667$	$loglkhd = 570.7317$	$loglkhd = 390.8922$
	$AIC = -9.2292$	$AIC = -3.1334$	$AIC = -1139.4634$	$AIC = -779.7843$
Gumbel	$\theta = 1.5146$	$\theta = 1.4312$	$\theta = 47$	$\theta = 10.4893$
	$loglkhd = 45.1896$	$loglkhd = 35.7616$	$loglkhd = 1026.308$	$loglkhd = 537.4991$
	$AIC = -88.3791$	$AIC = -69.5232$	$AIC = -2050.616$	$AIC = -1072.9982$
Gaussian	$\theta_1 = 1.1393$	$\theta_1 = 1.1377$	$\theta_1 = 0.0076$	$\theta_1 = 0.0471$
	$\theta_2 = 0.7167$	$\theta_2 = 0.8578$	$\theta_2 = 0.0401$	$\theta_2 = 0.1365$
	$\theta_3 = 0.9738$	$\theta_3 = 0.9779$	$\theta_3 = 0.0082$	$\theta_3 = 0.1997$
	$loglkhd = 67.7544$	$loglkhd = 48.6699$	$loglkhd = 1124.5854$	$loglkhd = 698.9416$
	$AIC = -129.5088$	$AIC = -91.3399$	$AIC = -2243.1709$	$AIC = -1391.8832$
Student-t	$\nu = 4$	$\nu = 11$	$\nu = 1$	$\nu = 1$
	$\theta_1 = -1.1554$	$\theta_1 = 1.1192$	$\theta_1 = -0.01$	$\theta_1 = 0.0252$
	$\theta_2 = 0.7193$	$\theta_2 = 0.8522$	$\theta_2 = 0.04$	$\theta_2 = 0.1581$
	$\theta_3 = -1.0125$	$\theta_3 = 0.9686$	$\theta_3 = -0.1$	$\theta_3 = 0.0825$
	$loglkhd = 72.6679$	$loglkhd = 49.7269$	$loglkhd = 1228.5991$	$loglkhd = 805.5994$
	$AIC = -137.3358$	$AIC = -91.4539$	$AIC = -2449.1982$	$AIC = -1603.1988$

Table 6.13: Trivariate copulas fitted to annual changes in model 1Y, 5Y and 10Y par rates.

Copula	CIRSR BSDGOP	CIRSR MMMDGOP	3/2SR BSDGOP	3/2SR MMMDGOP
Clayton	$\theta = 47$	$\theta = 14.1019$	$\theta = 13.0915$	$\theta = 9.2733$
	$loglkhd = 855.0759$	$loglkhd = 490.3732$	$loglkhd = 470.0468$	$loglkhd = 403.5421$
	$AIC = -1708.1518$	$AIC = -978.7465$	$AIC = -938.0936$	$AIC = -805.0843$
Frank	$\theta = 37$	$\theta = 37$	$\theta = 37$	$\theta = 37$
	$loglkhd = 570.9289$	$loglkhd = 424.4498$	$loglkhd = 516.5334$	$loglkhd = 405.3621$
	$AIC = -1139.8579$	$AIC = -846.8997$	$AIC = -1031.0668$	$AIC = -808.7243$
Gumbel	$\theta = 47$	$\theta = 12.0522$	$\theta = 12.2471$	$\theta = 8.3109$
	$loglkhd = 1040.1886$	$loglkhd = 575.8707$	$loglkhd = 586.0822$	$loglkhd = 484.503$
	$AIC = -2078.3771$	$AIC = -1149.7413$	$AIC = -1170.1644$	$AIC = -967.006$
Gaussian	$\theta_1 = 0.0073$	$\theta_1 = 0.0452$	$\theta_1 = 0.1134$	$\theta_1 = 0.1222$
	$\theta_2 = 0.0392$	$\theta_2 = 0.1229$	$\theta_2 = 0.0762$	$\theta_2 = 0.1531$
	$\theta_3 = 0.0073$	$\theta_3 = 0.1746$	$\theta_3 = 0.1482$	$\theta_3 = 0.1953$
	$loglkhd = 1133.7178$	$loglkhd = 719.3838$	$loglkhd = 657.3643$	$loglkhd = 549.7155$
	$AIC = -2261.4357$	$AIC = -1432.7676$	$AIC = -1308.7287$	$AIC = -1093.431$
Student-t	$\nu = 1$	$\nu = 1$	$\nu = 1$	$\nu = 1$
	$\theta_1 = -0.01$	$\theta_1 = -0.0232$	$\theta_1 = -0.1218$	$\theta_1 = -0.1614$
	$\theta_2 = 0.03$	$\theta_2 = 0.1336$	$\theta_2 = 0.0796$	$\theta_2 = 0.1699$
	$\theta_3 = -0.06$	$\theta_3 = -0.0678$	$\theta_3 = -0.1563$	$\theta_3 = -0.2227$
	$loglkhd = 1270.5773$	$loglkhd = 832.3876$	$loglkhd = 679.7185$	$loglkhd = 563.4594$
	$AIC = -2533.1546$	$AIC = -1656.7753$	$AIC = -1351.4369$	$AIC = -1118.9189$

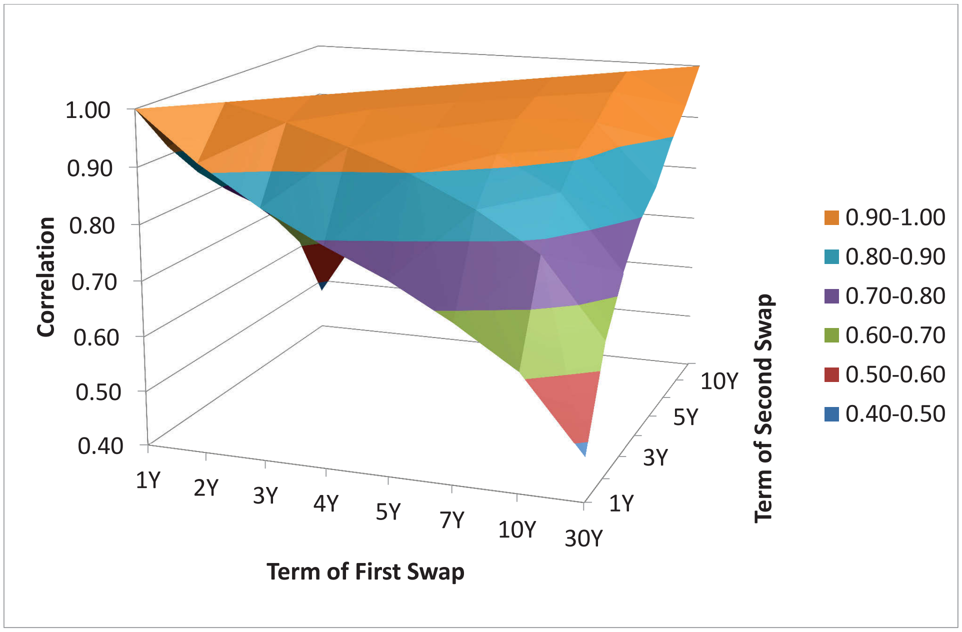
Table 6.14: Trivariate copulas fitted to monthly changes in actual 1Y, 5Y and 10Y swap rates.

Copula	MLEs
Clayton	$\theta = 1.8162$
	$loglkhd = 141.0854$
	$AIC = -284.1708$
Frank	$\theta = 5.6891$
	$loglkhd = 61.9667$
	$AIC = -125.9334$
Gumbel	$\theta = 2.1144$
	$loglkhd = 146.4812$
	$AIC = -294.9624$
Gaussian	$\theta_1 = 0.7643$
	$\theta_2 = 0.2693$
	$\theta_3 = 0.8960$
	$loglkhd = 275.9155$
	$AIC = -557.8310$
Student-t	$\nu = 7.0001$
	$\theta_1 = 5.5122$
	$\theta_2 = 3.4126$
	$\theta_3 = 2.2403$
	$loglkhd = 281.1129$
	$AIC = -570.2258$

Table 6.15: Trivariate copulas fitted to monthly changes in actual 1Y, 5Y and 10Y US Treasury bond yields.

Copula	MLEs
Clayton	$\theta = 2.5326$
	$loglkhd = 772.9970$
	$AIC = -1547.9939$
Frank	$\theta = 8.9333$
	$loglkhd = 501.1447$
	$AIC = -1004.2893$
Gumbel	$\theta = 2.7699$
	$loglkhd = 889.7526$
	$AIC = -1781.5052$
Gaussian	$\theta_1 = 0.5196$
	$\theta_2 = 0.3243$
	$\theta_3 = 0.6093$
	$loglkhd = 1149.4969$
	$AIC = -2304.9938$
Student-t	$\nu = 4.0000$
	$\theta_1 = 0.5290$
	$\theta_2 = 2.8414$
	$\theta_3 = 3.7820$
	$loglkhd = 1209.9819$
	$AIC = -2427.9637$

Figure 6.8: Correlations between daily moves in US swap rates of various terms (July 2000 to June 2014).



6.12 Conclusions

Several key conclusions can be drawn from the preceding analysis. Firstly, the market model composed of the CIR short rate and the MMM discounted GOP appears to best fit the historical data and best explain the level of ten-year bond yields[3]. Secondly, the long-term asymptotic forward rate is approximately 8.1%, this being implied by the best fitting market model. Thirdly, the considered market models give rise to leptokurtic forward rates and swap rates, a stylised feature of the interest rate market. Lastly, the correlation and dependency structure of forward rates and swap rates is modelled best by market models where the discounted GOP is modelled under the MMM. Improvements in the correlation and dependence structure of forward and swap rates under the market models could be made by adding a third factor, whereby the mean reverting level is modelled by a stochastic process such as the Ornstein-Uhlenbeck process, as done in models of Hull and White [1994] and Chen [1996]. Alternatively, Platen and Rendek [2009] have included a market activity component to the discounted GOP, which gives desired stylised features.

[3]By inspecting the parameter values and their corresponding standard errors in Table 6.1, it is clear that the parameters are well determined in these specific models and, therefore, the models are not employing parameters which result in overfitting the data.

Chapter 7

Hedging Zero-Coupon Bonds

7.1 Introduction

In this chapter we describe the hedging strategy used to replicate the payoffs of interest rate derivatives at maturity, in particular payoffs corresponding to a zero-coupon bond and a swaption. Crucially, we calculate the cost of purchasing a hedge portfolio at the outset of hedging the zero-coupon bond (ZCB), which ensures that the ZCB payoff is affordable by the hedge strategy with high probability. We compute the costs of hedging ZCBs across the considered market models and all ZCB terms to maturity and identify the best performing models[1].

Pricing and hedging of long-dated derivative payoffs remains a difficult and unsolved problem in financial and actuarial industries. Previous work on hedging long-dated zero-coupon bonds was published in Platen [2006a] and Bruti-Liberati and Platen [2010] where they make use of the growth optimal portfolio (GOP) which is maximising the logarithmic utility of expected terminal wealth. Platen employs the minimal market model (MMM) stock index dynamics to obtain low cost replicating hedge portfolios for zero-coupon bonds. The analysis in the current chapter extends this strategy to market models with stochastic interest rates. It is our intention to demonstrate that for market models employing the MMM discounted GOP the costs of hedging are significantly cheaper than for those employing the Black-Scholes form of the discounted GOP.

The market models examined here are specified by the stochastic differential equation (SDE) of the short rate r_t and the SDE of the discounted GOP $\bar{S}_t^{\delta_*}$. The short rate models considered are the deterministic short rate model, where the short rate is known for all times,

$$r_t = r(t), \tag{7.1.1}$$

[1]The results have been published in Fergusson and Platen [2014a]

the Vasicek short rate model described by Vasicek [1977],

$$dr_t = \kappa(\bar{r} - r_t)dt + \sigma dZ_t, \tag{7.1.2}$$

the Cox-Ingersoll-Ross (CIR) short rate model described by Cox et al. [1985],

$$dr_t = \kappa(\bar{r} - r_t)dt + \sigma\sqrt{r_t}dZ_t \tag{7.1.3}$$

and the 3/2 short rate model described by Ahn and Gao [1999],

$$dr_t = (pr_t + qr_t^2)dt + \sigma r_t^{3/2}dZ_t. \tag{7.1.4}$$

The discounted GOP models which are considered are the Black-Scholes model, equivalently the lognormal stock price model, employed by Black and Scholes [1973]

$$d\bar{S}_t^{\delta_*} = \bar{S}_t^{\delta_*}\theta^2 dt + \bar{S}_t^{\delta_*}\theta dW_t, \tag{7.1.5}$$

and the minimal market model described by Platen [2001]

$$d\bar{S}_t^{\delta_*} = \bar{\alpha}_t dt + \sqrt{\bar{\alpha}_t \bar{S}_t^{\delta_*}}dW_t, \tag{7.1.6}$$

where $\bar{\alpha}_t = \bar{\alpha}_0 \exp(\eta t)$. Here Z_t and W_t are independent Wiener processes, $r(t)$ is the realised value of the short rate at time t and $\bar{r}$, κ, σ, p, q, θ, $\bar{\alpha}_0$ and η are constants.

The fair prices of ZCBs under each market model are given in Section 5.2.

7.2 Description of Methodology

In respect of a derivative security, a hedging strategy is a trading strategy involving a portfolio of hedge securities whose value at a prescribed payoff date is intended to replicate the value of the derivative security.

When the market values of securities are driven by a deterministic short rate and stochastic discounted GOP then we have only one random factor in our market and we can hedge a suitable derivative security using a managed self-financing portfolio π of cash (the savings account) and the GOP. The value of the hedge portfolio can be written as

$$V_t^{(\pi)} = \delta_t^{(0)} B_t + \delta_t^{(1)} S_t^{\delta_*}, \tag{7.2.1}$$

where $\delta_t^{(0)}$ is the number of units of the cash account and $\delta_t^{(1)}$ is the number of units of the GOP account at time $t \in [0, T]$. The respective fractions invested at time $t \geq 0$ are $\pi_t = (\pi_t^{(0)}, \pi_t^{(1)})$ with $\pi_t^{(0)} = \delta_t^{(0)} B_t / V_t^{(\pi)}$ and $\pi_t^{(1)} = 1 - \pi_t^{(0)} = \delta_t^{(1)} S_t^{\delta_*} / V_t^{(\pi)}$. We have some flexibility in our choice of hedge securities and we could have used instead the savings account and futures on the GOP, for example.

When the market values of securities are driven by a stochastic short rate and a stochastic discounted GOP then we have two random factors in our market and we can hedge any derivative security using a managed portfolio of cash B_t, the GOP index $S_t^{\delta_*}$ and, for instance, a $(T-t)$-year zero-coupon bond $P(t,T)$. The value of the hedge portfolio π can be written as

$$V_t^{(\pi)} = \delta_t^{(0)} B_t + \delta_t^{(1)} S_t^{\delta_*} + \delta_t^{(2)} P(t,T), \tag{7.2.2}$$

where $\delta_t^{(0)}$ and $\delta_t^{(1)}$ describe numbers of units as before, and $\delta_t^{(2)}$ is the number of units of the T-maturity zero-coupon bond at time $t \in [0,T]$.

The cost C_t at time t of hedging a derivative since initial time 0 is equal to the cost of the derivative at time t less any gains from trading the hedge portfolio. We write

$$C_t = V_t^{\delta_{H_T}} - \int_0^t \delta_u^{(0)} dB_u - \int_0^t \delta_u^{(1)} dS_u^{\delta_*} = V_t^{\delta_{H_T}} - \int_0^t dV_u^{(\pi)} \tag{7.2.3}$$

where $V_t^{\delta_{H_T}}$ is the value of the derivative at time t and $V_t^{(\pi)}$ is the value of the hedge portfolio at time t.

This equation can be rewritten as

$$C_t = V_t^{\delta_{H_T}} - (V_t^{(\pi)} - V_0^{(\pi)}) \tag{7.2.4}$$

$$= V_0^{(\pi)} + (V_t^{\delta_{H_T}} - V_t^{(\pi)}) \tag{7.2.5}$$

and we can see that the cost of hedging can be expressed alternatively as the cost of the hedge portfolio at outset, namely $V_0^{(\pi)}$, plus additional funds needed at time t to purchase the derivative in excess of the value of the hedge portfolio.

At the payoff date T the cost of hedging is

$$C_T = V_T^{\delta_{H_T}} - \int_0^T dV_u^{(\pi)}. \tag{7.2.6}$$

Because we are interested in the real-world price of hedging, as given in (5.1.1), we consider the benchmarked cost of hedging, computed as

$$\hat{C}_T = \frac{C_T}{S_T^{\delta_*}} = \hat{V}_T^{\delta_{H_T}} - \int_t^T d\hat{V}_u^{(\pi)} = \hat{V}_t^{(\pi)} + \hat{V}_T^{\delta_{H_T}} - \hat{V}_T^{(\pi)}. \tag{7.2.7}$$

According to (5.1.1) the average of the benchmarked costs of hedging performed over a large number of backtests ought to approximate the real-world price of the derivative with payoff H_T.

Given a fully specified model with known parameters, we backtest hedging of the derivative over the time interval $[0,T]$ by setting the $n-1$ rebalancing times

$$t_1 < t_2 < ... < t_{n-1}$$

satisfying $0 = t_0 < t_1$ and $t_{n-1} < t_n = T$. The hedge portfolio $V^{(\pi)}$ is adjusted at the rebalancing times and is computed iteratively using the formula

$$V_{t_i}^{(\pi)} = \delta_{t_{i-1}}^{(0)} B_{t_i} + \delta_{t_{i-1}}^{(1)} S_{t_i}^{\delta_*} + \delta_{t_{i-1}}^{(2)} P(t_i, T) \tag{7.2.8}$$

for $i = 1, 2, \ldots, n$ with initial condition

$$V_0^{(\pi)} = V_0^{\delta_{H_T}}, \tag{7.2.9}$$

where, for $i = 1, 2, \ldots n-1$, the numbers of units held in the GOP and the ZCB at time t_i are computed as

$$\delta_{t_i}^{(1)} = \frac{\partial}{\partial S_s^{\delta_*}} V_s^{\delta_{H_T}}(r_s, S_s^{\delta_*})\big|_{s=t_i} - \delta_{t_i}^{(2)} \frac{\partial}{\partial S_s^{\delta_*}} P(s,T)\big|_{s=t_i} \tag{7.2.10}$$

$$\delta_{t_i}^{(2)} = \frac{\partial}{\partial r_s} V_s^{\delta_{H_T}}(r_s, S_s^{\delta_*})\big|_{s=t_i} \Big/ \frac{\partial}{\partial r_s} P(s,T)\big|_{s=t_i}$$

and the number of units held in the cash account at time t_i is computed as

$$\delta_{t_i}^{(0)} = \left(V_{t_i}^{(\pi)} - \delta_{t_i}^{(1)} S_{t_i}^{\delta_*} - \delta_{t_i}^{(2)} P(t_i, T) \right) / B_{t_i}. \tag{7.2.11}$$

7.3 Assessing a Hedging Strategy

A perfect hedge strategy is one for which

$$C_t = V_0^{(\pi)} \tag{7.3.1}$$

for all times $t \in [0, T]$. That is to say, the hedge portfolio replicates the value of the derivative over the life of the hedging strategy.

However, perfect hedging is not possible for many reasons and we are interested in strategies which generate the payoff at expiry date T, with “minimum” cost.

Therefore, for a given market model, a given data set and a given ZCB term to maturity we compute the benchmarked costs of hedging a ZCB at maturity over all possible periods within the data set. From this the p-th percentile of the set of benchmarked costs is computed. The best hedge strategy is derived from the market model which gives the minimum percentile benchmarked cost of hedging. Consequently, our task in this article is to compare the percentile benchmarked costs of hedging across all mentioned market models.

7.4 Hedge Securities

As stated earlier, when the market values of securities are driven by a stochastic short rate and a stochastic discounted GOP, we have two random factors in our market and we can theoretically hedge a suitable derivative security using a managed portfolio of cash, the GOP index and a ten year coupon bond, for example. Because liquidity is essential for any hedge strategy we would choose to hedge using a managed portfolio of cash, S&P 500 Index Futures and 10Y US Treasury Bonds.

7.5 Market Data and Fitting the Models

The data set used for backtesting has been the annual series of US 1Y deposit rates, 10Y treasury bond yields and S&P Composite Stock Index from 1871 to 2012, shown in Chapter 26 of Shiller [1989] and subsequently updated on Shiller's website http://aida.wss.yale.edu/ shiller/data/chapt26.xls (see Data Set A in Section L.1 of Appendix L). The 141 year length of this data series makes it a most useful series for analysing the hedging of long-dated ZCBs because we are able to backtest any given hedge strategy over the large term to maturity of the ZCB. Also, because there are 10Y bond yields accompanying the 1Y deposit rates and stock index values we are able to construct and backtest a hedge portfolio which immunises against movements in both the stock index and short rate. The MLEs of the parameters of all models fitted to US data are shown in Table 6.1.

The backtests of the hedging strategies were performed using in-sample estimation of parameters. Of course in reality one would backtest a hedge strategy using out-of-sample parameter estimates but by employing in-sample estimates any poorly performing model is readily falsified.

7.6 Hedging Costs under a Deterministic Short Rate

We present the costs of hedging ZCBs under deterministic short rate models.

In Table 7.1 the percentile benchmarked costs of hedging ZCBs of various terms to maturity and percentiles are shown for the deterministic short rate and Black-Scholes discounted GOP model.

In Table 7.2 the percentile benchmarked costs of hedging ZCBs of various terms to maturity and percentiles are shown for the deterministic short rate and MMM discounted GOP model.

For hedging ZCBs with terms to maturity shorter than 10 years the BS discounted GOP model and the MMM discounted GOP model perform similarly. At or beyond a 10 year term to maturity the MMM discounted GOP model significantly outperforms the BS discounted GOP model. For example, hedging a 50Y ZCB at the 99-th percentile incurs a cost of 0.068859 under the MMM discounted GOP model, which is about a quarter of the corresponding cost of 0.24457 under the BS discounted GOP model.

7.7 Hedging Costs under a Vasicek Short Rate

We present the costs of hedging ZCBs under Vasicek short rate models.

In Table 7.3 the percentile benchmarked costs of hedging ZCBs of various terms to maturity and percentiles are shown for the Vasicek short rate and Black-Scholes discounted GOP model.

In Table 7.4 the percentile benchmarked costs of hedging ZCBs of various terms to maturity and percentiles are shown for the Vasicek short rate and MMM discounted GOP model.

For hedging ZCBs with terms to maturity at or shorter than 15 years the BS discounted GOP model and the MMM discounted GOP model perform similarly. However, beyond a 15 year term to maturity the MMM discounted GOP model significantly outperforms the BS discounted GOP model. In particular, the cost of hedging a 50Y ZCB at the 99-th percentile is 0.12039 under the MMM discounted GOP model, which is about two-thirds of the corresponding cost of 0.17771 under the BS discounted GOP model.

7.8 Hedging Costs under a CIR Short Rate

We present the costs of hedging ZCBs under CIR short rate models.

In Table 7.5 the percentile benchmarked costs of hedging ZCBs of various terms to maturity and percentiles are shown for the CIR short rate and Black-Scholes discounted GOP model.

In Table 7.6 the percentile benchmarked costs of hedging ZCBs of various terms to maturity and percentiles are shown for the CIR short rate and MMM discounted GOP model.

For terms to maturity shorter than 15 years the BS discounted GOP model and the MMM discounted GOP model provide similar costs of hedging ZCBs. However, at or beyond a 15 year term to maturity the MMM discounted GOP model significantly outperforms the BS discounted GOP model. For example, the cost of hedging a ZCB is significantly reduced for a 50 year term to maturity at the 99-th percentile, the cost being 0.13086 under the MMM discounted GOP model, which is about a half of the corresponding cost of 0.23041 under the BS discounted GOP model.

7.9 Hedging Costs under a 3/2 Short Rate

We present the costs of hedging ZCBs under 3/2 short rate models.

In Table 7.7 the percentile benchmarked costs of hedging ZCBs of various terms to maturity and percentiles are shown for the 3/2 short rate and Black-Scholes discounted GOP model.

In Table 7.8 the percentile benchmarked costs of hedging ZCBs of various terms to maturity and percentiles are shown for the 3/2 short rate and MMM discounted GOP model.

For ZCB terms to maturity at or shorter than 10 years the BS discounted GOP model and the MMM discounted GOP model provide similar costs of hedging ZCBs. But beyond a term to maturity of 10 years the MMM discounted GOP model significantly outperforms the BS discounted GOP model. For example, the cost of hedging a 50Y ZCB at the 99-th percentile is 0.14498 under the MMM discounted GOP model, which is roughly a quarter of the corresponding cost of 0.51234 under the BS discounted GOP model.

7.10 Conclusions on Hedging ZCBs

In Figure 7.1 the 99-th percentile costs of hedging ZCBs of varying terms to maturity are graphed. Each model for which the discounted GOP is modelled by the MMM has significantly lower costs of hedging long-dated ZCBs, that is, ZCBs with maturities beyond 15 years. In particular, we find that among the considered market models having a stochastic short rate, the Vasicek short rate and MMM discounted GOP model provides the cheapest hedging strategy for long-dated ZCBs. In Chapter 8, Sections 8.2, 8.3 and 8.4 we compare model performances on hedging swaptions.

Figure 7.1: Percentile costs of hedging ZCBs of varying terms to maturity.

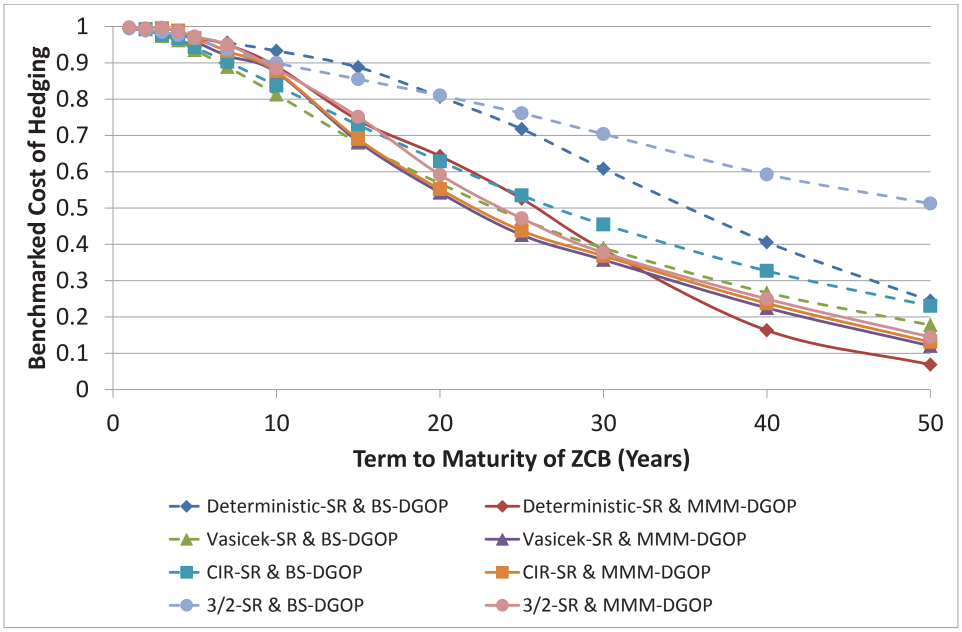


Table 7.1: Percentile costs of hedging ZCBs under a deterministic short rate & Black-Scholes discounted GOP based on US data 1871 - 2012.

Term to Maturity of ZCB	99-th Percentile	95-th Percentile	90-th Percentile	85-th Percentile	80-th Percentile
1Y	0.99473	0.99256	0.99	0.98445	0.97523
2Y	0.98889	0.98546	0.97681	0.96485	0.95167
3Y	0.983	0.97716	0.96162	0.94172	0.92744
4Y	0.97626	0.96988	0.94909	0.92018	0.89415
5Y	0.96928	0.96266	0.94193	0.89956	0.86139
7Y	0.95547	0.94773	0.91388	0.86023	0.81143
10Y	0.93309	0.92335	0.86542	0.80451	0.75071
15Y	0.88791	0.85433	0.79846	0.71259	0.65539
20Y	0.80645	0.78216	0.7149	0.65548	0.58733
25Y	0.71779	0.67893	0.64009	0.58695	0.49921
30Y	0.60825	0.59331	0.5553	0.50713	0.43517
40Y	0.40573	0.39554	0.3682	0.34926	0.32832
50Y	0.24457	0.2396	0.23292	0.23088	0.21696

Table 7.2: Percentile costs of hedging ZCBs under a deterministic short rate & MMM discounted GOP based on US data 1871 - 2012.

Term to Maturity of ZCB	99-th Percentile	95-th Percentile	90-th Percentile	85-th Percentile	80-th Percentile
1Y	0.99473	0.99256	0.99	0.98445	0.97523
2Y	0.98889	0.98546	0.97681	0.96485	0.95167
3Y	0.98302	0.97716	0.96162	0.94172	0.92744
4Y	0.97649	0.96952	0.94906	0.92013	0.89414
5Y	0.96964	0.95494	0.92906	0.89953	0.86139
7Y	0.9499	0.91168	0.88088	0.84826	0.81098
10Y	0.89031	0.83149	0.79108	0.75737	0.73169
15Y	0.74213	0.67106	0.61237	0.57877	0.56427
20Y	0.64319	0.49652	0.4638	0.41004	0.38913
25Y	0.52576	0.3897	0.31444	0.29971	0.29182
30Y	0.38523	0.29224	0.22849	0.22021	0.21614
40Y	0.16308	0.12845	0.11989	0.11917	0.11351
50Y	0.068659	0.065715	0.05967	0.057967	0.054139

Table 7.3: Percentile costs of hedging ZCBs under a Vasicek short rate & Black-Scholes discounted GOP based on US data 1871 - 2012.

Term to Maturity of ZCB	99-th Percentile	95-th Percentile	90-th Percentile	85-th Percentile	80-th Percentile
1Y	1.0013	0.98871	0.98544	0.98271	0.97847
2Y	0.99264	0.9739	0.96921	0.9637	0.9467
3Y	0.97404	0.95791	0.94726	0.93674	0.9253
4Y	0.96205	0.93717	0.92472	0.91721	0.90395
5Y	0.9355	0.91767	0.90768	0.89717	0.87567
7Y	0.88889	0.87213	0.86141	0.83933	0.82671
10Y	0.8127	0.79847	0.78845	0.77038	0.74663
15Y	0.67968	0.67295	0.66559	0.649	0.63895
20Y	0.56756	0.55927	0.55355	0.54078	0.52354
25Y	0.46926	0.46426	0.4608	0.44949	0.43604
30Y	0.38976	0.38519	0.38057	0.37245	0.36232
40Y	0.26676	0.2641	0.26108	0.25494	0.24617
50Y	0.17771	0.17666	0.17453	0.17411	0.16753

Table 7.4: Percentile costs of hedging ZCBs under a Vasicek short rate & MMM discounted GOP based on US data 1871 - 2012.

Term to Maturity of ZCB	99-th Percentile	95-th Percentile	90-th Percentile	85-th Percentile	80-th Percentile
1Y	1.0051	0.99447	0.9874	0.98391	0.97682
2Y	1.0014	0.98466	0.97472	0.96521	0.94847
3Y	0.99603	0.96746	0.95574	0.93713	0.92878
4Y	0.98577	0.95668	0.93736	0.92271	0.9063
5Y	0.95988	0.94028	0.92605	0.91038	0.88276
7Y	0.91948	0.89163	0.88089	0.85599	0.83274
10Y	0.87379	0.81513	0.77244	0.75994	0.7546
15Y	0.6831	0.65907	0.64572	0.63447	0.6218
20Y	0.54225	0.529	0.51811	0.50842	0.49915
25Y	0.42601	0.41738	0.40556	0.39134	0.38235
30Y	0.3573	0.33452	0.3185	0.29529	0.2832
40Y	0.22491	0.18573	0.17552	0.16199	0.14944
50Y	0.12039	0.1056	0.090466	0.084736	0.076245

Table 7.5: Percentile costs of hedging ZCBs under a CIR short rate & Black-Scholes discounted GOP based on US data 1871 - 2012.

Term to Maturity of ZCB	99-th Percentile	95-th Percentile	90-th Percentile	85-th Percentile	80-th Percentile
1Y	1.0001	0.98956	0.98631	0.98302	0.97828
2Y	0.99272	0.97622	0.97054	0.9648	0.94575
3Y	0.97723	0.96202	0.95097	0.93704	0.92666
4Y	0.96774	0.94217	0.93011	0.92037	0.90335
5Y	0.9425	0.92712	0.91128	0.90274	0.87842
7Y	0.90316	0.88562	0.86989	0.84718	0.82948
10Y	0.83723	0.82484	0.80931	0.78952	0.75835
15Y	0.72898	0.72066	0.70421	0.67551	0.65364
20Y	0.62868	0.61946	0.60705	0.58503	0.56702
25Y	0.53487	0.52786	0.5194	0.50455	0.47967
30Y	0.45474	0.44968	0.44021	0.42824	0.40364
40Y	0.32668	0.32162	0.31693	0.30465	0.29126
50Y	0.23041	0.22724	0.22549	0.22036	0.20748

Table 7.6: Percentile costs of hedging ZCBs under a CIR short rate & MMM discounted GOP based on US data 1871 - 2012.

Term to Maturity of ZCB	99-th Percentile	95-th Percentile	90-th Percentile	85-th Percentile	80-th Percentile
1Y	1.0033	0.99453	0.98864	0.98375	0.97739
2Y	1.0006	0.98628	0.97576	0.96508	0.94935
3Y	0.99508	0.971	0.95914	0.93814	0.92994
4Y	0.9886	0.96117	0.94117	0.92378	0.90995
5Y	0.96787	0.94869	0.93089	0.90962	0.88337
7Y	0.93207	0.90247	0.89243	0.86901	0.83723
10Y	0.8765	0.82884	0.79637	0.7777	0.75824
15Y	0.68994	0.67036	0.65193	0.6415	0.62889
20Y	0.55194	0.53916	0.52856	0.5136	0.50601
25Y	0.43752	0.42795	0.41449	0.40905	0.39516
30Y	0.3682	0.34461	0.33042	0.30899	0.29755
40Y	0.23746	0.19375	0.17902	0.17116	0.16419
50Y	0.13086	0.11298	0.099064	0.093498	0.083863

Table 7.7: Percentile costs of hedging ZCBs under a 3/2 short rate & Black-Scholes discounted GOP based on US data 1871 - 2012.

Term to Maturity of ZCB	99-th Percentile	95-th Percentile	90-th Percentile	85-th Percentile	80-th Percentile
1Y	0.99498	0.99158	0.98821	0.98359	0.97967
2Y	0.98935	0.98128	0.97437	0.96548	0.94832
3Y	0.98466	0.96941	0.96053	0.94512	0.92963
4Y	0.97618	0.95459	0.94238	0.93253	0.91251
5Y	0.96757	0.94845	0.93463	0.91505	0.88228
7Y	0.93604	0.91496	0.89838	0.86453	0.83134
10Y	0.89952	0.88432	0.84687	0.81012	0.78084
15Y	0.85513	0.83077	0.77158	0.71565	0.67678
20Y	0.81023	0.77288	0.71775	0.6346	0.58291
25Y	0.76106	0.71113	0.6639	0.57766	0.51403
30Y	0.70388	0.65474	0.60165	0.52649	0.45337
40Y	0.59249	0.54902	0.50104	0.44657	0.37547
50Y	0.51234	0.47976	0.44259	0.39518	0.33001

Table 7.8: Percentile costs of hedging ZCBs under a 3/2 short rate & MMM discounted GOP based on US data 1871 - 2012.

Term to Maturity of ZCB	99-th Percentile	95-th Percentile	90-th Percentile	85-th Percentile	80-th Percentile
1Y	0.99785	0.99392	0.98956	0.98369	0.97805
2Y	0.995	0.98708	0.97712	0.96662	0.94873
3Y	0.99639	0.97634	0.96354	0.94541	0.93078
4Y	0.98765	0.96792	0.95093	0.93281	0.91262
5Y	0.9722	0.95564	0.94524	0.92132	0.89381
7Y	0.95036	0.92299	0.90941	0.86955	0.83559
10Y	0.88323	0.86718	0.82947	0.80948	0.77803
15Y	0.7512	0.71025	0.68236	0.67631	0.65572
20Y	0.59172	0.55556	0.54667	0.53749	0.52085
25Y	0.47171	0.44381	0.43052	0.41659	0.4072
30Y	0.37711	0.35252	0.34447	0.32434	0.3118
40Y	0.24959	0.20988	0.20229	0.19247	0.1888
50Y	0.14498	0.12691	0.12073	0.11236	0.10697

Chapter 8

Hedging Swaptions

8.1 Introduction

Swaptions are interest rate derivatives which protect the owner of such an asset against a rise or fall in swap rates and are therefore used by many pension funds and life insurers seeking to hedge their exposure to interest rates. For example, low interest rates may be bad for some life insurers because it becomes expensive to invest in fixed income products which match their bond-like liabilities.

In this chapter we describe the hedging strategy used to replicate the payoff of a 3%-strike payer swaption at expiry having an underlying semi-annual ten-year swap and unit notional amount. As in Section 7.1 we calculate the cost of purchasing a hedge portfolio at the outset of hedging the swaption which ensures that the swaption payoff is affordable by the hedge strategy with high probability. We compute the costs of hedging swaptions across all market models and all swaption times to expiry and identify the best performing models[1]. We deliberately focus on swaptions having an underlying ten-year swap rate because our US data set contains ten-year swap rates and therefore allows us to calculate a payoff at expiration of the swaption.

We describe how we price swaptions in this chapter. The payoff at time T of a payer swaption with unit notional and strike rate R is

$$H_T = \sum_{i=1}^{n} P(T, T_i)(T_i - T_{i-1})(SW_T - R)^+, \tag{8.1.1}$$

where $T_1, \ldots, T_n$ are the payment times of the underlying swap, SW_T is the corresponding swap rate at time T and we take $T_0 = T$. This payoff is the same as that of a put option on a coupon bond with coupon rate R and having strike

[1] The results have been published in Fergusson and Platen [2014a]

price equal to one, that is,

$$H_T = \left(1 - \sum_{i=1}^{n} (R(T_i - T_{i-1}) + \mathbf{1}_{i=n}) P(T, T_i)\right)^+, \tag{8.1.2}$$

see for example Hull [1997]. Here $\mathbf{1}_{i=n}$ denotes the indicator function which equals one if $i = n$ and zero otherwise.

8.1.1 Model with Black-Scholes Discounted GOP

Applying (5.1.1) to this payoff for the deterministic short rate and Black-Scholes discounted GOP market model, the real-world swaption pricing formula simplifies to the intrinsic value of the swaption, namely

$$V_t^{\delta_{H_T}} = \left(\exp\left\{-\int_t^{T_0} r_s ds\right\} - \sum_{i=1}^{n} (R(T_i - T_{i-1}) + \mathbf{1}_{i=n}) \exp\left\{-\int_t^{T_i} r_s ds\right\}\right)^+. \tag{8.1.3}$$

For a mean reverting Gaussian interest rate model Jamshidian [1989] proves that the price of a coupon bond option with strike price K, and corresponding strike rate R, is equal to the sum of options on constituent zero-coupon bonds each having its strike price calculated from the common strike rate R. The derivation of the formula relies on the observation that the monotonicity of the zero-coupon bond price as a function of the short rate implies that the exercise short rate of the portfolio of ZCBs is the same as each of the exercise short rates of the options on the component ZCBs, see for example Hull and White [1990]. Jamshidian's formula suffices for the Vasicek short rate and Black-Scholes discounted GOP model and in this instance we have the real-world swaption pricing formula

$$\begin{aligned} V_t^{\delta_{H_T}} = \sum_{i=1}^{n} \big(R(T_i - T_{i-1}) + \mathbf{1}_{i=n}\big)\big(&- A(t, T_i)\exp(-r_t B(t, T_i)) N(-d_i^{(1)}) \\ &+ A(t, T)\exp(-r_t B(t, T)) K_i N(-d_i^{(2)})\big), \end{aligned} \tag{8.1.4}$$

where A and B are given in (3.2.49) and (3.2.48) and $d_i^{(1)}$ and $d_i^{(2)}$ are given by

$$d_i^{(1)} = \frac{1}{\sigma_i} \log\left(\frac{A(t, T_i)\exp(-r_t B(t, T_i))}{A(t, T)\exp(-r_t B(t, T)) K_i}\right) + \frac{1}{2}\sigma_i \tag{8.1.5}$$

$$d_i^{(2)} = \frac{1}{\sigma_i} \log\left(\frac{A(t, T_i)\exp(-r_t B(t, T_i))}{A(t, T)\exp(-r_t B(t, T)) K_i}\right) - \frac{1}{2}\sigma_i \tag{8.1.6}$$

with σ_i given by

$$\sigma_i = \sigma B(T, T_i) \sqrt{\frac{1}{2\kappa}(1 - \exp(-2\kappa(T - t)))} \tag{8.1.7}$$

and K_i given by

$$K_i = A(T, T_i) \exp(-xB(T, T_i)). \tag{8.1.8}$$

Here x is the solution to the equation

$$1 = \sum_{i=1}^{n} (R(T_i - T_{i-1}) + \mathbf{1}_{i=n}) A(T, T_i) \exp(-xB(T, T_i)). \tag{8.1.9}$$

8.1.2 Model with MMM Discounted GOP

Also, Jamshidian's method can be adapted to the deterministic short rate and MMM discounted GOP model giving the real-world swaption pricing formula

$$\begin{aligned} V_t^{\delta_{H_T}} &= \sum_{i=1}^{n} (R(T_i - T_{i-1}) + \mathbf{1}_{i=n}) \Bigg(- \frac{B_t}{B_{T_i}} \big(\chi^2_{0,\lambda}(u_i^*) - \exp(-\lambda/2) \big) \\ &+ \frac{B_t}{B_{T_i}} \left(\exp\left(- \frac{\tau_i}{1 + 2\tau_i} \lambda \right) \chi^2_{0,\lambda/(1+2\tau_i)}((1 + 2\tau_i) u_i^*) - \exp(-\lambda/2) \right) \\ &+ K_i \frac{B_t}{B_T} \big(\chi^2_{0,\lambda}(u_i^*) - \exp(-\lambda/2) \big) \Bigg), \end{aligned} \tag{8.1.10}$$

where

$$\begin{aligned} u_i^* &= \begin{cases} 2 \frac{\varphi_{T_i} - \varphi_T}{\varphi_T - \varphi_t} \log \frac{1}{1 - K_i B_{T_i}/B_T} & \text{if} \quad 1 > K_i B_{T_i}/B_T; \\ \infty \quad \text{otherwise}, \end{cases} \\ \lambda &= \frac{\bar{S}_t^{\delta_*}}{\varphi_T - \varphi_t}, \\ \tau_i &= \frac{1}{2} \frac{\varphi_T - \varphi_t}{\varphi_{T_i} - \varphi_T} \end{aligned} \tag{8.1.11}$$

and K_i given by

$$K_i = \frac{B_T}{B_{T_i}} \big(1 - \exp(-x\tau_i) \big). \tag{8.1.12}$$

Here x is the solution to the equation

$$1 = \sum_{i=1}^{n} (R(T_i - T_{i-1}) + \mathbf{1}_{i=n}) \frac{B_T}{B_{T_i}} \big(1 - \exp(-x\tau_i) \big). \tag{8.1.13}$$

We have used the notation $\chi^2_{\nu,\lambda}(x)$ to denote the cumulative distribution function of a non-centrally distributed random variable having non-centrality parameter λ and ν degrees of freedom.

However, for the Vasicek short rate and MMM discounted GOP model we resort to the following theorem, previously stated in Section 5.6, but proven here.

Theorem 8.1.1 *Suppose that the short rate r_t obeys Vasicek's SDE* (7.1.2) *and suppose the short rate and the discounted GOP $\bar{S}_t^{\delta_*}$ are independent. Then the real-world price of a coupon bond put option is given by*

$$V_t^{\delta_{H_T}} = \int_0^\infty \frac{\lambda}{u} V_t^{\delta_{H_T}}(u) f_{\chi^2_{4,\lambda}}(u)\, du, \tag{8.1.14}$$

where

$$\begin{aligned} V_t^{\delta_{H_T}}(u) = \sum_{i=1}^n \Bigg(& \big(R(T_i - T_{i-1}) + \mathbf{1}_{i=n}\big) \times (1 - \exp(-\tau_i u)) \\ & \times \big(- A(t,T_i)\exp(-r_t B(t,T_i)) N(-d_i^{(1)}(u)) \\ & + A(t,T)\exp(-r_t B(t,T)) K_i(u) N(-d_i^{(2)}(u))\big)\Bigg), \end{aligned} \tag{8.1.15}$$

where A and B are given in (3.2.49) *and* (3.2.48) *and $d_i^{(1)}(u)$ and $d_i^{(2)}(u)$ are given by*

$$d_i^{(1)}(u) = \frac{1}{\sigma_i} \log\left(\frac{A(t,T_i)\exp(-r_t B(t,T_i))}{A(t,T)\exp(-r_t B(t,T)) K_i(u)} \right) + \frac{1}{2}\sigma_i \tag{8.1.16}$$

$$d_i^{(2)}(u) = \frac{1}{\sigma_i} \log\left(\frac{A(t,T_i)\exp(-r_t B(t,T_i))}{A(t,T)\exp(-r_t B(t,T)) K_i(u)} \right) - \frac{1}{2}\sigma_i \tag{8.1.17}$$

with σ_i given by

$$\sigma_i = \sigma B(T,T_i) \sqrt{\frac{1}{2\kappa}(1 - \exp(-2\kappa(T-t)))} \tag{8.1.18}$$

and $K_i(u)$ given by

$$K_i(u) = A(T,T_i)\exp(-x_u B(T,T_i)). \tag{8.1.19}$$

Here x_u is the solution to the equation

$$1 = \sum_{i=1}^n (R(T_i - T_{i-1}) + \mathbf{1}_{i=n})(1 - \exp(-\tau_i u)) A(T,T_i)\exp(-x_u B(T,T_i)) \tag{8.1.20}$$

and τ_i is given in (8.1.11).

Proof. See Appendix O.

8.2 Hedging Costs under a Deterministic Short Rate

We present the costs of hedging swaptions under market models having a deterministic short rate.

In Table 8.1 the percentile benchmarked costs of hedging swaptions of various terms to expiry and percentiles are shown for the deterministic short rate and Black-Scholes discounted GOP model.

In Table 8.2 the percentile benchmarked costs of hedging swaptions of various terms to expiry and percentiles are shown for the deterministic short rate and MMM discounted GOP model.

For hedging swaptions with terms to expiry at or shorter than 15 years the BS discounted GOP model and the MMM discounted GOP model perform similarly. Beyond swaption terms to expiry of 15 years the MMM discounted GOP model significantly outperforms the BS discounted GOP model. For example, hedging a 50Y swaption at the 99-th percentile incurs a cost of 0.031698 under the MMM discounted GOP model, which is roughly a quarter of the corresponding cost of 0.12804 under the BS discounted GOP model.

8.3 Hedging Costs under a Vasicek Short Rate

We present the costs of hedging swaptions under Vasicek short rate models.

In Table 8.3 the percentile benchmarked costs of hedging swaptions of various terms to expiry and percentiles are shown for the Vasicek short rate and Black-Scholes discounted GOP model.

In Table 8.4 the percentile benchmarked costs of hedging swaptions of various terms to expiry and percentiles are shown for the Vasicek short rate and MMM discounted GOP model.

For hedging swaptions with terms to expiry shorter than 7 years the MMM discounted GOP model and the BS discounted GOP model perform similarly. However, at or beyond swaption terms to expiry of 7 year the MMM discounted GOP model gives lower costs of hedging than that given by the Black-Scholes discounted GOP model.

8.4 Conclusions on Hedging Swaptions

In Figure 8.1 the percentile costs of hedging swaptions of varying terms to expiry are graphed. Each model for which the discounted GOP is modelled by the MMM has lower costs of hedging long-dated swaptions. In particular, we find that the Vasicek short rate and MMM discounted GOP model provides the cheapest hedging strategy for long-dated swaptions. The market model composed of the Vasicek short rate and MMM discounted GOP was also the best model for hedging zero-coupon bonds in Section 7.10.

Figure 8.1: Percentile costs of hedging swaptions of varying terms to expiry.

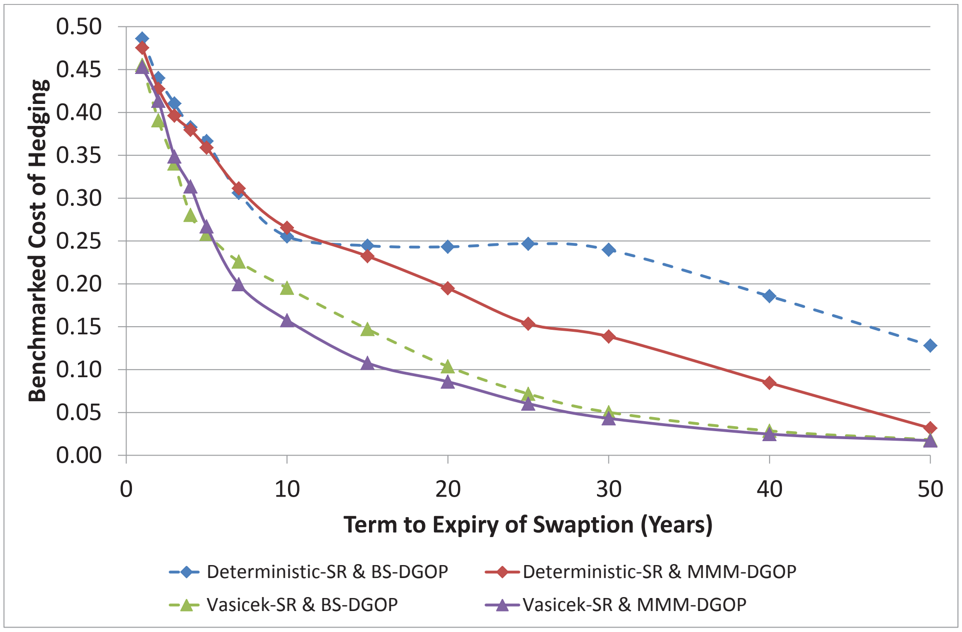


Table 8.1: Percentile costs of hedging swaptions under a deterministic short rate & Black-Scholes discounted GOP based on US data 1871 - 2012.

Term to Expiry of Swaption	99-th Percentile	95-th Percentile	90-th Percentile	85-th Percentile	80-th Percentile
1Y	0.48611	0.38003	0.31506	0.27528	0.20821
2Y	0.44005	0.33579	0.29116	0.25279	0.20378
3Y	0.4103	0.32175	0.27169	0.2263	0.19394
4Y	0.38279	0.29708	0.23888	0.19915	0.18646
5Y	0.36652	0.26234	0.22598	0.20163	0.18189
7Y	0.306	0.22095	0.20673	0.18653	0.17151
10Y	0.25529	0.21632	0.19987	0.17504	0.15582
15Y	0.24448	0.22389	0.20212	0.17748	0.14858
20Y	0.24321	0.22801	0.21065	0.16979	0.14934
25Y	0.24677	0.22338	0.21134	0.18003	0.15261
30Y	0.23978	0.22031	0.2028	0.17156	0.14505
40Y	0.18559	0.17428	0.17007	0.14754	0.12871
50Y	0.12804	0.11657	0.10833	0.1072	0.10012

Table 8.2: Percentile costs of hedging swaptions under a deterministic short rate & MMM discounted GOP based on US data 1871 - 2012.

Term to Expiry of Swaption	99-th Percentile	95-th Percentile	90-th Percentile	85-th Percentile	80-th Percentile
1Y	0.47544	0.37312	0.31076	0.25785	0.21711
2Y	0.42743	0.34451	0.29103	0.24248	0.19133
3Y	0.3962	0.31381	0.27224	0.22923	0.18142
4Y	0.37955	0.29242	0.25508	0.20019	0.18678
5Y	0.3588	0.26402	0.22649	0.19589	0.18186
7Y	0.31133	0.22974	0.2077	0.19088	0.17649
10Y	0.26536	0.22559	0.20246	0.19369	0.18583
15Y	0.23231	0.22473	0.21262	0.20827	0.18639
20Y	0.19473	0.18542	0.1812	0.17025	0.15873
25Y	0.15351	0.14079	0.12989	0.12479	0.11982
30Y	0.13853	0.11254	0.097728	0.084924	0.077339
40Y	0.084456	0.061148	0.046843	0.04057	0.036779
50Y	0.031698	0.025796	0.024644	0.02379	0.023148

Table 8.3: Percentile costs of hedging swaptions under a Vasicek short rate & Black-Scholes discounted GOP based on US data 1871 - 2012.

Term to Expiry of Swaption	99-th Percentile	95-th Percentile	90-th Percentile	85-th Percentile	80-th Percentile
1Y	0.45536	0.35855	0.30709	0.24918	0.21407
2Y	0.39077	0.3202	0.27933	0.23469	0.19518
3Y	0.34031	0.29422	0.25338	0.22579	0.16604
4Y	0.28045	0.25812	0.22965	0.21069	0.15451
5Y	0.2584	0.24	0.20927	0.18904	0.13578
7Y	0.22588	0.19141	0.17426	0.15609	0.12294
10Y	0.19525	0.15656	0.12256	0.11395	0.098862
15Y	0.1472	0.12391	0.095036	0.074855	0.06858
20Y	0.10367	0.09483	0.075293	0.061024	0.053613
25Y	0.071685	0.06604	0.058912	0.049244	0.045019
30Y	0.0502	0.045495	0.043199	0.040535	0.038422
40Y	0.028523	0.027775	0.026965	0.025942	0.025264
50Y	0.018271	0.017919	0.017433	0.016915	0.016747

Table 8.4: Percentile costs of hedging swaptions under a Vasicek short rate & MMM discounted GOP based on US data 1871 - 2012.

Term to Expiry of Swaption	99-th Percentile	95-th Percentile	90-th Percentile	85-th Percentile	80-th Percentile
1Y	0.45291	0.34929	0.29782	0.26397	0.20662
2Y	0.41355	0.32358	0.27757	0.24985	0.18266
3Y	0.3485	0.28436	0.24871	0.21766	0.16344
4Y	0.31349	0.24955	0.22659	0.19592	0.15913
5Y	0.26693	0.23529	0.20533	0.17569	0.14353
7Y	0.19943	0.18343	0.17188	0.15911	0.11132
10Y	0.15752	0.13175	0.12518	0.1185	0.094938
15Y	0.10778	0.10203	0.084549	0.070733	0.060173
20Y	0.085821	0.076264	0.068102	0.052175	0.045619
25Y	0.060256	0.057678	0.049517	0.04639	0.037491
30Y	0.043094	0.041715	0.038363	0.034956	0.032305
40Y	0.024757	0.024145	0.0237	0.023144	0.022709
50Y	0.017216	0.01655	0.016265	0.016035	0.014939

Chapter 9

Hedging Index Options

9.1 Introduction

Long-term savings products with embedded guarantees on capital, such as variable annuities, are popular among investors planning for retirement. Insurers who write such products are interested in hedging their risk exposure either through reinsurance, derivative markets or hedging programmes. Several frameworks of accounting standards such as US GAAP, IASB and IFRS prescribe that such products be marked-to-market and, therefore, hedging these products is paramount for insurers seeking stable earnings and high credit ratings.

Using the benchmark approach of Platen [2002a], Platen [2006c] and Platen and Heath [2006], pricing and hedging of long-dated claims on the S&P500 Total Return Index, when interest rates are deterministic, was demonstrated by Hulley and Platen [2012]. In this chapter we extend this work under the benchmark approach to price and hedge long-dated equity index options when interest rates are stochastic[1]. Some pricing and hedging of interest rate derivatives using the benchmark approach has been done by Fergusson and Platen [2014a]. The pricing and hedging of equity options when share prices and interest rates are stochastic has previously been done by Scott [1997] who also incorporates a jump diffusion component and stochastic volatility to the stock price dynamics. Many approaches to pricing equity options with models involving stochastic interest rates employ inverse Fourier transforms, as done in Lee [2004]. However, in this chapter we demonstrate less-expensive pricing and hedging of long-dated equity index options and provide approximate pricing formulae involving either the cumulative distribution functions of the normal distribution or the non-central chi-squared distribution. Furthermore, we compute the cost of hedging an equity index put option, whose strike price is an exponential function of the spot price, for each of the considered market models and various terms to expiry and identify the

[1]The results have been reported in Fergusson and Platen [2015a].

best performing models as those involving a discounted GOP being modelled as a squared Bessel process as in Platen's minimal market model (MMM).

9.2 Description of Hedging Methodology

Beyond pricing of long-dated put options on an equity index, our aim is to demonstrate cheaper costs of hedging such options. We focus on hedging a long-dated put option expiring at time T, whose strike price K keeps pace with the level of the equity index by way of the formula

$$K = S_t^{\delta_*} \exp\big((\eta + \mu_r)(T - t)\big). \tag{9.2.1}$$

Here t is the time at which the put option is written and $\eta = 0.045486$ is the net market growth rate given in Table 6.1 and $\mu_r = \frac{1}{141}\sum_s r(s) = 0.045726$ is the average of the one year continuously compounded cash rates over the 141 year period of the data. The hedging strategy is described in Section 7.2.

9.3 Hedging Costs under a Deterministic Short Rate

We present the costs of hedging GOP options under market models having a deterministic short rate.

In Table 9.1 the percentile benchmarked costs of hedging GOP options of various terms to expiry and levels of probability are shown for the deterministic short rate and Black-Scholes discounted GOP model.

In Table 9.2 the percentile benchmarked costs of hedging GOP options of various terms to expiry and levels of probability are shown for the deterministic short rate and MMM discounted GOP model.

For hedging GOP options with terms to expiry up to 10 years, the BS discounted GOP model and MMM discounted GOP model perform similarly. Beyond GOP option terms to expiry of 10 years, the MMM discounted GOP model outperforms the BS discounted GOP model. For example, hedging a 50Y GOP option at the 99% probability level incurs a cost of 5.0172 under the MMM discounted GOP model, which is significantly less than the corresponding cost of 23.995 under the BS discounted GOP model.

9.4 Hedging Costs under a Vasicek Short Rate

We present the costs of hedging GOP options under Vasicek short rate models.

In Table 9.3 the percentile benchmarked costs of hedging GOP options of various terms to expiry and levels of probability are shown for the Vasicek short rate and Black-Scholes discounted GOP model.

In Table 9.4 the percentile benchmarked costs of hedging GOP options of various terms to expiry and levels of probability are shown for the Vasicek short rate and MMM discounted GOP model.

For hedging GOP options with terms to expiry up to 15 years, the BS discounted GOP model and MMM discounted GOP model perform similarly. However, beyond 15 years the MMM discounted GOP model outperforms the BS discounted GOP model. In particular, the cost of hedging a 50Y GOP option at the 99% probability level is 10.537 under the MMM discounted GOP model, which is significantly less than the corresponding cost of 17.235 under the BS discounted GOP model.

9.5 Hedging Costs under a CIR Short Rate

We present the costs of hedging GOP options under CIR short rate models.

In Table 9.5 the percentile benchmarked costs of hedging GOP options of various terms to expiry and levels of probability are shown for the CIR short rate and Black-Scholes discounted GOP model.

In Table 9.6 the percentile benchmarked costs of hedging GOP options of various terms to expiry and levels of probability are shown for the CIR short rate and MMM discounted GOP model.

For GOP option terms to expiry up to 15 years, the BS discounted GOP model provides a significantly lower cost of hedging than under the MMM discounted GOP model. However, beyond a GOP option term to expiry of 15 years, the MMM discounted GOP model outperforms the BS discounted GOP model. For example, the cost of hedging a GOP option is significantly reduced for a 50 year term to expiry at the 99% probability level, the cost being 12.392 under the MMM discounted GOP model, which is significantly less than the corresponding cost of 22.827 under the BS discounted GOP model.

9.6 Hedging Costs under a $3/2$ Short Rate

We present the costs of hedging GOP options under 3/2 short rate models.

In Table 9.7 the percentile benchmarked costs of hedging GOP options of various terms to expiry and levels of probability are shown for the 3/2 short rate model and Black-Scholes discounted GOP model.

In Table 9.8 the percentile benchmarked costs of hedging GOP options of various terms to expiry and levels of probability are shown for the 3/2 short rate model and MMM discounted GOP model.

For GOP option terms to expiry shorter than 15 years the BS discounted GOP model is about the same as or lower than the MMM discounted GOP model. Beyond a GOP option term to expiry of 15 years the MMM discounted GOP model outperforms the BS discounted GOP model. For example, the cost of hedging a 50Y GOP option at the 99% probability level is 13.838 under the MMM discounted GOP model, which is significantly less than the corresponding cost of 51.438 under the BS discounted GOP model.

9.7 Conclusions on Hedging Index Options

In Figure 9.1 the percentile costs of hedging GOP options of varying terms to expiry are graphed. Each model for which the discounted GOP is modelled by the MMM has significantly cheaper costs of hedging long-dated GOP options. In particular, we find that among the models having a stochastic short rate, the Vasicek short rate and MMM discounted GOP model provides the cheapest hedging strategy for long-dated GOP put options. We remark on the effect of stochastic versus deterministic interest rates that Jensen's Inequality gives

$$E\bigg(\exp\big(-\int_t^T r_s\,ds\big)\bigg) \geq \exp\bigg(-E\big(\int_t^T r_s\,ds\big)\bigg), \tag{9.7.1}$$

since the function $f(x) = \exp(-x)$ is convex. This indicates what we have also seen and we see that stochastic interest rates will give rise to higher derivative prices than those from deterministic interest rates if everything else is modelled analogously. Improved MMM versions of GOP models with stochastic market volatility as in Platen and Rendek [2012b] can most likely improve results even further, in particular, for the shorter maturities. Forthcoming work will demonstrate this in detail.

Figure 9.1: Percentile Costs of Hedging GOP Put Options of Varying Terms to Expiry

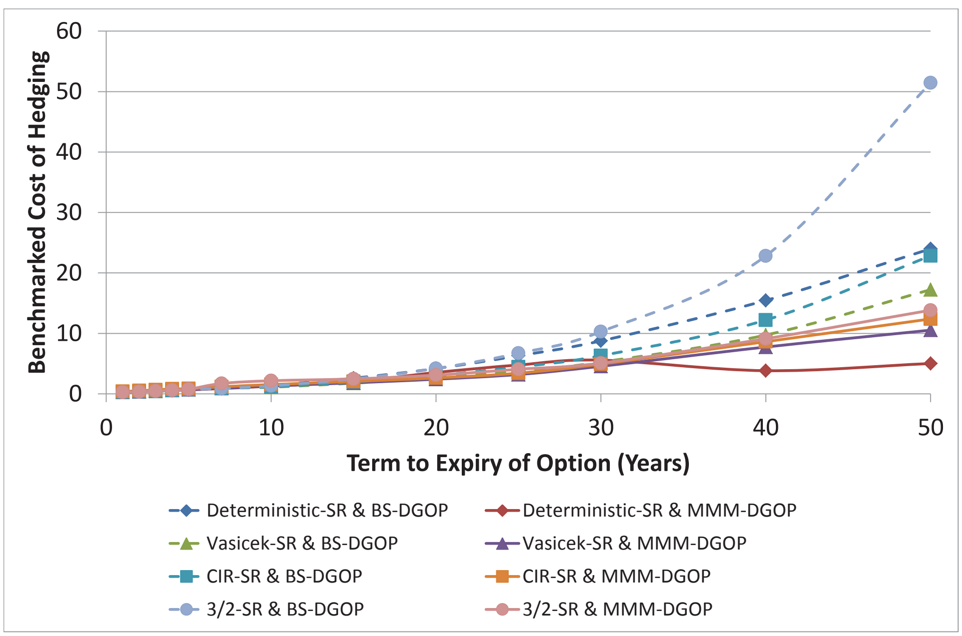


Table 9.1: Percentile costs of hedging put options under a deterministic short rate & Black-Scholes discounted GOP based on US data 1871 - 2012.

Term to Expiry of Put Option	99-th Percentile	95-th Percentile	90-th Percentile	85-th Percentile	80-th Percentile
1Y	0.27397	0.20183	0.18254	0.15424	0.13648
2Y	0.39842	0.29176	0.24874	0.22419	0.20839
3Y	0.44751	0.36636	0.34749	0.30787	0.29391
4Y	0.56623	0.48567	0.43258	0.41133	0.38396
5Y	0.68747	0.60918	0.54247	0.50127	0.46287
7Y	0.94346	0.89324	0.82469	0.73077	0.66711
10Y	1.4014	1.3563	1.2211	1.0961	1.0314
15Y	2.5887	2.4553	2.2317	1.9554	1.7858
20Y	4.184	3.9995	3.5928	3.2131	2.8099
25Y	6.3033	5.8811	5.5006	4.9965	4.1317
30Y	8.76	8.5493	7.9183	7.1568	6.0602
40Y	15.45	15.067	13.972	13.166	12.26
50Y	23.995	23.444	22.838	22.538	21.216

Table 9.2: Percentile costs of hedging put options under a deterministic short rate & MMM discounted GOP based on US data 1871 - 2012.

Term to Expiry of Put Option	99-th Percentile	95-th Percentile	90-th Percentile	85-th Percentile	80-th Percentile
1Y	0.26256	0.20971	0.17928	0.15455	0.13933
2Y	0.38844	0.28898	0.25646	0.22633	0.21036
3Y	0.44883	0.37276	0.34775	0.30562	0.28377
4Y	0.55028	0.50237	0.43545	0.41342	0.37323
5Y	0.6583	0.60488	0.54859	0.50968	0.45911
7Y	0.8764	0.82385	0.77834	0.73707	0.68101
10Y	1.2785	1.1621	1.0663	1.0022	0.95727
15Y	1.9878	1.7257	1.4987	1.3902	1.3662
20Y	3.5324	2.1929	1.7767	1.5438	1.4853
25Y	4.7702	2.117	1.9963	1.8997	1.8677
30Y	5.5783	2.6459	2.5039	2.4681	2.2429
40Y	3.8298	3.6076	3.5365	2.1097	1.3734
50Y	5.0172	3.217	2.3311	1.8324	1.2442

Table 9.3: Percentile costs of hedging put options under a Vasicek short rate & Black-Scholes discounted GOP based on US data 1871 - 2012.

Term to Expiry of Put Option	99-th Percentile	95-th Percentile	90-th Percentile	85-th Percentile	80-th Percentile
1Y	0.26209	0.20057	0.18518	0.15539	0.1384
2Y	0.38887	0.29589	0.25043	0.22619	0.2102
3Y	0.42445	0.38483	0.33276	0.30434	0.28799
4Y	0.56939	0.47183	0.42084	0.40336	0.36581
5Y	0.73917	0.5589	0.50985	0.49118	0.45009
7Y	0.9217	0.75279	0.72856	0.69505	0.67038
10Y	1.1019	1.0746	1.0463	1.0302	0.98135
15Y	1.7703	1.7526	1.7253	1.6592	1.6007
20Y	2.6505	2.6003	2.5506	2.4401	2.3668
25Y	3.7675	3.7148	3.6954	3.5111	3.411
30Y	5.2499	5.1718	5.127	4.9874	4.8277
40Y	9.7301	9.6595	9.5639	9.3504	9.0286
50Y	17.235	17.059	16.905	16.788	16.218

Table 9.4: Percentile costs of hedging put options under a Vasicek short rate & MMM discounted GOP based on US data 1871 - 2012.

Term to Expiry of Put Option	99-th Percentile	95-th Percentile	90-th Percentile	85-th Percentile	80-th Percentile
1Y	0.26891	0.21077	0.18833	0.15075	0.13911
2Y	0.34535	0.29305	0.26402	0.23253	0.21133
3Y	0.47236	0.37087	0.35022	0.3161	0.29035
4Y	0.59532	0.49272	0.43524	0.41383	0.3879
5Y	0.70438	0.59369	0.53894	0.50715	0.46685
7Y	0.9257	0.7902	0.75692	0.71941	0.69043
10Y	1.3062	1.089	1.0271	0.99164	0.96337
15Y	1.8235	1.6769	1.595	1.528	1.4794
20Y	2.4199	2.3152	2.2524	2.2031	2.1564
25Y	3.1984	3.153	2.9989	2.8215	2.7167
30Y	4.5578	4.2153	3.9152	3.4983	3.341
40Y	7.7437	6.1303	5.7859	5.2006	4.6009
50Y	10.537	8.7545	7.3966	6.9161	6.0211

Table 9.5: Percentile costs of hedging put options under a CIR short rate & Black-Scholes discounted GOP based on US data 1871 - 2012.

Term to Expiry of Put Option	99-th Percentile	95-th Percentile	90-th Percentile	85-th Percentile	80-th Percentile
1Y	0.26361	0.20099	0.18623	0.15548	0.13815
2Y	0.38399	0.29298	0.2529	0.22631	0.20976
3Y	0.42767	0.38364	0.3379	0.30484	0.28674
4Y	0.57874	0.47589	0.42838	0.413	0.37579
5Y	0.7401	0.57269	0.5157	0.50415	0.4559
7Y	0.91691	0.76812	0.74329	0.72131	0.67822
10Y	1.1679	1.1198	1.0954	1.0659	1.0108
15Y	1.9723	1.9333	1.8381	1.7431	1.6647
20Y	3.0453	2.9837	2.9111	2.6859	2.5507
25Y	4.4507	4.36	4.2831	4.0354	3.7726
30Y	6.3147	6.2158	6.0923	5.7914	5.5315
40Y	12.208	12.021	11.877	11.386	10.894
50Y	22.827	22.505	22.362	21.817	20.354

Table 9.6: Percentile costs of hedging put options under a CIR short rate & MMM discounted GOP based on US data 1871 - 2012.

Term to Expiry of Put Option	99-th Percentile	95-th Percentile	90-th Percentile	85-th Percentile	80-th Percentile
1Y	0.38674	0.27452	0.22708	0.1934	0.17269
2Y	0.52285	0.41077	0.35496	0.32561	0.28225
3Y	0.64996	0.55456	0.45693	0.39492	0.36086
4Y	0.81779	0.62385	0.57292	0.53832	0.46281
5Y	0.85147	0.79164	0.68805	0.60938	0.53912
7Y	1.1763	0.98336	0.88747	0.83298	0.7417
10Y	1.5361	1.3964	1.3001	1.131	1.0229
15Y	2.0624	1.9721	1.8385	1.7358	1.6192
20Y	2.6329	2.4387	2.3819	2.3443	2.2961
25Y	3.4261	3.3294	3.1933	3.1383	3.0093
30Y	4.9173	4.5382	4.3104	4.0591	3.8871
40Y	8.6307	6.8582	6.2608	6.0561	5.803
50Y	12.392	10.563	9.1388	8.5686	7.5826

Table 9.7: Percentile costs of hedging put options under a 3/2 short rate & Black-Scholes discounted GOP based on US data 1871 - 2012.

Term to Expiry of Put Option	99-th Percentile	95-th Percentile	90-th Percentile	85-th Percentile	80-th Percentile
1Y	0.2664	0.21034	0.18632	0.15655	0.13641
2Y	0.383	0.29462	0.2604	0.22775	0.21844
3Y	0.44263	0.37739	0.34415	0.32066	0.29338
4Y	0.60574	0.494	0.43825	0.41616	0.3963
5Y	0.7494	0.6122	0.53268	0.51847	0.48492
7Y	0.89428	0.82897	0.80913	0.75714	0.70008
10Y	1.3188	1.2501	1.1929	1.1476	1.0268
15Y	2.4691	2.3606	2.1138	1.8398	1.7279
20Y	4.1914	3.946	3.5957	2.8854	2.6338
25Y	6.715	6.1678	5.6977	4.8608	4.0434
30Y	10.283	9.5079	8.6169	7.4723	5.9803
40Y	22.815	21.099	19.133	17.088	14.239
50Y	51.438	48.154	44.121	39.216	32.847

Table 9.8: Percentile costs of hedging put options under a 3/2 short rate & MMM discounted GOP based on US data 1871 - 2012.

Term to Expiry of Put Option	99-th Percentile	95-th Percentile	90-th Percentile	85-th Percentile	80-th Percentile
1Y	0.26558	0.21062	0.18689	0.15239	0.14139
2Y	0.34809	0.29062	0.26405	0.22823	0.20994
3Y	0.47585	0.39805	0.35755	0.31779	0.29356
4Y	0.60302	0.51428	0.45006	0.43793	0.41451
5Y	0.75283	0.62405	0.56988	0.53439	0.51137
7Y	1.7135	0.84799	0.82889	0.7829	0.75554
10Y	2.1669	1.2285	1.1974	1.1496	1.1172
15Y	2.4737	1.9552	1.8517	1.7445	1.7174
20Y	3.102	2.7326	2.5879	2.4996	2.4446
25Y	4.0677	3.6911	3.5268	3.3924	3.2666
30Y	5.0805	4.7434	4.5609	4.4722	4.2388
40Y	9.1205	7.6594	7.2593	6.944	6.8233
50Y	13.838	11.988	11.422	10.898	9.9248

Chapter 10

Conclusion

This thesis has systematically applied the Benchmark Approach to pricing and hedging long-dated derivatives under well studied models for stochastic interest rates and stochastic index volatility.

When the discounted index obeys the standard Black-Scholes dynamics, the real-world and risk neutral probability measures are equivalent and the Radon-Nikodym derivative is a martingale. Thus, we recover in this case the classical risk neutral pricing formulae. Yet the Black-Scholes model and many of its popular relatives are unable to capture several stylised empirical features of the market such as the leverage effect and the volatility hump of cap term structures.

The advantage of the Benchmark Approach is that equivalence of the real-world measure to the risk neutral measure is not required. Therefore, a richer class of models can be explored, which can reflect more realistically the stylised facts of the market, in particular, long-term properties.

The thesis has developed original analytic and semi-analytic pricing formulae for zero-coupon bonds, options on the index, options on zero-coupon bonds, caps and swaptions under several market models with stochastic interest rates and stochastic volatility[1]. The models each provide a consistent framework for pricing and hedging caps and swaptions. Furthermore, the models are able to capture important features of the long-term behaviour of equity and interest rate derivatives which allows less-expensive and efficient hedging of long-dated equity and interest rate derivatives. It has been demonstrated that the minimal market model for the discounted index provides significantly cheaper hedge strategies across interest rate derivatives and equity index options. Among the short rate models, the 3/2 is able to capture the hump-shaped volatility feature at short maturities but does not appear to provide any advantage in hedging long-dated derivatives over the Vasicek and CIR short rate models.

[1] The results for interest rate derivatives have been published in Fergusson and Platen [2014a] and those for equity index options have been reported in Fergusson and Platen [2015a].

There is scope to generalise the market models in this thesis in several ways. Firstly, the market model can be extended to include foreign currencies which would allow exotic cross-currency derivatives to be priced and hedged. The works of Platen and Heath [2010] and Baldeaux et al. [2013] on the multi-currency minimal market model have trodden such a path. Secondly, the discounted index model can be enhanced by incorporating a random intrinsic time process, which would have the effect of reflecting better the short-term behaviour of volatility. An example of such work is that of Baldeaux et al. [2013] which has incorporated random time scaling. Thirdly, the short rate models in this thesis can be generalised by making the parameters time-dependent and correlating their uncertainty with that of the index, which would permit better calibration of the market model to the interest rate term structure and volatility surfaces.

Appendix A

Proofs of Results on the Vasicek Model

Here we present proofs of results in Section 3.2.3.

Proof.[of Lemma 3.2.4] Integrating the SDE (3.2.3) gives

$$r_t = r_s + \int_s^t \kappa(\bar{r} - r_u)du + \int_s^t \sigma dZ_u \tag{A.1}$$

and taking expectations conditioned on r_s gives

$$m_s(t) = r_s + \int_s^t \kappa(\bar{r} - m_s(u))du. \tag{A.2}$$

This can be written as a first order ordinary differential equation in $m_s(t)$

$$m_s(t)' = \kappa(\bar{r} - m_s(t)) \tag{A.3}$$

with initial condition $m_s(s) = r_s$, the solution of which is straightforward. Now the SDE of r_t^2 is, by Ito's Lemma,

$$dr_t^2 = (\sigma^2 + 2\kappa\bar{r}r_t - 2\kappa r_t^2)dt + 2\sigma r_t dZ_t \tag{A.4}$$

and integrating this SDE gives

$$r_t^2 = r_s^2 + \int_s^t (\sigma^2 + 2\kappa\bar{r}r_u - 2\kappa r_u^2)du + \int_s^t 2\sigma r_u dZ_u. \tag{A.5}$$

Taking expectations conditioned on r_s, and defining $m_s^{(2)}(t) = \mathrm{E}(r_t^2|\mathcal{A}_s)$, gives

$$m_s^{(2)}(s) = r_s^2 + \int_s^t (\sigma^2 + 2\kappa\bar{r}m_s(u) - 2\kappa m_s^{(2)}(u))du \tag{A.6}$$

from which we have the ordinary differential equation

$$m_s^{(2)\prime}(t) = \sigma^2 + 2\kappa\bar{r}m_s(t) - 2\kappa m_s^{(2)}(t). \tag{A.7}$$

Multiplying both sides by $\exp(2\kappa t)$ and rearranging gives

$$\begin{aligned}
\frac{d}{dt}\left(\exp(2\kappa t)m_t^{(2)}\right) &= \sigma^2\exp(2\kappa t) + 2\kappa\bar{r}m_s(t)\exp(2\kappa t) && \text{(A.8)}\\
&= \sigma^2\exp(2\kappa t) + 2\kappa\bar{r}\exp(2\kappa t)\left(\kappa\bar{r}B(s,t) + r_s(1-\kappa B(s,t))\right) && \text{(A.9)}\\
&= (\sigma^2 + 2\kappa\bar{r}r_s)\exp(2\kappa t) + 2\kappa^2\bar{r}(\bar{r}-r_s)\exp(2\kappa t)B(s,t).
\end{aligned}$$

We note that

$$\begin{aligned}
\int_s^t \exp(2\kappa u)B(s,u)du &= \frac{1}{2}\exp(2\kappa t)B(s,t)^2 && \text{(A.10)}\\
\int_s^t \exp(2\kappa u)du &= \frac{1}{2}\exp(2\kappa t)(2B(s,t) - \kappa B(s,t)^2).
\end{aligned}$$

Therefore, integrating both sides of (A.8) from s to t gives

$$\begin{aligned}
\exp(2\kappa t)m_s^{(2)}(t) &= \exp(2\kappa s)r_t^2 + (\sigma^2 + 2\kappa\bar{r}r_s)\frac{1}{2}\exp(2\kappa t)(2B(s,t) - \kappa B(s,t)^2) && \text{(A.11)}\\
&+ 2\kappa^2\bar{r}(\bar{r}-r_s)\frac{1}{2}\exp(2\kappa t)B(s,t)^2\\
&= \exp(2\kappa s)r_t^2 + (\sigma^2 + 2\kappa\bar{r}r_s)\exp(2\kappa t)B(s,t)\\
&+ \frac{1}{2}\exp(2\kappa t)B(s,t)^2\left(2\kappa^2\bar{r}(\bar{r}-r_s) - \kappa(\sigma^2 + 2\kappa\bar{r}r_s)\right)\\
&= \exp(2\kappa s)r_t^2 + (\sigma^2 + 2\kappa\bar{r}r_s)\exp(2\kappa t)B(s,t)\\
&+ \frac{1}{2}\exp(2\kappa t)B(s,t)^2\left(2\kappa^2\bar{r}^2 - \kappa\sigma^2 - 4\kappa^2\bar{r}r_s\right).
\end{aligned}$$

and dividing both sides by $\exp(2\kappa t)$ gives

$$\begin{aligned}
m_s^{(2)}(t) &= r_s^2\exp(-2\kappa(t-s)) + (\sigma^2 + 2\kappa\bar{r}r_s)B(s,t) && \text{(A.12)}\\
&+ \frac{1}{2}B(s,t)^2\left(2\kappa^2\bar{r}^2 - \kappa\sigma^2 - 4\kappa^2\bar{r}r_s\right).
\end{aligned}$$

The variance is computed as $v_s(t) = m_s^{(2)}(t) - (m_s(t))^2$, that is

$$\begin{aligned} v_s(t) &= r_s^2 \exp(-2\kappa(t-s)) + (\sigma^2 + 2\kappa\bar{r}r_s)B(s,t) \\ &+ \frac{1}{2}B(s,t)^2\Big(2\kappa^2\bar{r}^2 - \kappa\sigma^2 - 4\kappa^2\bar{r}r_s\Big) \\ &- \Big(\bar{r}\kappa B(s,t) + r_s(1-\kappa B(s,t))\Big)^2 \\ &= r_s^2 \exp(-2\kappa(t-s)) + (\sigma^2 + 2\kappa\bar{r}r_s)B(s,t) \\ &+ \frac{1}{2}B(s,t)^2\Big(2\kappa^2\bar{r}^2 - \kappa\sigma^2 - 4\kappa^2\bar{r}r_s\Big) \\ &- \bar{r}^2\kappa^2 B(s,t)^2 - r_s^2(1-\kappa B(s,t))^2 - 2\kappa\bar{r}r_s(B(s,t) - \kappa B(s,t)^2) \\ &= \sigma^2 B(s,t) - \frac{1}{2}\kappa\sigma^2 B(s,t)^2 \end{aligned} \tag{A.13}$$

as required. **Q.E.D.**

Proof.[of Theorem 3.2.6] Differentiating (3.2.17) with respect to $\bar{r}$ we have

$$\begin{aligned} \frac{\partial}{\partial\bar{r}}\ell(\bar{r},\kappa,\sigma) &= -\frac{1}{2}\sum_{i=1}^{n}\frac{2(r_{t_i} - m_{t_{i-1}}(t_i))}{v_{t_{i-1}}(t_i)} \times -\frac{\partial m_{t_i}}{\partial\bar{r}} \\ &= \kappa\sum_{i=1}^{n}\frac{(r_{t_i} - m_{t_{i-1}}(t_i))B(t_{i-1},t_i)}{v_{t_{i-1}}(t_i)}, \end{aligned} \tag{A.14}$$

where we have used the fact that $\frac{\partial m_{t_{i-1}}(t_i)}{\partial\bar{r}} = \kappa B(t_{i-1},t_i)$. Equating (A.14) to zero gives the equation

$$\sum_{i=1}^{n}(r_{t_i} - m_{t_{i-1}}(t_i)) = 0 \tag{A.15}$$

which, using (3.2.13), is equivalent to

$$\sum_{i=1}^{n}(r_{t_i} - \bar{r}) = \sum_{i=1}^{n}(r_{t_{i-1}} - \bar{r})(1 - \kappa B(t_{i-1},t_i)). \tag{A.16}$$

Since the sampling times t_i are equidistant we can solve for κ, giving the solution

$$\kappa = \frac{1}{\Delta}\log\frac{\sum_{i=1}^{n}(r_{t_{i-1}} - \bar{r})}{\sum_{i=1}^{n}(r_{t_i} - \bar{r})}. \tag{A.17}$$

Differentiating (3.2.17) with respect to σ we have

$$
\begin{aligned}
\frac{\partial}{\partial\sigma}\ell(\bar{r},\kappa,\sigma) &= -\frac{1}{2}\sum_{i=1}^{n}\left(\frac{1}{v_{t_{i-1}}(t_i)} - \frac{(r_{t_i}-m_{t_{i-1}}(t_i))^2}{v_{t_{i-1}}(t_i)^2}\right)\frac{\partial v_{t_{i-1}}(t_i)}{\partial\sigma} \qquad \text{(A.18)}\\
&= -\sum_{i=1}^{n}\left(\frac{1}{v_{t_{i-1}}(t_i)} - \frac{(r_{t_i}-m_{t_{i-1}}(t_i))^2}{v_{t_{i-1}}(t_i)^2}\right)\frac{v_{t_{i-1}}(t_i)}{\sigma}\\
&= -\frac{1}{\sigma}\sum_{i=1}^{n}\left(1 - \frac{(r_{t_i}-m_{t_{i-1}}(t_i))^2}{v_{t_{i-1}}(t_i)}\right),
\end{aligned}
$$

where we have used the fact that $\frac{\partial v_{t_{i-1}}(t_i)}{\partial\sigma} = 2\frac{v_{t_{i-1}}(t_i)}{\sigma}$. Equating (A.18) to zero gives the equation

$$
\sum_{i=1}^{n}\frac{(r_{t_i}-m_{t_{i-1}}(t_i))^2}{v_{t_{i-1}}(t_i)} = \sum_{i=1}^{n} 1 \qquad \text{(A.19)}
$$

which, using the equation for $v_{t_{i-1}}(t_i)$ in (3.2.13), is equivalent to

$$
\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}\frac{(r_{t_i}-m_{t_{i-1}}(t_i))^2}{B(t_{i-1},t_i)(1-\frac{1}{2}\kappa B(t_{i-1},t_i))}. \qquad \text{(A.20)}
$$

Since the sampling times t_i are equidistant we can simplify the equation for σ^2 to

$$
\sigma^2 = \frac{1}{n\beta(1-\frac{1}{2}\kappa\beta)}\sum_{i=1}^{n}(r_{t_i}-m_{t_{i-1}}(t_i))^2. \qquad \text{(A.21)}
$$

Differentiating (3.2.17) with respect to κ we have

$$
\begin{aligned}
\frac{\partial}{\partial\kappa}\ell(\bar{r},\kappa,\sigma) &= -\frac{1}{2}\sum_{i=1}^{n}\frac{1}{v_{t_i}}\frac{\partial v_{t_i}}{\partial\kappa} + \frac{\partial}{\partial\kappa}\left(\frac{(r_{t_i}-m_{t_{i-1}}(t_i))^2}{v_{t_{i-1}}(t_i)}\right) \qquad \text{(A.22)}\\
&= -\frac{1}{2}\sum_{i=1}^{n}\left\{\frac{1}{v_{t_{i-1}}(t_i)}\frac{\partial v_{t_{i-1}}(t_i)}{\partial\kappa}\right.\\
&+ \frac{2(r_{t_i}-m_{t_{i-1}}(t_i))}{v_{t_{i-1}}(t_i)} \times \left(-\frac{\partial m_{t_{i-1}}(t_i)}{\partial\kappa}\right)\\
&\left.- \frac{(r_{t_i}-m_{t_{i-1}}(t_i))^2}{v_{t_{i-1}}(t_i)^2} \times \frac{\partial v_{t_{i-1}}(t_i)}{\partial\kappa}\right\}\\
&= -\frac{1}{2}\sum_{i=1}^{n}\frac{1}{v_{t_{i-1}}(t_i)}\frac{\partial v_{t_{i-1}}(t_i)}{\partial\kappa}\left(1 - \frac{(r_{t_i}-m_{t_{i-1}}(t_i))^2}{v_{t_{i-1}}(t_i)}\right)\\
&- \frac{2(r_{t_i}-m_{t_{i-1}}(t_i))}{v_{t_{i-1}}(t_i)}\frac{\partial m_{t_{i-1}}(t_i)}{\partial\kappa}.
\end{aligned}
$$

To simplify (A.22) we determine expressions for $\frac{\partial v_{t_{i-1}}(t_i)}{\partial\kappa}$ and $\frac{\partial m_{t_{i-1}}(t_i)}{\partial\kappa}$. Firstly,

$$\begin{aligned}\frac{\partial v_{t_i}}{\partial\kappa} &= \frac{\partial}{\partial\kappa}\Big\{\sigma^2\Big(B(t_{i-1},t_i)-\frac{1}{2}\kappa B(t_{i-1},t_i)^2\Big)\Big\} \qquad (A.23)\\ &= \sigma^2\Big(\frac{\partial B(t_{i-1},t_i)}{\partial\kappa}-\frac{1}{2}B(t_{i-1},t_i)^2-\kappa B(t_{i-1},t_i)\frac{\partial B(t_{i-1},t_i)}{\partial\kappa}\Big).\end{aligned}$$

We note that

$$\begin{aligned}\frac{\partial(\kappa B(t_{i-1},t_i))}{\partial\kappa} &= \frac{\partial(1-\exp(-\kappa(t_i-t_{i-1})))}{\partial\kappa} \qquad (A.24)\\ &= (t_i-t_{i-1})\exp(-\kappa(t_i-t_{i-1}))\\ &= (t_i-t_{i-1})(1-\kappa B(t_{i-1},t_i))\end{aligned}$$

and, therefore,

$$\begin{aligned}\frac{\partial B(t_{i-1},t_i)}{\partial\kappa} &= \frac{1}{\kappa}\Big(\frac{\partial(\kappa B(t_{i-1},t_i))}{\partial\kappa}-B(t_{i-1},t_i)\Big) \qquad (A.25)\\ &= \frac{1}{\kappa}\Big((t_i-t_{i-1})(1-\kappa B(t_{i-1},t_i))-B(t_{i-1},t_i)\Big).\end{aligned}$$

Hence (A.23) becomes

$$\begin{aligned}\frac{\partial v_{t_i}}{\partial\kappa} &= \sigma^2\Big(-\frac{1}{2}B(t_{i-1},t_i)^2+(1-\kappa B(t_{i-1},t_i))\frac{\partial B(t_{i-1},t_i)}{\partial\kappa}\Big) \qquad (A.26)\\ &= \sigma^2\Big\{-\frac{1}{2}B(t_{i-1},t_i)^2\\ &+(1-\kappa B(t_{i-1},t_i))\frac{1}{\kappa}\Big((t_i-t_{i-1})(1-\kappa B(t_{i-1},t_i))-B(t_{i-1},t_i)\Big)\Big\}\\ &= \sigma^2\Big(-\frac{1}{2}B(t_{i-1},t_i)^2\\ &+\frac{t_i-t_{i-1}}{\kappa}(1-\kappa B(t_{i-1},t_i))^2-\frac{1}{\kappa}B(t_{i-1},t_i)(1-\kappa B(t_{i-1},t_i))\Big)\\ &= \sigma^2\Big(-\frac{1}{2}B(t_{i-1},t_i)^2\\ &+\frac{t_i-t_{i-1}}{\kappa}(1-\kappa B(t_{i-1},t_i))^2-\frac{1}{\kappa}B(t_{i-1},t_i)+B(t_{i-1},t_i)^2)\Big)\\ &= \sigma^2\Big(\frac{t_i-t_{i-1}}{\kappa}(1-\kappa B(t_{i-1},t_i))^2-\frac{1}{\kappa}B(t_{i-1},t_i)+\frac{1}{2}B(t_{i-1},t_i)^2)\Big)\\ &= \sigma^2\frac{t_i-t_{i-1}}{\kappa}(1-\kappa B(t_{i-1},t_i))^2-\frac{\sigma^2}{\kappa}\Big(B(t_{i-1},t_i)-\frac{1}{2}\kappa B(t_{i-1},t_i)^2\Big)\\ &= \sigma^2\frac{t_i-t_{i-1}}{\kappa}(1-\kappa B(t_{i-1},t_i))^2-\frac{1}{\kappa}v_{t_i}.\end{aligned}$$

Secondly,

$$\begin{aligned}\frac{\partial m_{t_{i-1}}(t_i)}{\partial \kappa} &= \frac{\partial}{\partial \kappa}\Big(r_s + (\bar{r} - r_s)\kappa B(t_{i-1}, t_i)\Big) \\ &= (\bar{r} - r_s)(t_i - t_{i-1})(1 - \kappa B(t_{i-1}, t_i)) \\ &= -(t_i - t_{i-1})(m_{t_{i-1}}(t_i) - \bar{r}).\end{aligned} \tag{A.27}$$

Substituting (A.26) and (A.27) into (A.22) gives

$$\begin{aligned}\frac{\partial}{\partial \kappa}\ell(\bar{r}, \kappa, \sigma) = -\frac{1}{2}\sum_{i=1}^{n}\Bigg\{&\frac{1}{v_{t_{i-1}}(t_i)}\Big(\sigma^2\frac{t_i - t_{i-1}}{\kappa}(1 - \kappa B(t_{i-1}, t_i))^2 - \frac{1}{\kappa}v_{t_{i-1}}(t_i)\Big) \\ &\times\Big(1 - \frac{(r_{t_i} - m_{t_{i-1}}(t_i))^2}{v_{t_{i-1}}(t_i)}\Big) \\ &+ \frac{2(r_{t_i} - m_{t_{i-1}}(t_i))}{v_{t_{i-1}}(t_i)}(t_i - t_{i-1})(m_{t_{i-1}}(t_i) - \bar{r})\Bigg\}.\end{aligned} \tag{A.28}$$

If the right hand side of (A.18) is zero, then (A.28) simplifies to

$$\begin{aligned}\frac{\partial}{\partial \kappa}\ell(\bar{r}, \kappa, \sigma) &= -\frac{1}{2}\sum_{i=1}^{n}\frac{2(r_{t_i} -_{t_{i-1}}(t_i))}{v_{t_{i-1}}(t_i)}(t_i - t_{i-1})(m_{t_{i-1}}(t_i) - \bar{r}) \\ &= \Delta\frac{1}{\sigma^2(\beta + \frac{1}{2}\kappa\beta^2)}\sum_{i=1}^{n}(r_{t_i} - m_{t_{i-1}}(t_i))(m_{t_{i-1}}(t_i) - \bar{r}).\end{aligned} \tag{A.29}$$

Hence $\frac{\partial}{\partial \kappa}\ell(\bar{r}, \kappa, \sigma) = 0$ is equivalent to

$$\begin{aligned}0 &= \sum_{i=1}^{n}(r_{t_i} - m_{t_{i-1}}(t_i))(m_{t_{i-1}}(t_i) - \bar{r}) \\ &= \sum_{i=1}^{n}(r_{t_i} - \bar{r} - (1 - \kappa\beta)(r_{t_{i-1}} - \bar{r}))(r_{t_{i-1}} - \bar{r})(1 - \kappa\beta)\end{aligned} \tag{A.30}$$

from which we have

$$1 - \kappa\beta = \frac{\sum_{i=1}^{n}(r_{t_i} - \bar{r})(r_{t_{i-1}} - \bar{r})}{\sum_{i=1}^{n}(r_{t_{i-1}} - \bar{r})^2}. \tag{A.31}$$

Thus in addition to (A.16), we have an expression for $1 - \kappa\beta$ in (A.31) and this allows us to solve explicitly for $\bar{r}$. **Q.E.D.**

Proof.[of Theorem 3.2.7] The log-likelihood function in (3.2.17) can be rewritten as

$$\ell(\bar{r}, \kappa, \sigma) = -\frac{1}{2}\sum_{i=1}^{n}\ell_i(m_i, v_i) \tag{A.32}$$

where for $i = 1, 2, \ldots, n$,

$$\ell_i(m_i, v_i) = \log(2\pi v_i) + \frac{(r_{t_i} - m_i)^2}{v_i} \tag{A.33}$$

and where

$$\begin{aligned} m_i \equiv m_{t_{i-1}}(t_i) &= \bar{r}\kappa B(t_{i-1}, t_i) + r_{t_{i-1}}(1 - \kappa B(t_{i-1}, t_i)) \\ v_i \equiv v_{t_{i-1}}(t_i) &= \sigma^2\Big(B(t_{i-1}, t_i) - \frac{1}{2}\kappa B(t_{i-1}, t_i)^2 \Big), \end{aligned} \tag{A.34}$$

with $B(s, t)$ as in (3.2.14). Rewriting (A.34) explicitly in $\bar{r}$, κ and σ we have

$$\begin{aligned} m_i &= \bar{r}(1 - \exp(-\kappa\Delta)) + r_{t_{i-1}} \exp(-\kappa\Delta) \\ v_i &= \frac{\sigma^2}{2\kappa}\kappa B(t_{i-1}, t_i)\Big(2 - \kappa B(t_{i-1}, t_i) \Big) \\ &= \frac{\sigma^2}{2\kappa}(1 - \exp(-\kappa\Delta))(1 + \exp(-\kappa\Delta)) \\ &= \frac{\sigma^2}{2\kappa}(1 - \exp(-2\kappa\Delta)) \equiv v. \end{aligned} \tag{A.35}$$

Our aim is to compute the second order partial derivatives of $\ell(\bar{r}, \kappa, \sigma)$ with respect of $\bar{r}$, κ and σ by employing the chain rule

$$\begin{aligned} \frac{\partial \ell(\bar{r}, \kappa, \sigma)}{\partial \bar{r}} &= -\frac{1}{2}\sum_{i=1}^{n} \frac{\partial \ell_i(m_i, v_i)}{\partial m_i}\frac{\partial m_i}{\partial \bar{r}} \\ \frac{\partial \ell(\bar{r}, \kappa, \sigma)}{\partial \kappa} &= -\frac{1}{2}\sum_{i=1}^{n} \frac{\partial \ell_i(m_i, v_i)}{\partial m_i}\frac{\partial m_i}{\partial \kappa} + \frac{\partial \ell_i(m_i, v_i)}{\partial v_i}\frac{\partial v_i}{\partial \kappa} \\ \frac{\partial \ell(\bar{r}, \kappa, \sigma)}{\partial \sigma} &= -\frac{1}{2}\sum_{i=1}^{n} \frac{\partial \ell_i(m_i, v_i)}{\partial v_i}\frac{\partial v_i}{\partial \sigma}. \end{aligned} \tag{A.36}$$

Then, applying the chain rule again, the second order partial derivatives can be written as

$$\frac{\partial^2 \ell(\bar{r}, \kappa, \sigma)}{\partial \bar{r}^2} = -\frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2 \ell_i(m_i, v_i)}{\partial m_i^2}\left(\frac{\partial m_i}{\partial \bar{r}}\right)^2 \tag{A.37}$$

$$\begin{aligned} \frac{\partial^2 \ell(\bar{r}, \kappa, \sigma)}{\partial \kappa^2} &= -\frac{1}{2}\sum_{i=1}^{n} \left\{ \frac{\partial^2 \ell_i(m_i, v_i)}{\partial m_i^2}\left(\frac{\partial m_i}{\partial \kappa}\right)^2 + 2\frac{\partial^2 \ell_i(m_i, v_i)}{\partial m_i \partial v_i}\frac{\partial m_i}{\partial \kappa}\frac{\partial v_i}{\partial \kappa} \right. \\ &\quad \left. + \frac{\partial^2 \ell_i(m_i, v_i)}{\partial v_i^2}\left(\frac{\partial v_i}{\partial \kappa}\right)^2 \right\} \end{aligned} \tag{A.38}$$

$$\frac{\partial^2 \ell(\bar{r},\kappa,\sigma)}{\partial \sigma^2} = -\frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2 \ell_i(m_i,v_i)}{\partial v_i^2}\left(\frac{\partial v_i}{\partial \sigma}\right)^2 \tag{A.39}$$

$$\frac{\partial^2 \ell(\bar{r},\kappa,\sigma)}{\partial \bar{r}\partial \kappa} = -\frac{1}{2}\sum_{i=1}^{n} \left\{\frac{\partial^2 \ell_i(m_i,v_i)}{\partial m_i^2}\frac{\partial m_i}{\partial \bar{r}}\frac{\partial m_i}{\partial \kappa} + \frac{\partial^2 \ell_i(m_i,v_i)}{\partial m_i \partial v_i}\frac{\partial m_i}{\partial \bar{r}}\frac{\partial v_i}{\partial \kappa}\right\} \tag{A.40}$$

$$\frac{\partial^2 \ell(\bar{r},\kappa,\sigma)}{\partial \bar{r}\partial \sigma} = -\frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2 \ell_i(m_i,v_i)}{\partial m_i \partial v_i}\frac{\partial m_i}{\partial \bar{r}}\frac{\partial v_i}{\partial \sigma} \tag{A.41}$$

$$\frac{\partial^2 \ell(\bar{r},\kappa,\sigma)}{\partial \sigma\partial \kappa} = -\frac{1}{2}\sum_{i=1}^{n} \left\{\frac{\partial^2 \ell_i(m_i,v_i)}{\partial v_i^2}\frac{\partial v_i}{\partial \sigma}\frac{\partial v_i}{\partial \kappa} + \frac{\partial^2 \ell_i(m_i,v_i)}{\partial m_i \partial v_i}\frac{\partial m_i}{\partial \kappa}\frac{\partial v_i}{\partial \sigma}\right\}. \tag{A.42}$$

From (A.33) we have the partial derivatives

$$\begin{aligned}
\frac{\partial \ell_i}{\partial m_i} &= -2\frac{r_{t_i} - m_i}{v_i} \qquad \text{(A.43)}\\
\frac{\partial \ell_i}{\partial v_i} &= \frac{1}{v_i} - \frac{(r_{t_i} - m_i)^2}{v_i^2}\\
\frac{\partial^2 \ell_i}{\partial m_i^2} &= \frac{2}{v_i} \equiv L_{mm}\\
\frac{\partial^2 \ell_i}{\partial m_i \partial v_i} &= 2\frac{r_{t_i} - m_i}{v_i^2}\\
\frac{\partial^2 \ell_i}{\partial v_i^2} &= -\frac{1}{v_i^2} + 2\frac{(r_{t_i} - m_i)^2}{v_i^3}.
\end{aligned}$$

From (A.35) we have the first order partial derivatives

$$\begin{aligned}
\frac{\partial m_i}{\partial \bar{r}} &= 1 - \exp(-\kappa\Delta) \equiv m_{\bar{r}} \qquad \text{(A.44)}\\
\frac{\partial m_i}{\partial \kappa} &= \Delta\ \exp(-\kappa\Delta)\,(\bar{r} - r_{t_{i-1}})\\
\frac{\partial v_i}{\partial \kappa} &= -\frac{\sigma^2}{2\kappa^2}(1 - \exp(-2\kappa\Delta)) + \frac{\sigma^2}{2\kappa}\,2\Delta\ \exp(-2\kappa\Delta)\\
&= -\frac{1}{\kappa}v_i - 2\Delta\, v_i + \frac{\sigma^2}{\kappa}\Delta \equiv v_\kappa\\
\frac{\partial v_i}{\partial \sigma} &= 2\frac{\sigma}{2\kappa}(1 - \exp(-2\kappa\Delta)) = \frac{2}{\sigma}v_i \equiv v_\sigma.
\end{aligned}$$

The following six summations, most of which emanate from the maximum likelihood conditions, namely each equation in (A.36) equalling zero, will prove useful in simplifying the formulae for the second order partial derivatives given in (A.37) to (A.42).

Firstly,

$$\sum_{i=1}^{n} \frac{\partial^2 \ell_i(m_i, v_i)}{\partial m_i \partial v_i} = 0 \tag{A.45}$$

stems from the maximum likelihood equation $\frac{\partial \ell(\bar{r},\kappa,\sigma)}{\partial \bar{r}} = 0$.

Secondly,

$$\sum_{i=1}^{n} \frac{\partial^2 \ell_i(m_i, v_i)}{\partial v_i^2} = \frac{n}{v^2} \tag{A.46}$$

stems from the maximum likelihood equation $\frac{\partial \ell(\bar{r},\kappa,\sigma)}{\partial \sigma} = 0$.

Thirdly,

$$\sum_{i=1}^{n} \frac{\partial \ell_i(m_i, v_i)}{\partial v_i} = 0 \tag{A.47}$$

also stems from the maximum likelihood equation $\frac{\partial \ell(\bar{r},\kappa,\sigma)}{\partial \sigma} = 0$.

Fourthly,

$$\sum_{i=1}^{n} \frac{\partial^2 \ell_i(m_i, v_i)}{\partial m_i \partial v_i} \frac{\partial m_i}{\partial \kappa} = 0 \tag{A.48}$$

stems from the maximum likelihood equations $\frac{\partial \ell(\bar{r},\kappa,\sigma)}{\partial \sigma} = 0$ and $\frac{\partial \ell(\bar{r},\kappa,\sigma)}{\partial \kappa} = 0$.

Fifthly,

$$\sum_{i=1}^{n} \frac{\partial m_i}{\partial \kappa} = \Delta \, \exp(-\kappa\Delta) \sum (\bar{r} - r_{t_{i-1}}) = n\Delta \, \exp(-\kappa\Delta)(\bar{r} - S_0), \tag{A.49}$$

where $S_0 = \sum r_{t_{i-1}}$.

Sixthly,

$$\sum_{i=1}^{n} \left(\frac{\partial m_i}{\partial \kappa}\right)^2 = \big(\Delta \, \exp(-\kappa\Delta)\big)^2 \sum (\bar{r} - r_{t_{i-1}})^2 = n\big(\Delta \, \exp(-\kappa\Delta)\big)^2 (\bar{r}^2 - 2\bar{r}S_0 + S_{00}). \tag{A.50}$$

where $S_{00} = \sum r_{t_{i-1}}^2$.

In the light of these summations we now simplify the formulae for the second order partial derivatives given in (A.37) to (A.42).

Using (A.43) and (A.44), the second order partial derivative (A.37) simplifies as follows

$$\begin{aligned}\frac{\partial^2 \ell(\bar{r},\kappa,\sigma)}{\partial \bar{r}^2} &= -\frac{1}{2}\sum_{i=1}^{n} L_{mm} m_{\bar{r}}^2 \\ &= -\frac{1}{2} n\, L_{mm} m_{\bar{r}}^2 \\ &= -\frac{n\kappa^2\beta}{\sigma^2(1-\frac{1}{2}\kappa\beta)}.\end{aligned} \tag{A.51}$$

Using (A.43) and (A.44), the second order partial derivative (A.38) simplifies as follows

$$\begin{aligned}\frac{\partial^2 \ell(\bar{r},\kappa,\sigma)}{\partial \kappa^2} &= -\frac{1}{2}\sum_{i=1}^{n}\left\{ L_{mm}\left(\frac{\partial m_i}{\partial \kappa}\right)^2 + 2\frac{\partial^2 \ell_i(m_i,v_i)}{\partial m_i \partial v_i}\frac{\partial m_i}{\partial \kappa} v_\kappa \right. \\ &\quad \left. + \frac{\partial^2 \ell_i(m_i,v_i)}{\partial v_i^2} v_\kappa^2 \right\} \\ &= -\frac{n}{v}\big(\Delta\, \exp(-\kappa\Delta)\big)^2(\bar{r}^2 - 2\bar{r}S_0 + S_{00}) - \frac{n}{2v^2}v_\kappa^2.\end{aligned} \tag{A.52}$$

Using (A.43) and (A.44), the second order partial derivative (A.39) simplifies as follows

$$\begin{aligned}\frac{\partial^2 \ell(\bar{r},\kappa,\sigma)}{\partial \sigma^2} &= -\frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2 \ell_i(m_i,v_i)}{\partial v_i^2} v_\sigma^2 \\ &= -\frac{2n}{\sigma^2}.\end{aligned} \tag{A.53}$$

Using (A.43) and (A.44), the second order partial derivative (A.40) simplifies as follows

$$\begin{aligned}\frac{\partial^2 \ell(\bar{r},\kappa,\sigma)}{\partial \bar{r}\partial \kappa} &= -\frac{1}{2}\sum_{i=1}^{n}\left\{ L_{mm} m_{\bar{r}}\frac{\partial m_i}{\partial \kappa} + \frac{\partial^2 \ell_i(m_i,v_i)}{\partial m_i \partial v_i} m_{\bar{r}} v_\kappa \right\} \\ &= -\frac{m_{\bar{r}}}{v}\Delta\, \exp(-\kappa\Delta)(\bar{r} - S_0).\end{aligned} \tag{A.54}$$

Using (A.43) and (A.44), the second order partial derivative (A.41) simplifies as follows

$$\frac{\partial^2 \ell(\bar{r},\kappa,\sigma)}{\partial \bar{r}\partial \sigma} = -\frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2 \ell_i(m_i,v_i)}{\partial m_i \partial v_i} m_{\bar{r}} v_\sigma = 0. \tag{A.55}$$

Using (A.43) and (A.44), the second order partial derivative (A.42) simplifies as follows

$$\begin{aligned}\frac{\partial^2 \ell(\bar{r},\kappa,\sigma)}{\partial \sigma\partial \kappa} &= -\frac{1}{2}\sum_{i=1}^{n}\left\{ \frac{\partial^2 \ell_i(m_i,v_i)}{\partial v_i^2} v_\sigma v_\kappa + \frac{\partial^2 \ell_i(m_i,v_i)}{\partial m_i \partial v_i}\frac{\partial m_i}{\partial \kappa} v_\sigma \right\} \\ &= -\frac{1}{2} v_\sigma v_\kappa \frac{n}{v^2}.\end{aligned} \tag{A.56}$$

The entries in the Fisher Information Matrix are simply the opposite of those in the Hessian matrix. **Q.E.D.**

Appendix B

Proofs of Expectations for the Vasicek Model

Here we present proofs of results in Section 3.2.7.

Proof.[of Lemma 3.2.18] Let us write the random variable Y in the form $Y = \mu + \sigma Z$, where Z is a standard normal random variable and σ is a positive real number. Clearly, $\mathrm{E}(Y) = \mu$ and $\mathrm{Var}(Y) = \sigma^2$. Also $\mathrm{E}(\exp(Y)) = \exp(\mu + \frac{1}{2}\sigma^2)$. Then

$$\begin{aligned}\mathrm{E}(\exp(Y)\mathbf{1}_{Y\le y}) &= \mathrm{E}(\exp(\mu+\sigma Z)\mathbf{1}_{\mu+\sigma Z\le y}) \\ &= \exp(\mu)\mathrm{E}(\exp(\sigma Z)\mathbf{1}_{Z\le(y-\mu)/\sigma})\end{aligned} \tag{B.1}$$

and from Lemma 4.2.8 we have

$$\begin{aligned}\mathrm{E}(\exp(Y)\mathbf{1}_{Y\le y}) &= \exp(\mu)\exp(\frac{1}{2}\sigma^2)\mathrm{E}(\mathbf{1}_{Z\le(y-\mu)/\sigma-\sigma}) \\ &= \mathrm{E}(\exp(Y)) \times \mathrm{E}(\mathbf{1}_{Z\le(y-\mu)/\sigma-\sigma}) \\ &= \mathrm{E}(\exp(Y)) \times \mathrm{E}(\mathbf{1}_{Y\le y-\sigma^2})\end{aligned} \tag{B.2}$$

as required. Also the second equality emerges after applying the relation

$$\mathrm{E}(\exp(Y)\mathbf{1}_{Y>y}) = \mathrm{E}(\exp(Y)(1-\mathbf{1}_{Y\le y})) = \mathrm{E}(\exp(Y)) - \mathrm{E}(\exp(Y)\mathbf{1}_{Y\le y}) \tag{B.3}$$

to the first equality. **Q.E.D.**

Proof.[of Lemma 3.2.19] We let

$$Y_2' = Y_2 - \beta Y_1, \tag{B.4}$$

where $\beta = \mathrm{Cov}(Y_1, Y_2)/\mathrm{Var}(Y_1)$. This allows us to write Y_2 as a linear combination of two uncorrelated random variables Y_1 and Y_2' as follows:

$$Y_2 = \beta Y_1 + Y_2'. \tag{B.5}$$

If $\beta = 0$, then Y_1 and Y_2 are uncorrelated and, because both are normally distributed random variables, are therefore independent which gives, by Lemma 3.2.18, the result. Henceforth we assume $\beta \neq 0$ and we have, by Lemma 3.2.18,

$$\begin{aligned} \mathrm{E}(\exp(Y_1)\mathbf{1}_{Y_2 \le y}) &= \mathrm{E}(\exp(Y_1)\mathbf{1}_{\beta Y_1 + Y_2' \le y}) \\ &= \mathrm{E}(\mathrm{E}(\exp(Y_1)\mathbf{1}_{\beta Y_1 + Y_2' \le y}|Y_2')) \\ &= \mathrm{E}(\mathrm{E}(\exp(Y_1)\mathbf{1}_{Y_1 \le \frac{1}{\beta}(y - Y_2')}|Y_2')) \text{ for } \beta > 0. \end{aligned} \tag{B.6}$$

We remark that for $\beta < 0$ the inequality is reversed in the indicator function above, yet an identical result to that which follows is obtained. We apply Lemma 3.2.18 to evaluate the inner expectation, giving

$$\begin{aligned} &\mathrm{E}(\mathrm{E}(\exp(Y_1)\mathbf{1}_{Y_1 \le \frac{1}{\beta}(y - Y_2')}|Y_2')) \\ &= \mathrm{E}(\mathrm{E}(\exp(Y_1))\mathrm{E}(\mathbf{1}_{Y_1 \le \frac{1}{\beta}(y - Y_2') - \mathrm{Var}(Y_1)}|Y_2')) \\ &= \mathrm{E}(\exp(Y_1))\mathrm{E}(\mathrm{E}(\mathbf{1}_{\beta Y_1 + Y_2' \le y - \beta \mathrm{Var}(Y_1)}|Y_2')) \\ &= \mathrm{E}(\exp(Y_1))\mathrm{E}(\mathbf{1}_{Y_2 \le y - \beta \mathrm{Var}(Y_1)}) \\ &= \mathrm{E}(\exp(Y_1))\mathrm{E}(\mathbf{1}_{Y_2 \le y - \mathrm{Cov}(Y_1, Y_2)}) \end{aligned} \tag{B.7}$$

which is the first equality. The second equality also follows similarly. **Q.E.D.**

Proof.[of Lemma 3.2.21] Because $\int_0^t r_s ds$ is normally distributed we have

$$\begin{aligned} G_{\bar{T}}(T) &= \mathrm{E}\left(\exp\left(-\int_T^{\bar{T}} r_s ds\right)\middle|\mathcal{A}_T\right) \\ &= \mathrm{E}\left(\exp\left(-\int_t^{\bar{T}} r_s ds + \int_t^T r_s ds\right)\middle|\mathcal{A}_T\right) \\ &= \exp\left(\mathrm{E}\left(-\int_t^{\bar{T}} r_s ds + \int_t^T r_s ds\middle|\mathcal{A}_T\right)\right. \\ &\quad \left. + \frac{1}{2}\mathrm{Var}\left(\int_t^{\bar{T}} r_s ds\middle|\mathcal{A}_T\right)\right), \end{aligned} \tag{B.8}$$

where we have used Condition 3.2.20, namely that $\mathrm{Var}\left(\int_t^T r_s ds\middle|\mathcal{A}_T\right) = 0$, and the properties of the lognormal distribution. Therefore, the conditional random variable L given the information available at time t is given by

$$L = \log G_{\bar{T}}(T) = -\mathrm{E}\left(\int_t^{\bar{T}} r_s ds\middle|\mathcal{A}_T\right) + \int_t^T r_s ds + \frac{1}{2}\mathrm{Var}\left(\int_t^{\bar{T}} r_s ds\middle|\mathcal{A}_T\right) \tag{B.9}$$

and is normally distributed.

Its expected value is

$$\begin{aligned}\mathrm{E}(L|\mathcal{A}_t) = &-\mathrm{E}\bigg(\int_t^{\bar{T}} r_s ds\bigg|\mathcal{A}_t\bigg) + \mathrm{E}\bigg(\int_t^{T} r_s ds\bigg|\mathcal{A}_t\bigg)\\ &+ \frac{1}{2}\mathrm{E}\bigg(\mathrm{Var}\bigg(\int_t^{\bar{T}} r_s ds\bigg|\mathcal{A}_T\bigg)\bigg|\mathcal{A}_t\bigg),\end{aligned} \tag{B.10}$$

which simplifies to

$$\begin{aligned}\mathrm{E}(L|\mathcal{A}_t) = &-\mathrm{E}\bigg(\int_t^{\bar{T}} r_s ds\bigg|\mathcal{A}_t\bigg) + \mathrm{E}\bigg(\int_t^{T} r_s ds\bigg|\mathcal{A}_t\bigg)\\ &+ \frac{1}{2}\mathrm{Var}\bigg(\int_t^{\bar{T}} r_s ds\bigg|\mathcal{A}_T\bigg),\end{aligned} \tag{B.11}$$

because the variance $\mathrm{Var}\bigg(\int_t^{\bar{T}} r_s ds\bigg|\mathcal{A}_T\bigg)$ is deterministic, as demonstrated by

$$\begin{aligned}\mathrm{Var}\bigg(\int_t^{\bar{T}} r_s ds\bigg|\mathcal{A}_T\bigg) &= \mathrm{Var}\bigg(\int_t^{T} r_s ds + \int_T^{\bar{T}} r_s ds\bigg|\mathcal{A}_T\bigg)\\ &= \mathrm{Var}\bigg(\int_T^{\bar{T}} r_s ds\bigg|\mathcal{A}_T\bigg)\\ &= v(T,\bar{T}).\end{aligned} \tag{B.12}$$

To simplify (B.11) we note that

$$\begin{aligned}\log G_{\bar{T}}(t) &= -\mathrm{E}\bigg(\int_t^{\bar{T}} r_s ds\bigg|\mathcal{A}_t\bigg) + \frac{1}{2}\mathrm{Var}\bigg(\int_t^{\bar{T}} r_s ds\bigg|\mathcal{A}_t\bigg)\\ \log G_{T}(t) &= -\mathrm{E}\bigg(\int_t^{T} r_s ds\bigg|\mathcal{A}_t\bigg) + \frac{1}{2}\mathrm{Var}\bigg(\int_t^{T} r_s ds\bigg|\mathcal{A}_t\bigg)\end{aligned} \tag{B.13}$$

and that, by virtue of the Law of Total Variance,

$$\mathrm{Var}(X) = \mathrm{Var}(\mathrm{E}(X|Y)) + \mathrm{E}(\mathrm{Var}(X|Y)), \tag{B.14}$$

we have

$$\mathrm{Var}\bigg(\int_t^{\bar{T}} r_s ds\bigg|\mathcal{A}_t\bigg) = \mathrm{Var}\bigg(\mathrm{E}\bigg(\int_t^{\bar{T}} r_s ds\bigg|\mathcal{A}_T\bigg)\bigg|\mathcal{A}_t\bigg) + \mathrm{Var}\bigg(\int_t^{\bar{T}} r_s ds\bigg|\mathcal{A}_T\bigg). \tag{B.15}$$

Therefore, we can rewrite (B.11) as

$$\begin{aligned}
\mathrm{E}(L|\mathcal{A}_t) &= \log G_{\bar{T}}(t) - \frac{1}{2}\mathrm{Var}\bigg(\int_t^{\bar{T}} r_s ds\bigg|\mathcal{A}_t\bigg) \\
&- \log G_T(t) + \frac{1}{2}\mathrm{Var}\bigg(\int_t^{T} r_s ds\bigg|\mathcal{A}_t\bigg) \\
&+ \frac{1}{2}\mathrm{Var}\bigg(\int_t^{\bar{T}} r_s ds\bigg|\mathcal{A}_T\bigg) \\
&= \log G_{\bar{T}}(t)/G_T(t) \\
&- \frac{1}{2}\mathrm{Var}\bigg(\mathrm{E}\bigg(\int_t^{\bar{T}} r_s ds\bigg|\mathcal{A}_T\bigg)\bigg|\mathcal{A}_t\bigg) - \frac{1}{2}\mathrm{Var}\bigg(\int_t^{\bar{T}} r_s ds\bigg|\mathcal{A}_T\bigg) \\
&+ \frac{1}{2}\mathrm{Var}\bigg(\int_t^{T} r_s ds\bigg|\mathcal{A}_t\bigg) + \frac{1}{2}\mathrm{Var}\bigg(\int_t^{\bar{T}} r_s ds\bigg|\mathcal{A}_T\bigg) \\
&= \log G_{\bar{T}}(t)/G_T(t) \\
&- \frac{1}{2}\mathrm{Var}\bigg(\mathrm{E}\bigg(\int_t^{\bar{T}} r_s ds\bigg|\mathcal{A}_T\bigg)\bigg|\mathcal{A}_t\bigg) + \frac{1}{2}\mathrm{Var}\bigg(\int_t^{T} r_s ds\bigg|\mathcal{A}_t\bigg).
\end{aligned} \tag{B.16}$$

Transposing (B.9) gives

$$\mathrm{E}\bigg(\int_t^{\bar{T}} r_s ds\bigg|\mathcal{A}_T\bigg) = -L + \int_t^{T} r_s ds + \frac{1}{2}\mathrm{Var}\bigg(\int_t^{\bar{T}} r_s ds\bigg|\mathcal{A}_T\bigg) \tag{B.17}$$

and taking the variance of both sides gives

$$\begin{aligned}
\mathrm{Var}\bigg(\mathrm{E}\bigg(\int_t^{\bar{T}} r_s ds\bigg|\mathcal{A}_T\bigg)\bigg|\mathcal{A}_t\bigg) &= \mathrm{Var}(L|\mathcal{A}_t) + \mathrm{Var}\bigg(\int_t^{T} r_s ds\bigg|\mathcal{A}_t\bigg) \\
-2\mathrm{Cov}\bigg(L, \int_t^{T} r_s ds\bigg|\mathcal{A}_t\bigg).
\end{aligned} \tag{B.18}$$

Substituting this variance formula into (B.16) gives (3.2.70). The formula for the variance (3.2.71) is easily deduced by rewriting (B.9) as

$$L = -\mathrm{E}\bigg(\int_T^{\bar{T}} r_s ds\bigg|\mathcal{A}_T\bigg) + \frac{1}{2}\mathrm{Var}\bigg(\int_t^{\bar{T}} r_s ds\bigg|\mathcal{A}_T\bigg) \tag{B.19}$$

and taking variances of both sides. **Q.E.D.**

Proof.[of Theorem 3.2.22] The price of the first order asset binary call option on $G_{\bar{T}}$ is

$$\begin{aligned}
f_1(t, T, K, \bar{T}) &= \mathrm{E}\bigg(\exp\bigg\{-\int_t^{T} r_s ds\bigg\} G_{\bar{T}}(T)\mathbf{1}_{G_{\bar{T}}(T)>K}\bigg|\mathcal{A}_t\bigg) \\
&= \mathrm{E}\bigg(\exp\bigg\{-\int_t^{T} r_s ds\bigg\}\mathrm{E}\bigg(\exp\bigg\{-\int_T^{\bar{T}} r_s ds\bigg\}\bigg|\mathcal{A}_T\bigg)\mathbf{1}_{L>\log K}\bigg|\mathcal{A}_t\bigg) \\
&= \mathrm{E}\bigg(\exp\bigg\{-\int_t^{\bar{T}} r_s ds\bigg\}\mathbf{1}_{L>\log K}\bigg|\mathcal{A}_t\bigg).
\end{aligned} \tag{B.20}$$

We can apply Lemma 3.2.19 to the right hand side of (B.20) to give

$$\begin{aligned} &f_1(t,T,K,\bar{T}) \\ &= \mathrm{E}\bigg(\exp\bigg\{-\int_t^{\bar{T}} r_s ds\bigg\}\bigg|\mathcal{A}_t\bigg) \times \mathrm{E}\bigg(\mathbf{1}_{L>\log K - \mathrm{Cov}(-\int_t^{\bar{T}} r_s ds, L|\mathcal{A}_t)}\bigg|\mathcal{A}_t\bigg) \\ &= G_{\bar{T}}(t)\mathrm{E}(\mathbf{1}_{Z>z_1}) \end{aligned} \tag{B.21}$$

for a standard normal random variable Z where

$$z_1 = \frac{1}{\sqrt{\mathrm{Var}(L)}}\bigg(\log K + \mathrm{Cov}\bigg(\int_t^{\bar{T}} r_s ds, L\bigg|\mathcal{A}_t\bigg) - \mathrm{E}(L)\bigg). \tag{B.22}$$

The expression $\log K + \mathrm{Cov}\Big(\int_t^{\bar{T}} r_s ds, L\Big|\mathcal{A}_t\Big) - \mathrm{E}(L)$ can be simplified using (B.16) to give

$$\begin{aligned} &\log KG_T(t)/G_{\bar{T}}(t) + \mathrm{Cov}\bigg(\int_t^{\bar{T}} r_s ds, L\bigg|\mathcal{A}_t\bigg) \\ &+ \frac{1}{2}\mathrm{Var}(L|\mathcal{A}_t) - \mathrm{Cov}\bigg(L, \int_t^{T} r_s ds\bigg|\mathcal{A}_t\bigg) \\ &= \log KG_T(t)/G_{\bar{T}}(t) + \mathrm{Cov}\bigg(\int_T^{\bar{T}} r_s ds, L\bigg|\mathcal{A}_t\bigg) \\ &+ \frac{1}{2}\mathrm{Var}(L|\mathcal{A}_t). \end{aligned} \tag{B.23}$$

From (B.19) we have

$$\begin{aligned} &\mathrm{Cov}\bigg(\int_T^{\bar{T}} r_s ds, L\bigg|\mathcal{A}_t\bigg) \\ &= -\mathrm{Cov}\bigg(\int_T^{\bar{T}} r_s ds, \mathrm{E}\bigg(\int_T^{\bar{T}} r_s ds\bigg|\mathcal{A}_T\bigg)\bigg|\mathcal{A}_t\bigg) \\ &= -\mathrm{E}\bigg(\int_T^{\bar{T}} r_s ds \times \mathrm{E}\bigg(\int_T^{\bar{T}} r_s ds\bigg|\mathcal{A}_T\bigg)\bigg|\mathcal{A}_t\bigg) \\ &+ \mathrm{E}\bigg(\int_T^{\bar{T}} r_s ds\bigg|\mathcal{A}_t\bigg) \times \mathrm{E}\bigg(\mathrm{E}\bigg(\int_T^{\bar{T}} r_s ds\bigg|\mathcal{A}_T\bigg)\bigg|\mathcal{A}_t\bigg). \end{aligned} \tag{B.24}$$

Using the law of total covariance, we have

$$\begin{aligned}
&\mathrm{Cov}\bigg(\int_T^{\bar{T}} r_s ds, L\bigg|\mathcal{A}_t\bigg) && \text{(B.25)}\\
&= -\mathrm{E}\bigg(\mathrm{E}\bigg(\int_T^{\bar{T}} r_s ds \times \mathrm{E}\bigg(\int_T^{\bar{T}} r_s ds\bigg|\mathcal{A}_T\bigg)\bigg|\mathcal{A}_T\bigg)\bigg|\mathcal{A}_t\bigg) + \bigg\{\mathrm{E}\bigg(\int_T^{\bar{T}} r_s ds\bigg|\mathcal{A}_t\bigg)\bigg\}^2\\
&= -\mathrm{E}\bigg(\bigg\{\mathrm{E}\bigg(\int_T^{\bar{T}} r_s ds\bigg|\mathcal{A}_T\bigg)\bigg\}^2\bigg|\mathcal{A}_t\bigg) + \bigg\{\mathrm{E}\bigg(\int_T^{\bar{T}} r_s ds\bigg|\mathcal{A}_t\bigg)\bigg\}^2\\
&= -\mathrm{Var}\bigg(\mathrm{E}\bigg(\int_T^{\bar{T}} r_s ds\bigg|\mathcal{A}_T\bigg)\bigg|\mathcal{A}_t\bigg)\\
&= -\mathrm{Var}(L|\mathcal{A}_t).
\end{aligned}$$

Thus (B.23) becomes

$$\log KG_T(t)/G_{\bar{T}}(t) - \frac{1}{2}\mathrm{Var}(L|\mathcal{A}_t) \tag{B.26}$$

and, therefore, (B.22) becomes

$$z_1 = \frac{1}{\sqrt{\mathrm{Var}(L)}}\bigg(\log KG_T(t)/G_{\bar{T}}(t) - \frac{1}{2}\mathrm{Var}(L)\bigg). \tag{B.27}$$

Thus

$$\mathrm{E}(\mathbf{1}_{Z>z_1}) = 1 - N(z_1) = N(-z_1) = N(d_1), \tag{B.28}$$

where

$$d_1 = \frac{1}{\sqrt{\mathrm{Var}(L)}}\bigg(\log G_{\bar{T}}(t)/KG_T(t) + \frac{1}{2}\mathrm{Var}(L)\bigg), \tag{B.29}$$

as specified in statement of the lemma. The formula for the first order asset binary put option is derived using call-put parity. **Q.E.D.**

Proof.[of Theorem 3.2.23] The price of the first order bond binary call option on $G_{\bar{T}}$ is

$$f_3(t,T,K,\bar{T}) = \mathrm{E}\bigg(\exp\bigg\{-\int_t^T r_s ds\bigg\}\mathbf{1}_{G_{\bar{T}}(T)>K}\bigg|\mathcal{A}_t\bigg). \tag{B.30}$$

We can apply Lemma 3.2.19 to the right hand side of (B.30) to give

$$\begin{aligned}
f_3(t,T,K,\bar{T}) &= \mathrm{E}\bigg(\exp\bigg\{-\int_t^T r_s ds\bigg\}\bigg|\mathcal{A}_t\bigg) \times \mathrm{E}\bigg(\mathbf{1}_{L>\log K - \mathrm{Cov}(-\int_t^T r_s ds, L|\mathcal{A}_t)}\bigg|\mathcal{A}_t\bigg) && \text{(B.31)}\\
&= G_T(t)\mathrm{E}(\mathbf{1}_{Z>z_2})
\end{aligned}$$

for a standard normal random variable Z where

$$z_2 = \frac{1}{\sqrt{\mathrm{Var}(L)}}\bigg(\log K + \mathrm{Cov}\bigg(\int_t^T r_s ds, L\bigg|\mathcal{A}_t\bigg) - \mathrm{E}(L)\bigg). \tag{B.32}$$

The expression $\log K + \mathrm{Cov}\left(\int_t^T r_s ds, L \middle| \mathcal{A}_t\right) - \mathrm{E}(L)$ can be simplified using (B.16) to give

$$\begin{aligned}&\log KG_T(t)/G_{\bar{T}}(t) + \mathrm{Cov}\left(\int_t^T r_s ds, L \middle| \mathcal{A}_t\right) \\ &+ \frac{1}{2}\mathrm{Var}(L|\mathcal{A}_t) - \mathrm{Cov}\left(L, \int_t^T r_s ds \middle| \mathcal{A}_t\right) \\ &= \log KG_T(t)/G_{\bar{T}}(t) + \frac{1}{2}\mathrm{Var}(L|\mathcal{A}_t).\end{aligned} \tag{B.33}$$

Therefore, (B.32) becomes

$$z_2 = \frac{1}{\sqrt{\mathrm{Var}(L)}}\left(\log KG_T(t)/G_{\bar{T}}(t) + \frac{1}{2}\mathrm{Var}(L)\right) \tag{B.34}$$

and

$$\mathrm{E}(\mathbf{1}_{Z>z_2}) = 1 - N(z_2) = N(-z_2) = N(d_2), \tag{B.35}$$

where

$$d_2 = \frac{1}{\sqrt{\mathrm{Var}(L)}}\left(\log G_{\bar{T}}(t)/KG_T(t) - \frac{1}{2}\mathrm{Var}(L)\right), \tag{B.36}$$

as specified in statement of the lemma. The formula for the first order bond binary put option is derived using call-put parity. **Q.E.D.**

Proof.[of Theorem 3.2.24] Expressing the call option as a combination of a first order asset binary call option and a first order bond binary call option, we have

$$f_5(t,T,K,\bar{T}) = f_1(t,T,K,\bar{T}) - Kf_3(t,T,K,\bar{T}). \tag{B.37}$$

Inserting (B.21) and (B.31) into (B.37) gives

$$\begin{aligned}c_{T,K,G_{\bar{T}}} &= G_{\bar{T}}(t)\mathrm{E}(\mathbf{1}_{Z>z_1}) - KG_T(t)\mathrm{E}(\mathbf{1}_{Z>z_2}) \\ &= G_{\bar{T}}(t)N(-z_1) - KG_T(t)N(-z_2)\end{aligned} \tag{B.38}$$

as required. The formula for the put option is derived from (B.38) and call-put parity. **Q.E.D.**

Proof.[of Corollary 3.2.25] Using Theorem 3.2.24 we must compute

$$\sigma_G^2 = \mathrm{Var}\left(\mathrm{E}\left(\int_T^{\bar{T}} r_s ds \middle| \mathcal{A}_T\right) \middle| \mathcal{A}_t\right). \tag{B.39}$$

Firstly, from Lemma 3.2.9,

$$\int_T^{\bar{T}} r_s ds = r_T B(T,\bar{T}) + \bar{r}(\bar{T} - T - B(T,\bar{T})) + \sigma\int_T^{\bar{T}} B(u,\bar{T})dZ_u \tag{B.40}$$

and therefore

$$\mathrm{E}\left(\int_T^{\bar{T}} r_s ds \middle| \mathcal{A}_T\right) = r_T B(T,\bar{T}) + \bar{r}(\bar{T} - T - B(T,\bar{T})) \tag{B.41}$$

from which we have

$$\sigma_G^2 = B(T,\bar{T})^2 \mathrm{Var}\left(r_T \middle| \mathcal{A}_t\right). \tag{B.42}$$

Secondly, from Lemma 3.2.4 we have

$$\mathrm{Var}(r_T|\mathcal{A}_t) = \sigma^2\left(B(t,T) - \frac{1}{2}\kappa B(t,T)^2\right), \tag{B.43}$$

which simplifies to

$$\mathrm{Var}(r_T|\mathcal{A}_t) = \sigma^2 \frac{1-\exp(-2\kappa(T-t))}{2\kappa}. \tag{B.44}$$

It follows that

$$\sigma_G^2 = B(T,\bar{T})^2\sigma^2\frac{1-\exp(-2\kappa(T-t))}{2\kappa} \tag{B.45}$$

and we arrive at the result. **Q.E.D.**

Appendix C

Proofs of Results on the CIR Model

We provide the proof to a lemma in Section 3.3.1.

Proof.[of Lemma 3.3.3] From (3.3.7) we have the SDE

$$\begin{aligned} dY_t &= X_{\varphi_t} dz_t + z_t dX_{\varphi_t} \\ &= \frac{Y_t}{z_t} dz_t + z_t(\nu d\varphi_t + 2\sqrt{X_{\varphi_t}} dZ_{\varphi_t}) \\ &= Y_t b_t dt + z_t(\nu \frac{c_t^2}{4z_t} dt + 2\sqrt{X_{\varphi_t}}\, dZ_{\varphi_t}) \\ &= (Y_t b_t + \nu \frac{c_t^2}{4}) dt + 2z_t \sqrt{X_{\varphi_t}}\, dZ_{\varphi_t}. \end{aligned} \tag{C.1}$$

We define the process $U_t = \int_0^t \frac{1}{\sqrt{\varphi'_u}} dZ_{\varphi_u}$ and note that $d[U]_t = dt$, which means that U is a standard Wiener process. Here $\varphi'(t) = \frac{c_t^2}{4z_t}$ and so

$$dU_t = \frac{1}{\sqrt{\varphi'_t}} dZ_{\varphi_t} = \frac{2\sqrt{z_t}}{c_t} dZ_{\varphi_t}. \tag{C.2}$$

Hence

$$dY_t = (Y_t b_t + \nu \frac{c_t^2}{4}) dt + c_t \sqrt{Y_t}\, dU_t \tag{C.3}$$

which proves (3.3.8). **Q.E.D.**

This lemma leads to the proof of a theorem in Section 3.3.1.

Proof.[of Theorem 3.3.4] Comparing the corresponding terms of (3.3.1) and

(3.3.8) gives

$$\kappa\bar{r}_t = \frac{\nu c_t^2}{4} \tag{C.4}$$

$$-\kappa = b_t \tag{C.5}$$

$$\sigma_t = c_t \tag{C.6}$$

from which we have

$$c_t = \sigma_t = \varsigma\sqrt{\bar{r}_t} \tag{C.7}$$

$$\nu = \frac{4\kappa\bar{r}_t}{c_t^2} \tag{C.8}$$

$$= \frac{4\kappa}{\varsigma^2}$$

$$b_t = -\kappa \tag{C.9}$$

$$\varphi_t = \varphi_0 + \frac{1}{4}\int_0^t \frac{c_u^2}{z_u}du = \varphi_0 + \frac{1}{4}\varsigma^2\int_0^t \frac{\bar{r}_u \exp(\kappa u)}{z_0}du \tag{C.10}$$

$$z_t = z_0 \exp(-\kappa t). \tag{C.11}$$

Q.E.D.

We give here the proof of a lemma pertaining to moments of the CIR short rate.

Proof.[of Lemma 3.3.17] Integrating the SDE (3.3.1) gives

$$r_t = r_s + \int_s^t \kappa(\bar{r} - r_u)du + \int_s^t \sigma\sqrt{r_u}dZ_u \tag{C.12}$$

and taking expectations conditioned on r_s gives

$$m_s(t) = r_s + \int_s^t \kappa(\bar{r} - m_s(u))du. \tag{C.13}$$

This can be written as a first order ordinary differential equation in $m_s(t)$

$$m_s(t)' = \kappa(\bar{r} - m_s(t)) \tag{C.14}$$

with initial condition $m_s(s) = r_s$, the solution of which is straighforward. Now the SDE of r_t^2 is, by Ito's Lemma,

$$dr_t^2 = ((2\kappa\bar{r} + \sigma^2)r_t - 2\kappa r_t^2)dt + 2\sigma r_t^{3/2}dZ_t \tag{C.15}$$

and integrating this gives

$$r_t^2 = r_s^2 + \int_s^t ((2\kappa\bar{r} + \sigma^2)r_u - 2\kappa r_u^2)du + \int_s^t 2\sigma r_u^{3/2}dZ_u. \tag{C.16}$$

Taking expectations conditioned on r_s, and defining $m_s^{(2)}(t) = \mathrm{E}(r_t^2|\mathcal{A}_s)$, gives

$$m_s^{(2)}(t) = r_s^2 + \int_s^t ((2\kappa\bar{r} + \sigma^2)m_s(u) - 2\kappa m_s^{(2)}(u))du \qquad \text{(C.17)}$$

from which we have the ordinary differential equation

$$m_s^{(2)}(t)' = (2\kappa\bar{r} + \sigma^2)m_s(t) - 2\kappa m_s^{(2)}(t) \qquad \text{(C.18)}$$

the solution of which is

$$m_s^{(2)}(t) = r_s^2 \exp(-2\kappa(t-s)) + (2\kappa\bar{r} + \sigma^2)\Bigg(\frac{1 - \exp(-2\kappa(t-s))}{2\kappa}\bar{r} \qquad \text{(C.19)}$$
$$+ (r_s - \bar{r})\frac{\exp(-\kappa(t-s)) - \exp(-2\kappa(t-s))}{\kappa}\Bigg).$$

The variance is computed as $v_s(t) = m_s^{(2)}(t) - m_s(t)^2$. **Q.E.D.**

Appendix D

Moments of CIR Savings Account

The proofs in this appendix pertain to calculations of the moments of the savings account under the CIR model.

Proof.[of Lemma 3.3.6] The moment generating function is

$$\begin{aligned} MGF_Y(t) &= \mathrm{E}\big(\exp(tY)\big) \\ &= \prod_{i=1}^{\nu} \mathrm{E}\big(\exp(t(\mu^{(i)} + Z^{(i)})^2)\big) \\ &= \exp(\lambda t) \prod_{i=1}^{\nu} \mathrm{E}\big(\exp(2t\mu^{(i)} Z^{(i)} + t(Z^{(i)})^2)\big). \end{aligned} \tag{D.1}$$

Now, for a standard normal random variable Z the expectation of the quadratic exponential simplifies as follows:

$$\begin{aligned} &\mathrm{E}\big(\exp(aZ^2 + bZ)\big) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-\frac{1}{2}(z^2 - 2az^2 - 2bz))dz \\ &= \frac{1}{\sqrt{2\pi}} \exp(\frac{1}{2}b^2/(1-2a)) \int_{-\infty}^{\infty} \exp(-\frac{1}{2}((1-2a)z^2 - 2bz + b^2/(1-2a)))dz \\ &= \frac{1}{\sqrt{2\pi}} \exp(\frac{1}{2}b^2/(1-2a)) \int_{-\infty}^{\infty} \exp(-\frac{1}{2}(1-2a)(z - b/(1-2a))^2)dz \\ &= \frac{1}{\sqrt{1-2a}} \exp(\frac{1}{2}b^2/(1-2a)). \end{aligned} \tag{D.2}$$

Therefore,

$$\begin{aligned} MGF_Y(t) &= \exp(\lambda t) \prod_{i=1}^{\nu} \frac{1}{\sqrt{1-2t}} \exp(\frac{1}{2}(2t\mu^{(i)})^2/(1-2t)) \qquad (D.3) \\ &= \exp(\lambda t)(1-2t)^{-\nu/2} \exp(2\lambda t^2/(1-2t)) \\ &= \exp(\lambda t)(1-2t)^{-\nu/2} \exp(\lambda(2t^2 - t + \frac{1}{2}(2t-1) + \frac{1}{2})/(1-2t)) \\ &= \exp(\lambda t)(1-2t)^{-\nu/2} \exp(\lambda(-t - \frac{1}{2} + \frac{1}{2}/(1-2t)) \\ &= \exp(\lambda t)(1-2t)^{-\nu/2} \exp(-\lambda t - \frac{1}{2}\lambda + \frac{1}{2}\lambda/(1-2t)) \\ &= \exp(-\frac{1}{2}\lambda)(1-2t)^{-\nu/2} \exp(\frac{1}{2}\lambda/(1-2t)). \end{aligned}$$

Q.E.D.

Proof.[of Lemma 3.3.8] We rewrite the MGF of Y in (3.3.12) as

$$\begin{aligned} MGF_Y(t) &= (1-2t)^{-\nu/2} \exp(-\frac{1}{2}\lambda) \sum_{k=0}^{\infty} \frac{1}{k!} (\frac{1}{2}\lambda(1-2t)^{-1})^k \qquad (D.4) \\ &= \exp(-\frac{1}{2}\lambda) \sum_{k=0}^{\infty} \frac{1}{k!} (\frac{1}{2}\lambda)^k (1-2t)^{-k-\nu/2}. \end{aligned}$$

Observing that the MGF of the chi-squared distributed random variable with n degrees of freedom is $(1-2t)^{-n/2}$ allows us to write the MGF of Y as

$$MGF_Y(t) = \exp(-\frac{1}{2}\lambda) \sum_{k=0}^{\infty} \frac{1}{k!} (\frac{1}{2}\lambda)^k MGF_{\chi^2_{\nu+2k}}(t). \qquad (D.5)$$

Taking inverse Laplace transforms gives the result. **Q.E.D.**

Proof.[of Corollary 3.3.10] Observing that the probability density function of the chi-squared distribution with n degrees of freedom is

$$f(x) = \frac{1}{2^{n/2}\Gamma(n/2)} x^{n/2-1} \exp(-x/2) \qquad (D.6)$$

and applying Lemma 3.3.8 gives

$$\begin{aligned} f_{\chi^2_{\nu,\lambda}}(x) &= \exp(-\lambda/2) \sum_{i=0}^{\infty} \frac{(\lambda/2)^i}{i!} \frac{1}{2^{i+\nu/2}\Gamma(i+\nu/2)} x^{i+\nu/2-1} \exp(-x/2) \qquad (D.7) \\ &= \exp(-\lambda/2) \exp(-x/2) x^{\nu/2-1} \frac{1}{2^{\nu/2}} \sum_{i=0}^{\infty} \frac{(\lambda x/4)^i}{i!} \frac{1}{\Gamma(i+\nu/2)} \end{aligned}$$

which is the result. **Q.E.D.**

We now provide the proof to Lemma 3.3.18 in Section 3.3.5.

Proof.[of Lemma 3.3.18] From Cox et al. [1985] we obtain directly the formula (3.3.40) for the case $k = -1$.

By (3.3.38) and (3.3.39) we have

$$\exp(kL) = (B_T/B_t)^k = \exp\big\{k \int_t^T r_u\, du\big\} = \exp\{\int_t^T \tilde{r}_u\, du\big\}, \tag{D.8}$$

where $\tilde{r}_u = k\, r_u$ for $u \geq 0$. By the Ito formula we have from (3.3.1) the SDE

$$\begin{aligned} d\tilde{r}_u = kdr_u &= k\kappa(\bar{r} - r_u)\, du + k\sigma\sqrt{r_u}\, dZ_u \\ &= \kappa(k\bar{r} - \tilde{r}_u)\, du + \sqrt{k}\,\sigma\sqrt{\tilde{r}_u}\, dZ_u \end{aligned} \tag{D.9}$$

for $u \geq 0$ with $\tilde{r}_0 = k\, r_0$.

By taking (3.3.40) with the choice $k = -1$ and replacing σ by $\sqrt{k}\,\sigma$, $\bar{r}$ by $k\,\bar{r}$ and r_t by kr_t we obtain directly formula (3.3.40). **Q.E.D.**

Appendix E

Proofs of Expectations for the CIR Model

Here we present proofs of results in Section 3.3.8.

Proof.[of Theorem 3.3.26] We commence by computing the expectation

$$A^+_{T,K,G_{\bar{T}}}(t) = \mathrm{E}\biggl(\exp\biggl(- \int_t^T r_s ds \biggr) G_{\bar{T}}(T) \mathbf{1}_{G_{\bar{T}}(T)>K} \bigg| \mathcal{A}_t \biggr). \tag{E.1}$$

The function $\mathbf{1}_{G_{\bar{T}}(T)>K}$ is equal to the function $\mathbf{1}_{r_T<R}$ where the strike rate R is determined from the equation

$$K = A(T,\bar{T}) \exp(-B(T,\bar{T})R). \tag{E.2}$$

Also we have that

$$G_{\bar{T}}(T) = \mathrm{E}\biggl(\exp\biggl(- \int_T^{\bar{T}} r_s ds \biggr) \bigg| \mathcal{A}_T \biggr). \tag{E.3}$$

Therefore, we consider the expectation

$$\mathrm{E}\biggl(\exp\biggl(- \int_t^{\bar{T}} r_s ds \biggr) \mathbf{1}_{r_T<R} \bigg| \mathcal{A}_t \biggr), \tag{E.4}$$

which can be rewritten in the form

$$\mathrm{E}\biggl(\exp\biggl(- \int_t^{\bar{T}} r_s ds \biggr) \bigg| \mathcal{A}_t \biggr) \times \mathrm{E}\left(\frac{\exp\bigl(- \int_t^{\bar{T}} r_s ds \bigr)}{\mathrm{E}\Bigl(\exp\bigl(- \int_t^{\bar{T}} r_s ds \bigr) \Big| \mathcal{A}_t \Bigr)} \mathbf{1}_{r_T<R} \middle| \mathcal{A}_t \right). \tag{E.5}$$

Define the random variable r'_T as having the probability density function

$$f_{r'_T}(x) = \frac{\exp\bigl(- \int_t^{\bar{T}} r_s ds \bigr)}{\mathrm{E}\Bigl(\exp\bigl(- \int_t^{\bar{T}} r_s ds \bigr) \Big| \mathcal{A}_t \Bigr)} f_{r_T}(x). \tag{E.6}$$

Then our expectation becomes

$$\begin{aligned} &= G_{\bar{T}}(t)\mathrm{E}\Big(\mathbf{1}_{r'_T<R}\Big|\mathcal{A}_t\Big) \\ &= G_{\bar{T}}(t)P(r'_T<R), \end{aligned} \tag{E.7}$$

where $P(r'_T < R)$ denotes the probability that the random variable r'_T is below the strike rate R determined in (E.2).

If we can compute the moment generating function of r'_T with the purpose of identifying the distribution of r'_T, then it will be straightforward to compute $P(r'_T < R)$.

So we compute the moment generating function of r'_T as follows,

$$\begin{aligned} \mathrm{E}\Big(\exp(r'_T u)\Big|\mathcal{A}_t\Big) &= \mathrm{E}\Big(\frac{\exp\big(-\int_t^{\bar{T}} r_s ds\big)}{\mathrm{E}\big(\exp\big(-\int_t^{\bar{T}} r_s ds\big)\big|\mathcal{A}_t\big)}\exp(r_T u)\Big|\mathcal{A}_t\Big) \\ &= \frac{1}{G_{\bar{T}}(t)}\mathrm{E}\Big(\exp\Big(r_T u - \int_t^{\bar{T}} r_s ds\Big)\Big|\mathcal{A}_t\Big) \\ &= \frac{1}{G_{\bar{T}}(t)}\mathrm{E}\Big(\exp\Big(r_T u - \int_t^{T} r_s ds - \int_T^{\bar{T}} r_s ds\Big)\Big|\mathcal{A}_t\Big) \\ &= \frac{1}{G_{\bar{T}}(t)}\mathrm{E}\Big(\exp\Big(r_T u - \int_t^{T} r_s ds\Big)\exp\Big(-\int_T^{\bar{T}} r_s ds\Big)\Big|\mathcal{A}_t\Big) \\ &= \frac{1}{G_{\bar{T}}(t)}\mathrm{E}\Big(\exp\Big(r_T u - \int_t^{T} r_s ds\Big)G_{\bar{T}}(T)\Big|\mathcal{A}_t\Big) \\ &= \frac{A(T,\bar{T})}{G_{\bar{T}}(t)}\mathrm{E}\Big(\exp\Big(r_T(u - B(T,\bar{T})) - \int_t^{T} r_s ds\Big)\Big|\mathcal{A}_t\Big). \end{aligned} \tag{E.8}$$

From Lemma 3.3.25 we have

$$\begin{aligned} &\mathrm{E}\Big(\exp\Big(r_T u - \int_t^{T} r_s ds\Big)\Big|\mathcal{A}_t\Big) \\ &= \Big(\frac{h\exp(\frac{1}{2}\kappa(T-t))}{(\kappa-\sigma^2 u)\sinh\frac{1}{2}h(T-t) + h\cosh\frac{1}{2}h(T-t)}\Big)^{2\kappa\bar{r}/\sigma^2} \\ &\times \exp\Big(\frac{hu\cosh\frac{1}{2}h(T-t) - (\kappa u + 2)\sinh\frac{1}{2}h(T-t)}{(\kappa-\sigma^2 u)\sinh\frac{1}{2}h(T-t) + h\cosh\frac{1}{2}h(T-t)} r_t\Big). \end{aligned} \tag{E.9}$$

It follows that

$$\begin{aligned} &\mathrm{E}\Big(\exp\Big(r_T(u - B(T,\bar{T})) - \int_t^{T} r_s ds\Big)\Big|\mathcal{A}_t\Big) \\ &= \Big(\frac{h\exp(\frac{1}{2}\kappa(T-t))}{(\kappa-\sigma^2(u-B(T,\bar{T})))\sinh\frac{1}{2}h(T-t) + h\cosh\frac{1}{2}h(T-t)}\Big)^{2\kappa\bar{r}/\sigma^2} \\ &\times \exp\Big(\frac{h(u-B(T,\bar{T}))\cosh\frac{1}{2}h(T-t) - (\kappa(u-B(T,\bar{T}))+2)\sinh\frac{1}{2}h(T-t)}{(\kappa-\sigma^2(u-B(T,\bar{T})))\sinh\frac{1}{2}h(T-t) + h\cosh\frac{1}{2}h(T-t)} r_t\Big). \end{aligned} \tag{E.10}$$

We can rewrite the expression inside the exponential as

$$\frac{h(u-B(T,\bar{T}))\cosh\frac{1}{2}h(T-t)-(\kappa(u-B(T,\bar{T}))+2)\sinh\frac{1}{2}h(T-t)}{(\kappa-\sigma^2(u-B(T,\bar{T}))\sinh\frac{1}{2}h(T-t)+h\cosh\frac{1}{2}h(T-t)}r_t \tag{E.11}$$
$$= r_t\frac{(h\cosh\frac{1}{2}h(T-t)-\kappa\sinh\frac{1}{2}h(T-t))u}{((\kappa+\sigma^2B(T,\bar{T}))\sinh\frac{1}{2}h(T-t)+h\cosh\frac{1}{2}h(T-t))-\sigma^2u\sinh\frac{1}{2}h(T-t)}$$
$$+ r_t\frac{(\kappa B(T,\bar{T})-2)\sinh\frac{1}{2}h(T-t)-hB(T,\bar{T})\cosh\frac{1}{2}h(T-t)}{((\kappa+\sigma^2B(T,\bar{T}))\sinh\frac{1}{2}h(T-t)+h\cosh\frac{1}{2}h(T-t))-\sigma^2u\sinh\frac{1}{2}h(T-t)}$$
$$= r_t\frac{a_1u+a_2}{a_3-\sigma^2u\sinh\frac{1}{2}h(T-t)}$$

where

$$a_1 = h\cosh\frac{1}{2}h(T-t)-\kappa\sinh\frac{1}{2}h(T-t), \tag{E.12}$$

$$\begin{aligned} a_2 &= (\kappa B(T,\bar{T})-2)\sinh\frac{1}{2}h(T-t)-hB(T,\bar{T})\cosh\frac{1}{2}h(T-t) \\ &= B(T,\bar{T})(\kappa\sinh\frac{1}{2}h(T-t)-h\cosh\frac{1}{2}h(T-t))-2\sinh\frac{1}{2}h(T-t) \\ &= -B(T,\bar{T})a_1-2\sinh\frac{1}{2}h(T-t) \end{aligned} \tag{E.13}$$

and

$$\begin{aligned} a_3 &= (\kappa+\sigma^2B(T,\bar{T}))\sinh\frac{1}{2}h(T-t)+h\cosh\frac{1}{2}h(T-t) \\ &= (\kappa\sinh\frac{1}{2}h(T-t)+h\cosh\frac{1}{2}h(T-t))+\sigma^2B(T,\bar{T})\sinh\frac{1}{2}h(T-t) \\ &= C(t,T)\left(1+\frac{1}{2}\sigma^2B(t,T)B(T,\bar{T})\right). \end{aligned} \tag{E.14}$$

Continuing, we have that the expression inside the exponential becomes

$$r_t\frac{1}{a_3}\times\left(\frac{a_1+2\frac{\frac{1}{2}\sigma^2\sinh\frac{1}{2}h(T-t)}{a_3}a_2}{1-2\frac{\frac{1}{2}\sigma^2\sinh\frac{1}{2}h(T-t)}{a_3}u}\times u+a_2\right). \tag{E.15}$$

From (E.10) we now have the simplified expression

$$\begin{aligned} &\mathrm{E}\left(\exp\left(r_T(u-B(T,\bar{T}))-\int_t^T r_s ds\right)\middle|\mathcal{A}_t\right) \\ &= \left(\frac{h\exp(\frac{1}{2}\kappa(T-t))}{a_3-\sigma^2u\sinh\frac{1}{2}h(T-t)}\right)^{2\kappa\bar{r}/\sigma^2} \\ &\times\exp\left(r_t\frac{1}{a_3}\times\left(\frac{a_1+2\frac{\frac{1}{2}\sigma^2\sinh\frac{1}{2}h(T-t)}{a_3}a_2}{1-2\frac{\frac{1}{2}\sigma^2\sinh\frac{1}{2}h(T-t)}{a_3}u}\times u+a_2\right)\right). \end{aligned} \tag{E.16}$$

From (E.8) and (E.16) we have

$$
\begin{aligned}
&\mathrm{E}\left(\exp(r'_T u) \middle| \mathcal{A}_t \right) \\
&= \frac{A(T,\bar{T})}{G_{\bar{T}}(t)} \mathrm{E}\left(\exp\left(r_T(u - B(T,\bar{T})) - \int_t^T r_s ds \right) \middle| \mathcal{A}_t \right) \\
&= \frac{A(T,\bar{T})}{G_{\bar{T}}(t)} \left(\frac{h \exp(\frac{1}{2}\kappa(T-t))}{a_3 - \sigma^2 u \sinh \frac{1}{2}h(T-t)} \right)^{2\kappa\bar{r}/\sigma^2} \\
&\times \exp\left(r_t \frac{1}{a_3} \times \left(\frac{a_1 + 2\frac{\frac{1}{2}\sigma^2 \sinh \frac{1}{2}h(T-t)}{a_3} a_2}{1 - 2\frac{\frac{1}{2}\sigma^2 \sinh \frac{1}{2}h(T-t)}{a_3} u} \times u + a_2 \right) \right) \\
&= \frac{A(T,\bar{T})}{G_{\bar{T}}(t)} \left(\frac{h \exp(\frac{1}{2}\kappa(T-t))}{a_3 - \sigma^2 u \sinh \frac{1}{2}h(T-t)} \right)^{2\kappa\bar{r}/\sigma^2} \\
&\times \exp\left\{ r_t \frac{1}{a_3} \times \left(a_1 + 2\frac{\frac{1}{2}\sigma^2 \sinh \frac{1}{2}h(T-t)}{a_3} a_2 \right) \right. \\
&\times \left. \left(1 - 2\frac{\frac{1}{2}\sigma^2 \sinh \frac{1}{2}h(T-t)}{a_3} u \right)^{-1} \times u \right\} \\
&\times \exp\left(r_t \frac{a_2}{a_3} \right).
\end{aligned}
\tag{E.17}
$$

Using (E.12), (E.13) and (E.14) we can simplify the term

$$
\begin{aligned}
&\left(\frac{h \exp(\frac{1}{2}\kappa(T-t))}{a_3 - \sigma^2 u \sinh \frac{1}{2}h(T-t)} \right)^{2\kappa\bar{r}/\sigma^2} \\
&= \left(\frac{h \exp(\frac{1}{2}\kappa(T-t))}{C(t,T)} \right)^{2\kappa\bar{r}/\sigma^2} \times \left(\frac{C(t,T)}{a_3 - \sigma^2 u \sinh \frac{1}{2}h(T-t)} \right)^{2\kappa\bar{r}/\sigma^2} \\
&= A(t,T) \times \left(1 + \frac{1}{2}\sigma^2 B(t,T) B(T,\bar{T}) \right)^{-2\kappa\bar{r}/\sigma^2} \\
&\times \left(1 - \frac{\sigma^2 u}{a_3} \sinh \frac{1}{2}h(T-t) \right)^{-2\kappa\bar{r}/\sigma^2} \\
&= A(t,T) \times \left(1 + \frac{1}{2}\sigma^2 B(t,T) B(T,\bar{T}) \right)^{-2\kappa\bar{r}/\sigma^2} \\
&\times \left(1 - \frac{1}{2}\sigma^2 u B(t,T) (1 + \frac{1}{2}\sigma^2 B(t,T) B(T,\bar{T}))^{-1} \right)^{-2\kappa\bar{r}/\sigma^2}
\end{aligned}
\tag{E.18}
$$

Also the latter terms in (E.17) simplify as follows,

$$\begin{aligned}
&\exp\left\{r_t\frac{1}{a_3}\times\left(a_1+2\frac{\frac{1}{2}\sigma^2\sinh\frac{1}{2}h(T-t)}{a_3}a_2\right)\right.\\
&\left.\times\left(1-2\frac{\frac{1}{2}\sigma^2\sinh\frac{1}{2}h(T-t)}{a_3}u\right)^{-1}\times u\right\}\\
&\times\exp\left(r_t\frac{a_2}{a_3}\right)\\
&=\exp\left(r_t\frac{1}{a_3}\times\left(a_1+\frac{1}{2}\sigma^2a_2B(t,T)(1+\frac{1}{2}\sigma^2B(t,T)B(T,\bar{T}))^{-1}\right)\right.\\
&\left.\times\left(1-\frac{1}{2}\sigma^2uB(t,T)(1+\frac{1}{2}\sigma^2B(t,T)B(T,\bar{T}))^{-1}\right)^{-1}\times u\right)\\
&\times\exp\left(r_t\frac{a_2}{a_3}\right).
\end{aligned}\tag{E.19}$$

Further we note that

$$\begin{aligned}
\frac{a_1}{a_3}&=B(t,T)\left(1+\frac{1}{2}\sigma^2B(t,T)B(T,\bar{T})\right)^{-1}\\
&\times\frac{1}{2\sinh\frac{1}{2}h(T-t)}\left(h\cosh\frac{1}{2}h(T-t)-\kappa\sinh\frac{1}{2}h(T-t)\right)\\
&=B(t,T)\left(1+\frac{1}{2}\sigma^2B(t,T)B(T,\bar{T})\right)^{-1}\times\left(\frac{1}{2}h\coth\frac{1}{2}h(T-t)-\frac{1}{2}\kappa\right)\\
\frac{a_2}{a_3}&=\frac{-B(T,\bar{T})a_1-2\sinh\frac{1}{2}h(T-t)}{a_3}\\
&=-B(T,\bar{T})\frac{a_1}{a_3}-B(t,T)(1+\frac{1}{2}\sigma^2B(t,T)B(T,\bar{T}))^{-1}
\end{aligned}\tag{E.20}$$

and that

$$\begin{aligned}
g&=\frac{a_1}{a_3}+\frac{1}{2}\sigma^2\frac{a_2}{a_3}B(t,T)(1+\frac{1}{2}\sigma^2B(t,T)B(T,\bar{T}))^{-1}\\
&=\frac{a_1}{a_3}+\frac{1}{2}\sigma^2\left(-B(T,\bar{T})\frac{a_1}{a_3}-B(t,T)(1+\frac{1}{2}\sigma^2B(t,T)B(T,\bar{T}))^{-1}\right)\\
&\times B(t,T)(1+\frac{1}{2}\sigma^2B(t,T)B(T,\bar{T}))^{-1}\\
&=\frac{a_1}{a_3}-\frac{1}{2}\sigma^2B(T,\bar{T})\frac{a_1}{a_3}B(t,T)(1+\frac{1}{2}\sigma^2B(t,T)B(T,\bar{T}))^{-1}\\
&-\frac{1}{2}\sigma^2B(t,T)(1+\frac{1}{2}\sigma^2B(t,T)B(T,\bar{T}))^{-1}B(t,T)(1+\frac{1}{2}\sigma^2B(t,T)B(T,\bar{T}))^{-1}\\
&=\frac{a_1}{a_3}\left(1-\frac{1}{2}\sigma^2B(T,\bar{T})B(t,T)(1+\frac{1}{2}\sigma^2B(t,T)B(T,\bar{T}))^{-1}\right)\\
&-\frac{1}{2}\sigma^2\left(B(t,T)(1+\frac{1}{2}\sigma^2B(t,T)B(T,\bar{T}))^{-1}\right)^2.
\end{aligned}\tag{E.21}$$

For ease of legibility of the formulae which follow we define

$$\gamma = \frac{1}{4}\sigma^2 B(t,T)(1 + \frac{1}{2}\sigma^2 B(t,T)B(T,\bar{T}))^{-1}. \tag{E.22}$$

This allows us to rewrite (E.21) as

$$g = \frac{a_1}{a_3}\Big(1 - 2B(T,\bar{T})\gamma\Big) - \frac{8}{\sigma^2}\gamma^2 \tag{E.23}$$

which, upon substituting the expression for a_1/a_3 in (E.20), simplifies to

$$g = \frac{4}{\sigma^2}\gamma\Big(\frac{1}{2}h\coth\frac{1}{2}h(T-t) - \frac{1}{2}\kappa\Big)\Big(1 - 2B(T,\bar{T})\gamma\Big) - \frac{8}{\sigma^2}\gamma^2 \tag{E.24}$$

Therefore, we can write

$$\begin{aligned} &\mathrm{E}\Big(\exp(r'_T u)\Big|\mathcal{A}_t\Big) && \text{(E.25)}\\ &\propto \Big(1 - \frac{1}{2}\sigma^2 uB(t,T)(1 + \frac{1}{2}\sigma^2 B(t,T)B(T,\bar{T}))^{-1}\Big)^{-2\kappa\bar{r}/\sigma^2}\\ &\times \exp\Big(r_t g u\Big(1 - \frac{1}{2}\sigma^2 uB(t,T)(1 + \frac{1}{2}\sigma^2 B(t,T)B(T,\bar{T}))^{-1}\Big)^{-1}\Big). \end{aligned}$$

From (3.3.12) the moment generating function of a non-central chi-squared random variable X with non-centrality parameter ω and ν degrees of freedom is

$$MGF_X(t) = (1-2t)^{-\nu/2}\exp\Big(\frac{\omega t}{1-2t}\Big) \tag{E.26}$$

and the moment generating function of the random variable $Y = \gamma X$ is

$$MGF_Y(t) = MGF_X(\gamma t) = (1-2\gamma t)^{-\nu/2}\exp\Big(\frac{\omega\gamma t}{1-2\gamma t}\Big). \tag{E.27}$$

Comparing this MGF with the right hand side of (E.25) we see that

$$\begin{aligned} \nu &= 4\kappa\bar{r}/\sigma^2 && \text{(E.28)}\\ \gamma &= \frac{1}{4}\sigma^2 B(t,T)(1 + \frac{1}{2}\sigma^2 B(t,T)B(T,\bar{T}))^{-1}\\ \omega &= r_t g/\gamma\\ &= r_t\frac{4}{\sigma^2}\Big(\frac{1}{2}h\coth\frac{1}{2}h(T-t) - \frac{1}{2}\kappa\Big)\Big(1 - 2B(T,\bar{T})\gamma\Big) - r_t\frac{8}{\sigma^2}\gamma \end{aligned}$$

where g is as in (E.24).

Thus r'_T is distributed as a scaled non-central chi-squared random variable with parameters ν, ω, γ as in (E.28). Therefore, we have

$$P(r'_T < R) = \chi^2_{\nu,\omega}(R/\gamma) \tag{E.29}$$

where R is determined from (E.2). It follows from (E.5) that

$$A^{+}_{T,K,G_{\bar{T}}}(t) = G_{\bar{T}}(t)\chi^2_{\nu,\omega}(R/\gamma), \tag{E.30}$$

which is the result for the asset binary call option. Now for the asset binary put option we have similarly

$$\begin{aligned} A^{-}_{T,K,G_{\bar{T}}}(t) &= \mathrm{E}\bigg(\exp\bigg(-\int_t^T r_s ds \bigg) G_{\bar{T}}(T) \mathbf{1}_{G_{\bar{T}}(T) \leq K} \bigg| \mathcal{A}_t \bigg) \\ &= \mathrm{E}\bigg(\exp\bigg(-\int_t^T r_s ds \bigg) G_{\bar{T}}(T)(1 - \mathbf{1}_{G_{\bar{T}}(T) > K}) \bigg| \mathcal{A}_t \bigg) \\ &= G_{\bar{T}}(t) - A^{+}_{T,K,G_{\bar{T}}}(t) \\ &= G_{\bar{T}}(t)\Big(1 - \chi^2_{\nu,\omega}(R/\gamma) \Big). \end{aligned} \tag{E.31}$$

Q.E.D.

Appendix F

Proofs on CIR Short Rate Contributions to Bond Yields and Forward Rates

The following proof corresponds to the short rate contribution to the long bond yield.

Proof.[of Corollary 3.3.21] The short rate contribution to the long ZCB yield is given by the formula

$$
\begin{aligned}
h_\infty(t) &= -\lim_{\bar{T}\to\infty} \frac{1}{\bar{T}-t} \log G_{\bar{T}}(t) \qquad \text{(F.1)}\\
&= \lim_{\bar{T}\to\infty} \frac{1}{\bar{T}-t}(-\log A(t,\bar{T}) + r_t B(t,\bar{T})).
\end{aligned}
$$

But $\lim_{\bar{T}\to\infty} B(t,\bar{T}) = \frac{1}{\kappa}$ and, therefore, $\lim_{\bar{T}\to\infty} \frac{1}{\bar{T}-t}(r_t B(t,\bar{T})) = 0$.

$$
\begin{aligned}
h_\infty(t) &= \lim_{\bar{T}\to\infty} -\frac{1}{\bar{T}-t} \log A(t,\bar{T}) \qquad \text{(F.2)}\\
&= \lim_{\bar{T}\to\infty} -\frac{1}{\bar{T}-t}\frac{2\kappa\bar{r}}{\sigma^2} \log \frac{h\exp(\frac{1}{2}\kappa(T-t))}{\kappa \sinh \frac{1}{2}h(T-t) + h\cosh\frac{1}{2}h(T-t)}\\
&= \lim_{\bar{T}\to\infty} -\frac{2\kappa\bar{r}}{\sigma^2}\left(\frac{1}{2}\kappa - \frac{1}{2}h\right)\\
&= \frac{\kappa\bar{r}}{\sigma^2}(h-\kappa)
\end{aligned}
$$

as required. **Q.E.D.**

The following proof corresponds to the short rate contribution to the $\bar{T}$-forward rate.

Proof.[of Lemma 3.3.22] Taking logarithms on both sides of (3.3.44) gives

$$\log G_{\bar{T}}(t) = \log A(t, \bar{T}) - B(t, \bar{T}) r_t \tag{F.3}$$

and then taking the negative of the partial derivative of this with respect to $\bar{T}$ gives

$$\begin{aligned} g_{\bar{T}}(t) &= -\frac{\partial}{\partial \bar{T}} \log G_{\bar{T}}(t) \\ &= -\frac{\partial}{\partial \bar{T}} \log A(t, \bar{T}) + r_t \frac{\partial}{\partial \bar{T}} B(t, \bar{T}) \\ &= -\frac{2\kappa\bar{r}}{\sigma^2} \frac{\partial}{\partial \bar{T}} L(t, \bar{T}) + r_t \frac{\partial}{\partial \bar{T}} B(t, \bar{T}), \end{aligned} \tag{F.4}$$

where

$$\begin{aligned} L(t, \bar{T}) &= \left(\frac{2\kappa\bar{r}}{\sigma^2} \right)^{-1} \log A(t, \bar{T}) \\ &= \log \frac{h \exp\left(\frac{1}{2}\kappa(\bar{T} - t)\right)}{C(t, \bar{T})} \\ &= \log h + \frac{1}{2}\kappa(\bar{T} - t) - \log C(t, \bar{T}). \end{aligned} \tag{F.5}$$

Now

$$\begin{aligned} \frac{\partial}{\partial \bar{T}} B(t, \bar{T}) &= \frac{h \cosh \frac{1}{2} h(\bar{T} - t)}{C(t, \bar{T})} \\ &- \frac{2 \sinh \frac{1}{2} h(\bar{T} - t)}{C(t, \bar{T})^2} \frac{h}{2} (\kappa \cosh \frac{1}{2} h(\bar{T} - t) + h \sinh \frac{1}{2} h(\bar{T} - t)) \\ &= \frac{h \cosh \frac{1}{2} h(\bar{T} - t)}{C(t, \bar{T})} - \frac{h}{C(t, \bar{T})^2} (\kappa \sinh \frac{1}{2} h(\bar{T} - t) \cosh \frac{1}{2} h(\bar{T} - t) \\ &+ h \cosh^2 \frac{1}{2} h(\bar{T} - t) - h) \\ &= \frac{h \cosh \frac{1}{2} h(\bar{T} - t)}{C(t, \bar{T})} + \frac{h^2}{C(t, \bar{T})^2} - \frac{h \cosh \frac{1}{2} h(\bar{T} - t)}{C(t, \bar{T})} \\ &= \frac{h^2}{C(t, \bar{T})^2} \end{aligned} \tag{F.6}$$

and

$$\begin{aligned}
\frac{\partial}{\partial \bar{T}} L(t,\bar{T}) &= \frac{1}{2}\kappa - \frac{1}{C(t,\bar{T})}\frac{\partial C(t,\bar{T})}{\partial \bar{T}} \qquad \text{(F.7)}\\
&= \frac{1}{2}\kappa - \frac{1}{C(t,\bar{T})}(\frac{1}{2}h\kappa\cosh\frac{1}{2}h(\bar{T}-t) + \frac{1}{2}h^2\sinh\frac{1}{2}h(\bar{T}-t))\\
&= \frac{1}{2}\kappa - \frac{h}{2\sinh\frac{1}{2}h(\bar{T}-t))C(t,\bar{T})}\\
&\times (\kappa\sinh\frac{1}{2}h(\bar{T}-t)\cosh\frac{1}{2}h(\bar{T}-t)) - h + h\cosh^2\frac{1}{2}h(\bar{T}-t)))\\
&= \frac{1}{2}\kappa - \frac{h}{2\sinh\frac{1}{2}h(\bar{T}-t))C(t,\bar{T})}(\cosh\frac{1}{2}h(\bar{T}-t))C(t,\bar{T}) - h)\\
&= \frac{1}{2}\kappa - \frac{h}{2}\coth\frac{1}{2}h(\bar{T}-t)) + \frac{1}{2}h^2\frac{1}{\sinh\frac{1}{2}h(\bar{T}-t))C(t,\bar{T})}.
\end{aligned}$$

Both (F.6) and (F.7) inserted into (F.4) lead to (3.3.50). **Q.E.D.**

Appendix G

Laplace Transform for the $3/2$ Short Rate

We supply a proof of the formula for the Laplace transform of $\int_t^T r_s\, ds$ when r_t obeys (3.4.1). This proof is based on the proof given for Theorem 3 in Carr and Sun [2007]. Furthermore, this formula for the Laplace transform agrees with that given in Section 5.5 of Baldeaux and Platen [2013].

Proof.[of Lemma 3.4.4] From Section 9.7 of Platen and Heath [2006], the Feynman-Kac Theorem states that if

$$u(x,t) = \mathrm{E}\bigg(\int_t^T \exp\bigg\{ -\int_t^s V(X_\tau,\tau)\, d\tau \bigg\} f(X_s,s)\, ds \qquad \text{(G.1)}$$
$$+ \exp\bigg\{ -\int_t^T V(X_\tau,\tau)\, d\tau \bigg\} \psi(X_T) \bigg| X_t = x \bigg),$$

where $X = \{X_t : t \geq 0\}$ is a stochastic process satisfying

$$dX_t = \mu(X_t,t)dt + \sigma(X_t,t)dW_t \qquad \text{(G.2)}$$

then $u(x,t)$ satisfies the partial differential equation (PDE)

$$\frac{\partial u}{\partial t} + \mu(x,t)\frac{\partial u}{\partial x} + \frac{1}{2}\sigma(x,t)^2\frac{\partial^2 u}{\partial x^2} - V(x,t)\, u(x,t) - f(x,t) = 0, \qquad \text{(G.3)}$$

with the boundary condition

$$u(x,T) = \psi(x). \qquad \text{(G.4)}$$

Applying this theorem to the problem of computing the Laplace transform

$$u(r_t,t) = \mathrm{E}\bigg(\exp\bigg\{ -s\int_t^T r_\tau\, d\tau \bigg\} \bigg| \mathcal{A}_t \bigg), \qquad \text{(G.5)}$$

where $r = \{r_t : t \geq 0\}$ is the stochastic process given in (3.4.1), we must solve the PDE (G.3). We have

$$\begin{aligned} \mu(x,t) &= px + qx^2 \\ \sigma(x,t) &= \sigma x^{3/2} \\ \psi(x) &= 1 \\ V(x,t) &= s\,x \\ f(x,t) &= 0 \end{aligned} \tag{G.6}$$

and (G.3) becomes

$$\frac{\partial u}{\partial t} + (px + qx^2)\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2 x^3 \frac{\partial^2 u}{\partial x^2} - s\,x\,u(x,t) = 0. \tag{G.7}$$

The boundary conditions for the solution to (G.7) are

$$\begin{aligned} u(x,T) &= 1 \\ u(0,t) &= 1 \\ \lim_{x\to\infty} u(x,t) &= 0. \end{aligned} \tag{G.8}$$

As done in Carr and Sun [2007], we guess that

$$u(x,t) = w(y), \tag{G.9}$$

where

$$y = x\{\exp(p(T-t)) - 1\}/p. \tag{G.10}$$

The boundary conditions in (G.8) become

$$\begin{aligned} w(0) &= 1 \\ \lim_{y\to\infty} w(y) &= 0. \end{aligned} \tag{G.11}$$

We have

$$\begin{aligned} \frac{\partial u}{\partial t} &= w'(y)\,(-x - py) \\ \frac{\partial u}{\partial x} &= w'(y)\,\frac{\exp(p(T-t)) - 1}{p} = w'(y)\frac{y}{x} \\ \frac{\partial^2 u}{\partial x^2} &= w''(y)\left(\frac{\exp(p(T-t)) - 1}{p}\right)^2 = w''(y)\frac{y^2}{x^2} \end{aligned} \tag{G.12}$$

and (G.7) becomes

$$w'(y)\,(-x + q\,x\,y) + \frac{1}{2}\sigma^2 x y^2 w''(y) - s\,x\,w(y) = 0, \tag{G.13}$$

which upon dividing both sides by x becomes

$$w'(y)\,(-1 + q\,y) + \frac{1}{2}\sigma^2 y^2 w''(y) - s\,w(y) = 0. \tag{G.14}$$

Write $w(y) = z^\alpha h(z)$ where $z = \beta/y$, for some constants α and β to be chosen later to effect some simplification. Then

$$w'(y) = -\frac{\alpha}{\beta} z^{\alpha+1} h(z) - \frac{1}{\beta} z^{\alpha+2} h'(z) \tag{G.15}$$
$$w''(y) = \frac{\alpha(\alpha+1)}{\beta^2} z^{\alpha+2} h(z) + \frac{2\alpha+2}{\beta^2} z^{\alpha+3} h'(z) + \frac{1}{\beta^2} z^{\alpha+4} h''(z)$$

and substituting into (G.14) gives

$$\frac{1}{2}\sigma^2 z^{\alpha+2} h''(z) + \left(\frac{1}{2}(2\alpha+2)\sigma^2 z^{\alpha+1} - \frac{1}{\beta} z^{\alpha+1}(-z+q\beta)\right) h'(z) \tag{G.16}$$
$$+ \left(-s\, z^\alpha - \frac{\alpha}{\beta} z^\alpha(-z+q\beta) + \frac{1}{2}\alpha(\alpha+1)\sigma^2 z^\alpha\right) h(z) = 0.$$

Dividing both sides by z^α gives

$$\frac{1}{2}\sigma^2 z^2 h''(z) + \left(\frac{1}{2}(2\alpha+2)\sigma^2 z - \frac{1}{\beta} z(-z+q\beta)\right) h'(z) \tag{G.17}$$
$$+ \left(-s + \frac{\alpha}{\beta} z - \alpha q + \frac{1}{2}\alpha(\alpha+1)\sigma^2\right) h(z) = 0.$$

Choosing α such that

$$-s - \alpha q + \frac{1}{2}\alpha(\alpha+1)\sigma^2 = 0 \tag{G.18}$$

and dividing both sides of (G.17) by $\frac{1}{2}\sigma^2 z$ gives

$$zh''(z) + \left((2\alpha+2) + \frac{2}{\sigma^2\beta} z - \frac{2}{\sigma^2} q\right) h'(z) + \frac{2\alpha}{\sigma^2\beta} h(z) = 0. \tag{G.19}$$

Choosing β such that the coefficient of z in the multiplier of $h'(z)$ in (G.19) is -1, that is,

$$\beta = -\frac{2}{\sigma^2}, \tag{G.20}$$

and writing

$$\gamma = 2\alpha + 2 - \frac{2}{\sigma^2} q \tag{G.21}$$

gives

$$zh''(z) + (\gamma - z)h'(z) - \alpha h(z) = 0. \tag{G.22}$$

This is Kummer's differential equation, see Chapter 4 of Watson [1966], whose general solution, when γ is not an integer, is expressible in terms of the confluent hypergeometric functions of the first and second kinds, namely

$$h(z) = a\, M(\alpha, \gamma, z) + b\, U(\alpha, \gamma, z), \tag{G.23}$$

where

$$M(\alpha, \gamma, z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\gamma)_n} \frac{z^n}{n!} \tag{G.24}$$

and

$$U(\alpha,\gamma,z)=\frac{\pi}{\sin\pi\gamma}\left(\frac{M(\alpha,\gamma,z)}{\Gamma(1+\alpha-\gamma)\Gamma(\gamma)}-z^{1-\gamma}\frac{M(1+\alpha-\gamma,2-\gamma,z)}{\Gamma(\alpha)\Gamma(2-\gamma)}\right). \quad \text{(G.25)}$$

Therefore

$$w(y)=z^{\alpha}h(z)=a\,z^{\alpha}M(\alpha,\gamma,z)+b\,z^{\alpha}U(\alpha,\gamma,z), \quad \text{(G.26)}$$

where $z=-2/(\sigma^2 y)$. The boundary conditions (G.11) are met when $b=0$ and α equals the upper root of (G.18), that is,

$$\alpha=\frac{-(-q+\frac{1}{2}\sigma^2)+\sqrt{(-q+\frac{1}{2}\sigma^2)^2+2s\,\sigma^2}}{\sigma^2}. \quad \text{(G.27)}$$

So we have

$$w(y)=a\,z^{\alpha}M(\alpha,\gamma,z). \quad \text{(G.28)}$$

As $z\to 0$, $y\to\infty$ and $w(y)\to a\,z^{\alpha}M(\alpha,\gamma,0)=0$. Also, as $z\to-\infty$, $y\to 0^+$ and

$$M(\alpha,\gamma,z)\sim\frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)}(-z)^{-\alpha}, \quad \text{(G.29)}$$

so that letting

$$a=\frac{\Gamma(\gamma-\alpha)}{\Gamma(\gamma)}(-1)^{\alpha} \quad \text{(G.30)}$$

makes $w(y)=a\,z^{\alpha}M(\alpha,\gamma,z)\to 1$ as $z\to-\infty$. Thus, from (G.9) we have

$$u(x,t)=\frac{\Gamma(\gamma-\alpha)}{\Gamma(\gamma)}\left(\frac{2}{\sigma^2 y}\right)^{\alpha}M\big(\alpha,\gamma,-\frac{2}{\sigma^2 y}\big), \quad \text{(G.31)}$$

where y is given by (G.10), α is given by (G.27) and γ is given by (G.21). **Q.E.D.**

Appendix H

Proofs on 3/2 Short Rate Contributions to Bond Yields and Forward Rates

The following proof corresponds to the short rate contribution to the long bond yield.

Proof.[of Corollary 3.4.6] We write the logarithm of (3.4.22) as

$$\log G_{\bar{T}}(t) = \log \Gamma(\gamma_1 - \alpha_1) - \log \Gamma(\gamma_1) + \alpha_1 \log z + \log M(\alpha_1, \gamma_1, -z), \tag{H.1}$$

where $z = \frac{2}{\sigma^2 y(t,r_t)}$. We note that

$$\begin{aligned}
\lim_{\bar{T}\to\infty} \frac{\log y(t,r_t)}{\bar{T}-t} &= \lim_{\bar{T}\to\infty} \frac{\log r_t/p + \log(\exp(p(\bar{T}-t)) - 1))}{\bar{T}-t} \\
&= \lim_{\bar{T}\to\infty} \frac{\log(\exp(p(\bar{T}-t)) - 1))}{\bar{T}-t} \\
&= \lim_{\bar{T}\to\infty} \frac{p\log(\exp(p(\bar{T}-t)) - 1))}{\log\exp(p(\bar{T}-t))} \\
&= p.
\end{aligned} \tag{H.2}$$

The short rate contribution to the long ZCB yield is given by the formula

$$\begin{aligned}
h_\infty(t) &= -\lim_{\bar{T}\to\infty} \frac{1}{\bar{T}-t} \log G_{\bar{T}}(t) \\
&= \lim_{\bar{T}\to\infty} \frac{-1}{\bar{T}-t} (\log \Gamma(\gamma_1 - \alpha_1) - \log \Gamma(\gamma_1) + \alpha_1 \log z + \log M(\alpha_1, \gamma_1, -z)) \\
&= \lim_{\bar{T}\to\infty} \frac{-\alpha_1 \log z}{\bar{T}-t} \\
&= \lim_{\bar{T}\to\infty} \frac{\alpha_1 \log y(t,r_t)}{\bar{T}-t} \\
&= \alpha_1 p
\end{aligned} \tag{H.3}$$

as required. **Q.E.D.**

The following proof corresponds to the short rate contribution to the $\bar{T}$-forward rate.

Proof.[of Lemma 3.4.7] We write the logarithm of (3.4.22) as

$$\log G_{\bar{T}}(t) = \log \Gamma(\gamma_1 - \alpha_1) - \log \Gamma(\gamma_1) + \alpha_1 \log z + \log M(\alpha_1, \gamma_1, -z) \tag{H.4}$$

where $z = \frac{2}{\sigma^2 y(t,r_t)}$. We note that

$$\begin{aligned} \frac{\partial z}{\partial \bar{T}} &= \frac{2p}{\sigma^2 r_t} \times \frac{-1}{(\exp(p(\bar{T}-t))-1)^2} \times p \exp(p(\bar{T}-t)) \\ &= -z \frac{p \exp(p(\bar{T}-t))}{\exp(p(\bar{T}-t))-1} \\ &= -zp\left(1 + \frac{1}{\exp(p(\bar{T}-t))-1}\right). \end{aligned} \tag{H.5}$$

Also we have straightforwardly

$$\frac{\partial M(\alpha_1, \gamma_1, z)}{\partial z} = \frac{\alpha_1}{\gamma_1} M(\alpha_1 + 1, \gamma_1 + 1, z). \tag{H.6}$$

From (3.2.58) and the above relations we have

$$\begin{aligned} g_{\bar{T}}(t) &= -\frac{\partial}{\partial \bar{T}} \log G_{\bar{T}}(t) \\ &= -\frac{\alpha_1}{z}\frac{\partial z}{\partial \bar{T}} - \frac{1}{M(\alpha_1, \gamma_1, -z)} \frac{\partial M(\alpha_1, \gamma_1, -z)}{\partial z} \frac{\partial z}{\partial \bar{T}} \\ &= -\frac{\partial z}{\partial \bar{T}}\left(\frac{\alpha_1}{z} + \frac{1}{M(\alpha_1, \gamma_1, -z)} \frac{\partial M(\alpha_1, \gamma_1, -z)}{\partial z}\right) \\ &= zp\left(1 + \frac{1}{\exp(p(\bar{T}-t))-1}\right)\left(\frac{\alpha_1}{z} - \frac{\alpha_1}{\gamma_1}\frac{M(\alpha_1+1, \gamma_1+1, -z)}{M(\alpha_1, \gamma_1, -z)}\right) \end{aligned} \tag{H.7}$$

and simplifying gives the result. **Q.E.D.**

Appendix I

Approximate Pricing Formulae for Equity Index Options

The calculation of inverse Fourier transforms is computationally intensive on a computer and a faster computational method approximates the distribution of the GOP with a probability distribution having the same moments up to the second or third order and having the same form as the distribution of the discounted GOP. Another approach is to employ the Edgeworth series expansion up to second or third order which we do not describe here. So for a model involving a BS discounted GOP the distribution of the GOP is approximated by a lognormal distribution that matches the first two moments of the GOP. Also, for a model involving a MMM discounted GOP the distribution of the GOP is approximated by a noncentral gamma distribution that matches the first three moments of the GOP.

Because of the independence of the driving Wiener processes Z and W of the short rate and the discounted GOP, respectively, the moments of the GOP $S_T^{\delta_*}$ are the product of the corresponding moments of the savings account B_T and discounted GOP $\bar{S}_T^{\delta_*}$. Also, the k-th moment of the related random variable $R_T^{\delta_*}$ is

$$\frac{\mathrm{E}\left(\frac{S_t^{\delta_*}}{S_T^{\delta_*}}(S_T^{\delta_*})^k\right)}{\mathrm{E}\left(\frac{S_t^{\delta_*}}{S_T^{\delta_*}}\right)} = \frac{S_t^{\delta_*}}{P(t,T)}\mathrm{E}\big((S_T^{\delta_*})^{k-1}\big) \tag{I.1}$$

and therefore can be computed from the $(k-1)$-th moment of $S_T^{\delta_*}$.

The k-th moment of B_T is computed as

$$B_t^k MGF_L(k), \tag{I.2}$$

where $L = \log B_T/B_t$. When the discounted GOP obeys the BS model, the k-th moment of $\bar{S}_T^{\delta_*}$ is

$$(\bar{S}_t^{\delta_*})^k \exp\big(\frac{k}{2}\theta^2(T-t) + \frac{k^2}{2}\theta^2(T-t)\big). \tag{I.3}$$

When the discounted GOP obeys the MMM model, the first, second and third moments of $\bar{S}_T^{\delta_*}$, which is distributed as $NCG(2, 1/(2(\varphi_T - \varphi_t)), \lambda)$, are

$$\begin{aligned} \mathrm{E}\big(\bar{S}_T^{\delta_*}\big) &= (\varphi_T - \varphi_t)(4+\lambda) \\ \mathrm{E}\big((\bar{S}_T^{\delta_*})^2\big) &= (\varphi_T - \varphi_t)^2(8+4\lambda) \\ \mathrm{E}\big((\bar{S}_T^{\delta_*})^3\big) &= (\varphi_T - \varphi_t)^3(32+24\lambda). \end{aligned} \tag{I.4}$$

Having computed the moments of $S_T^{\delta_*}$ as the product of corresponding moments of B_T and $\bar{S}_T^{\delta_*}$ the lognormal approximations to the distributions of $S_T^{\delta_*}$ and $R_T^{\delta_*}$ are

$$\begin{aligned} S_T^{\delta_*} &\sim LN(m, v) \\ R_T^{\delta_*} &\sim LN(m-v, v), \end{aligned} \tag{I.5}$$

where

$$\begin{aligned} v &= \log\big(1 + \mathrm{Var}\big(S_T^{\delta_*}\big)/\mathrm{E}\big(S_T^{\delta_*}\big)^2\big) \\ m &= \log \mathrm{E}\big(S_T^{\delta_*}\big) - \frac{1}{2}v. \end{aligned} \tag{I.6}$$

Also, the noncentral gamma approximations to the distributions of $S_T^{\delta_*}$ and $R_T^{\delta_*}$ are

$$\begin{aligned} S_T^{\delta_*} &\sim NCG(\alpha, \gamma, \lambda) \\ R_T^{\delta_*} &\sim NCG(\alpha', \gamma', \lambda'), \end{aligned} \tag{I.7}$$

where α, γ, λ are given by

$$\begin{aligned} \gamma &= 2\frac{\mathrm{Var}\big(S_T^{\delta_*}\big)}{\mathrm{Skew}\big(S_T^{\delta_*}\big)} + \sqrt{4\bigg(\frac{\mathrm{Var}\big(S_T^{\delta_*}\big)}{\mathrm{Skew}\big(S_T^{\delta_*}\big)}\bigg)^2 - 2\frac{\mathrm{E}\big(S_T^{\delta_*}\big)}{\mathrm{Skew}\big(S_T^{\delta_*}\big)}} \\ \alpha &= 2\gamma\mathrm{E}\big(S_T^{\delta_*}\big) - \gamma^2\mathrm{Var}\big(S_T^{\delta_*}\big) \\ \lambda &= -2\gamma\mathrm{E}\big(S_T^{\delta_*}\big) + 2\gamma^2\mathrm{Var}\big(S_T^{\delta_*}\big) \end{aligned} \tag{I.8}$$

and α', γ', λ' have the corresponding formulae in terms of moments of $R_T^{\delta_*}$.

Using these approximations we can compute prices of various options straightforwardly from (5.3.7).

Appendix J

Proofs of Expectations Involving a Standard Normal Random Variable

This appendix provides the proofs of the results in Section 4.2.5.

Proof.[of Lemma 4.2.8] We have

$$\begin{aligned} \mathrm{E}(\exp(\alpha Z)) &= \int_{-\infty}^{\infty} \exp(\alpha u) n(u) du \\ &= \int_{-\infty}^{\infty} \exp(\frac{1}{2}\alpha^2) n(u-\alpha) du \\ &= \exp(\frac{1}{2}\alpha^2) \int_{-\infty}^{\infty} n(v) dv \\ &= \exp(\frac{1}{2}\alpha^2). \end{aligned} \tag{J.1}$$

Next we have

$$\begin{aligned} \mathrm{E}(\exp(\alpha Z)\mathbf{1}_{Z>z}) &= \int_{z}^{\infty} \exp(\alpha u) n(u) du \\ &= \int_{z}^{\infty} \exp(\frac{1}{2}\alpha^2) n(u-\alpha) du \\ &= \exp(\frac{1}{2}\alpha^2) \int_{z-\alpha}^{\infty} n(v) dv \\ &= \exp(\frac{1}{2}\alpha^2) \mathrm{E}(\mathbf{1}_{Z>z-\alpha}) \\ &= \exp(\frac{1}{2}\alpha^2)\big(1 - N(z-\alpha)\big), \end{aligned} \tag{J.2}$$

which is the second result. The third result is obtained by transposing the identity

$$\mathrm{E}(\exp(\alpha Z)\mathbf{1}_{Z>z}) + \mathrm{E}(\exp(\alpha Z)\mathbf{1}_{Z\leq z}) = \mathrm{E}(\exp(\alpha Z)) \tag{J.3}$$

and applying the first two results. **Q.E.D.**

Proof.[of Lemma 4.2.9] We have

$$\begin{aligned}
\mathrm{E}(N(\alpha Z+\beta)) &= \mathrm{E}(\mathrm{E}(\mathbf{1}_{Y\le\alpha Z+\beta}|Z)) \\
&= \mathrm{E}(\mathbf{1}_{Y\le\alpha Z+\beta}) \\
&= \mathrm{E}(\mathbf{1}_{Y\le\alpha Z+\beta}|Y) \\
&= \mathrm{E}(\mathbf{1}_{(Y-\alpha Z)/\sqrt{1+\alpha^2}\le\beta/\sqrt{1+\alpha^2}}) \\
&= \mathrm{E}(\mathbf{1}_{X\le\beta/\sqrt{1+\alpha^2}}) \\
&= N\left(\frac{\beta}{\sqrt{1+\alpha^2}}\right),
\end{aligned} \tag{J.4}$$

where we have made use of the facts $Y \sim N(0,1)$ and $X = (Y-\alpha Z)/\sqrt{1+\alpha^2} \sim N(0,1)$. **Q.E.D.**

Proof.[of Lemma 4.2.10] For standard normal random variables X and Y, and making use of Lemma 4.2.9, we have

$$\begin{aligned}
\mathrm{E}(\exp(\gamma Z)N(\alpha Z+\beta)) &= \mathrm{E}(\exp(\gamma Z)\mathrm{E}(\mathbf{1}_{Y\le\alpha Z+\beta}|Z)) \\
&= \mathrm{E}(\exp(\gamma Z)\mathbf{1}_{Y\le\alpha Z+\beta}) \\
&= \mathrm{E}(\mathrm{E}(\exp(\gamma Z)\mathbf{1}_{Y\le\alpha Z+\beta}|Y)) \\
&= \mathrm{E}(\mathrm{E}(\exp(\gamma Z)\mathbf{1}_{(Y-\beta)/\alpha\le Z}|Y)) \\
&= \mathrm{E}(\mathrm{E}(\exp(\frac{1}{2}\gamma^2)(1-N((Y-\beta)/\alpha-\gamma))|Y)) \\
&= \exp(\frac{1}{2}\gamma^2)\mathrm{E}(1-N((Y-\beta)/\alpha-\gamma)) \\
&= \exp(\frac{1}{2}\gamma^2)\mathrm{E}(1-N((Y-\beta)/\alpha-\gamma)) \\
&= \exp(\frac{1}{2}\gamma^2)\left(1-N\left(\frac{-\gamma-\beta/\alpha}{\sqrt{1+1/\alpha^2}}\right)\right) \\
&= \exp(\frac{1}{2}\gamma^2)N\left(\frac{\alpha\gamma+\beta}{\sqrt{1+\alpha^2}}\right),
\end{aligned} \tag{J.5}$$

which is the result. **Q.E.D.**

Appendix K

Proofs Involving Non-Central Chi-Squared Random Variables

Here we provide proofs to the results in Section 4.3.6.

Proof.[of Lemma 4.3.17]

$$
\begin{aligned}
\mathrm{E}\Big(\frac{1}{Z+1}f(Z)\Big) &= \exp(-\mu)\sum_{z=0}^{\infty}\frac{1}{z!}\mu^z\frac{1}{z+1}f(z) \qquad \text{(K.1)}\\
&= \frac{1}{\mu}\exp(-\mu)\sum_{z=0}^{\infty}\frac{1}{(z+1)!}\mu^{z+1}f(z)\\
&= \frac{1}{\mu}\exp(-\mu)\Big(-f(-1)+\sum_{z=0}^{\infty}\frac{1}{z!}\mu^z f(z-1)\Big)\\
&= -\frac{1}{\mu}\exp(-\mu)f(-1)+\frac{1}{\mu}\mathrm{E}\Big(f(Z-1)\Big).
\end{aligned}
$$

Q.E.D.

Proof.[of Lemma 4.3.18]

$$
\begin{aligned}
\mathrm{E}\Big(\frac{1}{X}\mathbf{1}_{X\le x}\Big) &= \int_0^x \frac{1}{u}f_{\chi^2_\nu}(u)du \qquad \text{(K.2)}\\
&= \frac{2^{-\nu/2}}{\Gamma(\nu/2)}\int_0^x u^{\nu/2-2}\exp(-u/2)du\\
&= \frac{2^{-1}}{\nu/2-1}\frac{2^{-(\nu-2)/2}}{\Gamma((\nu-2)/2)}\int_0^x u^{\nu/2-2}\exp(-u/2)du\\
&= \frac{2^{-1}}{\nu/2-1}\int_0^x f_{\chi^2_{\nu-2}}(u)du\\
&= \frac{1}{\nu-2}\mathrm{E}\Big(\mathbf{1}_{Y\le x}\Big).
\end{aligned}
$$

Q.E.D.

Proof.[of Lemma 4.3.19] We prove the first expectation. From Remark 3.3.9 we write $U = \chi^2_{4+2Z}$ where $Z \sim \text{Poisson}(\lambda/2)$, giving

$$\begin{aligned} \mathrm{E}\left(\frac{\lambda}{U}\mathbf{1}_{U\le x}\right) &= \mathrm{E}\left(\frac{\lambda}{\chi^2_{4+2Z}}\mathbf{1}_{\chi^2_{4+2Z}\le x}\right) \\ &= \mathrm{E}\left(\mathrm{E}\left(\frac{\lambda}{\chi^2_{4+2Z}}\mathbf{1}_{\chi^2_{4+2Z}\le x}\middle| Z\right)\right). \end{aligned} \tag{K.3}$$

Now from Lemma 4.3.18 we have that

$$\mathrm{E}\left(\frac{\lambda}{\chi^2_{4+2Z}}\mathbf{1}_{\chi^2_{4+2Z}\le x}\middle| Z\right) = \frac{\lambda}{2Z+2}\mathrm{E}\left(\mathbf{1}_{\chi^2_{2+2Z}\le x}\right) \tag{K.4}$$

and inserting this into (K.3) gives

$$\begin{aligned} \mathrm{E}\left(\frac{\lambda}{U}\mathbf{1}_{U\le x}\right) &= \mathrm{E}\left(\frac{\lambda}{2Z+2}\mathrm{E}\left(\mathbf{1}_{\chi^2_{2+2Z}\le x}\middle| Z\right)\right) \\ &= \frac{\lambda}{2}\mathrm{E}\left(\frac{1}{Z+1}\mathrm{E}\left(\mathbf{1}_{\chi^2_{2+2Z}\le x}\middle| Z\right)\right). \end{aligned} \tag{K.5}$$

Employing Lemma 4.3.17 with $f(Z) = \mathrm{E}\left(\mathbf{1}_{\chi^2_{2+2Z}\le x}\middle| Z\right)$ gives

$$\begin{aligned} \mathrm{E}\left(\frac{\lambda}{U}\mathbf{1}_{U\le x}\right) &= \mathrm{E}\left(\mathrm{E}\left(\mathbf{1}_{\chi^2_{2Z}\le x}\middle| Z\right)\right) - \exp(-\lambda/2)\mathrm{E}\left(\mathbf{1}_{\chi^2_0\le x}\right) \\ &= \mathrm{E}\left(\mathbf{1}_{\chi^2_{0,\lambda}\le x}\right) - \exp(-\lambda/2) \\ &= \chi^2_{0,\lambda}(x) - \exp(-\lambda/2). \end{aligned} \tag{K.6}$$

The second and third expectations have similar proofs. **Q.E.D.**

Proof.[of Lemma 4.3.20]

$$\begin{aligned} \mathrm{E}\left(\frac{1}{Z+1}\gamma^Z f(Z)\right) &= \exp(-\mu)\sum_{z=0}^{\infty}\frac{1}{z!}\mu^z\gamma^z\frac{1}{z+1}f(z) \\ &= \frac{1}{\mu\gamma}\exp(-\mu)\sum_{z=0}^{\infty}\frac{1}{(z+1)!}(\mu\gamma)^{z+1}f(z) \\ &= \frac{1}{\mu\gamma}\exp(-\mu)\left(-f(-1)+\sum_{w=0}^{\infty}\frac{1}{w!}(\mu\gamma)^w f(w-1)\right) \\ &= -\frac{1}{\mu\gamma}\exp(-\mu)f(-1)+\frac{1}{\mu\gamma}\exp(-\mu)\exp(\mu\gamma)\mathrm{E}\Big(f(W-1)\Big) \end{aligned} \tag{K.7}$$

where $W \sim \text{Poisson}(\mu\gamma)$. **Q.E.D.**

Proof.[of Lemma 4.3.21] The first expectation is evaluated straightforwardly as

$$
\begin{aligned}
\mathrm{E}\bigg(\frac{1}{X}\exp(-\tau X)\mathbf{1}_{X\leq x}\bigg) &= \int_0^x \frac{1}{u}\exp(-\tau u) f_{\chi^2_\nu}(u)du \qquad \text{(K.8)}\\
&= \frac{2^{-\nu/2}}{\Gamma(\nu/2)}\int_0^x u^{\nu/2-2}\exp(-\frac{u}{2}(2\tau+1))du\\
&= \frac{2^{-1}}{\nu/2-1}\frac{2^{-(\nu-2)/2}}{\Gamma((\nu-2)/2)}\int_0^x u^{\nu/2-2}\exp(-\frac{u}{2}(2\tau+1))du\\
&= \frac{2^{-1}}{\nu/2-1}\frac{1}{(2\tau+1)^{\nu/2-1}}\int_0^{x(2\tau+1)} f_{\chi^2_{\nu-2}}(v)dv\\
&= \frac{1}{\nu-2}\frac{1}{(2\tau+1)^{\nu/2-1}}\mathrm{E}\bigg(\mathbf{1}_{Y\leq(2\tau+1)x}\bigg).
\end{aligned}
$$

The second expectation follows similarly. **Q.E.D.**

Proof.[of Lemma 4.3.22] From Remark 3.3.9 we write $U = \chi^2_{4+2Z}$ where $Z \sim$ Poisson$(\lambda/2)$, giving

$$
\begin{aligned}
\mathrm{E}\bigg(\frac{\lambda}{U}\exp(-\tau U)\mathbf{1}_{U\leq x}\bigg) &= \mathrm{E}\bigg(\frac{\lambda}{\chi^2_{4+2Z}}\exp(-\tau\chi^2_{4+2Z})\mathbf{1}_{\chi^2_{4+2Z}\leq x}\bigg) \qquad \text{(K.9)}\\
&= \mathrm{E}\bigg(\mathrm{E}\bigg(\frac{\lambda}{\chi^2_{4+2Z}}\exp(-\tau\chi^2_{4+2Z})\mathbf{1}_{\chi^2_{4+2Z}\leq x}\bigg|Z\bigg)\bigg).
\end{aligned}
$$

Now from Lemma 4.3.21 we have that

$$
\mathrm{E}\bigg(\frac{\lambda}{\chi^2_{4+2Z}}\exp(-\tau\chi^2_{4+2Z})\mathbf{1}_{\chi^2_{4+2Z}\leq x}\bigg|Z\bigg) = \frac{\lambda}{2Z+2}\frac{1}{(2\tau+1)^{Z+1}}\mathrm{E}\bigg(\mathbf{1}_{\chi^2_{2+2Z}\leq(2\tau+1)x}\bigg) \qquad \text{(K.10)}
$$

and inserting this into (K.9) gives

$$
\begin{aligned}
\mathrm{E}\bigg(\frac{\lambda}{U}\exp(-\tau U)\mathbf{1}_{U\leq x}\bigg) &= \mathrm{E}\bigg(\frac{\lambda}{2Z+2}\frac{1}{(2\tau+1)^{Z+1}}\mathrm{E}\bigg(\mathbf{1}_{\chi^2_{2+2Z}\leq(2\tau+1)x}\bigg|Z\bigg)\bigg) \qquad \text{(K.11)}\\
&= \frac{\lambda}{2}\frac{1}{(2\tau+1)}\mathrm{E}\bigg(\frac{\lambda}{Z+1}\frac{1}{(2\tau+1)^{Z}}\mathrm{E}\bigg(\mathbf{1}_{\chi^2_{2+2Z}\leq(2\tau+1)x}\bigg|Z\bigg)\bigg).
\end{aligned}
$$

Employing Lemma 4.3.20 with $\gamma = 1/(2\tau+1)$ and $f(Z) = \mathrm{E}\bigg(\mathbf{1}_{\chi^2_{2+2Z}\leq(2\tau+1)x}\bigg|Z\bigg)$ gives

$$
\begin{aligned}
\mathrm{E}\bigg(\frac{\lambda}{U}\exp(-\tau U)\mathbf{1}_{U\leq x}\bigg) &= \exp(-\frac{1}{2}\lambda(1-1/(2\tau+1)))\mathrm{E}\bigg(\mathrm{E}\bigg(\mathbf{1}_{\chi^2_{2W}\leq(2\tau+1)x}\bigg|Z\bigg)\bigg) \qquad \text{(K.12)}\\
&\quad - \exp(-\lambda/2)\mathrm{E}\bigg(\mathbf{1}_{\chi^2_{0}\leq(2\tau+1)x}\bigg)
\end{aligned}
$$

where $W \sim \text{Poisson}(\lambda/(2\tau+1))$. Continuing we have

$$\begin{aligned}
&\mathrm{E}\left(\frac{\lambda}{U}\exp(-\tau U)\mathbf{1}_{U\leq x}\right) \qquad \text{(K.13)}\\
&= \exp(-\frac{1}{2}\lambda(1-1/(2\tau+1)))\mathrm{E}\left(\mathbf{1}_{\chi^2_{0,\lambda/(2\tau+1)}\leq(2\tau+1)x}\right) - \exp(-\lambda/2)\\
&= \exp(-\frac{1}{2}\lambda(1-1/(2\tau+1)))\chi^2_{0,\lambda/(2\tau+1)}((2\tau+1)x) - \exp(-\lambda/2).
\end{aligned}$$

Simplifying the exponential term gives the first expectation. The proof of the other expectation is similar. **Q.E.D.**

Appendix L

US Market Data Sets Used in this Thesis

Several data sets composed of short-term interest rates and stock market index values have been employed in this thesis. Each data set is in respect of the US market and have annual, monthly and daily observation frequencies.

L.1 Data Set A: Shiller's Annual US Data Set 1871 to 2012

Our first data set is the annual data set given by Shiller [1989] and has 141 observations, making it amenable to fast parameter estimation on the models used in this thesis. The cash account $B(t)$ commences with a value of one and is iteratively computed using the formula

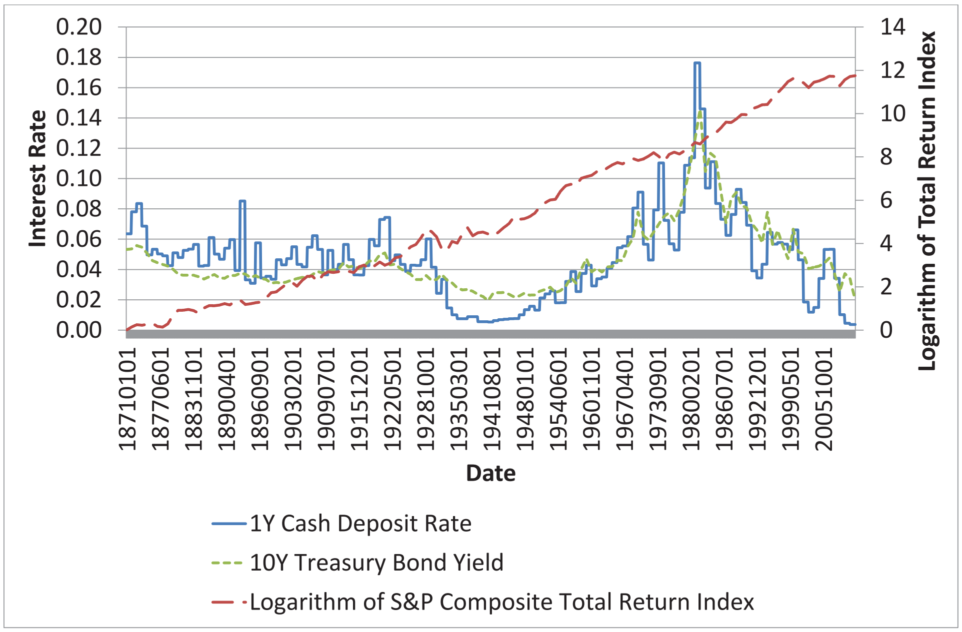
$$B(t+1) = (1 + r_t^{(1)})B(t)$$

where $r_t^{(1)}$ is the one-year cash rate. The stock index, used as a proxy for the growth optimal portfolio, commences with a value of one and is iteratively computed using the formula

$$S^{\delta_*}(t+1) = S^{\delta_*}(t) \times (I(t+1) + D(t+1))/I(t)$$

where $I(t)$ is the stock price index at the end of year t and $D(t)$ is the gross value of dividends paid during year t. We assume that a financial institution's hedging profits can be offset against its product offerings and therefore attract zero taxation. The ten-year semi-annual government bond yield is denoted by $y_t^{(10)}$ and is assumed to be a par bond yield.

Figure L.1: Graph of Interest Rates and Logarithm of S&P 500 Composite Total Return Index (Data Set B).



L.2 Data Set B: Shiller's Monthly US Data Set 1871 to 2012

The following monthly data set is interpolated from the annual data set given by Shiller [1989] in respect of the US and is shown in Figure L.1. The cash account $B(t)$ commences with a value of one and is iteratively computed using the formula

$$B(t + 1/12) = (1 + r_t^{(1)})^{1/12} B(t)$$

where $r_t^{(1)}$ is the one-year cash rate. The stock index, used as a proxy for the growth optimal portfolio, commences with a value of one and is iteratively computed using the formula

$$S^{\delta_*}(t + 1/12) = S^{\delta_*}(t) \times (I(t + 1/12) + D(t + 1/12))/I(t)$$

where $I(t)$ is the stock price index at time t and $D(t)$ is the gross value of dividends paid during the period $(t - 1/12, t)$. As before, the ten-year semi-annual government bond yield is denoted by $y_t^{(10)}$ and is assumed to be a par bond yield.

L.3 Data Set C: Shiller's Monthly US Data Set 1871 to 2017

This is the extended from Shiller's monthly US data set of cash rates and S&P500 total return values over the period from 1871 to 2012, as given in Shiller [1989] and his website. Values from 2013 to 2017 have been spliced using data obtained from Bloomberg data services.

L.4 Data Set D: Monthly Data Set January 1962 to June 2014

The cash rates and ten-year bond yields of the following data set are the one-year and ten-year Treasury bond yields sourced from the US Federal Reserve Bank website whereas the S&P 500 Total Return Index is sourced from Global Financial Data. The cash account $B(t)$, stock index $S^{\delta_*}(t)$ and ten-year semi-annual government bond yield $y_t^{(10)}$ have the same meaning as for Shiller's Annual Data Set, Data Set A in Section L.1.

L.5 Data Set E: Daily Data Set January 1970 to May 2017

The daily US data set comprises the Federal Funds Rate and S&P 500 total return index values over the period from January 1970 to May 2017, sourced from Bloomberg data services.

Appendix M

Algorithm for MLEs of SGH Parameters

The symmetric generalised hyperbolic (SGH) distribution is specified by the probability density function

$$f_L(x;\mu,\delta,\bar{\alpha},\lambda) = \frac{1}{\delta K_\lambda(\bar{\alpha})}\sqrt{\frac{\bar{\alpha}}{2\pi}}\left(1+\frac{(x-\mu)^2}{\delta^2}\right)^{\frac{1}{2}(\lambda-\frac{1}{2})} K_{\lambda-\frac{1}{2}}\left(\bar{\alpha}\sqrt{1+\frac{(x-\mu)^2}{\delta^2}}\right) \tag{M.1}$$

for $x \in (-\infty,\infty)$, location parameter $\mu \in (-\infty,\infty)$ and two shape parameters $\lambda \in (-\infty,\infty)$ and $\bar{\alpha} = \alpha\delta \in [0,\infty)$, defined so that they are invariant under scale transformations. The scale parameter is $\delta \in [0,\infty)$. The parameters α and δ are such that $\bar{\alpha} = \alpha\delta$ with $\alpha, \delta \in [0,\infty)$ and

$$\delta > 0,\ \alpha \geq 0, \qquad \text{if } \lambda < 0, \tag{M.2}$$

$$\delta > 0,\ \alpha > 0, \qquad \text{if } \lambda = 0, \tag{M.3}$$

$$\delta \geq 0,\ \alpha > 0, \qquad \text{if } \lambda > 0. \tag{M.4}$$

Also

$$K_\lambda(\omega) = \frac{1}{2}\int_0^\infty u^{\lambda-1}\exp(-\frac{1}{2}\omega(u+u^{-1}))du \tag{M.5}$$

is the modified Bessel function of the third kind with index λ, as given in Section 6.22 of Watson [1966].

Our methodology for obtaining the maximum likelihood estimates of the four parameters $\mu, \delta, \bar{\alpha}, \lambda$ is:

Input: For a given data set $\{x_i : i = 1, 2, \ldots, n\}$ we compute the maximum likelihood estimates of the SGH distribution;

Initialise: Prescribe grid step parameters $\Delta\bar{\alpha}$ and $\Delta\lambda$ and grid size parameter k;

Outer Loop: For each pair $(\bar{\alpha}, \lambda)$ in the grid specified by $\{(\bar{\alpha}_0 + i\Delta\bar{\alpha}, \lambda_0 + j\Delta\lambda) : i, j = -k, -k+1, \ldots, k\}$ compute the maximum likelihood estimates of μ and δ conditioned on the parameter pair $(\bar{\alpha}, \lambda)$;

Inner Loop: Compute the maximum likelihood estimates of μ and δ conditioned on the parameter pair $(\bar{\alpha}, \lambda)$ using the Newton-Raphson iterative scheme.

The outer loop of the algorithm is simply a grid search, whereas the inner loop of the algorithm is a two-dimensional Newton-Raphson iteration scheme. We now describe in more detail the workings of the inner loop.

We write the logarithm of the SGH probability density function in (6.2.1) as

$$\begin{aligned} g(x) &= \log f_L(x; \mu, \delta, \bar{\alpha}, \lambda) \\ &= \log c(\delta, \bar{\alpha}, \lambda) + \frac{1}{2}\left(\lambda - \frac{1}{2}\right) \log\left(1 + \frac{(x-\mu)^2}{\delta^2}\right) \\ &+ \log K_{\lambda - \frac{1}{2}}\left(\bar{\alpha}\sqrt{1 + \frac{(x-\mu)^2}{\delta^2}}\right) \end{aligned} \tag{M.6}$$

where the constant c is defined as

$$c(\delta, \bar{\alpha}, \lambda) = \frac{1}{\delta K_\lambda(\bar{\alpha})}\sqrt{\frac{\bar{\alpha}}{2\pi}} \tag{M.7}$$

and further, by letting $t = (x - \mu)/\delta$, we rewrite (M.6) as

$$h(t) = g(x) = \log c(\delta, \bar{\alpha}, \lambda) + \frac{1}{2}\left(\lambda - \frac{1}{2}\right)\log(1 + t^2) + \log K_{\lambda - \frac{1}{2}}(\bar{\alpha}\sqrt{1 + t^2}). \tag{M.8}$$

From (6.2.5) we can derive straightforwardly the relations

$$K_{\lambda+1}(\omega) = \frac{2\lambda}{\omega}K_\lambda(\omega) + K_{\lambda-1}(\omega) \tag{M.9}$$

and

$$\begin{aligned} \frac{\partial K_\lambda(\omega)}{\partial \omega} &= -\frac{1}{2}\big(K_{\lambda+1}(\omega) + K_{\lambda-1}(\omega)\big) \\ &= -\frac{\lambda}{\omega}K_\lambda(\omega) - K_{\lambda-1}(\omega) \end{aligned} \tag{M.10}$$

which allow us to more easily differentiate (M.8) with respect to t, giving

$$\begin{aligned} h'(t) &= \big(\lambda - \frac{1}{2}\big)\frac{t}{1+t^2} + \frac{t}{\sqrt{1+t^2}} \times \frac{\bar{\alpha}K'_{\lambda-\frac{1}{2}}(\bar{\alpha}\sqrt{1+t^2})}{K_{\lambda-\frac{1}{2}}(\bar{\alpha}\sqrt{1+t^2})} \qquad \text{(M.11)} \\ &= \big(\lambda - \frac{1}{2}\big)\frac{t}{1+t^2} - \frac{\bar{\alpha}t}{\sqrt{1+t^2}} \times \Big(\frac{\lambda}{\bar{\alpha}\sqrt{1+t^2}} + \frac{K_{\lambda-\frac{3}{2}}(\bar{\alpha}\sqrt{1+t^2})}{K_{\lambda-\frac{1}{2}}(\bar{\alpha}\sqrt{1+t^2})}\Big)) \\ &= \frac{t}{\sqrt{1+t^2}}\Bigg(\frac{\lambda - \frac{1}{2}}{\sqrt{1+t^2}} - \frac{\lambda}{\sqrt{1+t^2}} - \frac{\bar{\alpha}K_{\lambda-\frac{3}{2}}(\bar{\alpha}\sqrt{1+t^2})}{K_{\lambda-\frac{1}{2}}(\bar{\alpha}\sqrt{1+t^2})}\Bigg) \\ &= -\frac{t}{\sqrt{1+t^2}}\Bigg(\frac{1}{2\sqrt{1+t^2}} + \frac{\bar{\alpha}K_{\lambda-\frac{3}{2}}(\bar{\alpha}\sqrt{1+t^2})}{K_{\lambda-\frac{1}{2}}(\bar{\alpha}\sqrt{1+t^2})}\Bigg). \end{aligned}$$

For a given data set $X = \{x_i : i = 1, 2, \dots, n\}$ the log likelihood function is

$$g_X(\mu, \delta, \bar{\alpha}, \lambda) = \sum_{i=1}^{n} h(t_i; \mu, \delta, \bar{\alpha}, \lambda) \qquad \text{(M.12)}$$

where $t_i = (x_i - \mu)/\delta$, for $i = 1, 2, \dots, n$. The maximum likelihood estimates of μ and δ, conditional on $\bar{\alpha}$ and λ, are the solutions to the equations

$$\begin{aligned} \frac{\partial g_X}{\partial \mu} &= 0 \qquad \text{(M.13)} \\ \frac{\partial g_X}{\partial \delta} &= 0. \end{aligned}$$

We have that

$$\begin{aligned} \frac{\partial g_X}{\partial \mu} &= \sum_{i=1}^{n} h'(t_i)\frac{\partial t_i}{\partial \mu} \qquad \text{(M.14)} \\ &= -\frac{1}{\delta}\sum_{i=1}^{n} h'(t_i) \end{aligned}$$

and that

$$\begin{aligned} \frac{\partial g_X}{\partial \delta} &= n\frac{\partial \log c(\delta, \bar{\alpha}, \lambda)}{\partial \delta} + \sum_{i=1}^{n} h'(t_i)\frac{\partial t_i}{\partial \delta} \qquad \text{(M.15)} \\ &= -\frac{n}{\delta} - \frac{1}{\delta}\sum_{i=1}^{n} t_i h'(t_i). \end{aligned}$$

We compute initial estimates (μ_0, δ_0) of (μ, δ) as

$$\begin{aligned} \mu_0 &= \frac{1}{n}\sum_{i=1}^{n} x_i \qquad \text{(M.16)} \\ \delta_0 &= \sqrt{\frac{\bar{\alpha}K_{\lambda}(\bar{\alpha})}{K_{\lambda+1}(\bar{\alpha})}} \times \sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \big(\frac{1}{n}\sum_{i=1}^{n} x_i\big)^2}. \end{aligned}$$

From (M.11) we write

$$h'(t) = -\frac{x-\mu}{\delta} h_1(t) \tag{M.17}$$

where

$$h_1(t) = \frac{1}{\sqrt{1+t^2}} \left(\frac{1}{2\sqrt{1+t^2}} + \frac{\bar{\alpha} K_{\lambda-\frac{3}{2}}(\bar{\alpha}\sqrt{1+t^2})}{K_{\lambda-\frac{1}{2}}(\bar{\alpha}\sqrt{1+t^2})} \right). \tag{M.18}$$

Then the condition $\partial g_X / \partial \mu = 0$ allows for an iterative formula for μ by virtue of the equivalence relations

$$\begin{aligned} \frac{\partial g_X}{\partial \mu} = 0 &\Leftrightarrow \sum h'(t_i) = 0 \\ &\Leftrightarrow \sum (x_i - \mu) h_1(t_i) = 0 \\ &\Leftrightarrow \mu = \frac{\sum x_i h_1(t_i)}{\sum h_1(t_i)}. \end{aligned} \tag{M.19}$$

So we arrive at the iterative formula

$$\mu_{j+1} = \frac{\sum x_i h_1(t_i)}{\sum h_1(t_i)} \tag{M.20}$$

where the right hand side uses the values μ_j and δ_j.

Also the condition $\partial g_X / \partial \delta = 0$ allows for an iterative formula for δ by virtue of the equivalence relations

$$\begin{aligned} \frac{\partial g_X}{\partial \delta} = 0 &\Leftrightarrow \sum t_i h'(t_i) = -n \\ &\Leftrightarrow \frac{1}{\delta^2} \sum (x_i - \mu)^2 h_1(t_i) = n \\ &\Leftrightarrow \delta^2 = \frac{\sum (x_i - \mu)^2 h_1(t_i)}{n}. \end{aligned} \tag{M.21}$$

So we arrive at the iterative formula

$$\delta_{j+1}^2 = \frac{\sum (x_i - \mu_{j+1})^2 h_1(t_i)}{n} \tag{M.22}$$

where the right hand side uses the values μ_{j+1} and δ_j.

We continue the iterative formulae (M.20) and (M.22) in turn until the convergence criteria

$$\begin{aligned} |\mu_{j+1} - \mu_j| &< \epsilon \\ |\delta_{j+1} - \delta_j| &< \epsilon \end{aligned} \tag{M.23}$$

are satisfied for a prespecified tolerance level $\epsilon > 0$, thereby concluding the execution of the inner loop of the parameter estimation algorithm.

Appendix N

Asymptotic Properties of Functions Involving the Modified Bessel Function of the First Kind

We give some asymptotic properties of the function $f(x) = \log\{\exp(-x)I_1(x)\}$ given in (4.3.27).

Lemma N.0.1 *For the function f in* (4.3.27) *we have the asymptotic formulae*

$$f'(x) = -\frac{1}{2x} + \frac{3}{8x^2} + \frac{3}{8x^3} + O\left(\frac{1}{x^4}\right) \tag{N.0.1}$$

$$f''(x) = \frac{1}{2x^2} - \frac{3}{4x^3} + O\left(\frac{1}{x^4}\right)$$

Proof. As given in Chapter 6 of Watson [1966], the modified Bessel function of the first kind can be written as

$$I_\nu(x) = \frac{1}{\pi}\int_0^\pi \exp(x\cos(t))\cos(\nu t)\,dt - \frac{\sin\nu\pi}{\pi}\int_0^\infty \exp(-x\cosh t - \nu t)\,dt, \tag{N.0.2}$$

which simplifies to

$$I_1(x) = \frac{1}{\pi}\int_0^\pi \exp(x\cos(t))\cos(t)\,dt \tag{N.0.3}$$

when $\nu = 1$. Given $f(x) = \log\{\exp(-x)I_1(x)\}$ we have

$$f'(x) = \frac{d}{dx}\left\{-x + \log I_1(x)\right\} = -1 + \frac{I_1'(x)}{I_1(x)} \tag{N.0.4}$$

$$f''(x) = \frac{d}{dx}\left\{-1 + \frac{I_1'(x)}{I_1(x)}\right\} = \frac{I_1''(x)}{I_1(x)} - \frac{I_1'(x)^2}{I_1(x)^2}.$$

Making the change of variables $u = 2\sqrt{x}\sin(t/2)$ in (N.0.3) gives

$$\begin{aligned}
I_1(x) &= \frac{1}{\pi}\exp(x)\int_0^{\pi}\exp(x\cos(t)-x)\cos(t)\,dt \qquad \text{(N.0.5)}\\
&= \frac{1}{\pi}\exp(x)\int_0^{\pi}\exp(-2x\sin^2(t/2))\cos(t)\,dt\\
&= \frac{1}{\pi}\exp(x)\int_0^{2\sqrt{x}}\exp(-u^2/2)\frac{\cos(t)}{\sqrt{x}\cos(t/2)}\,du\\
&= \frac{1}{\pi}\frac{\exp(x)}{\sqrt{x}}\int_0^{2\sqrt{x}}\exp(-u^2/2)(1-\frac{1}{2}u^2/x)(1-\frac{1}{4}u^2/x)^{-\frac{1}{2}}\,du\\
&= \frac{\exp(x)}{\sqrt{2\pi x}}\sqrt{\frac{2}{\pi}}\left\{\int_0^{\infty}-\int_{2\sqrt{x}}^{\infty}\right\}\exp(-u^2/2)(1-\frac{u^2}{2x})\\
&\times(1+\frac{u^2}{8x}+\frac{3u^4}{128x^2}+\frac{5u^6}{1024x^3}+O(\frac{u^8}{x^4}))\,du\\
&= \frac{\exp(x)}{\sqrt{2\pi x}}\sqrt{\frac{2}{\pi}}\int_0^{\infty}\exp(-u^2/2)(1-\frac{3u^2}{8x}-\frac{5u^4}{128x^2}-\frac{7u^6}{1024x^3}+O(\frac{u^8}{x^4}))\,du\\
&= \frac{\exp(x)}{\sqrt{2\pi x}}\left\{1-\frac{3}{8x}-\frac{15}{128x^2}-\frac{105}{1024x^3}+O\left(\frac{1}{x^4}\right)\right\},
\end{aligned}$$

where we have made use of

$$\sqrt{\frac{2}{\pi}}\int_0^{\infty}\exp(-u^2/2)u^{2k}\,du = (2k-1)(2k-3)\dots 1, \qquad \text{(N.0.6)}$$

for $k \in \{0, 1, 2, \dots\}$ and

$$\int_{2\sqrt{x}}^{\infty}\exp(-u^2/2)\,du \leq \int_{2\sqrt{x}}^{\infty}\frac{u}{2\sqrt{x}}\exp(-u^2/2)\,du = \frac{\exp(-2x)}{2\sqrt{x}}. \qquad \text{(N.0.7)}$$

Similarly we have

$$\begin{aligned}
I_1'(x) &= \frac{1}{\pi}\exp(x)\int_0^{\pi}\exp(x\cos(t)-x)\cos^2(t)\,dt \qquad \text{(N.0.8)}\\
&= \frac{1}{\pi}\frac{\exp(x)}{\sqrt{x}}\int_0^{2\sqrt{x}}\exp(-u^2/2)(1-\frac{1}{2}u^2/x)^2(1-\frac{1}{4}u^2/x)^{-\frac{1}{2}}\,du\\
&= \frac{\exp(x)}{\sqrt{2\pi x}}\sqrt{\frac{2}{\pi}}\int_0^{\infty}\exp(-u^2/2)(1-\frac{7u^2}{8x}+\frac{19u^4}{128x^2}+\frac{13u^6}{1024x^3}+O(\frac{u^8}{x^4}))\,du\\
&= \frac{\exp(x)}{\sqrt{2\pi x}}\left\{1-\frac{7}{8x}+\frac{57}{128x^2}+\frac{195}{1024x^3}+O\left(\frac{1}{x^4}\right)\right\}
\end{aligned}$$

and

$$I_1''(x) = \frac{1}{\pi}\exp(x)\int_0^{\pi}\exp(x\cos(t)-x)\cos^3(t)\,dt \tag{N.0.9}$$
$$= \frac{1}{\pi}\frac{\exp(x)}{\sqrt{x}}\int_0^{2\sqrt{x}}\exp(-u^2/2)(1-\frac{1}{2}u^2/x)^3(1-\frac{1}{4}u^2/x)^{-\frac{1}{2}}\,du$$
$$= \frac{\exp(x)}{\sqrt{2\pi x}}\sqrt{\frac{2}{\pi}}\int_0^{\infty}\exp(-u^2/2)(1-\frac{11u^2}{8x}+\frac{75u^4}{128x^2}-\frac{63u^6}{1024x^3}+O(\frac{u^8}{x^4}))\,du$$
$$= \frac{\exp(x)}{\sqrt{2\pi x}}\left\{1-\frac{11}{8x}+\frac{225}{128x^2}-\frac{945}{1024x^3}+O\left(\frac{1}{x^4}\right)\right\}.$$

From these expressions we have

$$\frac{I_1'(x)}{I_1(x)} = 1-\frac{1}{2x}+\frac{3}{8x^2}+\frac{3}{8x^3}+O\left(\frac{1}{x^4}\right) \tag{N.0.10}$$
$$\frac{I_1'(x)^2}{I_1(x)^2} = 1-\frac{1}{x}+\frac{1}{x^2}+\frac{3}{8x^3}+O\left(\frac{1}{x^4}\right)$$
$$\frac{I_1''(x)}{I_1(x)} = 1-\frac{1}{x}+\frac{3}{2x^2}-\frac{3}{8x^3}+O\left(\frac{1}{x^4}\right)$$

and the result follows. **Q.E.D.**

Appendix O

Proofs on Real-World Pricing of Swaptions

Proof.[of Theorem 8.1.1] We employ the real-world pricing formula (5.1.1) to the payoff H_T given in (8.1.2). The real-world price of the bond put option is

$$\begin{aligned} V_t^{\delta_{H_T}} &= \mathrm{E}\left(\frac{S_t^{\delta_*}}{S_T^{\delta_*}} H_T \middle| \mathcal{A}_t \right) \\ &= \mathrm{E}\left(\frac{\bar{S}_t^{\delta_*}}{\bar{S}_T^{\delta_*}} \frac{B_t}{B_T} H_T \middle| \mathcal{A}_t \right) \\ &= \mathrm{E}\left(\frac{\bar{S}_t^{\delta_*}}{\bar{S}_T^{\delta_*}} \mathrm{E}\left(\frac{B_t}{B_T} H_T \middle| \bar{S}_T^{\delta_*}, \mathcal{A}_t \right) \middle| \mathcal{A}_t \right). \end{aligned} \tag{O.0.1}$$

Now let $U = \bar{S}_T^{\delta_*}/(\varphi_T - \varphi_t)$ which is a non-central chi-squared random variable having four degrees of freedom and non-centrality parameter λ, see Platen and Heath [2006]. We can write the inner expectation above as

$$\begin{aligned} & V_t^{\delta_{H_T}}(u) \\ &= \mathrm{E}\left(\frac{B_t}{B_T} H_T \middle| \bar{S}_T^{\delta_*} = (\varphi_T - \varphi_t)u, \mathcal{A}_t \right) \\ &= \mathrm{E}\left(\frac{B_t}{B_T} H_T \middle| U = u, \mathcal{A}_t \right) \\ &= \mathrm{E}\left(\frac{B_t}{B_T} \left(1 - \sum_{i=1}^{n} (R(T_i - T_{i-1}) + \mathbf{1}_{i=n})(1 - \exp(-\tau_i u)) \mathrm{E}\left(\frac{B_T}{B_{T_i}} \middle| \mathcal{A}_T \right) \right)^+ \middle| \mathcal{A}_t \right), \end{aligned} \tag{O.0.2}$$

which we see is the value of a put option on a portfolio of zero-coupon bonds. By analogy to (8.1.4) we have

$$\begin{aligned} V_t^{\delta_{H_T}}(u) = \sum_{i=1}^{n} \Bigg(& \big(R(T_i - T_{i-1}) + \mathbf{1}_{i=n}\big) \times (1 - \exp(-\tau_i u)) \\ & \times \big(- A(t,T_i) \exp(-r_t B(t,T_i)) N(-d_i^{(1)}(u)) \\ & + A(t,T) \exp(-r_t B(t,T)) K_i(u) N(-d_i^{(2)}(u)) \big) \Bigg), \end{aligned} \tag{O.0.3}$$

where A and B are given in (3.2.49) and (3.2.48) and $d_i^{(1)}(u)$ and $d_i^{(2)}(u)$ are given by

$$d_i^{(1)}(u) = \frac{1}{\sigma_i} \log\left(\frac{A(t,T_i)\exp(-r_t B(t,T_i))}{A(t,T)\exp(-r_t B(t,T))K_i(u)} \right) + \frac{1}{2}\sigma_i \tag{O.0.4}$$

$$d_i^{(2)}(u) = \frac{1}{\sigma_i} \log\left(\frac{A(t,T_i)\exp(-r_t B(t,T_i))}{A(t,T)\exp(-r_t B(t,T))K_i(u)} \right) - \frac{1}{2}\sigma_i \tag{O.0.5}$$

with σ_i given by

$$\sigma_i = \sigma B(T,T_i) \sqrt{\frac{1}{2\kappa}(1 - \exp(-2\kappa(T-t)))} \tag{O.0.6}$$

and $K_i(u)$ given by

$$K_i(u) = A(T,T_i)\exp(-x_u B(T,T_i)). \tag{O.0.7}$$

Here x_u is the solution to the equation

$$1 = \sum_{i=1}^{n} (R(T_i - T_{i-1}) + \mathbf{1}_{i=n})(1 - \exp(-\tau_i u)) A(T,T_i)\exp(-x_u B(T,T_i)). \tag{O.0.8}$$

Therefore

$$\begin{aligned} V_t^{\delta_{H_T}} &= \mathrm{E}\left(\frac{\bar{S}_t^{\delta_*}}{\bar{S}_T^{\delta_*}} V_t^{\delta_{H_T}}\left(\frac{\bar{S}_T^{\delta_*}}{\varphi_T - \varphi_t} \right) \Bigg| \mathcal{A}_t \right) \\ &= \int_0^\infty \frac{\lambda}{u} V_t^{\delta_{H_T}}(u) f_{\chi^2_{4,\lambda}}(u)\, du, \end{aligned} \tag{O.0.9}$$

as required. **Q.E.D.**

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