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- A unified polynomial expansion is established for interval model, random model and hybrid uncertain model;
 - The arbitrary polynomial chaos is extended for interval analysis and hybrid uncertain analysis;
 - The method is applied to structure-acoustic problem with interval/random variables involving complex probability distribution;
 - The proposed method has been compared with the hybrid perturbation method;
 - The proposed method for three uncertain models has been compared with several widely used polynomial chaos methods.

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Unified polynomial expansion for interval and random response analysis of uncertain structure-acoustic system with arbitrary probability distribution

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Abstract

For structure-acoustic system with uncertainties, the interval model, the random model and the hybrid uncertain model have been introduced. In the interval model and the random model, the uncertain parameters are described as either the random variable with well defined *probability density function*(PDF) or the interval variable without any probability information, whereas in the hybrid uncertain model both interval variable and random variable exist simultaneously. For response analysis of these three uncertain models of structure-acoustic problem involving arbitrary PDFs, a unified polynomial expansion method named as the *Interval and Random Arbitrary Polynomial Chaos method*(IRAPCM) is proposed. In IRAPCM, the response of the structure-acoustic system is approximated by APC expansion in a unified form. Particularly, only the weight function of polynomial basis is required to be changed to construct the APC expansion for the response of different uncertain models. Through the unified APC expansion, the uncertain properties of the response of three uncertain

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models can be efficiently obtained. As the APC expansion can provide a free choice of the polynomial basis, the optimal polynomial basis for the random variable with arbitrary PDFs can be obtained by using the proposed IRAPCM. The IRAPCM has been employed to solve a mathematical problem and a structure-acoustic problem, and the effectiveness of the unified IRAPCM for response analysis of three uncertain models is demonstrated by fully comparing it with the hybrid first-order perturbation method and several existing polynomial chaos methods.

Key words: Interval model; Random model; Hybrid uncertain model; Arbitrary Polynomial Chaos; Gauss integration; Structure-acoustic system.

1 Introduction

The response analysis of structural-acoustic system is a key procedure for the control and optimization of the vibration and noise behaviors of engineering products, such as automobiles, steamships, aircrafts, submarines and spacecrafts. Traditional methods for response analysis of structural-acoustic system are deterministic numerical methods by assuming that all input parameters are fixed[1]. However, uncertainties related to material properties, boundary conditions and surrounding environment are unavoidable in the real engineering practices. Without considering these uncertainties, the results obtained by using deterministic numerical methods may be unreliable. Therefore, there is a growing interest for developing numerical methods for the response analysis of structural-acoustic system with uncertainties[2-6].

The most widely used technique for uncertainty quantification is the probabilistic method, in which the uncertain parameter is represented by the random variable with well defined *probability density function*(PDF). During past decades, lots of methods have been proposed for random uncertainty quantification, such as the Monte Carlo method[7-9], the perturbation probabilistic method[10-13] and the polynomial chaos method[14,15]. Among these methods, the Monte Carlo method is the simplest and the most versatile method for uncertain problems. However, the Monte Carlo method

1 suffers from tremendous computational cost for large-scale engineering systems[7].
2 The perturbation **probabilistic method** is a very efficient way for random analysis, but
3 it is only accurate for uncertain problems with small uncertainty level[10]. The
4 polynomial chaos method is proposed based on orthogonal polynomial theory, which
5 is free from small perturbation assumption and the efficiency is much higher than
6 Monte Carlo method[14]. Thus, the polynomial chaos method has been widely used to
7 solve random engineering problems[16-18].

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10 The **probabilistic method** is established based on the condition that the precise
11 probability distribution is obtained. However, at the early stage of design, the PDF of
12 random variables may be not available due to the limited information. To model the
13 uncertain problems with limited information, various of non-probabilistic
14 mathematical frameworks have been developed, such as the interval analysis[19-21],
15 the fuzzy theory[22, 23], the evidence theory[24-26] and the p-box set[27,28]. All
16 these non-probabilistic mathematical frameworks have their own merit in application.
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18 **The fuzzy theory is an effectively technique to model the subjective probability**
19 **derived from the expert opinions.** The evidence theory and the p-box set are suitable
20 to represent imprecise probability. In the interval analysis, only the lower and upper
21 bounds of an uncertain parameter are required. Thus, the interval analysis is most
22 suitable to describe the uncertainties whose probability information is completely
23 missing. As the determination of bounds for an interval may be easier and more
24 straightforward than the identification of an imprecise probability distribution, the
25 interval analysis is also a popular mathematical framework to deal with the
26 uncertainties in engineering problems. Researches on **uncertainty quantification** of
27 interval model is rather mature and different methods have been proposed, including
28 the interval perturbation method[29-31], the interval Chebyshev method[32], the
29 interval Legendre method[33], the interval factor method[34], the vertex method[35],
30 the rational expansion method[36,37] and et al. More detailed review of interval
31 methods can be found from Ref.[38].

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Obviously, the interval methods and the random methods aforementioned are
focused on uncertain problem with either random or interval parameters. However,

1 the random and interval parameters may exist simultaneously in some engineering
2 problems. To represent the hybrid uncertainties, Elishakoff and Colombi developed a
3 hybrid uncertain model, in which some uncertain parameters with well defined PDFs
4 are treated as random variables, whereas the others are described as interval
5 variables[39]. The **uncertainty quantification** of the hybrid uncertain model is more
6 challenging than the interval uncertainty quantification and the random uncertainty
7 quantification, as the approximation for the response related to different types of
8 uncertainty in the hybrid uncertain model should be properly integrated[40]. Up to
9 now, the studies for uncertainty quantification of the hybrid uncertain model are
10 relatively small. The perturbation technique is a general choice for the hybrid
11 uncertain analysis in the last decades, but it is limited to hybrid uncertain problems
12 with small uncertainty level[41-44]. Recently, the polynomial chaos method has been
13 developed for hybrid uncertain analysis. By integrating the Chebyshev polynomial
14 with the *generalized Polynomial Chaos*(gPC), Wu et al. proposed a hybrid method for
15 uncertainty quantification and robust topology optimization [45, 46]. Subsequently,
16 Yin et al. employed the Gegenbauer polynomial of gPC to construct a unified
17 polynomial chaos expansion for structure-acoustic problems with interval and/or
18 random uncertainties[47]. Wang et.al developed a response surface method for
19 structural-acoustic systems with random and interval parameters based on the
20 gPC[48]. To improve the computational efficiency for interval analysis of gPC
21 expansion, Xu et. al developed a hybrid uncertainty analysis method by introducing
22 the dimension wise analysis[49]. Compared with the perturbation technique based
23 method, these gPC based methods have shown better accuracy for hybrid uncertain
24 problem with large uncertainty level.

25 The random model, the interval model and the hybrid uncertain model listed
26 above can be used to describe the uncertain system with interval and/or random
27 variables in different cases according to the available information. For the uncertainty
28 quantification of these three uncertain models, the polynomial chaos method can be
29 effectively used for the uncertain problem with large uncertainty level and the
30 efficiency is much higher than the Monte Carlo method. Thus, this paper will focus on
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1 the application of polynomial chaos method for uncertainty quantification of these
2 three uncertain models. From the overall perspective, though the polynomial chaos
3 method has gained a great achievement for uncertainty analysis, some important
4 issues still remain unresolved. Firstly, as we mentioned before, the polynomial chaos
5 methods for hybrid uncertain model are generally developed based on the polynomial
6 basis of gPC. However, the accuracy and efficiency of these gPC based methods may
7 be deteriorated for hybrid uncertain problem with the probability distribution out of
8 Askey scheme, as the optimal polynomial basis of polynomial chaos expansion for
9 uncertainty analysis with the probability distribution out of Askey scheme cannot be
10 obtained by using gPC[50]. Secondly, there is little research on developing the unified
11 polynomial expansion method for interval model, random model and hybrid uncertain
12 model, especially when the random parameter of these uncertain models is following
13 an arbitrary probability distribution. Recently, the Gegenbauer polynomial has been
14 developed to construct the unified polynomial expansion for interval model, random
15 model and hybrid uncertain model[47]. By using the unified Gegenbauer expansion,
16 the response for these three uncertain models can be obtained by using a common
17 numerical algorithm. However, unified Gegenbauer expansion method is only suitable
18 for the uncertain problem with the bounded random variable following mono-valley
19 or mono-peak probability distributions[47]. As regarding the engineering application,
20 the PDF of random variable can be an arbitrary function, sometimes may be very
21 complex. Therefore, it is desirable to develop new unified polynomial expansion
22 method that can be used for three uncertain models with interval variable and/or
23 random variable following arbitrary probability distributions.

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48 The aim of the present study is to develop a new unified polynomial expansion
49 method for response analysis of structure-acoustic systems with interval and/or
50 random variables. For structure-acoustic systems with interval and/or random
51 variables, three uncertain models will be considered, namely the interval model, the
52 random model and the hybrid uncertain model. In order to construct the unified
53 polynomial expansion for these three uncertain models, the Arbitrary Polynomial
54 Chaos(APC) which has been successfully applied to uncertainty analysis with random
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variable following arbitrary probability distributions[51-53], will be developed for the uncertainty quantification of interval model and hybrid uncertain model. With this development, the unified *Interval and Random Arbitrary Polynomial Chaos method*(IRAPCM) is proposed to predict the response of three uncertain models of structure-acoustic system. In IRAPCM, the response of three uncertain models is approximated by the APC expansion in a unified form. For different uncertain models, only the weight function of polynomial basis is changed to construct the APC expansion. The coefficients of APC expansion are calculated though the Gauss integration. Once the APC expansion for uncertain models is obtained, the uncertain properties of response can be easily computed. The proposed IRAPCM is applied to a simple mathematical problem and a structure-acoustic problem. The effectiveness of IRAPCM for response analysis of interval model, random model and hybrid uncertain model has been investigated by comparing it with **the hybrid first-order perturbation method and** several existing polynomial chaos methods.

2 Fundamentals of the arbitrary polynomial chaos expansion

This section will briefly summarize the fundamentals of APC theory. Besides, the Gauss integration will be introduced to compute the coefficient of APC expansion due to its robustness and good efficiency. Furthermore, in order to efficiently calculate the weights and nodes of Gauss integration, the polynomial basis of APC expansion is constructed based on the recursive relations of the monic orthogonal polynomial.

2.1 Arbitrary polynomial chaos expansion for a function

A function $Y(\xi)$ approximated by the APC expansion can be expressed as follows

$$Y(\xi) = \sum_{i=0}^N y_i \varphi_i(\xi) \quad (1)$$

where N is the retained order of APC expansion, y_i represents the expansion

coefficient to be estimated, $\varphi_i(\xi)$ denotes the polynomial basis of order i , which satisfied the following orthogonality relation

$$\langle \varphi_i(\xi), \varphi_j(\xi) \rangle = h_i \delta_{ij} \quad (2)$$

where $h_i = \langle \varphi_i^2(\xi) \rangle$, δ_{ij} denotes the Kronecker delta and $\langle \cdot, \cdot \rangle$ denotes the inner product with respect to the weight function in a specific domain Ω . $\langle \varphi_i(\xi), \varphi_j(\xi) \rangle$ can be expressed as

$$\langle \varphi_i(\xi), \varphi_j(\xi) \rangle = \int_{\Omega} \varphi_i(\xi) \varphi_j(\xi) w(\xi) d\xi \quad (3)$$

where, $w(\xi)$ is the weight function. $w(\xi)$ in the framework of APC theory can be an arbitrary continuous or discrete function, such as the piecewise function. The free choice of the weight function of polynomial basis is the main advantage of APC expansion.

For multi-dimension uncertain problems, $Y(\xi)$ can be approximated by using the tensor order APC expansion as follows

$$Y(\xi) = \sum_{i_1=0}^{N_1} \cdots \sum_{i_L=0}^{N_L} y_{i_1, \dots, i_L} \varphi_{i_1, \dots, i_L}(\xi) \quad (4)$$

where, $\xi = [\xi_1, \xi_2, \dots, \xi_L]$ is a L -dimension vector, $N_i (i=1, 2, \dots, L)$ denotes the retained order of APC expansion related to ξ_i , y_{i_1, \dots, i_L} is the expansion coefficient to be estimated, $\varphi_{i_1, \dots, i_L}$ is the L -dimension polynomial basis, which is given by

$$\varphi_{i_1, \dots, i_L}(\xi) = \prod_{j=1}^L \varphi_{i_j}(\xi_j), \quad j=1, 2, \dots, L, \quad i_j=1, 2, \dots \quad (5)$$

In the above equation, $\varphi_{i_j}(\xi_j) (j=1, 2, \dots, L)$ denotes the polynomial basis related to ξ_j , i_j denotes the order of the polynomial basis $\varphi_{i_j}(\xi_j)$.

2.2 Construction of polynomial basis for arbitrary given weight functions

In APC expansion, the polynomial basis for a given weight function can be

numerically obtained based on several numerical theories, such as the Gram-Schmidt orthogonalization[51] and the recursive relations of monic orthogonal polynomials [54]. Gram-Schmidt orthogonalization is the most widely used technique to construct the polynomial basis of APC expansion. However, the polynomial basis obtained by using Gram-Schmidt orthogonalization is not unique for a given weight function. As a comparison, the unique polynomial basis that is orthogonalized to a given weight function can be obtained based on the recursive relations of monic orthogonal polynomials. In addition, the Gauss integration formula for calculating the coefficients of APC expansion can be easily computed according to the coefficients of recursive relations of monic orthogonal polynomials. Therefore, the polynomial basis of APC expansion will be constructed based on the recursive relations of monic orthogonal polynomials in this paper.

Suppose $w(\xi)$ is a positive measure supported on an interval such that all moments $\mu^k = \int_{\Omega} \xi^k w(\xi) d\xi$ exist and are finite. Then, there always exist a set of monic orthogonal polynomials that satisfied the following three-term recurrence relations[54]

$$\begin{aligned} \varphi_{-1}(\xi) &= 0, \\ \varphi_0(\xi) &= 1, \\ \varphi_{k+1}(\xi) &= (\xi - a_k)\varphi_k(\xi) - b_k\varphi_{k-1}(\xi), \quad k = 0, 1, 2, \dots \end{aligned} \quad (6)$$

Where, a_k and $b_k (k=1, 2, \dots)$ are the recurrence coefficients of the orthogonal polynomials. In the framework of gPC, the recurrence coefficient of the orthogonal polynomial from the Askey scheme is well defined. As a comparison, the recurrence coefficient of the orthogonal polynomials of the APC expansion should be estimated. According to the theory of orthogonal polynomial, a_k and $b_k (k=1, 2, \dots)$ of the APC expansion can be determined by[44]

$$a_k = \frac{\langle \xi \varphi_k(\xi), \varphi_k(\xi) \rangle}{\langle \varphi_k(\xi), \varphi_k(\xi) \rangle}, \quad k = 0, 1, 2, \dots \quad (7)$$

$$b_k = \frac{\langle \varphi_k(\xi), \varphi_k(\xi) \rangle}{\langle \varphi_{k-1}(\xi), \varphi_{k-1}(\xi) \rangle}, \quad k = 1, 2, \dots \quad (8)$$

with the coefficient b_0 being arbitrary and set by convention such that

$$b_0 = \int w(x)dx.$$

2.3 Calculation of the expansion coefficient by using the Gauss integration

Owing the orthogonality of the polynomial basis, the expansion coefficient y_i in Eq. (1) can be obtained via the following expression[54]

$$y_i = \frac{\langle Y(\xi), \varphi_i(\xi) \rangle}{\langle \varphi_i(\xi), \varphi_i(\xi) \rangle} = \frac{1}{h_i} \int_{\Omega} Y(\xi) \varphi_i(\xi) w(\xi) d\xi \quad (9)$$

Lots of integration techniques can be employed to calculate the integral in the above equation, such as the Gauss integration technique[54], the Clenshaw–Curtis integration technique[58] and the Newton–Cotes integration technique[59]. The Gauss integration technique is a widely used integration method for calculating the coefficient of the tensor-order polynomial chaos expansion[16]. This is because the Gauss integration technique can generally achieve high accuracy for determining the integral of the polynomial function, when the number of Gauss nodes is up to a certain value[54]. In this paper, the Gauss integration technique is introduced to calculate the integral in Eq.(9) due to its robustness.

By using Gauss integration rule, y_i in Eq.(9) can be expressed as a weighted sum of a finite set of function evaluations, that is[54]

$$y_i = \frac{1}{h_i} \int_{\Omega} Y(\xi) \varphi_i(\xi) w(\xi) d\xi = \frac{1}{h_i} \sum_{i=1}^m Y(\hat{\xi}_i) \varphi_i(\hat{\xi}_i) w_i \quad (10)$$

Where, \hat{x}_i and w_i are the nodes and weights of the Gauss integration rule, respectively; m is the total number of integration nodes. $\hat{\xi}_i$ and w_i of the m -point Gauss integration only depend on $w(\xi)$. When $w(\xi)$ is a weight function of the orthogonal polynomial from the Askey scheme, $\hat{\xi}_i$ and w_i of the m -point Gauss integration can be determined by an explicit formula[14]. However, there is no

explicit formula to determine $\hat{\xi}_i$ and w_i of the m -point Gauss integration for arbitrary weight functions. According to Ref.[54], $\hat{\xi}_i$ and w_i of the Gauss integration with regard to an arbitrary weight function should be obtained from the eigenvalue decomposition of the Jacobi matrix \mathbf{J}_n . The Jacobi matrix \mathbf{J}_n assembled with the recurrence coefficients a_i and b_i can be expressed as[54]

$$\mathbf{J}_n = \begin{bmatrix} a_0 & \sqrt{b_1} & & & & \\ \sqrt{b_1} & a_1 & \sqrt{b_2} & & & \\ & \sqrt{b_2} & \ddots & \ddots & & \\ & & \ddots & a_{n-2} & \sqrt{b_{n-1}} & \\ & & & \sqrt{b_{n-1}} & a_{n-1} & \end{bmatrix} \quad (11)$$

In particular, if $\mathbf{V}^T \mathbf{J}_n \mathbf{V} = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ and $\mathbf{V}^T \mathbf{V} = \mathbf{I}$, in which \mathbf{I} is the $n \times n$ dimension identity matrix. Then, the desired x_i and \hat{w}_i can be determined by

$$x_i = \lambda_i, \quad \hat{w}_i = b_0 v_{i,1}^2, \quad i = 1, 2, \dots \quad (12)$$

where $v_{i,1}$ is the first component of the i -th column vector of \mathbf{V} .

Similarly, the expansion coefficient y_{i_1, \dots, i_L} shown in Eq.(4) can be determined according to the orthogonality of the polynomial basis and expressed as

$$y_{i_1, \dots, i_L} = \frac{\langle Y(\xi), \varphi_{i_1, \dots, i_L}(\xi) \rangle}{\langle \varphi_{i_1, \dots, i_L}(\xi), \varphi_{i_1, \dots, i_L}(\xi) \rangle} = \frac{1}{h_{i_1, \dots, i_L}} \int_{\Omega} Y(\xi) \varphi_{i_1, \dots, i_L}(\xi) w_{i_1, \dots, i_L}(\xi) d\xi \quad (13)$$

where

$$h_{i_1, \dots, i_L} = \prod_{j=1}^L h_{i_j}, \quad w_{i_1, \dots, i_L}(\xi) = \prod_{j=1}^L w_{i_j}(\xi_j) \quad (14)$$

By using the Gauss integration, y_{i_1, \dots, i_L} can be obtained and expressed as

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$$\begin{aligned}
y_{i_1, \dots, i_L} &= \frac{1}{h_{i_1, \dots, i_L}} \int_{\Omega} Y(\xi) \varphi_{i_1, \dots, i_L}(\xi) w_{i_1, \dots, i_L}(\xi) d\xi \\
&= \frac{1}{h_{i_1, \dots, i_L}} \sum_{j_1=1}^{m_1} \dots \sum_{j_L=1}^{m_L} Y(\hat{\xi}_{j_1, \dots, j_L}) \varphi_{i_1}(\hat{\xi}_{j_1, \dots, j_L}) w_{j_1, \dots, j_L}
\end{aligned} \tag{15}$$

where

$$\hat{\xi}_{j_1, \dots, j_L} = [\hat{\xi}_{j_1}, \hat{\xi}_{j_2}, \dots, \hat{\xi}_{j_L}], \quad w_{j_1, \dots, j_L} = \prod_{k=1}^L w_{j_k} \tag{16}$$

In the above equations, $\hat{\xi}_{j_k}$ denotes the j_k th integration nodes for ξ_k , and w_{j_k} denotes the weight of Gauss integration related to $\hat{\xi}_{j_k}$, $m_k (k=1, 2, \dots, L)$ denotes the total number of integration nodes related to ξ_k .

3 Three uncertain models of structure-acoustic systems with interval and/ or random variables

3.1 Dynamic equilibrium equation for structure-acoustic system with uncertain parameters

Without considering the structural damping, the dynamic equilibrium equation of the structure-acoustic system under the time harmonic external excitation derived from finite element analysis can be expressed as

$$\begin{bmatrix} \mathbf{K}_s - \omega^2 \mathbf{M}_s & -\mathbf{H} \\ \rho_f \omega^2 \mathbf{H}^T & \mathbf{K}_f - \omega^2 \mathbf{M}_f \end{bmatrix} \begin{Bmatrix} \mathbf{u}_s \\ \mathbf{p} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}_s \\ \mathbf{F}_f \end{Bmatrix} \tag{17}$$

where ω is the angular frequency of external excitation; ρ_f is the density of fluid in the acoustic cavity; \mathbf{K}_s and \mathbf{M}_s are the stiffness matrix and the mass matrix of the vibrating structure; \mathbf{K}_f and \mathbf{M}_f are the stiffness matrix and the mass matrix of the acoustic cavity; \mathbf{H} is the spatial coupled matrix; \mathbf{F}_s and \mathbf{F}_f are the generalized force vectors loading on the vibrating structure and the acoustic cavity, respectively;

\mathbf{u}_s and \mathbf{p} are the displacement vector of the vibrating structure and the sound pressure vector in the acoustic cavity, respectively.

For the sake of simplicity, Eq.(17) can be rewritten as

$$\mathbf{Z}\mathbf{U} = \mathbf{F} \quad (18)$$

where

$$\mathbf{Z} = \begin{bmatrix} \mathbf{K}_s - \omega^2 \mathbf{M}_s & -\mathbf{H} \\ \rho_f \omega^2 \mathbf{H}^T & \mathbf{K}_f - \omega^2 \mathbf{M}_f \end{bmatrix}, \mathbf{U} = \begin{Bmatrix} \mathbf{u}_s \\ \mathbf{p} \end{Bmatrix}, \mathbf{F} = \begin{Bmatrix} \mathbf{F}_s \\ \mathbf{F}_f \end{Bmatrix} \quad (19)$$

In the above equations, \mathbf{Z} , \mathbf{U} and \mathbf{F} denote the dynamic stiffness matrix, the response vector and the force vector of the structure-acoustic system, respectively.

Due to the unpredictable environment and the manufacturing tolerance, the structure-acoustic system always involved uncertainties. By using the vector $\mathbf{x} = [x_1, x_2, \dots, x_L]$ to represent the uncertain parameters, the dynamic equilibrium equation of the structure-acoustic system can be rewritten as

$$\mathbf{Z}(\mathbf{x})\mathbf{U}(\mathbf{x}) = \mathbf{F}(\mathbf{x}) \quad (20)$$

Where $\mathbf{Z}(\mathbf{x})$ and $\mathbf{F}(\mathbf{x})$ denote the uncertain structure-acoustic dynamic stiffness matrix and the uncertain force vector, respectively.

3.2 Definition of three uncertain models for uncertain structure-acoustic system

In this paper, the uncertain parameter of structure-acoustic system is treated as either random or interval variable. When there is sufficient data to construct the PDF of x_i , x_i can be modeled by a random variable x_i^R and denoted as $x_i = x_i^R$. When the PDF of x_i is not available due to the limited information, the variable x_i can be described by an interval variable x_i^I and denoted as $x_i = x_i^I \in [\underline{x}_i, \bar{x}_i]$. According to the available PDF of uncertain parameters, the interval model, the random model and the hybrid uncertain model will be introduced to treat with the uncertain parameters.

Case1: the interval model

In the interval model, each of the uncertain parameters is described as the interval variable. Accordingly, the uncertain vector \mathbf{x} can be described as an interval vector and expressed as

$$\mathbf{x} = \mathbf{x}^I = [x_1^I, x_2^I, \dots, x_L^I] \quad (21)$$

Case2: the random model

In the random model, all of the uncertain parameters are described as the random variables and the uncertain vector \mathbf{x} can be the expressed as

$$\mathbf{x} = \mathbf{x}^R = [x_1^R, x_2^R, \dots, x_L^R] \quad (22)$$

Case3: the hybrid uncertain model

In the hybrid uncertain model, the interval variable and the random variable exist simultaneously. In this case, the uncertain vector \mathbf{x} can be expressed as a hybrid vector, which can be expressed as

$$\mathbf{x} = [\mathbf{x}^I, \mathbf{x}^R] = [x_1^I, x_2^I, \dots, x_{L_1}^I, x_{L_1+1}^R, \dots, x_L^R] \quad (23)$$

Where, L_1 denotes the number of interval variables of the hybrid uncertain model. From Eqs.(21)~(23), one can find that the interval model is a special case of the hybrid uncertain model when $L_1=L$, while the random model can be viewed as a special case of the hybrid uncertain model when $L_1=0$.

4 Arbitrary polynomial chaos expansion for response analysis of structure-acoustic system with interval and random variables

The APC has been previously applied for random analysis[18]. In this section,

1 the APC will be developed for response analysis of hybrid uncertain model of
2 structure-acoustic system. As we mentioned in Section 3.2, both the interval model and
3 the random model can be viewed as the special case of the hybrid uncertain model.
4 Therefore, through the extension of APC expansion for hybrid uncertainty analysis, a
5 unified polynomial expansion approach is consequently established for response
6 analysis of the interval model, the random model and the hybrid uncertain model of
7 structure-acoustic system. In the following subsections, the procedure of APC
8 expansion for hybrid uncertainty analysis of structure-acoustic problem will be
9 deduced in detail.
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20 *4.1 Determine the polynomial basis with respect to the random variable*

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23 Polynomial chaos method is an uncertainty propagation approach which has been
24 used in many engineering problems. The key idea of polynomial chaos method for
25 random analysis is to approximate the random response by a sum of orthogonal
26 polynomials. In the infinite amount of orthogonal polynomials, there always exists an
27 optimal orthogonal polynomial for a given random variable. In particular, the
28 orthogonal polynomial whose weight function is identical to the PDF of random
29 variable can be viewed as the optimal polynomial basis of the polynomial chaos
30 expansion for the random variable[50]. When the optimal polynomial basis is
31 obtained, the polynomial chaos method can achieve exponentially convergence rate
32 for random problem. Thus, in this paper, the PDF is used as the weight function of the
33 polynomial basis with related to the random variable. Once the weight function
34 related to a random variable is determined, the polynomial basis can be calculated
35 through Eqs.(6)~(8).
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50 Note that the widely used gPC can only provide the optimal orthogonal
51 polynomial for the probability distribution in the Askey scheme[14]. As a comparison,
52 the APC can provide the optimal polynomial basis for any probability distribution,
53 which is the main advantage of the APC expansion for uncertainty analysis with
54 random variable.
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4.2 Determine the polynomial basis with respect to the interval variable

Theoretically, an arbitrary orthogonal polynomial that is defined on a closed interval can be used as the polynomial basis of APC expansion for the approximation of response of uncertain system with interval variable. However, the accuracy of APC expansion for the interval problem may change with different polynomial bases. **Therefore, it is necessary to determine a suitable polynomial basis of APC expansion for interval analysis.** According to Section 2.2, the polynomial basis of APC expansion is determined by its corresponding weight function. **In order to determine the polynomial basis of APC expansion** for the interval problem, the effect of the weight function of polynomial basis on the accuracy of the APC expansion will be firstly investigated and discussed by a simple example as follows.

Example. Consider $y = f(x) = e^{x^2}$, where $x \in [-1, 1]$. The APC expansions with different weight functions are used to approximate the original function $f(x)$. The Legendre polynomial and the Chebyshev polynomial are widely used for interval analysis, thus the weight functions of Legendre polynomial and Chebyshev polynomial will be considered to construct the APC expansion in this numerical example. According to ref.[55], the weight functions of both Legendre polynomial and Chebyshev polynomial can be viewed as the special case of the λ function. The λ function is defined as follows[55]

$$\rho(x, \lambda) = \frac{\Gamma(\lambda + 1)}{\Gamma(1/2)\Gamma(\lambda + 1/2)} (1 - x^2)^{\lambda - 1/2}, \quad -1 \leq x \leq 1 \quad (24)$$

In particular, the weight function of Legendre polynomial and Chebyshev polynomial are $\rho(x, 0.5)$ and $\rho(x, \lambda \rightarrow 0)$, respectively. In this paper, $\rho(x, \lambda = 0.01)$ is used instead of $\rho(x, \lambda \rightarrow 0)$. Thus, APC expansions with the weight function $\rho(x, \lambda = 0.01)$ and $\rho(x, \lambda = 0.5)$ will be used for the approximation of y . For a comparison the APC expansion with the weight function $\rho(x, \lambda = 3)$ will also be considered. Three weight functions, namely $\rho(x, \lambda)$ with

$\lambda = 0.01, 0.5$ and 3 , are plotted in Fig.1.

Define the Relative error(Re) as

$$\text{Re} = \left| \frac{P(x) - f(x)}{f(x)} \right| \quad (25)$$

Where, $P(x)$ denotes the APC expansion. The relative error of the fifth-order APC expansion with different weight functions is plotted in Fig.2.

It can be found from Fig.2 that that the errors yielded by the APC expansions with the weight functions $\rho(x, \lambda = 0.01)$ and $\rho(x, \lambda = 0.5)$ are more uniformly distributed over the interval than the APC expansion with the weight function $\rho(x, \lambda = 3)$. Especially, the accuracy of the APC expansion with the weight function $\rho(x, \lambda = 3)$ will be seriously deteriorated around the bounds of the interval. The main reason is that the values of $\rho(x, \lambda = 3)$ at the neighborhood of bounds are very small. Note that the weight function is always used to minimize the residual error of APC expansion[56]. Thus the error of APC expansion may be relatively large on the region where the values of the weight function approach to zeros. Besides, we can find that the errors of APC expansion with the weight function $\rho(x, \lambda = 0.01)$ around the bounds of interval are relatively smaller than those of APC expansion with the weight function $\rho(x, \lambda = 0.5)$. This is mainly because the values of $\rho(x, \lambda = 0.01)$ around the bounds of interval are larger than those of $\rho(x, \lambda = 0.5)$. Thus, the APC expansion with the weight function $\rho(x, \lambda = 0.01)$ can achieve relatively high accuracy around the bounds of interval.

The interval analysis is to search the maximum and minimum of a function over the whole closed interval of uncertain input, and the maximum and minimum of a function may be obtained at any value of the closed interval of uncertain input. Thus,

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it is ideal to use the approximation technique that can achieve the same accuracy over the whole interval of uncertain input for interval analysis. However, as is shown in Fig.2, the accuracy of APC expansion with each weight function will fluctuate in the interval. Namely, the ideal approximation for interval problems may be not available by using the APC expansion. But from an overall point of view, the accuracy of the APC expansions with the weight functions $\rho(x, \lambda = 0.01)$ and $\rho(x, \lambda = 0.5)$ is more uniformly distributed in the interval than the APC expansion with the weight functions $\rho(x, \lambda = 3)$. It indicates that it is more suitable to select $\rho(x, \lambda = 0.01)$ or $\rho(x, \lambda = 0.5)$ rather than $\rho(x, \lambda = 3)$ as the weight function of the polynomial basis of APC expansion for interval analysis. On the other hand, as regarding engineering problems, the maximum or minimum of the response is more likely to be obtained at the bounds of interval. Thus, the APC expansion with the weight function $\rho(x, \lambda = 0.01)$, which can achieve relatively high accuracy at the bounds of interval, will be used for interval analysis in this paper. In other words, $\rho(x, \lambda = 0.01)$, namely the λ function with $\lambda = 0.01$, will be adopted as the weight function of the polynomial basis of APC expansion for interval analysis.

4.3 Construct the arbitrary polynomial chaos expansion for response with interval and random variables

Based on the APC expansion, the response of the hybrid uncertain structure-acoustic system can be approximated as

$$U_k = U_k(\mathbf{x}^I, \mathbf{x}^R) = \sum_{i_1=0}^{N_1} \dots \sum_{i_L=0}^{N_L} f_{i_1, \dots, i_L}^k \varphi_{i_1, \dots, i_{i_1}}(\mathbf{x}^I) \varphi_{i_{i_1+1}, \dots, i_L}(\mathbf{x}^R) \quad k = 1, 2, \dots, N_{tot} \quad (26)$$

In the above equations, $U_k (k = 1, 2, \dots, N_{tot})$ denotes k -th element of the response vector \mathbf{U} , N_{tot} denotes the dimension of \mathbf{U} ; $N_j (j = 1, 2, \dots, L)$ denotes the retained order of APC expansion related to x_j , f_{i_1, \dots, i_L}^k denotes the expansion coefficient to

be estimated. The coefficients f_{i_1, \dots, i_L}^k can be calculated according to section 2.3, which can be determined by

$$f_{i_1, \dots, i_L}^k \approx \frac{1}{h_1 \times \dots \times h_L} \sum_{j_1=1}^{m_1} \dots \sum_{j_L=1}^{m_L} U_k(\hat{\mathbf{x}}^I, \hat{\mathbf{x}}^R) \varphi_{i_1, \dots, i_{L_1}}(\hat{\mathbf{x}}^I) \varphi_{i_{L_1+1}, \dots, i_L}(\hat{\mathbf{x}}^R) w_{i_1, \dots, i_L} \quad (27)$$

In the above equation, $\hat{\mathbf{x}}^I$ and $\hat{\mathbf{x}}^R$ denote the integration nodes related to the interval variables and random variables, respectively. w_{i_1, \dots, i_L} denotes the weight with respect to the integration nodes. The integration nodes and their corresponding weight can be calculated through Eqs.(11) and (12); $U_k(\hat{\mathbf{x}}^I, \hat{\mathbf{x}}^R)$ denotes the responses of structure-acoustic system at the integration nodes, which can be calculate through Eq.(18).

4.4 Evaluate the uncertainty property of sound pressure of structure-acoustic system

The response analysis of uncertain structure-acoustic system with interval and random variables includes two main steps. In the first step, the interval variables are regarded as constant parameters, and the response of hybrid uncertain structure-acoustic system can be rewritten as the following form

$$\begin{aligned} U_k &= \sum_{i_{L_1+1}=0}^{N_{L_1+1}} \dots \sum_{i_L=0}^{N_L} \left(\sum_{i_1=0}^{N_1} \dots \sum_{i_{L_1}=0}^{N_{L_1}} f_{i_1, \dots, i_{L_1}}^k \varphi_{i_1, \dots, i_{L_1}}(\mathbf{x}^I) \right) \varphi_{i_{L_1+1}, \dots, i_L}(\mathbf{x}^R) \\ &= \sum_{i_{L_1+1}=0}^{N_{L_1+1}} \dots \sum_{i_L=0}^{N_L} z_{i_{L_1+1}, \dots, i_L}^k \varphi_{i_{L_1+1}, \dots, i_L}(\mathbf{x}^R) \end{aligned} \quad (28)$$

Where

$$z_{i_{L_1+1}, \dots, i_L}^k = \sum_{i_1=0}^{N_1} \dots \sum_{i_{L_1}=0}^{N_{L_1}} f_{i_1, \dots, i_{L_1}}^k \varphi_{i_1, \dots, i_{L_1}}(\mathbf{x}^I) \quad (29)$$

Based on the APC expansion, the expectation of U_k can be determined by

$$\begin{aligned} \mu_{U_k} &= \mathbb{E} \left[\sum_{i_{L_1+1}=0}^{N_{L_1+1}} \dots \sum_{i_L=0}^{N_L} z_{i_{L_1+1}, \dots, i_L}^k \varphi_{i_{L_1+1}, \dots, i_L}(\mathbf{x}^R) \right] \\ &= \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \left(\sum_{i_{L_1+1}=0}^{N_{L_1+1}} \dots \sum_{i_L=0}^{N_L} z_{i_{L_1+1}, \dots, i_L}^k \varphi_{i_{L_1+1}, \dots, i_L}(\mathbf{x}^R) P_{X_{L_1+1}}(x_{L_1+1}^R) \dots P_{X_L}(x_L^R) \right) dx_{L_1+1}^R \dots dx_L^R \end{aligned}$$

(30)

In the above equation, $P_{x_j}(x_j^R)$ ($j = L_1 + 1, \dots, L$) is the PDF of x_j^R , and there is $P_{x_j}(x_j^R) = w_j(x_j^R)$, where $w_j(x_j^R)$ is the weight function of the polynomial basis related to x_j^R . As the polynomial basis is orthogonal with respect to the PDF of the random variable, the analytical solution of the expectation of the APC expansion can be readily obtained[14]. According to Ref.[14], the expectation of the response approximated by APC expansion can be expressed as

$$\mu_{U_k} = z_{0, \dots, 0}^k \quad (31)$$

Before calculating the variance of the response, the expectation of mean square response should be obtained, which can be written as

$$E[(U_k)^2] = E\left[\left(\sum_{i_{L_1+1}=0}^{N_{L_1+1}} \dots \sum_{i_L=0}^{N_L} z_{i_{L_1+1}, \dots, i_L}^k \varphi_{i_{L_1+1}, \dots, i_L}(\mathbf{x}^R)\right)^2\right] \quad (32)$$

Based on the orthogonal relationship of polynomial basis, the expectation of mean square response can be finally obtained and written as [14]

$$E[(U_k)^2] = \sum_{i_{L_1+1}=0}^{N_{L_1+1}} \dots \sum_{i_L=0}^{N_L} (z_{i_{L_1+1}, \dots, i_L}^k)^2 h_{i_{L_1+1}}, \dots, h_{i_L} \quad (33)$$

Consequently, the variance of the response can be obtained and expressed as

$$\begin{aligned} \sigma_{U_k}^2 &= E[(U_k)^2] - (\mu_{U_k})^2 \\ &= \sum_{i_{L_1+1}=0}^{N_{L_1+1}} \dots \sum_{i_L=0}^{N_L} (z_{i_{L_1+1}, \dots, i_L}^k)^2 h_{i_{L_1+1}}, \dots, h_{i_L} - (z_{0, \dots, 0}^k)^2 \end{aligned} \quad (34)$$

Owing to the orthogonality of the polynomial basis, the expectation and variance of the response can be determined and expressed as

$$\mu_{U_k} = E\left[\sum_{i_{L_1+1}=0}^{N_{L_1+1}} \dots \sum_{i_L=0}^{N_L} z_{i_{L_1+1}, \dots, i_L}^k \varphi_{i_{L_1+1}, \dots, i_L}(\mathbf{x}^R)\right] = z_{0, \dots, 0}^k \quad (35)$$

$$\begin{aligned}
\sigma_{U_k}^2 &= \mathbb{E} \left[\left(\sum_{i_{L+1}=0}^{N_{L+1}} \dots \sum_{i_L=0}^{N_L} z_{i_{L+1}, \dots, i_L}^k \varphi_{i_{L+1}, \dots, i_L}(\mathbf{x}^R) \right)^2 \right] - (\mu_{U_k})^2 \\
&= \sum_{i_{L+1}=0}^{N_{L+1}} \dots \sum_{i_L=0}^{N_L} (z_{i_{L+1}, \dots, i_L}^k)^2 h_{i_{L+1}} \dots h_{i_L} - (z_{0, \dots, 0}^k)^2
\end{aligned} \tag{36}$$

Substituting Eq. (29) into Eq. (35) and Eq. (36), the expectation and variance of the response can be rewritten as

$$\mu_{U_k} = \mu_{U_k}(\mathbf{x}^I) = \sum_{i_1=0}^{N_1} \dots \sum_{i_L=0}^{N_L} f_{i_1, \dots, i_L, 0, \dots, 0}^k \varphi_{i_1, \dots, i_L}(\mathbf{x}^I) \tag{37}$$

$$\begin{aligned}
\sigma_{U_k}^2 &= \sigma_{U_k}^2(\mathbf{x}^I) = \sum_{i_{L+1}=0}^{N_{L+1}} \dots \sum_{i_L=0}^{N_L} \left(\sum_{i_1=0}^{N_1} \dots \sum_{i_L=0}^{N_L} f_{i_1, \dots, i_L}^k \varphi_{i_1, \dots, i_L}(\mathbf{x}^I) \right)^2 h_{i_{L+1}} \dots h_{i_L} \\
&\quad - \left(\sum_{i_1=0}^{N_1} \dots \sum_{i_L=0}^{N_L} f_{i_1, \dots, i_L, 0, \dots, 0}^k \varphi_{i_1, \dots, i_L}(\mathbf{x}^I) \right)^2
\end{aligned} \tag{38}$$

In the second step, the lower and upper bounds of the expectation and variance can be calculated by the Monte Carlo simulation and expressed as

$$\begin{aligned}
[\underline{\mu}_{U_k}, \bar{\mu}_{U_k}] &= \left[\min_{\mathbf{x}^I \in [\underline{\mathbf{x}}, \bar{\mathbf{x}}]} \{ \mu_{U_k}(\mathbf{x}^I) \}, \max_{\mathbf{x}^I \in [\underline{\mathbf{x}}, \bar{\mathbf{x}}]} \{ \mu_{U_k}(\mathbf{x}^I) \} \right], \\
[\underline{\sigma}_{U_k}^2, \bar{\sigma}_{U_k}^2] &= \left[\min_{\mathbf{x}^I \in [\underline{\mathbf{x}}, \bar{\mathbf{x}}]} \{ \sigma_{U_k}^2(\mathbf{x}^I) \}, \max_{\mathbf{x}^I \in [\underline{\mathbf{x}}, \bar{\mathbf{x}}]} \{ \sigma_{U_k}^2(\mathbf{x}^I) \} \right]
\end{aligned} \tag{39}$$

To obtain the maximum and minimum of the APC expansion shown in Eq.(39), various methods can be employed, such as the conventional optimization method[54], the Monte Carlo method[47], the interval arithmetic[37], and the dimension wise analysis[49]. The Monte Carlo method is the most accurate approach for interval analysis. However, a large number of sampling points is required to achieve a prescribed accuracy by using the Monte Carlo method. The interval arithmetic is the most efficient method for interval analysis, but its accuracy can hardly be evaluated due to the wrapping effect. The dimension wise analysis can also achieve high efficiency for interval analysis. However, the main potential limitation for dimension wise analysis is that the cooperative effects of multiple interval parameters acting together upon the system response are ignored. Thus, the accuracy of dimension wise analysis may be decreased in some cases[49]. The **Genetic Algorithm(GA)** algorithm

1 is a widely used method for solving complex optimization problems. Generally, the
2 GA algorithm can achieve a prescribed accuracy for the interval analysis through an
3 iterative process, and the computational efficiency of GA algorithm is much higher
4 than the Monte Carlo method. Due to the good accuracy of GA algorithm, the GA
5 algorithm will be employed to calculate the maximum and minimum of the APC
6 expansion in this paper. Note that the APC expansion is a simple function, thus the
7 computational cost suffering the GA algorithm is acceptable.
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4.5 Procedure of arbitrary polynomial chaos expansion for uncertainty analysis with interval and random variables

This paper employs the APC expansion to approximate the response of uncertain structure-acoustic systems with interval and random variables. Based on the APC expansion, the interval and random analysis of uncertain structure-acoustic systems can be easily implemented. For structure-acoustic systems with different type of uncertain variables, only the weight function should be changed to construct the APC expansion. The proposed method, which can provide a unified approximation for the response of structure-acoustic systems with interval and random variables, is termed as the Interval and Random Arbitrary Polynomial Chaos Method(IRAPCM). The procedures of IRAPCM for structure-acoustic systems with interval and random variables can be summarized as follows

Step1. Determine the weight function of polynomial basis with respect to each variable. For the random variable, the weight function is the same as the PDF; for the interval variable, the weight function is given as the λ function with $\lambda = 0.01$;

Step2. Construct the polynomial basis that is orthogonalized with respect to the weight function related to each variable through Eqs.(6)~(8);

Step3. Compute the integration nodes and weights through Eq.(12);

Step4. Calculate the response of structure-acoustic system at the interpolation points through Eq.(20);

Step5. Calculate the coefficients of APC expansion through Eq.(27);

1 *Step6.* Calculate the response of structure-acoustic systems with interval and
2 random variables through Eq.(39).
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4 The main difference between the proposed IRAPCM and the conventional gPC
5 based method is that different types of orthogonal polynomials are used for the
6 polynomial chaos expansion. In the gPC based method, the orthogonal polynomial is
7 selected from the Askey scheme, while the orthogonal polynomial in the proposed
8 IRAPCM is numerically generated. As the choice of polynomials in Askey scheme is
9 limited to some well known orthogonal polynomials, the optimal polynomial basis of
10 polynomial chaos expansion for a wide range of complex probability distributions is
11 not available by using the gPC based methods[50]. As a comparison, the optimal
12 polynomial basis for an arbitrary PDF can be constructed by using IRAPCM. In other
13 words, the proposed IRAPCM has the ability to provide the optimal polynomial basis
14 for the uncertain problem involving arbitrary probability distribution.
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28 **5 Numerical examples**

29 In previous years, lots of polynomial chaos methods have been proposed for
30 uncertainty quantification of interval model, random model and hybrid uncertain
31 model. To verify the good accuracy of the proposed IRAPCM, several widely used
32 polynomial chaos methods have been introduced for comparison. For the response
33 analysis of interval model, the widely used Interval Legendre method(ILM[33]) and
34 the Interval Chebyshev method(ICM[5]) are introduced. For the response analysis of
35 random model, the gPC method(gPCM[14]) is introduced. For the hybrid uncertain
36 analysis, the hybrid gPC and Interval Chebyshev method(gPC-ICM[46]) and the
37 hybrid gPC and dimension wise analysis method(gPC-DWM[49]) are introduced.
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51 *5.1 Mathematical problem*

52 Consider a function as follows
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$$55 \quad y = e^{x_1^2} + x_1 e^{x_2} \quad (40)$$

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x_1 and x_2 are assumed as uncertain parameters. Table1 listed three uncertain models to describe x_1 and x_2 .

In the interval model, both x_1 and x_2 are described as interval variables. As the PDF of the interval variable is not available, only the range of variation is given for the interval variable. In the random model, both x_1 and x_2 are described as random variables. In the hybrid uncertain model, x_1 is described as random variable, while x_2 is described as interval variable. In Table1, the PDFs of ξ_1 and ξ_2 are given as follows

$$p_{\xi_1}(\xi) = \begin{cases} a_0(1-\xi^2)^3 & -1 \leq \xi \leq -0.1 \\ a_0(1-0.1^2)^3 & -0.1 < \xi < 0.1 \\ a_0(1-\xi^2)^3 & 0.1 \leq \xi \leq 1 \end{cases} \quad (41)$$

$$p_{\xi_2}(\xi) = \frac{2\Gamma(\lambda+1)}{\Gamma(1/2)\Gamma(\lambda+1/2)} (4\xi-3)^{\frac{1}{2}\lambda-\frac{3}{4}} (2-\sqrt{4\xi-3})^{\lambda-\frac{1}{2}}, \quad \xi \in [0.75, 1.75] \quad (42)$$

In the above equations, a_0 can be determined by $\int_{-1}^1 p_{\xi_1}(\xi) d\xi = 1$; λ is the distribution parameter, which can be any value of $\lambda \geq 0$. In this numerical example, λ is taken as 2.5. Both PDFs of ξ_1 and ξ_2 are out of the Askey scheme. ξ_1 cannot be represented by using the random variable with the PDF from the Askey scheme, while ξ_2 can be represented by using the second order polynomial function of the random variable with λ distribution from the Askey scheme[52], that is

$$\xi_2 = 1 + 0.5\xi^\lambda + 0.25(\xi^\lambda)^2, \quad \xi_\lambda \in [-1, 1] \quad (43)$$

Specially, ξ^λ with the λ distribution is related to the Gegenbauer polynomial in the Askey scheme, and the PDF of ξ^λ is given by

$$p_{\xi^\lambda}(\xi) = \frac{\Gamma(\lambda+1)}{\Gamma(1/2)\Gamma(\lambda+1/2)} (1-\xi^2)^{\lambda-\frac{1}{2}}, \quad -1 \leq \xi \leq 1 \quad (44)$$

In the proposed method, the Gauss integration method is adopted to calculate the

1 coefficient of APC expansion. The accuracy of the Gauss integration method depends
2 on the number of integration nodes. Generally, the error of Gauss integration method
3 can be reduced by increasing the number of integration nodes. However, the increase
4 of the number of integration nodes will lead to the increase of computational burdens.
5 Thus, to guarantee the accuracy of the Gauss integration and reduce the computational
6 cost, it is important to determine the minimal number of required integration nodes. In
7 order to determine the number of required integration nodes of Gauss integration for
8 calculating the expansion coefficient, the effect of the number of integration nodes on
9 the accuracy of the proposed IRAPCM will be firstly investigated. When the retained
10 order of APC expansion of the IRAPCM is $n=1, 2$ and 3 , the relative errors of the
11 lower and upper bounds of interval model yielded by IRAPCM with different number
12 of integration nodes are plotted in Fig.3. The reference result of the lower and upper
13 bounds of y is calculated by using the GA algorithm.

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Fig.3 shows that the relative error of IRAPCM at a certain retained order is gradually decreased with the increase of the number of integration nodes. When the number of integration nodes is up to $n+1$, the relative error yielded by IRAPCM under a certain retained order almost no longer changed. In other words, the accuracy of IRAPCM at a certain retained order can be hardly improved by increasing the number of integration nodes when the number of integration nodes is up to $n+1$. Therefore, to minimize the computational burden without losing the accuracy, the number of integration nodes is set as $n+1$ in this paper.

The proposed IRAPCM is employed to calculate the response of three uncertain models of the mathematical problem. In order to compare the accuracy of the proposed method to other polynomial chaos based uncertainty method, the ILM, the ICM, the gPCM, the gPC-ICM and the gPC-DWM will also be introduced for response analysis of different uncertain models. The relative errors yielded by different methods are plotted in Fig.4 for the interval model, in Fig.5 for the random model and in Figs.6~7 for the hybrid uncertain model. The reference result of the expectation and variance of y related to the random variable is obtained by using the

1 integration method[54], and the reference result of the lower and upper bounds of y
2 related to the interval variable is calculated by using the GA algorithm[57].
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6 From Fig.4, we can find that the accuracy of IRAPCM is the same as that of the
7 widely used ICM for interval analysis. The main reason may be that the weight
8 function of the APC expansion in IRAPCM for interval analysis is approximately the
9 same as the weight function of the Chebyshev polynomial in ICM. However, it should
10 be noted that the polynomial basis of APC expansion in IRAPCM is different from
11 that of the Chebyshev polynomial in ICM. For instance, the APC expansion is
12 constructed based on the monic polynomial, while the high-order Chebyshev
13 polynomial is not the monic polynomial[32]. Therefore, the Chebyshev polynomial
14 cannot be viewed as a special case of the polynomial basis of the APC expansion.
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25 When compared with the ILM, it can be observed from Fig.4 that the error of the
26 lower bound obtained by using IRAPCM is smaller than that of the ILM, while the
27 error of the upper bound obtained by using IRAPCM is slightly larger than that of
28 the ILM at several retained orders. This is mainly because that the upper bound of y is
29 obtained at the bounds of x , where the APC expansion can achieve higher accuracy
30 than the Legendre expansion(Refer to Fig.2 in Section 4.2); while the lower bound of
31 y is obtained around the mid-point of x , where the accuracy of APC expansion may
32 be lower than that of the Legendre expansion. Therefore, the accuracy of IRAPCM is
33 higher than that of ILM for calculating the upper bound of y , but is slightly lower than
34 that of ILM for calculating the lower bound of y .
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48 From Fig.5, we can find that IRAPCM can converge exponentially. As a
49 comparison, the convergence rate of gPCM is much slower than that of IRAPCM, and
50 the accuracy of gPCM remains no longer changed when the retained order is up to 3.
51 This is mainly because the weight function of polynomial basis of gPCM can not
52 accurately represent the random variable whose PDF is a piecewise function. In other
53 words, some errors have been introduced for the PDF of random variable by using
54 gPCM. Consequently, the results obtained by gPCM cannot converge to the exact
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1 result. In addition, the nonlinear transformation of the random variable may also
2 degrade the convergence rate of the gPCM. Therefore, the accuracy of gPCM is much
3 lower than that of the IRAPCM for random problems when the PDF of random
4 parameters is out of the Askey scheme.
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10 From Figs.6 and 7, we can find that IRAPCM can also converge exponentially
11 for hybrid uncertain analysis, while the gPC-ICM and gPC-WDM converges very
12 slowly, especially when the retained order is up to 3. In other words, the IRAPCM can
13 achieve much higher accuracy than the gPC-ICM and the gPC-WDM for hybrid
14 uncertain analysis. As is addressed before, the accuracy of IRAPCM is the same as
15 that of ICM for interval analysis, but is much higher than that of gPCM for random
16 analysis. It indicates that the deterioration of the accuracy of gPC-ICM may be mainly
17 caused by the use of the gPCM for random analysis. Therefore, it is more desirable to
18 use the APC in the proposed method rather than the gPC for uncertainty quantification
19 involving random variables.
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28 As a conclusion from Figs.4~7, the proposed IRAPCM can achieve the same
29 accuracy as the widely used ICM for interval analysis, whereas the accuracy of
30 IRAPCM is much higher than that of the gPC based methods for random analysis and
31 hybrid uncertain analysis. In other words, the proposed IRAPCM can not only keep
32 the good accuracy of ICM for interval analysis, but can also improve the accuracy of
33 the gPC based method for random analysis and hybrid uncertain analysis.
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38 In addition, we can find from Figs.4~7 that the relative error of the proposed
39 IRAPCM for response analysis of three uncertain models is gradually reduced with
40 the increase of the retained order. It indicates that the proposed IRAPCM can achieve
41 a high accuracy for response analysis of three uncertain models if the retained order of
42 APC expansion is sufficiently large.
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48 5.2 Structure-acoustic problem

49 5.2.1 Description of four cases of structure-acoustic system with interval and random 50 variables 51

52 Fig. 8 shows a shell structure-acoustic system, in which the shell is located at the
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top of the acoustic cavity. The flexible shell is made of steel ($E = 2.1 \times 10^5 \text{ MPa}$, $\nu = 0.3$, $\rho_s = 7850 \text{ kg/m}^3$). The thickness of the shell is 2mm. The acoustic cavity is filled with air ($\rho_f = 1.225 \text{ kg/m}^3$ and $c = 340.5 \text{ m/s}$). All edges of the shell are fixed, while the walls of the acoustic cavity are rigid. The shell is excited by a unit normal harmonic point force at middle point denoted as Node B in Fig.8. **The Finite Element(FE) method is used to analyze the response of structure-acoustic system. Particularly, the acoustic cavity and the shell structure are discretized by using the quadrilateral elements and the hexahedral elements, respectively. The total number of elements and nodes of the FE model of acoustic cavity are 1024 and 1337. The total number of elements and nodes of the FE model of shell structure are 128 and 153.**

Considering the unpredictability of the environment temperature and the manufacturing errors of materials, the Young's modulus E , the thickness t , the density of air ρ_f and the speed of air c are considered as the independent uncertain parameters. To validate the accuracy of the proposed method for uncertain problem in different cases, four cases of uncertain structure-acoustic system is considered. The uncertainty information of four cases is listed in Table2. In case1, all uncertain parameters are described as interval variables. In case2, all uncertain parameters are described as random variables. For convenience, the random variables are assumed as a linear function of the unitary random variable defined on $[-1, 1]$. In case3 and case4, both interval variables and random variables existed simultaneously. The uncertainty level of the uncertain variable of case4 is much smaller than that of cases1~3. In Table2, the unitary random variables ξ^{λ_1} and ξ^{λ_2} obey the λ distribution[55]. The value of λ for ξ^{λ_1} and ξ^{λ_2} are $\lambda_1 = 4.5$ and $\lambda_2 = 0.5$, respectively. The PDF of ξ_3 is assumed as follows

$$p_{\xi_3}(\xi) = \begin{cases} 0.84075 \times (1 - \xi^2)^3 & -1 \leq \xi \leq -0.2 \\ 0.84075 \times (1 - 0.1^2)^3 & -0.2 < \xi < 0.2 \\ 0.84075 \times (1 - \xi^2)^3 & 0.2 \leq \xi \leq 1 \end{cases} \quad (50)$$

5.2.2 Compared with the hybrid perturbation method

In the last decade, the perturbation method and the polynomial chaos method have been widely used for uncertainty analysis of structure-acoustic system with interval and random variables[3,5,33,40]. Both the perturbation method and the polynomial chaos method have their own merits and application scope. The perturbation method can achieve high computational efficiency, but it is limited to uncertain problem with small uncertainty level. The polynomial chaos method can be employed to solve uncertain problem with large uncertainty level. However, the computational efficiency of the polynomial chaos method is lower than that of the perturbation method. The comparison between the perturbation method and the polynomial chaos method for structure-acoustic system with pure interval uncertainty has been fully discussed in the previous study[5,33]. In this paper, the proposed IRAPCM will be compared to the perturbation method for uncertainty analysis with both interval and random variables. Particularly, the Hybrid First-order Perturbation Method(HFPM) in Ref.[3] will be introduced for comparison. For uncertainty analysis of structure-acoustic system with interval and random variables, two cases with different uncertainty level will be considered. In case 4, the uncertainty level of the interval and random variables is very small, while the uncertainty level of the interval and random variables of case3 is much larger than that of case4. The first-order IRAPCM and the HFPM is employed to calculate the response of case3 and case4 at $f=300\text{Hz}$. In the first-order IRAPCM, the retained order of APC expansion for each uncertain variable is one. The lower and upper bounds of the expectation and variance of sound pressure distributing on the middle section obtained by the HFPM and the first-order IRAPCM are plotted in Figs.9 and 10. The reference results are obtained by using the Monte Carlo simulation. In Monte Carlo simulation, the sampling points for the random variables are 100000, and 10 uniformly distributed sampling points are used for each interval variable.

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From Fig.9, we can find that the results obtained by the HFPM and the first-order IRAPCM are very close to the reference results. It indicates that both the HFPM and the first-order IRAPCM can achieve high accuracy for hybrid uncertainty analysis of structure-acoustic problem with small uncertainty level.

From Fig.10, we can find that both the HFPM and the first-order IRAPCM will lead to large errors. Namely, the HFPM and the first-order IRAPCM are not suitable to solve the structure-acoustic problem with large uncertainty level. It can be seen from Figs.6 and 7 that the accuracy of the IRAPCM for hybrid uncertainty analysis can be improved by increasing the retained order. To reduce the computational error of IRAPCM, the high-order IRAPCM will be employed to calculate the response of case3. In the high-order IRAPCM, the retained orders of the APC expansion of IRAPCM are 3, 2, 1 and 5 for E , t , ρ_f and c , respectively. The results obtained by the high-order IRAPCM are plotted in Fig.11.

It can be seen from Fig.11 that the result obtained by high-order IRAPCM is very close to the reference result. It indicates that the proposed IRAPCM can achieve high accuracy for uncertainty analysis with large uncertainty level if the retained order is sufficiently large.

Theoretically, the accuracy of the hybrid perturbation method can also be improved by using high-order expansion. However, the computation of the derivatives of the high-order expansion of perturbation method for engineering problem is rather difficult and extremely cumbersome. Thus, the perturbation method for most of engineering problems is developed by using the low-order expansion, such as the first-order expansion and the second-order expansion. For uncertainty analysis of structure-acoustic problem with large uncertainty level, the accuracy of perturbation method cannot be significantly improved by using the second-order expansion instead of the first-order expansion. Therefore, up to now, the hybrid perturbation method is limited for structure-acoustic problem with small uncertainty level.

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The computational time of the HFPM and the first-order IRAPCM is 12.8s and 75.2s, respectively. Namely, the efficiency of the HFPM is much higher than that of the first-order IRAPCM for structure-acoustic problem with interval and random uncertainties. Note that the HFPM can achieve high accuracy for uncertain structure-acoustic problem with small uncertainty level. Therefore, to save the computational cost, it is more reasonable to use the HFPM rather than the IRAPCM for response analysis of structure-acoustic problem with small uncertainty level.

5.2.3 Compared with several widely used polynomial chaos methods

In this subsection, the proposed IRAPCM will be compared to several widely used polynomial chaos methods for response analysis of cases1~3. In particular, ICM and ILM are introduced to calculate the response of case1; the gPCM is introduced to calculate the response of case2; and the gPC-ICM and the gPC-WDM are introduced to calculate the response of case3. The retained orders of the polynomial expansion in the polynomial chaos methods are 3, 2, 1 and 5 for E , t , ρ_f and c , respectively. The reference results are obtained by using the Monte Carlo simulation. In Monte Carlo simulation, the sampling points for the random variables are 100000, and 10 uniformly distributed sampling points are used for each interval variable. The uncertainty property of the sound pressure distributing on the middle section at $f=300\text{Hz}$ obtained by the proposed method and other methods are plotted in Fig.12 for case1, Fig. 13 for case2 and Fig.14 for case3.

From Fig.12, we can find that all these three polynomial chaos based interval methods, including the proposed IRAPCM, the ICM and the ILM, can achieve high accuracy for the response analysis of case1, and the accuracy of these three interval methods are almost the same. It verifies that IRAPCM can be successfully used for the interval analysis of structure-acoustic problems. From Figs.13 and 14, we can see that the IRAPCM can achieve much higher accuracy than the other polynomial chaos

1 based methods for case2 and case3, which further verifies merits of the proposed
2 method in accuracy for uncertainty quantification involving random variables.
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4 Computational efficiency is another important index to evaluate the performance
5 of the numerical methods. For response analysis of three uncertain models of
6 structure-acoustic system, there are three main steps in the proposed IRAPCM. Firstly,
7 the polynomial basis is numerically constructed according to the PDF of random
8 variables. Secondly, the polynomial chaos expansion is established for the
9 approximation of the response of the uncertain structure-acoustic system. In particular,
10 the coefficients of polynomial expansion are obtained through the response reanalysis
11 of the structure-acoustic system. For brevity, the response reanalysis of the
12 structure-acoustic system is denoted as RRSS. Finally, uncertainty property of
13 response can be obtained through the interval and random analysis of *polynomial*
14 *chaos expansion*(PCE). To illustrate the computational burdens of the proposed
15 method in detail, the total computational time and the computational time of three
16 main steps of the proposed method for three cases are listed in Table3. As a
17 comparison, the computational time of ILM and ICM for Case1, the computational
18 time of gPCM for Case2 and the computational time of gPC-ICM and the gPC-WDM
19 for Case3 are also listed in Table3. All of the computational results are obtained by
20 using MATLAB R2014a on a 3.20GHz Intel(R) Core (TM) CPU i5-3470.
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42 From Table3, we can find that the computational time of the RRSS of each
43 polynomial chaos method is close to its total computational time. It indicates that the
44 computational costs of the polynomial chaos methods for three cases mainly suffer
45 from the RRSS. Besides, we can find from Table3 that the computational time of the
46 RRSS by using different polynomial chaos methods are almost the same. This is
47 mainly because the same retained order is used in these polynomial chaos methods.
48 According to Eqs.(24) and (25), the total number of RRSS is determined by the
49 retained order of polynomial chaos expansion. Therefore, computational time of
50 RRSS by using different polynomial chaos methods will be very close when the same
51 retained order is used in polynomial chaos expansion.
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Furthermore, we can find that the execution time of the proposed IRAPCEM is relatively longer than that of the other polynomial chaos based methods under the same retained order. There are two main reasons for the increase of computational cost by using the proposed IRAPCEM. First, the interval analysis of PCE in the proposed method is processed by using the GA algorithm. Generally, the computational efficiency of the GA algorithm is lower than the dimension wise analysis(in gPC-WDM and ILM). Second, the orthogonal polynomials in IRAPCEM are constructed numerically, while the analytical expression of orthogonal polynomials of the other polynomial chaos methods has been well defined. Thus, in IRAPCEM, additional computational time will encountered by constructing the orthogonal polynomial for random variable and interval variable. However, compared with the computational burden suffering from the response reanalysis of structure-acoustic system, the additional computational burdens for both the uncertainty analysis of PCE by using GA algorithm and the computation of polynomial basis are much less.

6 Conclusion

Through an extension of the APC expansion for interval analysis and hybrid uncertain analysis, a unified polynomial chaos method named as IRAPCEM, is proposed for response analysis of the interval model, random model and hybrid uncertain model of structure-acoustic system. In IRAPCEM, the response of three uncertain models is approximated by the APC expansion in a unified form. Based on the unified APC expansion, the uncertainty property of the response of structure-acoustic system can be efficiently obtained. In the procedure to construct the APC expansion for different uncertain models, only the weight function of polynomial basis is required to be changed. In particular, the λ function with a small value of λ is used as the weight function of polynomial basis for the interval variable, while the weight function of polynomial basis for the random variable is the same as the PDF. For a given weight function, the polynomial basis of APC expansion

1 is determined based on the recursive relations of monic orthogonal polynomials. As
2 the weight function of polynomial basis of the APC expansion can be an arbitrary
3 continuous or discrete function, the unified APC expansion can be effectively used for
4 response analysis of three uncertain models of structure-acoustic system involving the
5 random variable with arbitrary PDFs.
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10 The proposed IRAPCM has been employed to calculate the response of three
11 uncertain models of a mathematical problems and a structure-acoustic problem.
12 Different uncertainty properties have been obtained for three uncertain models,
13 including the bounds of the response of interval model, the expectation and variance
14 of the response of random model, and the bounds of the expectation and variance of
15 the response of hybrid interval and random model. The merits of the proposed method
16 is demonstrated by comparing it with the hybrid first-order perturbation method and
17 several widely used polynomial chaos methods, including the interval Chebyshev
18 method(ICM), the interval Legendre method(ILM), the generalized Polynomial Chaos
19 method(gPCM), the hybrid generalized Polynomial Chaos and Interval Chebyshev
20 method(gPC-ICM) and the hybrid generalized Polynomial Chaos and dimension wise
21 analysis method(gPC-WDM). Numerical results have shown that: (1) the proposed
22 IRAPCM can achieve high accuracy for interval and random analysis with large
23 uncertainty level if the retained order is sufficiently large; (2)the accuracy of the
24 proposed IRAPCM is the same as that of the widely used ICM for interval analysis; (3)
25 the proposed IRAPCM can achieve higher accuracy than the gPC based methods for
26 random analysis and hybrid uncertain analysis; (4) the computational efficiency of
27 IRAPCM is lower than that of the hybrid first-order perturbation method, but the
28 hybrid first-order perturbation method is limited for structure-acoustic problem with
29 small uncertainty level.
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51 As a conclusion, the proposed method can not only provide a unified polynomial
52 expansion for the response analysis of three uncertain models of structure-acoustic
53 system with interval and/or random variable, but also can achieve better accuracy than
54 the gPC based method for response analysis of random model and hybrid uncertain
55 model of structure-acoustic problems with large uncertainty level. Note that the
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1 computational cost of the proposed IRAPCM is relatively larger than that of the gPC
2 based methods under the same retained order. However, compared with the
3 improvement in accuracy, the increase of computational effort by using the proposed
4 IRAPCM is deemed acceptable.
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Figure1

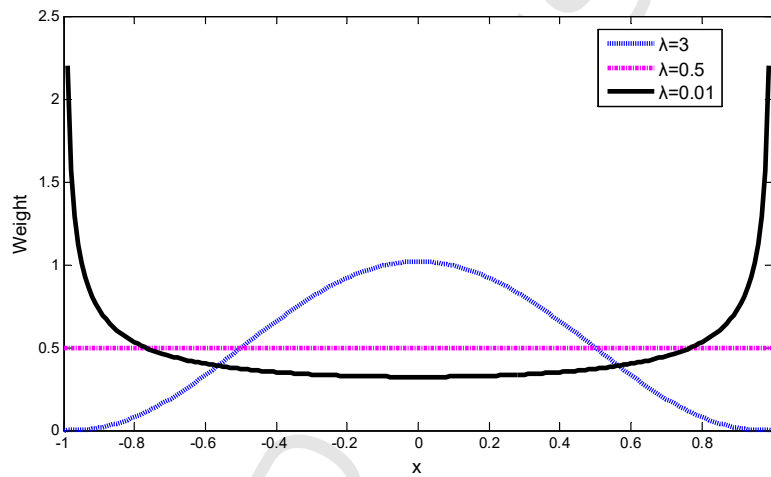
**Fig.1** Three weight functions of the polynomial basis

Figure2

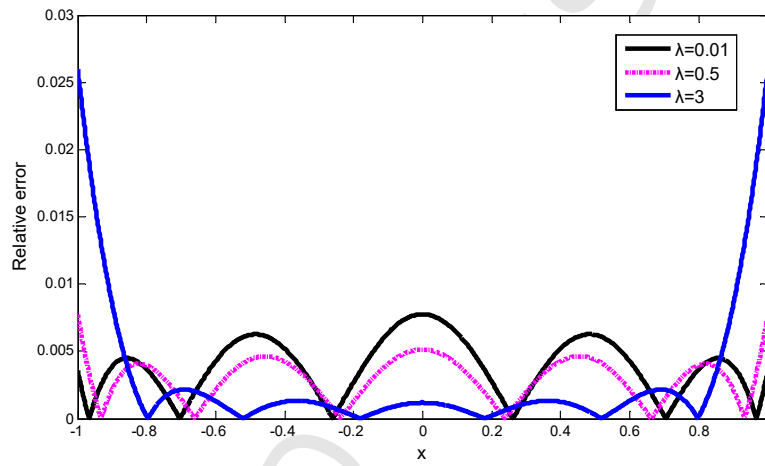


Fig.2 The relative error yielded by the fifth-order APC expansion with different weight functions over the interval

Figure 3

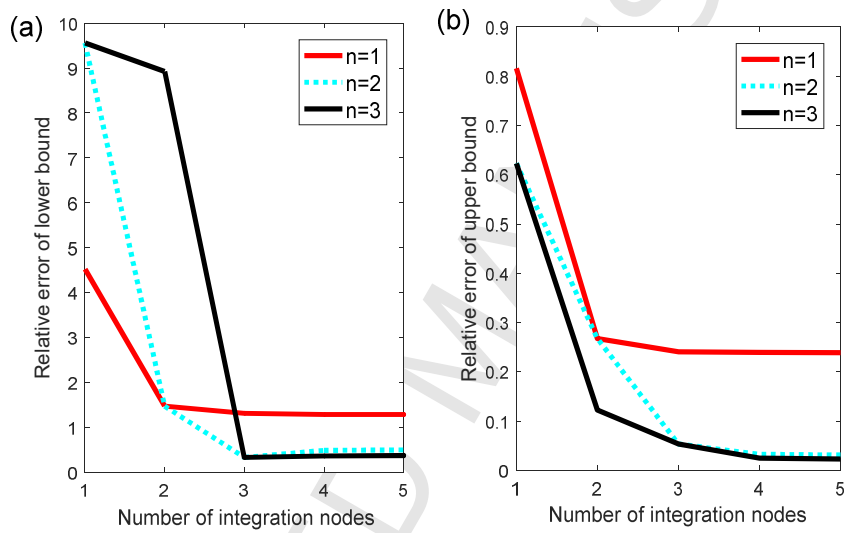


Fig.3 Effect of the number of integration nodes on the accuracy of IRAPCM at different retained orders

Figure 4

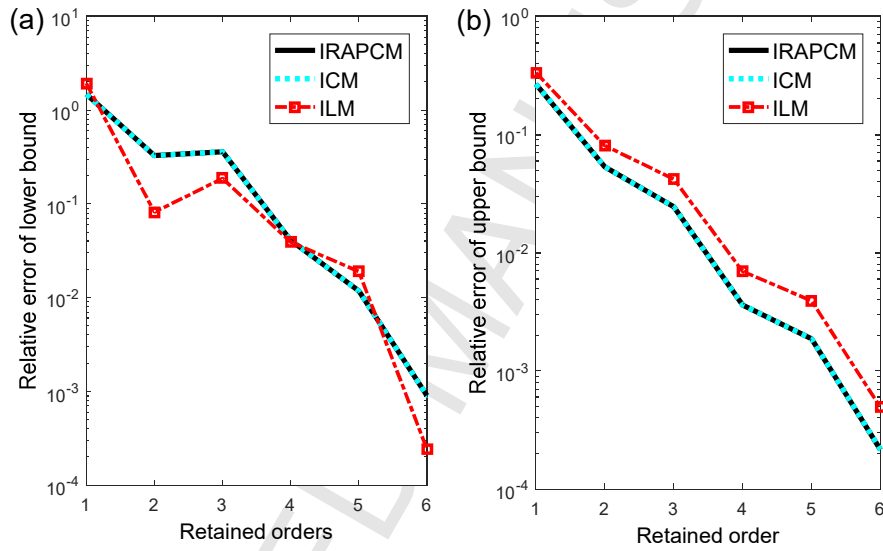


Fig.4 Relative error of the lower and upper bounds of y : (a) lower bound; (b) upper bound.

Figure5

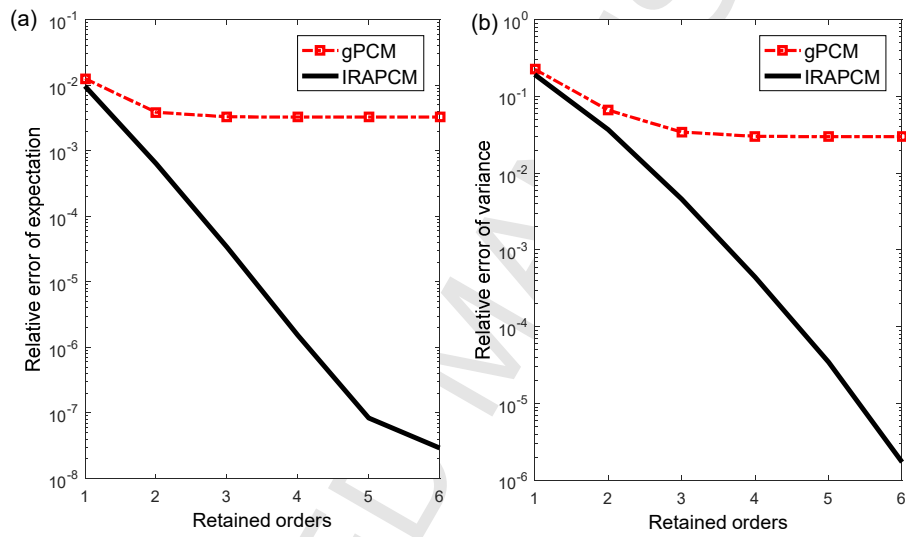
**Fig.5** Relative error of the expectation and variance of y : (a) expectation; (b) variance.

Figure6

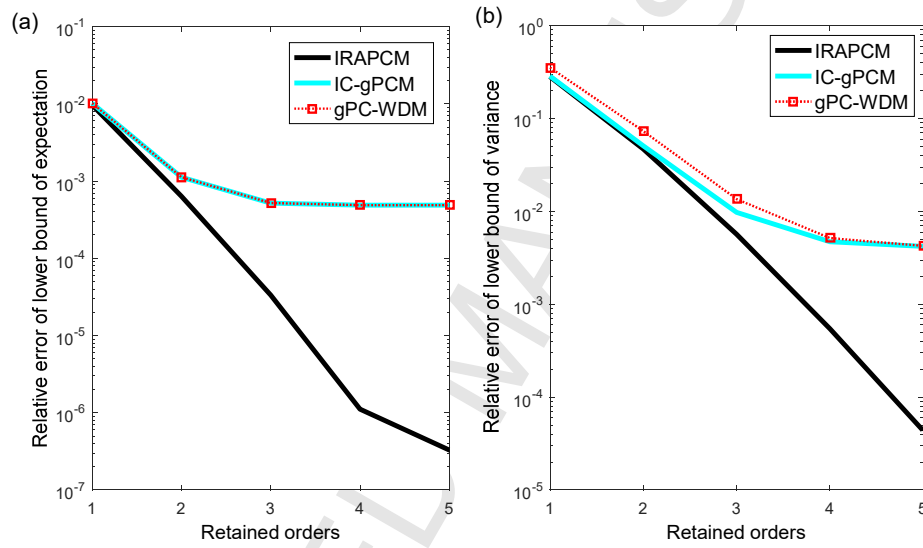


Fig.6 Relative error of the lower bound of the expectation and variance of y : (a) lower bound of expectation; (b) lower bound of variance.

Figure7

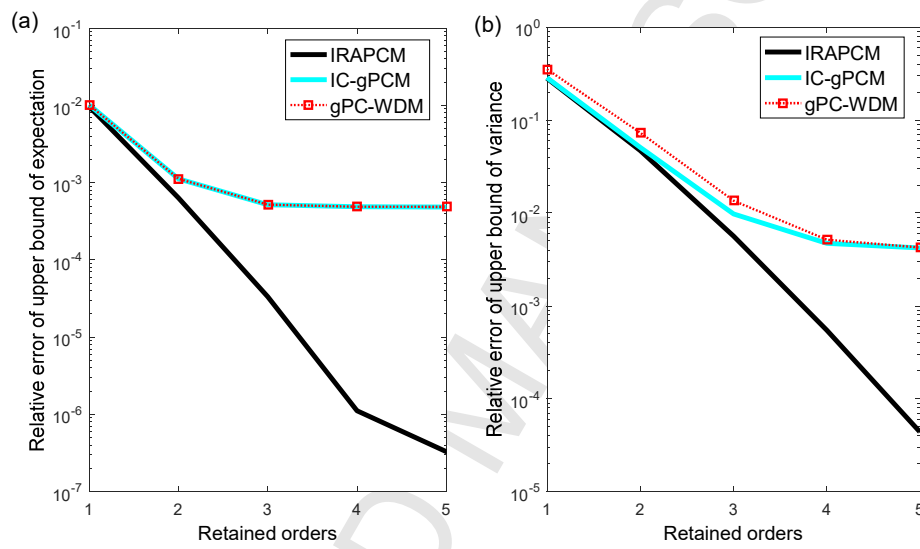


Fig.7 Relative error of the upper bound of the expectation and variance of y : (a) upper bound of expectation; (b) upper bound of variance.

Figure8

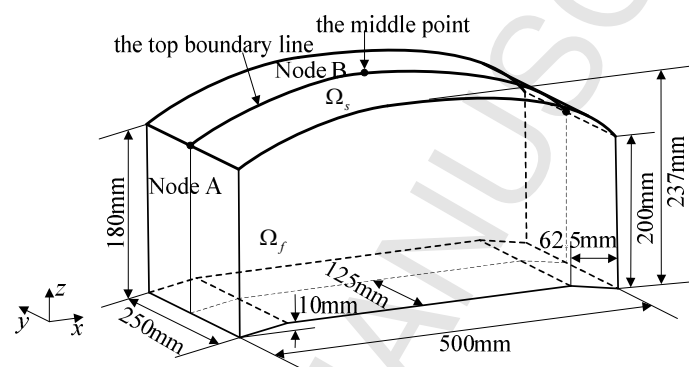


Fig.8 A shell structure-acoustic system

Figure9

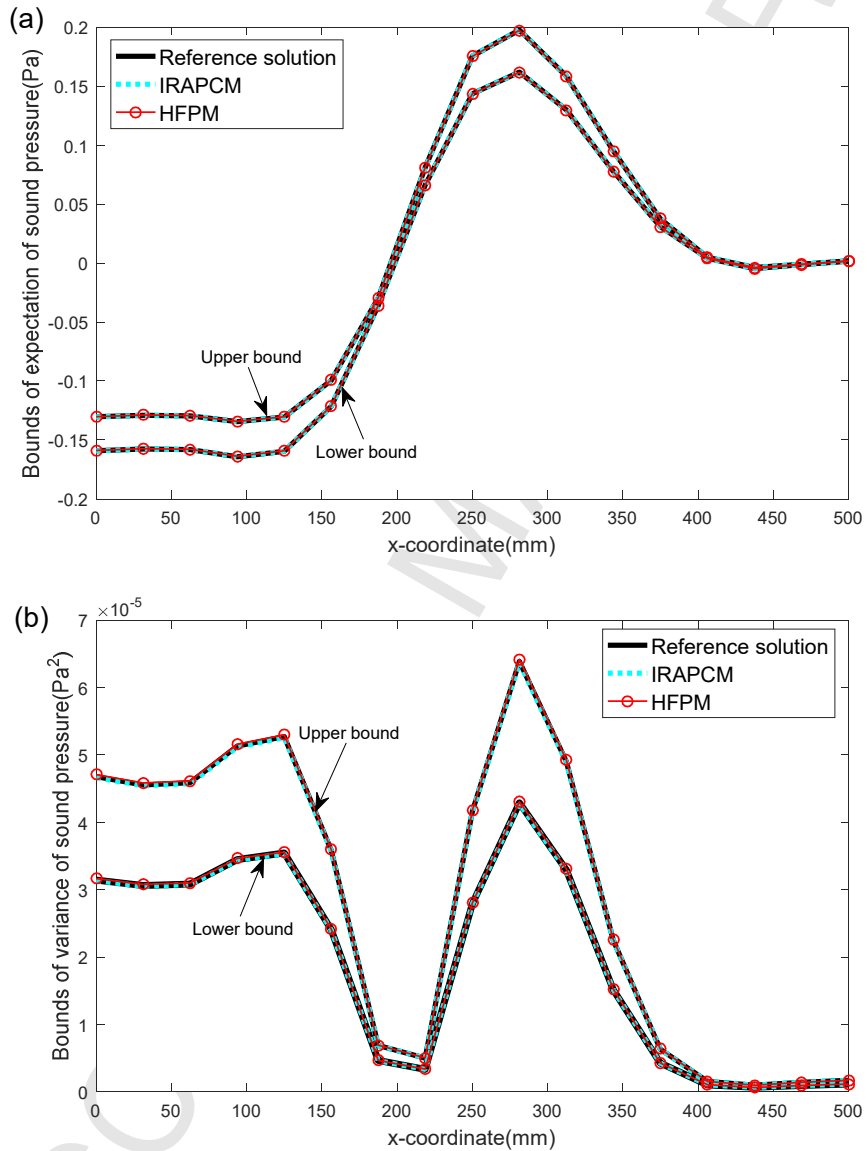


Fig.9 The bounds of expectation and variance of the sound pressure distributing along the top boundary line of case4 calculated by the first-order IRAPCM and the HFPM: (a) bounds of expectation, (b) bounds of variance.

Figure10

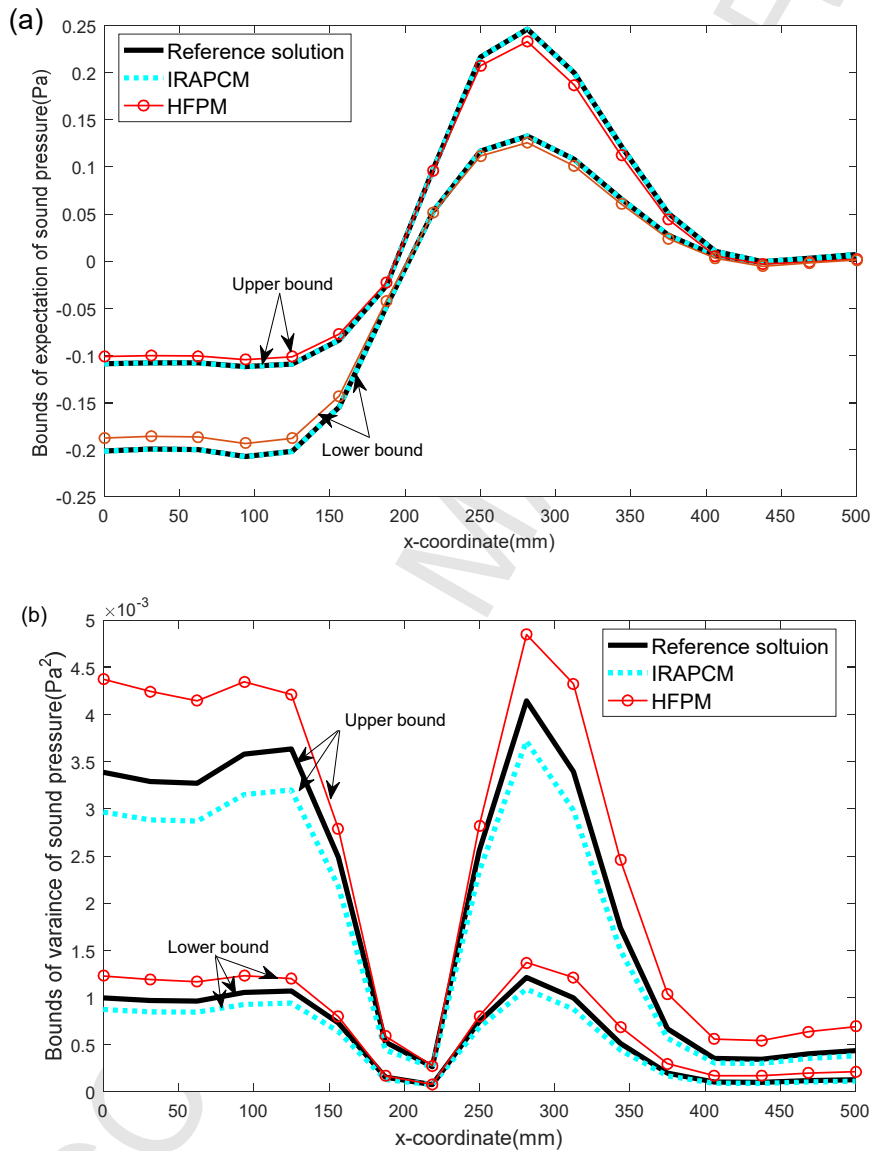


Fig.10 The bounds of expectation and variance of the sound pressure distributing along the top boundary line of case3 calculated by the first-order IRAPCM and the HFPM: (a) bounds of expectation, (b) bounds of variance.

Figure 11

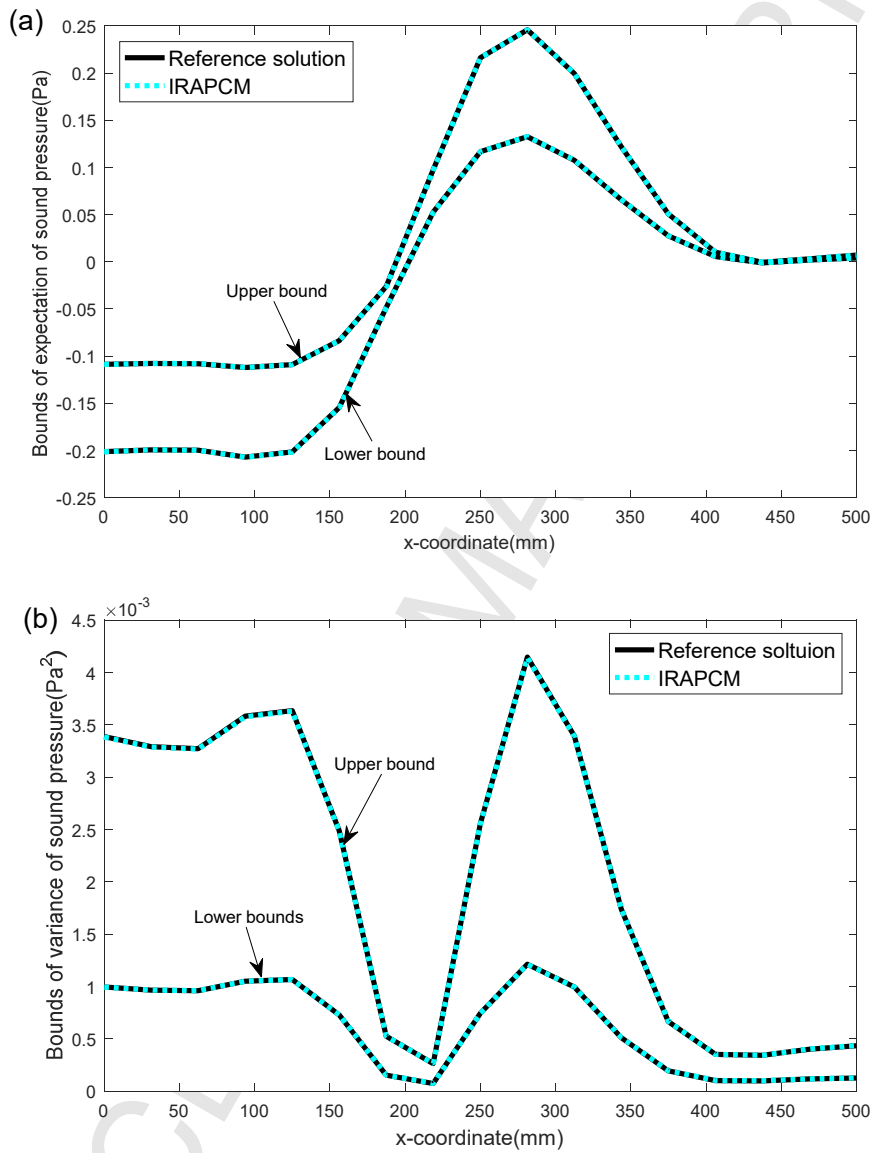


Fig.11 The bounds of expectation and variance of the sound pressure distributing along the top boundary line of case3 calculated by the high-order IRAPCM (a) bounds of expectation, (b) bounds of variance.

Figure12

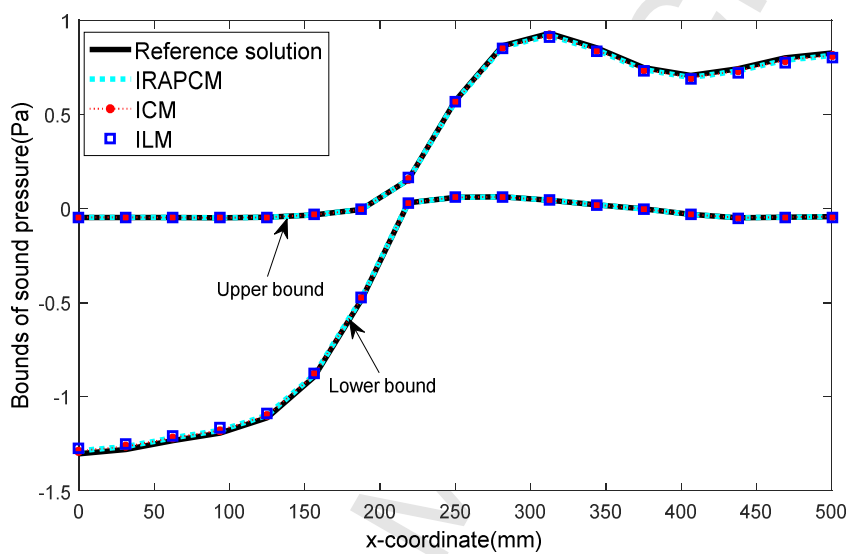


Fig.12 The bounds of the sound pressure distributing along the top boundary line of case1

Figure13

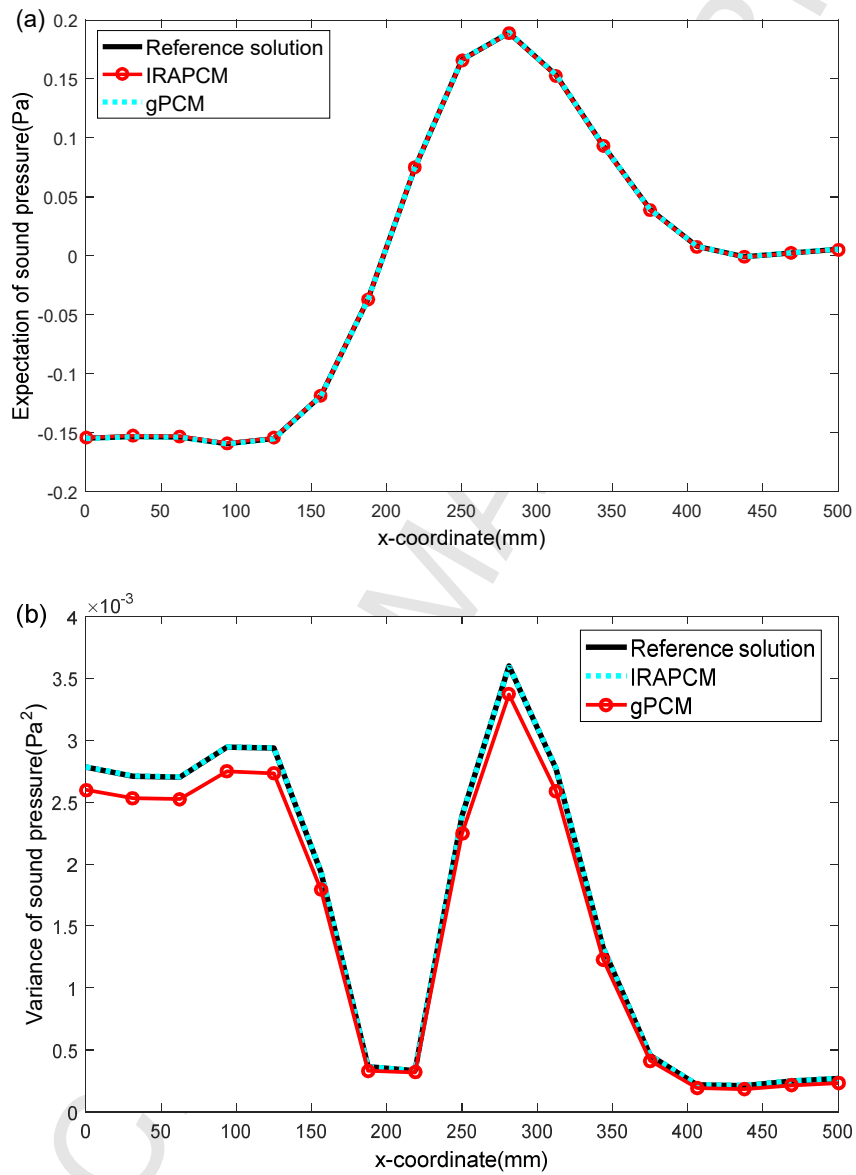


Fig.13 The expectation and variance of the sound pressure distributing along the top boundary line of case2: (a) expectation, (b) variance.

Figure14

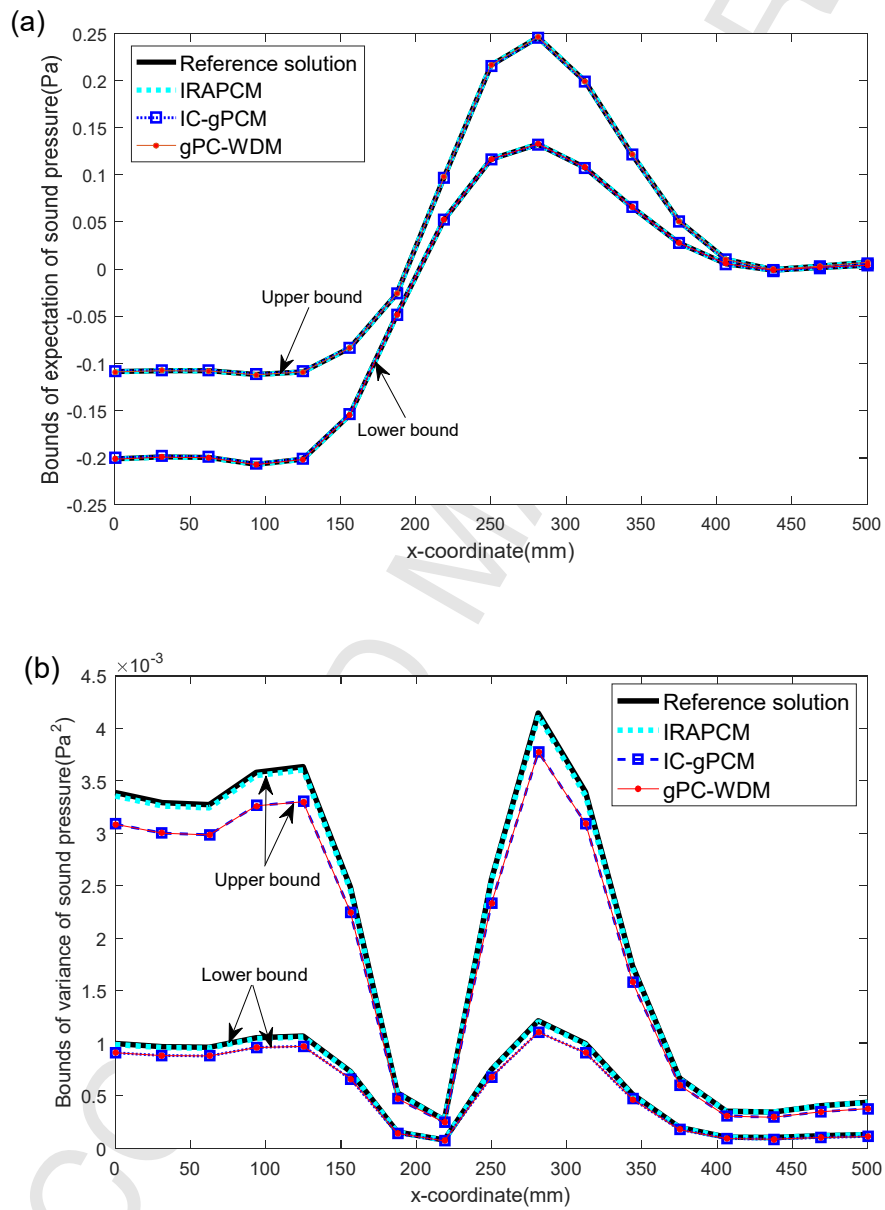


Fig.14 The bounds of expectation and variance of the sound pressure distributing along the top boundary line of case3: (a) bounds of expectation, (b) bounds of variance.

Table1

Table 1 Three uncertain models for x_1 and x_2

Interval model	Random model	Hybrid uncertain model
$x_1 \in [-1, 1]$	$x_1 = \xi_1$	$x_1 = \xi_1$
$x_2 \in [-1, 1]$	$x_2 = 2\xi_2 - 2.5$	$x_2 \in [-1, 1]$

Table2

Table2 Uncertain parameters of four cases of uncertain shell structure-acoustic systems

Uncertain parameters	Case1	Case2	Case3	Case4
$t(\text{mm})$	[1.6, 2.4]	$2+0.6\xi_3$	$2+0.6\xi_1$	$2+0.1\xi_1$
$c(\text{m/s})$	[306, 364]	$340+34\xi_3$	$340+34\xi_1$	$340+4\xi_1$
$E(\text{GPa})$	[168, 252]	$210+63\xi_1^{\lambda_1}$	$210+63\xi_1^{\lambda_1}$	$210+10\xi_1^{\lambda_1}$
$\rho_f(\text{kg/m}^3)$	[0.96, 1.44]	$1.2+0.36\xi_2^{\lambda_2}$	[0.96, 1.44]	[1.08, 1.32]

Table3

Table3 Execution time of different methods for response analysis of three cases

Uncertain models	Methods	Time for construction of polynomial basis	Time of RRSS	Time for uncertainty analysis of PCE	Total time
Case1	IRAPCM	4.1s	431.3 s	4.2s	439.6s
	ICM	0s	431.5s	3.1s	434.6s
	ILM	0s	431.5s	0.7s	432.2s
Case2	IRAPCM	4.3s	431.6 s	0s	435.9s
	gPCM	0s	431.2s	0s	431.2s
	IRAPCM	4.1s	431.6 s	1.9s	437.6s
Case3	gPC-ICM	0s	431.2s	1.1s	432.3s
	gPC-WDM	0s	431.5s	0.5s	432.0s