ON THE EQUIVALENCE OF LIE SYMMETRIES AND GROUP REPRESENTATIONS

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Abstract. We consider families of linear, parabolic PDEs in n dimensions which possess Lie symmetry groups of dimension at least four. We identify the Lie symmetry groups of these equations with the $2n+1$ dimensional Heisenberg group and $SL(2,\mathbb{R})$. We then show that for PDEs of this type, the Lie symmetries may be regarded as global projective representations of the symmetry group. We construct explicit intertwining operators between the symmetries and certain classical projective representations of the symmetry groups. Banach algebras of symmetries are introduced.

1. Introduction

In the 1880's, Sophus Lie published his pioneering work on the symmetries of systems of ordinary and partial differential equations. Specifically, he developed a method of computing local groups of transformations which transform solutions of a system of DEs to other solutions. Following Lie's remarkable achievements, the modern theory of Lie groups was developed.

The applications of Lie's theory to the study of differential equations are extensive, and an exhaustive list is not possible here. The books by Olver [18] and Bluman and Kumei [2], contain excellent modern introductions to the theory of Lie symmetry groups. Both these works present many applications of Lie symmetry analysis.

After Lie's initial contributions, however, Lie symmetry analysis was comparatively neglected till Birkhoff began the modern revival of the subject in the 1950s with his famous work on the equations of hydrodynamics, [1]. The apparently local nature of the transformations provided by Lie's method, was generally held to place them outside the realm of modern Lie group theory, with its emphasis on representation theory and global analysis, see the remarks in the introduction to Olver's book on this subject.

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However, in [5] and [6], Craddock considered a number of important equations, specifically the heat equation, the zero potential Schrodinger equation, a Fokker-Planck equation, the two dimensional Laplace equation and the axially symmetric wave equation. For each of these equations, the Lie symmetry group can be identified with a well known classical Lie group and the Lie symmetries, restricted to a rich solution space, are actually equivalent to an irreducible representation of the group. The intertwining operators were explicitly constructed.

This initial work naturally leads to the question: Can the Lie symmetries of every linear PDE be identified with global group group representations in a similar way? The current paper is the first in a series of articles which aims to give a positive answer to this question for, second order linear equations in any dimension, which possess a fundamental solution.

In the current work, we consider equations of the form

$$cu_t = \Delta u, \; cu_t = \Delta u + \sum_{k=1}^{n} a_k x_k u,$$
$$cu_t = \Delta u + A\|x\|^2 u, \; cu_t = \Delta u + \frac{A}{\|x\|^2} u,$$
$$cu_t = \Delta u + \left(\frac{A}{\|x\|^2} + B\|x\|^2 \right) u,$$

where $\{a_k\}_{k=1}^n$, A and B are nonzero constants, $c = 1$ or i, $\|x\|^2 = \sum_{k=1}^n x_k^2$ and Δ is the usual $\mathbb{R}^n$ Laplacian. We demonstrate that for each of these equations, the classical Lie symmetry group may be realised as a global group, and that the symmetries may be intertwined with irreducible representations of the group. The first three equations we treat in complete generality. For the fourth we restrict attention to the $n = 2$ case. The final case we deal with only briefly.

In the case where $n = 1$, the above equations are the only equations of the form $cu_t = u_{xx} + Q(x)u$, with symmetry group of dimension at least four. (We exclude superposition of solutions in our discussion). One very important corollary for the $n = 1$ case will be that any PDE on the line of the form

$$u_t = A(x,t)u_{xx} + B(x,t)u_x + C(x,t)u, \; x \in \Omega \subseteq \mathbb{R}, \tag{1.1}$$

with Lie symmetry group of dimension at least four, the Lie symmetries, restricted to a certain solution space, are equivalent to an irreducible representation the semi-direct product of $SL(2,\mathbb{R})$ and the Heisenberg group, or the direct product of $SL(2,\mathbb{R})$ with $\mathbb{R}$.

The results presented in the current work extend to general second order, linear, parabolic equations in higher dimensions, but the extension is non trivial, and will be treated in a subsequent article. For linear hyperbolic and elliptic equations of order two, we have analagous results, which need their own separate treatments.

A well known application of Lie symmetries is to the construction of fundamental solutions. For example, in [18], the constant solution of the heat equation is transformed to the usual heat kernel. More generally, previous work of the authors has shown that for the heat equation on any nilpotent Lie group, we can find a group transformation which maps the constant solution to the heat kernel, [8]. Craddock has also shown that for classes of linear parabolic PDEs on the line, it is possible to transform a stationary solution, to a Laplace or Fourier transform of a fundamental solution, by the application of a symmetry, [7].

The fact that the application of a symmetry to a stationary solution yields a classical integral transform of a fundamental solution, suggests a close connection between Lie symmetries and harmonic analysis. In this paper we will show how this method for finding fundamental solutions for such PDEs arises naturally from representation theory, and extend Craddock's specialised results to any PDE on the line with at least a four dimensional symmetry group.

Section 2 gives general background on symmetries and representations, Section 3 deals with the Schrödinger equation with zero potential, and Sections 4 to Section 6 with more general Schrödinger-type equations, culminating in Theorem(4.10). The remaining three sections deal with the generalised heat equations, where we can no longer use unitary methods. Finally, in Section 7 we construct spaces of distributions for the representations in the case where the symmetry groups are four dimensional and the action is nonunitary. The final section presents applications of our results to integral transforms on the line and the Heisenberg group.

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2. Some Background on Lie Symmetry Groups and Representation Theory

We present here the basic facts that we need. For simplicity, consider a single nondegenerate PDE

$$P(x, D^\alpha u) = 0, \quad x \in \Omega \subseteq \mathbb{R}^m, \tag{2.1}$$

of order n. Here $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial_{x_m}^{\alpha_m}}$, with α a multi-index. We introduce vector fields, of the form

$$\mathbf{v} = \sum_{i=1}^{m} \xi_i(x, u)\partial_{x_i} + \phi(x, u)\partial_u, \tag{2.2}$$

in which $\partial_{x_i} = \partial/\partial x_i$ etc. Lie derived conditions on ξ_k and ϕ which guarantee that (2.2) generates a local group of transformations preserving solutions of (2.1).The resulting vector field is called an *infinitesimal symmetry*, and the set of all infinitesimal symmetries form a Lie algebra. See [18] for the details.

Every vector field has a so called evolutionary representative, written $\mathbf{v}_Q = Q(x,t,u)\partial_u$ which generates the same symmetry. The evolutionary representative for (2.2) is $\mathbf{v}_Q = \left(\phi(x,u) - \sum_{i=1}^m \xi_i(x,u)u_{x_i}\right)\partial_u$. The function Q is known as the characteristic of the vector field.

Lie established many results about the infinitesimal symmetries of a PDE. The following result (See [18]) is of fundamental important to us.

Theorem 2.1 (Lie)**.** *Any two PDEs which are equivalent under an invertible change of variables have isomorphic Lie symmetry algebras and Lie symmetry groups.*

2.0.1. *Representations and Global Symmetries.* Now we focus our attention on a general linear PDE (2.1), where $P(x, D^\alpha)$ is a linear partial differential operator and Ω is a domain in $\mathbb{R}^n$. We suppose that the semigroup e^{tP} has a kernel $K(x,y,t)$ defined on $\Omega\times\Omega$, and that there exists a topological vector space $V(\Omega)$ of measurable functions on Ω such that

$$u(x,t) = \mathcal{A}f(x) = \int_\Omega f(y)K(x,y,t)dy, \tag{2.3}$$

defines a solution of (2.1) satisfying the initial condition $u(x,0) = f(x)$, for all $f\in V(\Omega)$.

At this level of generality, we assume only that $V(\Omega)$ is such that the integral (2.1) is convergent and defines solutions of (2.1) for all $f\in V$. We are not concerned with questions of regularity or uniqueness. In order to guarantee non-triviality, we shall require that the elements of $V(\Omega)$ separate points of Ω, in the sense that for all $x\neq y\in\Omega$ there exist $f,g\in V(\Omega)$ with $\mathrm{supp} f\cap \mathrm{supp} g = \emptyset$, and $x\in \mathrm{supp} f$, $y\in\mathrm{supp} g$.

Now suppose that we have a group G, which we may take to be a Lie group, (though this is not essential), which has a representation (ρ, V). We do not require that ρ be unitary, or even bounded. We use this representation to define a group symmetry σ of (2.1), a technique first presented in [4]. We will define a symmetry operator $\sigma(g)$ according to the rule

$$\sigma(g)u(x) = \int_\Omega (\rho(g)f)(y)K(x,y,t)dy. \tag{2.4}$$

Since $\rho(g): V\to V$, the integral (2.4) converges, so it is a solution of (2.1). It therefore follows that $\sigma(g)$ is a symmetry. We are transferring the action of the representation ρ from V to the solution space $\mathcal{H} =$

$\{u : u = \mathcal{A}f\}$. In the language of representation theory, the operator $\mathcal{A}$ intertwines ρ and σ. That is

$$\sigma\mathcal{A} = \mathcal{A}\rho. \tag{2.5}$$

A key consequence of this approach is that G is automatically a *global* group of symmetries of (2.1).

Now let $Q(t, D^\alpha)$ be a second partial differential operator such that the PDE

$$w_t = Q(t, D^\alpha)w,\ x \in \overline{\Omega} \subseteq \mathbb{R}^n,\ t \geq 0, \tag{2.6}$$

is equivalent to (2.1) under an invertible change of variables, given by an invertible mapping $E : \Omega \times [0, \infty) \to \overline{\Omega} \times [0, \infty)$. Then it is clear that G is also a group of symmetries of (2.6). Indeed, given a solution u of (2.1) we may obtain a solution w of (2.6) by $w = Eu$. Then we define $\overline{\sigma}$ by

$$(\overline{\sigma}(g))w(x) = (E\sigma(g)E^{-1}w)(x), \tag{2.7}$$

and this is a symmetry of (2.6). Thus $\overline{\sigma} = E\sigma E^{-1}$. The relationship between $\overline{\sigma}$ and ρ is then

$$\overline{\sigma} = E\sigma E^{-1} = E\mathcal{A}\rho\mathcal{A}^{-1}E^{-1}, \tag{2.8}$$

from which we have $\overline{\sigma}E\mathcal{A} = E\mathcal{A}\rho$. Thus the operator $E\mathcal{A}$ intertwines $\overline{\sigma}$ and ρ.

There are obvious questions to ask about this construction. Suppose that we are given a PDE, the intertwining operator $\mathcal{A}$ and an appropriate representation (ρ, V) of some Lie group G. Does it follow that the operator σ constructed by (2.4) acts on solutions by Lie point symmetry transformations? The answer turns out to be no, (see [8]). However, any representation of a group G defined on $V(\Omega)$ must lead to group symmetries of some type.

In [8] it was pointed out that many more symmetries for a PDE than the standard Lie point symmetries can be obtained via the construction (2.5). For example, we can obviously find principal series representations of $SL(2, \mathbb{R})$ which preserve $L^2(\mathbb{R})$. Since the integral in $u(x,t) = \int_{-\infty}^{\infty} f(y)\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}dy$ converges for $f \in L^2(\mathbb{R})$, whenever $t \geq c > 0$, it follows that any principal series representation of $SL(2, \mathbb{R})$ which preserves L^2, will automatically yield symmetries of the heat equation, although these will not in general be point symmetries.

Suppose now that we have a group G of point symmetries. A natural question to ask is whether this can be described in terms of the representation theory of G? More precisely, assume that the PDE (2.1) has Lie point symmetry group G, which acts on solutions by $\sigma(g)u$. We ask whether or not it is possible to find a representation (ρ, V) of G and an operator $\mathcal{A} : V \to \mathcal{H}$ such that $\sigma(g)\mathcal{A}f = \mathcal{A}\rho(g)f$, at least whenever

$\sigma(g)\mathcal{A}f$ is defined? By the preceding argument, if we can find such a ρ and $\mathcal{A}$, then the equivalence extends automatically to any other PDE which can be obtained from the original PDE by an invertible change of variables. It is also immediate that the symmetry group G, which is *a priori* only a local group, can be extended to a global group of symmetries if we restrict the action to the solution space $\mathcal{H}$.

3. The Schrödinger Equation With Zero Potential

The equation

$$iu_t = \Delta u. \tag{3.1}$$

is the Schrödinger equation with zero potential. For the one dimensional case, it was shown in [5] that the Lie symmetries of (3.1) are unitarily equivalent to a projective representation of the group $H_3 \rtimes SL(2,\mathbb{R})$. Here $g \in SL(2,\mathbb{R})$ acts on $h \in H_3$ by $g.(a\,b\,c)^T = (g\begin{pmatrix} a \\ b \end{pmatrix}, c)^T$. We briefly recall one of the main results of that paper. We adopt here a slightly different notation. We introduce the operator

$$u(x,t) = \mathcal{A}f(x,t) = \lim \int_{-\infty}^{\infty} \widehat{f}(y)K(x-y,t)dy, \tag{3.2}$$

where $\widehat{f}$ is the Fourier transform of f, $K(x-y,t) = \frac{1}{\sqrt{-4\pi i t}}e^{\frac{(x-y)^2}{4it}}$, and lim denotes the limit in the L^2 mean. For all $f \in L^2(\mathbb{R})$ the operator (3.2) defines solutions of (3.1). Usually we will omit the lim prefix, but it is to be understood.

Equation (3.1) possesses a six dimensional Lie symmetry algebra spanned by the vector fields

$$\mathbf{v}_1 = \partial_x, \mathbf{v}_2 = \partial_t,\ \mathbf{v}_3 = iu\partial_u,\ \mathbf{v}_4 = x\partial_x + 2t\partial_t - \frac{1}{2}iu\partial_u$$
$$\mathbf{v}_5 = 2t\partial_x - ixu\partial_u,\ \mathbf{v}_6 = 4xt\partial_x + 4t^2\partial_t - (ix^2 + 2t)u\partial_u.$$

The Lie point symmetry operators obtained from exponentiating the $\mathbf{v}_i$ we denote by σ.

Recall the Schrödinger representation of the three dimensional Heisenberg group H_3, which we realise as $\mathbb{R}^2 \ltimes \mathbb{R}$, acting in $L^2(\mathbb{R})$ by:

$$(\rho_\lambda(a,b,c)f)(z) = e^{i\lambda(c+a(z-\frac{1}{2}b)}f(z-b), \tag{3.3}$$

for all $(a,b,c) \in H_3$. Details of the construction of these representations may be found in Folland, [12]. Observe that for $\lambda = 1$

$$(\rho_1(0,0,\epsilon)f)(z) = e^{i\epsilon}f(z). \tag{3.4}$$

Now recall that for the Schrödinger equation (3.1)

$$\sigma(\exp \epsilon\mathbf{v}_3)u(x,t) = e^{i\epsilon}u(x,t). \tag{3.5}$$

The vector fields $\{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_5\}$ form a basis for the Heisenberg Lie algebra and $\mathbf{v}_3$ generates the centre of the Heisenberg group. Thus the Stone-Von Neumann Theorem, (see [12]), suggests that ρ_1 and σ should be equivalent via some unitary operator. This is true and the unitary operator turns out to be $\mathcal{A}$. We shall denote ρ_1 by ρ.

The vector fields $\{\mathbf{v}_2, \mathbf{v}_4, \mathbf{v}_6\}$ generate $SL(2,\mathbb{R})$. To realise the full symmetry group action in terms of representations, we need a representation for $SL(2,\mathbb{R})$, and we shall use the Segal-Shale-Weil representation (SSW representation). Strictly speaking, it is a projective representation of $SL(2,\mathbb{R})$ which can be lifted to a representation of the double cover $SL(2,\mathbb{R}) \times \mathbb{Z}_2$ by a $\mathbb{Z}_2$-valued cocycle, see [5]. The group $\mathbb{Z}_2$ appears because (3.1) has a discrete symmetry d_- which acts by

$$d_- u(x,t) = u(-x,t).$$

We introduce the following family of operators:

$$R_\lambda \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} f(z) = e^{-i\lambda b z^2} f(z) \tag{3.6}$$

$$R_\lambda \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} f(z) = \sqrt{|a|} f(az) \tag{3.7}$$

$$R_\lambda \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} f(z) = \sqrt{|\lambda|} \widehat{f}(\lambda z). \tag{3.8}$$

We extend R_λ to the whole of $SL(2,\mathbb{R})$ by the *Bruhat decomposition* of $SL(2,\mathbb{R})$, see [16]. Suppose that $\alpha \neq 0$ and $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbb{R})$. Then $\delta = \frac{1}{\alpha} + \frac{\beta\gamma}{\alpha}$. Let $p = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Using the fact $p^{-1} = p^3$, we have $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = p^{-1} \begin{pmatrix} 1 & -c \\ 0 & 1 \end{pmatrix} p \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$. We therefore define

$$R_\lambda \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} f = R_\lambda(p^3) R_\lambda \begin{pmatrix} 1 & -c \\ 0 & 1 \end{pmatrix} R_\lambda(p) R_\lambda \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} R_\lambda \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} f.$$

The case when $\alpha = 0$ is similar.

Remark 3.1. The action of R_λ on $L^2(\mathbb{R})$ is not irreducible, since it preserves even and odd functions.

In what follows we will consider the (projective) representation T_1 of $G = H_3 \rtimes SL(2,\mathbb{R})$ acting on $L^2(\mathbb{R})$ given by

$$T_1(h,g) f(z) = \rho(h) R(g) f(z), \;\; f \in L^2(\mathbb{R}), \tag{3.9}$$

where $h \in H_3$ and $g \in SL(2,\mathbb{R})$ and $R(g) = R_1(g)$. This projective representation may be lifted to a true representation of $G \rtimes \mathbb{Z}_2$. In [5] the following result was established.

Theorem 3.2. *Let σ be the Lie symmetry of $iu_t = u_{xx}$ given by exponentiating $\{\mathbf{v}_1, .., \mathbf{v}_6\}$. Let $(T_1, L^2(\mathbb{R}))$ be the projective representation of $G = H_3 \rtimes SL(2,\mathbb{R})$ given by (3.9) and let the operator $\mathcal{A}$ be defined by (3.2). Then G is a globally defined symmetry group when it is restricted to*

$$\mathcal{V} = \{u(x,t) : iu_t = u_{xx}, u(x,t) = \mathcal{A}f(x,t), f \in L^2(\mathbb{R})\} \tag{3.10}$$

and for all $h \in H_3, g \in SL(2,\mathbb{R})$

$$(\sigma(h,g)\mathcal{A}f)(x,t) = (\mathcal{A}T_1(g,h)f)(x,t). \tag{3.11}$$

The discrete symmetry d_- is dealt with as follows

Proposition 3.3. *Let $\mathbb{Z}_2$ act on $L^2(\mathbb{R})$ by*

$$l(1)f(y) = f(y), \tag{3.12}$$

$$l(-1)f(y) = f(-y). \tag{3.13}$$

Then $\mathcal{A}l(-1)f(x,t) = d_-\mathcal{A}f(x,t)$.

Proof. We have

$$(d_-)u(x,t) = \int_{-\infty}^{\infty} \widehat{f}(y)K(-x-y,t)dy = \int_{-\infty}^{\infty} \widehat{f}(-y)K(x-y,t)dy.$$

□

The representation l of $\mathbb{Z}_2$ defined here is not irreducible, since it preserves even functions. However the representation $(T, L^2(\mathbb{R}))$ is an irreducible projective unitary representation of G. The irreducibility is a consequence of the irreducibility of the Heisenberg group representation, see [12].

3.1. **Higher Dimensional Schrôdinger Equations.** For the n dimensional Schrödinger equation (3.1) the following holds.

Proposition 3.4. *The Lie symmetry algebra of (3.1) is spanned by*

$$\mathbf{v}_k = \partial_{x_k},\ \mathbf{v}_{n+1} = \partial_t, \mathbf{v}_{n+2} = iu\partial_u,$$

$$\mathbf{v}_{n+2+l} = x_k\partial_{x_j} - x_j\partial_{x_k}, k \neq j, j = 1, ..., n, l = 1, ..., \frac{n(n-1)}{2},$$

$$\mathbf{v}_{n+2+\frac{n(n-1)}{2}} = \sum_{k=1}^{n} x_k\partial_{x_k} - \frac{n}{2}iu\partial_u,\ \mathbf{v}_{n+3+\frac{n(n-1)}{2}+k} = 2t\partial_{x_k} - ix_ku\partial_u,$$

$$\mathbf{v}_{2n+4+\frac{n(n-1)}{2}} = \sum_{k=1}^{n} 4x_kt\partial_{x_k} + 4t^2\partial_t - (i\|x\|^2 + 2nt)u\partial_u,\ \mathbf{v}_\beta = \beta(x,t)\partial_u$$

where $k = 1, ..., n$, $\|x\|^2 = x_1^2 + \cdots + x_n^2$, β is any solution of (3.1).

It is straightforward to identify the Lie group generated by these vector fields.

Proposition 3.5. *The vector fields* $\{\mathbf{v}_1, ..., \mathbf{v}_{2n+4+\frac{1}{2}n(n-1)}\}$ *span a Lie algebra isomorphic to the Lie algebra of the semidirect product*

$$G = H_{2n+1} \rtimes (SL(2,\mathbb{R}) \times SO(n)),$$

where: $SL(2,\mathbb{R})$ *is embedded in* $SL(2n,\mathbb{R})$ *by the mapping*

$$k \in SL(2,\mathbb{R}) \to g_k \in SL(2n,\mathbb{R}),$$

where

$$k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} aI_n & bI_n \\ cI_n & dI_n \end{pmatrix}; \tag{3.14}$$

and $SO(n)$ *is embedded in* $SO(2n)$ *by*

$$A \in SO(n) \mapsto \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \in SO(2n).$$

Here I_n *is the* $n \times n$ *identity matrix and* $SL(2,\mathbb{R}) \times SO(n)$ *acts on* H_{2n+1} *by matrix multiplication. More precisely, we let* $h = (x, y, z) \in H_{2n+1}$ *where* $x, y \in \mathbb{R}^n$*. Then* $g = (k, A) \in SL(2,\mathbb{R}) \times SO(n)$ *acts on* h *by* $g.h = (kA(x, y)^T, z)$.

The representation theory of the group $G = (H_{2n+1} \rtimes (SL(2,\mathbb{R}) \times SO(n))$ is known.

Theorem 3.6. *Let* $G = (SL(2,\mathbb{R}) \times SO(n)) \rtimes H_{2n+1}$*. For each* $\lambda \in \mathbb{R}^*$ *there is an irreducible unitary projective representation of* G*, acting on* $L^2(\mathbb{R}^n)$ *given by* $T_\lambda = \pi_\lambda \otimes R_\lambda$*, where*

$$\left(R_\lambda\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, e_{SO(n)}\right) f\right)(\xi) = e^{-i\lambda b|\xi|^2} f(\xi) \tag{3.15}$$

$$\left(R_\lambda\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, e_{SO(n)}\right) f\right)(\xi) = |a|^{n/2} f(a\xi), \tag{3.16}$$

$$\left(R_\lambda\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, e_{SO(n)}\right) f\right)(\xi) = |\lambda|^{n/2} \widehat{f}(\lambda\xi) \tag{3.17}$$

$$(R_\lambda(e_{SL(2,\mathbb{R})}, A)f)(\xi) = f(A\xi), \tag{3.18}$$

where $\widehat{f}$ *is the Fourier transform of* f*. For the Heisenberg group we have*

$$(\pi_\lambda(x, y, z)f)(\xi) = e^{i\lambda(z+(x,\xi-\frac{1}{2}y))} f(\xi - y). \tag{3.19}$$

It is worth noting that if $n = 4k$, then $(R_\lambda, L^2(\mathbb{R}^n))$ is a genuine representation, see [17]. The following theorem may be proved by the same means as the one dimensional case. See [4] for the details.

Theorem 3.7. *Let ρ be the Lie symmetry operator for equation (3.1) obtained from exponentiating the vector fields of Proposition 3.4. Let $K(x,t) = (-4\pi i t)^{-n/2} e^{-\frac{i|x|^2}{4t}}$ and define the operator*

$$Bf(x,t) = \lim \int_{\mathbb{R}^n} \widehat{f}(y) K(x-y,t)dy,$$

where $f \in L^2(\mathbb{R}^n)$. Then for all $g \in G, f \in L^2(\mathbb{R}^n)$, we have

$$(\rho(g)Bf)(x,t) = (BT_1(g)f)(x,t). \tag{3.20}$$

4. The Case of a Four Dimensional Symmetry Group on the Line

We turn our attention to the Schrödinger equation

$$iu_t = u_{xx} - \left(\frac{B}{x^2} + C\right) u, \ x > 0, \tag{4.1}$$

where $B \in \mathbb{C}$. We take $C = 0$ since taking $w = e^{Ct}u$ reduces (4.1) to the equation $iw_t = w_{xx} - \frac{B}{x^2}w$. We will prove the equivalence of the Lie symmetries of this PDE and a variant of the SSW representation, whose action is defined on $L^2(\mathbb{R}^+)$. The role played by the Fourier transform in the previous section will be taken by the radial Fourier transform, which is of course the Hankel transform.

The Hankel transform of $f \in L^1(\mathbb{R}^+)$ is

$$\tilde{f}_\nu(y) = \mathfrak{H}_\nu f(y) = \int_0^\infty f(x)\sqrt{xy} J_\nu(xy)dy,$$

where $J_\nu(z)$ is the Bessel function of the first kind of order $\nu \in \mathbb{N}$. The Hankel transform can be extended to the whole of $L^2(\mathbb{R}^+)$. For the Hankel transform we have the following Plancherel Theorem.

Theorem 4.1. *Let $f \in L^2(\mathbb{R}^+)$. Then the Hankel transform of f exists and $\int_0^\infty |f(y)|^2 dy = \int_0^\infty |\tilde{f}_\nu(y)|^2 dy$.*

See Sneddon [22] for properties of the Hankel transform. We will define a variant of the SSW representation, which acts on $L^2(\mathbb{R}^+)$. In contrast to the SSW representation, this representation is irreducible.

Definition 4.2. For $\Re(\nu) > -2$, $\lambda \in \mathbb{R}^*$ and $f \in L^2(\mathbb{R}^+)$ we define the modified Segal-Shale-Weil representation by

$$R_\lambda^\nu \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} f(z) = e^{-i\lambda b z^2} f(z) \tag{4.2}$$

$$R_\lambda^\nu \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} f(z) = \sqrt{|a|} f(az) \tag{4.3}$$

$$R_\lambda^\nu \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} f(z) = \sqrt{|\lambda|} \tilde{f}_\nu(\lambda z). \tag{4.4}$$

Here

$$\tilde{f}_\nu(y) = \int_0^\infty f(x)\sqrt{xy}J_\nu(xy)dy. \tag{4.5}$$

Theorem 4.3. *The pair $(R^\nu_\lambda, L^2(\mathbb{R}^+))$, where R^ν_λ is given by Definition 4.2, is an irreducible projective unitary representation of $SL(2,\mathbb{R})$.*

Proof. The fact that the operators define a projective representation is clear. We only need to prove irreducibility. Let N be a closed, R^ν_λ invariant subspace of $L^2(\mathbb{R}^+)$ and let $h \in N^\perp$. Then it follows that $\int_0^\infty \overline{h(z)}(R^\nu_\lambda(g)f)(z)dz = 0$ for all $h \in N^\perp$. In particular

$$\int_0^\infty \overline{h(z)}e^{-i\lambda bz^2}a^{1/2}\tilde{f}(az)dz = 0, \tag{4.6}$$

for all b. But if we let $z = \sqrt{y}$, then this becomes

$$\int_0^\infty \phi(x)e^{-ix\lambda b}dx = 0, \tag{4.7}$$

where $\phi(x) = \frac{1}{2\sqrt{x}}\overline{h(\sqrt{x})}\tilde{f}(a\sqrt{x})$. Now $\widehat{\phi}(y) = 0$ a.e. implies that $\phi = 0$ a.e. But f is nonzero, so we must have $h = 0$ in $L^2(\mathbb{R}^+)$ and hence $N^\perp = \{0\}$. Since $L^2(\mathbb{R}^+) = N \oplus N^\perp$ it follows that $N = L^2(\mathbb{R}^+)$ and the representation is irreducible. □

Next we introduce our intertwining operator.

Theorem 4.4. *Let $f \in L^2(\mathbb{R}^+)$ and $u = \mathcal{A}f(x,t)$ where,*

$$\mathcal{A}f(x,t) = \lim \int_0^\infty \tilde{f}_\nu(y)\frac{\sqrt{xy}}{2it}\exp\left(-\frac{(x^2+y^2)}{4it} - i\frac{\nu\pi}{2}\right)J_\nu\left(\frac{xy}{2t}\right)dy, \tag{4.8}$$

where $\nu = \frac{1}{2}\sqrt{1+4B}$. Then u is a solution of $iu_t = u_{xx} - \frac{B}{x^2}u$. The integral is defined as the limit in the L^2 mean.

Proof. The kernel $p(x,y,t) = \frac{\sqrt{xy}}{2it}\exp\left(-\frac{(x^2+y^2)}{4it}\right)J_\nu\left(\frac{xy}{2t}\right)$ is a fundamental solution (4.1). See [7] for its construction. □

Observe that for all $B \in \mathbb{C}$, we may choose a branch of the square root so that $\Re(\nu) > 0$, since if $b^2 = \nu$, then $(-b)^2 = \nu$.

Remark 4.5. The inclusion of the factor of $e^{-i\frac{\nu\pi}{2}}$ is only for notational convenience. We will write $R(g) = R^\nu_1(g)$ with the value of ν being understood.

The infinitesimal symmetries of the PDE (4.1) form a subalgebra of the Lie algebra of symmetries of (3.1). Specifically the Lie symmetry group is $G = SL(2,\mathbb{R}) \times \mathbb{R}$. The action of the symmetries is given by

$$\sigma(\exp \epsilon \mathbf{v}_3)u(x,t) = u(x, t-\epsilon) \tag{4.9}$$

$$\sigma(\exp \epsilon \mathbf{v}_1)u(x,t) = e^{\frac{1}{2}\epsilon}u(e^{\epsilon}x, e^{2\epsilon}t) \tag{4.10}$$

$$\sigma(\exp \epsilon \mathbf{v}_2)u(x,t) = \frac{1}{\sqrt{1+4\epsilon t}} \exp\left\{\frac{-i\epsilon x^2}{1+4\epsilon t}\right\} u\left(\frac{x}{1+4\epsilon t}, \frac{t}{1+4\epsilon t}\right) \tag{4.11}$$

$$\sigma(\exp \epsilon \mathbf{v}_3)u(x,t) = e^{i\epsilon}u(x,t). \tag{4.12}$$

Here $\mathbf{v}_3 = \partial_t$, $\mathbf{v}_1 = x\partial_x + 2t\partial_t + \frac{1}{2}u\partial_u$, $\mathbf{v}_2 = 4xt\partial_x + 4t^2 - (ix^2+2t)u\partial_u$ and $\mathbf{v}_4 = iu\partial_u$. The main result is the following.

Theorem 4.6. *Consider the PDE (4.1). Let σ denote the Lie symmetry operator,(4.9)-(4.12). For all $B \in \mathbb{C}$ and $g \in SL(2,\mathbb{R})$ we have the equivalence*

$$(\sigma(g)\mathcal{A}f)(x,t) = (\mathcal{A}R(g)f))(x,t), \tag{4.13}$$

in which the intertwining operator is given by (4.8).

Proof. The idea of the proof is to establish the equivalence relationship for the one parameter groups generated by the basis vectors. We then use the Bruhat decomposition to extend the result to the whole of $SL(2,\mathbb{R})$. A basis for the Lie Algebra $\mathfrak{sl}_2$ is given by

$$X_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ X_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ X_3 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}. \tag{4.14}$$

There is a Lie algebra isomorphishm sending $\mathbf{v}_i$ to X_i, $i = 1,2,3$. So we need to show that $\sigma(\exp(\epsilon \mathbf{v}_i)\mathcal{A}f)(x,t) = \mathcal{A}R(\exp(\epsilon X_i)f(x,t)$. The proofs of these identities for $i = 1,2,4$ require only simple algebra. For example,

$$\sigma(\exp \epsilon \mathbf{v}_2)\mathcal{A}f(x,t) = \frac{1}{\sqrt{1+4\epsilon t}} \exp\left\{\frac{-i\epsilon x^2}{1+4\epsilon t}\right\} \mathcal{A}f\left(\frac{x}{1+4\epsilon t}, \frac{t}{1+4\epsilon t}\right).$$

Now if we apply the symmetry to $p(x,y,t)$ we have

$$\frac{1}{\sqrt{1+4\epsilon t}} \exp\left\{\frac{-i\epsilon x^2}{1+4\epsilon t}\right\} p\left(\frac{x}{1+4\epsilon t}, y, \frac{t}{1+4\epsilon t}\right) = e^{-i\epsilon y^2}p(x,y,t).$$

Thus

$$\begin{aligned}\sigma(\exp \epsilon \mathbf{v}_2)\mathcal{A}f(x,t) &= \int_0^\infty e^{-i\epsilon y^2}\tilde{f}_\nu(y)p(x,y,t) \\ &= \mathcal{A}R(\exp(\epsilon X_2))f(x,t).\end{aligned}$$

The final identity to be established is in the case $i = 3$. The key to the proof is the following Hankel transform, valid for all $\Re(\nu) > -2$.

$$\mathfrak{H}_\nu\left(\frac{\sqrt{y}}{2it}\exp\left(\frac{i(y^2+x^2)}{4t} - \frac{i\nu\pi}{2}\right) J_\nu\left(\frac{xy}{2t}\right)\right)(z) = \sqrt{z}e^{-itz^2}J_\nu(xz).$$

This may be found in standard tables, such as the Bateman manuscript [10] equation 8.11 (26) and (27) (page 51).

We note that the following identity holds for the Hankel transform:

$$\int_0^\infty (\mathfrak{H}_\nu f)(y)g(y)dy = \int_0^\infty f(y)(\mathfrak{H}_\nu g)(y)dy. \tag{4.15}$$

We thus have

$$\begin{aligned}
\mathcal{A}R(\exp(\epsilon X_3))f(x,t) &= \int_0^\infty \mathfrak{H}_\nu\left(e^{i\epsilon y^2}f(y)\right)\frac{\sqrt{xy}}{2it}e^{-\frac{(x^2+y^2)}{4it}-i\frac{\nu\pi}{2}}J_\nu\left(\frac{xy}{2t}\right)dy \\
&= \int_0^\infty e^{i\epsilon z^2}f(z)\sqrt{xz}e^{-itz^2}J_\nu(xz)dz \\
&= \int_0^\infty f(z)\sqrt{xz}e^{-iz^2(t-\epsilon)}J_\nu(xz)dz \\
&= \int_0^\infty \tilde{f}_\nu(y)\frac{\sqrt{xy}}{2i(t-\epsilon)}e^{-\frac{(x^2+y^2)}{4i(t-\epsilon)}-i\frac{\nu\pi}{2}}J_\nu\left(\frac{xy}{2(t-\epsilon)}\right)dy \\
&= u(x,t-\epsilon).
\end{aligned}$$

Note that the first line in this case is well defined for all ϵ, so we can define $\sigma(\exp(\epsilon)\mathbf{v}_4)\mathcal{A}f(x,t)$ to be equal to $\mathcal{A}R(\exp(\epsilon X_3))f(x,t)$, in the case when $\sigma(\exp(\epsilon\mathbf{v}_4)\mathcal{A}f(x,t)$ may not be a-priori defined. This completes the proof. □

Remark 4.7. The reader will observe that letting $B \to 0$ we recover the zero potential Schrodinger equation, giving a somewhat different realisation of the $SL(2,\mathbb{R})$ symmetries of that equation.

The representation $(R, L^2(\mathbb{R}^+))$ is an irreducible unitary projective representation of $G = SL(2,\mathbb{R})$, from which we conclude:

Corollary 4.8. *Let*

$$\mathcal{V} = \{u(x,t) : iu_t = u_{xx} - \frac{B}{x^2}u, u(x,t) = \mathcal{A}f(x,t), f \in L^2(\mathbb{R}^+)\}.$$

Then the restriction to $SL(2,\mathbb{R})$ of the Lie symmetry $(\sigma,\mathcal{V})$ of the PDE (4.1) is an irreducible unitary projective representation of $G = SL(2,\mathbb{R})$.

Another easy corollary is the following.

Corollary 4.9. *The group $G = SL(2,\mathbb{R})$ is a global Lie group of symmetries of $iu_t = u_{xx} - \frac{B}{x^2}u$ when its action is restricted to $\mathcal{V}$.*

So we have established that the Lie symmetries of the PDEs $iu_t = u_{xx}$ and $iu_t = u_{xx} - \frac{B}{x^2}u$ are in fact projective unitary representations of their respective symmetry groups. The equation $iu_t = \Delta u - \frac{A}{\|x\|^2}u$ will be treated below. The equation $iu_t = \Delta u$ may be found in [4].

Before continuing we prove an important result.

Theorem 4.10. *Let the PDE*

$$iu_t = A(x,t)u_{xx} + B(x,t)u_x + C(x,t)u,$$

have a four dimensional Lie algebra of symmetries. Then the Lie symmetries are unitarily equivalent to an action of the modified Segal-Shale-Weil representation, defined in 4.2 and $SL(2,\mathbb{R})\times\mathbb{R}$ is a global Lie group of symmetries. If the PDE has a six dimensional Lie algebra of symmetries, then the continuous Lie symmetries are unitarily equivalent to an action of the projective representation $(T_1, L^2(\mathbb{R}))$ of $G = H_3 \rtimes SL(2,\mathbb{R})$ and G is a global group of symmetries.

Proof. Lie proved that at any PDE of the given form with six dimensional Lie algebra of symmetries can be reduced to $iu_t = u_{xx}$. (See [14]). If the PDE has symmetry group of dimension 4, then it can be reduced to $iu_t = u_{xx} - \frac{B}{x^2}u$. From this and Theorem 2.2, the result follows. □

A consequence of Theorem 4.10 is the equivalence of the symmetries of $iu_t = u_{xx} - \left(Ax^2 + \frac{B}{x^2}\right)u$, and the representation $(R, L^2(\mathbb{R}^+))$. The reader can also prove the equivalence directly. Here the intertwining operator is given by $\mathcal{A}f(x,t) = \lim \int_0^\infty \tilde{f}_\nu(y)K(x,y,it)dy$ where $\tilde{f}_\nu$ is the Hankel transform of f, $\nu = \frac{1}{2}\sqrt{1+4A}$ and

$$K(x,y,t) = \frac{\sqrt{xy}}{2\sinh(2\sqrt{B}t)}\exp\left(-\frac{x^2+y^2}{2\tanh(2\sqrt{B}t)}\right) I_\nu\left(\frac{2\sqrt{B}xy}{\sinh(2\sqrt{B}t)}\right).$$

This fundamental solution is well known and can be constructed by the methods in [7]. For the n dimensional version of this equation, the equivalent result holds, but we will also defer a proof of this to a later article.

We next consider the two dimensional Schrödinger equation

$$iu_t = u_{xx} + u_{yy} - \frac{A}{x^2+y^2}u, \ (x,y)\in\mathbb{R}^2. \tag{4.16}$$

The Lie symmetries of this equation are unitarily equivalent to a version of the modified Segal-Shale-Weil representation.

Proposition 4.11. *The Lie algebra of symmetries of (4.16) is spanned by the vector fields*

$$\begin{aligned}
&\mathbf{v}_1 = \partial_t,\ \mathbf{v}_2 = x\partial_x + y\partial_y + 2t\partial_t - u\partial_u,\\
&\mathbf{v}_3 = 4xt\partial_x + 4yt\partial_y + 4t^2\partial_t - (i(x^2+y^2)+4t)u\partial_u,\\
&\mathbf{v}_4 = y\partial_x - x\partial_y, \mathbf{v}_5 = iu\partial_u, \mathbf{v}_\beta = \beta(x,y,t)\partial_u.
\end{aligned}$$

The following result is easily established.

Proposition 4.12. *The vector fields $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ generate $\mathfrak{sl}_2$.*

Exponentiation of the infinitesimal symmetries leads to the following proposition.

Proposition 4.13. *The Lie representation for (4.16) obtained by exponentiating the vector fields of 4.11 is*

$$(\pi(\exp\epsilon\mathbf{v}_1)u(x,y,t) = u(x,y,t-\epsilon)$$
$$(\pi(\exp\epsilon\mathbf{v}_2)u(x,y,t) = e^{\epsilon}u(e^{\epsilon}x, e^{\epsilon}y, e^{2\epsilon}t)$$
$$(\pi(\exp\epsilon\mathbf{v}_3)u(x,y,t) = \frac{1}{1+4\epsilon t}e^{\frac{-i\epsilon(x^2+y^2)}{1+4\epsilon t}}u\left(\frac{x}{1+4\epsilon t}, \frac{y}{1+4\epsilon t}, \frac{t}{1+4\epsilon t}\right)$$
$$(\pi(\exp\epsilon\mathbf{v}_4)u)(x,y,t) = u(x\cos\epsilon + y\sin\epsilon, y\cos\epsilon - x\sin\epsilon, t)$$
$$(\pi(\exp\epsilon\mathbf{v}_5)u)(x,y,t) = e^{i\epsilon}u(x,y,t).$$

For an intertwining operator, it is sufficient to use the following result.

Proposition 4.14. *Let $f \in L^2(\mathbb{R}^+)$ and $J_\mu(z)$ be a Bessel function of the first kind of order μ. Then*

$$\begin{aligned} u(x,y,t) &= \frac{1}{4\pi i t}\int_{\mathbb{R}^2}\tilde{f}(\rho)e^{-\frac{i\left(x^2+y^2+\eta^2+\xi^2\right)}{4t}}J_\mu\left(\frac{r\rho}{2t}\right)d\xi d\eta \\ &= (\mathcal{B}f)(x,y,t), \end{aligned} \tag{4.17}$$

where $\mu = \sqrt{1+A}$, $r = \sqrt{x^2+y^2}$, $\rho = \sqrt{\eta^2+\xi^2}$ is a solution of (4.16). Here $\tilde{f}$ is the Hankel transform of f.

Remark 4.15. The solution u defined above is *not* in general a solution of the Cauchy problem for (4.16) with initial data f. It can be shown that

$$p(x,y,\xi,\eta,t) = \frac{1}{4\pi i t}\sum_{n\in\mathbb{Z}} e^{-\frac{i\left(r^2+\rho^2\right)}{4t}}\left(\frac{(x+iy)(\eta-i\xi)}{(x-iy)(\eta+i\xi)}\right)^{n/2}J_{\sqrt{n^2+A}}\left(\frac{r\rho}{2t}\right),$$

with $r = \sqrt{x^2+y^2}$, $\rho = \sqrt{\eta^2+\xi^2}$, is a fundamental solution of (4.16). The derivation will be presented elsewhere, [9].

Let us now restrict attention to radial functions. The representation of $SL(2,\mathbb{R})$ that we need was introduced previously. The following result may be established by the methods we have introduced here.

Theorem 4.16. *Let $\bar{\pi}$ be the Lie representation of Proposition 4.13 restricted to $SL(2,\mathbb{R})$ and R^ν_λ be the modified Segal-Shale-Weil representation of Definition 4.2. Then for all $f \in L^2(\mathbb{R}^+), g \in SL(2,\mathbb{R})$ we have*

$$(\bar{\pi}(g)\mathcal{B}f)(x,y,t) = (\mathcal{B}R^\nu_1(g)f)(x,y,t), \tag{4.18}$$

where $\mathcal{B}$ is given by (4.17).

Remark 4.17. We can include the rotation group, by taking a tensor product of the SSW representation and a regular representation of $SO(2)$ and the obvious extension of the theorem holds. The n dimensional version of this result is also true, but the proof is slightly more involved and we defer it to a later publication.

5. Equations with Linear Potentials

We will consider the n dimensional PDE

$$iu_t = \Delta u + \left(\sum_{i=1}^n a_i x_i + c\right) u, \tag{5.1}$$

which has a fundamental solution

$$p(t,x,y) = \frac{e^{ct}}{(-4\pi i t)^{\frac{n}{2}}} \exp\left(\frac{i}{12}\sum_{i=1}^n a_i^2 t^3 - \frac{\|x-y\|^2}{4it} - i\frac{t}{2}\sum_{i=1}^n a_i(x_i+y_i)\right). \tag{5.2}$$

The PDE (5.1) has $G = H_{2n+1} \rtimes (SL(2,\mathbb{R}))$ as its Lie symmetry group. We prove the following result.

Theorem 5.1. *Let $u(x,t) = \mathcal{A}f(x,t)$ be a solution of (5.1), with $f \in L^2(\mathbb{R}^n)$, $\mathcal{A}f(x,t) = \lim \int_{\mathbb{R}^n} \widehat{f}(y)p(t,x,y)dy$, with p given by (5.2). Let $(T_1, L^2(\mathbb{R}^n))$ be as in Theorem 3.6 and $G = H_{2n+1} \rtimes SL(2,\mathbb{R})$. The group G is a global group of symmetries of (5.1) and if σ is the corresponding Lie symmetry operator, we have*

$$(\sigma(g)\mathcal{A}f)(x,t) = (\mathcal{A}T_1(g)f)(x,t), \tag{5.3}$$

for all $g \in G$ and $f \in L^2(\mathbb{R}^n)$.

Proof. We prove the $n=1$ case directly. The proof of the general case is simply the $n=1$ case repeated n times. Without loss of generality, we set $c=0$. The infinitesimal symmetries are given by

$$\begin{aligned}
&\mathbf{v}_1 = 2t\partial_x - i(x+At^2)u\partial_u, \mathbf{v}_2 = \partial_t + 2At\partial_x - iA(x+At^2)u\partial_u,\\
&\mathbf{v}_3 = iu\partial_u,\ \mathbf{v}_4 = (x+3At^2)\partial_x + 2t\partial_t + (\frac{1}{2} - i(A^2t^3 + 3Axt))u\partial_u,\\
&\mathbf{v}_5 = t\partial_x - iAtu\partial_u,\\
&\mathbf{v}_6 = (4xt + 4At^3)\partial_x + 4t^2\partial_t - (ix^2 + 2t + 6iAxt^2 + iA^2t^4)u\partial_u.
\end{aligned}$$

and the symmetry operators are

$$\begin{aligned}
\sigma(\exp(\epsilon\mathbf{v}_1))u(x,t) &= e^{i\epsilon x + i\epsilon^2 t - i\epsilon A t^2} u(x - 2\epsilon t, t)\\
\sigma(\exp(\epsilon\mathbf{v}_2))u(x,t) &= e^{\frac{-iA\epsilon}{3}\left(3x + A\left(3t^2 - 6t\epsilon + 2\epsilon^2\right)\right)}\\
&\quad \times u(x - 2A\epsilon t + A\epsilon^2, t-\epsilon)\\
\sigma(\exp(\epsilon\mathbf{v}_3))u(x,t) &= e^{i\epsilon}u(x,t)
\end{aligned}$$

$$\sigma(\exp(\epsilon\mathbf{v}_4))u(x,t) = e^{\frac{1}{2}\epsilon}e^{\frac{i}{3}\left(e^{3\epsilon}-1\right)\left(3Axt+\left(2e^{3\epsilon}-1\right)A^2t^3\right)}$$
$$\times\, u(e^{\epsilon}x+\left(e^{4\epsilon}-e^{\epsilon}\right)At^2, e^{2\epsilon}t)$$
$$\sigma(\exp(\epsilon\mathbf{v}_5))u(x,t) = e^{-iA\epsilon t}u(x-\epsilon t,t)$$
$$\sigma(\exp(\epsilon\mathbf{v}_6))Af(x,t) = e^{-\frac{i\epsilon x^2}{1+4\epsilon t}+\frac{\frac{i}{3}A\epsilon t^2\left(At^2(-3+8\epsilon t(3+2\epsilon t))-6(1+4\epsilon t)(3+4\epsilon t)x\right)}{(1+4\epsilon t)^3}}$$
$$\times\, u\left(\frac{x-4A\epsilon t^3+4\epsilon xt}{(1+4\epsilon t)^2}, \frac{t}{1+4\epsilon t}\right).$$

It is easy to show that $\mathbf{v}_1, \mathbf{v}_3$ and $\mathbf{v}_5$ generate the Heisenberg group, and $\mathbf{v}_2 - A\mathbf{v}_5, \mathbf{v}_4$ and $\mathbf{v}_6$ span $\mathfrak{sl}_2$. We set $u(x,t) = \mathcal{A}f(x,t)$. We let e_S, e_H denote the identity elements of $SL(2,\mathbb{R})$ and the Heisenberg group respectively. The central group element is trivial to check. Now

$$\begin{aligned}\mathcal{A}T_1((\epsilon,0,0),e_S)f(x,t) &= \int_{-\infty}^{\infty}\frac{\widehat{f}(y-\epsilon)}{\sqrt{-4\pi it}}e^{\frac{iA^2t^3}{12}+\frac{(x-y)^2}{4it}-\frac{iAt}{2}(x+y)}dy\\ &= \int_{-\infty}^{\infty}\frac{\widehat{f}(y)}{\sqrt{-4\pi it}}e^{\frac{iA^2t^3}{12}+\frac{(x-y-\epsilon)^2}{4it}-\frac{iAt}{2}(x+\epsilon+y)}dy\\ &= e^{-iA\epsilon t}u(x-\epsilon t,t)\\ &= \sigma(\exp(\epsilon\mathbf{v}_5))u(x,t).\end{aligned}$$

The remaining symmetries follow in a similar manner. We have

$$\mathcal{A}T_1((0,\epsilon,0),e_S)f(x,t) = \int_{-\infty}^{\infty}e^{-i\epsilon y}\frac{\widehat{f}(y)}{\sqrt{-4\pi it}}e^{\frac{iA^2t^3}{12}+\frac{(x-y)^2}{4it}-\frac{iAt}{2}(x+y)}dy \tag{5.4}$$
$$\begin{aligned}&= e^{i\epsilon x+i\epsilon^2t-i\epsilon At^2}\int_{-\infty}^{\infty}\frac{\widehat{f}(y)}{\sqrt{-4\pi it}}e^{\frac{iA^2t^3}{12}+\frac{(x-2\epsilon t-y)^2}{4it}-\frac{iAt}{2}(x-2\epsilon t+y)}dy\\ &= e^{i\epsilon x+i\epsilon^2t-i\epsilon At^2}u(x-2\epsilon t,t)\\ &= \sigma(\exp(\epsilon\mathbf{v}_1))u(x,t).\end{aligned}$$

The equivalence of the scaling symmetries follows from

$$\begin{aligned}\mathcal{A}T_1\left(e_H,\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix}\right)f(x,t) &= \int_{-\infty}^{\infty}a^{\frac{1}{2}}\frac{\widehat{f}(ay)}{\sqrt{-4\pi it}}e^{\frac{iA^2t^3}{12}+\frac{(x-y)^2}{4it}-\frac{iAt}{2}(x+y)}dy\\ &= \int_{-\infty}^{\infty}\frac{a^{\frac{1}{2}}\widehat{f}(y)}{\sqrt{-4\pi ia^2t}}e^{\frac{iA^2t^3}{12}+\frac{(ax-y)^2}{4ia^2t}-\frac{iAt}{2a}(ax+y)}dy.\end{aligned}$$

To evaluate the integral in terms of $u(x,t)$ we need the following identity:

$$\frac{iA^2t^3}{12}+\frac{(ax-y)^2}{4ia^2t}-\frac{iAt}{2a}(ax+y) = \frac{i}{3}\left(a^3-1\right)\left(3Axt+\left(2a^3-1\right)A^2t^3\right)$$
$$+\frac{iA^2a^6t^3}{12}+\frac{\left(ax+\left(a^4-a\right)At^2-y\right)^2}{4ia^2t}-\frac{iAa^2t}{2}\left(ax+\left(a^4-a\right)At^2+y\right).$$

It therefore follows that

$$\mathcal{A}T_1\left(e_H,\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix}\right)f(x,t)=\sqrt{a}\,e^{\frac{i}{3}\left(a^3-1\right)\left(3Axt+\left(2a^3-1\right)A^2t^3\right)}$$
$$\times\, u(ax+\left(a^4-a\right)At^2,a^2\,t).$$

Then $a=e^\epsilon$ gives $\mathcal{A}T_1\left(e_H,\begin{pmatrix}e^\epsilon & 0\\ 0 & e^{-\epsilon}\end{pmatrix}\right)f(x,t)=\sigma(\exp(\epsilon\mathbf{v}_4))Af(x,t)$, as required.

Omitting the somewhat laborious algebra we next find that

$$\mathcal{A}T_1\left(e_H,\begin{pmatrix}1 & \epsilon\\ 0 & 1\end{pmatrix}\right)f(x,t)=\int_{-\infty}^{\infty}e^{-i\epsilon y^2}\frac{\widehat{f}(y)}{\sqrt{-4\pi it}}e^{\frac{iA^2t^3}{12}+\frac{(x-y)^2}{4it}-\frac{iAt}{2}(x+y)}dy$$
$$=e^{-\frac{i\epsilon x^2}{1+4\epsilon t}+\frac{\frac{i}{3}A\epsilon t^2\left(At^2(-3+8\epsilon t(3+2\epsilon t))-6(1+4\epsilon t)(3+4\epsilon t)x\right)}{(1+4\epsilon t)^3}}$$
$$\times\, u\left(\frac{x-4A\epsilon t^3+4\epsilon xt}{(1+4\epsilon t)^2},\frac{t}{1+4\epsilon t}\right)=\sigma(\exp(\epsilon\mathbf{v}_6))u(x,t).$$

The final equivalence involves a combined $SL(2,\mathbb{R})$ and H_3 symmetry. To construct it, we evaluate $\mathcal{A}T_1\left(e_H,\begin{pmatrix}1 & 0\\ -\epsilon & 1\end{pmatrix}\right)f(x,t)$. We use the fact that

$$\int_{-\infty}^{\infty}\frac{1}{\sqrt{-4\pi it}}e^{\frac{iA^2t^3}{12}+\frac{(x-y)^2}{4it}-\frac{iAt}{2}(x+y)-iy\xi}dy=e^{it\xi^2+i(At^2-x)\xi+\frac{iA^2t^3}{3}-iAtx}.$$

From which

$$\mathcal{A}T_1\left(e_H,\begin{pmatrix}1 & 0\\ -\epsilon & 1\end{pmatrix}\right)f(x,t)=\int_{-\infty}^{\infty}\frac{\widehat{e^{-i\epsilon y^2}f(y)}}{\sqrt{-4\pi it}}e^{\frac{iA^2t^3}{12}+\frac{(x-y)^2}{4it}-\frac{iAt}{2}(x+y)}dy$$
$$=\int_{-\infty}^{\infty}e^{-i\epsilon y^2}f(y)e^{ity^2+i(At^2-x)y+\frac{iA^2t^3}{3}-iAtx}dy$$
$$=\int_{-\infty}^{\infty}\frac{\widehat{f}(y)}{\sqrt{-4\pi i(t-\epsilon)}}e^{\frac{iA^2t^3(t-\epsilon)}{12(t-\epsilon)}+\frac{(x-y)^2}{4i(t-\epsilon)}-\frac{iAt(t(x+y)-2x\epsilon)}{2(t-\epsilon)}}dy.$$

Again we express this in terms of the original solution $u(x,t)=\mathcal{A}f(x,t)$. After some algebra we find

$$\mathcal{A}T_1\left(e_H,\begin{pmatrix}1 & 0\\ -\epsilon & 1\end{pmatrix}\right)f(x,t)=e^{\frac{-iA\epsilon}{3}\left(3x+A\left(3t^2-6t\epsilon+2\epsilon^2\right)\right)}$$
$$\times\, u(x-2A\epsilon t+A\epsilon^2,t-\epsilon).$$

With $\mathbf{v}_2=\partial_t+2At\partial_x-iA(x+At^2)u\partial_u$ we therefore have

$$\sigma(\exp(\epsilon\mathbf{v}_2))\mathcal{A}f(x,t)=\mathcal{A}T\left(e_H,\begin{pmatrix}1 & 0\\ -\epsilon & 1\end{pmatrix}\right)f(x,t).$$

□

Remark 5.2. If we define $\mathbf{v}_2' = \partial_t$ then $\{\mathbf{v}_2', \mathbf{v}_4, \mathbf{v}_6\}$ form a basis for $\mathfrak{sl}_2$, but the basis we have used here is slightly more convenient.

6. The Harmonic Oscillator

We now turn to the equation for the harmonic oscillator,

$$iu_t = u_{xx} - A\|x\|^2 u, \;\; x \in \mathbb{R}^n, \;\; A > 0. \tag{6.1}$$

We prove the following.

Theorem 6.1. *Let $u(x,t) = \mathcal{A}f(x,t)$ be a solution of (6.1), with $f \in L^2(\mathbb{R}^n)$ and $\mathcal{A}f$ given by $\mathcal{A}f(x,t) = \lim \int_{\mathbb{R}^n} \widehat{f}(y)p(x,y,t)dy$ where*

$$p(x,y,t) = \frac{A^{\frac{n}{4}}}{\sqrt{(-2\pi i \sin(2\sqrt{A}t))^n}} e^{\frac{\sqrt{A}(\|x\|^2+\|y\|^2)}{2i\tan(2\sqrt{A}t)} - \frac{\sqrt{A}x\cdot y}{i\sin(2\sqrt{A}t)}}.$$

is Mehler's formula for the harmonic oscillator. Let $(T_1, L^2(\mathbb{R}^n))$ be given by Theorem 3.6 and $G = H_3 \rtimes (SL(2,\mathbb{R}) \times SO(n))$. The group G is a global group of symmetries of (5.1) and if σ is the Lie symmetry operator obtained by Lie's method, we have

$$(\sigma(g)\mathcal{A}f)(x,t) = (\mathcal{A}T_1(g)f)(x,t), \tag{6.2}$$

for all $g \in G$ and $f \in L^2(\mathbb{R}^n)$.

Proof. As in the case of the linear potential, we treat the $n = 1$ case first. The Lie symmetry algebra is spanned by

$$\begin{aligned}
\mathbf{v}_1 &= -(\cos(2\sqrt{A}t)\partial_x + ix\sqrt{A}\sin(2\sqrt{A}t)u\partial_u), \;\; \mathbf{v}_2 = \sin(2\sqrt{A}t) \\
&\quad - ix\sqrt{A}\cos(2\sqrt{A}t)u\partial_u, \mathbf{v}_3 = iu\partial_u, \mathbf{v}_4 = \partial_t, \mathbf{v}_5 = -2\sqrt{A}x\sin(4\sqrt{A}t)\partial_x \\
&\quad + \cos(4\sqrt{A}t)\partial_t + (2iAx^2\cos(4\sqrt{A}t) + \sqrt{A}\sin(4\sqrt{A}t))u\partial_u, \\
\mathbf{v}_6 &= 2\sqrt{A}x\cos(4\sqrt{A}t)\partial_x + \sin(4\sqrt{A}t)\partial_t + (2iAx^2\sin(4\sqrt{A}t) \\
&\quad - \sqrt{A}\cos(4\sqrt{A}t))u\partial_u.
\end{aligned}$$

Now

$$\begin{aligned}
&\mathcal{A}T_1((\epsilon,0,0),e_S)f(x,t) = \int_{-\infty}^{\infty} \widehat{f}(y)p(x,y-\epsilon,t)dy \\
&= e^{\frac{\sqrt{A}\epsilon^2}{2i\tan(2\sqrt{A}t)} + \frac{\sqrt{A}\epsilon x}{i\sin(2\sqrt{A}t)}} \int_{-\infty}^{\infty} \frac{A^{\frac{1}{4}}\widehat{f}(y)}{\sqrt{-2\pi i\sin(2\sqrt{A}t)}} e^{\frac{\sqrt{A}(x^2+y^2)}{2i\tan(2\sqrt{A}t)} - \frac{\sqrt{A}y(x+\epsilon\cos(2\sqrt{A}t))}{i\sin(2\sqrt{A}t)}} dy \\
&= e^{-i\frac{\sqrt{A}\epsilon^2}{2}\sin(2\sqrt{A}t)\cos(2\sqrt{A}t) - i\sqrt{A}\epsilon x\sin(2\sqrt{A}t)} \\
&\times \int_{-\infty}^{\infty} \frac{A^{\frac{1}{4}}\widehat{f}(y)}{\sqrt{-2\pi i\sin(2\sqrt{A}t)}} e^{\frac{\sqrt{A}((x+\epsilon\cos(2\sqrt{A}t)^2+y^2)}{2i\tan(2\sqrt{A}t)} - \frac{\sqrt{A}y(x+\epsilon\cos(2\sqrt{A}t))}{i\sin(2\sqrt{A}t)}} dy \\
&= e^{-i\frac{\sqrt{A}\epsilon^2}{2}\sin(2\sqrt{A}t)\cos(2\sqrt{A}t) - i\sqrt{A}\epsilon x\sin(2\sqrt{A}t)} u(x+\epsilon\cos(2\sqrt{A}t,t) \\
&= \sigma(\exp(\epsilon\mathbf{v}_1))\mathcal{A}f(x,t).
\end{aligned}$$

The $\mathbf{v}_3$ case is trivial. Similarly we have

$$\mathcal{A}T((0,\epsilon,0),e_S)f(x,t) = \int_{-\infty}^{\infty} e^{i\epsilon y}\widehat{f}(y)p(x,y,t)dy$$
$$= \int_{-\infty}^{\infty} \frac{A^{\frac{1}{4}}\widehat{f}(y)}{\sqrt{-2\pi i\sin(2\sqrt{A}t)}} e^{\frac{\sqrt{A}(x^2+y^2)}{2i\tan(2\sqrt{A}t)} - \frac{\sqrt{A}(x+\frac{\epsilon}{\sqrt{A}}\sin(2\sqrt{A}t))}{i\sin(2\sqrt{A}t)}} dy.$$

After simplification this is

$$\mathcal{A}T_1((0,e,0),e_S)f(x,t) = e^{i\epsilon x\cos(2\sqrt{A}t)+i\frac{\epsilon^2}{2\sqrt{A}}\cos(2\sqrt{A}t)\sin(2\sqrt{A}t)}$$
$$\times \int_{-\infty}^{\infty} \frac{A^{\frac{1}{4}}\widehat{f}(y)}{\sqrt{-2\pi i\sin(2\sqrt{A}t)}} e^{\frac{\sqrt{A}((x+\frac{\epsilon}{\sqrt{A}}\sin(2\sqrt{A}t))^2+y^2)}{2i\tan(2\sqrt{A}t)} - \frac{\sqrt{A}(x+\frac{\epsilon}{\sqrt{A}}\sin(2\sqrt{A}t))}{i\sin(2\sqrt{A}t)}} dy$$
$$= e^{i\epsilon x\cos(2\sqrt{A}t)+i\frac{\epsilon^2}{2\sqrt{A}}\cos(2\sqrt{A}t)\sin(2\sqrt{A}t)}\mathcal{A}f(x+\frac{\epsilon}{\sqrt{A}}\sin(2\sqrt{A}t)),t).$$

This establishes that $\mathcal{A}T_1((0,\epsilon,0),e_S)f(x,t) = \sigma(\exp(\frac{\epsilon}{\sqrt{A}}\mathbf{v}_2)\mathcal{A}f(x,t)$.

The $SL(2,\mathbb{R})$ symmetries are rather more complicated to deal with. The problem is to find the right basis to work with, but a useful trick allows us to do this. We observe that

$$\frac{d}{d\epsilon}\mathcal{A}T_1\left(e_H,\begin{pmatrix}1 & \epsilon\\ 0 & 1\end{pmatrix}\right)f(x,t)\bigg|_{\epsilon=0} = -i\int_{-\infty}^{\infty} y^2\widehat{f}(y)p(x,y,t)dy. \quad (6.3)$$

Now

$$p_x = i\left(-\frac{\sqrt{A}x}{\tan(2\sqrt{A}t)} + \frac{\sqrt{A}y}{\sin(2\sqrt{A}t)}\right)p,$$

and $p_t = \left(\frac{iA(x^2+y^2)}{\sin^2(2\sqrt{A}t)} - \frac{2iAxy\cos(2\sqrt{A}t)}{\sin^2(2\sqrt{A}t)} - \frac{\sqrt{A}\cos(2\sqrt{A}t)}{\sin(2\sqrt{A}t)}\right)p$. From this we obtain:

$$iy^2p = \frac{1}{A}\Bigg(\sqrt{A}x\sin(4\sqrt{A}t)p_x + \frac{1}{2}(1-\cos(4\sqrt{A}t))p_t$$
$$+ (iAx^2\cos(4\sqrt{A}t) + \frac{\sqrt{A}}{2}\sin(4\sqrt{A}t))p\Bigg). \quad (6.4)$$

The right hand side of (6.4) is easily seen to be the characteristic for the vector field $\frac{1}{2A}(\mathbf{v}_5-\mathbf{v}_4)$. So we let $\mathbf{v}_5' = \frac{1}{2A}(\mathbf{v}_5-\mathbf{v}_4)$. It is now a straightforward exercise to prove that

$$\mathcal{A}T_1\left(e_H,\begin{pmatrix}1 & \epsilon\\ 0 & 1\end{pmatrix}\right)f(x,t) = \sigma(\exp(\epsilon\mathbf{v}_5')\mathcal{A}f(x,t). \quad (6.5)$$

The details of this are similar to our previous calculations and are left to the reader.

Turning to the scaling symmetries we have:

$$\frac{d}{d\epsilon}\mathcal{A}T\left(e_H,\begin{pmatrix}e^{\epsilon} & 0\\ 0 & e^{-\epsilon}\end{pmatrix}\right)f(x,t)\bigg|_{\epsilon=0} = -\int_{-\infty}^{\infty}\widehat{f}(y)(\frac{1}{2}p+yp_y)dy. \quad (6.6)$$

We find that

$$-\frac{1}{2}p - yp_y = x\cos(4\sqrt{A}t)p_x + \frac{1}{2\sqrt{A}}\sin(4\sqrt{A}t)p_t$$
$$- (i\sqrt{A}x^2\sin(4\sqrt{A}t) - \cos(4\sqrt{A}t))p. \qquad (6.7)$$

This is the characteristic for $-\mathbf{v}_6$. One then easily shows that

$$\mathcal{A}T_1\left(e_H, \begin{pmatrix} e^{\epsilon} & 0 \\ 0 & e^{-\epsilon}\end{pmatrix}\right) f(x,t) = \sigma(\exp(-\epsilon\mathbf{v}_6))\mathcal{A}f(x,t). \qquad (6.8)$$

The final equivalence requires the Fourier transform of p in the y variable. Using the Fresnel integrals we obtain the Fourier transform

$$\int_{-\infty}^{\infty} e^{-iy\xi}p(x,y,t)dy = \frac{e^{\frac{i\left(Ax^2+\xi^2\right)\tan\left(2\sqrt{A}t\right)}{2\sqrt{A}} - ix\xi\sec\left(2\sqrt{A}t\right)}}{\sqrt{2\cos\left(2\sqrt{A}t\right)}}. \qquad (6.9)$$

Now we use the fact that

$$\mathcal{A}T_1\left(e_H, \begin{pmatrix} 1 & 0 \\ -\epsilon & 1\end{pmatrix}\right) f(x,t) = \int_{-\infty}^{\infty} e^{-i\epsilon\xi^2} f(\xi)\frac{e^{\frac{i\left(Ax^2+\xi^2\right)}{2\sqrt{A}\cot\left(2\sqrt{A}t\right)} - \frac{ix\xi}{\cos\left(2\sqrt{A}t\right)}}}{\sqrt{2\cos\left(2\sqrt{A}t\right)}}d\xi.$$

From which

$$\frac{d}{d\epsilon}\mathcal{A}T_1\left(e_H, \begin{pmatrix} 1 & 0 \\ -\epsilon & 1\end{pmatrix}\right) f(x,t)\Big|_{\epsilon=0} = -i\int_{-\infty}^{\infty} \xi^2 f(\xi)\frac{e^{\frac{i\left(Ax^2+\xi^2\right)}{2\sqrt{A}\cot\left(2\sqrt{A}t\right)} - \frac{ix\xi}{\cos\left(2\sqrt{A}t\right)}}}{\sqrt{2\cos\left(2\sqrt{A}t\right)}}d\xi.$$

We denote the Fourier transform in y of $p(x,y,t)$ by $\widehat{p}$. Then we find

$$-i\xi^2\widehat{p} = \sqrt{A}x\sin(4\sqrt{A}t)p_x - \frac{1}{2}(1+\cos(4\sqrt{A}t))p_t$$
$$+ (iAx^2\cos(4\sqrt{A}t) + \frac{\sqrt{A}}{2}\sin(4\sqrt{A}t))\widehat{p}. \qquad (6.10)$$

This is the characteristic for $V'' = \frac{1}{2A}(\mathbf{v}_5 + \mathbf{v}_4)$. We may then establish that

$$\mathcal{A}T_1\left(e_H, \begin{pmatrix} 1 & 0 \\ -\epsilon & 1\end{pmatrix}\right) f(x,t) = \exp(\exp(\epsilon\mathbf{v}''))\mathcal{A}f(x,t). \qquad (6.11)$$

Now $\{\frac{1}{2A}(\mathbf{v}_5 - \mathbf{v}_4), \frac{1}{2A}(\mathbf{v}_5 + \mathbf{v}_4), -\mathbf{v}_6\}$ is also a basis for $\mathfrak{sl}_2$. So we have established that the Lie symmetries of (6.1) are unitarily equivalent to the representation $(T_1, L^2(\mathbb{R}))$.

The calculations for the general n dimensional case are similar. The only difference is that the symmetry group now contains the rotation group $SO(n)$. But the calculations for the rotation group are easily handled by the methods we have been using. □

Remark 6.2. The PDE $iu_t = \Delta u + \sum_{i=1}^n A_i x_i^2 u$ has H_{2n+1} as a symmetry group, but does not have $SL(2,\mathbb{R})$ or $SO(n)$ symmetries unless $A_k = A$, for all k. In the case when only Heisenberg group symmetries exist, the obvious variation of Theorem 6.1 holds, with the intertwining operator being given by $\mathcal{A}f(x,t) = \int_{\mathbb{R}^n} \widehat{f}(y)K(x,y,t)dy$ and

$$K(x,y,t) = \prod_{k=1}^n \frac{A_k^{\frac{1}{4}}}{\sqrt{-2\pi i \sin(2\sqrt{A_k}t)}} e^{\frac{\sqrt{A_k}(x^2+y^2)}{2i\tan(2\sqrt{A_k}t)} - \frac{\sqrt{A_k}xy}{i\sin(2\sqrt{A_k}t)}}.$$

6.1. Unitary Symmetries of Diffusion Equations With Real Coefficients.

It was shown in [5] that the classical Lie point symmetries of the one dimensional heat equation are not unitary, but they can be realised as nonunitary representations on a space of distributions.

We can however use our general method for defining symmetries to construct *unitary symmetries* of the heat equation, if we allow the independent variables to be complex valued. Consider the projective unitary representation (3.9) of $G = H_3 \rtimes SL(2,\mathbb{R})$ defined in Section Three. We define $\mathcal{A}f(x,t) = \int_{-\infty}^{\infty} \widehat{f}(y) \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{(x-y)^2}{4t}\right) dy$. As before, for $(h,g) \in G$ we let

$$\sigma(h,g)\mathcal{A}f(x,t) = \int_{-\infty}^{\infty} (T_1(h,g)\widehat{f})(y) \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{(x-y)^2}{4t}\right) dy.$$

This leads to the following result.

Theorem 6.3. *Let u be a solution of the heat equation, given by $u(x,t) = \mathcal{A}f(x,t)$ for some $f \in L^2(\mathbb{R})$. Then*

$$\tilde{u}_\epsilon(x,t) = \sigma(\exp(\epsilon \mathbf{v}_k))u(x,t)$$

is also a solution, for all $\epsilon \in \mathbb{R}$ and $k = 1, ..., 6$. Here $\mathbf{v}_1, ..., \mathbf{v}_6$ are the infinitesimal symmetries

$$\mathbf{v}_1 = \partial_x,\ \mathbf{v}_2 = i\partial_t,\ \mathbf{v}_3 = iu\partial_u, \mathbf{v}_4 = x\partial_x + 2t\partial_t - \frac{1}{2}u\partial_u.$$
$$\mathbf{v}_5 = i(2t\partial_x - xu\partial_u),\ \mathbf{v}_6 = i(4xt\partial_x + 4t^2\partial_t - (x^2+2t)u\partial_u).$$

Moreover, for all $f \in L^2(\mathbb{R})$, $g, h \in g$ we have the equivalence

$$\sigma(g,h)\mathcal{A}f(x,t) = \mathcal{A}T(g,h)f(x,t). \tag{6.12}$$

Proof. Hölder's inequality establishes the convergence of the integral for $f \in L^2(\mathbb{R})$. The proof then proceeds along the same lines as in previous section. □

If we define a norm by $\|u\| = \|u(x,0)\|_2$, then these symmetries are unitary. So what this theorem says is that it is possible to analytically continue the standard Lie symmetries of the heat equation to symmetries which preserve solutions with L^2 initial data. Similar results can be obtained for all equations of the form $u_t = u_{xx} + Q(x)u$, with non trivial Lie symmetries.

However the standard Lie symmetries are not unitary. In order to analyse them we need to introduce a new type of representation and a new representation space. This was done in [5]. It was shown that the representation $\{T_\lambda, L^2(\mathbb{R})\}$ of $SL(2,\mathbb{R}) \rtimes H_3$, given by Definition 3.9, may be analytically continued in λ to a space of distributions, specifically $\mathcal{D}'(\mathbb{R})$ and that this representation is equivalent to the symmetries of the heat equation for $\lambda = i$. The n dimensional case was handled in [4]. Moreover this representation is topologically completely irreducible. We refer the reader to the references for a fuller discussion. Equivalent results for the PDEs

$$u_t = \Delta u + \sum_{k=1}^{n} a_k x_k u \tag{6.13}$$

$$u_t = \Delta u - B\|x\|^2 u. \tag{6.14}$$

may be quite easily established by the methods of [5]. We present an example. The operator in Theorem 6.4 is Mehler's formula; see [7].

Theorem 6.4. *Let $f \in L^2(\mathbb{R}^n)$ and set*

$$\begin{aligned} u(x,t) &= \int_{\mathbb{R}^n} \frac{B^{\frac{n}{4}} \widehat{f}(y)}{\sqrt{(2\pi \sinh(2\sqrt{B}t))^n}} e^{\frac{-\sqrt{A}(\|x\|^2+\|y\|^2)}{2\tanh(2\sqrt{B}t)} - \frac{\sqrt{B}x\cdot y}{\sinh(2\sqrt{B}t)}} dy \\ &= (Af)(x,t). \end{aligned} \tag{6.15}$$

Then u is a solution of $u_t = \Delta u - B\|x\|^2 u$. Moreover A can be extended to an operator $\mathcal{A}$ acting on $\mathcal{D}'(\mathbb{R}^n)$ which also defines (distributional) solutions of this PDE.

Theorem 6.5. *Let $G = H_{2n+1} \rtimes (SL(2,\mathbb{R}) \times SO(n))$. The representation T_λ of Theorem 3.6 may be analytically continued in λ to $\mathcal{D}'(\mathbb{R}^n)$. Moreover, G is a global group of symmetries of $u_t = \Delta u - B\|x\|^2 u$ for all B and the Lie symmetry operator σ satisfies*

$$\sigma(g)\mathcal{A}f(x,t) = (\mathcal{A}T_i(g)f)(x,t) \tag{6.16}$$

for all $f \in \mathcal{D}'(\mathbb{R}^n)$ and $g \in G$. The operator $\mathcal{A}$ is given by Theorem 6.4.

For brevity, we will give full technical details only for one equation in the nonunitary case. Letting $A \to 0$ will provide an alternative model for the $SL(2,\mathbb{R})$ symmetries of the heat equation.

7. Equations with an Inverse Quadratic Potential

We now wish to show that the Lie point symmetries of the equation

$$u_t = u_{xx} - \frac{A}{x^2} u, \ x > 0, \tag{7.1}$$

are equivalent to a representation of $G = SL(2,\mathbb{R}) \times \mathbb{R}$. A very different approach to making the Lie point symmetries of this equation global was given in [21]. The technique we use here follows in the spirit of [5] and the material of the previous sections. The Lie symmetries we require are as follows.

Proposition 7.1. *The Lie point symmetries of (7.1) are*

$$\sigma(\exp(\epsilon\mathbf{v}_1))u(x,t) = u(x,t-\epsilon)$$
$$\sigma(\exp(\epsilon\mathbf{v}_2))u(x,t) = e^{-\frac{1}{2}\epsilon}u(e^{-\epsilon}x, e^{-2\epsilon}t)$$
$$\sigma(\exp(\epsilon\mathbf{v}_3))u(x,t) = \frac{1}{\sqrt{1+4\epsilon t}}\exp\left(-\frac{\epsilon x^2}{1+4\epsilon t}\right)u\left(\frac{x}{1+4\epsilon t}, \frac{t}{1+4\epsilon t}\right)$$
$$\sigma(\exp(\epsilon\mathbf{v}_4))u(x,t) = e^{\epsilon}u(x,t).$$

Here $\mathbf{v}_1 = \partial_t, \mathbf{v}_2 = x\partial_x + 2t\partial_t - \frac{1}{2}u\partial_u$, $\mathbf{v}_3 = 4xt\partial_x + 4t^2\partial_t - (x^2+2t)u\partial_u$ *and* $\mathbf{v}_4 = u\partial_u$. *The vector fields* $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ *form a basis for* $\mathfrak{sl}_2$.

This Lie representation σ will be fixed for this section. The first step to establishing a global realisation of the Lie symmetries of (7.1) is to construct an appropriate space upon which our representation acts. The representation which is equivalent to the Lie symmetries, acts by scalings, multiplication by Gaussians and Hankel transform. Specifically, we want a vector space that is preserved by the actions given by Definition 4.2. The complication here is that we are forced to choose $\lambda = i$.

We consider functions of the form $f(x) = x^{\nu+\frac{1}{2}}e^{-\beta x^2}L_n^{(\nu)}(ax^2)$, where $Re(\beta) > 0$ and $L_n^{(\nu)} = \frac{1}{n!}e^x x^{-\nu}\frac{d^n}{dx^n}(e^{-x}x^{n+\nu})$ is the nth Laguerre polynomial. (See p990 of [15] for properties of the Laguerre polynomials). The Hankel transform of f is known, (see [10] formula 8.9.5, p43).

$$\mathfrak{H}_\nu(f) = \frac{\beta^{-n}(\beta-a)^n}{(2\beta)^{\nu+1}}x^{\nu+\frac{1}{2}}e^{-\frac{x^2}{4\beta}}L_n^{(\nu)}\left(\frac{ax^2}{4\beta(a-\beta)}\right). \tag{7.2}$$

Notice that if $f(x) = x^{\nu+\frac{1}{2}}e^{-\beta x^2}L_n^{(\nu)}(ax^2)$, then

$$R_i^\nu\left(\begin{pmatrix}1 & -\beta\\ 0 & 1\end{pmatrix}\right)f(x) = e^{\beta x^2}f(x) = x^{\nu+\frac{1}{2}}L_n^{(\nu)}(ax^2). \tag{7.3}$$

So if we build our representation space out of these functions, we will need the Hankel transform of $x^{\nu+\frac{1}{2}}L_n^{(\nu)}(ax^2)$. We could define these as distributions, and this will lead to one particular model for the symmetries. However an alternative approach is to require $Im(\beta) \neq 0$. If $Im(\beta) \neq 0$ and $\beta_2 > Re(\beta)$, then

$$R_i^\nu\left(\begin{pmatrix}1 & -\beta_2\\ 0 & 1\end{pmatrix}\right)f(x) = e^{\beta x^2}f(x) = x^{\nu+\frac{1}{2}}e^{\beta' x^2}L_n^{(\nu)}(ax^2), \tag{7.4}$$

and $Re(\beta') > 0$. We need to calculate the Hankel transform of (7.4).

We define Hankel transforms of functions of this type by treating them as a functional in $\mathcal{D}'(\mathbb{R}^+)$. This is based upon a method of defining the Fourier transform of distributions in $\mathcal{D}(\mathbb{R})$ that was introduced by Gelfand and Shilov, see [5] and [13]. For a function of type (7.4) we will define

$$\mathfrak{H}_\nu(f) = \int_0^{i\infty} f(x)\sqrt{xy}J_\nu(xy)dx. \tag{7.5}$$

Lemma 7.2. *Let $Re(\beta) > 0$. Then if $f(x) = x^{\nu+\frac{1}{2}}e^{\beta x^2}L_n^{(\nu)}(ax^2)$,*

$$\int_0^{i\infty} f(x)\sqrt{xy}J_\nu(xy)dx = (-1)^{\nu+1}\gamma y^{\nu+\frac{1}{2}}e^{\frac{y^2}{4\beta}}L_n^{(\nu)}\left(\frac{-ay^2}{4\beta(a+\beta)}\right), \tag{7.6}$$

where $\gamma = \frac{\beta^{-n}(\beta+a)^n}{(2\beta)^{\nu+1}}$.

Proof. We require two identities for Laguerre polynomials, namely

$$L_n^{(\nu)}\left(\frac{x}{1+s}\right) = \frac{1}{(1+s)^n}\sum_{k=0}^n s^{n-k}\binom{n+\nu}{n-k}L_k^{(\nu)}(x), \tag{7.7}$$

$$L_n^{(\nu)}(x) = \sum_{m=0}^n (-1)^m\binom{n+\nu}{n-m}\frac{x^m}{m!}, \tag{7.8}$$

see [20]. We also require the following identities for the confluent hypergeometric function: $\binom{k+\nu}{n}{}_1F_1(-k,\nu+1,z) = L_k^{(\nu)}(z)$ for k a positive integer and ${}_1F_1(a,b,z) = e^z{}_1F_1(b-a,b,-z)$, both of which can be found in [15]. The change of variables $x = i\xi$ leads to

$$\begin{aligned}\mathfrak{H}_\nu(f) &= (-1)^{\nu+1}\int_0^\infty \xi^{\nu+\frac{1}{2}}e^{-\beta\xi^2}L_n^{(\nu)}(-a\xi^2)\sqrt{\xi y}I_\nu(\xi y)d\xi \\ &= (-1)^{\nu+1}\int_0^\infty \xi^{\nu+\frac{1}{2}}e^{-\beta\xi^2}\sqrt{\xi y}\sum_{k=0}^n(-1)^k\binom{n+\nu}{n-k}\frac{(-a\xi^2)^k}{k!}I_\nu(\xi y)d\xi.\end{aligned}$$

The integral $\int_0^\infty \xi^r e^{-c\xi^2}I_\nu(\xi y)d\xi$ is in [15]. Let $s = \frac{a}{\beta}$. Then

$$\begin{aligned}\mathfrak{H}_\nu(f) &= \sum_{k=0}^n \frac{(-1)^{\nu+1}a^k y^{\nu+\frac{1}{2}}\Gamma(n+\nu+1)}{k!(n-k)!2^{\nu+1}\beta^{k+\nu+1}\Gamma(\nu+1)}{}_1F_1(k+\nu+1,\nu+1,\frac{y^2}{4\beta}) \\ &= \frac{(-1)^{\nu+1}y^{\nu+\frac{1}{2}}}{(2\beta)^{\nu+1}\Gamma(\nu+1)}e^{\frac{y^2}{4\beta}}\sum_{k=0}^n s^k\frac{\Gamma(n+\nu+1)}{k!(n-k)!}{}_1F_1(-k,\nu+1,-\frac{y^2}{4\beta}) \\ &= \frac{(-1)^{\nu+1}y^{\nu+\frac{1}{2}}e^{\frac{y^2}{4\beta}}}{(2\beta)^{\nu+1}\Gamma(\nu+1)}\sum_{k=0}^n s^k\frac{\Gamma(n+\nu+1)}{k!(n-k)!}\frac{\Gamma(\nu+1)\Gamma(k+1)}{\Gamma(k+\nu+1)}L_k^{(\nu)}\left(-\frac{y^2}{4\beta}\right),\end{aligned}$$

So that

$$\begin{aligned}\mathfrak{H}_\nu(f) &= \frac{(-1)^{\nu+1}y^{\nu+\frac{1}{2}}e^{\frac{y^2}{4\beta}}}{(2\beta)^{\nu+1}\Gamma(\nu+1)}\sum_{k=0}^{n} s^k \binom{n+\nu}{n-k} L_k^{(nu)}\left(-\frac{y^2}{4\beta}\right) \\ &= (-1)^{\nu+1}\frac{1}{(2\beta)^{\nu+1}}y^{\nu+\frac{1}{2}}e^{\frac{y^2}{4\beta}}\left(\frac{1+s}{s}\right)^n L_n^{(\nu)}\left(\frac{-y^2}{1+s}\right) \\ &= (-1)^{\nu+1}\frac{\beta^{-n}(a+\beta)^n}{(2\beta)^{\nu+1}}y^{\nu+\frac{1}{2}}e^{\frac{y^2}{4\beta}}L_n^{(\nu)}\left(\frac{-ay^2}{4\beta(a+\beta)}\right).\end{aligned}$$

□

The Hankel transform defined in this way is a continuous operator, on this space of functions, since letting $\beta \to -\beta$ transforms (7.6) into (7.2). We can therefore define the Hankel transform in the case that $Re(\beta) = 0$ by taking the limit $Re(\beta) \to 0$.

The preceding observations are used in the construction of the representation space.

Definition 7.3. Fix a nonnegative integer n. Then let

$$W^{\nu,n} = \left\{\sum_{k=1}^{N}\sum_{j=1}^{n} f_{jk}, N \in \mathbb{N}\right\}, \tag{7.9}$$

where $f_{jk}(x) = b_{jk}x^{\nu+\frac{1}{2}}e^{-\beta_{jk}x^2}L_j^{(\nu)}(a_{jk}x^2)$, $b_{jk,}, a_{jk} \in \mathbb{R}, Im(\beta_{jk}) \neq 0$. This splits as $W^{\nu,n} = W_1^{\nu,n} \oplus W_2^{\nu,n} \oplus W_3^{\nu,n}$, with

$$W_1^{\nu,n} = \{f \in W^{\nu,n}, Re(\beta_{jk}) > 0\}, \tag{7.10}$$

$$W_2^{\nu,n} = \{f \in W^{\nu,n}, Re(\beta_{jk}) < 0\}, \tag{7.11}$$

$$W_3^{\nu,n} = \{f \in W^{\nu,n}, Re(\beta_{jk}) = 0\}. \tag{7.12}$$

In the unitary case, it is easy to show that $R_\lambda(p)^2$ is the identity, since the Hankel transform is its own inverse. For the nonunitary case here, the action of the Weyl element on $W^{\nu,n}$ given above has to be slightly modified if we are to obtain an operator which is of order two.

Definition 7.4. Let $f \in W^{\nu,n}$. Then we set

$$(R_\lambda^\nu(p)f)(iy) = \lambda^{1/2}\tilde{f}(\lambda y), \tag{7.13}$$

in which $\tilde{f}$ is the Hankel transform of f.

The following result is elementary.

Proposition 7.5. *For $g \in SL(2,\mathbb{R})$ the action of $R_\lambda^\nu(g)$ defined by*

$$R_\lambda^\nu\left(\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\right)f(x) = e^{-\lambda ibx^2}f(x), \tag{7.14}$$

$$R_\lambda^\nu\left(\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix}\right)f(x) = |a|^{1/2}f(ax), \tag{7.15}$$

and $R_\lambda^\nu(p)$, given by definition 7.4, preserves $W^{\nu,n}$, for $\lambda \in \mathbb{C} - \{0\}$.

Now we specialise to the case when $\lambda = i$.

Lemma 7.6. *If* $f \in W^{\nu,n}$, *then* $R_i^\nu(p)^2 f = f$.

Proof. It is sufficient to prove the claim for $f(x) = x^{\nu+\frac{1}{2}} e^{-\beta x^2} L_n^\nu(ax^2)$. We do the case where $Re(\beta) > 0$. The case $Re(\beta) < 0$ is identical and the case $Re(\beta) = 0$ follows from taking left and right limits. We have

$$(R_i^\nu(p)f)(x) = c\frac{\beta^{-n}(\beta-a)^n}{(2\beta)^{\nu+1}}(ix)^{\nu+\frac{1}{2}} e^{\frac{x^2}{4\beta}} L_n^{(\nu)}\left(\frac{-ax^2}{4\beta(a-\beta)}\right), \quad (7.16)$$

$c = e^{\frac{i\pi}{4}}$. Applying $R_i^\nu(p)$ again we have

$$(R_i^\nu(p)^2 f)(x) = c^2(-1)^{\nu+1}(-1)^{\nu+\frac{1}{2}} f(x) = i(-1)^{\nu+1}(-1)^\nu i f(x) = f(x),$$

where we have used (7.6). The result extends to arbitrary $f \in W^{\nu,n}$ by linearity. □

It thus follows that $R_i^\nu(p)$ has order two and so $R_i^\nu(p)^{-1} = R_i^\nu(p)$. Now we introduce the intertwining operator.

Definition 7.7. Let $f = \sum_{j=1}^3 f_j \in W^{\nu,n}$, $f_j \in W_j^{\nu,n}$, $\nu = \frac{1}{2}\sqrt{1+4A}$ and set

$$(\mathcal{A}f)(x,t) = \sum_{j=1}^3 \int_\gamma (\tilde{f}_j)_\nu(y) \frac{\sqrt{xy}}{2t} e^{-\frac{x^2+y^2}{4t}} I_\nu\left(\frac{xy}{2t}\right) dx, \quad (7.17)$$

where the contour of integration is chosen as either $\gamma(t) = t, t \geq 0$ or $\gamma(t) = it, t \geq 0$ according as whether $(\tilde{f}_j)_\nu(y)e^{-\frac{x^2+y^2}{4t}}$ is in $W_1^{\nu,n} \oplus W_3^{\nu,n}$ or $W_2^{\nu,n}$. An equivalent expression is

$$(\mathcal{A}f)(x,t) = \sum_{j=1}^3 \int_\gamma f_j(y)\sqrt{xy}e^{-ty^2} J_\nu(xy)dy. \quad (7.18)$$

The equivalent form follows from the well known Hankel transform $\mathfrak{H}_\nu(x^{1/2}e^{-\beta x^2} J_\nu(ax))(y) = \frac{y^{1/2}}{2\beta} e^{-\frac{a^2+y^2}{4\beta}} I_\nu\left(\frac{ay}{2\beta}\right), \Re(\beta) > 0$, see formula 8.11.23 on page 51 of [10].

Proposition 7.8. *The function* $u(x,t) = (\mathcal{A}f)(x,t)$, *given in Definition 7.7, is a solution of the equation* $u_t = u_{xx} - \frac{A}{x^2}u$ *for all* $f \in W^{\nu,n}$.

The proof can be carried out by differentiation under the integral sign. We now have the first major result of this section.

Theorem 7.9. *Let* $f \in W^{\nu,n}$ *and let* σ *be the Lie representation of* $SL(2,\mathbb{R}) \times \mathbb{R}$ *given in Proposition 7.1. Let* $\bar{R}_\lambda^\nu = R_\lambda^\nu \otimes \xi$ *be the tensor product of the representation of* $SL(2,\mathbb{R})$ *of Proposition 7.5 and the representation of* $\mathbb{R}$ *given by* $(\xi_\lambda(\epsilon)f)(x) = e^{-i\lambda\epsilon}f(x)$. *Then* $G = SL(2,\mathbb{R}) \times \mathbb{R}$ *is a global group of symmetries and*

$$(\sigma(g)\mathcal{A}f)(x,t) = (\mathcal{A}\bar{R}_i^\nu(g)f)(x,t), \quad (7.19)$$

for all $g \in G$ *and* $f \in W^{\nu,n}$.

Proof. For convenience we will drop the ν subscript from the Hankel transform $\tilde{f}_\nu$, so that $\tilde{f}_\nu = \tilde{f}$ is to be understood. The case for $\mathbf{v}_4$ is trivial. For the remaining cases, we easily find that

$$
\begin{aligned}
(\sigma(\exp \epsilon \mathbf{v}_1)\mathcal{A}f)(x,t) &= \sum_{j=1}^{3}\int_\gamma f_j(y)\sqrt{xy}e^{-(t-\epsilon)y^2}J_\nu(xy)dy \\
&= \sum_{j=1}^{3}\int_\gamma e^{\epsilon y^2}f_j(y)e^{-ty^2}J_\nu(xy)dy \\
&= \left(\mathcal{A}\bar{R}_i^\nu\left(\begin{pmatrix}1 & \epsilon\\ 0 & 1\end{pmatrix}, e_{\mathbb{R}}\right)f\right)(x,t).
\end{aligned}
$$

Turning to the second vector field, we see that if $a = e^{-\epsilon}$,

$$
\begin{aligned}
(\sigma(\exp \epsilon \mathbf{v}_2)\mathcal{A}f)(x,t) &= e^{-\frac{1}{2}\epsilon}\sum_{j=1}^{3}\int_\gamma \tilde{f}_j(y)\frac{\sqrt{e^{-\epsilon}xy}}{2e^{-2\epsilon}t}e^{-\frac{e^{-2\epsilon}x^2+y^2}{4e^{-2\epsilon}t}}I_\nu\left(\frac{e^{-\epsilon}xy}{2e^{-2\epsilon}t}\right)dy \\
&= e^{\frac{1}{2}\epsilon}\sum_{j=1}^{3}\int_\gamma \tilde{f}_j(y)\frac{\sqrt{xe^{\epsilon}y}}{2t}e^{-\frac{x^2+e^{2\epsilon}y^2}{4t}}I_\nu\left(\frac{xe^{\epsilon}y}{2t}\right)dy \\
&= e^{-\frac{1}{2}\epsilon}\sum_{j=1}^{3}\int_\gamma \tilde{f}_j(e^{-\epsilon}y)\frac{\sqrt{xy}}{2t}e^{-\frac{x^2+y^2}{4t}}I_\nu\left(\frac{xy}{2t}\right)dy \\
&= \left(\mathcal{A}\bar{R}_i^\nu\left(\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix}, e_{\mathbb{R}}\right)f\right)(x,t).
\end{aligned}
$$

To complete the proof we consider the action for $\exp(\epsilon\mathbf{v}_3)$. We have $R_i^\nu\left(\begin{pmatrix}1 & 0\\ -\epsilon & 1\end{pmatrix}\right)f = R_i^\nu\left(p^3\begin{pmatrix}1 & \epsilon\\ 0 & 1\end{pmatrix}p\right)f$. But $p^3 = p^{-1}$, so that $R_i^\nu(p^3)f = R_i^\nu(p)f$. Consequently, for $f(x) = x^{\nu+\frac{1}{2}}e^{-\beta x^2}L_n^{(\nu)}(ax^2)$, we have

$$
\left(R_i^\nu\left(\begin{pmatrix}1 & 0\\ -\epsilon & 1\end{pmatrix}\right)f\right)(x) = \gamma c R_i^\nu(p)(ix)^{\nu+\frac{1}{2}}e^{(\epsilon+\frac{1}{4\beta})x^2}L_n^{(\nu)}(lx^2), \tag{7.20}
$$

$\gamma = \frac{\beta^{-n}(\beta-a)^n}{(2\beta)^{\nu+1}}$, $l = \frac{-a}{4\beta(a-\beta)}$. There are three cases, $Re(\epsilon+\frac{1}{4\beta}) > 0$, $Re(\epsilon+\frac{1}{4\beta}) = 0$ and $Re(\epsilon+\frac{1}{4\beta}) < 0$. Suppose that $Re(\epsilon+\frac{1}{4\beta}) > 0$. Then, with $m = (\epsilon+\frac{1}{4\beta})$

$$
\begin{aligned}
&\left(R_i^\nu\left(\begin{pmatrix}1 & 0\\ -\epsilon & 1\end{pmatrix}\right)f\right)(y) = \mu\int_0^{i\infty}x^{\nu+\frac{1}{2}}e^{mx^2}L_n^{(\nu)}(lx^2)\sqrt{ixy}J_\nu(ixy)dx \\
&= i\gamma(-1)^{\nu+\frac{1}{2}}(-1)^{\nu+1}\frac{m^{-n}(m+l)^n y^{\nu+\frac{1}{2}}}{(2m)^{\nu+1}}e^{-\frac{y^2}{4m}}L_n^{(\nu)}\left(\frac{lx^2}{4m(l+m)}\right) \\
&= \gamma\frac{m^{-n}(m+l)^n y^{\nu+\frac{1}{2}}}{(2m)^{\nu+1}}e^{-\frac{y^2}{4m}}L_n^{(\nu)}\left(\frac{lx^2}{4m(l+m)}\right),
\end{aligned}
$$

with $\mu = c^2\gamma i^{\nu+\frac{1}{2}}$. Now as the Hankel transform is it's own inverse, we find

$$\mathfrak{H}_\nu\left(\alpha y^{\nu+\frac{1}{2}}e^{-\frac{y^2}{4m}}L_n^{(\nu)}\left(\frac{lx^2}{4m(l+m)}\right)\right) = \gamma y^{\nu+\frac{1}{2}}e^{-(\epsilon+\frac{1}{4\beta})y^2}L_n^{(\nu)}(ly^2)$$
$$= e^{-\epsilon y^2}\tilde{f}(y),$$

where $\alpha = \gamma\frac{m^{-n}(m+l)^n}{(2m)^{\nu+1}}$. Suppose next that $Re(\epsilon + \frac{1}{4\beta}) < 0$, then

$$\left(R_i^\nu\left(\begin{pmatrix}1 & 0\\ -\epsilon & 1\end{pmatrix}\right)f\right)(y) = \mu\int_0^\infty x^{\nu+\frac{1}{2}}e^{-mx^2}L_n^{(\nu)}(lx^2)\sqrt{ixy}J_\nu(ixy)dx$$
$$= i\gamma(-1)^{\nu+\frac{1}{2}}\frac{m^{-n}(l-m)^n y^{\nu+\frac{1}{2}}}{(2m)^{\nu+1}}e^{\frac{y^2}{4m}}L_n^{(\nu)}\left(\frac{-lx^2}{4m(l-m)}\right)$$
$$= \gamma(-1)^{\nu+1}\frac{m^{-n}(l-m)^n y^{\nu+\frac{1}{2}}}{(2m)^{\nu+1}}e^{\frac{y^2}{4m}}L_n^{(\nu)}\left(\frac{-lx^2}{4m(l-m)}\right),$$

with $(\epsilon + \frac{1}{4\beta}) = -m$. This lies in $W_2^{\nu,n}$. So taking the Hankel transform once more we have

$$\mathfrak{H}_\nu((-1)^{\nu+1}\alpha y^{\nu+\frac{1}{2}}e^{\frac{y^2}{4m}}L_n^{(\nu)}\left(\frac{-lx^2}{4m(l-m)}\right)) = e^{-\epsilon y^2}\tilde{f}(y). \tag{7.21}$$

We obtain the same result when we take $Re(\epsilon + \frac{1}{4\beta}) = 0$ by taking limits as the the real part of $\epsilon + \frac{1}{4\beta} \to 0$. The calculations for $f \in W_2^{\nu,n}$ are essentially the same and so by linearity we have that for all $f \in W^{\nu,n}$

$$\mathfrak{H}_\nu(\bar{R}_i^\nu\left(\begin{pmatrix}1 & 0\\ -\epsilon & 1\end{pmatrix}\right)f)(y) = e^{-\epsilon y^2}\tilde{f}(y). \tag{7.22}$$

Thus

$$\mathcal{A}\left(\bar{R}_i^\nu\left(\begin{pmatrix}1 & 0\\ -\epsilon & 1\end{pmatrix}\right), e_{\mathbb{R}}\right)f)(y) = \sum_{j=1}^{3}\int_\gamma e^{-\epsilon y^2}\tilde{f}_j(y)p(x,y,t)dy$$
$$= \sigma(\exp\epsilon\mathbf{v}_3)\mathcal{A}f(x,t),$$

where $p(x,y,t) = \frac{\sqrt{xy}}{2t}e^{-\frac{x^2+y^2}{4t}}I_\nu\left(\frac{xy}{2t}\right)$. □

As in the case of the heat equation, it is possible to introduce larger representation spaces.

Theorem 7.10. *The representation $\{\bar{R}_i^\nu, W^{\nu,n}\}$ and the intertwining operator $\mathcal{A}$ of Theorem (7.9) can be extended to $\mathcal{D}'(\mathbb{R}^+)$ and for all $g \in SL(2,\mathbb{R})\times\mathbb{R}$ and all $f \in \mathcal{D}'(\mathbb{R}^+)$*

$$(\sigma(g)\mathcal{A}f)(x,t) = (\mathcal{A}\bar{R}_i^\nu(g)f)(x,t). \tag{7.23}$$

Moreover, $\{\bar{R}_i^\nu, \mathcal{D}'(\mathbb{R}^+)\}$ is topologically, completely irreducible.

Proof. We work in the weak* topology. Any distribution in $\mathcal{D}'(\mathbb{R}^+)$ can be approximated by the sum of a locally integrable function and a Dirac measure, (see Rudin [19] for weak* convergence of sequences of distributions), and these can be approximated by functions in $W^{\nu,n}$ by general properties of Laguerre polynomials, see [20]. Now R_i^ν is continuous in the weak* sense and so extends to distributions in $\mathcal{D}'(\mathbb{R}^+)$. The intertwining operator also extends to $\mathcal{D}'(\mathbb{R}^+)$ by the same argument, and the equivalence relation holds by uniqueness of limits. Specifically, $(\sigma(g)\mathcal{A}f_k)(x,t) = (\mathcal{A}\bar{R}_i^\nu(g)f_k)(x,t)$, for each $f_k \in W^{\nu,n}$ and both sides converge as $k \to \infty$.

To prove irreducibility, we suppose that X is a closed invariant subspace of $\mathcal{D}'(\mathbb{R}^+)$. Observe that $\mathcal{D}'(\mathbb{R}^+)$ is reflexive, see [23]. Let $Y = \mathcal{D}'(\mathbb{R}^+) \ominus X$. Then let X^0 and Y^0 be the annihilators of X and Y respectively. We know that $\mathcal{D}(\mathbb{R}^+) = X^0 \oplus Y^0$. Pick $\phi \in \mathcal{D}(\mathbb{R}^+)$ such that $\mu(\phi) = 0$ for all $\mu \in X$. Any μ can be approximated by $f \in W^{\nu,n}$ for some n and by the invariance of X we then have $R_i(g)f \in X$ for all X, so

$$\int_0^\infty e^{-bx^2}\tilde{f}(ax)\phi(x)dx = 0 \tag{7.24}$$

for all b, a. But letting $x^2 = y$ gives $\mathcal{L}(h) = 0$, $h(x) = \frac{1}{2\sqrt{y}}\tilde{f}(ax)\phi(x)$ and $\mathcal{L}$ denotes the Laplace transform. But since f is nonzero, by elementary properties of Laplace transform, we must have $\phi = 0$. So we conclude that $X^0 = \{0\}$ and hence $X = \mathcal{D}'(\mathbb{R}^+)$. □

We finish this section with a corollary of our work so far.

Theorem 7.11. *Suppose that $u_t = A(x,t)u_{xx} + B(x,t)u_x + C(x,t)u$ has a four dimensional Lie algebra of infinitesimal symmetries. Then the symmetries are equivalent to the representation $\{\bar{R}_i^\nu, \mathcal{D}'(\mathbb{R}^+)\}$ of $G = SL(2,\mathbb{R})\times\mathbb{R}$ and the this is also a global group of symmetries of the PDE. If the PDE has a six dimensional Lie algebra of symmetries, then $G = H_3 \rtimes SL(2,\mathbb{R})$ is a global group of symmetries and the symmetries are equivalent to $\{T_i, \mathcal{D}'(\mathbb{R})\}$.*

It is possible to establish a nonunitary analogue of Theorem 4.16, but we will leave this to the interested reader. The equation $u_t = \Delta u - (\frac{A}{\|x\|^2} + B\|x\|^2)u$ will be treated in detail elsewhere.

8. Applications

8.1. Generalised Integral Transforms of Fundamental Solutions.

In [7], it was shown that for certain classes of parabolic PDE on the line, it is possible to obtain an integral transform of a fundamental solution by the application of a symmetry to a stationary solution. To connect this to the representation theory of the symmetry group, consider the initial value problem

$$u_t = A(x,t)u_{xx} + B(x,t)u_x + C(x,t)u, \ x \in \Omega, \tag{8.1}$$
$$u(x,0) = h(x).$$

Suppose that (8.1) has a six dimensional Lie algebra of infinitesimal symmetries. We know that it may be reduced to the PDE

$$v_t = v_{yy} + Q(y,t)v, \tag{8.2}$$

where $Q(y,t) = q_1(t) + q_2(t)y + q_3(t)y^2$, by a change of variables of the form

$$v(y,t) = u(\phi^{-1}(y,t))e^{F(y,t)}, \tag{8.3}$$

with $y = \phi(x,t) = \int_{x_0}^{x} \frac{dk}{\sqrt{A(k,t)}}$. (See [3] for the change of variables). The given initial value problem for (8.1) is transformed to

$$v_t = v_{yy} + Q(y,t)v, \ y \in \mathbb{R} \tag{8.4}$$
$$v(y,0) = h(\phi^{-1}(y,0))e^{F(y,0)}.$$

Note that by construction, the change of variables maps the domain Ω to the line.

Suppose that $K(y,\xi,t)$ is a fundamental solution of (8.2), then a solution of the given initial value problem for (8.4) is

$$\begin{aligned} v(y,t) &= \int_{-\infty}^{\infty} h(\phi^{-1}(\xi,0))e^{F(\xi,0)}K(y,\xi,t)d\xi \\ &= \int_{\Omega} h(z)e^{F(\phi(z,0))}K(y,\phi(z,0),t)\phi_z(z,0),t)dz. \end{aligned} \tag{8.5}$$

Here $\phi_z(z,0) = \frac{d}{dz}\phi(z,0)$.

It follows that the original problem has solution

$$\begin{aligned} u(x,t) &= \int_{\Omega} h(z)e^{F(\phi(z,0))-F(\phi(x,t))}K(\phi(x,t),\phi(z,0),t)\phi_z(z,0),t)dz \\ &= \int_{\Omega} h(z)p(x,z,t)dz, \end{aligned} \tag{8.6}$$

and $p(x,z,t)$ is a fundamental solution of (8.4).

Now we know that there is an infinitesimal symmetry $\mathbf{v}$ of (8.4) such that

$$\rho(\exp(i\epsilon\mathbf{v})v(y,t) = \int_{-\infty}^{\infty} e^{-i\epsilon\xi}h(\phi^{-1}(\xi,0))e^{F(\xi,0)}K(y,\xi,t)d\xi. \tag{8.7}$$

It therefore follows that there is an infinitesimal symmetry $\bar{\mathbf{v}}$ of (8.1) such that

$$\rho(\exp(i\epsilon\bar{\mathbf{v}}))u(x,t) = \int_{\Omega} e^{-i\epsilon\phi(z,0)}h(z)p(x,z,t)dz. \tag{8.8}$$

The right hand side of (8.8) is an integral transform of $h(z)p(x,z,t)$ which can be inverted by converting it to a Fourier transform, by setting

$\eta = \phi(z, 0)$. For this reason we term it a *generalised Fourier transform.* We have thus proved the following generalisation of the results of [7].

Theorem 8.1. *Let*

$$u_t = A(x,t)u_{xx} + B(x,t)u_x + C(x,t)u, \ x \in \Omega, \tag{8.9}$$

have a six dimensional Lie algebra of symmetries and suppose that $u(x,t) = \int_\Omega u_0(z)p(x,z,t)dz$ *is a nonzero solution of (8.9). Then there is a Lie Symmetry which maps solutions* $u(x,t)$ *to a generalised Fourier transform of a product of* u_0 *and a fundamental solution* $p(x,z,t)$.

A similar argument proves the following.

Theorem 8.2. *Let*

$$u_t = A(x,t)u_{xx} + B(x,t)u_x + C(x,t)u, \ x \in \Omega, \tag{8.10}$$

have a four dimensional Lie algebra of symmetries and suppose that $u(x,t) = \int_\Omega u_0(z)p(x,z,t)dz$ *is a nonzero solution of (8.10). Then there is a Lie symmetry which maps solutions* $u(x,t)$ *to a generalised Laplace transform of a product of* u_0 *and a fundamental solution* $p(x,z,t)$.

By a generalised Laplace transform we mean an integral transform of the form $F(s) = \int_\Omega e^{-s\phi(z,0)} f(z)dz$, where the change of variables $\eta = \phi(z, 0)$ converts the integral to a Laplace transform.

Thus the methods developed in [7] extend to any linear, second order parabolic PDE on the line, which has at least a four dimensional Lie algebra of symmetries. This apparently remarkable fact is nothing more than a consequence of the representation theory of the group.

8.2. **Heisenberg Group Fourier Transforms.** We restrict attention to the $n = 1$ case, but we may perform the same analysis here for any PDE which possesses the Heisenberg group acting unitarily.

Definition 8.3. Let $f \in \mathcal{S}(H_3)$, where $\mathcal{S}(H_3)$ is the Schwartz space of smooth, rapidly decreasing functions, which we identify with $\mathcal{S}(\mathbb{R}^3)$. The Fourier transform on the Heisenberg group is the operator

$$(\pi_\lambda(f)\phi)(\xi) = \int_{H_3} f(a,b,c)\pi_\lambda(a,b,c)\phi(\xi)dadbdc, \tag{8.11}$$

for $\phi \in L^2(\mathbb{R})$.

The operator $\pi_\lambda(f)$ is a Hilbert-Schmidt operator of trace class. The following Fourier inversion theorem may be found in the paper by Fabec, [11].

Theorem 8.4. *Let* $f \in \mathcal{S}(H_3)$. *Then*

(1)

$$f(e) = \frac{1}{2\pi^2}\int_{-\infty}^{\infty} \mathrm{tr}(\pi_\lambda(f))|\lambda|d\lambda. \tag{8.12}$$

(2)

$$\|f\|_2^2 = \frac{1}{2\pi^2}\int_{-\infty}^{\infty} \|\pi_\lambda(f)\|_2^2 |\lambda| d\lambda. \tag{8.13}$$

(3)

$$f(h) = \frac{1}{2\pi^2}\int_{-\infty}^{\infty} \mathrm{tr}(\pi_\lambda(h^{-1})\pi_\lambda(f))|\lambda| d\lambda. \tag{8.14}$$

Here $\|T\| = \mathrm{tr}(T^*T)^{1/2}$ *is the Hilbert-Schmidt norm.*

One easily shows the following.

Theorem 8.5. *The Fourier transform on the Heisenberg group extends to* $L^1(H_1)$.

Now, for any $\phi \in L^2(\mathbb{R})$, $u(x,t) = \mathcal{A}\phi(x,t) = \int_{-\infty}^{\infty}\phi(y)K(x-y,y)dy$, where $K(x,t) = \frac{1}{\sqrt{-4\pi i t}}e^{x^2/4it}$, is a solution of $iu_t = u_{xx}$ with initial data $u(x,0) = \phi(x)$. We define σ_λ by the rule

$$\sigma_\lambda(h)\mathcal{A}\phi = \mathcal{A}\pi_\lambda(h)\phi \tag{8.15}$$

for all $\phi \in H_3$. Then for all $h = (a,b,c) \in H_3$ we have:

$$\sigma_\lambda(a,0,0)u(x,t) = u(x-\lambda a,t) \tag{8.16}$$

$$\sigma_\lambda(0,b,0)u(x,t) = e^{-ibx+ib^2t}u(x-2bt,t) \tag{8.17}$$

$$\sigma_\lambda(0,0,c)u(x,t) = e^{i\lambda c}u(x,t). \tag{8.18}$$

Using the decomposition $(a,b,c) = (a,0,0)(0,b,0)(0,0,c-\frac{1}{2}ab)$ we easily obtain

$$\sigma_\lambda(a,b,c)u(x,t) = e^{-ibx+ib^2t+i\lambda(c+\frac{1}{2}ab)}u(x-\lambda a-2bt,t). \tag{8.19}$$

Thus we have a family of unitary representations of H_3 unitarily equivalent to $(\pi_\lambda, L^2(\mathbb{R}))$, which act on solutions of the zero potential Schrödinger equation. The Hilbert space $\mathcal{H}$ on which σ_λ acts is the image Hilbert space of $L^2(\mathbb{R})$ under the unitary mapping $\mathcal{A}$. We can therefore use σ_λ to define a Fourier transform. This will produce operators which map solutions to solutions. In essence, we can realise $L^1(H_3)$ as a Banach algebra of symmetries of the zero potential Schrödinger equation. The next result makes this statement more precise.

Corollary 8.6. *For* $f \in L^1(H_3)$ *we have the Heisenberg group Fourier transform*

$$\sigma_\lambda(f)u(x,t) = \int_{H_3} f(a,b,c)\sigma_\lambda(a,b,c)u(x,t)dadbdc \tag{8.20}$$

which acts on solutions of $iu_t = u_{xx}$. *The following properties hold. Let* $f \in L^1(H_3)$. *Then*

(1)

$$f(e) = \frac{1}{2\pi^2}\int_{-\infty}^{\infty} \mathrm{tr}(\sigma_\lambda(f))|\lambda| d\lambda. \tag{8.21}$$

(2)
$$\|f\|_2^2 = \frac{1}{2\pi^2}\int_{-\infty}^{\infty} \|\sigma_\lambda(f)\|_2^2 |\lambda| d\lambda. \tag{8.22}$$

(3)
$$f(h) = \frac{1}{2\pi^2}\int_{-\infty}^{\infty} \mathrm{tr}(\sigma_\lambda(h^{-1})\sigma_\lambda(f))|\lambda| d\lambda. \tag{8.23}$$

Here $\|T\| = \mathrm{tr}(T^*T)^{1/2}$ *is the Hilbert-Schmidt norm.*

The result for $\{\sigma_\lambda, \mathcal{H}\}$ follows from the corresponding result for $\{\pi_\lambda, L^2(\mathbb{R})\}$ by unitary equivalence. This Fourier transform maps solutions to solutions, and its eigenfunctions form an orthonormal basis for $\mathcal{H}$.

The following lemma is an easy calculation.

Lemma 8.7. *For each* $f \in L^1(H_3)$*,* $\sigma_\lambda(f)$ *is a mapping from* $\mathcal{H}$ *into itself. Thus* $L^1(H_3)$ *is a Banach algebra of symmetries of* $iu_t = u_{xx}$*. Further we may write*

$$\sigma_\lambda(f)u(x,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \widehat{f}^3(a,b,c-\frac{1}{2}ab)e^{-ibx+ib^2t}u(x-\lambda a-2bt,t)dadb.$$

Here $\widehat{f}^3$ *denotes the classical Fourier transform of* f *in the third variable. We take* $\widehat{f}(y) = \int_{-\infty}^{\infty} f(x)e^{iyx}dx$*. If* $u = A\phi$ *then we also have*

$$\sigma_\lambda(f)u(x,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \widehat{f}^{1,3}(\lambda\frac{(\xi+b)}{2}, \xi-b, \lambda)\phi(\xi)e^{i(\xi+b)^2t-i(\xi+b)x}\frac{d\xi}{2\pi}db.$$

Here $\widehat{f}^{1,3}$ *denotes the classical Fourier transform in the first and third variables.*

Example 8.1. Let us take $f(a,b,c) = e^{-\frac{1}{2}(a^2+b^2+c^2)}$ and the solution $u = 1$. The integration is straightforward and we have

$$\begin{aligned}\sigma_\lambda(f)u(x,t) &= \int_{H_3} e^{-\frac{1}{2}(a^2+b^2+c^2)}e^{-ibx+ib^2+i\lambda(c+\frac{1}{2}ab)}dadbdc \\ &= \frac{(2\pi)^{3/2}e^{-\frac{1}{2}\lambda^2}}{\sqrt{1+\frac{\lambda^2}{4}-2it}}\exp\left\{\frac{-x^2}{2(1+\frac{\lambda^2}{4}-2it)}\right\}. \end{aligned}\tag{8.24}$$

From the solution $u = 1$ we have obtained a family of solutions of the Schrôdinger equation, indexed by a parameter λ. We have thus a new notion of symmetry, which may potentially be fruitful in the analysis of solutions of PDEs. It is possible to construct Fourier transforms on solution spaces of other PDEs. The proof follows immediately from our previous considerations.

Theorem 8.8. *The PDEs* $iu_t = \Delta u - \sum_{k=1}^n a_k x_k^2 u$ *and* $iu_t = \Delta u + \sum_{k=1}^n a_k x_k u$*,* $a_k \in \mathbb{R}$*, each possess* $L^1(H_{2n+1})$ *as a Banach algebra of symmetries.*

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