Quantum Information and Computation, Vol. 10, No. 11&12 (2010) 0925–0935

WHEN IS THERE A MULTIPARTITE MAXIMUM ENTANGLED STATE?[a]

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Received January 19, 2010
Revised August 4, 2010

For a multipartite quantum system of the dimension $d_1 \otimes d_2 \otimes \cdots \otimes d_n$, where $d_1 \geq d_2 \geq \cdots \geq d_n \geq 2$, is there an entangled state *maximum* in the sense that all other states in the system can be obtained from the state through local quantum operations and classical communications (LOCC)? When $d_1 \geq \Pi_{i=2}^{n} d_i$, such state exists. We show that this condition is also necessary. Our proof, somewhat surprisingly, uses results from algebraic complexity theory.

Keywords: quantum information theory, quantum communication protocol,tensor rank, maximum entangled state, stochastic entanglement transformation
Communicated by: D Wineland & B Terhal

1 Background and the statement of the main result

A quantum system consisting of several subsystems may be in an *entangled* state, such that measurements on the subsystems may produce outcome statistics fundamentally different from those produced through a classical process. Since its discovery by Einstein, Podolsky, and Rosen [1], quantum entanglement has been found to be central for non-classical properties of quantum systems. In particular, it plays a fundamental role in quantum information processing applications such as unconditional secure key distribution and super fast quantum algorithms. It is therefore of fundamental importance to understand the nature of entan-

[a]This work was supported in part by the National Science Foundation of the United States under Awards 0347078, 0622033, and 1017335.
[b]Email: runyao.duan@uts.edu.au. Supported in part by the QCIS of UTS, the National Natural Science Foundation of China (Grant Nos. 60702080 and 60621062), the FANEDD under Grant No. 200755, and the Hi-Tech Research and Development Program of China (863 project) (Grant No. 2006AA01Z102). Part of this work was done while the author was visiting the University of Michigan, Ann Arbor.
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glement. Indeed, the past two decades have witnessed the rapid development of a theory of quantum information, at the heart of which is the theory of quantum entanglement. Horodecki *et al.* [2] and Gühne and Tóth [3] are recent surveys on the subject.

Our study is motivated by the following objective, which is important for the practical applications of quantum entanglement: how do we establish quantum entanglement between multiple parties separated spatially? One straightforward solution is for one party to prepare the desired state $|\phi\rangle$, and send the others their corresponding portion of the state. The problem of this solution is that moving quantum objects around without corrupting them is difficult and expensive, especially when the parties are remotely separated.

The celebrated *quantum teleportation protocol* [4] provides an alternative approach: the parties initially share some special but fixed entangled state $|\phi_0\rangle$, which will then be transformed to $|\phi\rangle$ through local quantum operations and classical communications (LOCC). Ideally, $|\phi_0\rangle$ should work for all possible $|\phi\rangle$ desired. The question we address is: for which dimensions of the system is there such an initial state that can generate all other states in the system?

Let $n, d_1, d_2, \cdots, d_n$ be integers with $n \geq 2$, and $d_1 \geq d_2 \geq \cdots \geq d_n \geq 2$. We denote by $d_1 \otimes d_2 \otimes \cdots d_n$ the tensor product of n Hilbert spaces, each of the dimension d_1, d_2, ..., d_n, respectively. We refer to the whole system and its associated Hilbert space by $\mathcal{H}$, and each subsystem by A, B, C, ..., Z, respectively. We use superscripts $A, B, C, \cdots, Z$ on states or operators to indicate the space they are associated with. Let $|\phi_1\rangle$ and $|\phi_2\rangle$ be two states in $d_1 \otimes \cdots \otimes d_n$. We write $|\phi_2\rangle \leq_{\text{LOCC}} |\phi_1\rangle$ if $|\phi_1\rangle$ can be transformed to $|\phi_2\rangle$ through a LOCC protocol. A state $|\phi_0\rangle$ is said to be a *maximum entangled state* (MES) if $|\phi\rangle \leq_{\text{LOCC}} |\phi_0\rangle$ for all $|\phi\rangle$ in the same space. Thus our problem is, which space $d_1 \otimes \cdots \otimes d_n$ contains a maximum entangled state?

Besides the practical motivation described above, our question is also among the most basic questions in the framework of entanglement manipulations, which is to study properties of entanglement under LOCC transformations. This is a major paradigm for studying entanglement where many central results were obtained. A particular task in this paradigm is to classify entangled states through their conversion relations. Note that classical communication should not increase any reasonable notion quantifying entanglement — indeed, this monotonicity under LOCC transformation is considered the only natural requirement for entanglement measures [5]. Therefore, the relation $\leq_{\text{LOCC}}$ induces a natural partial ordering of quantum states (or more precisely, of the LOCC equivalence classes) by the amount of entanglement. Our problem, which is to ask when a maximum element exists, is thus among the very basic questions regarding the structure of this ordering. We stress that the definition of "maximum" in this paper is restricted to the LOCC ordering. There may be other definitions of maximum entangled states with respect to other orderings.

When $n = 2$, the answer to our question is well known through the use of the teleportation protocol with the generalized EPR state (commonly referred to as the maximum entangled state for bipartite systems)

$$|\Phi_{d_2}\rangle^{AB} \stackrel{\text{def}}{=} \sum_{i=0}^{d_2-1} |i\rangle^A |i\rangle^B, \tag{1}$$

where $\{|i\rangle^A : i = 0, \cdots, d_2 - 1\}$ and $\{|i\rangle^B : i = 0, \cdots, d_2 - 1\}$ are orthonormal in A and B, respectively. The teleportation protocol can be generalized to arbitrary n, as long as

$$d_1 \geq \Pi_{i=2}^{n} d_i. \tag{2}$$

On the other hand, not all spaces have a maximum entangled state. For example, Dür, Vidal and Cirac showed that there is no MES in the $2 \otimes 2 \otimes 2$ space [6]. The main result of this paper is that a MES exists only if Eqn. (2) holds.

Our result is actually slightly stronger. Following the notation of Bennett *et al.* [7], if $|\phi_1\rangle$ can be transformed to $|\phi_2\rangle$ with a non-zero probability, we write $|\phi_2\rangle \leq_{\text{SLOCC}} |\phi_1\rangle$, where "SLOCC" stands for *Stochastic* Local Operations and Classical Communications. Similarly, $|\phi_1\rangle$ is called a *stochastic maximum entangled state* (SMES) if $|\phi_2\rangle \leq_{\text{SLOCC}} |\phi_1\rangle$ for all $|\phi_2\rangle$ in the same space. The partial ordering $\leq_{\text{SLOCC}}$ was introduced by Bennett et al. [7] in order to provide a simpler classification of multipartite entanglement (there are infinitely number of LOCC equivalence classes even for 2 qubits), and has been subsequently studied by many authors. Clearly a MES is also a SMES; thus if Eqn. (2) holds then a SMES exists. We now state our main theorem.

Theorem 1 *If $d_1 < \Pi_{i=2}^{n} d_i$, there is no stochastic maximum entangled state in the state space $d_1 \otimes d_2 \otimes \cdots \otimes d_n$, where $d_1 \geq d_2 \geq \cdots \geq d_n \geq 2$.*

Our proof uses the notion of tensor rank from algebraic complexity theory (C.f. Chapter 14 in [8]). The tensor rank of $|\phi\rangle \in \mathcal{H}$, $\text{Sch}(\phi)$, is the minimum number of product vectors that can linearly express $|\phi\rangle$. That is, $\text{Sch}(\phi)$ is the minimum integer ℓ such that there exists product vectors $|\phi_i\rangle^A \otimes |\phi_i\rangle^B \otimes \cdots |\phi_i\rangle^Z \in d_1 \otimes d_2 \otimes \cdots \otimes d_n$ such that

$$|\phi\rangle \stackrel{\text{def}}{=} \sum_{i=1}^{\ell} |\phi_i\rangle^A \otimes |\phi_i\rangle^B \otimes \cdots |\phi_i\rangle^Z. \tag{3}$$

The quantity $\text{Sch}(\phi)$ is also called *Schmidt rank* or *Schmidt number* [9], and when $n = 2$, is precisely the rank of the reduced density matrix $Tr_A(|\phi\rangle\langle\phi|)$. In general, the tensor rank is the minimum number of multiplications to compute a set of linear forms determined by $|\phi\rangle$. For example, the minimum number of non-scalar multiplications for multiplying two n by n matrices is precisely the tensor rank of the following element in $n^2 \otimes n^2 \otimes n^2$:

$$\sum_{i,j,k=0}^{n-1} |i,j\rangle|i,k\rangle|k,j\rangle,$$

where each component space has a product orthonormal basis $\{|i,j\rangle : i,j = 0, \cdots, n-1\}$. It was observed in [10] that the above state is precisely the tripartite state $|\Psi_n\rangle^{ABC} = |\Phi_n\rangle^{AB} \otimes |\Phi_n\rangle^{BC} \otimes |\Phi_n\rangle^{CA}$. This connection enables us and a co-author to show the equivalence between the computational complexity of matrix multiplication and efficiency of a certain entanglement transformation that produces EPR pairs [10].

The tensor rank of a Hilbert space $\mathcal{H}$ is

$$\text{Sch}(\mathcal{H}) \stackrel{\text{def}}{=} \max\{\text{Sch}(\phi) : |\phi\rangle \in \mathcal{H}\}.$$

Many works have been done to determine the tensor rank of specific tensors and of various spaces. We will use the following results.

Theorem 2 *Consider* $\mathrm{Sch}(\mathcal{H})$ *for* $\mathcal{H} = d_1 \otimes d_2 \otimes d_3$*. Let* $k = d_2 d_3 - d_1$*.*

(i) (Theorem 6(ii) of [11]) If $k \geq 1$*, then* $\mathrm{Sch}(\mathcal{H}) \geq d_1 + \lfloor\sqrt{2k+2}\rfloor - 2$*.*

(ii) (Theorem 3 of [12]) If $k \leq \max\{d_2, d_3\}$ *and* $0 \leq k \leq 4$*, then* $\mathrm{Sch}(\mathcal{H}) = d_2 d_3 - \lceil \frac{k}{2} \rceil$*.*

2 Proof of the Main Theorem

We now turn to the proof of the main result. We only need to focus on the following case

$$d_1 < \Pi_{i=2}^n d_i. \tag{4}$$

To simplify our discussions, we shall first obtain some structural results about SLOCC and the induced ordering on the states under this special condition. We say that $|\phi_1\rangle$ and $|\phi_2\rangle$ are SLOCC equivalent if $|\phi_1\rangle \leq_{\mathrm{SLOCC}} |\phi_2\rangle$ and $|\phi_2\rangle \leq_{\mathrm{SLOCC}} |\phi_1\rangle$. Then $\leq_{\mathrm{SLOCC}}$ defines a partial oder on SLOCC equivalence classes. We will often identify a state with its equivalence class. A state $|\phi\rangle$ is said to be SLOCC *maximal* if for any $|\psi\rangle$, $|\phi\rangle \leq_{\mathrm{SLOCC}} |\psi\rangle$ implies $|\psi\rangle \leq_{\mathrm{SLOCC}} |\phi\rangle$. For the rest of the paper, we may omit "SLOCC" when referring to equivalence, equivalence classes, maximal state, etc. We know the following fact about SLOCC [6].

Lemma 1 *Let* $|\phi\rangle$ *and* $|\psi\rangle \in d_1 \otimes d_2 \otimes \cdots \otimes d_n$*. Then* $|\psi\rangle \leq_{SLOCC} |\phi\rangle$ *if and only if there are linear operators* $L_1, \cdots, L_n$ *such that* $(L_1 \otimes \cdots \otimes L_n)|\phi\rangle = |\psi\rangle$*. In particular,* $|\phi\rangle$ *and* $|\psi\rangle$ *are equivalent under SLOCC if and only if* $L_1, \cdots, L_n$ *can be invertible.*

Since local linear operators cannot increase tensor rank, we have the following fact that relates tensor rank and SLOCC [13].

Proposition 3 *If* $|\psi\rangle \leq_{SLOCC} |\phi\rangle$*,* $\mathrm{Sch}(\phi) \geq \mathrm{Sch}(\psi)$*.*

For distinct indices $i_1, i_2, ..., i_\ell \in \{1, 2, \cdots, n\}$, denote by $\rho_\Phi^{i_1, i_2, ..., i_\ell}$ the reduced density operator of $|\Phi\rangle\langle\Phi|$ with sub-systems not in $\{i_1, ..., i_\ell\}$ traced-out. We say that $|\Phi\rangle \in \mathcal{H}$ is of *full local ranks* if $\mathrm{rank}(\rho_\Phi^i) = \min\{d_i, \Pi_{j \neq i} d_j\}$ for any i, $1 \leq i \leq n$. If Assumption (4) holds, $|\Phi\rangle$ is of full local ranks if and only if $\mathrm{rank}(\rho_\Phi^i) = d_i$ for all i. Note that local ranks cannot increase under local linear operators, either.

Proposition 4 *If* $|\psi\rangle \leq_{SLOCC} |\phi\rangle$*,* $\mathit{rank}(\rho_\psi^i) \leq \mathit{rank}(\rho_\phi^i)$*, for any* $i = 1, \cdots, n$*.*

The following lemma will be useful. Denote by $\mathrm{supp}(\rho)$ the *support* of a Hermitian operator ρ, i.e., $\mathrm{supp}(\rho)$ is the space spanned by eigenvectors of ρ corresponding to non-zero eigenvalues.

Lemma 2 *If* $|\Phi\rangle$ *is not of full local ranks, there exists* $|\Psi\rangle$ *of full total ranks such that* $|\Phi\rangle \leq_{SLOCC} |\Psi\rangle$*.*

Proof: Without loss of generality, assume that $\mathrm{rank}(\mathrm{supp}(\rho_\Phi^1)) < \min\{d_1, d_2 d_3 \cdots d_n\}$. Let $|\alpha\rangle$ and $|\beta\rangle$ be (normalized) states in d_1 and $d_2 \otimes \cdots d_n$ orthogonal to $\mathrm{supp}(\rho_\Phi^1)$ and $\mathrm{supp}(\rho_\Phi^{2,3,\ldots,n})$, respectively. Consider $|\Phi'\rangle = |\Phi\rangle + |\alpha\rangle \otimes |\beta\rangle$. We have that $|\Phi\rangle = ((I - |\alpha\rangle\langle\alpha|) \otimes I \otimes \cdots I)\ |\Phi'\rangle$. Thus $|\Phi\rangle \leq_{\mathrm{SLOCC}} |\Phi'\rangle$. Furthermore, for any $i = 1, 2, ..., n$, $\rho_{\Phi'}^i = \rho_\Phi^i + \gamma_i$ for some positive semidefinite γ_i. In particular for $i = 1$, $\gamma_i = |\alpha\rangle\langle\alpha|$. Thus we have $\mathrm{rank}(\mathrm{supp}(\rho_{\Phi'}^i)) \leq \mathrm{rank}(\mathrm{supp}(\rho_\Phi^i))$ for all i, and $\mathrm{rank}(\mathrm{supp}(\rho_{\Phi'}^1)) = \mathrm{rank}(\mathrm{supp}(\rho_\Phi^1)) + 1$. After a finite number of repetitions of this process we arrive at a state $|\Psi\rangle$ of full total ranks and $|\Phi\rangle \leq_{\mathrm{SLOCC}} |\Psi\rangle$. $\square$

We characterize maximal states below.

Lemma 3 *A state is maximal if and only if it has full local ranks.*

Proof: We prove the result for $n = 3$. The other cases are similar. Suppose that $|\Phi\rangle$ is of full local ranks. Let $|\Psi\rangle \in \mathcal{H}$ be such that $|\Phi\rangle \leq_{\text{SLOCC}} |\Psi\rangle$. Then there exists linear operators L_1, L_2, L_3 such that $(L_1 \otimes L_2 \otimes L_3)|\Psi\rangle = |\Phi\rangle$. As $|\Phi\rangle$ is of full local ranks, and local ranks do not increase under local linear operators, we must have that $|\Psi\rangle$ is also of full local ranks. Since for each i, $i = 1, 2, 3$, $\text{supp}(\rho^i_\Phi)$ is a subspace of $L_i\text{supp}(\rho^i_\Psi)$, those two linear spaces must be the same (as they are of the same dimension). Thus $L_i : \text{supp}(\rho^i_\Psi) \to \text{supp}(\rho^i_\Phi)$ has an inverse $L_i^{-1} : \text{supp}(\rho^i_\Phi) \to \text{supp}(\rho^i_\Psi)$, and $(L_1^{-1} \otimes L_2^{-1} \otimes L_3^{-1})|\Phi\rangle = |\Psi\rangle$. Therefore $|\Phi\rangle$ is maximal.

For the other direction, assume that $|\Phi\rangle$ is maximal. By Lemma 2, there exists $|\Psi\rangle$ of full total ranks and $|\Phi\rangle \leq_{\text{SLOCC}} |\Psi\rangle$. Since $|\Phi\rangle$ is maximal, we have $|\Psi\rangle \leq_{\text{SLOCC}} |\Phi\rangle$. It follows from Proposition 4 that $|\Phi\rangle$ is of full total ranks.

The following lemmas show that there are at least two general ways of constructing a maximal state.

Lemma 4 *There is a maximal state* $|\Phi\rangle$ *such that* $\text{Sch}(\Phi) = \text{Sch}(\mathcal{H})$.

Proof: By definition, there exists $|\Phi\rangle$ such that $\text{Sch}(\Phi) = \text{Sch}(\mathcal{H})$. Let $|\Psi\rangle$ be a state of full total ranks and $|\Phi\rangle \leq_{\text{SLOCC}} |\Psi\rangle$. The existence of $|\Psi\rangle$ follows from Lemma 2. Then $|\Psi\rangle$ is maximal by Lemma 3. Furthermore, from Proposition 3, $\text{Sch}(\Psi) \geq \text{Sch}(\Phi) = \text{Sch}(\mathcal{H})$. Thus $\text{Sch}(\Psi) = \text{Sch}(\mathcal{H})$. □

Lemma 5 *Under Assumption (4), there is a maximal state with tensor rank* d_1.

Proof: We construct a maximal state with the tensor rank d_1 as follows. Take a basis $\{|a_i\rangle : i = 1, \cdots, d_1\}$ of A and a set of d_1 linearly independent product vectors $\{|b_i\rangle|c_i\rangle : 1 \leq i \leq d_1\}$ of $B \otimes C$ and then construct

$$|\Psi\rangle = \sum_{i=1}^{d_1} |a_i\rangle|b_i\rangle|c_i\rangle.$$

Clearly $\text{Sch}(\Psi) = d_1$. However, we cannot guarantee that $\text{Sch}(\Psi)$ is of full local rank at the sides of B and C. For instance, $\{|b_i\rangle : 1 \leq i \leq d_1\}$ may not span B. A simple example is $|0\rangle|00\rangle + |1\rangle|01\rangle$, which is of tensor rank 2 but is not of full local ranks. One can avoid this problem by using the special construction presented in Ref. [15]. An alternative construction is as follows. Let $\{|0\rangle, \cdots, |d-1\rangle\}$ be an orthonormal basis for d-dimensional state space. Consider the following (unnormalized) state

$$|\Phi_1\rangle = \sum_{i=0}^{d_3-1} |i,i,i\rangle + \sum_{i=d_3}^{d_2-1} |i,i,0\rangle + \sum_{i=d_2}^{d_1-1} |i,b_i,c_i\rangle,$$

where (b_i, c_i)'s are distinct elements that do not appear in the first two terms. By construction, we have $\text{Sch}(\Phi_1) = d_1$. We will verify the above state is of full local ranks, thus maximal by Lemma 3.

By direct computation, $\rho^A_{\Phi_1} = I_A$, thus of full rank. If we now discard the third term by applying a local operator $\sum_{i=0}^{d_2-1} |i\rangle\langle i|^A \otimes I_B \otimes I_C$ to $|\Phi_1\rangle$, we obtain the following state

$$|\Phi_2\rangle = \sum_{i=0}^{d_3-1} |i,i,i\rangle + \sum_{i=d_3}^{d_2-1} |i,i,0\rangle.$$

By direction computation, $\rho^B_{\Phi_2} = \sum_{i=0}^{d_2-1} |i\rangle\langle i|^B = I_B$, and is of full rank. Thus $|\Phi_1\rangle^B$ is also of full local rank on B's sub-system. That $|\Phi_1\rangle$ is of the full local rank on the third sub-system can be similarly shown. Therefore $|\Phi_1\rangle$ is maximal and has tensor rank d_1. $\square$

We will now show that Under (4), there are at least two incomparable maximal states. We will focus on $n = 3$ and return to the general case later. Let $k = d_2 d_3 - d_1$.

First, we prove the result for the case $\mathrm{Sch}(\mathcal{H}) > d_1$. We then show if $d_1 < d_2 d_3 - 1$ then $\mathrm{Sch}(\mathcal{H}) > d_1$. Finally we show that for $d_1 = d_2 d_3 - 1$, there are precisely $\min\{d_2, d_3\} = d_3 \geq 2$ number of maximal equivalence classes.

By Lemma 4 and 5, if $\mathrm{Sch}(\mathcal{H}) \neq d_1$ then there are two incomparable maximal states. This is indeed the case when $k > 1$.

Lemma 6 *There are at least two incomparable maximal states in $d_1 \otimes d_2 \otimes d_3$ if $k > 1$.*

Proof: By Theorem 2, we have $\mathrm{Sch}(\mathcal{H}) \geq d_1 + 1$ for $k \geq 4$, by Item (i), and for $k = 2, 3$ by Item (ii) (note that when $k = 3$, $\max\{d_2, d_3\} \geq k$, since otherwise $d_1 = d_2 = d_3 = k = 2$). Therefore, when $k > 4$, $\mathrm{Sch}(\mathcal{H}) \neq d_1$. Since any two equivalent states must have the same tensor rank (by Proposition 3), Theorem 1 implies that there are two incomparable maximal states.

We now focus on the case $d_1 = d_2 d_3 - 1$. By Theorem 2(ii), $\mathrm{Sch}(\Phi) = d_1$. Since a maximal state has full local ranks thus having a tensor rank $\geq d_1$, its tensor rank must be precisely d_1. $\square$

Lemma 7 *If $d_1 = d_2 d_3 - 1$, there are precisely d_3 inequivalent maximal states.*

Proof: We establish a one-to-one correspondence π between the SLOCC equivalence classes of maximal states in $d_1 \otimes d_2 \otimes d_3$ and the SLOCC equivalence classes of $d_2 \otimes d_3$. Note that the latter have $\min\{d_2, d_3\} = d_3$ elements, each of which is represented by a bipartite state of Schmidt rank i, $i = 1, 2, ..., d_3$.

Let $|\Phi\rangle$ be a maximal state in $d_1 \otimes d_2 \otimes d_3$. By Lemma 3, $|\Phi\rangle$ is of full local ranks. Since $d_1 \leq d_2 \cdot d_3$, $\mathrm{rank}(\rho^A_\Phi) = \mathrm{rank}(\mathrm{supp}(\rho^{BC}_\Phi)) = d_1$. Let $|\Phi'\rangle$ be the unique, up to a non-zero scaler, state in $d_2 \otimes d_3$ perpendicular to $\mathrm{supp}(\rho^{BC}_\Phi)$. We set π to map the class represented by $|\Phi\rangle$ to that represented by $|\Phi'\rangle$.

We show that π is well-defined in that it does not depend on the choice of the representative state $|\Phi\rangle$. If $|\Psi\rangle = (L_1 \otimes L_2 \otimes L_3)|\Phi\rangle$ be a state equivalent to $|\Phi\rangle$, where L_i, $i = 1, 2, 3$, is invertible. Then

$$(L_2 \otimes L_3)\ \mathrm{supp}(\rho^{BC}_\Phi) = \mathrm{supp}(\rho^{BC}_\Psi). \tag{5}$$

This is equivalent to

$$|\Phi'\rangle^{BC} = ((L_2^\dagger)^{-1} \otimes (L_3^\dagger)^{-1})|\Psi'\rangle^{BC}. \tag{6}$$

In other words, $|\Phi'\rangle$ and $|\Psi'\rangle$ are equivalent under SLOCC. Therefore π is well defined.

To see that π is on-to, fix an arbitrary bipartite state $|\Phi'\rangle^{BC}$. Extend $|\Phi'\rangle^{BC}$ to a basis of $d_2 \otimes d_3$ by $|\phi_i\rangle$, $i = 1, 2, ..., d_1$. The state

$$|\Phi\rangle = \sum_{i=1}^{d_1} |i\rangle \otimes |\phi_i\rangle \tag{7}$$

has full local rank on the sub-system A. It must also have full local rank at the other two sub-systems, as otherwise it would be in the space $d_1 \otimes d_2 \otimes (d_3 - 1)$ or $d_1 \otimes (d_2 - 1) \otimes d_3$. In either case, $\mathrm{rank}(\mathrm{supp}(\rho_\Phi^{BC})) \leq d_2(d_3-1) < d_2d_3 - 1$, a contradiction. Thus $|\Phi\rangle$ is maximal. Clearly, the equivalence class of $|\Phi\rangle$ is mapped by π to that of $|\Phi'\rangle$. Thus π is on-to.

To see that π is in-to, assume that bipartite states $|\Phi'\rangle$ and $|\Psi'\rangle$ have the same Schmidt rank, and that for invertible L_2 and L_3, Eqn. (6) holds. Consequently, Eqn. (5) holds. Thus $I_A \otimes L_2 \otimes L_3|\Psi\rangle = \sum_{i=0}^{d_1-1}|\alpha_i\rangle^A|\phi_i\rangle$, for some states $|\alpha_i\rangle^A$, $0 \leq i \leq d_1-1$. Those states $|\alpha_i\rangle$ must be linearly independent, since $L_2 \otimes L_3$ does not change the local rank of $|\Psi\rangle$. Thus there exists basis $|\hat{\alpha}_i\rangle$ such that $\langle\hat{\alpha}_i|\alpha_j\rangle = \delta_{ij}$, $0 \leq i,j \leq d_1-1$, where δ_{ij} is the Kronecker delta. Then setting $L_1 = \sum_{i=0}^{d_1-1}|i\rangle\langle\hat{\alpha}_i|$, we have that L_1 is invertible and $|\Phi\rangle = L_1 \otimes L_2 \otimes L_3|\Psi\rangle$. Thus $|\Phi\rangle$ and $|\Psi\rangle$ are equivalent. Consequently, π is a one-to-one correspondence, implying there are precisely d_3 number of maximal equivalence class. □

An example to illustrate Lemma 7 is the state space $\mathcal{H} = 3\otimes2\otimes2$. Miyake has obtained all eight equivalence class of this space [16]. Two of these equivalence classes are maximal. The above lemma provides an alternative method to characterize the maximal states in this space. By the Lemma, there is a one-to-one correspondence between the maximal equivalence class of $\mathcal{H}$ and the equivalence class of $\mathcal{H}' = 2 \otimes 2$. The latter space has precisely two equivalence classes with the representatives $|\Phi_1'\rangle = |10\rangle$ and $|\Phi_2'\rangle = |01\rangle - |10\rangle$. As a result, there are only two maximal equivalence classes in $\mathcal{H}$, which can be constructed according to $|\Phi_1'\rangle$ and $|\Phi_2'\rangle$ as follows:

$$\begin{aligned} |\Phi_1\rangle &= |0\rangle|00\rangle + |1\rangle|01\rangle + |2\rangle|11\rangle, \\ |\Phi_2\rangle &= |0\rangle|00\rangle + |1\rangle(|01\rangle + |10\rangle) + |2\rangle|11\rangle. \end{aligned} \tag{8}$$

Lemma 6 and 7 together imply Theorem 1 for $n = 3$. We deal with the general case below (that is to show that there is no maximum state in $d_1 \otimes d_2 \otimes \cdots \otimes d_n$ if $d_1 < d_2d_3\cdots d_n$).

Proof of Theorem 1: We need only consider $n > 3$. Suppose that $n = 4$ and $d_1 < d_2d_3d_4$. Consider the tripartite state space $d_1 \otimes d_2 \otimes d_3d_4$. There are two cases:

Case 1. $d_3d_4 = d_1d_2$. Since $d_1 \geq d_2 \geq d_3 \geq d_4$ we have $d_1 = d_2 = d_3 = d_4 = d$. One can easily verify that $|\Phi_d\rangle^{AB} \otimes |\Phi_d\rangle^{CD}$ and $|\Phi_d\rangle^{AC} \otimes |\Phi_d\rangle^{BD}$ both are of full local rank d, where $|\Phi_d\rangle$ is the generalized EPR state defined in Eqn. 1. Observe that with respect to the $AC : BD$ partition, the former is entangled yet the latter is not, and with respect to the $AB : CD$ partition, the opposite holds. Since no LOCC protocol can create entanglement, the two states are incomparable under SLOCC.

Case 2. $d_3d_4 < d_1d_2$. Any maximal state in this tripartite space $d_1 \otimes d_2 \otimes (d_3d_4)$ must be of d_3d_4 rank on the $(d_3 \otimes d_4)$ sub-system, thus as a state in $d_1 \otimes d_2 \otimes d_3 \otimes d_4$, it must be of the maximumly possible local rank on the d_3 and d_4 spaces. It follows that it remains of full local rank in the four-partite system. Therefore, applying the result for $n = 3$ we know that there are at least two inequivalent maximal states in $d_1 \otimes d_2 \otimes d_3d_4$. They remain maximal states in the four-partite system, and are incomparable under SLOCC with respect to this refined partition.

Suppose that the theorem is correct for $n = \ell$, $\ell \geq 4$. Consider $n = \ell + 1$. Since $\ell \geq 4$, $d_\ell d_{\ell+1} < d_1d_2d_3\cdots d_{\ell-1}$. By the inductive hypothesis, there are two incomparable maximum states in $d_1 \otimes d_2 \otimes \cdots \otimes d_{\ell-1} \otimes d_\ell d_{\ell+1}$. By an argument similar to that in Case 2 of $n = 4$, they remain maximal and incomparable in the refinement $\mathcal{H}$. Thus the theorem is correct for

$n = \ell + 1$, therefore correct for all $n \geq 4$. □

3 Correspondence between maximal equivalence classes and SLOCC equivalence classes

In this section we further study state spaces such that $d_1 < d_2 \cdots d_n$. So it is impossible to find one state from which one can locally prepare any other state even probabilistically.

An alternative goal is to characterize all maximal equivalence classes. In particular, we ask when a multipartite state space $\mathcal{H}$ has only a finite number of maximal stochastic equivalence classes. Suppose that $\mathcal{H}$ has a finite number of maximal equivalence classes with the representative states $|\Phi_1\rangle, \cdots, |\Phi_N\rangle$. Then for any state $|\psi\rangle \in \mathcal{H}$, there exists i, $1 \leq i \leq N$ such that $|\Phi_i\rangle$ can be converted into $|\psi\rangle$ by SLOCC. So the set of states $\{|\Phi_1\rangle, \cdots, |\Phi_N\rangle\}$ is able to locally prepare any other state in $\mathcal{H}$ with nonzero probability. In practice, we only need to prepare the set of maximal states $\{|\Phi_i\rangle\}$ and then create other states using SLOCC. Thus identifying the maximal equivalence classes for a given space is highly desirable.

For the sake of convenience, from now on we mainly focus on tripartite state space. Most of our results are also valid for the case of $n > 3$. We assume that $d_1 = d_2 d_3 - k$, where $k < d_2 d_3/2$. We shall employ a correspondence between the maximal equivalence classes of $d_1 \otimes d_2 \otimes d_3$ and the equivalence classes of $k \otimes d_2 \otimes d_3$.

Definition 1 *Let $|\Phi\rangle \in d_1 \otimes d_2 \otimes d_3$ and rank$(\rho_\Phi^{A_1}) = d_1$, write $|\Phi\rangle = \sum_{i=1}^{d_1} |i\rangle^{A_1} |\phi_i\rangle^{A_2A_3}$, where $\{|i\rangle^{A_1} : 1 \leq i \leq d_1\}$ is any orthonormal basis for d_1. Let $\mathcal{T}^{A_1}(\Phi)$ be the SLOCC equivalence class of $k \otimes d_2 \otimes d_3$ with representative state $|\Phi'\rangle = \sum_{i=1}^{k} |i\rangle^{A_1} |\phi_i^\perp\rangle^{A_2A_3}$, where $\{|i\rangle^{A_1} : 1 \leq i \leq k\}$ is a basis for the space k and $\{|\phi_i^\perp\rangle : 1 \leq i \leq k\}$ is any basis for* $\text{span}^\perp\{|\phi_i\rangle^{A_2A_3} : 1 \leq i \leq d_1\}$.

It is easy to see that $\mathcal{T}^{A_1}(\Phi)$ is well-defined in the sense that it does not depend on which basis of $\text{span}^\perp\{|\phi_i\rangle^{A_2A_3} : 1 \leq i \leq d_1\}$ we choose. It is also worth noting that by construction any state in $\mathcal{T}^{A_1}(\Phi)$ should have a full local rank k at A_1's side.

The importance of the map $\mathcal{T}^{A_1}$ is due to the following lemma, which can be treated as a generalization of Lemma 7.

Lemma 8 *Let $|\Phi\rangle$ and $|\Psi\rangle$ be two vectors in $\mathcal{H}$ such that rank$(\rho_\Phi^{A_1}) =$ rank$(\rho_\Psi^{A_1}) = d_1$. Then $|\Phi\rangle$ and $|\Psi\rangle$ are equivalent under SLOCC if and only if $\mathcal{T}^{A_1}(\Phi) = \mathcal{T}^{A_1}(\Psi)$.*

Proof: The proof idea is similar to Lemma 7. For completeness, we present a detailed proof here. By Lemma 1, $|\Phi\rangle$ and $|\Psi\rangle$ are equivalent under SLOCC if and only if there are invertible linear operators L_1, L_2, L_3 such that $|\Phi\rangle = (L_1 \otimes L_2 \otimes L_3)|\Psi\rangle$. More explicitly, we have

$$\sum_{i=1}^{d_1} |i\rangle^{A_1} |\phi_i\rangle^{A_2A_3} = (L_1 \otimes L_2 \otimes L_3) \sum_{i=1}^{d_1} |i\rangle^{A_1} |\psi_i\rangle^{A_2A_3}. \tag{9}$$

Applying $\langle j|^{A_1} \otimes I^{A_2A_3}$ to both sides of the above equation, we have

$$|\phi_j\rangle^{A_2A_3} = (L_2 \otimes L_3) \sum_{i=1}^{d_1} \langle j|L_1|i\rangle |\psi_i\rangle^{A_2A_3}.$$

That means

$$|\phi_j\rangle^{A_2A_3} \in (L_2 \otimes L_3)\text{span}\{|\psi_i\rangle^{A_2A_3} : 1 \leq i \leq d_1\}$$

for each $1 \leq j \leq d_1$. Noticing further that L_1 is invertible, we have

$$\mathrm{span}\{|\phi_i\rangle^{A_2A_3}\} = (L_2 \otimes L_3)\mathrm{span}\{|\psi_i\rangle^{A_2A_3}\}. \tag{10}$$

Conversely, we can readily show that the existence of invertible linear operators L_2 and L_3 such that Eqn. (10) holds also implies the SLOCC equivalence between $|\Phi\rangle$ and $|\Psi\rangle$. It is easy to verify Eqn. (10) can be rewritten into the following

$$\mathrm{span}^{\perp}\{|\phi_i\rangle^{A_2A_3}\} = ((L_2^{\dagger})^{-1} \otimes (L_3^{\dagger})^{-1})\mathrm{span}^{\perp}\{|\psi_i\rangle^{A_2A_3}\}. \tag{11}$$

Using a similar argument, we can show the above equation means that $|\Phi'\rangle = \sum_{i=1}^{k} |i\rangle^{A_1}|\phi_i^{\perp}\rangle^{A_2A_3}$ and $|\Psi'\rangle = \sum_{i=1}^{k} |i\rangle^{A_1}|\psi_i^{\perp}\rangle^{A_2A_3}$ are equivalent. In other words, $\mathcal{T}^{A_1}(\Phi)$ and $\mathcal{T}^{A_1}(\Psi)$ coincide. □

When $k < d_2d_3/2$, we have $k < d_1$. It may be much easier to decide the SLOCC equivalence between $\mathcal{T}^{A_1}(\Phi)$ and $\mathcal{T}^{A_1}(\Psi)$ than that between $|\Phi\rangle$ and $|\Psi\rangle$. However, $\mathcal{T}^{A_1}$ is not a one-to-one correspondence between the maximal equivalence classes of $d_1 \otimes d_2 \otimes d_3$ and the equivalence classes of $k \otimes d_2 \otimes d_3$. In general, the image of $\mathcal{T}^{A_1}$ is only a proper subset of $k \otimes d_2 \otimes d_3$. Fortunately, in the special case of $k = 1$, we do have a one-to-one correspondence as stated below. The proof is exactly the same as the case of $n = 3$, which has been proven in Lemma 7.

Theorem 5 *There is a one-to-one correspondence between the maximal equivalence classes in $d_1 \otimes \cdots \otimes d_n$ and the stochastic equivalence classes of $d_2 \otimes \cdots \otimes d_n$, where $d_1 = d_2 \cdots d_n - 1$.*

The following theorem also follows directly from Lemma 8.

Theorem 6 *If $k \otimes d_2 \otimes \cdots \otimes d_n$ has a finite number of equivalence classes, $d_1 \otimes d_2 \otimes \cdots \otimes d_n$ also has a finite number of maximal equivalence classes.*

Using the known result that there are a finite number of equivalence classes for tripartite systems of the dimensions $d_3 = 2$, $d_2 \leq 3$ [14], we have the following corollary.

Corollary 1 *Each of the following spaces has a finite number of maximal equivalence classes: $(2n-2) \otimes n \otimes 2$, $(2n-3) \otimes n \otimes 2$ and, when $2 \leq \min\{m, n\} \leq 3$, $(2mn-1) \otimes m \otimes n \otimes 2$.*

For $\mathcal{H} = 7 \otimes 2 \otimes 2 \otimes 2$, it follows from the above corollary that $\mathcal{H}$ has a finite number of maximal equivalence classes. In contrast, $\mathcal{H}$ has an infinite number of equivalence classes [18]. Another notable case is $\mathcal{H} = 4 \otimes 3 \otimes 2$. We know from [16] that $\mathcal{H}' = 2 \otimes 3 \otimes 2$ has 8 equivalence classes. Thus by Theorem 6, $\mathcal{H}$ has at most 8 different maximal equivalence classes. However, the exact number is strictly smaller than 8 as some equivalence classes do not correspond to any equivalence classes. A careful investigation shows that $4 \otimes 3 \otimes 2$ has exactly 5 maximal equivalence classes.

4 Discussions and open problems

We showed as our main result that a multipartite quantum system is allowed to have a maximum entangled state only when there is a subsystem whose dimension is no less than the total dimension of the rest of the system. When this condition does not hold, there are multiple distinct maximal equivalence classes. A complete classification of those maximal states would be of great value, both theoretically and practically. To this end, we provided a connection between the maximal equivalence classes in a state space with the stochastic equivalence classes in another state space of a smaller dimension. In particular, we proved that when $d_1 = d_2 \cdots d_n - 1$, there is a one-to-one correspondence between the maximal equivalence classes of $d_1 \otimes \cdots \otimes d_n$ and the stochastic equivalence classes of $d_2 \otimes \cdots \otimes d_n$. Various examples are studied to demonstrate the applications of these results.

We conclude by proposing two directions for further investigations that we consider of both theoretical and practical importance. The first is to understand deeper the structure of partial orders on LOCC and equivalence class. Structural results will not only deepen our understanding of entanglement, but will also find applications for establishing multipartite entanglement when there is no maximum state.

For example, which spaces have an infinite number of SLOCC equivalence classes, or an infinite number of maximal classes? For those spaces having a finite number of maximal equivalence classes, the parties can share some number of each maximal states, and use them later to generate arbitrary desired states. Note that in this case the ratio of the output states and the initial states will not be as efficient as the case when a maximum state exists, unless the distribution of the output states is known in advance. A second and related question is, given a space that does not admit a maximum state, what is the "smallest" state outside the specified space yet is able to generate an arbitrary state in that space? For instance, there are two maximal equivalence classes in $2 \otimes 2 \otimes 3$, represented by the states $|\Phi_1\rangle$ and $|\Phi_2\rangle$ in Eqn. (8). Either state, however, can generate any state from $2 \otimes 2 \otimes 2$ through SLOCC.

A second direction is to consider approximate generation of entangled states. Are there spaces that do not have a maximum state but have an "approximate" maximum state in the sense that all other states can be approximated to an arbitrary small precision through an LOCC protocol on that state? Such an approximate state is as good as the precise state in practice. Consider another setting where the parties wish to generate a large number of a target state. A solution is for them to share in bulk some initial state, since many copies of a fixed state are likely to be cheaper to manufacture. A natural question is, which initial state will offer the most efficient rate of conversion in the worst case (over all possible target states)? In particular, which spaces admit the best possible ratio of 1 asymptotically? Perhaps the notion of "border rank" (C.f. Chapter 15 in [8]), the approximate version of tensor rank, in algebraic complexity theory may be useful for tackling those intriguing problems.

Acknowledgment

We are grateful to Eric Chitambar and Zhengfeng Ji for helpful discussions and comments.

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