Chapter 1

INTRODUCTION

Integer Structure Analysis in number theory has commonly been applied to primes, powers, various algebraic equations, and so on. However, any system which features integers by means of integer structure can reveal simple relationships that might be obscured by more sophisticated techniques. This is illustrated here where we start to transfer converging infinite series into modular ring language. We also indicate the role of number theory as an important part of a genuine liberal education accessible to all students in a way that education in the ancient *quadrivium* was confined to a small section of society. The topics of number theory in the hands of well-educated teachers can inspire a love of learning beyond passing fads.

For this reason we have embedded relevant issues on liberal education as a foundation for education in the 21st century, particularly in fostering creativity through the inspiration and passion of teachers – something which seems to wax and wane in pedagogical fashion every half-century or so (cf. Dewey 1916; Shannon 1968; Wang & Huang 2017). These issues are teased out to some extent in the first and last chapters, with the mathematics as the meat in the sandwich! One of these topics is the golden section to which the Fibonacci numbers are related. Mathematically the golden ratio is a humble surd; by replacing its argument we are able to show that it has an infinity of close relatives which can be a source of further exploration.

The context of this book is the teaching and learning of mathematics. This happens in historical and sociological contexts, though not in the way that those who are wedded to identity politics would now have us believe (*cf*. Gutiérrez, 2018). These are only one side of the mathematics education coin: the other side is in the improvement of instructional based strategies for teaching and learning (Stephens 2011).

There are sufficient historical and philosophical allusions in the development here in this short monograph for anyone to see that the mathematics *per se* transcends race, religion, history and geography.

1.1. What the Book Is About

This book is a collection of current research ideas on classical problems linked to the sequence of Fibonacci numbers. The book goes further in that it uses 'elegance' (number theory beauty) to relate some of the topics to the wider curriculum. In fact, number theory as we now know it is, in a real sense, one part of the modern version of the ancient (mathematical based) *quadrivium*. It serves as a foundation for the modern *trivium* of history, philosophy and language, all of which are of concern to the authors in what at first glance appear to be solely mathematical problems. Some of the implicit properties which we shall consider seem to have been known in classical times in Greece (Deakin 2013) and India (Hall 2008). In that sense, some familiarity of major historical landmarks and philosophical questions of number theory is part of everyone's cultural heritage of a truly liberating education.

The unifying and underlying parallel theme is the structure of the integers stripped bare by modular rings, but with references to the relevant history, philosophy and language: *history* because mathematical insights are made at a particular time; *philosophy* because of the use of logic to provide elegant solutions; *language* so that there is precision in the elaboration of ideas. That said, the book is not essentially philosophical, but the broader context cannot be ignored (Schinzel 1990).

In a mathematical sense, the book is fundamentally elementary. A consequence of this is that the problems are generally easily stated even if their elaboration requires detailed exposition. Thus the problems are simple, but the attempted solutions are neither simple nor easy. Some though are trivial, and some are tedious, but all can provide enrichment material from high school levels (if the teachers know about them) through to early postgraduate research. The topics here range from generalizations of Pythagorean problems through continued fraction algorithms to extensions of the golden section.

The classical conjectures in number theory are, by and large, inherently easy to state and intrinsically enjoyable to solve, whether it be by "mucking around" with trial and error or by systematic use of sophisticated technical tools (Franklin 2016). For these reasons, number theory can be made attractive to a wide variety of students to inspire them with notation as a tool of thought (Iverson, 1980; Halmos 1985; Hardy 1940).

These are part of general mathematical education when it goes beyond moving symbols around in an almost blind attempt to get the answer at the back of the book! As the British mathematician Peter Larcombe (2018) has vividly expressed it: "…the emotions that bubble up from connecting with a subject whose grace and allure lies at its core and awakens the spirits. Henri Poincaré's areas of interest matched broadly those of Weyl, and the French academic thought that mathematics had a triple end, one of which was that its innate aesthetic properties. In addition to stimulating enquiry within nature and philosophy should touch practitioners in ways that painting and music do." These sentiments echo in turn the thrust of Howard Gardner (2011): "Truth and beauty are fundamentally different: whereas truth is a *property of statements*, beauty reveals itself in the course of an *experience with an object.*

Although some of this material has already been published elsewhere, the difference here is that we are shedding new light on old material by connecting items hitherto diverse and by comments on what is not otherwise obvious. In particular, the two themes of the book have previously not been inter-connected in quite this way or collated this way within each of the themes. There is also a substantial quantity of new material related to the themes of the book which has not appeared before. In fact, each section has new material.

Each of the theoretical developments is accompanied by practical examples, including frequent use of tabular demonstrations so that the reader can obtain immediate reinforcement of the fundamental ideas being elaborated and extended. The book is not intended to be a stand-alone textbook as is outlined in Section 1.2, but the examples proposed in each section lend themselves to other generalizations, extensions and connections. In the words of Phillips (2005): "In the world of music, there are *two* sets of people; active musicians who play musical instruments or sing, and the much larger set of passive musicians who listen to the sounds produced by the first set. However, in the world of mathematics, we contend that there is only *one* set of mathematicians: the active set."

Thus this book is particularly suited for those teachers with a passion for the subject; teachers who wish to enlarge the horizons of their students through enrichment material either in specific mathematics classes or more general liberal arts programs.

The foregoing does raise the question "does Mathematics have a Place in the Liberal Arts" which can only be answered after we have considered the nature and scope of the liberal arts. "Such concepts as unity, truth, beauty and causality arise naturally in mathematics and an understanding of them is necessary for mastery of philosophy and theology. Perhaps the most surprising aspect of mathematics is its beauty. It is a subject which deals with absolute truth" (Townsend, 2015).

The Liberal Arts deal with the human being as a whole and hence with what lies at the essence of being human. As a result, the Liberal Arts have a far greater capacity to do good than other fields of study, for their foundation in philosophy and theology enables them to bring students into contact with the ultimate questions which they are free to accept (or reject). Even if these questions have little or no "market value," it should be obvious that the way they are taught and learned is going to have a powerful impact upon the future of the students and society.

1.2. HOW THE BOOK FLOWS

The book is partly whimsical in the style of the previous section in that it connects with the wider panorama of the beauty and elegance of number theory and its connections with the cultural heritage of mathematics in particular, but education in general.

It is a common mistake to blame the current materialism and moral decline of the Western world on its extraordinary technological achievements, as if a scientific-technological outlook on life were incompatible with belief in God and the supremacy of spiritual values. The decline has occurred because of the gradual displacement and internal disorientation of the properly conceived liberal arts program which should occupy the center of secondary and university education.

The late Rev Dr Austin Woodbury *sm* provocatively considered both *liber* (free) and *libra* (balance) in the etymology of "liberal." These words can be applied to Liberal Education which should free the minds of students to be open to a balanced view of the things that matter in life so that they can make decisions with an informed conscience (Newman, 1992). Nussbaum puts it well when she says that "cultivated capacities for critical thinking and reflection are crucial in keeping democracies alive and wide awake" (Nussbaum, 2010).

According to Professor David J. Walsh of the Catholic University of America, one finds in American higher education no clear idea of the end result to be aimed at. In most universities there is "an assemblage of incoherent, fragmentary disciplines and sub-disciplines... without any clearer guidance than some vague commitment to methodological requirements within the separate fields" (Walsh, 1985). In short, there is no "unifying sense of direction." But this phenomenon is not only a crisis of educators; it is a "crisis of knowledge" (Newman, 1996). Contemporary education is hopelessly at sea because, despite often vast knowledge in particular fields, many scholars lack knowledge of what matters most of all – the purpose of human existence. Again according to Walsh, "the clearest evocation of paradigmatic excellence" has traditionally been found within the cluster of disciplines called the Liberal Arts. Today, however, "education in the Liberal Arts has sadly very little to do with the formation of existential purpose; ... it has generally devolved into an increasingly irrelevant discussion of 'ideas,' 'theories,' methods and techniques." In other words, teachers of the Liberal Arts have lost their sense of vocation. Too often, so-called liberal studies can merely be a smorgasbord of subjects from which students can choose and which they tick off as they meet degree requirements. There is little sense of their inter-relations and intimate connections.

Perhaps Professor Walsh is claiming too much for the liberal arts? What does he mean by their traditionally providing "the clearest evocation of paradigmatic excellence?"

Great literature speaks both to the heart and to the mind, as do all the arts when true to their proper nature. Great literature conveys a vision of truth and beauty and moral excellence capable of raising the spirit of the reader to unsuspected heights, even in the most unpromising circumstances (Gardner, 2011). This has been demonstrated time and time again. One recalls the dramatic effect the reading of Cicero's (now lost) work, Hortensius, had on the young Augustine, kindling in him a passion for wisdom that was to inspire his whole life. For Cicero the liberal arts formed the basis of one's "humanitas" (Augustine, 1997).

In ancient Greek and Roman cultures the liberal arts referred to intellectual arts as distinct from the mechanical arts, the arts of the hand. It may be that the Greeks, in particular, exaggerated the distinction between the two kinds of art, and that the ancients in general demeaned manual work as servile. This appeared to be a mistake to many who came later, but they should not forget that the Greek conviction that reality is intelligible made possible the modern scientific revolution of which we are the heirs (Whitehead, 1994). In ancient times the seven liberal arts were the trivium and the quadrivium united by metaphysics and theology.

Contrary to the popular impression that the arts are a "soft option," true liberal education is actually very demanding. For this reason few people have attained so noble and so realistic an understanding of human affairs as for instance, Thucydides and Aristotle. Educational reformers would do well to consider including The Peloponnesian War, The Nicomachean

Ethics, and The Politics in the reading program of students who are bent upon careers in public service which, fundamentally, is a noble vocation (Maritain and Adler, 1940).

The *trivium* consisted of • Grammar→ basic systematic knowledge • Rhetoric →the 'how' • Logic →the 'why' →	The *quadrivium* consisted of • Arithmetic→Number in itself • Geometry→ Number in space • Music→ Number in time • Astronomy → Number in space and time
Illuminated by • Metaphysics → ultimate reality • Theology→ ultimate end →	Permeated by • Truth • Beauty • Goodness

Figure 1.2.1. Trivium and Quadrivium.

In answer to the question which was posed at the beginning of this section it should be clear by now that the answer is "yes," but it depends on how mathematics is taught. Thus we get to the structure of the book which itself defies mathematical ordering, whether linear, partial or quasi (Shrőder, 2002). The following diagram (Figure 1.2.1) tries to give some coherence to the topics which actually have many inter-twined links:

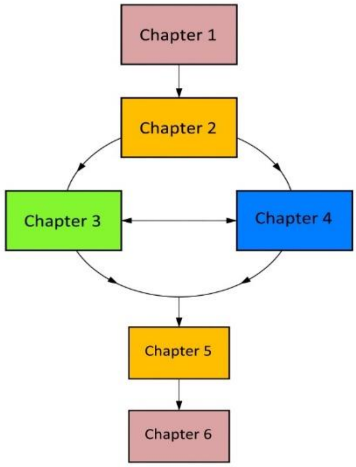

Figure 1.2.2. Flowchart of main ideas.

There is necessarily some repetition, such as repeating the Fibonacci recurrence relation [Equation 1.3.1], because we do not assume that readers will move from Chapter 1 to Chapter 6 in any particular order. Some sections will have more immediate appeal to some readers depending on their background and interests.

1.3. PRELIMINARY REMARKS[1]

As an introduction to the two major themes of this book, various Fibonacci number identities are [8] outlined here in terms of their underlying integer structure in the modular ring Z_5.

Over eight centuries ago in Chapter XII of his book, *Liber Abaci*, Leonardo of Pisa (nicknamed Fibonacci) presented and solved his famous problem on the reproduction of rabbits in terms of the famous sequence which bears his name. Four centuries later, Albert Girard (1634) gave the notation for the recurrence relation for the terms of the sequence in use today, namely

$$F_{n+1} = F_n + F_{n-1}. \tag{1.3.1}$$

The Fibonacci sequence as such was recorded before 200 BC by Pingala, an Indian Sanskrit grammarian and mathematician in his book *Chandahsutra* (*cf.* Hall 2008 who looks at the combinatorics and she also links mathematics and poetry).

Over the centuries since the Fibonacci sequence of integers has been applied to a myriad of mathematical applications, especially in number theory (Leyendekkers et al. 2007). In particular, Kepler (Livio, 2006) observed that the ratio of consecutive Fibonacci numbers converges to the Golden Ratio φ. He also showed that the square of any term differs by unity from the product of the two adjacent terms in the sequence (Simson's or Cassini's Identity (1.3.3) below).

[1] Based on Leyendekkers &.Shannon (2013a).

Table 1.3.1. Rows of modular ring Z_5

Class	$\overline{0}_5$	$\overline{1}_5$	$\overline{2}_5$	$\overline{3}_5$	$\overline{4}_5$
Row	$5r_0$	$5r_1+1$	$5r_2+2$	$5r_3+3$	$5r_4+4$
0	0	1	2	3	4
1	5	6	7	8	9
2	10	11	12	13	14
3	15	16	17	18	19
4	20	21	22	23	24

In this section we discuss the structure of the Fibonacci sequence in the context of the modular ring Z_5 (Table 1.3.1) (Leyendekkers & Shannon 2012b). The underlying structure accounts for many of the unique properties of this fascinating sequence, particularly their congruence properties (Shannon et al. 1974).

There are many characteristics of the Fibonacci sequence that are directly related to this structure. We consider some of them here. They serve as examples for further analysis. Another approach would be to consider the algebra of $F(\sqrt{5})$ where $F(x)$ is the characteristic polynomial associated with the recurrence relation (1.1) and is irreducible in the field F of its coefficients (Simons and Wright 2008).

The pattern of the Fibonacci numbers in Z_5 is displayed in Table 1.3.2.

Table 1.3.2. Fibonacci numbers in Z_5

n	1	2	3	4	5	6	7	8	9
Z_5	$\overline{1}_5$	$\overline{1}_5$	$\overline{2}_5$	$\overline{3}_5$	$\overline{0}_5$	$\overline{3}_5$	$\overline{3}_5$	$\overline{1}_5$	$\overline{4}_5$
10	11	12	13	14	15	16	17	18	19
$\overline{0}_5$	$\overline{4}_5$	$\overline{4}_5$	$\overline{3}_5$	$\overline{2}_5$	$\overline{0}_5$	$\overline{2}_5$	$\overline{2}_5$	$\overline{4}_5$	$\overline{1}_5$
n	20	21	22	23	24	25	26	27	28
Z_5	$\overline{0}_5$	$\overline{1}_5$	$\overline{1}_5$	$\overline{2}_5$	$\overline{3}_5$	$\overline{0}_5$	$\overline{3}_5$	$\overline{3}_5$	$\overline{1}_5$
29	30	31	32	33	34	35	36	37	38
$\overline{4}_5$	$\overline{0}_5$	$\overline{4}_5$	$\overline{4}_5$	$\overline{3}_5$	$\overline{2}_5$	$\overline{0}_5$	$\overline{2}_5$	$\overline{2}_5$	$\overline{4}_5$

Table 1.3.3. Details of the patterns (F_n^*: Class of F_n)

F_n^*	n for $\overline{N}_5$	n for $\overline{M}_5$
(1,6) $\bar{1}_5$	19, 39, 59, 79, ... 1, 21, 41, 61, ... 2, 22, 42, 62, ...	8, 28, 48, 68, ...
(2,7) $\bar{2}_5$	14, 34, 54, 74, ... 16, 36, 56, 76, ... 17, 37, 57, 77, ...	3, 23, 43, 63, ...
(3,8) $\bar{3}_5$	4, 24, 44, 64, ... 6, 26, 46, 66, ... 7, 27, 47, 67, ...	13, 33, 53, 73, ...
(4,9) $\bar{4}_5$	9, 29, 49, 69, ... 11, 31, 51, 71, ... 12, 32, 52, 72, ...	18, 38, 58, 78, ...
(0,5) $\bar{0}_5$	0,5,10,15,20,...	

Table 1.3.4. Data from Tables 1.3.2, 1.3.3

n	F_n^*	$\left(F_n^2\right)^*$	F_{n+1}^*	F_{n-1}^*	$\left(F_{n+1}F_{n-1}\right)^*$
4	3,8	9,4	0,5	2,7	0,5
41	1,6	1,6	1,6	0,5	0,5
77	2,7	4,9	4,9	2,7	3,8
92	4,9	6,1	3,8	4,9	7,2

The patterns of the modular residues follow the form $\overline{N}_5 \bar{0}_5 \overline{N}_5 \overline{N}_5 \overline{M}_5$ in which the numbers $\overline{N}_5$ have the pattern $\bar{1}_5 \bar{3}_5 \bar{4}_5 \bar{2}_5$ and the interstitial numbers $\overline{M}_5$ have the pattern $\bar{2}_5 \bar{1}_5 \bar{3}_5 \bar{4}_5$. These patterns allow prediction of the class of F_n, and hence the right-end-digit (RED) from n (Table 1.3.3).

If the measure of a line AB is given by F_{n+1},nd AB is divided into two different sized segments, AC and CB, with CB>AC, then AB/CB = CB/AC approximately defines φ, the Golden Ratio if CB = F_n and AC = F_{n-1}, so that approximately

$$F_{n+1}F_{n-1} \cong F_n^2 \tag{1.3.2}$$

However, as first noted by Kepler the two sides of (1.3.2) always differ by unity as we can see from the class structures (Table 1.3.4).

The equality is expressed in Simson's Identity

$$F_{n+1}F_{n-1} = F_n^2 + (-1)^n \tag{1.3.3}$$

$$F_{n+1} / F_n = F_n / F_{n-1} + (-1)^n / \left(F_n F_{n-1}\right) \tag{1.3.4}$$

of which the second term on the right hand side is very small for large *n*; this is the error term in the Fibonacci approximation for φ (Leyendekkers and Shannon 2014a).

$$F_{n+6} = 4F_{n+3} + F_n \tag{1.3.5}$$

so that

$$F_{n+6} / F_n - 4F_{n+3} / F_n = 1, \tag{1.3.6}$$

which when substituted into (1.3.3) yield

$$F_{n+1}F_nF_{n-1} = \begin{cases} F_n^3 + F_{n+6} - 4F_{n+3}, & n \text{ even}, \\ F_n^3 - F_{n+6} + 4F_{n+3,} & n \text{ odd}. \end{cases} \tag{1.3.7}$$

An odd number of golden rectangles with sides equal to successive Fibonacci numbers can appear to fit into squares as "demonstrated" in Figure 1.3.1.

This is not drawn to scale, but essentially a golden rectangle of sides F_5 and F_7 units, and hence of area 65 square units, is transformed into a square of side F_6 units and hence of area 64 square units. Of course, while the eye might just be deceived, Simson is not! From Simson's identity we get for odd n that

$$F_{n-1}^2 + F_n F_{n-1} + F_n F_{n+1} = F_{n+1}^2 \tag{1.3.8}$$

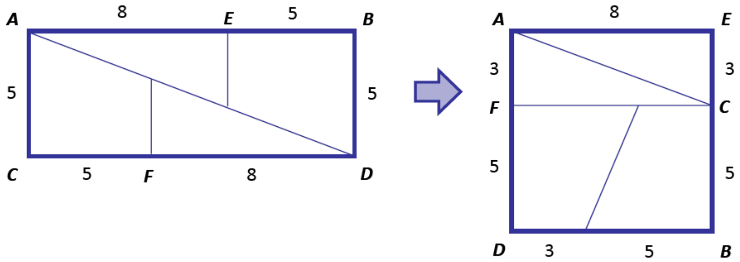

Figure 1.3.1. "Squaring" the Golden Rectangle.

Table 1.3.5. Classes of Sums

Number of Products n	Classes of		Class of
	F_{n+1}	F_{n+1}^2	$F_{n-1}^2 + F_n F_{n-1} + F_n F_{n+1}$
3	$\overline{3}_5$	$\overline{4}_5$	$\overline{1}_5 + \overline{2}_5 + \overline{1}_5 = \overline{4}_5$
5	$\overline{3}_5$	$\overline{4}_5$	$\overline{4}_5 + \overline{0}_5 + \overline{0}_5 = \overline{4}_5$
7	$\overline{1}_5$	$\overline{1}_5$	$\overline{4}_5 + \overline{4}_5 + \overline{3}_5 = \overline{1}_5$
9	$\overline{0}_5$	$\overline{0}_5$	$\overline{1}_5 + \overline{4}_5 + \overline{0}_5 = \overline{0}_5$
11	$\overline{4}_5$	$\overline{1}_5$	$\overline{0}_5 + \overline{0}_5 + \overline{1}_5 = \overline{1}_5$
13	$\overline{2}_5$	$\overline{4}_5$	$\overline{1}_5 + \overline{2}_5 + \overline{1}_5 = \overline{4}_5$
15	$\overline{2}_5$	$\overline{4}_5$	$\overline{4}_5 + \overline{0}_5 + \overline{0}_5 = \overline{4}_5$
17	$\overline{4}_5$	$\overline{1}_5$	$\overline{4}_5 + \overline{4}_5 + \overline{3}_5 = \overline{1}_5$
19	$\overline{0}_5$	$\overline{0}_5$	$\overline{1}_5 + \overline{4}_5 + \overline{0}_5 = \overline{0}_5$
21	$\overline{1}_5$	$\overline{1}_5$	$\overline{0}_5 + \overline{0}_5 + \overline{1}_5 = \overline{1}_5$

The structure of the Fibonacci sequence in Z_5 above shows that this square sum depends on the constraints of the squares which occur only in Classes $\bar{0}_5$, $\bar{1}_5$ and $\bar{4}_5$, and the sums are also confined to these classes in harmony with this square (Table 1.3.5).

The result

$$\lim_{n\to\infty} F_{n+6} / F_n = 8\varphi + 5$$

from (Leyendekkers and Shannon 2014) was obtained from the above characteristics of the sequence. Moreover, the Class of the sum of ten consecutive integers is the same as the class of the seventh number in the ten. The seventh number times 11 equals the sum of the ten. This is consistent with $11 \in \bar{1}_5$ and $\bar{1}_5 \times \bar{a}_5 = \bar{a}_5$ (Table 6). Note that the RED of the sum is the same as the RED of the seventh number, and since the RED of 11 is 1 it is the only integer to satisfy.

Table 1.3.6. Class structure in sets of 10 integers

Range of n	Class of sum	Class of 7th Integer, N_7	Class of $\bar{1}_5 \times N_7$
1 – 10	$\bar{3}_5$	$\bar{3}_5$	$\bar{3}_5$
2 – 11	$\bar{1}_5$	$\bar{1}_5$	$\bar{1}_5$
3 – 12	$\bar{4}_5$	$\bar{4}_5$	$\bar{4}_5$
4 – 13	$\bar{0}_5$	$\bar{0}_5$	$\bar{0}_5$
5 – 14	$\bar{4}_5$	$\bar{4}_5$	$\bar{4}_5$
6 – 15	$\bar{4}_5$	$\bar{4}_5$	$\bar{4}_5$
7 – 16	$\bar{3}_5$	$\bar{3}_5$	$\bar{3}_5$
8 – 17	$\bar{2}_5$	$\bar{2}_5$	$\bar{2}_5$
9 – 18	$\bar{0}_5$	$\bar{0}_5$	$\bar{0}_5$
10 – 19	$\bar{2}_5$	$\bar{2}_5$	$\bar{2}_5$

Table 1.3.7. Periodicities of Fibonacci REDs

Class of F_n	F_n^*	n^*	n	Δn
$\overline{0}_5$	0	0	30, 60, 90, 120, 150	30, 30, 30, 30
	5	5	15, 45, 75, 105, 135	30, 30, 30, 30
$\overline{1}_5$	6	1	21, 81, 141, 201	60, 60, 60, 60
		2	42, 102, 162, 222	60, 60, 60, 60
		8	48, 108, 168, 228	60, 60, 60, 60
		9	39, 99, 159, 219	60, 60, 60, 60
	1	1	1, 41, 61, 101, 121	40, 20, 40, 20
		2	2, 22, 62, 82, 122	20, 40, 20, 40
		8	8, 28, 68, 88, 128	20, 40, 20, 40
		9	19, 59, 79, 119, 139	40, 20, 40, 20
$\overline{2}_5$	2	3	3, 63, 123, 183	60, 60, 60, 60
		4	54, 114, 174, 234	60, 60, 60, 60
		6	36, 96, 156, 216	60, 60, 60, 60
		7	57, 117, 177, 237	60, 60, 60, 60
	7	3	23, 43, 83, 103, 143	20, 40, 20, 40
		4	14, 34, 74, 94, 134	20, 40, 20, 40
		6	16, 56, 76, 116, 136	40, 20, 40, 20
		7	17, 37, 77, 97, 137	20, 40, 20, 40
$\overline{3}_5$	8	3	33, 93, 153, 213	60, 60, 60, 60
		4	24, 84, 144, 204	60, 60, 60, 60
		6	6, 66, 126, 186	60, 60, 60, 60
		7	27, 87.147, 207	60, 60, 60, 60
	3	3	13, 53, 73, 113, 133	40, 20, 40, 20
		4	4, 44, 64, 104, 124	40, 20, 40, 20
		6	26, 46, 86, 106, 146	20, 40, 20, 40
		7	7, 47, 67, 107, 127	40, 20, 40, 20
$\overline{4}_5$	4	1	51, 111, 171, 231	60, 60, 60, 60
		2	12, 72, 132, 192	60, 60, 60, 60
		8	18, 78, 138, 198	60, 60, 60, 60
		9	9, 69, 129, 189	60, 60, 60, 60
	9	1	11, 31, 71, 91, 131	20, 40, 20, 40
		2	32, 52, 92, 112, 152	20, 40, 20, 40
		8	38, 58, 98, 118, 158	20, 40, 20, 40
		9	29, 49, 89, 109, 149	20, 40, 20, 40

The class structure of the 7th number in each set of ten integers is:

$\bar{3}_5$ $\bar{1}_5$ $\bar{4}_5$ $\bar{0}_5$ $\bar{4}_5$ $\bar{4}_5$ $\bar{3}_5$ $\bar{2}_5$ $\bar{0}_5$ $\bar{2}_5$

which corresponds to F_7 on the F_n class pattern above.

A RED periodicity of 60 for integers was discovered in general in 1774 by Joseph Louis Lagrange (Livio 2002). However, this periodicity pattern is more complicated than previously assumed for the Fibonacci sequence. For even REDs the interval is 60, but for odd REDs the intervals can be 20 or 40 which indeed sum to 60, and for Class $\bar{0}_5$ the intervals are 30 (Table 1.3.7).

Finally, the structure of the F_n^2 sequence is

$$\bar{1}_5\,\bar{1}_5\,\bar{4}_5\,\bar{4}_5\,\bar{0}_5\,\bar{4}_5\,\bar{4}_5\,\bar{1}_5\,\bar{1}_5\,\bar{0}_5\,\bar{1}_5\,\bar{1}_5\,\bar{4}_5\,\bar{4}_5\,\bar{0}_5\,\bar{4}_5\,\bar{4}_5$$

which follows from the restricted distribution of the squares in Z_5. This simple structure facilitates the formation of Pythagorean triples from F_n. and known results such as

$$F_{2n-1} = F_n^2 + F_{n-1}^2, n > 1, \quad \text{and} \quad F_{n+2} = 1 + \sum_{j=1}^{n} F_j \ .$$

These can also be related to the structure which we shall do in this book as well as links with primes and the golden section. Mario Livio's book provides a vivid picture of Fibonacci's world, the ramifications of the Fibonacci sequence, and the wider ramifications of the liberating arts! "A curriculum based on great books, ordered by the chronological development of Western civilisation and a treatment of its great themes, approached from the perspectives of multiple disciplines, will ensure that the sense of being part of a shared quest will be genuinely experienced by students and staff, making good conversations and deep inquiry possible. Such programs in Western civilisation within our universities, based on a genuine communal desire to discover 'the nature of things' (to recall the title of Lucretius's great poem), may even be an essential step for any genuine restoration of civic and political discourse in our time" (McInerney 2017).