Received August 3, 2017; | Revised ; | Accepted

DOI: xxx/xxxx

RESEARCH ARTICLE

Novel frameworks for the design of fault tolerant control using optimal sliding mode control

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Summary

This paper describes two schemes for fault tolerant control using a novel optimal sliding mode control, which can also be employed as actuator redundancy management for over-actuated uncertain linear systems. By using the effectiveness level of the actuators in the performance indexes, two schemes for redistributing the control effort among remaining (redundant or non-faulty) set of actuators are constructed based on an $\mathscr{H}_2$ based optimal sliding mode control. In contrast to the current sliding mode fault tolerant control design schemes, in this new method the level of control effort required to maintain sliding is penalised. The proposed method for the design of optimal sliding mode fault tolerant control is implemented in two stages. In the first stage, a state feedback gain is derived using an LMI-based scheme that can assign a number of the closed-loop eigenvalues to a known value whilst satisfying performance specifications. The sliding function matrix related to the particular state feedback derived in the first stage is obtained in the second stage. The difference between the two schemes proposed for sliding mode fault tolerant control is that the second one includes a separate control allocation module which makes it easier to apply actuator constraints to the problem. Moreover, it will be shown that with the second scheme, we can deal with actuator faults or even failures without controller reconfiguration. We further discuss the advantages and disadvantages of the two schemes in more details. The effectiveness of the proposed schemes are illustrated with a flight control example.

KEYWORDS:
$\mathscr{H}_2$ synthesis, partial eigenstructure assignment, control allocation, fault tolerant control, sliding surface selection.

1 | INTRODUCTION

Actuator or effector redundancy is a critical issue in most systems whose safety is crucial; e.g. passenger aircraft and modern fighter aircraft, which should be considered when designing controllers (1). This can introduce a design freedom for constructing a control system, which can handle actuator faults and failures while ensuring the closed-loop stability as well as having an acceptable performance. This is referred to as fault tolerant control (FTC). One possible approach is to employ an optimal control scheme (e.g. $\mathscr{H}_2/\mathscr{H}_\infty$), whose control input weighting matrix is modified based on the identified actuator faults and failures, to design the controller and control distribution simultaneously (1). This approach can basically be seen as a reconfigurable control system which compensates for actuator faults and failures in real-time, and thus, it requires to derive new control configurations in an on-line manner while guaranteeing system stability and minimising the performance degradation of the closed-loop system relative to non-faulty situation. Control allocation (CA) is another approach which can be employed to design control distribution, whereas different control strategies can be exploited for handling actuator faults (2, 3). In CA method, the regulation task and the control distribution task are performed by two separate sections, which means that the control design is modular based. The first module determines the total control efforts required to stabilise the closed-loop dynamics and ensure specified control performances. The second module will distribute the control efforts among the actuators according to the information received from fault detection and isolation section. An extensive literature exists for control allocation strategy in which different algorithms and applications are discussed (1, 4, 5, 6, 7, 8, 9). For example, in (10), based on linear and quadratic programming, two schemes are proposed for CA. A comparison between optimal control design and CA for distributing control effort among redundant actuators is given in (1). In (11), a specific method for CA is proposed in which actuators are divided into two separate subsets (primary and secondary actuators). In this method, referred to as daisy chaining, if the primary actuators reach saturation the control system will then exploit the secondary actuators. While most of the above-cited references lies within the scope of aerospace control and marine vehicles control, similar manner is also used in yaw stability control for cars (12) as well as bio-mechanical muscle control (13).

Sliding mode control (SMC) is a control method which, due to its robustness properties against matched uncertainties, has progressively been used in different applications (14, 15, 16, 17, 18, 19). The combination of SMC and CA has been considered in the literature. In (20) and (21), this idea is studied through practical examples as an FTC method. Moreover, having enough actuator redundancy in the control system, it is stated in (22) that SMC schemes can handle the total actuator failures. Nevertheless, the work in (22) assumes exact duplication of actuators for achieving redundancy. Furthermore, a more rigorous framework for the design of sliding mode controller with CA, which can be seen as a sliding mode FTC (SMFTC) mechanism, is proposed in (5). It is worth noting that the SMC schemes proposed in the aforementioned references are unable to limit the available control action required for satisfying the control objective. Indeed, roughly speaking, all the traditional SMC design methods suffer from this drawback. Note that traditional SMC design methods consist of two separate stages. In the first stage, an appropriate sliding surface is chosen so that it can guarantee a reduced-order sliding motion with suitable dynamics. Many approaches have been developed for this goal; for example, pole placement and optimal quadratic (16), and linear matrix inequality (LMI) methods (23). Following this, the second stage designs a controller to persuade and retain the sliding motion. As during the switching function synthesis, there is no sense of the level of the control action required to persuade and retain sliding (17), a very impractical switching surface and thereby a control law may always be derived which requires high level of control effort to reach the switching surface and maintain there thereafter.

This paper proposes a different way for the sliding surface design in which the control effort associated with the linear part of the control law is optimised. This approach is a middle-of-the-road method in that it uses a specific partial eigenstructure assignment method to assign some arbitrary stable real eigenvalues while an appropriate sliding motion dynamics will be ensured by enforcing different Lyapunov-type constraints. It is shown that this method can be used for both over-actuated and non-over-actuated systems and can be set the stage for the design of an optimal SMFTC. The proposed method for the design of optimal SMFTC is implemented in two stages. In the first stage, a state feedback gain is derived using an LMI-based scheme that can assign a number of the closed-loop eigenvalues to a known desired value whilst satisfying performance specifications. The sliding function matrix related to the particular state feedback derived in the first stage is obtained in the second stage. This paper proposes two distinct methods for the optimal SMFTC design problem. While the first proposed SMFTC requires to reconfigure the controller to deal with faults and failures, the second SMFTC scheme includes a separate control allocation module which makes it easier to apply actuator constraints to the problem and can handle faults and failures without controller reconfiguration. This paper further discusses the advantages and disadvantages of the two schemes in more details. The effectiveness of the proposed schemes are illustrated with a flight control example. The advantages of the proposed approach for the design of SMFTC compared to all the aforementioned references are threefold: *i*) it can set the stage for the design of SMFTC while the level of control efforts is taken into account; *ii*) it makes it possible to integrate several Lyapunov-type constraints, e.g. regional pole placement constraints, in the SMFTC design problem; *iii*) the controller can be computed in a numerically very efficient method.

The structure of the paper is as follows. Section 2 is devoted to the problem statement and preliminaries. Sections 3 and 4 explain respectively Approaches 1 and 2 for the design of $\mathscr{H}_2$ based SMFTC. In Section 5, we show that the two design approaches can offer precisely the same design freedom and further given the parameters of the second approach we present how to set the parameters of the first approach to obtain the same control law. Some comments are given in Section 6. Section 7 evaluates the proposed approaches via considering the flight control problem. Section 8 will finally conclude the paper.

Notation: herm(Σ), where Σ is a square matrix, stands for $\Sigma + \Sigma^*$ where Σ^* denotes the transpose conjugate of Σ.

2 | PROBLEM STATEMENT AND PRELIMINARIES

Consider the following uncertain linear time invariant (LTI) continuous-time system:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B_2\Phi(t)[u(t) + f(x,u,t)],\\ z_2(t) &= C_2x(t) + D_2(t)u(t),\end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $z_2(t) \in \mathbb{R}^q$ are the state vector, control input vector and $\mathscr{H}_2$ performance output vector of the system, respectively. Moreover, the matrix $\Phi(t) := \mathrm{diag}(\phi_1(t), \cdots, \phi_m(t))$ is referred to as the effectiveness gain (27) with $\phi_i(t)$ denoting a scalar satisfying $0 \le \phi_i(t) \le 1$ and is a slowly varying piecewise constant sequence representing the effectiveness of the i-th actuator; i.e. $\phi_i(t) = 1$ implies that the i-th actuator is fault-free, whilst $0 < \phi_i(t) < 1$ represents a fault in the i-th actuator, and $\phi_i(t) = 0$ shows the actuator failure. Assuming the system in Equation (1) involves multiplicative faults, an actuator fault reconstruction framework such as the one in (24) or Kalman filter-based fault reconstruction frameworks (25) can be employed to compute $\Phi(t)$ in a real-time manner.
It is also assumed that the matrices A and C_2 in Equation (1) are constant and have appropriate dimensions. The unknown signal $f(x,u,t) : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_+ \to \mathbb{R}^m$ denotes matched uncertainty in Equation (1) whose Euclidean norm is bounded by a known function $\varrho(x,u,t)$.

Remark 1. Let us reformulate the system in Equation (1) as

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B_2[\bar{u}(t) + \bar{f}(x,u,t)],\\ z_2(t) &= C_2x(t) + D_2\Phi^{-1}(t)\bar{u}(t),\end{aligned} \tag{2}$$

where

$$\begin{aligned}\bar{u}(t) &= \Phi(t)u(t),\\ \bar{f}(t) &= \Phi(t)f(t).\end{aligned} \tag{3}$$

As can be seen from the formulation above, although during the design of the control law $u(t)$ in Equation 1, the matrix $\Phi(t)$ can be assumed as a part of input distribution matrix B_2, i.e. $B_\phi(t) \triangleq B_2\Phi(t)$, there is another alternative that is introducing a new control signal, i.e $\bar{u}(t) := \Phi(t)u(t)$ and reformulating the system description as Equation 2. In such a case, the control weighting matrix will be $D_\phi := D_2\Phi^{-1}$. Note that if $\phi_i \to 0$, then $\phi_i^{-1} \to \infty$, and hence, the associated component $\bar{u}_i$ is weighted heavily in the $\mathscr{H}_2$ performance output.

We further assume that the matrix B_2 does not have full column rank (i.e. $\mathrm{rank}(B_2) = l < m$), and we let

$$B_2 = \bar{B}_2B, \tag{4}$$

where $\bar{B}_2 \in \mathbb{R}^{n\times l}$, $B \in \mathbb{R}^{l\times m}$ and $\mathrm{rank}(\bar{B}_2) = \mathrm{rank}(B) = l$. Now, we construct the following system descriptor:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + \bar{B}_2[\tilde{u}(t) + \tilde{f}(x,u,t)],\\ z_2(t) &= C_2x(t) + \tilde{D}_2(t)\tilde{u}(t),\\ \tilde{u}(t) &= B\Phi(t)u(t),\\ \tilde{f}(x,u,t) &= B\Phi(t)f(x,u,t),\end{aligned} \tag{5}$$

where $\tilde{D}_2(t)$ is a matrix to be discussed later. Note that $\tilde{u}(t) \in \mathbb{R}^l$ in the system above is known as virtual control input and can be regarded as the total control effort generated by the system actuators (1). Hence, we may obtain

$$u(t) = (B\Phi(t))^\dagger\tilde{u}(t) = \Phi(t)B^T(B\Phi^2(t)B^T)^{-1}\tilde{u}(t). \tag{6}$$

This paper aims to design two SMFTCs for the system in Equation (1). While in the first approach, an $\mathscr{H}_2$ based SMC is designed directly in terms of u, the second approach designs an $\mathscr{H}_2$ based SMC in terms of virtual control input $\tilde{u}$ and then the obtained result is mapped onto u using control allocation strategy.

Assumption 1. The control system in Equation 1 remains controllable in the event of failures and faults, i.e. the controllability matrix associated to the matrix pair $(A, B_2\Phi(t))$ is full rank for different $\Phi(t)$.

Note that nearly most of the work in the literature of fault tolerant control implicitly assume that the underlying control systems still remain controllable in the event of failures and faults; cf. (1, 4, 5).

Assumption 2. The rank of the control input distribution matrix in Equation 1 remains constant in the event of failures and faults, i.e. for all $\Phi(t)$, $\mathrm{rank}(B_2\Phi(t)) = l$.

The above assumption implies that the number of failed actuators should always be less than or equal to $m-l$.

3 | APPROACH 1: DIRECT $\mathscr{H}_2$ BASED SMFTC

Consider a linear switching surface as:

$$\mathcal{S} = \{x : \ \sigma(t) \triangleq Sx(t) = 0\}, \tag{7}$$

where $S \in \mathbb{R}^{l \times n}$ is the full rank sliding matrix to be designed later so that the associated reduced order sliding motions have suitable dynamics.

Let us also consider the following controller:

$$u(t) = -(B\Phi(t))^{\dagger}(S\bar{B}_2)^{-1}(SA - \Lambda S)x(t) + \vartheta(t), \tag{8}$$

where $(B\Phi(t))^{\dagger}$ denotes the pseudo inverse of $B\Phi(t)$; i.e., $(B\Phi(t))^{\dagger} := \Phi(t)B^T(B\Phi^2(t)B^T)^{-1}$, $\Lambda \in \mathbb{R}^{l\times l}$ is a stable matrix, and $\vartheta(t) \in \mathbb{R}^l$ is used to denote the nonlinear part of the sliding mode controller which has the following form

$$\vartheta(t) = -(B\Phi(t))^{\dagger}(S\bar{B}_2)^{-1}\varrho(x,u,t)\frac{\sigma(t)}{\|\sigma(t)\|} \quad \text{if } \sigma(t) \neq 0, \tag{9}$$

in which the scalar function $\varrho(\cdot)$ satisfies $\varrho(x,u,t) \geq \left\|SB_2\Phi(t)f(x,u,t)\right\|$.

Remark 2. It can readily be shown that for all $\Phi(t) = \mathrm{diag}(\phi_1(t), \cdots, \phi_m(t))$, with $0 \leq \phi_i(t) \leq 1$, $\left\|SB_2\Phi(t)f(x,u,t)\right\|$ is bounded by a scalar function $\varrho(\cdot)$ which is finite and independent of $\Phi(t)$. One trivial choice is $\varrho(x,u,t) = \left\|SB_2\right\| \|f(x,u,t)\|$, as $0 \leq \|\Phi(t)\| \leq 1, \forall t \geq 0$.

Letting $\Lambda = \lambda I_l$, with λ a known negative scalar, the control law $u(k)$ in (8) can be reformulated as

$$u(t) = (B\Phi(t))^{\dagger}(S\bar{B}_2)^{-1}SA_{\lambda}x(t) + \vartheta(t), \tag{10}$$

where $A_\lambda = \lambda I_n - A$. Now let $f = 0$ and $\vartheta = 0$ in (1). We then assume the controller in (10) contains only the linear part, therefore

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_2\Phi(t)u(t) + B_1w(t), \\ z_2(t) &= C_2x(t) + D_2(t)u(t), \\ u(t) &= (B\Phi(t))^{\dagger}(S\bar{B}_2)^{-1}SA_{\lambda}x(t), \end{aligned} \tag{11}$$

where $w(t)$ is an artificial mismatched disturbance and the distribution matrix B_1 is of appropriate dimension. The objective here is to find a sliding matrix S so that the resulting reduced order motion, when restricted to $\mathcal{S}$, is stable and meets $\mathscr{H}_2$ performance specification. In order to design a direct $\mathscr{H}_2$-based SMFTC, we need to address the following two problems:

Problem 1. Blend the $\mathscr{H}_2$ problem with the eigenstructure assignment method, i.e. design a state feedback F enforcing the $\mathscr{H}_2$ constraints while ensuring $l = \mathrm{rank}(B_2\Phi(t))$ poles of the closed-loop system; i.e. $A + B_2\Phi(t)F$, are precisely located at λ.

Indeed, provided by the LMI characterisation in (A1), (A2) and (A3) in the Appendix section, the problem above can be set as an optimisation problem in the variables $X > 0$, $Z > 0$, and $\gamma > 0$:

$$\begin{aligned} &\text{minimise} \quad \gamma \\ &\text{subject to (A1), (A2), (A3),} \\ &\qquad \text{while } l \text{ poles of the closed-loop system are precisely located at } \lambda. \end{aligned} \tag{12}$$

Remark 3. It should be emphasised that the specific LMI characterisation in (A1) sets the stage for utilising different Lyapunov matrices in different LMI constraints involved in the problem. Further in (A1), the product term between the system matrix A and the Lyapunov matrix (X) disappears, and the Lyapunov matrix plays no direct role in the control gain. This feature can substantially reduce the conservatism of the quadratic approach proposed for multi-objective control synthesis schemes (26).

Problem 2. Obtain the sliding matrix S associated with the particular state feedback F, derived in Problem 1.

The above-mentioned problems are dealt with in the following two subsections.

3.1 | Solving Problem (1): Designing the linear part of the control law in (10)

Assigning $l < m$ of the closed-loop eigenvalues to a certain negative value is performed in this subsection. The problem is to partially assign the set of eigenvalues

$$\{\overbrace{\lambda, \ \cdots, \ \lambda}^{l \ times}\}, \tag{13}$$

by state feedback. This problem can be dealt with in two steps:

1) compute the base $\begin{bmatrix} M_\lambda \\ N_\lambda \end{bmatrix}$ of nullspace of $[A - \lambda I \ B_\phi]$, where $B_\phi := B_2\Phi(t)$, with conformable partitioning;
2) with arbitrary $\eta_k \in \mathbb{R}^m$, $k = 1, \ \cdots, l$, the state feedback can be derived as $F = YG^{-1}$ with

$$Y = N\Sigma_N, \ G = M\Sigma_M, \tag{14}$$

in which

$$N := [\overbrace{N_\lambda, \ \cdots, \ N_\lambda}^{l \ times}, \overbrace{I, \ \cdots, \ I}^{(n-l) \ times}],$$
$$M := [\overbrace{M_\lambda, \ \cdots, \ M_\lambda}^{l \ times}, \overbrace{I, \ \cdots, \ I}^{(n-l) \ times}],$$
$$\Sigma_N := \mathrm{diag}\left(\eta_1, \ \cdots, \ \eta_l, \ \kappa_1, \ \cdots, \ \kappa_{(n-l)}\right),$$
$$\Sigma_M := \mathrm{diag}\left(\eta_1, \ \cdots, \ \eta_l, \ \iota_1, \ \cdots, \ \iota_{(n-l)}\right) \tag{15}$$

with $\kappa_k \in \mathbb{R}^n$ and $\iota_k \in \mathbb{R}^n$. Note that only vectors η_k are related to the assignment of the l eigenvalues to λ. In other words, other vectors (κ_k and ι_k) are not exploited in the pole placement purposes and thereby can be employed to meet other Lyapunov-type constraints.

It is worth noting that we have not yet shown that the set of closed-loop eigenvalues encompasses (13).

Lemma 1. The set (13) is a subset of the closed-loop system eigenvalues, acquired by applying the state feedback $F = YG^{-1}$ with Y and G presented in (14), to the system in (1) in the absence of uncertainty, i.e. $f = 0$.

Proof. From (14), we can write

$$\begin{aligned}
&\left(A + B_\phi F\right) M_\lambda \eta_k \\
&= \left[A + B_\phi (N\Sigma_N)(M\Sigma_M)^{-1}\right] M_\lambda \eta_k \\
&= \left[A + B_\phi (N\Sigma_N)(M\Sigma_M)^{-1}\right] \left(M\Sigma_M\right) e_k \\
&= \left[A\left(M\Sigma_M\right) + B_\phi\left(N\Sigma_N\right)\right] e_k \\
&= AM_\lambda \eta_k + B_\phi N_\lambda \eta_k \\
&= \lambda M_\lambda \eta_k \qquad\qquad k = 1, \ \cdots, l,
\end{aligned}$$

where $B_\phi := B_2\Phi(t)$ and e_k denotes the canonical basis of $\mathbb{R}^n$. □

Now, the Problem 1 can be recast as an LMI program in the variables $X > 0$, $Z > 0$, Σ_M, Σ_N and $\gamma > 0$:

$$\begin{aligned} &\text{minimise} \quad \gamma \\ &\text{subject to (A1), (A2), (A3), and (14).} \end{aligned} \tag{16}$$

3.2 | Solving Problem 2: Obtaining The Switching Function Matrix

Consider the system description in (5) and the linear switching surface in (7).

3.2.1 | Method 1

We represent the first method to obtain the sliding matrix, associated with the state feedback F, in the following theorem.

Theorem 1. Assume that $(A, B_2\Phi(t))$ is a controllable matrix pair where B_2 satisfies the equality in (4). Then *i)* $\forall\ \lambda \in \mathbb{R}_{-}$, there always exists a feedback matrix F such that l of the eigenvalues of $A + B_2\Phi(t)F$ are equivalent to λ, and $A + B_2\Phi(t)F$ has l independent eigenvectors associated with λ.

ii) Define $S = [v_1, \cdots, v_l]^T$, where v_i is a left eigenvector of $A + B_2\Phi(t)F$ associated with the eigenvalue λ, then, $S(A_\lambda - B_2\Phi(t)F) = 0$ and $S\bar{B}_2$ is invertible.

Proof. *i)* Since $(A, B_2\Phi(t))$ is controllable, $(\lambda I - A, B_2\Phi(t))$ is also controllable $\forall \lambda \in \mathbb{R}_{-}$. Then, it is easy to show that it is always possible to find F such that the null space of $A_\lambda - B_2\Phi(t)F$ has dimension l, which implies that $A + B_2\Phi(t)F$ has l independent eigenvectors associated with λ.
ii) Define $S = [v_1, \cdots, v_l]^T$, simply can be shown that $S(A_\lambda - B_2\Phi(t)F) = 0$. Now, assume

$$S\bar{B}_2 := \begin{bmatrix} v_1^T \\ \vdots \\ v_m^T \end{bmatrix} \bar{B}_2 = \Gamma,$$

where $\Gamma \in \mathbb{R}^{l\times l}$. If Γ is not full rank, then there exists a nonsingular matrix Ξ such that the first row of $\Xi\Gamma$ is zero. This is equivalent to

$$\Xi \begin{bmatrix} v_1^T \\ \vdots \\ v_l^T \end{bmatrix} \bar{B}_2 := \begin{bmatrix} \tilde{v}_1^T \\ \vdots \\ \tilde{v}_l^T \end{bmatrix} \bar{B}_2 = \Xi\Gamma,$$

i.e. there exists a vector $\tilde{v}_1$ such that $\tilde{v}_1^T \bar{B}_2 = 0$. On the other hand, we know $\tilde{v}_1^T[A_\lambda - \bar{B}_2 B\Phi(t)F] = 0$, and thus

$$\operatorname{rank}\left(\begin{bmatrix} \lambda I - (A + B_2\Phi(t)F) & B_2\Phi(t) \end{bmatrix}\right) < n.$$

This is clearly in contradiction with the controllability of $(A, B_2\Phi(t))$. In other words, if there exists a left eigenvector of $A + B_2\Phi(t)F$ associated with λ that is orthogonal to $\bar{B}_2$, $(A, B_2\Phi(t))$ must be uncontrollable, which is a contradiction. □

It can be realised from the theorem above that the switching function matrix S, associated with the state feedback F, can be selected as the set of l linearly independent left eigenvectors of $A + B_2\Phi(t)F$ associated with the (arbitrarily selected) l repeated eigenvalue $\lambda \in \mathbb{R}_{-}$.

3.2.2 | Method 2

In order to obtain the sliding matrix, we propose an alternative method by directly solving the equality

$$(S\bar{B}_2)^{-1} S A_\lambda = B\Phi(t)F, \tag{17}$$

utilising an LMI optimisation approach as follows.
Note that as the spectrum of the closed-loop matrix $A + \bar{B}_2 B\Phi(t)F$ includes l repeated eigenvalues λ, the spectrum of $\tilde{A} - \lambda I_n + \bar{B}_2 B\Phi(t)F$ includes l repeated zero eigenvalue. Hence, it can be shown that there always exists a matrix S such that $S(\tilde{A} - \lambda I_n + \bar{B}_2 B\Phi(t)F) = 0$. This statement is equivalent to the equality (17), employed for obtaining the switching matrix.
As the matrix S should be chosen such that $S\bar{B}_2$ is invertible, let us suppose $S = \bar{B}_2^T P$, with P a symmetric positive definite matrix which will be obtained in this subsection. The condition in (17) can be dealt with a simple relaxation method as:

$$\text{minimise } \alpha \ \text{ subject to } \ \left\| \bar{B}_2^T P(A_\lambda - B_2\Phi(t)F) \right\| < \alpha,$$

where $\alpha > 0$ is a scalar variable and F is a given state feedback matrix, obtained in the previous subsection, ensuring l of the closed-loop eigenvalues are equal to λ. It can simply be shown that the above optimisation problem is equivalent to the following LMI minimisation problem:

$$\text{minimise } \alpha \ \text{ subject to } \ \begin{bmatrix} -\alpha I & \star \\ \bar{B}_2^T P(A_\lambda - B_2\Phi(t)F) & -\alpha I \end{bmatrix} < 0. \tag{18}$$

Hence, the $\mathscr{H}_2$ based SMFTC problem is to find the global solution of the above minimisation problem and then the switching matrix is $S = \bar{B}_2^T P$. In the case of feasibility, this problem will enforce α to be an extremely small number associated with the precision of the computational unit.

3.3 | The Summary of Approach 1

The proposed $\mathscr{H}_2$ based SMFTC is summarised in the following theorem.

Theorem 2. Assume that the optimisation problem in (16) has a solution F for some $\gamma > 0$. Then the $\mathscr{H}_2$ performance constraint $\left\|T_{wz_2}\right\|_2^2 < \gamma$ is ensured, and the resulting reduced order sliding mode dynamics, derived by the control law

$$u(t) = Fx(t) + \vartheta(t), \tag{19}$$

where $\vartheta(t)$ is the nonlinear part of the controller introduced in (9), is asymptotically stable.

Proof. Note that the state feedback F is designed such that it guarantees the stability of $A + B_2\Phi(t)F$, the $\mathscr{H}_2$ performance constraint $\left\|T_{wz_2}\right\|_2^2 < \gamma$, and further it ensures that the null space of $A_\lambda - B_2\Phi(t)F$ has dimension l; see proof of Theorem 1. The latter statement implies that $A_\lambda - B_2\Phi(t)F$ is an asymptotically stable reduced $n-l$ order dynamics. In addition, by taking the time derivative of (7), substituting $\dot{x}$ as the state equation (1), and using the controller (10), (9), we obtain that

$$\dot{\sigma}(t) = \lambda\sigma(t) - \varrho(x,u,t)\frac{\sigma(t)}{\|\sigma(t)\|} + SB_2\Phi(t)f(x,u,t). \tag{20}$$

Finally, it follows from $\left\|SB_2\Phi(t)f(x,u,t)\right\| \leq \varrho(x,u,t)$ that the reachability condition $\frac{\sigma^T\dot{\sigma}}{\|\sigma\|} < 0$ holds. □

4 | APPROACH 2: $\mathscr{H}_2$ BASED SMFTC AND CONTROL ALLOCATION

Consider the system description in Equation 5 and a switching function as:

$$\tilde{\mathscr{S}} = \{x : \ \tilde{\sigma}(t) \triangleq \tilde{S}x(t) = 0\}, \tag{21}$$

where $\tilde{S} \in \mathbb{R}^{l\times n}$ is a full rank sliding matrix to be designed specifically for the second approach of SMFTC such that the associated reduced order sliding motions have suitable dynamics and $\tilde{S}\bar{B}_2$ is invertible.

Now rather than determining the controller $u(t)$ directly by the method given in the previous section, we can determine firstly $\tilde{u}(t)$ as

$$\tilde{u}(t) = (\tilde{S}\bar{B}_2)^{-1}\tilde{S}A_\lambda + \tilde{\vartheta}(t), \tag{22}$$

where

$$\tilde{\vartheta}(t) = -(\tilde{S}\bar{B}_2)^{-1}\tilde{\varrho}(x,u,t)\frac{\tilde{\sigma}(t)}{\|\tilde{\sigma}(t)\|} \quad \text{if } \tilde{\sigma}(t) \neq 0, \tag{23}$$

where $\tilde{\varrho}(\cdot)$ is a scalar function satisfying $\tilde{\varrho}(x,u,t) \geq \left\|\tilde{S}\bar{B}_2\tilde{f}(x,u,t)\right\|$, and then derive the actual SMFTC as $u(t) = (B\Phi(t))^\dagger\tilde{u}(t)$. The latter step can be seen as a control allocation scheme which can be exploited to map the virtual control input $\tilde{u}(t)$ onto $u(t)$. Similar to the first approach proposed for designing $\mathscr{H}_2$ based SMFTC, the second approach consists of two steps.

Problem 3. Design a state feedback $\tilde{F}$ enforcing the $\mathscr{H}_2$ constraints while ensuring l poles of the virtual closed-loop system; i.e. $A + \bar{B}_2\tilde{F}$, are precisely located at λ.

Problem 4. Obtain the sliding matrix S associated with the state feedback $\tilde{F}$, derived in Problem 3.

The linear part of the virtual controller $\tilde{u}(t)$, i.e. the state feedback $\tilde{F}$ in Problem 3, can be determined by solving an optimisation problem as:

$$\text{minimise} \quad \tilde{\gamma} \tag{24}$$

$$\text{subject to} \begin{bmatrix} -(\tilde{G}+\tilde{G}^T) & \star & \star \\ A\tilde{G}+\bar{B}_2\tilde{Y}+\tilde{X}+\tilde{G} & -2\tilde{X} & \star \\ C_2\tilde{G}+\tilde{D}_2\tilde{Y} & 0 & -\tilde{\gamma}I \end{bmatrix} < 0, \tag{25}$$

$$\begin{bmatrix} -\tilde{Z} & \star \\ B_1 & -\tilde{X} \end{bmatrix} < 0, \tag{26}$$

$$\text{trace}(\tilde{Z}) < 1, \tag{27}$$

$$\tilde{Y} = \tilde{N}\tilde{\Sigma}_N, \ \tilde{G} = \tilde{M}\tilde{\Sigma}_M, \tag{28}$$

where

$$\tilde{N} := [\overbrace{\tilde{N}_\lambda, \cdots, \tilde{N}_\lambda}^{l\ times}, \overbrace{I, \cdots, I}^{(n-l)\ times}],$$
$$\tilde{M} := [\overbrace{\tilde{M}_\lambda, \cdots, \tilde{M}_\lambda}^{l\ times}, \overbrace{I, \cdots, I}^{(n-l)\ times}],$$
$$\tilde{\Sigma}_N := \mathrm{diag}\left(\tilde{\eta}_1, \cdots, \tilde{\eta}_l,\ \tilde{\kappa}_1, \cdots, \tilde{\kappa}_{(n-l)}\right),$$
$$\tilde{\Sigma}_M := \mathrm{diag}\left(\tilde{\eta}_1, \cdots, \tilde{\eta}_l,\ \tilde{\iota}_1, \cdots, \tilde{\iota}_{(n-l)}\right),$$

in which $\tilde{\eta}_k \in \mathbb{R}^l$, $\tilde{\kappa}_k \in \mathbb{R}^n$ and $\tilde{\iota}_k \in \mathbb{R}^n$, $k = 1, \cdots, l$ are arbitrary variables. Note that $\begin{bmatrix} \tilde{M}_\lambda \\ \tilde{N}_\lambda \end{bmatrix}$ is the base of nullspace of $[A - \lambda I\ \bar{B}_2]$.
Problem 4 can also be addressed by employing one of the two methods explained in Subsection 3.2, albeit by replacing B_ϕ and F by $\bar{B}_2$ and $\tilde{F}$, respectively, i.e. selecting $\tilde{S}$ as the set of l linearly independent left eigenvectors of $A + \bar{B}_2\tilde{F}$ associated with the (arbitrarily selected) l repeated eigenvalue $\lambda \in \mathbb{R}_-$, or directly solving the equality

$$(\tilde{S}\bar{B}_2)^{-1}\tilde{S}A_\lambda = \tilde{F}. \tag{29}$$

5 | RELATION BETWEEN APPROACHES 1 AND 2

We aim now to discuss the two approaches proposed for SMFTC in the previous sections.

Theorem 3. Consider Approaches 1 and 2. By assuming

$$D_2 = \tilde{D}_2 B\Phi(t), \tag{30}$$

and letting $G = \tilde{M}\tilde{\Sigma}_M$ and $Y = (B\Phi(t))^\dagger \tilde{N}\tilde{\Sigma}_N$, then the following statements hold.

- The LMIs associated with Approaches 1 and 2 are identical. Moreover, if one of the optimisation problems, used to design the switching function and thereby SMFTC, has a solution, the other one will also.
- If $F^\star$ and $\tilde{F}^\star$ are the state feedback gains from Approaches 1 and 2, respectively, then

$$\tilde{F}^\star = B\Phi(t)F^\star, \tag{31}$$

and thereby identical corresponding switching functions, i.e. $S^\star = \tilde{S}^\star$, can be derived such that

$$\tilde{u}^\star(t) = B\Phi(t)u^\star(t), \tag{32}$$

where $u^\star(t)$ and $\tilde{u}^\star(t)$ are obtained by exploiting $S^\star$ or $\tilde{S}^\star$ in (10), (9) and (22), (23), respectively, and hence the obtained system state trajectories by applying $\tilde{u}^\star(t)$ and $u^\star(t)$ are identical.

Proof. Let $F^\star$ and $\tilde{F}^\star$ be the state feedback gains corresponding respectively to the optimisation problems in (16) and (24). Letting D_2 be as in (30), the term D_2Y in (A1) can be rewritten as

$$\begin{aligned} D_2Y =& \tilde{D}_2B\Phi(t)Y = \tilde{D}_2B\Phi(t)(B\Phi(t))^\dagger\tilde{N}\tilde{\Sigma}_N \\ =& \tilde{D}_2\tilde{Y}. \end{aligned}$$

Since $G = \tilde{M}\tilde{\Sigma}_M$, $Y = (B\Phi(t))^\dagger\tilde{N}\tilde{\Sigma}_N$, $\tilde{D}_2\tilde{Y} = D_2Y$ and $\bar{B}_2\tilde{Y} = B_2\Phi(t)Y$, we conclude that the both set of LMIs in (16) and (24) are identical and so

$$\begin{aligned} \tilde{F}^\star &= \tilde{Y}\tilde{G}^{-1} = B\Phi(t)YG^{-1} \\ &:= B\Phi(t)F^\star. \end{aligned} \tag{33}$$

Suppose that $\tilde{S}^\star$ and $S^\star$ are the switching matrices associated with the state feedback gains $\tilde{F}^\star$ and $F^\star$, respectively. According to Theorem 1, it is evident that

$$S^\star(A_\lambda - \bar{B}_2B\Phi(t)F^\star) = 0,$$

and from (33) we can show that

$$S^\star(A_\lambda - \bar{B}_2\tilde{F}^\star) = 0.$$

Furthermore, as it is already guaranteed that there exists an $\tilde{S}^\star$ satisfying $\tilde{S}^\star(A_\lambda - \bar{B}_2\tilde{F}^\star) = 0$, it is therefore possible to select a unique switching matrix corresponding to both $F^\star$ and $\tilde{F}^\star$. Thereby, an identical switching function; i.e. $\sigma^\star(t) = \tilde{\sigma}^\star(t)$, can be employed in both SMFTCs. Also, as $\tilde{S}^\star = S^\star$, $\tilde{S}^\star \tilde{B}_2 \tilde{f}(x,u,t) = S^\star B_2 \Phi(t) f(x,u,t)$ and consequently the upper bounds scalar functions $\varrho(\cdot)$ and $\tilde{\varrho}(\cdot)$ can be identical. Now as $\tilde{\vartheta}^\star(t) = B\Phi(t)\vartheta^\star(t)$ and $\tilde{F}^\star = B\Phi(t)F^\star$, it can be concluded that $\tilde{u}^\star(t) = B\Phi(t)u^\star(t)$ and the state trajectories arising from the both SMFTCs are the same. $\square$

Remark 4. It is worth noting that the base $\begin{bmatrix} M_\lambda \\ N_\lambda \end{bmatrix}$ of nullspace of $\begin{bmatrix} A-\lambda I & B_\phi \end{bmatrix}$ and the base $\begin{bmatrix} \tilde{M}_\lambda \\ \tilde{N}_\lambda \end{bmatrix}$ of nullspace of $\begin{bmatrix} A-\lambda I & \tilde{B}_2 \end{bmatrix}$ are related as follows:

$$\begin{bmatrix} M_\lambda \\ N_\lambda \end{bmatrix} = \mathrm{diag}[I, B\Phi(t)] \begin{bmatrix} \tilde{M}_\lambda \\ \tilde{N}_\lambda \end{bmatrix} E, \tag{34}$$

where $E \in \mathbb{R}^{l\times m}$ is an arbitrary full rank matrix.

Theorem 3 relates the first and second approaches of optimal SMFTC designed previously. Indeed, according to Theorem 3, the both approaches have identical potential to stabilise the closed-loop dynamics and to distribute the control effort among the available actuators. However, it is worth noting that the computation time that the proposed LMI-based SMFTC require to be solved can be quite large, specifically, if the order of the system under study is high. By keeping $\tilde{D}_2$ constant in the second approach, if $\Phi(t)$ is changed by the fault reconstruction scheme, for calculating the control law, we do not need to recalculate $\tilde{u}$ in (22), but recalculate u by (6). According to Theorem 3, under some conditions this second step can be done without $\mathscr{H}_2$ performance degradation.

Additionally, by modification of $\Phi(t)$, B_ϕ in Approach 1 will be changed. This can thus have effect on the control distribution and the closed-loop behaviour of the system. However, as can be seen in Approach 2, modification of $\Phi(t)$ does not affect the closed loop dynamics. In other words, in the second approach, the control synthesis part is separated from the design of the control distribution part. In summary, the second approach that includes a CA module has the advantage of obviating the need to reconfigure the control structure in the case of faults and failures.

6 | COMMENTS

It is worth pointing out that most of the developed frameworks for the design of SMC as well as sliding mode fault tolerant control are not able to take into account the control action that is necessary to induce and maintain sliding; cf. (5, 4). This is due to this fact that in these SMC design schemes a switching function is designed in the first step so that a desired reduced order motion is achieved and then a controller is derived to steer the system states onto the sliding surface and keep them thereafter. Consequently, when synthesising the switching function, it is not possible to penalise the required control action. For instance, the optimal quadratic method used in (4) deliberately does not penalise the control effort and hence adopts the principle of cheap control. This manuscript proposed a method for the design of SMC in which a linear controller is designed while optimising an index function of system states as well as control signals and more importantly assigning some of the closed-loop eigenvalues to a known value in order to ensure the dimension of the null space of the closed-loop system is equal to the rank of the input distribution matrix. This is a useful method for the design of SMC specifically in FTC applications, as including weighting matrices can provide the possibility to obtain different control costs for different control surfaces and hence prioritise among them.

As can be seen in the second approach, in the absence of actuator constraints, the actual control signal $u(t)$ is obtained from the virtual control signal $\tilde{u}(t)$ by means of solving the following optimisation problem:

$$\min_u \|u\|,$$
$$\text{subject to } B_2\Phi u = \tilde{B}_2\tilde{u}.$$

It is well-known that the above optimisation problem has a simple explicit solution (i.e. analytical pseudo-inverse-based solution). However, in real cases, there may exist some constraints on the system actuators to be met during solving the control allocation problem. Let us assume that the input signal $u(t) \in \Gamma$, where Γ is a compact set as:

$$\Gamma \triangleq \left\{ u \in \mathbb{R}^m | \ \underline{u} \le u \le \bar{u} \right\},$$

where $\underline{u} = [\underline{u}_1, \cdots, \underline{u}_m]^T$ and $\bar{u} = [\bar{u}_1, \cdots, \bar{u}_m]^T$ and the inequalities are element-wise. Now, the control allocation problem can be set as the following optimisation problem (27, 28):

$$\min_{u,d} \|d\|_Q^2 + \|u\|_R^2,$$
$$\text{subject to } B_2\Phi u = \tilde{B}_2\tilde{u} + d, \ u \in \Gamma,$$

where $Q > 0$ and $R > 0$ are suitable weighting matrices and d is a slack variable used to penalise the difference between $B_2\Phi u$ and $\tilde{B}_2\tilde{u}$, as a perfect allocation of the virtual command may not be attainable due to actuator constraints. It is worth noting that the penalty term imposed on the control signal u in the above optimisation problem is optional and can be used to prioritise among the actuators. Note that as the main objective is to minimise the slack variable d, naturally $Q \gg R$ should be chosen.

Some computationally efficient constrained CA methods are proposed in the literature including the active set method (29) and interior point method (30). Additionally, in (28), by using multi-parametric programming approach, explicit solutions are obtained to constrained CA problems.

As stated in (4), even if no explicit rate or position limit is considered in the FTC synthesis problem, practically the difference between the commanded actuator position and the actual one is observed by the built-in fault reconstruction scheme and the SMFTC scheme proposed will then attempt to reduce the burden on the actuator involving with saturation and redistribute the control effort to other available actuators.

An advantage of using SMFTC over other linear FTC schemes proposed in the literature is that SMC has inherently the ability to cope with a certain degree of plant–model mismatch occurred due to varying operating conditions.

7 | NUMERICAL EXAMPLES

The effectiveness and application of the proposed novel schemes for the design of SMFTC are evaluated by a numerical example. Consider the B747 aircraft (4) whose 12 rigid body states can be split into two separate axes: 6 longitudinal axis states and 6 lateral and directional axes states. The same as in (4) we only consider the first four states of the lateral axis which are the roll rate p, yaw rate r, sideslip angle β, and roll angle ϕ. Considering an operating condition of 263,000 Kg, 92.6 m/s true airspeed, and 600 m altitude at 25.6 % of maximum thrust and at a 20° flap position, a linear model can be obtained. In this case, the lateral system, about the trim condition, can be represented as:

$$
\begin{aligned}
A &= \begin{bmatrix} -1.0579 & 0.1718 & -1.6478 & 0.0004 \\ -0.1186 & -0.2066 & 0.2767 & -0.0019 \\ 0.1014 & -0.9887 & -0.0999 & 0.1055 \\ 1.0000 & 0.0893 & 0 & 0 \end{bmatrix}, \\
B_2 &= \begin{bmatrix} -0.0832 & 0.0832 & -0.2285 & 0.2285 & -0.2625 & -0.0678 & 0.0678 & 0.2625 & 0.1187 & 0.0246 & 0.0140 & -0.0140 & -0.0246 \\ -0.0154 & 0.0154 & -0.0123 & 0.0123 & -0.0180 & -0.0052 & 0.0052 & 0.0180 & -0.2478 & 0.1269 & 0.0724 & -0.0724 & -0.1269 \\ 0 & 0 & 0 & 0 & 0.0017 & 0.0006 & -0.0006 & -0.001 & 0.0174 & 0.0005 & 0.0005 & -0.0005 & -0.0005 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}
\tag{35}
$$

As can be seen B_2 can be written as $\bar{B}_2 B$ where

$$\bar{B}_2 = \begin{bmatrix} I_{3\times3} \\ 0_{1\times3} \end{bmatrix}$$

and B contains the first three nonzero rows of B_2. Note that the lateral control surfaces are

$$\delta_{lat} = \begin{bmatrix} \delta_{air} & \delta_{ail} & \delta_{aor} & \delta_{aol} & \delta_{sp1-4} & \delta_{sp5} & \delta_{sp8} & \delta_{sp9-12} & \delta_r & e_1 & e_2 & e_3 & e_4 \end{bmatrix},$$

denoting aileron deflection (right and left - inner and outer)(rad), spoiler deflections (left: 1-4 and 5, right: 8 and 9-12) (rad), rudder deflection (rad) and lateral contributions to the engine pressure ratios (EPR), respectively. We let the system output be sideslip angle β, i.e. the output distribution matrix is

$$C_k = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}. \tag{36}$$

Exploiting an integral action, we include a tracking facility in the problem. Defining

$$\dot{\eta}(t) = \mathrm{r}(t) - y_k(t), \tag{37}$$

where $\mathrm{r}(t)$ is the input reference to be tracked by $y_k(t) = C_k x(t)$, with $C_k \in \mathbb{R}^{k\times n}$, and η represents the integral of the tracking error, i.e. $\mathrm{r}(t) - y_k(t)$, and introducing $\hat{x} := \left[\begin{smallmatrix} \eta \\ x \end{smallmatrix}\right]$, two augmented systems can be derived as:

$$\text{Augmented System 1:} \begin{cases} \dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}\Phi(t)u(t) + \hat{B}_1 w(t) + B_{\mathrm{r}}\mathrm{r}(t) \\ \hat{y}(t) = \hat{C}\hat{x}(t), \\ \hat{z}(t) = \hat{C}_2\hat{x}(t) + \hat{D}_2 B\Phi(t)u(t), \end{cases} \tag{38}$$

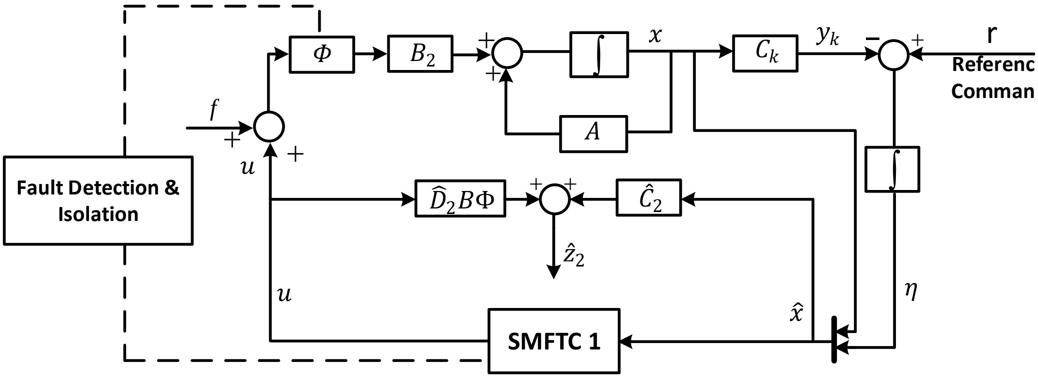

FIGURE 1 Schematic of the first proposed reference tracking SMFTC

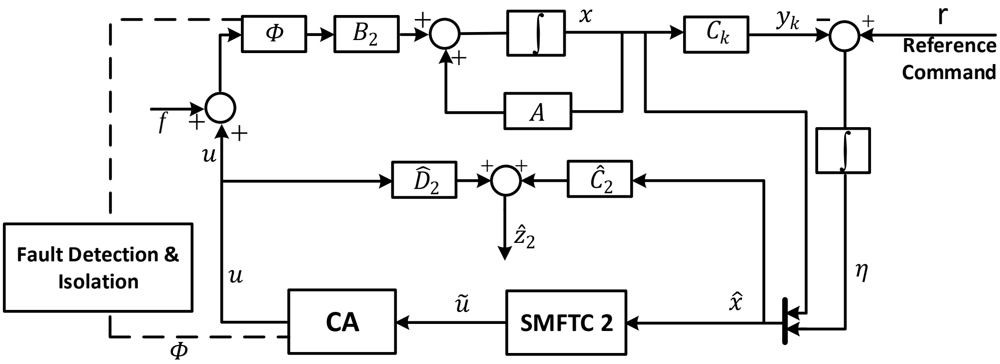

FIGURE 2 Schematic of the second proposed reference tracking SMFTC

$$\text{Augmented System 2: } \begin{cases} \dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \bar{B}\tilde{u}(t) + \hat{B}_1 w(t) + B_{\mathrm{r}}\mathrm{r}(t) \\ \hat{y}(t) = \hat{C}\hat{x}(t), \\ \hat{z}(t) = \hat{C}_2\hat{x}(t) + \hat{D}_2\tilde{u}(t), \end{cases} \tag{39}$$

with

$$\hat{A} = \begin{bmatrix} 0 & -C_k \\ 0 & A \end{bmatrix}, \ \hat{B} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \ \bar{B} = \begin{bmatrix} 0 \\ \bar{B}_2 \end{bmatrix}, \ B_{\mathrm{r}} = \begin{bmatrix} I_k \\ 0 \end{bmatrix}, \ \hat{C} = \begin{bmatrix} C_k & 0 \\ 0 & I_n \end{bmatrix}, \tag{40}$$
$$\hat{B}_1 = \begin{bmatrix} 0 \\ B_1 \end{bmatrix} = \begin{bmatrix} 0_{1\times 4} \\ I_4 \end{bmatrix}, \ \hat{C}_2 = \begin{bmatrix} \mathrm{diag}(0.1, 50, 50, 10, 10) \\ 0_{13\times 5} \end{bmatrix},$$
$$\hat{D}_2 = D_0 B^T (BB^T)^{-1}, \ D_0 = \begin{bmatrix} 0_{5\times 13} \\ \mathrm{diag}(\sqrt{2} I_8, I_5) \end{bmatrix}, \ k = 1, \ n = 4, \ m = 13,$$

Note that the first term of $\hat{C}_2$ is associated with the integral action and is less heavily weighted. In addition, the second and third diagonal elements of $\hat{C}_2$ have strongly been weighted in comparison with the fourth and fifth diagonal elements to provide an adequate quick closed-loop response in terms of the angular acceleration in roll and yaw. It is worth noting that if the matrix triplet (A, B_2, C_k) has no zeros at the origin and the matrix pair (A, B_2) is controllable, then $(\hat{A}, \hat{B})$ is controllable (4). Moreover, it can be shown that (A, B_2) is controllable if and only if $(A, \bar{B}_2)$ is controllable and thus if the matrix triplet $(A, \bar{B}_2, C_k)$ has no zeros at the origin, then $(\hat{A}, \bar{B})$ is controllable. The schematics of the two proposed reference tracking SMFTCs are shown in Figure 1 and 2 .

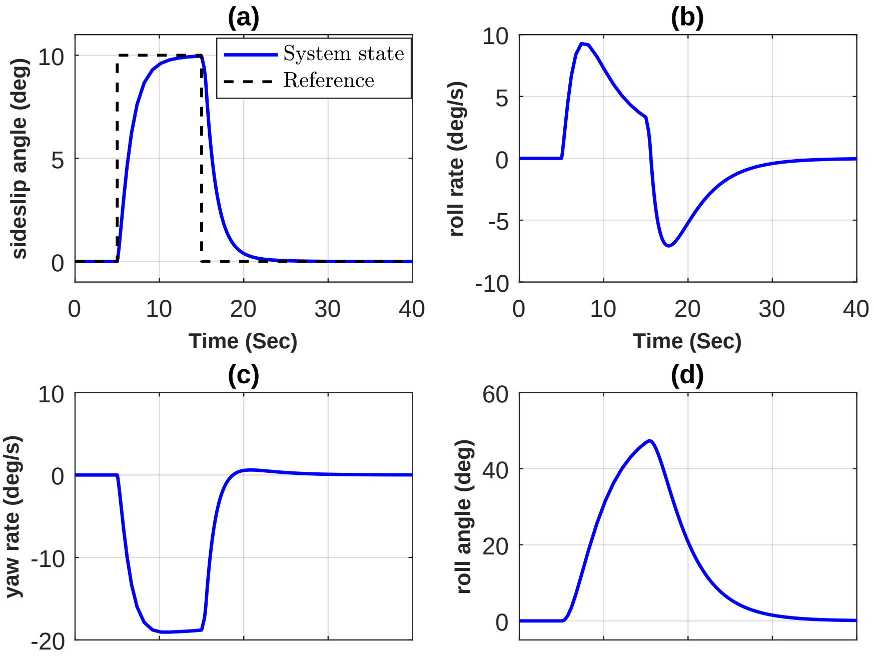

FIGURE 3 Rudder fault evolution: system states

Assuming the whole system states are available to the controller, the linear part of the control law in Approaches 1 and 2 can be considered as:

$$u_l(t) = \begin{bmatrix} F_r & F \end{bmatrix} \begin{bmatrix} \eta(t) \\ x(t) \end{bmatrix} \triangleq \mathbf{F}\hat{x}(t), \tag{41}$$

$$\tilde{u}_l(t) = \begin{bmatrix} \tilde{F}_r & \tilde{F} \end{bmatrix} \begin{bmatrix} \eta(t) \\ x(t) \end{bmatrix} \triangleq \tilde{\mathbf{F}}\bar{x}(t), \tag{42}$$

where $F \in \mathbb{R}^{m\times n}$ ($\tilde{F} \in \mathbb{R}^{l\times n}$) is the state feedback gain, $F_r \in \mathbb{R}^{m\times k}$ ($\tilde{F}_r \in \mathbb{R}^{l\times k}$) is the feed-forward gain due to the reference signal $\mathrm{r}(t)$. By selecting $\lambda = -2$ and according to the conditions in Theorem 3, the poles associated with the reduced order sliding motion in both approaches are $\{-0.7797, -0.2661\}$.

For the purpose of β tracking, we assume that at least one of the control surfaces associated to β tracking, i.e. the rudder and the two engine thrusts, is available when a fault or failure occurs. We further assume that the matched uncertainty term in (1) is as $f_i(x,u,t) = 0.2\sin(t)\beta(t)$, $i = 1,\cdots,m$. When the actuators are not subject to saturation, we solve the LMI sets given in Approaches 1 and 2 in accordance with Theorem 3 and with different $\Phi(t)$. It is seen that both Approaches 1 and 2 generate exactly the same control signals, i.e. identical state feedback gains $\tilde{\mathbf{F}} = B\Phi(t)\mathbf{F}$ and identical switching matrices $S = \tilde{S}$. We consider a step of 10 degrees for β during 5 to 15 s. Note that the discontinuity in the nonlinear control terms $\vartheta(t)$ and $\tilde{\vartheta}(t)$ in (9) and (23), respectively, are smoothed by using a sigmoidal approximation (4) as

$$\vartheta_\epsilon(t) = -(B\Phi(t))^\dagger(S\bar{B}_2)^{-1}\varrho(x,u,t)\frac{\sigma(t)}{\epsilon + \|\sigma(t)\|}, \tag{43}$$

$$\tilde{\vartheta}_\epsilon(t) = -(\tilde{S}\bar{B}_2)^{-1}\tilde{\varrho}(x,u,t)\frac{\tilde{\sigma}(t)}{\epsilon + \|\tilde{\sigma}(t)\|}, \tag{44}$$

with the scalar $\epsilon = 0.01$, $\varrho(x,u,t) = 0.5$ and $\tilde{\varrho}(x,u,t) = 0.5$.

Figure 3 shows the responses of the closed-loop system under 7 different rudder fault conditions ranging from non-faulty to total failure ($\phi_9 =$ 1, 0.75, 0.5, 0.25, 0.1, 0.05, 0). It is evident from Figure 4 that the control signal is systematically re-routed to two engine thrusts. According to the tracking responses shown in Figure 3 , no degradation occurs in the performance of the controller.

In the next simulations again a step of 10 degrees for β tracking during 5 to 15 s is considered. However, we assume that there is a saturation limit on the rudder actuator ($\delta_r \in [-10, 10]\cdot\frac{\pi}{180}$). Figures 5 and 6 show respectively the responses of the closed-loop system

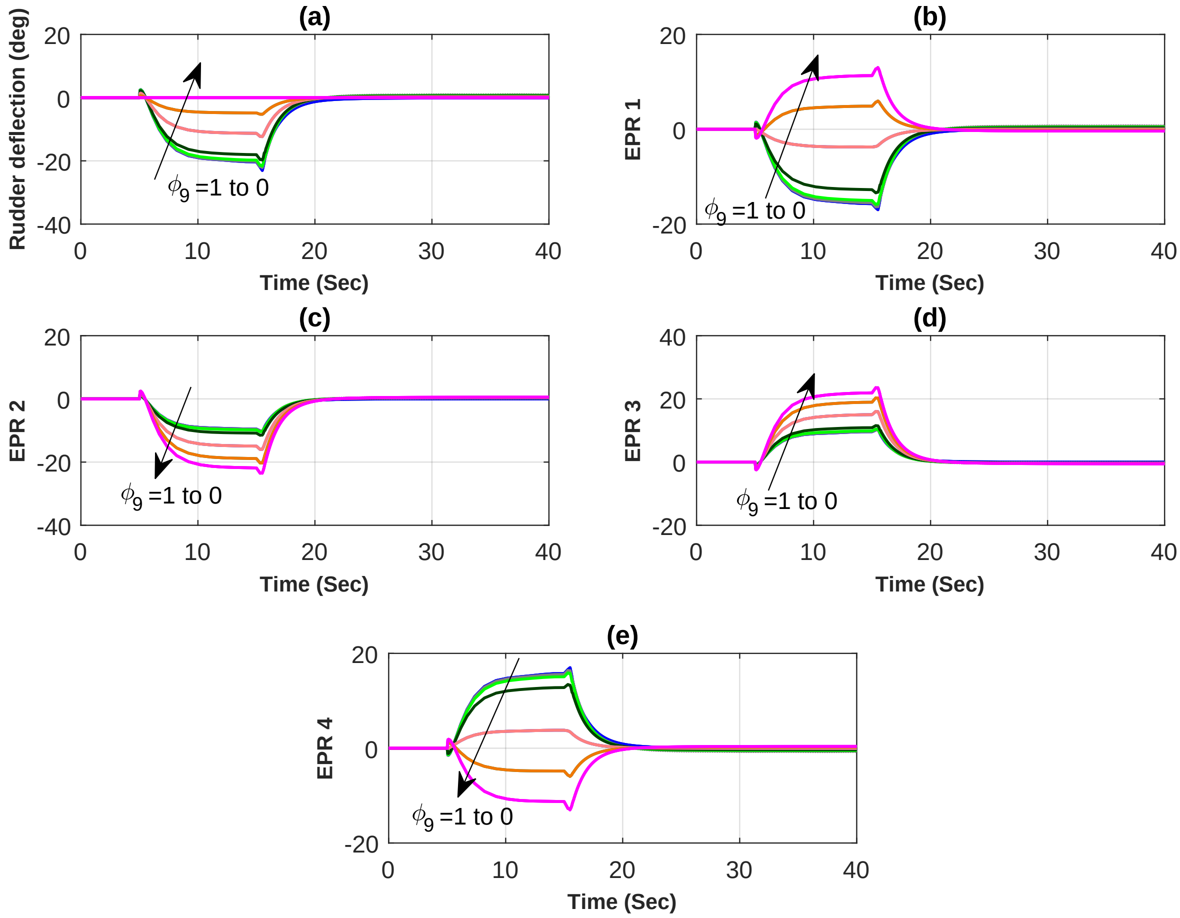

FIGURE 4 Rudder fault evolution: actuator deflection

and the control effort by considering 3 different scenarios for SMFTC: 1) using Approach 2 without constraint and exploiting the pseudo inverse CA method, 2) using Approach 2 without constraint, letting $\phi_9 = 0.08$, and exploiting the pseudo inverse CA method, 3) using Approach 2 with constraint and exploiting the interior point CA method (30). It can be seen that while the control signal in the first scenario does not satisfy the limit on the rudder actuator, the second scenario, in which the control signal is not limited directly but the effectiveness matrix Φ is revised, is able to meet the control objective while not exceeding the saturation limit. Additionally, it can be seen from these figures that although the interior point CA method can avoid the rudder to reach its limits, this leads to an overshoot in the sideslip angle variable.

8 | CONCLUSIONS

In this paper, two approaches for the design of sliding mode fault tolerant control have been proposed. The core of these two approaches is a novel optimal sliding mode control constructed based on a convex partial eigenstructure assignment method. In contrast to the current sliding mode fault tolerant control design schemes, in these two approaches the control effort required to induce and maintain sliding is taken into account. The advantage of the second SMFTC over the first one is that it can deal with actuator faults or even failures without controller reconfiguration. This is an important advantage as LMI-based control synthesis approaches are not scalable and consequently solving large LMIs may be time consuming. This fact can make Approach 1 an undesirable method for using as an online FTC. Moreover, while in both approaches an optimal scheme has been used to design SMFTC, the second approach, which contains a separate CA module, can handle actuator magnitude constraints (which has explicitly been imposed on the problem) to some extent. Nevertheless, it is worth noting that in the case of exceeding a position limit or rate limit, the occurred difference between the commanded actuator and expected

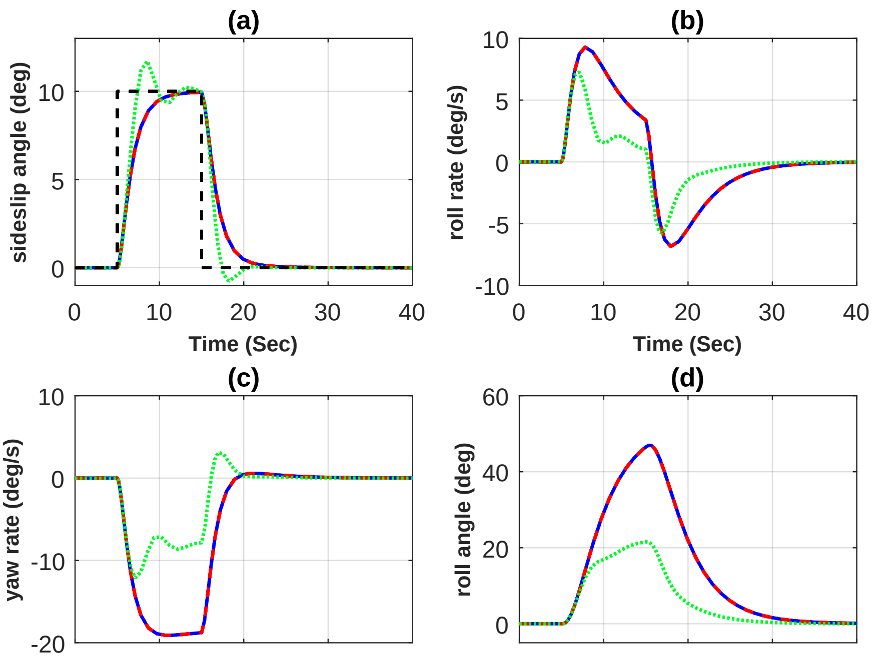

FIGURE 5 System states: Approach 2 without constraint (blue solid), Approach 2 without constraint and with $\phi_9 = 0.08$ (red dashed), and Approach 2 with constraint and with interior point CA method (green dotted).

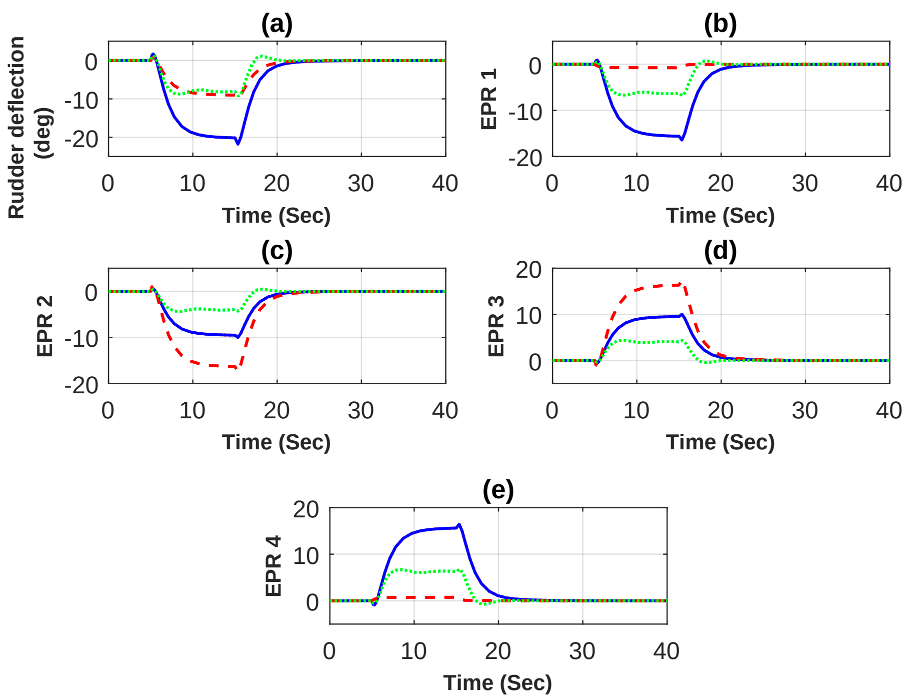

FIGURE 6 Actuator deflection: Approach 2 without constraint and with $\Phi = I_{13}$ (blue solid), Approach 2 without constraint and with $\phi_9 = 0.08$ (red dashed), and Approach 2 with constraint and with interior point CA method (green dotted).

one can be thought of as a fault. The proposed $\mathscr{H}_2$-based SMFTCs can then redistribute the control signals to other available actuators, so that the effect of the actuator saturation is diminished. The effectiveness of the proposed approaches for SMFTC design is illustrated with a flight control example.

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$\square$

APPENDIX A: $\mathscr{H}_2$ LMI CHARACTERISATION

Consider the system in Equation (11) and define $B_\phi := B_2\Phi(t)$.

Lemma 2. The following statements are equivalent:

i) $\exists\ F$ such that $A + B_\Phi F$ is stable and $\left\|(C_2 + D_2 F)\left[sI - (A + B_\Phi F)\right]^{-1} B_1\right\|_2^2 < \gamma$.

ii) $\exists\ X > 0$ and $Z > 0$ such that

$$\begin{bmatrix} AX + B_\Phi Y + XA^T + Y^T B_\Phi^T & \star \\ C_2 X + D_2 Y & -\gamma I \end{bmatrix} < 0,$$

$$\begin{bmatrix} -Z & \star \\ B_1 & -X \end{bmatrix} < 0,$$

$$\text{trace}(Z) < 1,$$

where $Y = FX$.

iii) $\exists\ X > 0,\ Z > 0$ and G such that

$$\begin{bmatrix} -(G + G^T) & \star & \star \\ AG + B_\Phi Y + X + G & -2X & \star \\ C_2 G + D_2 Y & 0 & -\gamma I \end{bmatrix} < 0, \tag{A1}$$

$$\begin{bmatrix} -Z & \star \\ B_1 & -X \end{bmatrix} < 0, \tag{A2}$$

$$\text{trace}(Z) < 1, \tag{A3}$$

where $Y = FG$,

in which $X > 0$, $Z > 0$ are s.p.d matrices, and G is a general matrix variable.

Proof. Note that the equivalence between i) and ii) is a standard $\mathscr{H}_2$ state feedback synthesis (31). Using the Schur complement, it can simply be shown that the first LMI in iii) can be reformulated as

$$\begin{bmatrix} -(G+G^T)+\gamma^{-1}(C_2G+D_2Y)^T(C_2G+D_2Y) & \star \\ AG+B_\Phi Y+X+G & -2X \end{bmatrix} < 0.$$

Note that as $G^T+G>0$, G is nonsingular. Performing the congruence transformation $\begin{bmatrix} G^{-T} & 0 \\ 0 & X^{-1} \end{bmatrix}$ in the above inequality leads to

$$\begin{bmatrix} -(\tilde{G}+\tilde{G}^T)+\gamma^{-1}(C_2^TC_2+F^TD_2^TD_2F) & \star \\ \tilde{X}(A+B_\Phi F)+\tilde{X}+\tilde{G} & -2\tilde{X} \end{bmatrix} < 0.$$

where $\tilde{G}=G^{-1}$, $\tilde{X}=X^{-1}$, $F=YG^{-1}$ and $C_2D_2^T=0$. The above inequality can be written as

$$\begin{bmatrix} \gamma^{-1}(C_2^TC_2+F^TD_2^TD_2F) & \star \\ \tilde{X}(A+B_\Phi F)+\tilde{X} & -2\tilde{X} \end{bmatrix} + \text{herm}\left(\begin{bmatrix} -I \\ I \end{bmatrix} \tilde{G} \begin{bmatrix} I & 0 \end{bmatrix} \right) < 0.$$

Based on the projection lemma, the above inequality holds iff the following inequalities are satisfied:

$$\begin{bmatrix} I \\ I \end{bmatrix}^T \begin{bmatrix} \gamma^{-1}(C_2^TC_2+F^TD_2^TD_2F) & \star \\ \tilde{X}(A+B_\Phi F)+\tilde{X} & -2\tilde{X} \end{bmatrix} \begin{bmatrix} I \\ I \end{bmatrix} < 0, \tag{A4}$$

$$\begin{bmatrix} 0 \\ I \end{bmatrix}^T \begin{bmatrix} \gamma^{-1}(C_2^TC_2+F^TD_2^TD_2F) & \star \\ \tilde{X}(A+B_\Phi F)+\tilde{X} & -2\tilde{X} \end{bmatrix} \begin{bmatrix} 0 \\ I \end{bmatrix} < 0. \tag{A5}$$

As can be seen, the inequality (A5) implies the trivial inequality $-\tilde{X} < 0$ and the equation (A4) is

$$\tilde{X}(A+B_\Phi F)+(A+B_\Phi F)^T\tilde{X}+\gamma^{-1}(C_2^TC_2+F^TD_2^TD_2F) < 0.$$

Pre- and post-multiplying the above inequality by $X=\tilde{X}^{-1}$ leads to

$$AX+B_\Phi Y+(AX+B_\Phi Y)^T+\gamma^{-1}(XC_2^TC_2X+Y^TD_2^TD_2Y) < 0,$$

where $Y=FX$. Using the Schur complement and recalling this fact that $C_2^TD_2=0$, it is readily demonstrated that the above inequality can be written as item ii). □