Solutions sets to systems of equations in hyperbolic groups are EDT0L in PSPACE

Laura Ciobanu
Heriot-Watt University, Edinburgh EH14 4AS, Scotland
l.ciobanu@hw.ac.uk

Murray Elder
University of Technology Sydney, Ultimo NSW 2007, Australia
murray.elder@uts.edu.au

Abstract

We show that the full set of solutions to systems of equations and inequations in a hyperbolic group, with or without torsion, as shortlex geodesic words, is an EDT0L language whose specification can be computed in $\mathsf{NSPACE}(n^2 \log n)$ for the torsion-free case and $\mathsf{NSPACE}(n^4 \log n)$ in the torsion case. Our work combines deep geometric results by Rips, Sela, Dahmani and Guirardel on decidability of existential theories of hyperbolic groups, work of computer scientists including Plandowski, Jeż, Diekert and others on PSPACE algorithms to solve equations in free monoids and groups using compression, and an intricate language-theoretic analysis.

The present work gives an essentially optimal formal language description for all solutions in all hyperbolic groups, and an explicit and surprising low space complexity to compute them.

2012 ACM Subject Classification Theory of computation → Formal languages and automata theory; Theory of computation → Grammars and context-free languages; Theory of computation → Complexity classes; Mathematics of computing → Combinatorics on words

Keywords and phrases Hyperbolic group, Existential theory, EDT0L language, PSPACE

Digital Object Identifier 10.4230/LIPIcs.ICALP.2019.150

Category Track B: Automata, Logic, Semantics, and Theory of Programming

Related Version https://arxiv.org/abs/1902.07349

Funding Research supported by Australian Research Council (ARC) Project DP160100486, and a Follow-On Grant from the International Centre of Mathematical Sciences (ICMS), Edinburgh
Laura Ciobanu: Supported by EPSRC grant EP/R035814/1

Acknowledgements The authors wish to thank Yago Antolín, Alex Bishop, François Dahmani, Volker Diekert, Michal Ferov and Jim Howie for extremely helpful conversations, and the anonymous reviewers for their feedback and corrections.

1 Introduction

Hyperbolic groups were introduced by Gromov in 1987 [25], and play a significant role in group theory and geometry [12, 33, 40]. Virtually free groups, small cancellation groups, and the fundamental groups of extensive classes of negative curvature manifolds are important examples (see [1] for background). In a certain probabilistic sense made precise in [26, 37, 41], almost all finitely generated groups are hyperbolic. They admit very efficient solutions to the word and conjugacy problems [21, 27, 28], and extremely nice language-theoretic properties, for example the set of all geodesics over any generating set is regular (see Lemma 16), and forms a biautomatic structure [22]. They are exactly the groups which admit context-free multiplication tables [23], and have a particularly simple characterisation in terms of rewriting systems [6, 35] (see Lemma 13).

In this paper we consider systems of equations and inequations in hyperbolic groups, building on and generalising work done in the area of solving equations over various groups and monoids in PSPACE. Starting with work of Plandowski [38], many prominent researchers have given PSPACE algorithms [7, 14, 16, 17, 18, 30, 31] to find (all) solutions to systems of equations over free monoids, free groups, partially commutative monoids and groups, and virtually free groups (that is, groups which have a free subgroup of finite index).

The satisfiability of equations over torsion-free hyperbolic groups is decidable by the work of Rips and Sela [39], who reduced the problem in hyperbolic groups to solving equations in free groups, and then calling on Makanin's algorithm [36]. Kufleitner proved PSPACE for decidability in the torsion-free case [34], without an explicit complexity bound, by following Rips-Sela and then using Plandowski's result [38]. Dahmani and Guirardel radically extended Rips and Sela's work to all hyperbolic groups (with torsion), by reducing systems of equations to systems over virtually free groups, which they then reduced to systems of *twisted* equations over free monoids [11]. In terms of describing solution sets, Grigorchuck and Lysionok gave efficient algorithms for the special case of quadratic equations [24].

Here we combine Rips, Sela, Dahmani and Guirardel's approach with recent work of the authors with Diekert [7, 14, 15] to obtain the following results.

▶ **Theorem 1** (Torsion-free). *Let G be a torsion-free hyperbolic group with finite symmetric generating set S. Let Φ be a system of equations and inequations of size n (see Section 2 for a precise definition of input size). Then the set of all solutions, as tuples of shortlex geodesic words over S, is EDT0L. Moreover there is an* $\mathsf{NSPACE}(n^2 \log n)$ *algorithm which on input Φ prints a description for the EDT0L grammar.*

▶ **Theorem 2** (Torsion). *Let G be a hyperbolic group with torsion with finite symmetric generating set S. Let Φ be a system of equations and inequations of size n (see Section 2 for a precise definition of input size). Then the set of all solutions, as tuples of shortlex geodesic words over S, is EDT0L. Moreover there is an* $\mathsf{NSPACE}(n^4 \log n)$ *algorithm which on input Φ prints a description for the EDT0L grammar.*

A corollary of Theorems 1 and 2 is that the existential theory for hyperbolic groups can be decided in $\mathsf{NSPACE}(n^2 \log n)$ for torsion-free and $\mathsf{NSPACE}(n^4 \log n)$ for groups with torsion. Another consequence of our work is that we can decide in the same space complexity as above whether or not the solution set is empty, finite or infinite (see [8]).

EDT0L is a surprisingly low language complexity for this problem. EDT0L languages are playing an increasingly useful role in group theory, not only in describing solution sets to equations in groups [7, 14, 17], but more generally [4, 5, 9].

The paper is organised as follows. We briefly set up some notation for solution sets and input size in Section 2. We then give an informal description of the entire argument for the torsion-free case in Section 3. This overview uses various concepts which are defined more carefully afterwards, but we hope that having the entire argument in one place is useful for the reader to understand the 'big picture' before descending into the details. Section 4 develops necessary material on EDT0L and space complexity. Section 5 covers the necessary background on hyperbolic groups, including the key step to obtain a full solution set (as tuples of shortlex geodesics) from a *covering solution set* (see Definition 3(iii)). In Section 6 we use Rips and Sela's *canonical representatives* (see [8, Appendix A]) in torsion-free hyperbolic groups, to reduce the problem of finding solutions in a torsion-free hyperbolic group to finding solutions in the free group on the same generators as the hyperbolic one. We show that if the input system has size n then the resulting system in the free group has size $O(n^2)$. Applying [7] produces a covering solution set in $O(n^2 \log n)$ nondeterministic space, from which we obtain the full set of solutions as shortlex geodesics in the original group, as an EDT0L language, in the same space complexity. In Section 7 we prove the general case for hyperbolic groups with torsion, following Dahmani and Guirardel who construct canonical representatives in a graph containing the Cayley graph of the hyperbolic group, and working in an associated virtually-free group.

2 Notations for equations and solution sets

Let G be a fixed group with finite symmetric generating set S. Let $\pi\colon S^* \to G$ be the natural projection map. Let $\{X_1, \dots, X_m\}$, $m \geqslant 1$, be a set of variables to which we adjoin their formal inverses X_i^{-1} and denote by $\mathcal{X}$ the union $\{X_i, X_i^{-1} \mid 1 \leqslant i \leqslant m\}$. Let $\mathcal{C} = \{a_1, \dots, a_k\} \subseteq G$ be a set of constants and

$$\Phi = \{\varphi_j(\mathcal{X}, \mathcal{C}) = 1\}_{j=1}^{h} \cup \{\varphi_j(\mathcal{X}, \mathcal{C}) \neq 1\}_{j=h+1}^{s} \tag{1}$$

be a set of s equations and inequations in G, where the length of each (in)equation is l_i. Then the total length of the equations is $n = \sum_{i=1}^{s} l_i$, and we take $|\Phi| = n$ as the input size in the remainder of the paper.

A tuple $(g_1, \dots, g_m) \in G^m$ *solves* an equation [resp. inequation] φ_j in Φ if replacing each variable X_i by g_i (and X_i^{-1} by g_i^{-1}) produces an identity [resp. inequality] in the group as follows:

$$\varphi_j(g_1, \dots, g_m, a_1, \dots, a_k) = 1 \text{ [resp. } \varphi_j(g_1, \dots, g_m, a_1, \dots, a_k) \neq 1].$$

A tuple $(g_1, \dots, g_m) \in G^m$ solves Φ if it simultaneously solves φ_j for all $1 \leqslant j \leqslant s$.

▶ **Definition 3.** *(i) The* group element solution set *to Φ is the set*

$$Sol_G(\Phi) = \{(g_1, \dots, g_m) \in G^m \mid (g_1, \dots, g_m) \text{ solves } \Phi\}.$$

(ii) Let $T \subseteq S^*$ and $\#$ a symbol not in S. The full set of T-solutions *is the set*

$$Sol_{T,G}(\Phi) = \{w_1 \# \dots \# w_m \mid w_i \in T, (\pi(w_1), \dots, \pi(w_m)) \text{ solves } \Phi\}.$$

(iii) A set $L \subseteq \{w_1 \# \dots \# w_m \mid w_i \in S^*, 1 \leqslant i \leqslant m\}$ is a covering solution set *to Φ if*

$$\{(\pi(w_1), \dots, \pi(w_k)) \mid w_1 \# \dots \# w_m \in L\} = Sol_G(\Phi).$$

3 Overview of the proof

In a free group, the equation $xy = z$ has a solution in reduced words (that is, words which do not contain factors aa^{-1} for any $a \in S$) if and only if there exist words P, Q, R with $x = PQ, y = Q^{-1}R, z = PR$ in the free monoid with involution over S ([7, Lemma 4.1]). In a hyperbolic group this direct reduction to cancellation-free equations is no longer true: a triangle $xy = z$ where x, y, z are replaced by geodesics looks as in Figure 1a.

Rips and Sela [39] proved that in a torsion-free hyperbolic group one can define certain special words called *canonical representatives* so that a system of equations of the form $X_jY_j = Z_j, 1 \leqslant j \leqslant O(n)$ has solutions which are canonical representatives with the properties that their prefixes and suffixes coincide, as shown in Figure 1b, and the inner circle is the concatenation of three words with lengths in $O(n)$. Moreover, these canonical representatives are (λ, μ)-quasigeodesics (Definition 15) where the constants λ, μ depend only on the group.

We use these facts to devise the following algorithm, presented here for the torsion-free case. We treat the hyperbolic group G with finite generating set S as a constant. On input a system of equations and inequations as in (1) of size n:

(a) Using geodesics

(b) Using canonical representatives

Figure 1 Solutions to $xy = z$ in the Cayley graph of a hyperbolic group.

1. Replace inequations by equations (by using a new variable and requiring that this variable is not trivial in the group, as explained in Section 6.3).
2. Triangulate the system, so that all equations have the form $X_jY_j = Z_j$. The size of the resulting system is still in $O(n)$. Suppose there are $q \in O(n)$ such equations.
3. Enumerate, one at a time, all possible tuples $\mathbf{c} = (c_{11}, c_{12}, c_{13}, \ldots, c_{q1}, c_{q2}, c_{q3})$ of words (say, in lex order) so that the length $\ell(c_{ji})$ with respect to S is bounded by a constant in $O(n)$. Note that the size of each tuple (the sum of the lengths of the c_{ij}) is in $O(n^2)$.
4. For each tuple $\mathbf{c}$, run Dehn's algorithm to check $c_{j1}c_{j2}c_{j3} =_G 1$ for $1 \leqslant j \leqslant q$. If this holds for all j, write down a system of $3q$ equations

$$X_j = P_jc_{j1}Q_j, Y_j = Q_j^{-1}c_{j2}R_j, Z_j = P_jc_{j3}R_j.$$

   Note that the resulting system, $\Phi_{\mathbf{c}}$, has size in $O(n^2)$.
5. We now call the algorithm of the authors and Diekert [7] to find all solutions to $\Phi_{\mathbf{c}}$ in the free group generated by S. This algorithm, on input of size $O(n^2)$, runs in $\mathsf{NSPACE}(n^2 \log n)$, and prints a description of the EDT0L grammar which generates all tuples of solutions as reduced words in S^*. Specifically it prints nodes and edges of a trim NFA which is the rational control for the EDT0L grammar (see Definition 4 below). Modify the algorithm so that the nodes printed include the label $\mathbf{c}$ which has length $O(n^2)$ (so does not affect the complexity).
6. Delete the current system stored, and move to the next tuple $\mathbf{c}$.
7. At the end, print out a new start node and ϵ edges to the start node of the NFA for the system $\Phi_{\mathbf{c}}$ for all $\mathbf{c}$ already printed.

The NFA that is printed gives an EDT0L grammar that generates a language of tuples which is a covering solution to the original system in the hyperbolic group. To obtain the full set of solutions as shortlex geodesic words we need to perform further steps. Using the facts that canonical representatives are (λ, μ)-quasigeodesics, and

- the full set of (λ, μ)-quasigeodesics, $Q_{S,\lambda,\mu}$
- the set of all pairs $\{(u, v) \in Q_{S,\lambda,\mu} \mid u =_G v\}$
- the set of all shortlex geodesics in G

are all regular, we can obtain from the covering solution an ET0L language, in the same space complexity (by Proposition 9 below), which represents the full set of solutions in shortlex geodesic words. Then finally, because of the special form of our solutions, we can apply a version of the *Copying Lemma* of Ehrenfeucht and Rozenberg [19] to show that in fact the resulting language of shortlex representatives is EDT0L in $\mathsf{NSPACE}(n^2 \log n)$.

Details for handling the case of hyperbolic groups with torsion also follows this general scheme, however finding the analogue of canonical representatives is harder in this case, so further work is required, and we describe this in Section 7.

4 E(D)T0L in PSPACE

4.1 ET0L and EDT0L languages

Let C be an alphabet. A *table* for C is a finite subset of $C \times C^*$. If (c, v) is in some table t, we say that (c, v) is a *rule* for c. A table t is *deterministic* if for each $c \in C$ there is exactly one $v \in C^*$ with $(c, v) \in t$.

If t is a table and $u \in C^*$ then we write $u \longrightarrow^t v$ to mean that v is obtained by applying rules from t to each letter of u. That is, $u = a_1 \dots a_n$, $a_i \in C$, $v = v_1 \dots v_n$, $v_i \in C^*$, and $(a_i, v_i) \in t$ for $1 \leqslant i \leqslant n$. If H is a set of tables and $r \in H^*$ then we write $u \longrightarrow^r v$ to mean that there is a sequence of words $u = v_0, v_1, \dots, v_n = v \in C^*$ such that $v_{i-1} \longrightarrow^{t_i} v_i$ for $1 \leqslant i \leqslant n$ where $r = t_1 \dots t_n$. If $R \subseteq H^*$ we write $u \longrightarrow^R v$ if $u \longrightarrow^r v$ for some $r \in R$.

▶ **Definition 4** ([2]). *Let Σ be an alphabet. We say that $L \subseteq \Sigma^*$ is an* ET0L *language if there is an alphabet C with $\Sigma \subseteq C$, a finite set $H \subset \mathcal{P}(C \times C^*)$ of tables, a regular language $R \subseteq H^*$ and a letter $c_0 \in C$ such that*

$$L = \{w \in \Sigma^* \mid c_0 \longrightarrow^R w\}.$$

In the case when every table $h \in R$ is deterministic, i.e. each $h \in R$ is in fact a homomorphism, we write $L = \{r(c_0) \in \Sigma^* \mid r \in R\}$ and say that L is EDT0L. *The set R is called the* rational control, *the symbol c_0 the* start symbol *and C the* extended alphabet.

4.2 Space complexity for E(D)T0L

Let $f \colon \mathbb{N} \to \mathbb{N}$ be a function. Recall an algorithm is said to run in $\mathsf{NSPACE}(f(n))$ if it can be performed by a non-deterministic Turing machine with a read-only input tape, a write-only output tape, and a read-write work tape, with the work tape restricted to using $O(f(n))$ squares on input of size n. The following definition formalises the idea of producing some E(D)T0L language (such as the solution set of some system of equations) in $\mathsf{NSPACE}(f(n))$, where the language is the output of a computation with input (such as a system of equations) of size n. We use the notation $L(\mathcal{A})$ to denote the language accepted by an automaton $\mathcal{A}$.

▶ **Definition 5.** *Let Σ be a (fixed) alphabet and $f \colon \mathbb{N} \to \mathbb{N}$ a function. If there is an $\mathsf{NSPACE}(f(n))$ algorithm that on input Ω of size n outputs the specification of an ET0L language $L_\Omega \subseteq \Sigma^*$, then we say that L_Ω is* ET0L in $\mathsf{NSPACE}(f(n))$.

Here the specification of L_Ω consists of:

- *an extended alphabet $C \supseteq \Sigma$,*
- *a start symbol $c_0 \in C$,*
- *a finite list of nodes of a (trim) NFA $\mathcal{A}$, labeled by some data, some possibly marked as initial and/or final,*
- *a finite list $\{(u, v, h)\}$ of edges of $\mathcal{A}$ where u, v are nodes and $h \in \mathcal{P}(C \times C^*)$ is a table*

such that $L_\Omega = \{w \in \Sigma^* \mid c_0 \to^{L(\mathcal{A})} w\}$.

A language L_Ω is EDT0L in $\mathsf{NSPACE}(f(n))$ *if, in addition, every table h labelling an edge of $\mathcal{A}$ is deterministic.*

Note that the entire print-out is not required to be in $O(f(n))$ space. Previous results of the authors with Diekert can now be restated as follows.

▶ **Theorem 6** ([7, Theorem 2.1]). *The set of all solutions to a system of size n of equations (with rational constraints), as reduced words, in a free group is EDT0L in* $\mathsf{NSPACE}(n \log n)$.

▶ **Theorem 7** ([15, Theorem 45]). *The set of all solutions to a system of size n of equations (with rational constraints), as words in a particular quasigeodesic normal form over a certain finite generating set, in a virtually free group is EDT0L in* $\mathsf{NSPACE}(n^2 \log n)$.

▶ Remark 8. In our applications below we have Ω representing some system of equations and inequations, with $|\Omega| = n$, and we construct algorithms where the extended alphabet C has size $|C| \in O(n)$ in the torsion-free case and $|C| \in O(n^2)$ in the torsion case. This means we can write down the entire alphabet C as binary strings within our space bounds. Moreover, each element (c, v) of any table we construct has v of (fixed) bounded length, so we can write down entire tables within our space bounds.

4.3 Closure properties

It is well known [32, Theorem 2.8] that ET0L is a full AFL (closed under homomorphism, inverse homomorphism, finite union, intersection with regular languages). Here we show the space complexity of an ET0L language is not affected by these operations.

▶ **Proposition 9.** *Let Σ, Γ be finite alphabets of fixed size, M an NFA of constant size with $L(M) \subseteq \Sigma^*$, and $\varphi\colon \Gamma^* \to \Sigma^*$, $\psi\colon \Sigma^* \to \Gamma^*$ homomorphisms. If $L_{\Omega_1}, L_{\Omega_2} \subseteq \Sigma^*$ are E(D)T0L in* $\mathsf{NSPACE}(f(n))$ *(on inputs Ω_1, Ω_2, respectively, with $|\Omega_1|, |\Omega_2| \in O(n)$) then*

- *(homomorphism) $\psi(L_{\Omega_1})$ is E(D)T0L in* $\mathsf{NSPACE}(f(n))$,
- *(intersection with regular) $L_{\Omega_1} \cap L(M)$ is E(D)T0L in* $\mathsf{NSPACE}(f(n))$,
- *(union) $L_{\Omega_1} \cup L_{\Omega_2}$ is E(D)T0L in* $\mathsf{NSPACE}(f(n))$,
- *(inverse homomorphism) $\varphi^{-1}(L_{\Omega_1})$ is ET0L in* $\mathsf{NSPACE}(f(n))$.

The proof is straightforward keeping track of complexity in the standard proofs [3, 10]. Note EDT0L is not closed under inverse homomorphism [20].

▶ **Proposition 10** (Projection onto a factor). *If $L_\Omega \subseteq \Sigma^*$ is E(D)T0L in* $\mathsf{NSPACE}(f(n))$ *on an input Ω of size n, and for some fixed integer s all words in L_Ω have the form $u_1\#\ldots\#u_s$ with $u_i \in (\Sigma \setminus \{\#\})^*$, and $1 \leqslant i \leqslant j \leqslant s$, then*

$$L = \{u_i\#\ldots\#u_j \mid u_1\#\ldots\#u_i\#\ldots\#u_j\#\ldots\#u_s \in L_\Omega\}$$

is E(D)T0L in $\mathsf{NSPACE}(f(n))$.

4.4 From ET0L to EDT0L

In computing the full solution set to equations as shortlex geodesic words, we will need to take inverse homomorphism. Even though in general the image under an inverse homomorphism of an EDT0L language is just ET0L, because of the special structure of solution sets we can apply the *Copying Lemma* of Ehrenfeucht and Rozenberg [19] to show the following.

▶ **Proposition 11.** *Let S be an alphabet and $h : S \to S'$ be a homomorphism of from S to a disjoint alphabet $S' = \{s' \mid s \in S\}$ defined by $h(s) = s'$. Let $\wr$ be a symbol not in $S \cup S'$ and define $h(\wr) = \wr$. Let L_1 be a set of words of the form $w \wr h(w)$ where $w \in S^*$. If L_1 is ET0L in* $\mathsf{NSPACE}(f(n))$, *then $L_2 = \{w \mid w \wr h(w) \in L_1\}$ is EDT0L in* $\mathsf{NSPACE}(f(n))$.

Proof. By [19], any nondeterministic table in the grammar for L_1 can be replaced by a finite number of deterministic tables (essentially, if nondeterminism allowed some letter $c \in C$ to produce two different results, then some word in L_1 would not have the form $w \wr h(w)$). So without loss of generality we can replace a table f containing $(c, v_1), \ldots, (c, v_k)$ by k tables f_i containing (c, v_i) only). This modification is clearly in the same space bound. Project onto the prefix using Proposition 10. ◀

5 Hyperbolic groups

5.1 Definitions

Recall the *Cayley graph* for a group G with respect to a finite symmetric generating set S is a directed graph $\Gamma(G, S)$ with vertices labeled by $g \in G$ and a directed edge (g, h) labeled by $s \in S$ whenever $h =_G gs$. Let $\ell(p)$, $i(p)$ and $f(p)$ resp. be the length, initial and terminal vertices of a path p in the Cayley graph. A path p is *geodesic* if $\ell(p)$ is minimal among the lengths of all paths q with the same endpoints. If x, y are two points in $\Gamma(G, S)$, we define $d(x, y)$ to be the length of a shortest path from x to y in $\Gamma(G, S)$.

▶ **Definition 12** (δ-hyperbolic group (Gromov)). *Let G be a group with finite symmetric generating set S, and let $\delta \geqslant 0$ be a fixed real number. If p, q, r are geodesic paths in $\Gamma(G, S)$ with $f(p) = i(q), f(q) = i(r), f(r) = i(p)$, we call $[p, q, r]$ a* geodesic triangle*. A geodesic triangle is δ-*slim *if p is contained in a δ-neighbourhood of $q \cup r$, that is, every point on one side of the triangle is within δ of some point on one of the other sides. (See for example Figure 1a.) We say (G, S) is δ-*hyperbolic *if every geodesic triangle in $\Gamma(G, S)$ is δ-slim. We say (G, S) is* hyperbolic *if it is δ-hyperbolic for some $\delta \geqslant 0$.*

It is a straightforward to show that being hyperbolic is independent of choice of finite generating set. Thus we say G is hyperbolic if (G, S) is for some finite generating set S.

▶ **Lemma 13** (Dehn presentation). *G is hyperbolic if and only if there is a finite list of pairs of words $(u_i, v_i) \in S^* \times S^*$ with $|u_i| > |v_i|$ and $u_i =_G v_i$ such that the following holds: if $w \in S^*$ is equal to the identity of G then it contains some u_i as a factor.*

This gives an algorithm to decide whether or not a word $w \in S^*$ is equal to the identity: while $\ell(w) > 0$, look for some u_i factor. If there is none, then $w \neq_G 1$. Else replace u_i by v_i (which is shorter). This procedure is called *Dehn's algorithm*.

▶ **Lemma 14.** *Dehn's algorithm runs in (linear time and) linear space.*

▶ **Definition 15** (Quasigeodesic). *For $\lambda \geqslant 1, \mu \geqslant 0$ real numbers, a path p in $\Gamma(G, S)$ is a* (λ, μ)-quasigeodesic *if for any subpath q of p we have $\ell(q) \leqslant \lambda d(i(q), f(q)) + \mu$.*

Throughout this article, we assume G is a fixed hyperbolic group which we treat as a constant for complexity purposes. We assume we are given (G, S), the constant δ, the finite list of pairs for Dehn's algorithm, and any other constants depending only on the group.

5.2 Languages in hyperbolic groups

▶ **Proposition 16.** *Let G be a fixed hyperbolic group with finite generating set S, $\lambda \geqslant 1, \mu \geqslant 0$ constants with $\lambda \in \mathbb{Q}$ and μ sufficiently large. Then the following sets are regular languages.*

1. *The set of all geodesics over S.*
2. *The set of all shortlex geodesics over S.*
3. *The set of all (λ, μ)-quasigeodesics, $Q_{S,\lambda,\mu} \subseteq S^*$.*

Furthermore, the set of all pairs of words $(u, v) \in Q^2_{S,\lambda,\mu}$ such that $u =_G v$ is accepted by an asynchronous 2-tape automaton.

See [22, 29].

5.3 Main reduction result

Here is our key technical result.

▶ **Proposition 17** (Covering to full solution sets). *Let G be a hyperbolic group with finite symmetric generating set S. Let $h : S \to S'$ and $Q_{S,\lambda,\mu}$ be as defined above, $\#, \wr$ symbols not in $S \cup S'$, $h(\#) = \#, h(\wr) = \wr$, and $\mathcal{T} \subseteq Q_{S,\lambda,\mu}$ a regular set of quasigeodesic words in bijection with G. Suppose $L_1 \subseteq (S \cup S' \cup \{\#, \wr\})^*$ consists of words of the form*

$$u_1\# \dots \#u_r \wr h(v_1)\# \dots \#h(v_r),\ \ u_i, v_i \in Q_{S,\lambda,\mu}, u_i =_G v_i, 1 \leqslant i \leqslant r.$$

If L_1 is ET0L in $\mathsf{NSPACE}(f(n))$, *then*

1. $L_Q = \{w_1\# \dots \#w_r \wr h(z_1)\# \dots \#h(z_r) \mid \exists u_1\# \dots \#u_r \wr h(v_1)\# \dots \#h(v_r) \in L_1, w_i =_G z_i =_G u_i, w_i, z_i \in Q_{S,\lambda,\mu}\}$ *is ET0L in* $\mathsf{NSPACE}(f(n))$.
2. $L_{\mathcal{T}} = \{w_1\# \dots \#w_r \mid \exists u_1\# \dots \#u_r \wr h(v_1)\# \dots \#h(v_r) \in L_1, w_i =_G u_i, w_i \in \mathcal{T}\}$ *is EDT0L in* $\mathsf{NSPACE}(f(n))$.

The proof involves a series of operations as in Proposition 9–11, see [8] for further details. Note that the set of all shortlex geodesics is a suitable choice for $\mathcal{T}$ in the proposition.

6 Reduction from torsion-free hyperbolic to free groups

Section 3 contains an overview of the general algorithm for solving equations in torsion-free hyperbolic groups. Here we provide further details, and give a proof of the soundness and completeness of our algorithm. The algorithm relies on the existence and special properties of canonical representatives, whose construction is very technical (details are provided in [8]). Their existence guarantees that the solutions of a system in a torsion-free hyperbolic group generated by S can be found by solving an associated system in the free group on S, while the fact that they are quasigeodesics (see [8, Prop. 30]) allows us to apply the results of the previous sections to obtain the EDT0L characterisation of solutions in shortlex normal form.

▶ **Proposition 18.** *Let G be a torsion-free hyperbolic group, with finite symmetric generating set S. Let Φ be a system of equations and inequations of input size n as in Section 2. Let $h : S \to S', \#, \wr$ be as in Proposition 17. Then there exist $\lambda \geqslant 1, \mu \geqslant 0$ and*

$$L = \{w_1\# \dots \#w_m \wr h(w_1)\# \dots \#h(w_m) \mid w_i \in Q_{S,\lambda,\mu}, 1 \leqslant i \leqslant m\}$$

such that $\{w \mid w\wr h(w) \in L\}$ is a covering solution for Φ, and L is EDT0L in $\mathsf{NSPACE}(n^2 \log n)$.

Applying Proposition 17 immediately gives Theorem 1.

Proof. We produce a language L of quasigeodesic words over S such that the projection of any tuple in L is in the group element solution set $\mathrm{Sol}_G(\Phi)$ (soundness). We then prove (using [39, Corollary 4.4]) that any solution in $\mathrm{Sol}_G(\Phi)$ is the projection of some tuple in L (completeness). Our proof follows the outline presented in Section 3.

1. Preprocessing

- (Remove inequations) We first transform Φ into a system consisting entirely of equations by adding a variable x_D to $\mathcal{X}$ and replacing any inequation $\varphi_j(\mathcal{X}, \mathcal{A}) \neq 1$ by $\varphi_j(\mathcal{X}, \mathcal{A}) = x_D$, with the constraint $x_D \neq_G 1$.

- (Triangulation) We transform each equation into several equations of length 3, by introducing new variables. This can always be done (see the discussion in [7, Section 4]), and it produces approximately $\sum_{i=1}^{s} l_i \in O(n)$ triangular equations with set of variables $\mathcal{Z}$ where $m \leqslant |\mathcal{Z}| \in O(n)$ and $\mathcal{X} \subset \mathcal{Z}$. From now on assume that the system Φ consists of $q \in O(n)$ equations of the form $X_jY_j = Z_j$ where $1 \leqslant j \leqslant q$.

2. Lifting Φ to the free group on S

In [39, Theorem 4.2] Rips and Sela define a constant, which they call '*bp*', that roughly bounds the circumference of the 'centres' of the triangles whose edges are canonical representatives. We denote here *bp* by ρ, and note that $\rho \in O(q) = O(n)$ depends on δ and linearly on q. As described in Section 3 we run in lex order through all possible tuples of words $\mathbf{c} = (c_{11}, c_{12}, c_{13}, \dots, c_{q1}, c_{q2}, c_{q3})$ with $c_{ji} \in S^*, \ell(c_{ji}) \leqslant \rho \in O(n)$. For each tuple $\mathbf{c}$ we use Dehn's algorithm to check $c_{j1}c_{j2}c_{j3} =_G 1$, and if this holds for all $1 \leqslant j \leqslant q$ we then construct a system $\Phi_{\mathbf{c}}$ of equations of the form

$$X_j = P_jc_{j1}Q_j,\ Y_j = Q_j^{-1}c_{j2}R_j,\ Z_j = P_jc_{j3}R_j,\ \ 1 \leqslant j \leqslant q, \tag{2}$$

which has size $O(n^2)$. In order to avoid an exponential size complexity we write down each system $\Phi_{\mathbf{c}}$ one at a time, so the space required for this step is $O(n^2)$. Let $\mathcal{Y} \supset \mathcal{Z} \supset \mathcal{X}$ be the new set of variables.

3. Some observations

We pause to make the following observations. Any solution to $\Phi_{\mathbf{c}}$ in the free group $F(S)$ is guaranteed to be a solution to Φ in the original hyperbolic group G. Thus if $S_1 \subseteq F(S)^m$ is a group element solution to $\Phi_{\mathbf{c}}$ then $\pi(S_1)$ is a group element solution to Φ in G. This will show soundness below.

Secondly, if $(g_1, \dots, g_m) \in G^m$ is a solution to Φ in the original hyperbolic group, [39, Theorem 4.2 and Corollary 4.4] (see Theorem 31 in [8]) guarantees that there exist canonical representatives $w_i \in Q_{S,\lambda,\mu}$ with $w_i =_G g_i$ for $1 \leqslant i \leqslant m$, which have reduced forms $u_i =_G w_i$ for $1 \leqslant i \leqslant m$, and our construction is guaranteed to capture any such collection of words. This will show completeness below.

Thirdly, note that the constraint that a word $w \in S^*$ must be a (λ, μ)-quasigeodesic and satisfy $w =_G 1$ implies that $\ell(w) \leqslant \mu$. Therefore we can construct a DFA $\mathcal{D}$ which accepts all words in S^* equal to 1 in the hyperbolic group G of length at most μ in constant space (using for example Dehn's algorithm). In our next step, we will use this rational constraint to handle the variable x_D added in the first step above (to remove inequalities).

Now let us complete the construction by finding the covering solution required.

4. Covering solution set

We now run the algorithm from [7] (which we will refer to as the CDE algorithm) which takes input $\Phi_{\mathbf{c}}$, which has size in $O(n^2)$, plus the rational constraint $x_D \notin L(\mathcal{D})$, plus for each $y \in \mathcal{Y}$ the rational constraint that the solution for y is a word in $Q_{S,\lambda,\mu}$. Since these constraints have constant size (depending only on the group G, not the system Φ), they do not contribute to the $O(n^2)$ size of the input to the CDE algorithm.

We make two modifications to the details of the CDE algorithm. First, every node printed by the algorithm should include the additional label $\mathbf{c}$. (This ensures the NFA we print for each system $\Phi_{\mathbf{c}}$ is distinct.) This does not affect the complexity since $\mathbf{c}$ has size in $O(n^2)$.

Second, so that we can apply Proposition 11 later, we modify the form of 'extended equations' in [7] by inserting the factor $\wr h(W)$ in the appropriate position(s). This simply increases the size of the nodes by a factor (of two).

We run the CDE algorithm to print an NFA (possibly empty) for each $\Phi_{\mathbf{c}}$, which is the rational control for an EDT0L grammar that produces all solutions as freely reduced words for elements of $F(S)$ which correspond to solutions as (λ, μ)-quasigeodesics to the same system $\Phi_{\mathbf{c}}$ in the hyperbolic group. If $(w_1, \ldots, w_m)$ is a solution in canonical representatives to Φ then $(u_1, \ldots, u_m, \ldots u_{|\mathcal{Y}|})$ will be included in the solution to $\Phi_{\mathbf{c}}$ output by the CDE algorithm, with u_i the reduced forms of w_i for $1 \leqslant i \leqslant m$. This shows completeness once we union the grammars from all systems $\Phi_{\mathbf{c}}$ together.

Adding a new start node with edges to each of the start nodes of the NFA's with label $\mathbf{c}$, we obtain a rational control for the EDT0L grammar generating L as required. The space required is exactly that required by the CDE algorithm on input $O(n^2)$, which is $\mathsf{NSPACE}(n^2 \log n)$. ◀

7 Reduction from hyperbolic with torsion to virtually free groups

In the case of a hyperbolic group G with torsion, the general approach of Rips and Sela can still be applied, but the existence of canonical representatives is not always guaranteed (see Delzant [13, Rem.III.1]). To get around this, Dahmani and Guirardel 'fatten' the Cayley graph $\Gamma(G, S)$ of G to a larger graph $\mathcal{K}$ which contains $\Gamma(G, S)$ (in fact $\Gamma(G, S)$ with midpoints of edges included), and solve equations in G by considering equalities of paths in $\mathcal{K}$. More precisely, $\mathcal{K}$ is the 1-skeleton of the barycentric subdivision of a Rips complex of G (see [8] for definitions).

▶ **Definition 19.** *Let γ, γ' be paths in $\mathcal{K}$.*

(i) We denote by $i(\gamma)$ the initial vertex of γ, by $f(\gamma)$ the final vertex of γ, and by $\overline{\gamma}$ the reverse of γ starting at $f(\gamma)$ and ending at $i(\gamma)$.

(ii) We say that γ is reduced *if it contains no backtracking, that is, no subpath of length 2 of the form $e\bar{e}$.*

(iii) We write $\gamma\gamma'$ for the concatenation of γ, γ' if $i(\gamma') = f(\gamma)$.

(iv) Two paths in $\mathcal{K}$ are homotopic if one can obtain a path from the other by adding or deleting backtracking subpaths. Each homotopy class has a unique reduced representative.

Let V be the set of all homotopy classes $[\gamma]$ of paths γ in $\mathcal{K}$ with $i(\gamma) = 1_G$, and $f(\gamma) \in G$. For $[\gamma]$, $[\gamma'] \in V$ define their product $[\gamma][\gamma'] = [\gamma^v\gamma']$, where $\gamma^v\gamma'$ denotes the concatenation of γ and the translate ${}^v\gamma'$ of γ' by $v = f(\gamma)$, and let $[\gamma]^{-1}$ be the homotopy class of ${}^{v^{-1}}\overline{\gamma}$. Then V is a group that projects onto G by the final vertex map f, that is, $f : V \twoheadrightarrow G$ is a surjective homomorphism. Moreover, since G has an action on $\mathcal{K}$ induced by the natural action on its Rips complex, V will act on $\mathcal{K}$ as well. This gives rise to an action of V onto the universal cover T (which is a tree) of $\mathcal{K}$, and [11, Lemma 9.9] shows that the quotient T/V is a finite graph (isomorphic to $\mathcal{K}/G$) of finite groups, and so V is virtually free.

We assume that the algorithmic construction (see [11, Lemma 9.9]) of a presentation for V is part of the preprocessing of the algorithm, will be treated as a constant, and will not be included in the complexity discussion.

The first step in solving a system Φ of equations in G is to translate Φ into identities between quasigeodesic paths (with start and end point in G) in $\mathcal{K}$, defined as $\mathcal{QG}_{\lambda_1,\mu_1}(V)$ in (5) in [8], paths which can be seen as the analogues of the canonical representatives from the torsion-free case. This can be done by Proposition 9.8 [11]. The second step in solving Φ is to express the equalities of quasigeodesic paths in $\mathcal{K}$ in terms of equations in the virtually free group V based on $\mathcal{K}$. Finally, Proposition 9.10 [11] shows it is sufficient to solve the systems of equations in V in order to obtain the solutions of the system Φ in G.

In the virtually free group V we will use the results from [15]. Let Y be the generating set of V and $T \subseteq Y^*$ the set of normal forms for V over Y as in [15, Remark 44, page 50], and let

$$\mathrm{Sol}_{T,V}(\Psi) = \{w_1\# \dots \# w_m) \in T^n \mid (\pi(w_1), \dots, \pi(w_m)) \text{ solves } \Psi \text{ in } V\}$$

be the language of T-solutions in V of a system Ψ of size $|\Psi| = O(k)$; by [15] the language $\mathrm{Sol}_{T,V}(\Psi)$ consists of (λ_Y, μ_Y)-quasigeodesics and is EDT0L in $\mathsf{NSPACE}(k^2 \log k)$ over Y.

▶ **Proposition 20.** *Let G be a hyperbolic group with torsion, with finite symmetric generating set S. Let Φ be a system of equations and inequations with $|\Phi| = n$ as in Section 2. Let $h : S \to S', \#, \wr$ be as in Proposition 17. Then there exist $\lambda \geqslant 1, \mu \geqslant 0$ and*

$$L = \{w_1\# \dots \# w_m \wr h(v_1)\# \dots \# h(v_m) \mid w_i, v_i \in Q_{S,\lambda,\mu}, w_i =_G v_i, 1 \leqslant i \leqslant m\}$$

such that $\{w \mid w \wr h(v) \in L\}$ is a covering solution for Φ, and L is ET0L in $\mathsf{NSPACE}(n^4 \log n)$.

Again applying Proposition 17 immediately gives Theorem 2.

Before proving Proposition 20 we need to show how one can translate between elements and words in V over the generating set Y, and elements and words in G over S via the graph $\mathcal{K}$, so that the EDT0L characterisation of languages is preserved.

▷ Notation. Let Z be some generating set of V and let $\pi \colon Z^* \to V$ be the standard projection map from words to group elements in V.

(i) For each $z_i \in Z$ there exists a unique reduced path p_i in $\mathcal{K}$ with $i(p_i) = 1_G$ and $f(p_i) \in G$; by concatenation for each word $w = z_{i_1} \dots z_{i_k}$ over Z there is then a unique path denoted

$$p_w = p_{i_1} \dots p_{i_k} \tag{3}$$

with $i(p_w) = 1_{\mathcal{K}} = 1_G$ and $f(p_w) \in G$.

(ii) For each $z_i \in Z$, assign a geodesic path γ_i in the Cayley graph $\Gamma(G, S)$ such that $i(\gamma_i) = 1_G$ and $f(\gamma_i) = f(p_i) \in G$, where p_i as in (i). Let $\sigma \colon Z^* \to S^*$ be the map/substitution given by $\sigma(z_i) = \gamma_i$; by concatenation one can associate to each word $w = z_{i_1} \dots z_{i_k}$ over Z a path in $\Gamma(G, S)$ denoted

$$\gamma_w = \gamma_{i_1} \dots \gamma_{i_k} = \sigma(w) \tag{4}$$

with $i(p_w) = 1_G$ and $f(\gamma_w) = f(w) \in G$.

(iii) There exists a unique reduced path, denoted $p_{\pi(w)}$, which is homotopic to p_w.

Proof of Proposition 20. The algorithm to produce the language of solutions for Φ is similar to that outlined in Section 3 and detailed in the proof of Proposition 18, but it applies to different groups. The triangulation of Φ and introduction of a variable with rational constraint to deal with the inequations proceeds in the same manner. Again, we suppose after preprocessing we have $q \in O(n)$ triangular equations.

Then for $\kappa \in O(n)$ as in [8, Prop 36] define $V_{\leqslant \kappa} = \{[\gamma] \in V \mid \gamma \text{ reduced and } \ell_{\mathcal{K}}(\gamma) \leqslant \kappa\}$. One lifts the system Φ in G to a finite set of systems $\Psi_{\mathbf{c}}$ in the virtually free group V, one system for each q-tuple $\mathbf{c}$ of triples (c_1, c_2, c_3) with $c_i \in V_{\leqslant \kappa}$ and such that $f(c_1c_2c_3) = 1_G$, as in [8, Prop 36]. We enumerate these tuples by enumerating triples of words (v_1, v_2, v_3) over the generating set Y of V with $\ell_Y(v_i) \leqslant \kappa_Y$, where $\kappa_Y \in O(q)$ is a constant depending on κ, as in Lemma 21(ii). By Lemma 21(ii) the tuples of path triples $(p_{v_1}, p_{v_2}, p_{v_3})$ (see (3)) in $\mathcal{K}$ contain all q-tuples of triples (c_1, c_2, c_3) with $c_i \in V_{\leqslant \kappa_Y}$, up to homotopy. Then for each triple (v_1, v_2, v_3) we check whether $f(v_1v_2v_3) = 1_G$, and this is done by checking whether $\sigma(v_1)\sigma(v_2)\sigma(v_3) =_G 1$ using the Dehn algorithm in G.

Then each system $\Psi_{\mathbf{c}}$ is obtained as in (2) in the proof of Proposition 18 and has input size $O(q^2) \in O(n^2)$ since it has $O(q)$ equations, each of length in $O(q)$, and the factors c_i inserted also have length in $O(q)$. For each system $\Psi_{\mathbf{c}}$ over V we apply the algorithm in [15] and obtain the set of solutions $\mathrm{Sol}_{T,V}(\Psi_{\mathbf{c}})$ as an EDT0L in $\mathsf{NSPACE}((q^2)^2 \log(q^2)) = \mathsf{NSPACE}(n^4 \log n)$ of (λ_Y, μ_Y)-quasigeodesics over Y.

Now let $\mathcal{Q}\mathrm{Sol}_{T,V}(\Psi_{\mathbf{c}})$ be the set of all (λ_1', μ_1')-quasigeodesics which represent solutions of $\Psi_{\mathbf{c}}$ in V over Y. By Proposition 17 this language is ET0L and by Corollary 22 it contains at least one word over Y for each solution in $\mathcal{QG}_{\lambda_1,\mu_1}(V)$.

Then $\sigma(\mathcal{Q}\mathrm{Sol}_{T,V}(\Psi_{\mathbf{c}}))$ is ET0L since ET0L languages are preserved by substitutions, and by [8, Prop 36] $\mathcal{S} = \cup_{\mathbf{c}} \sigma(\mathcal{Q}\mathrm{Sol}_{T,V}(\Psi_{\mathbf{c}}))$ contains $\mathrm{Sol}_G(\Phi)$, so it is a covering solution set of Φ. By Lemma 23 the set $\mathcal{S}$ consists of at least one (λ_G, μ_G)-quasigeodesic over S for each solution, and then by intersection with the regular set Q_{S,λ_G,μ_G} of quasigeodesics in G over S we obtain a set of solutions for Φ consisting of (λ_G, μ_G)-quasigeodesics.

Finally, we run the modified DE algorithm (inserting the additional $\wr h(W)$ and label $\mathbf{c}$ for each node printed) to print an NFA for each $\Phi_{\mathbf{c}}$ for the EDT0L grammar for $\mathrm{Sol}_{T,V}(\Psi_{\mathbf{c}})$, which we union using an extra start node as before. From the above work this grammar generates the language L as required. ◀

▶ **Lemma 21.** *(i) If $c \in V$ and the reduced path representing c in $\mathcal{K}$ is an (a, b)-quasigeodesic, then there exists a word w on Y representing c such that w is an (a', b')-quasigeodesic, where a', b' depend on a, b and Y.*

(ii) If $c \in V$ and the length of the reduced path representing c in $\mathcal{K}$ is $\leqslant L$, then there exists a word w on Y representing c such that $\ell_Y(w) \leqslant L_Y$, where L_Y depends on L and Y.

▶ **Corollary 22.** *For any element $v \in \mathcal{QG}_{\lambda_1,\mu_1}(V)$ there is a (λ_1', μ_1')-quasigeodesic word over Y representing v, where λ_1', μ_1' depend on λ_1, μ_1 and Y.*

▶ **Lemma 23.** *Let w be a (λ_1', μ_1')-quasigeodesic word over Y. Then if the reduced path $p_{\pi(w)}$ is (a, b)-quasigeodesic in $\mathcal{K}$ the (unreduced) path p_w is $(a_{\mathcal{K}}, b_{\mathcal{K}})$-quasigeodesic in $\mathcal{K}$, where $(a_{\mathcal{K}}, b_{\mathcal{K}})$ depend on a, b, λ_1', μ_1' and Y.*

Moreover, $\sigma(w)$ is a (λ_G, λ_G)-quasigeodesic over S in the hyperbolic group G, where λ_G, λ_G depend on λ_1, μ_1 and Y.

Proof. Consider the generating set $Z = Y \cup V_{\leqslant 3}$ for V and let λ_Z, μ_Z be such that any (λ_1', μ_1')-quasigeodesic over Y is (λ_Z, μ_Z)-quasigeodesic over Z. Let $M = \max\{l_{\mathcal{K}}(p_y) \mid y \in Y\}$. That is, M is the maximal length of a generator in Y with respect to the associated reduced path length in $\mathcal{K}$. We will show the statement in the lemma holds for $(a_{\mathcal{K}}, b_{\mathcal{K}}) = (a, b + M\mu_Z)$.

We say that a subpath s_w of p_w is a maximal backtrack if $p_w = ps_wp'$, s_w is homotopic to an empty path (via the elimination of backtrackings), and s_w is not contained in a longer subpath of p_w with the same property. This implies there is a point A on p_w such that s_w starts and ends at A, and such a maximal backtrack traces a tree in $\mathcal{K}$. We can then write $p_w = a_1s_1a_2 \dots s_{n-1}a_n$, where a_i are (possibly empty) subpaths of p_w and s_i are maximal backtracks; thus $p_{\pi(w)} = a_1a_2 \dots a_n$. If $l_{\mathcal{K}}(s_i) \leqslant M\mu_Z$ for all i, then the result follows immediately. Otherwise there exists an s_i with $l_{\mathcal{K}}(s_i) > M\mu_Z$, and we claim that we can write s_i in terms of a word over Z that is not a quasigeodesic, which contradicts the assumption that w is quasigeodesic.

To prove the claim, suppose $i(s_i) = f(s_i) = A$. We have two cases: in the first case $A \in G$ then $\pi(s_i) =_V 1$ and s_i corresponds to a subword v of w for which $l_{\mathcal{K}}(p_v) \geqslant M\mu_Z$. But v represents a word over Z, so $l_{\mathcal{K}}(p_v) \leqslant l_Z(v)M$, and altogether $M\mu_Z \leqslant l_{\mathcal{K}}(p_v) \leqslant l_Z(v)M$. Since $|v|_Z = 0$ and v is a (λ_Z, μ_Z)-quasigeodesic word over Z^*, $l_Z(v) \leqslant \mu_Z$, which contradicts $l_Z(v) \geqslant \mu_Z$ from above.

In the second case $A \notin G$, so take a point $B \in G$ at distance 1 from A in $\mathcal{K}$ (this can always be done), and modify the word w to get w' over Z so that $p_{w'}$ in $\mathcal{K}$ includes the backtrack $[AB, BA]$ off the path p_w. Also modify s_i to obtain a new backtrack s'_i. Clearly $\pi(p_w) = \pi(p_{w'})$ and $\pi(s_i) = \pi(s'_i)$, and s'_i becomes a maximal backtrack of $p_{w'}$ which can be written as a word over the generators Z that represents the trivial element in V. We can the apply the argument from the first case.

The fact that $\sigma(w)$ is a (λ_G, μ_G)-quasigeodesic over S in the hyperbolic group G follows from the fact that $\mathcal{K}$ and $\Gamma(G, S)$ are quasi-isometric. ◀

References

1 J. M. Alonso, T. Brady, D. Cooper, V. Ferlini, M. Lustig, M. Mihalik, M. Shapiro, and H. Short. Notes on word hyperbolic groups. In *Group theory from a geometrical viewpoint (Trieste, 1990)*, pages 3–63. World Sci. Publ., River Edge, NJ, 1991.

2 Peter R. J. Asveld. Controlled iteration grammars and full hyper-AFL's. *Information and Control*, 34(3):248–269, 1977.

3 Peter R.J. Asveld. *A Characterization of ET0L and EDT0L Languages*. Number 129 in Memorandum / Department of Applied Mathematics. Department of Applied Mathematics, University of Twente, 1976.

4 Alex Bishop and Murray Elder. Bounded automata groups are co-ET0L. In Carlos Martín-Vide, Alexander Okhotin, and Dana Shapira, editors, *Language and Automata Theory and Applications*, pages 82–94, Cham, 2019. Springer International Publishing.

5 Tara Brough, Laura Ciobanu, Murray Elder, and Georg Zetzsche. Permutations of context-free, ET0L and indexed languages. *Discrete Math. Theor. Comput. Sci.*, 17(3):167–178, 2016.

6 James W. Cannon. The theory of negatively curved spaces and groups. In *Ergodic theory, symbolic dynamics, and hyperbolic spaces (Trieste, 1989)*, Oxford Sci. Publ., pages 315–369. Oxford Univ. Press, New York, 1991.

7 Laura Ciobanu, Volker Diekert, and Murray Elder. Solution sets for equations over free groups are EDT0L languages. *Internat. J. Algebra Comput.*, 26(5):843–886, 2016. doi:10.1142/S0218196716500363.

8 Laura Ciobanu and Murray Elder. Solutions sets to systems of equations in hyperbolic groups are EDT0L in PSPACE. *arXiv e-prints*, February 2019.

9 Laura Ciobanu, Murray Elder, and Michal Ferov. Applications of L systems to group theory. *Internat. J. Algebra Comput.*, 28(2):309–329, 2018. doi:10.1142/S0218196718500145.

10 Karel Culik, II. On some families of languages related to developmental systems. *Internat. J. Comput. Math.*, 4:31–42, 1974. doi:10.1080/00207167408803079.

11 François Dahmani and Vincent Guirardel. Foliations for solving equations in groups: free, virtually free, and hyperbolic groups. *J. Topol.*, 3(2):343–404, 2010. doi:10.1112/jtopol/jtq010.

12 François Dahmani and Vincent Guirardel. The isomorphism problem for all hyperbolic groups. *Geom. Funct. Anal.*, 21(2):223–300, 2011. doi:10.1007/s00039-011-0120-0.

13 T. Delzant. L'image d'un groupe dans un groupe hyperbolique. *Comment. Math. Helv.*, 70(2):267–284, 1995. doi:10.1007/BF02566008.

14 Volker Diekert and Murray Elder. Solutions of twisted word equations, EDT0L languages, and context-free groups. In *44th International Colloquium on Automata, Languages, and Programming*, volume 80 of *LIPIcs. Leibniz Int. Proc. Inform.*, pages Art. No. 96, 14. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2017.

15 Volker Diekert and Murray Elder. Solutions to twisted word equations and equations in virtually free groups. *arXiv e-prints*, January 2017. arXiv:1701.03297.

16 Volker Diekert, Claudio Gutiérrez, and Christian Hagenah. The existential theory of equations with rational constraints in free groups is PSPACE-complete. In A. Ferreira and H. Reichel, editors, *Proc. 18th Annual Symposium on Theoretical Aspects of Computer Science (STACS'01), Dresden (Germany), 2001*, volume 2010 of *Lecture Notes in Computer Science*, pages 170–182. Springer-Verlag, 2001. doi:10.1007/3-540-44693-1_15.

17 Volker Diekert, Artur Jeż, and Manfred Kufleitner. Solutions of word equations over partially commutative structures. In *43rd International Colloquium on Automata, Languages, and Programming*, volume 55 of *LIPIcs. Leibniz Int. Proc. Inform.*, pages Art. No. 127, 14. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2016.

18 Volker Diekert, Artur Jeż, and Wojciech Plandowski. Finding all solutions of equations in free groups and monoids with involution. *Inform. and Comput.*, 251:263–286, 2016. doi:10.1016/j.ic.2016.09.009.

19 A. Ehrenfeucht and G. Rozenberg. On decomposing some ET0L languages into deterministic ET0L languages, 1974. CU-CS-045-74. Computer Science Technical Reports (44). URL: https://scholar.colorado.edu/csci_techreports/44.

20 A. Ehrenfeucht and G. Rozenberg. On inverse homomorphic images of deterministic ET0L languages. In *Automata, languages, development*, pages 179–189. North-Holland, Amsterdam, 1976.

21 David Epstein and Derek Holt. The linearity of the conjugacy problem in word-hyperbolic groups. *Internat. J. Algebra Comput.*, 16(2):287–305, 2006. doi:10.1142/S0218196706002986.

22 David B. A. Epstein, James W. Cannon, Derek F. Holt, Silvio V. F. Levy, Michael S. Paterson, and William P. Thurston. *Word processing in groups.* Jones and Bartlett Publishers, Boston, MA, 1992.

23 Robert H. Gilman. On the definition of word hyperbolic groups. *Math. Z.*, 242(3):529–541, 2002.

24 R. I. Grigorchuk and I. G. Lysionok. A description of solutions of quadratic equations in hyperbolic groups. *Internat. J. Algebra Comput.*, 2(3):237–274, 1992. doi:10.1142/S0218196792000153.

25 M. Gromov. Hyperbolic groups. In *Essays in group theory*, volume 8 of *Math. Sci. Res. Inst. Publ.*, pages 75–263. Springer, New York, 1987. doi:10.1007/978-1-4613-9586-7_3.

26 M. Gromov. Random walk in random groups. *Geom. Funct. Anal.*, 13(1):73–146, 2003. doi:10.1007/s000390300002.

27 Derek F. Holt. Word-hyperbolic groups have real-time word problem. *Internat. J. Algebra Comput.*, 10(2):221–227, 2000. doi:10.1142/S0218196700000078.

28 Derek F. Holt, Markus Lohrey, and Saul Schleimer. Compressed decision problems in hyperbolic groups. In *36th International Symposium on Theoretical Aspects of Computer Science (STACS 2019)*, volume 34 of *LIPIcs. Leibniz Int. Proc. Inform.* Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2019. doi:10.4230/LIPIcs.STACS.2019.34.

29 Derek F. Holt and Sarah Rees. Regularity of quasigeodesics in a hyperbolic group. *Internat. J. Algebra Comput.*, 13(5):585–596, 2003. doi:10.1142/S0218196703001560.

30 Artur Jeż. Recompression: a simple and powerful technique for word equations. *J. ACM*, 63(1):Art. 4, 51, 2016. doi:10.1145/2743014.

31 Artur Jeż. Word equations in nondeterministic linear space. In *44th International Colloquium on Automata, Languages, and Programming*, volume 80 of *LIPIcs. Leibniz Int. Proc. Inform.*, pages Art. No. 95, 13. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2017.

32 Lila Kari, Grzegorz Rozenberg, and Arto Salomaa. L systems. In *Handbook of formal languages, Vol. 1*, pages 253–328. Springer, Berlin, 1997.

33 Olga Kharlampovich and Alexei Myasnikov. Definable sets in a hyperbolic group. *Internat. J. Algebra Comput.*, 23(1):91–110, 2013. doi:10.1142/S021819671350001X.

34 Manfred Kufleitner. Wortgleichungen in hyperbolischen Gruppen. Diplomarbeit, Universität Stuttgart, Fakultät Informatik, Germany, Juli 2001. URL: `http://www2.informatik.uni-stuttgart.de/cgi-bin/NCSTRL/NCSTRL_view.pl?id=DIP-1922&engl=0`.

35 I. G. Lysionok. Some algorithmic properties of hyperbolic groups. *Izv. Akad. Nauk SSSR Ser. Mat.*, 53(4):814–832, 912, 1989. `doi:10.1070/IM1990v035n01ABEH000693`.

36 Gennadií Semyonovich Makanin. Equations in a free group. *Izv. Akad. Nauk SSR*, Ser. Math. 46:1199–1273, 1983. English transl. in Math. USSR Izv. 21 (1983).

37 A. Yu. Olʹ shanskiĭ. Almost every group is hyperbolic. *Internat. J. Algebra Comput.*, 2(1):1–17, 1992. `doi:10.1142/S0218196792000025`.

38 Wojciech Plandowski. Satisfiability of word equations with constants is in PSPACE. *J. ACM*, 51(3):483–496, 2004. `doi:10.1145/990308.990312`.

39 E. Rips and Z. Sela. Canonical representatives and equations in hyperbolic groups. *Invent. Math.*, 120(3):489–512, 1995. `doi:10.1007/BF01241140`.

40 Z. Sela. Diophantine geometry over groups and the elementary theory of free and hyperbolic groups. In *Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002)*, pages 87–92. Higher Ed. Press, Beijing, 2002.

41 L. Silberman. Addendum to: "Random walk in random groups" [Geom. Funct. Anal. **13** (2003), no. 1, 73–146; mr1978492] by M. Gromov. *Geom. Funct. Anal.*, 13(1):147–177, 2003. `doi:10.1007/s000390300003`.