UNIVERSITY OF TECHNOLOGY SYDNEY

DOCTORAL THESIS

Fundamental Solutions for Linear Parabolic Systems and Matrix Processes

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Declaration of Authorship

I, Alba SANTÍN GARCIA, declare that this thesis titled, "Fundamental Solutions for Linear Parabolic Systems and Matrix Processes" and the work presented in it are my own. I confirm that:

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- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
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"Mathematics is the music of reason. To do mathematics is to engage in an act of discovery and conjecture, intuition and inspiration; to be in a state of confusion—not because it makes no sense to you, but because you gave it sense and you still don't understand what your creation is up to; to have a break-through idea; to be frustrated as an artist; to be awed and overwhelmed by an almost painful beauty; to be alive, damn it."

Paul Lockhart

"Questa cosí vana prosunzione d'intendere il tutto non può aver principio da altro che dal non avere inteso mai nulla, perché, quando altri avesse esperimentato una volta sola a intender perfettamente una sola cosa ed avesse gustato veramente come è fatto il sapere, conoscerebbe come dell'infinità dell'altre conclusioni niuna ne intende."

Galileo Galilei

UNIVERSITY OF TECHNOLOGY SYDNEY

Abstract

Faculty of Science
School of Mathematical and Physical Sciences

Doctor of Philosophy

Fundamental Solutions for Linear Parabolic Systems and Matrix Processes

by Alba SANTÍN GARCIA

In this thesis we use Lie symmetry methods and integral transforms to obtain fundamental matrices for systems of PDEs of the form

$$\begin{cases} u_t = u_{xx} + g_1(x)v \\ v_t = v_{xx} + g_2(x)u \end{cases} \quad \text{and} \quad \begin{cases} v_t = \sigma x^\gamma v_{xx} + f_1(x)v_x - f_2(x)w_x \\ w_t = \sigma x^\gamma w_{xx} + f_2(x)v_x + f_1(x)w_x, \end{cases} \quad x, t > 0$$

for functions $g_i(x)$, and $f_i(x)$ satisfying some necessary conditions. We also provide the methodology to obtain these matrices for a wider range of systems.

We then turn to the Lie symmetry study of the Kolmogorov Backwards equation associated to the process of the eigenvalues of a Wishart process. We focus on 2-dimensional Wishart processes with eigenvalues $X_t > Y_t \geq 0$ for most of our work. We obtain the cosine transform of the transition density function of the difference $X_t - Y_t$, as well as some integral expressions for $E[X_t]$, $E[Y_t]$. We also obtain some bounds for the variances of X_t and Y_t and the expected values for a wide range of functions of these eigenvalues including, among many others, the expected value for any symmetric polynomial in the variables X_t, Y_t. These results are all new, to the best of our knowledge.

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During the course of my research, I have also had the chance to attend the AMSI Summer School in two occasions: in 2016 at RMIT University (Melbourne) to attend the courses of Calculus of Variations and Stochastic Modelling, and in 2017 at the University of Sydney to take a course in Harmonic Analysis. I am grateful to the UTS Faculty of Science for the student project funding and to the Australian Mathematical Sciences Institute (AMSI) for their travel grant, which gave me the opportunity to learn so much from excellent lecturers coming from all over the country.

I want to thank my fellow doctoral students for their support, understanding and encouragement in this very rewarding but also hard journey we have been on together for all these years. The shared laughs, jokes and all the evenings spent together have, without doubt, made even the most stressful times a lot more enjoyable. To the Physics/Chemistry group for adopting me as one of their own, even though they have never quite liked my equations.

Finally, I owe my deepest gratitude to my family and friends back home. To those who are still there and to those who long ago embarked on a new adventure. I would not be here now if it had not been for their support, encouragement and great efforts to help me achieve my goals during all these years. Special thanks to Marc for his unconditional support and for walking beside me on this path that

has been very steep at some points but has taken us to some places with the most rewarding views.
To all of you,

Moltes gràcies/ Muchas gracias/ Moitas grazas /Thank you.

Contents

To my family

Chapter 1

Introduction

Linear Parabolic Partial Differential Equations, such as the so called heat equation, $u_t(x,t) = \alpha u_{xx}(x,t)$, are fundamental in the modelling of many kinds of phenomena arising in areas such as Physics or Finance. In particular, they play an essential role in the study of diffusion processes (see Section 2.2) . It turns out that the transition probability density for a diffusion process is given by a fundamental solution of a particular parabolic PDE associated with the diffusion, the so called Kolmogorov backwards equation.

The transition probability density is essentially what allows us to determine the probability of the process transitioning between two states in a given time interval. An important example is the transition density for Brownian motion, which as it turns out, is given by a fundamental solution to the heat equation, widely known as the "heat kernel".

A lot of research has been done in developing methods for finding fundamental solutions for a given PDE. Some of the most widely spread techniques over the last 50 years include the obtention of such fundamental solutions as group invariant solutions of the relevant PDE. Authors like Bluman et al. [10, 7, 4, 5, 9] or Ibragimov et al. [41, 42, 43, 44, 45, 3] developed successful methods based on this group theoretic approach. They studied a wide range of boundary value problems including the heat equation, the wave equation and the Laplace equation, as well as some other examples like the one-dimensional Fokker-Planck equation, studied by Bluman in [5, 9]. Many examples of PDEs arising from financial Mathematics have also been studied using this range of techniques [32, 56].

Another approach to such problem is the reduction of the given PDE to some canonical form for which these fundamental solutions are known. Very interesting results have been published in this line of research by different authors such as Bluman [6], Goard [34] or Ibragimov [42].

The main limitations of the previously mentioned approaches often have to do with boundary conditions. Moreover, the fundamental solutions that can be obtained using these methods usually do not possess the required characteristics to be regarded as transition densities for an appropriate diffusion process.

Pioneering work by Craddock and his coauthors [17, 18, 19, 24, 25, 22, 20, 21, 26], particularly for one dimensional problems, offers an approach to the problem of finding fundamental solutions that indeed satisfy the necessary conditions to be regarded as transition densities for a given process. This new perspective relies on the fact that for a large class of PDEs, we can identify fundamental solutions with inverse integral transforms of a particular solution to the PDE, obtained through the application of an element of the Lie algebra of the PDE to a stationary solution that can be often obtained by inspection.

As an extension of the methods used for one-dimensional problems, some recent research has been conducted by Craddock and Lennox [21, 47] on the construction of explicit fundamental solutions for multi-dimensional parabolic equations of second order. In addition, in his recent research [18], Craddock also shows that fundamental solutions for certain parabolic systems of PDEs can be found by using only a scaling symmetry. However, the amount of existing work done in higher dimensional cases and systems of PDEs is very limited and it is definitely an area in which there is still room for further investigation.

Another limitation that we come across when trying to use Lie Symmetry methods for the computation of transition densities using integral transforms is that we rely on the fact that the relevant PDE has enough symmetries and that they are complex enough to allow us to obtain time-dependent solutions from stationary ones. However, this is not always the case, and there does not seem to be much research available that focuses on dealing with such cases.

The idea of this project is to follow up on the existing study of Lie symmetry methods and to extend its scope in two different directions:

- First, we wish to extend the use of these methods as a tool for computing fundamental solutions for single PDEs to one that allows us to deal with systems of PDEs.

- Second, we aim to provide a set of tools that allow us to obtain enough information about a particular diffusion process, even when the existing methods fail to produce a transition density due to lack of enough symmetry in the associated Kolmogorov Backward equation.

1.1 Overview and structure

In order to cover the aspects specified above, we have organised this thesis into different chapters, according to the following structure:

- Chapter 2 provides the reader with the necessary background knowledge and an overview of the research conducted up to date on the topics that will be later on covered in following chapters. This chapter also presents some examples to illustrate the main methodologies that have been used in this area of research throughout the past few years.

- Chapter 3 and Chapter 4 both deal with finding fundamental matrices for particular types of systems of PDEs.
  In particular, in Chapter 3 we study a family of systems for which we compute the Lie Algebra in different cases. We then proceed to develop a methodology to obtain fundamental matrices for the systems in each case.
  In Chapter 4, instead, we use the existing theory available for single PDEs to develop a method that deals with the computation of fundamental matrices for real systems arising from single PDEs concerning complex-valued functions.
  The results presented in these two chapters are all new unless otherwise specified. As far as we know, the techniques we develop in this thesis for the computation of fundamental matrices for systems of linear parabolic PDEs have not been used before.

- Lastly, in Chapter 5 we present a wide range of tools that we have developed to deal with diffusion processes for which the associated Kolmogorov Backward equation does not have enough symmetries to allow us to compute a fundamental solution. We illustrate all these tools through an example of a matrix diffusion process: the Wishart Process. These processes turn out to be of great interest in many areas such as financial Mathematics, where they are widely used as a tool to model stochastic volatility.
  In particular, we focus on the eigenvalues of such processes to present the research we have conducted. We compute the expected value for these eigenvalues, as well as some bounds for their variance. We also obtain an infinite series expression for the Fourier cosine transform of the density function of the difference of the eigenvalues for 2-dimensional Wishart processes. In addition, we compute the expected value of all sorts of functions of these eigenvalues, such as any symmetric polynomial in these eigenvalues. The techniques we use to compute all these expectations rely mostly on Lie symmetries and classical integral transforms.
  Again, to the best of our knowledge, all the results obtained in this chapter for the eigenvalues of a Wishart Process are new unless otherwise specified.

Chapter 2

Theoretical Background and Literature Review

For the purpose of giving a self-contained overview of our study, this chapter provides a brief explanation of the main topics on which our research is based, as well as an outline of the methodology we use to approach the problem of finding fundamental solutions for a parabolic PDE.

We also review some of the methods that have historically been used to obtain fundamental solutions through the computation of Lie symmetries. These methods include, for example, the use of group-invariant solutions or the reduction by symmetry to a canonical form.

We then focus on a completely different approach that allows us to obtain fundamental solutions for some classes of PDEs by simply inverting a classical integral transform such as the Laplace transform, the Fourier transform or the Mellin transform. It is precisely this method that we will mostly be concerned with and it relies on the fact that transforming a stationary solution of these PDEs via an appropriate Lie symmetry yields a new solution that can be expressed as a classical integral transform of the fundamental solution.

In order to make sense of all these methods, we will first give a brief introduction to Lie symmetries and how to compute them. We will then proceed to present some examples of the different methods we can use to obtain fundamental solutions and, after a very brief explanation on a few basic concepts in the field of Stochastic Processes, we will explain how we relate the computation of fundamental solutions of PDEs to the obtention of transition densities for a given Stochastic Process.

The main definitions in section 2.1 regarding the computations of Lie symmetries for a given system of differential equations have been taken from Olver's book [52] unless otherwise stated. Similarly, in section 2.2, the main theorems on Stochastic Differential Equations are from Øksendal's book [51] unless otherwise specified.

2.1 Lie Symmetries for Partial Differential Equations: computation and obtention of fundamental solutions

Given a PDE or a system of PDEs, one might be interested in computing their symmetries. It turns out Lie's method for the systematic computation of symmetries is a rather powerful and relatively simple method to do so. A detailed explanation on this subject can be found in Olver's book [52], as well as [4, 8, 40].

In what follows, we provide a general picture of how this method works. However, we do not include some proofs or go too much into detail, since these results can be easily found in the literature. Note that this method can be used for single PDEs but also for systems, as well as for ordinary differential equations.

Suppose we have a system $\mathscr{S}$ of n-th order differential equations in p independent and q dependent variables defined by

$$\Delta_\nu(x, u^{(n)}) = 0, \quad \nu = 1, \dots, l,$$

involving $x = (x^1, \dots, x^p)$ (the independent variables), $u = (u^1, \dots, u^q)$ (the dependent variables) and the derivatives of u with respect to x up to order n, where $\Delta(x, u^{(n)}) = (\Delta_1(x, u^{(n)}), \dots, \Delta_l(x, u^{(n)}))$ can be regarded as a smooth map from the jet space $X \times U^{(n)}$ to some l-dimensional Euclidean space

$$\Delta : X \times U^{(n)} \to \mathbb{R}^l$$

Observe that the differential equations determine a subvariety where the map Δ vanishes:

$$\mathscr{S}_\Delta = \{(x, u^{(n)}) : \Delta(x, u^{(n)}) = 0\} \subset X \times U^{(n)}$$

Our goal is to determine explicitly the symmetry group for the system $\mathscr{S}$ (or, in particular, the infinitesimal generators of such symmetry group). That is, we are looking for a local group of transformations G acting on the independent and dependent variables of the system that maps solutions of $\mathscr{S}$ to other solutions of the system. More precisely, let $\mathcal{H}_\Delta$ denote the space of all solutions of the system $\mathscr{S}$:

$$\Delta_\nu(x, u^{(n)}) = 0, \quad \nu = 1, \dots, l,$$

we are looking for a mapping $\mathcal{S}$ of $\mathcal{H}_\Delta$ into itself, so that if $u \in \mathcal{H}_\Delta$ then $\mathcal{S}u \in \mathcal{H}_\Delta$. Such a mapping $\mathcal{S} : \mathcal{H}_\Delta \to \mathcal{H}_\Delta$ is called a symmetry.

It turns out that the symmetries we will be looking for can be proved to possess group properties. In particular, they typically form a Lie group. We again refer the reader to Olver's book [52] for a more thorough explanation on this particular subject.

We will only deal with symmetries in which the transformations act solely on the independent and dependent variables x and u. These type of symmetries are known as *point symmetries*. However, there exist more complex *generalized symmetries*, which involve transformations acting on the derivatives of the dependent variables. We do not discuss these in this work.

For the computation of point symmetries, we will be dealing with right-invariant vector fields of the form:

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha(x,u)\frac{\partial}{\partial u^\alpha}, \tag{2.1}$$

defined on an open subset $M \subset X \times U$.

It is convenient to refresh the formal definition of the *Lie bracket* of two vector fields as well as the notion of a *Lie algebra*:

Definition 2.1.1. Let $\mathbf{v}$ and $\mathbf{w}$ be smooth vector fields on a manifold M. The *Lie bracket* $[\mathbf{v},\mathbf{w}]$ of $\mathbf{v}$ and $\mathbf{w}$ is the vector field defined as

$$[\mathbf{v},\mathbf{w}](f) := \mathbf{v}(\mathbf{w}(f)) - \mathbf{w}(\mathbf{v}(f))$$

for all $f \in C^\infty(M)$.

Definition 2.1.2. A Lie algebra $\mathfrak{g}$ is a vector space over a field F with an operation $[\cdot,\cdot] : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ called the *Lie bracket* (definition 2.1.1), which has the following properties:

(i) It is *bilinear*

(ii) It is *skew symmetric*: $[\mathbf{v},\mathbf{v}] = 0$, which implies $[\mathbf{v},\mathbf{w}] = -[\mathbf{w},\mathbf{v}]$ for all $\mathbf{v},\mathbf{w} \in \mathfrak{g}$

(iii) It satisfies the *Jacobi Identity*: $[\mathbf{u},[\mathbf{v},\mathbf{w}]] + [\mathbf{v},[\mathbf{w},\mathbf{u}]] + [\mathbf{w},[\mathbf{u},\mathbf{v}]] = 0$ for all $\mathbf{u},\mathbf{v},\mathbf{w} \in \mathfrak{g}$

The Lie algebra $\mathfrak{g}$ of a Lie group G can be identified with the tangent space to G at the identity element, i.e. $\mathfrak{g} \simeq TG|_e$. It can be shown that there is a one-to-one correspondence between one-dimensional subspaces of $\mathfrak{g}$ and (connected) one-parameter subgroups of G.

It is important to mention that, as remarked in Olver's book [52], although the usual approach to Lie algebras requires the vector fields defining $\mathfrak{g}$ to be left-invariant under the action of G, the fact that we are using right-invariant vector fields will not have any consequences for our purposes.

It is well known that the flow of a sufficiently well behaved vector field $\mathbf{v} = \sum \xi^i(x)\frac{\partial}{\partial x^i}$ (usually denoted as $\exp(\epsilon\mathbf{v})$) is a one-parameter local Lie group. We can recover the action generated by $\mathbf{v}$ by summing the so-called *Lie series*:

$$\exp(\epsilon\mathbf{v})x = x + \epsilon\xi(x) + \frac{\epsilon^2}{2}\mathbf{v}(\xi)(x) + \cdots = \sum_{k=0}^{\infty}\frac{\epsilon^k}{k!}\mathbf{v}^k(x)$$

However, this is not a practical way to do so. A much more convenient and efficient way to recover the action generated by a vector field of the form (2.1) on (x, u) is to think of the flow as a transformation of (x, u) into $(\tilde{x}, \tilde{u}) = \exp(\epsilon\mathbf{v})(x, u)$ and solve the following system of differential equations:

$$\frac{d\tilde{x}^i}{d\epsilon} = \xi^i(\tilde{x}, \tilde{u}), \quad \tilde{x}^i(0) = x^i, \quad i = 1, \dots p \tag{2.2}$$

$$\frac{d\tilde{u}^\alpha}{d\epsilon} = \phi_\alpha(\tilde{x}, \tilde{u}), \quad \tilde{u}^\alpha(0) = u^\alpha, \quad \alpha = 1, \dots q \tag{2.3}$$

In this case, the parameter of the group action generated by $\mathbf{v}$ is ϵ.

Let us now denote by $\mathcal{G}$ the group generated by a vector field $\mathbf{v}$. We need to introduce the notion of the n-th prolongation of $\mathcal{G}$, $\mathsf{pr}^{(n)}\mathcal{G}$ as the natural extension of the action of $\mathcal{G}$ to not only the dependent and independent variables x and u, but also the derivatives of u up to order n. The n-th prolongation of $\mathcal{G}$ is defined in such a way that applying $\mathsf{pr}^{(n)}\mathcal{G}$ to $(x, u^{(n)})$ is the same as first transforming (x, u) through the action of $\mathcal{G}$ and then computing the derivatives of the transformed dependent variables up to order n.

In a similar way to how a vector field generates a one-parameter group action, the n-th prolongation of $\mathcal{G}$, $\mathsf{pr}^{(n)}\mathcal{G}$, also has an infinitesimal generator, usually denoted by $\mathsf{pr}^{(n)}\mathbf{v}$. Its technical definition is the following:

Definition 2.1.3 ([52])**.** Let $M \subset X \times U$ be open and suppose $\mathbf{v}$ is a vector field on M, with corresponding local one-parameter group $\exp(\epsilon\mathbf{v})$. The n-th prolongation of $\mathbf{v}$, denoted by $\mathsf{pr}^{(n)}\mathbf{v}$, will be a vector field on the n-jet space $M^{(n)} \subset X \times U^{(n)}$, and is defined to be the infinitesimal generator of the corresponding prolonged one-parameter group $\mathsf{pr}^{(n)}[\exp(\epsilon\mathbf{v})]$, i.e.

$$\mathsf{pr}^{(n)}\mathbf{v}|_{(x,u^{(n)})} = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \mathsf{pr}^{(n)}[\exp(\epsilon\mathbf{v})](x, u^{(n)}), \tag{2.4}$$

for any $(x, u^{(n)}) \in M^{(n)}$

There exists a specific formula for the computation of $\mathsf{pr}^{(n)}\mathbf{v}$, which is often referred to as the *General Prolongation Formula*. The following result determines this specific expression for $\mathrm{pr}^{(n)}\mathbf{v}$, which makes its calculation only an exercise of computing a few derivatives:

Theorem 2.1.1 ([52]). *Let*

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha(x,u)\frac{\partial}{\partial u^\alpha}$$

be a vector field defined on an open subset $M \subset X \times U$. The n-th prolongation of $\mathbf{v}$ is the vector field

$$\mathrm{pr}^{(n)}\mathbf{v} = \mathbf{v} + \sum_{\alpha=1}^{q}\sum_{J} \phi_\alpha^J(x,u^{(n)})\frac{\partial}{\partial u_J^\alpha} \tag{2.5}$$

defined on the corresponding jet space $M^{(n)} \subset X \times U^{(n)}$, the second summation being over all (unordered) multi-indices $J = (j_1, \ldots, j_k)$, with $1 \le j_k \le p$, $1 \le k \le n$. The coefficient functions ϕ_α^J of $\mathrm{pr}^{(n)}\mathbf{v}$ are given by the following formula:

$$\phi_\alpha^J(x,u^{(n)}) = D_J\left(\phi_\alpha - \sum_{i=1}^{p}\xi^i u_i^\alpha\right) + \sum_{i=1}^{p}\xi^i u_{J,i}^\alpha, \tag{2.6}$$

where $u_i^\alpha = \partial u^\alpha/\partial x^i$, $u_{J,i}^\alpha = \partial u_J^\alpha/\partial x^i$ and D_J is the total differentiation operator.

With the above definitions, let us go back to our initial problem of computing explicit symmetries for a system of differential equations. We are now ready to present the main theorem that will allow us to systematically do so. This theorem, as well as its proof, can again be found in Olver's book [52] and it provides necessary and sufficient conditions for a Lie group with infinitesimal generator of the form (2.1) to be a symmetry group:

Theorem 2.1.2. (Lie's Theorem)
Suppose

$$\Delta_\nu(x,u^{(n)}) = 0, \quad \nu = 1,\ldots,l,$$

is a non-degenerate system of differential equations defined over $M \subset X \times U$. If G is a connected local group of transformations acting on M, then G is a symmetry group of the system if and only if

$$\mathrm{pr}^{(n)}\mathbf{v}[\Delta_\nu(x,u^{(n)})] = 0, \quad \nu = 1,\ldots,l \quad \textit{whenever} \quad \Delta_\nu(x,u^{(n)}) = 0, \tag{2.7}$$

for every infinitesimal generator $\mathbf{v}$ of G.

Note. We often refer to the vector fields satisfying (2.7) as *infinitesimal symmetries*. It turns out that the set of all infinitesimal symmetries of the system forms a Lie algebra of vector fields on M ([52])

In light of this result, the computation of the symmetries of a system of differential equations is reduced to applying Theorem 2.1.2 to our particular system, thus yielding a set of determining equations for ξ^i, ϕ_α that can usually be solved by inspection. Each of the Lie group symmetries will produce a continuous family of solutions parametrized by the group variable.

The next step of our work is to compute fundamental solutions. Let us begin by defining the concept of a fundamental solution:

Definition 2.1.4. Let L be a linear differential operator on a domain Ω. A fundamental solution for L is a distribution p defined on Ω with the property that

$$Lp = \delta(x).$$

Here δ refers to the so called *Dirac Delta function*. However, the Dirac Delta is technically speaking not a function but a *distribution* or a *generalised function*. More information on this topic can be found in [39, 33, 49]
There exist equivalent definitions of fundamental solutions for particular types of PDEs. For example, for parabolic PDEs the definition of a fundamental solution can be formulated as follows

Definition 2.1.5. Let $\frac{\partial}{\partial t} - L$ be a parabolic differential operator. A fundamental solution p_t of $\frac{\partial}{\partial t} - L$ can be defined to be a solution of the PDE

$$\left(\frac{\partial}{\partial t} - L\right) u = 0,$$

subject to the initial condition $p_0 = \delta(x)$.

A very useful property of fundamental solutions can be derived from definition 2.1.4. Observe that knowing a fundamental solution p for a differential operator L one may solve the equation $Lu = f$ for any appropriate function f. It is clear that $u = f * p$ will satisfy $Lu = f$:

$$\begin{aligned} Lu = L(f * p) &= L\left(\int_\Omega f(y)p(x-y)dy\right) \\ &= \int_\Omega f(y)Lp(x-y)dy \\ &= \int_\Omega f(y)\delta(x-y)dy = f(x) \end{aligned} \tag{2.8}$$

Here $f * p$ refers to the convolution[1] of f and p, defined as

Definition 2.1.6. Let $f, g \in L^1(\mathbb{R}^n)$. The convolution of f and g is defined by

$$f * g(x) = \int_{\mathbb{R}^n} f(y)g(x-y)dx.$$

The methodology we will use in most of our work to compute fundamental solutions consists essentially in using trivial or rather simple solutions of a system to ultimately construct complex solutions. Roughly speaking, the way we approach this problem is by transforming these trivial solutions through the action of an appropriate Lie group symmetry in order to produce other solutions which are non-trivial.

It has been seen that there are many effective approaches to the use of symmetry methods to compute fundamental solutions for a given PDE. One such approach is to use the fact that fundamental solutions to many PDEs can be obtained as group invariant solutions. In [25] for example, the authors first use the method of characteristics to find group invariant solutions for the particular PDE that is being considered and then use it to construct the fundamental solution for such PDE.

One possible method of obtaining fundamental solutions as group invariant solutions for a given boundary value problem (BVP) is described by Bluman and Cole in their joint work [10] or by Bluman and Anco in [7]. They deal with BVPs of the following form

Definition 2.1.7. Solve the n-th order PDE

$$P(x, u^{(n)}) = 0, \quad x \in \Omega \subset \mathbb{R}^m \tag{2.9}$$

subject to the boundary conditions

$$B_j(x, u, u^{(n-1)}) = 0, \tag{2.10}$$

when $\omega_j(x) = 0$, $j = 1, \ldots, k$.

For a BVP defined as above they present the following definition and result

Definition 2.1.8. A vector field $\mathbf{v}$ is admitted by an $n-$th order boundary value problem if

(i) $\text{pr}^{(n)}\mathbf{v}[P(x, D^\alpha u)] = 0$ when $P(x, D^\alpha u) = 0$.

(ii) $\mathbf{v}(\omega_j(x)) = 0$ when $\omega_j(x) = 0$

(iii) $\text{pr}^{(n-1)}\mathbf{v}[B_j(x, u, u^{(n-1)})] = 0$ when $B_j(x, u, u^{(n-1)}) = 0$ on the surface $\omega_j(x) = 0$.

Proposition 2.1.3. *Suppose that a boundary value problem admits a vector field* $\mathbf{v}$*. Then the solution of the boundary value problem is a group invariant solution with respect to the symmetries generated by* $\mathbf{v}$*.*

In [43], one may also find a group theoretic approach to finding fundamental solutions.

Let us now present a very basic but illustrative example of this first approach to finding fundamental solutions:

Example 2.1.1. Consider the one-dimensional heat equation $u_t = u_{xx}$ on the real line with $t \geq 0$, which has a well-known fundamental solution known as the *heat kernel*:

$$k(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \tag{2.11}$$

One may obtain this fundamental solution as a group invariant solution of the problem

$$u_t = u_{xx}, \quad x \in \mathbb{R}, \quad t \geq 0 \tag{2.12}$$

$$u(x,0) = \delta(x) \tag{2.13}$$

$$\lim_{x \to \pm\infty} u(x,t) = 0 \tag{2.14}$$

Using Lie's method for the systematic computation of symmetries one obtains that for this particular PDE (2.12) the finite dimensional part of the Lie algebra of point symmetries is six dimensional and is spanned by the vector fields

$$\begin{cases} \mathbf{v_1} = \frac{\partial}{\partial x}, & \mathbf{v_4} = 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u}, \\ \mathbf{v_2} = \frac{\partial}{\partial t}, & \mathbf{v_5} = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}, \\ \mathbf{v_3} = u\frac{\partial}{\partial u}, & \mathbf{v_6} = 4xt\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} - (x^2+2t)u\frac{\partial}{\partial u}. \end{cases} \tag{2.15}$$

Note that there is also an infinite dimensional ideal[2]within the Lie algebra consisting of vector fields of the form $\mathbf{v_f} = f(x,t)\frac{\partial}{\partial u}$, where $f_t = f_{xx}$. This ideal simply generates superposition of solutions.

However, not all the elements of the Lie algebra for this PDE will be admitted by our BVP. In their work [10],[7] Bluman et al. describe some methods that allow us to find the largest subalgebra admitted by our BVP, that is, the most general form of a Lie point symmetry of the heat equation that will preserve the boundary conditions of our BVP. But we won't do that in this example, since we are not here

[2] An ideal in a Lie algebra $\mathfrak{g}$ is a vector subspace $\mathcal{I} \subset \mathfrak{g}$ so that for all $\mathcal{X} \in \mathfrak{g}$ and $\mathcal{Y} \in \mathcal{I}$ we have $[\mathcal{X}, \mathcal{Y}] \in \mathcal{I}$.

concerned about the general case. For our purposes it will suffice to pick a suitable element in (2.15) that is indeed admitted by our BVP.

Let us turn our focus to the point symmetry generated by the vector field $\mathbf{v_6}$. It is not excessively hard to show that this vector field is admitted by our BVP. Note for example that $\mathbf{v_6}(t) = 4t^2$, which is equal to zero when $t = 0$. Note also that $\mathbf{v_6}(u - \delta) = -4xt\delta'(x) - (x^2 + 2t)u$, which on the surface $t = 0$ simplifies to $\mathbf{v_6}(u - \delta)|_{t=0} = -x^2 u(x, 0)$, which is zero when $u(x, 0) = \delta(x)$[3].

Again, there are many different methods to find the invariants under the action of the group generated by this particular vector field, one of which is to solve

$$\frac{dx}{4xt} = \frac{dt}{4t^2} = -\frac{du}{(x^2 + 2t)u}$$

Some simple calculations yield the functionally independent invariants $y = \frac{x}{t}$ and $v = e^{\frac{x^2}{4t}}\sqrt{t}u$. Note that the first equality gives $\int \frac{dx}{x} = \int \frac{dt}{t}$, from which we obtain that $\log x = \log t + C_1$ or $C_1 = \log x - \log t = \log \frac{x}{t}$. Then, if C_1 is a constant, so is $y = e^{C_1} = \frac{x}{t}$. Similarly, to find the second invariant, v, we must solve the second equality, which gives $\int (\frac{y^2}{4} + \frac{1}{2t})dt = -\int \frac{du}{u}$, yielding $\frac{y^2}{4}t + \log t^{\frac{1}{2}} = -\log u + C_2$ or $C_2 = \frac{y^2 t}{4} + \log \sqrt{t} + \log u = \frac{x^2}{4t} + \log \sqrt{t}u$. Again, exponentiation gives $v = e^{\frac{x^2}{4t}}\sqrt{t}u$. Applying the chain rule to $u = \frac{e^{-\frac{x^2}{4t}}}{\sqrt{t}}v$ gives

$$u_t = \frac{e^{-\frac{x^2}{4t}}}{t^{\frac{5}{2}}}\left(\left(\frac{x^2}{4} - \frac{t}{2}\right)v(y) - xv'(y)\right)$$

$$u_{xx} = \frac{e^{-\frac{x^2}{4t}}}{t^{\frac{5}{2}}}\left(\left(\frac{x^2}{4} - \frac{t}{2}\right)v(y) - xv'(y) + v''(y)\right)$$

So the heat equation becomes simply $v''(y) = 0$. Therefore we must have $v(y) = Ay + B$ and so

$$u(x,t) = \frac{e^{-\frac{x^2}{4t}}}{\sqrt{t}}v\left(\frac{x}{t}\right) = \frac{e^{-\frac{x^2}{4t}}}{\sqrt{t}}\left(A\frac{x}{t} + B\right)$$

Observe that in order to satisfy the boundary condition (2.13) we need

$$\int_{-\infty}^{\infty} \frac{e^{-\frac{x^2}{4t}}}{\sqrt{t}}\left(A\frac{x}{t} + B\right)dx = 1,$$

thus yielding the choice $B = \frac{1}{\sqrt{4\pi}}$. The choice of $A = 0$ comes from the behaviour and the properties of the delta function as a distribution. Note for example that

[3] See [49] for properties and further explanation on the Dirac delta function

the delta function is an even function (in the distribution sense), whereas $u(x,t)$ is odd unless we take $A = 0$. So with this choices of A and B our fundamental solution becomes $u(x,t) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right)$, which is exactly the expression for the heat kernel (2.11).

It is important to point out that in the above example the boundary and initial conditions were not given in the form that definitions 2.1.7 and 2.1.8 contemplate. Nevertheless, in this particular case the methodology described in [10],[7] does not fail to produce a fundamental solution. However, one must be very careful when generalizing these methods to more complicated forms of boundary conditions. Cherniha et al. consider extensions of the method presented by Bluman and his co-authors to more general and complicated boundary value problems in [14, 16, 15]. They look into problems with free boundaries and, in particular, they focus on BVPs of the Stefan type. Arrigo et al. [1] also discuss and extend the study of some invariance methods used by Bluman and Cole for a wider notion of invariant solutions, known as nonclassical solutions.

The method of using group-invariant solutions in the construction of fundamental solutions has been widely explored in the last 50 years (see for example [52, 41, 42, 43, 45]). In [3], Berest and Ibragimov explore these methods for the heat equation and in [44], Ibragimov studies the heat equation, the wave equation and the Laplace equation through this method. In [28], Finkel classifies the symmetries of the Fokker–Planck equation in two spatial dimensions with a constant positive-definite diffusion matrix. For the 1-dimensional case, in [9, 5], Bluman obtained fundamental solutions for the Fokker-Planck equation $u_t = u_{xx} + (f(x)u)_x$ in the case where f satisfies a certain Riccati equation. He also studied the n-dimensional wave equation and the Laplace equation amongst other examples in [4]. The method of group-invariant solutions has also been applied for finding fundamental solutions to some PDEs arising in financial mathematics, such as the so called Black-Scholes equation (see [32]). Laurence and Wang explored this method for a multi-dimensional case in [46], only obtaining fundamental solutions for some special cases.

Other methods obtain fundamental solutions through the reduction by symmetry of the given equation to a canonical form (see for example [6], where Bluman shows how to reduce a type of PDE to the heat equation by symmetry). Goard also used this approach to finding fundamental solutions in [34], where she reduces equations to either the heat equation or the equation $u_t = u_{xx} - \frac{A}{x^2}u$. A very thorough explanation on the method of reduction to canonical form can also be found in [42]. Using these methods it is possible to determine general types of differential equations that can be reduced to a particular canonical form. Nevertheless, this requires some additional background knowledge that is not relevant to our purposes so we will omit it here. We will however provide an example of how this method allows us to study rather complex PDEs by reducing them to one of the canonical forms. There are several known results that study particular types of PDEs and determine under what circumstances those PDEs can be reduced to one canonical form or another. One of such results is the following (see [42] or Goard's paper [34]):

Proposition 2.1.4. *Let the functions $P(x,t)$ and $R(x,t)$ be non-zero functions and consider the following evolution equation for an appropriate function $u(x,t)$:*

$$P(x,t)u_t + Q(x,t)u_x + R(x,t)u_{xx} + S(x,t)u = 0 \tag{2.16}$$

Then there exists a suitable transformation of the dependent and independent variables (i.e. x, t and u) that reduces equation (2.16) to

$$v_{\bar{t}} = v_{\bar{x}\bar{x}} + Z(\bar{t},\bar{x})v. \tag{2.17}$$

Furthermore, knowing the symmetry operators admitted by the PDE (2.16) (aside from the trivial ones $\frac{\partial}{\partial u}$ and $\phi(x,t)\frac{\partial}{\partial u}$, where ϕ is any solution of (2.16)) we can further reduce this PDE to one of the following forms:

(i) If the PDE (2.16) admits the additional symmetry operator $\frac{\partial}{\partial t}$ then it is reducible to

$$v_{\bar{t}} = v_{\bar{x}\bar{x}} + Z(\bar{x})v.$$

(ii) If the PDE has at least three additional symmetries is reducible to

$$v_{\bar{t}} = v_{\bar{x}\bar{x}} + \frac{\alpha}{\bar{x}^2}v,$$

where α is constant.

(iii) If it has at least five extra symmetries, the PDE (2.16) can be reduced to

$$v_{\bar{t}} = v_{\bar{x}\bar{x}}.$$

Let us now move on to show one illustrative example that can be derived from some of the results in Goard's paper:

Example 2.1.2. Suppose we are looking for a function $p(x,t;y,t')$ satisfying the PDE

$$p_t + \frac{1}{2}xp_{xx} + p_x = 0, \tag{2.18}$$

subject to

$$p(x, t; y, t') = \delta(x - y), \tag{2.19}$$

and such that for $x, y \geq 0$, the condition

$$\int_0^\infty p(x, t; y, t')dy = 1 \tag{2.20}$$

is satisfied. It turns out that this particular problem arises from the problem of finding the transition density function (TDF) of the the following Itô diffusion :

$$dX_t = dt + \sqrt{X_t}dW_t, \tag{2.21}$$

We will explain with more detail what an Itô diffusion is and how to derive such PDE later on in this chapter, but the main idea is that the expectations of any function of an Itô diffusion always satisfy a particular PDE that is known as the backward Kolmogorov equation. It turns out that for a diffusion of the type (2.21) the PDE we end up having to deal with is (2.18). With this, the density function $p(x, t; y, t')$ we are looking for will have the following meaning:

$$Pr(a < X_{t'} < b \quad | \quad X_t = x) = \int_a^b p(x, t; y, t')dy. \tag{2.22}$$

In [6] Bluman shows that equation (2.18) can be reduced to the form

$$q_\eta = q_{\xi\xi} + \alpha(\xi, \eta)q \tag{2.23}$$

through a transformation of the type

$$\begin{cases} \xi = \xi(x, t) \\ \eta = \eta(x, t) \\ q = \Phi(x, t)p. \end{cases} \tag{2.24}$$

In particular, it can be seen that in our case, the transformation

$$\begin{cases} \xi = 2\sqrt{x} \\ \eta = \frac{t'-t}{2} \\ q = x^{\frac{3}{4}}p \implies q(\xi, \eta; y, t') = \frac{\xi^{3/2}}{2\sqrt{2}}p\left(\frac{\xi^2}{4}, t' - 2\eta; y, t'\right) \end{cases} \tag{2.25}$$

reduces our initial problem to :

$$q_\eta = q_{\xi\xi} - \frac{3}{4}\frac{1}{\xi^2}q, \tag{2.26}$$

subject to

$$q(\xi, 0; y, t') = \frac{\xi^{3/2}}{2\sqrt{2}}\delta\left(\frac{\xi^2}{4} - y\right). \tag{2.27}$$

It turns out that the solution to the Cauchy problem (2.26)-(2.27) is known to be

$$\begin{aligned} q(\xi, \eta; y, t') &= \frac{\sqrt{\xi}}{2\eta}e^{-\frac{\xi^2}{4\eta}} \int_0^\infty e^{-\frac{s^2}{4\eta}} \frac{s^2}{2\sqrt{2}} I_1\left(\frac{\xi s}{2\eta}\right) \delta\left(\frac{s^2}{4} - y\right) ds \\ &= \frac{\sqrt{\xi}}{2\eta}e^{-\frac{\xi^2}{4\eta}} e^{-\frac{y}{\eta}} \sqrt{2}\sqrt{y} I_1\left(\frac{\xi\sqrt{y}}{\eta}\right) \\ &= \frac{\sqrt{\xi}\sqrt{y}}{\sqrt{2}\eta} e^{-\frac{\xi^2}{4\eta}-\frac{y}{\eta}} I_1\left(\frac{\xi\sqrt{y}}{\eta}\right), \end{aligned}$$

i.e. the solution to our problem will be

$$\begin{aligned} p\left(\frac{\xi^2}{4}, t' - 2\eta; y, t'\right) &= \frac{2\sqrt{2}}{\xi^{3/2}} \frac{\sqrt{\xi}\sqrt{y}}{\sqrt{2}\eta} e^{-\frac{\xi^2}{4\eta}-\frac{y}{\eta}} I_1\left(\frac{\xi\sqrt{y}}{\eta}\right) \\ &= \frac{2\sqrt{y}}{\eta\xi} e^{-\frac{\xi^2}{4\eta}-\frac{y}{\eta}} I_1\left(\frac{\xi\sqrt{y}}{\eta}\right). \end{aligned}$$

One need only transform the independent variables back to the original ones to get

$$p(x, t; y, t') = \frac{2}{(t' - t)} \sqrt{\frac{y}{x}} \exp\left(-2\frac{x + y}{t' - t}\right) I_1\left(\frac{4\sqrt{xy}}{t' - t}\right) \tag{2.28}$$

as the fundamental solution for (2.18) satisfying (2.19) and (2.20). It can be checked that this is precisely the transition density function for an Itô diffusion defined according to (2.21).

So by knowing the form solutions take for some canonical PDEs, we can derive fundamental solutions for a wide range of PDEs that can be reduced to such canonical forms via some transformation of the dependent and independent variables. However, these methods sometimes produce issues with the boundary conditions for the reduced equation. We illustrate this with some examples in what follows:

Example 2.1.3. Consider the problem

$$\begin{cases} u_t = x^4 u_{xx} + 2x^3 u_x \\ u(x, 0) = f(x) \\ u_x(0, t) = 0. \end{cases} \tag{2.29}$$

Let $y = \frac{1}{x}$ and $v(y,t) = u(x,t)$. Observe that

$$\begin{cases} u_t & = v_t \\ u_x & = -\frac{1}{x^2} v_y = -y^2 v_y \\ u_{xx} & = \frac{1}{x^4} v_{yy} + 2\frac{1}{x^3} v_y = y^4 v_{yy} + 2y^3 v_y \end{cases}$$

Hence the PDE in (2.29) in the new variables becomes $v_t = v_{yy}$, but the conditions become:

$$\begin{cases} f(x) = u(x,0) = v(y,0) = f\left(\frac{1}{y}\right) \\ u_x(0,t) = 0 \iff -\lim_{y\to\infty} y^2 v_y(y,t) = 0 \end{cases}$$

Therefore, we must now consider the problem

$$\begin{cases} v_t = v_{yy}, \quad y \in (0,\infty) \\ v(y,0) = f\left(\frac{1}{y}\right) \\ \lim_{y\to\infty} y^2 v_y(y,t) = 0, \end{cases}$$

which is not very convenient.

Example 2.1.4. Consider the problem

$$\begin{cases} u_t = x u_{xx} + \frac{1}{2} u_x, \quad x > 0 \\ u(x,0) = f(x) \\ u_x(0,t) = 0. \end{cases}$$

Let $y = 2\sqrt{x}$ and $v(y,t) = u(x,t)$. We now have

$$\begin{cases} u_t & = v_t \\ u_x & = \frac{1}{\sqrt{x}} v_y = 2\frac{v_y}{y} \\ u_{xx} & = \frac{1}{x} v_{yy} - \frac{1}{2x^{3/2}} v_y = \frac{4}{y^2} v_{yy} - \frac{2}{y^3} v_y \end{cases}$$

Therefore, the initial PDE in the new variables becomes $v_t = v_{yy}$, but the conditions become:

$$\begin{cases} f(x) = u(x,0) = v(y,0) = f\left(\frac{y^2}{4}\right) \\ u_x(0,t) = 0 \iff \lim_{y\to 0^+} \frac{v_y(y,t)}{y} = 0 \end{cases}$$

The new problem to consider is

$$\begin{cases} v_t = v_{yy} \\ v(y,0) = f\left(\frac{y^2}{4}\right) \\ \lim_{y\to 0^+} \frac{v_y(y,t)}{y} = 0, \end{cases}$$

which, again, is not very convenient.

Example 2.1.5. The equation $u_t = u_{xx} - xu$ can be reduced to $v_\tau = v_{yy}$ by a number of different variable changes. Some of these have quite ugly effects. For example, one possible choice for the new time variable is $\tau = -\frac{1}{16t}$. This maps $t = 0$ to $\tau = -\infty$ and $t = \infty$ to $\tau = 0$.

After some experimentation with the various choices of the constants of integration, the simplest choice we have found is

$$u = F(x,t)v\left(\frac{4T(x-t^2)}{1+16Tt}, \frac{16tT^2}{1+16Tt}\right), \tag{2.30}$$

where $F(x,t)e^{A(t)x^2+B(t)x+C(t)}$, and A, B and C are rather complex, but not relevant here. Let $y = \dfrac{4T(x-t^2)}{1+16Tt}$ and $\tau = \dfrac{16tT^2}{1+16Tt}$.

Then $t = \dfrac{\tau}{16T(T-\tau)}$ and $x = \dfrac{\tau^2 + 64T^2 y(T-\tau)}{256T^2(T-\tau)^2}$.

The problem

$$\begin{cases} u_t = u_{xx} - xu, \quad x > 0 \\ u(x,0) = f(x) \\ u(0,t) = \phi(t) \end{cases} \tag{2.31}$$

becomes

$$\begin{cases} v_\tau = v_{yy}, \quad y > \frac{-\tau^2}{64T^2(T-\tau)}, \quad \tau \in [0,T) \\ v(y,0) = f(\frac{y}{4}) \\ v(\frac{-\tau^2}{64T^2(T-\tau)}, \tau) = \phi(\frac{\tau}{16T(T-\tau)}) \end{cases} \tag{2.32}$$

This is a moving boundary problem which is much harder than the original problem.

A recent different approach to the same problem of finding fundamental solutions is linked to the fact that for some families of PDEs, we can apply an appropriate Lie symmetry to a stationary solution to obtain an integral transform of a

fundamental solution. Then we can recover the fundamental solution by inverting the integral transform. Of course, to be able to do so, we must be dealing with an integral transform possessing a known inversion integral. This method has been explored by Craddock and his co-authors in [18, 25, 20, 22, 19, 24].
In order to demonstrate how this method works, we first need to introduce a theoretical result for linear PDEs of the type

$$P(x, D^{\alpha})u = \sum_{|\alpha| \leq n} a_{\alpha}(x) D^{\alpha} u, \quad x \in \Omega \subseteq \mathbb{R}^m, \tag{2.33}$$

with $\alpha = (\alpha_1, \dots, \alpha_m)$, $\alpha_i \in \mathbb{N}$, $|\alpha| = \alpha_1 + \cdots + \alpha_m$ and

$$D^{\alpha} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_m^{\alpha_m}}.$$

For such PDEs, the following theorem holds:

Theorem 2.1.5. *(see theorem and its proof in [19]) Let $\tilde{u}_{\epsilon}(x)$ be the continuous one-parameter family of solutions of (2.33) obtained through the action of a one parameter group of symmetries G on a solution u of the system. Then for φ defined on an appropriate region and with sufficiently rapid decay, we have by continuity and linearity that*

$$U(x) = \int_{\Omega} \varphi(\epsilon) \tilde{u}_{\epsilon}(x) d\epsilon \tag{2.34}$$

is a solution of (2.33) for a suitable region of integration Ω. Further, if the PDE is time-dependent and $\tilde{u}_{\epsilon}(x,t)$ is the family of symmetry solutions, then

$$u(x,t) = \int_{\Omega} \varphi(\epsilon) \tilde{u}_{\epsilon}(x,t) d\epsilon \tag{2.35}$$

and

$$u(x,0) = \int_{\Omega} \varphi(\epsilon) \tilde{u}_{\epsilon}(x,0) d\epsilon. \tag{2.36}$$

Further $\frac{d^n \tilde{u}_{\epsilon}(x)}{d\epsilon^n}$ is also a solution for all $n = 1, 2, 3, \dots$

The idea is to identify (2.34) with an integral transform of a fundamental solution of the PDE. In [20, 25], for instance, the following methodology is suggested:

- Consider a linear PDE of the form

$$u_t = P(x, u^{(n)}), \quad x \in \Omega \subseteq \mathbb{R} \tag{2.37}$$

- First, we note that the action of G (generated by $\mathbf{v}$) on any solution u, can typically be expressed as

$$\tilde{u}_\epsilon(x,t) = \rho(\exp(\epsilon\mathbf{v}))u(x,t) = \sigma(x,t;\epsilon)u(a_1(x,t;\epsilon), a_2(x,t;\epsilon)), \tag{2.38}$$

for some functions σ, a_1, a_2. The functions a_1 and a_2 are called the *change of variables* of the symmetry, and σ is called the *multiplier*.

- Next, recall that by property (2.8) of fundamental solutions, if $p(t,x,y)$ is a fundamental solution of (2.37), then

$$u(x,t) = \int_\Omega f(y)p(t,x,y)dy \tag{2.39}$$

solves the initial value problem for (2.37) with appropriate initial data $u(x,0) = f(x)$

- Then we take a stationary (time independent) solution $u = u_0(x)$. So in this case

$$\rho(\exp(\epsilon\mathbf{v}))u_0(x) = \sigma(x,t;\epsilon)u_0(a_1(x,t;\epsilon)) \tag{2.40}$$

- Finally, setting $t = 0$ and considering (2.39) yields

$$\int_\Omega \sigma(y,0;\epsilon)u_0(a_1(y,0;\epsilon))p(t,x,y)dy = \sigma(x,t;\epsilon)u_0(a_1(x,t;\epsilon)) \tag{2.41}$$

It turns out that for large classes of PDEs (see [20]), this integral transform is a classic one such as the Fourier or Laplace transforms as well as the Whittaker, Hankel and other transforms that possess a known inversion formula. Therefore, we can recover the fundamental solution by inverting the transform.

In [26], Craddock and Dooley build up on the work from [19] regarding a PDE of the general type

$$u_t = A(x,t)u_{xx} + B(x,t)u_x + C(x,t)u, \quad u \in \Omega$$

thus proving a result that yields two theorems which ensure that if the lie algebra of a PDE of this particular type is at least four-dimensional, then we can use integral transform methods to compute a fundamental solution. In particular, for this type of PDEs and depending on the dimension of the lie algebra, the Fourier and Laplace transforms arise. Let us present here these two theorems that can be found in [26] and that will be relevant to our upcoming work:

Theorem 2.1.6. *Let*

$$u_t = A(x,t)u_{xx} + B(x,t)u_x + C(x,t)u, \quad x \in \Omega \tag{2.42}$$

have a six-dimensional Lie algebra of symmetries and suppose that $u(x,t) = \int_\Omega u_0(z)p(x,z,t)dz$ *is a non-zero solution of (2.42). Then there is a Lie symmetry which maps solutions* $u(x,t)$ *to a generalised Fourier transform of a product of* u_0 *and a fundamental solution* $p(x,z,t)$*.*

Similarly, we have

Theorem 2.1.7. *Let*

$$u_t = A(x,t)u_{xx} + B(x,t)u_x + C(x,t)u, \quad x \in \Omega \tag{2.43}$$

have a four-dimensional Lie algebra of symmetries and suppose that $u(x,t) = \int_\Omega u_0(z)p(x,z,t)dz$ *is a non-zero solution of (2.43). Then there is a Lie symmetry which maps solutions* $u(x,t)$ *to a generalised Laplace transform of a product of* u_0 *and a fundamental solution* $p(x,z,t)$*.*

Let us now illustrate these integral transform methods with some examples where the Fourier, Laplace and Mellin transforms arise respectively.

Example 2.1.6. Consider the PDE

$$u_t = u_{xx} - \frac{1}{x^2}u, \quad x > 0 \tag{2.44}$$

Although we do not include all the calculations in this example, it can be seen that applying Lie's theorem 2.1.2 to this example produces the following basis for the lie algebra of (2.44):

$$\begin{cases} \mathbf{v_1} = \frac{\partial}{\partial t}, & \mathbf{v_3} = u\frac{\partial}{\partial u}, \\ \mathbf{v_2} = t\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x}, & \mathbf{v_4} = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} - u\left(\frac{x^2}{4} + \frac{t}{2}\right)\frac{\partial}{\partial u}, \\ \mathbf{v}_\alpha = \alpha(x,t)\frac{\partial}{\partial u}, & \end{cases} \tag{2.45}$$

where $\alpha(x,t)$ is an arbitrary solution of (2.44).
Note that by Theorem 2.1.7, we can expect that this example can be dealt with using a generalised Laplace transform.
Consider the symmetry generated by the vector field $\mathbf{v_4}$. To find the specific form

of such symmetry we need to solve the system

$$\begin{cases} \frac{d\bar{t}}{d\epsilon} = \bar{t}^2, \quad \bar{t}(0) = t, \\ \frac{d\bar{x}}{d\epsilon} = \bar{t}\bar{x}, \quad \bar{x}(0) = x, \\ \frac{d\bar{u}}{d\epsilon} = -\bar{u}\left(\frac{\bar{x}^2}{4} + \frac{\bar{t}}{2}\right), \quad \bar{u}(0) = u, \end{cases} \tag{2.46}$$

The first equation in the above system gives

$$\begin{aligned} \int \frac{d\bar{t}}{\bar{t}^2} &= \int d\epsilon \\ \Leftrightarrow -\frac{1}{\bar{t}} &= \epsilon + C_1 \\ \Leftrightarrow \bar{t} &= \frac{1}{-C_1 - \epsilon} \end{aligned}$$

But the initial condition gives $\bar{t}(0) = \frac{1}{-C_1} = t$, so we must have $-C_1 = \frac{1}{t}$ or $\bar{t} = \frac{t}{1-\epsilon t}$. Next we have

$$\begin{aligned} \int \frac{d\bar{x}}{\bar{x}} &= \int \bar{t} d\epsilon = \int \frac{t}{1-\epsilon t} d\epsilon \\ \Leftrightarrow \log \bar{x} &= -\log(1-\epsilon t) + C_2 \\ \Leftrightarrow \bar{x} &= \frac{C_3}{1-\epsilon t} \end{aligned}$$

In this case, the initial condition translates to $\bar{x}(0) = C_3 = x$, so we get $\bar{x} = \frac{x}{1-\epsilon t}$. Finally, the last equation can be solved as

$$\begin{aligned} \int \frac{d\bar{u}}{\bar{u}} &= -\int \left(\frac{\bar{x}^2}{4} + \frac{\bar{t}}{2}\right) d\epsilon = -\int \left(\frac{x^2}{4(1-\epsilon t)^2} + \frac{t}{2(1-\epsilon t)}\right) d\epsilon \\ \Leftrightarrow \log \bar{u} &= -\frac{x^2}{4t(1-\epsilon t)} + \frac{1}{2}\log(1-\epsilon t) + C_4 \\ \Leftrightarrow \bar{u} &= C_5 e^{-\frac{x^2}{4t(1-\epsilon t)}} \sqrt{1-\epsilon t}, \end{aligned}$$

and applying the initial condition $\bar{u}(0) = C_5 e^{-\frac{x^2}{4t}} = u$ gives $C_5 = e^{\frac{x^2}{4t}} u$ and therefore $\bar{u} = u(x,t) e^{-\frac{\epsilon x^2}{4(1-\epsilon t)}} \sqrt{1-\epsilon t}$.

Observe that in terms of the new independent variables, $\bar{x}$ and $\bar{t}$, this can be written as:

$$\bar{u}(\bar{x}, \bar{t}) = u\left(\frac{\bar{x}}{1+\epsilon\bar{t}}, \frac{\bar{t}}{1+\epsilon\bar{t}}\right) \frac{1}{\sqrt{1+\epsilon\bar{t}}} \exp\left(-\frac{\epsilon\bar{x}^2}{4(1+\epsilon\bar{t})}\right) \tag{2.47}$$

Let us make the change $\epsilon \to 4\epsilon$ and drop the bars in the new variables for the sake of convenience in notation. We can make this change for ϵ because it is an arbitrary constant. We then have the symmetry:

$$u(x,t) = u\left(\frac{x}{1+4\epsilon t}, \frac{t}{1+4\epsilon t}\right)\frac{1}{\sqrt{1+4\epsilon t}}\exp\left(-\frac{\epsilon x^2}{1+4\epsilon t}\right) \tag{2.48}$$

This symmetry has multiplier $\sigma(x,t;\epsilon) = (1+4\epsilon t)^{-1/2}\exp\left(-\frac{\epsilon x^2}{1+4\epsilon t}\right)$ and changes of variables $a_1(x,t;\epsilon) = x(1+4\epsilon t)^{-1}$ and $a_2(x,t;\epsilon) = t(1+4\epsilon t)^{-1}$.

Now, to be able to use the relationship in (2.41), we need to find a stationary solution to our PDE (2.44), i.e. we need to solve $0 = u_{xx}(x) - \frac{1}{x^2}u(x)$. This is an Euler-type equation, so we look for solutions of the form $u(x) = x^\alpha$. We must have $0 = x^2u_{xx}(x) - u(x) = (\alpha^2-\alpha-1))x^\alpha$, so α needs to be a root of the polynomial $\alpha^2-\alpha-1$, thus giving $\alpha_1 = \frac{1}{2}+\frac{\sqrt{5}}{2}$ or $\alpha_2 = \frac{1}{2}-\frac{\sqrt{5}}{2}$. So we have two linearly independent stationary solutions, $u_1(x) = x^{\frac{1}{2}+\frac{\sqrt{5}}{2}}$ and $u_2(x) = x^{\frac{1}{2}-\frac{\sqrt{5}}{2}}$.

Let us use $u_1(x)$ and the symmetry (2.48) in the integral equation (2.41):

$$\int_0^\infty e^{-\epsilon y^2}u_1(y)p(t,x,y)dy = \frac{1}{\sqrt{1+4\epsilon t}}\exp\left(-\frac{\epsilon x^2}{1+4\epsilon t}\right)u_1\left(\frac{x}{1+4\epsilon t}\right) \tag{2.49}$$

or

$$\int_0^\infty e^{-\epsilon y^2}y^{\frac{1}{2}+\frac{\sqrt{5}}{2}}p(t,x,y)dy = \exp\left(-\frac{\epsilon x^2}{1+4\epsilon t}\right)\frac{x^{\frac{1}{2}+\frac{\sqrt{5}}{2}}}{(1+4\epsilon t)^{1+\frac{\sqrt{5}}{2}}}. \tag{2.50}$$

We realise that making the change of variables $z = y^2$ we get

$$\int_0^\infty e^{-\epsilon z}\frac{1}{2}z^{-\frac{1}{4}+\frac{\sqrt{5}}{4}}q(t,x,z)dz = \exp\left(-\frac{\epsilon x^2}{1+4\epsilon t}\right)\frac{x^{\frac{1}{2}+\frac{\sqrt{5}}{2}}}{(1+4\epsilon t)^{1+\frac{\sqrt{5}}{2}}}, \tag{2.51}$$

so we can recover the fundamental solution $q(t,x,z)$ as an inverse Laplace transform as follows:

$$\begin{aligned}
\frac{1}{2}z^{-\frac{1}{4}+\frac{\sqrt{5}}{4}}q(t,x,z) &= \mathcal{L}^{-1}\left(\exp\left(-\frac{\epsilon x^2}{1+4\epsilon t}\right)\frac{x^{\frac{1}{2}+\frac{\sqrt{5}}{2}}}{(1+4\epsilon t)^{1+\frac{\sqrt{5}}{2}}}\right)\\
&= x^{\frac{1}{2}+\frac{\sqrt{5}}{2}}\mathcal{L}^{-1}\left(\frac{\exp\left(-\frac{(\epsilon+\frac{1}{4t}-\frac{1}{4t})x^2}{4t(\epsilon+\frac{1}{4t})}\right)}{(4t)^{1+\frac{\sqrt{5}}{2}}(\epsilon+\frac{1}{4t})^{1+\frac{\sqrt{5}}{2}}}\right)\\
&= \frac{x^{\frac{1}{2}+\frac{\sqrt{5}}{2}}}{(4t)^{1+\frac{\sqrt{5}}{2}}}\exp\left(-\frac{x^2}{4t}\right)\mathcal{L}^{-1}\left(\frac{\exp\left(\frac{x^2}{(4t)^2(\epsilon+\frac{1}{4t})}\right)}{(\epsilon+\frac{1}{4t})^{1+\frac{\sqrt{5}}{2}}}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{x^{\frac{1}{2}+\frac{\sqrt{5}}{2}}}{(4t)^{1+\frac{\sqrt{5}}{2}}} \exp\left(-\frac{x^2}{4t}\right) \exp\left(-\frac{z}{4t}\right) \mathcal{L}^{-1}\left(\frac{\exp\left(\frac{x^2}{\epsilon(4t)^2}\right)}{\epsilon^{1+\frac{\sqrt{5}}{2}}}\right) \\
&= \frac{x^{\frac{1}{2}+\frac{\sqrt{5}}{2}}}{(4t)^{1+\frac{\sqrt{5}}{2}}} \exp\left(-\frac{x^2+z}{4t}\right) 2^{\sqrt{5}}\left(\frac{t}{x}\right)^{\frac{\sqrt{5}}{2}} z^{\frac{\sqrt{5}}{4}} I_{\frac{\sqrt{5}}{2}}\left(\frac{x\sqrt{z}}{2t}\right) \\
&= \frac{\sqrt{x}}{4t}\exp\left(-\frac{x^2+z}{4t}\right) z^{\frac{\sqrt{5}}{4}} I_{\frac{\sqrt{5}}{2}}\left(\frac{x\sqrt{z}}{2t}\right),
\end{aligned}
$$

where $\mathcal{L}^{-1}$ denotes the classical inverse Laplace transform. Hence

$$
q(t,x,z) = \frac{\sqrt{x}}{2t}\exp\left(-\frac{x^2+z}{4t}\right) z^{\frac{1}{4}} I_{\frac{\sqrt{5}}{2}}\left(\frac{x\sqrt{z}}{2t}\right),
$$

or

$$
p(t,x,y) = \exp\left(-\frac{x^2+y^2}{4t}\right)\frac{\sqrt{xy}}{2t} I_{\frac{\sqrt{5}}{2}}\left(\frac{xy}{2t}\right)
$$

is a fundamental solution of our PDE.

However, as we will remark later, fundamental solutions are not unique. To illustrate the non-uniqueness of fundamental solutions, let us see what would have happened if instead of the stationary solution $u_1(x)$ we had chosen $u_2(x)$:

$$
\int_0^\infty e^{-\epsilon y^2} y^{\frac{1}{2}-\frac{\sqrt{5}}{2}} \bar{p}(t,x,y)dy = \exp\left(-\frac{\epsilon x^2}{1+4\epsilon t}\right)\frac{x^{\frac{1}{2}-\frac{\sqrt{5}}{2}}}{(1+4\epsilon t)^{1-\frac{\sqrt{5}}{2}}}. \tag{2.52}
$$

The same change of variables we used before would produce

$$
\int_0^\infty e^{-\epsilon z}\frac{1}{2} z^{-\frac{1}{4}-\frac{\sqrt{5}}{4}} \bar{q}(t,x,z)dz = \exp\left(-\frac{\epsilon x^2}{1+4\epsilon t}\right)\frac{x^{\frac{1}{2}-\frac{\sqrt{5}}{2}}}{(1+4\epsilon t)^{1-\frac{\sqrt{5}}{2}}}, \tag{2.53}
$$

Therefore

$$
\begin{aligned}
\frac{1}{2} z^{-\frac{1}{4}-\frac{\sqrt{5}}{4}} \bar{q}(t,x,z) &= \mathcal{L}^{-1}\left(\exp\left(-\frac{\epsilon x^2}{1+4\epsilon t}\right)\frac{x^{\frac{1}{2}-\frac{\sqrt{5}}{2}}}{(1+4\epsilon t)^{1-\frac{\sqrt{5}}{2}}}\right) \\
&= \frac{x^{\frac{1}{2}-\frac{\sqrt{5}}{2}}}{(4t)^{1-\frac{\sqrt{5}}{2}}}\exp\left(-\frac{x^2}{4t}\right)\mathcal{L}^{-1}\left(\frac{\exp\left(\frac{x^2}{(4t)^2(\epsilon+\frac{1}{4t})}\right)}{(\epsilon+\frac{1}{4t})^{1-\frac{\sqrt{5}}{2}}}\right) \\
&= \frac{x^{\frac{1}{2}-\frac{\sqrt{5}}{2}}}{(4t)^{1-\frac{\sqrt{5}}{2}}}\exp\left(-\frac{x^2}{4t}\right)\exp\left(-\frac{z}{4t}\right)\mathcal{L}^{-1}\left(\frac{\exp\left(\frac{x^2}{\epsilon(4t)^2}\right)}{\epsilon^{1-\frac{\sqrt{5}}{2}}}\right)
\end{aligned} \tag{2.54}
$$

The inverse Laplace transform we are left with in (2.54) does not exist as an ordinary function but it does exist as a distribution (or generalised function). Therefore we need to consider a broader notion for the Laplace transform than that we usually deal with. We need to extend the notion of the classical Laplace transform to one that deals with not ordinary functions but distributions. Some background for this particular construction is given in Appendix A. For a much more detailed study of the distributional Laplace transform, see [58]. We will need to use the following result from [23] by Craddock and Platen in order to compute this particular distributional inverse Laplace transform:

Proposition 2.1.8. *The following Laplace transform inversion formula holds when n is a non-negative integer:*

$$\mathcal{L}^{-1}(\lambda^n e^{k/\lambda}) = \sum_{l=0}^{n} \frac{k^l}{l!}\delta^{(n-l)}(y) + \left(\frac{k}{y}\right)^{\frac{n+1}{2}} I_{n+1}(2\sqrt{ky}). \tag{2.55}$$

If $n-1<\mu<n$ then

$$\mathcal{L}^{-1}(\lambda^\mu e^{k/\lambda}) = y\left(\frac{k}{y}\right)^{\frac{\mu+1}{2}} I_{-\mu-1}(2\sqrt{ky}), \tag{2.56}$$

and the inverse Laplace transforms are to be regarded as distributions in the sense of Hadamard [37].

The proof of this result can be found in [23].
Using the above proposition we have that by (2.56) with $\mu = \frac{\sqrt{5}}{2}-1$, $k=\left(\frac{x}{4t}\right)^2$

$$\mathcal{L}^{-1}\left(\epsilon^{\frac{\sqrt{5}}{2}-1} e^{\frac{x^2}{\epsilon(4t)^2}}\right) = z\left(\frac{(x/4t)^2}{z}\right)^{\frac{\sqrt{5}}{4}} I_{-\frac{\sqrt{5}}{2}}\left(\frac{x\sqrt{z}}{2t}\right). \tag{2.57}$$

Then, substituting this generalised inverse Laplace transform we are left with the following result:

$$\begin{aligned}
\frac{1}{2}z^{-\frac{1}{4}-\frac{\sqrt{5}}{4}}\bar{q}(t,x,z) &= \frac{x^{\frac{1}{2}-\frac{\sqrt{5}}{2}}e^{-\frac{x^2}{4t}}e^{-\frac{z}{4t}}}{(4t)^{1-\frac{\sqrt{5}}{2}}}\mathcal{L}^{-1}\left(\frac{e^{\frac{x^2}{\epsilon(4t)^2}}}{\epsilon^{1-\frac{\sqrt{5}}{2}}}\right) \\
&= \frac{x^{\frac{1}{2}-\frac{\sqrt{5}}{2}}\exp\left(-\frac{x^2+z}{4t}\right)}{(4t)^{1-\frac{\sqrt{5}}{2}}}\left(z\left(\frac{x^2}{z(4t)^2}\right)^{\frac{\sqrt{5}}{4}} I_{-\frac{\sqrt{5}}{2}}\left(\frac{x\sqrt{z}}{2t}\right)\right) \\
&= \exp\left(-\frac{x^2+z}{4t}\right)\left(\frac{\sqrt{x}z^{1-\frac{\sqrt{5}}{4}}}{4t}\right) I_{-\frac{\sqrt{5}}{2}}\left(\frac{x\sqrt{z}}{2t}\right)
\end{aligned}$$

or

$$\bar{q}(t,x,z) = \exp\left(-\frac{x^2+z}{4t}\right)\left(\frac{z\sqrt{x\sqrt{z}}}{2t}\right) I_{-\frac{\sqrt{5}}{2}}\left(\frac{x\sqrt{z}}{2t}\right)$$

And going back to the y variable we get

$$\bar{p}(t,x,y) = y^2\frac{\sqrt{xy}}{2t}\exp\left(-\frac{x^2+y^2}{4t}\right) I_{-\frac{\sqrt{5}}{2}}\left(\frac{xy}{2t}\right) \tag{2.58}$$

as a second fundamental solution.

Example 2.1.7. Consider the PDE

$$u_t = u_{xx} - x^2 u \tag{2.59}$$

Lie's method for the computation of the symmetries for this PDE produces the following spanning set for its Lie algebra:

$$\begin{cases} \mathbf{v_1} = \frac{\partial}{\partial t}, & \mathbf{v_4} = u\frac{\partial}{\partial u}, \\ \mathbf{v_2} = e^{2t}\frac{\partial}{\partial x} - uxe^{2t}\frac{\partial}{\partial u}, & \mathbf{v_5} = e^{4t}\frac{\partial}{\partial t} + 2xe^{4t}\frac{\partial}{\partial x} - ue^{4t}(1+2x^2)\frac{\partial}{\partial u}, \\ \mathbf{v_3} = e^{-2t}\frac{\partial}{\partial x} + uxe^{-2t}\frac{\partial}{\partial u}, & \mathbf{v_6} = e^{-4t}\frac{\partial}{\partial t} - 2xe^{-4t}\frac{\partial}{\partial x} + ue^{-4t}(1-2x^2)\frac{\partial}{\partial u}, \\ \mathbf{v}_\alpha = \alpha(x,t)\frac{\partial}{\partial u}, \end{cases} \tag{2.60}$$

where $\alpha(x,t)$ is an arbitrary solution of (2.59).
Note that the form of these infinitesimal generators suggests that it might be convenient to combine $\mathbf{v_2}$ with $\mathbf{v_3}$ and $\mathbf{v_5}$ with $\mathbf{v_6}$ for the sake of simplicity in the calculations. Consider now the vector field

$$\begin{aligned} \mathbf{w} &= \frac{\mathbf{v_2} - \mathbf{v_3}}{2} \\ &= \left(\frac{e^{2t}-e^{-2t}}{2}\right)\frac{\partial}{\partial x} - ux\left(\frac{e^{2t}+e^{-2t}}{2}\right)\frac{\partial}{\partial u} \\ &= \sinh(2t)\frac{\partial}{\partial x} - ux\cosh(2t)\frac{\partial}{\partial u}, \end{aligned} \tag{2.61}$$

which is clearly in the lie algebra of (2.59).
We need to solve the system

$$\begin{cases} \frac{d\bar{t}}{d\epsilon} = 0, \quad \bar{t}(0) = t, \\ \frac{d\bar{x}}{d\epsilon} = \sinh(2\bar{t}), \quad \bar{x}(0) = x, \\ \frac{d\bar{u}}{d\epsilon} = -\bar{u}\bar{x}\cosh(2\bar{t}), \quad \bar{u}(0) = u, \end{cases} \tag{2.62}$$

From the first equation in (2.62) it is clear we must have $\bar{t} = t$. Substitution in the next equation gives:

$$\frac{d\bar{x}}{d\epsilon} = \sinh(2\bar{t}) = \sinh(2t) \Leftrightarrow \int d\bar{x} = \int \sinh(2t) d\epsilon \Leftrightarrow \bar{x} = \epsilon \sinh(2t) + C_1$$

and the initial condition $\bar{x}(0) = x$ gives $\bar{x} = x + \epsilon \sinh(2t)$.

The last equation in (2.62) is then

$$\begin{aligned}
\frac{d\bar{u}}{d\epsilon} &= -\bar{u}\bar{x}\cosh(2\bar{t}) = -\bar{u}(x + \epsilon \sinh(2t))\cosh(2t) \\
&= -\bar{u}(x\cosh(2t) + \epsilon \sinh(2t)\cosh(2t)) \\
&= -\bar{u}\left(x\cosh(2t) + \frac{\epsilon \sinh(4t)}{2}\right) \\
\Leftrightarrow \int \frac{d\bar{u}}{\bar{u}} &= -\int \left(x\cosh(2t) + \frac{\epsilon \sinh(4t)}{2}\right) d\epsilon \\
\Leftrightarrow \log \bar{u} &= -x\epsilon \cosh(2t) - \frac{\epsilon^2 \sinh(4t)}{4} + C_2 \\
\Leftrightarrow \bar{u} &= C_3 \exp\left(-x\epsilon\cosh(2t) - \frac{\epsilon^2}{4}\sinh(4t)\right)
\end{aligned}$$

Again, the condition $\bar{u}(0) = u$ gives the result $\bar{u} = u \exp\left(-x\epsilon \cosh(2t) - \frac{\epsilon^2}{4}\sinh(4t)\right)$, which can be expressed in terms of the new independent variables $\bar{x}$ and $\bar{t}$ as

$$\bar{u}(\bar{x}, \bar{t}) = \exp\left(-\bar{x}\epsilon\cosh(2\bar{t}) + \frac{\epsilon^2}{4}\sinh(4\bar{t})\right) u(\bar{x} - \epsilon \sinh(2\bar{t}), \bar{t}).$$

To avoid complicating the notation we will drop the bars from this point on. According to (2.38) we can express the action of $\mathbf{w}$ on any solution u as

$$\rho(\exp(\epsilon \mathbf{w}))u(x,t) = \sigma(x,t;\epsilon)u(a_1(x,t;\epsilon), a_2(x,t;\epsilon)),$$

so in our case it is clear that the changes of variables are $a_1(x,t;\epsilon) = x - \epsilon\sinh(2t)$, $a_2(x,t;\epsilon) = t$ and the multiplier is $\sigma(x,t;\epsilon) = \exp\left(-x\epsilon\cosh(2t) + \frac{\epsilon^2}{4}\sinh(4t)\right)$. Observe that $u(x,t) = \exp\left(-\left(\frac{x^2}{2} + t\right)\right)$ is a solution of our initial PDE (2.59), with initial condition $u(x,0) = \exp\left(-\frac{x^2}{2}\right)$.

So on the one hand we have

$$\begin{aligned}\int_{-\infty}^{\infty}\sigma(y,0;\epsilon)u(a_1(y,0;\epsilon),a_2(y,0;\epsilon))p(t,x,y)dy &= \int_{-\infty}^{\infty} e^{-y\epsilon}u(y,0)p(t,x,y)dy \\ &= \int_{-\infty}^{\infty} e^{-y\epsilon}e^{-\frac{y^2}{2}}p(t,x,y)dy,\end{aligned}$$

while on the other hand we have

$$\sigma(x,t;\epsilon)u(a_1(x,t;\epsilon),a_2(x,t;\epsilon)) = \exp\left(\frac{\epsilon^2\sinh(4t)}{4} - x\epsilon\cosh(2t)\right)u(x-\epsilon\sinh(2t),t)$$

$$= \exp\left(-x\epsilon\cosh(2t)+\frac{\epsilon^2}{4}\sinh(4t)\right)\exp\left(-\left(\frac{(x-\epsilon\sinh(2t))^2}{2}+t\right)\right)$$

$$= \exp\left(-x\epsilon\cosh(2t)+\frac{\epsilon^2}{4}\sinh(4t)-\left(\frac{x^2}{2}+t\right)+\epsilon x\sinh(2t)-\frac{\epsilon^2}{2}\sinh^2(2t)\right)$$

$$= \exp\left(-\left(\frac{x^2}{2}+t\right)-x\epsilon(\cosh(2t)-\sinh(2t))+\frac{\epsilon^2}{4}\frac{e^{4t}-e^{-4t}}{2}-\frac{\epsilon^2}{2}\frac{(e^{2t}-e^{-2t})^2}{4}\right)$$

$$= \exp\left(-\left(\frac{x^2}{2}+t\right)-x\epsilon\left(\frac{e^{2t}+e^{-2t}-e^{2t}+e^{-2t}}{2}\right)+\frac{\epsilon^2}{4}(-e^{-4t}+1)\right)$$

$$= \exp\left(-\left(\frac{x^2}{2}+t\right)-x\epsilon e^{-2t}+\frac{\epsilon^2}{4}(-e^{-4t}+1)\right) \quad (2.63)$$

We now aim to identify the above transformed solution with an integral transform of the fundamental solution $p(t,x,y)$. We have that:

$$\int_{-\infty}^{\infty} e^{-y\epsilon}e^{-\frac{y^2}{2}}p(t,x,y)dy = \exp\left(-\left(\frac{x^2}{2}+t\right)-x\epsilon e^{-2t}+\frac{\epsilon^2}{4}(-e^{-4t}+1)\right)$$

So by making the change of parameter $\epsilon \to i\lambda$ we will easily recognise a Fourier transform:

$$\int_{-\infty}^{\infty} e^{-i\lambda y}e^{-\frac{y^2}{2}}p(t,x,y)dy = \exp\left(-\left(\frac{x^2}{2}+t\right)-i\lambda x e^{-2t}+\frac{\lambda^2}{4}(e^{-4t}-1)\right),$$

and we need only invert this Fourier transform to get:

$$
\begin{aligned}
e^{-\frac{y^2}{2}} p(t,x,y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda y} \exp\left(-\left(\frac{x^2}{2}+t\right) - i\lambda x e^{-2t} + \frac{\lambda^2}{4}(e^{-4t}-1)\right) d\lambda \\
&= \frac{1}{\sqrt{\pi(1-e^{-4t})}} \exp\left(-\frac{x^2-2t+e^{4t}(2t+x^2+2y^2)-4e^{2t}xy}{2(-1+e^{4t})}\right)
\end{aligned}
$$

Finally, dividing by $e^{-\frac{y^2}{2}}$ and after some algebraic manipulation we obtain the fundamental solution

$$
p(t,x,y) = \frac{1}{\sqrt{2\pi\sinh(2t)}} \exp\left(\frac{xy}{\sinh(2t)} - \frac{x^2+y^2}{2\tanh(2t)}\right). \tag{2.64}
$$

Example 2.1.8. Consider the Cauchy problem given by the *forward-propagating* Black-Scholes PDE, with constant risk-free rate r and constant volatility σ

$$
u_t = -ru + rxu_x + \frac{1}{2}\sigma^2 x^2 u_{xx}, \tag{2.65}
$$

subject to the initial condition $u(x,0) = f(x)$. This problem is also considered in [25]. As remarked by Craddock et al. in their paper, this is not the usual form in which option-pricing problems are set up in financial mathematics since instead of a terminal value corresponding to the payoff at expiry, we are providing an initial condition and solving the PDE forward in time.
Using Lie's method for the systematic computation of symmetries one can obtain the following spanning set for the Lie algebra of (2.65)

$$
\begin{cases}
\mathbf{v_1} = x\frac{\partial}{\partial x}, & \mathbf{v_4} = 2t\frac{\partial}{\partial t} + (\ln x - (r-\frac{1}{2}\sigma^2)t)x\frac{\partial}{\partial x} - 2rtu\frac{\partial}{\partial u}, \\
\mathbf{v_2} = \frac{\partial}{\partial t}, & \mathbf{v_5} = -\sigma^2 tx\frac{\partial}{\partial x} + (\ln x + (r-\frac{1}{2}\sigma^2)t)u\frac{\partial}{\partial u}, \\
\mathbf{v_3} = u\frac{\partial}{\partial u}, & \mathbf{v_6} = \frac{\sigma^2 tx}{2}\ln x\frac{\partial}{\partial x} + 2\sigma^2 t^2\frac{\partial}{\partial t} - ((\ln x + (r-\frac{1}{2}\sigma^2)t)^2 + \sigma^2 t(1+2rt))u\frac{\partial}{\partial u}, \\
 & \mathbf{v}_\alpha = \alpha(x,t)\frac{\partial}{\partial u},
\end{cases} \tag{2.66}
$$

where $\alpha(x,t)$ is an arbitrary solution of (2.65).

In order to find a fundamental solution to this PDE, we will use a symmetry generated by the vector field $\mathbf{v_5}$ in (2.66). We need to solve the system:

$$\begin{cases} \frac{d\bar{x}}{d\epsilon} = -\sigma^2\bar{t}\bar{x}, \quad \bar{x}(0) = x, \\ \frac{d\bar{t}}{d\epsilon} = 0, \quad \bar{t}(0) = t, \\ \frac{d\bar{u}}{d\epsilon} = (\ln\bar{x} + (r - \frac{1}{2}\sigma^2)\bar{t})\bar{u}, \quad \bar{u}(0) = u, \end{cases} \tag{2.67}$$

so we must have $\bar{t} = t$ as well as

$$\begin{aligned} \int \frac{d\bar{x}}{\bar{x}} &= -\int \sigma^2\bar{t}d\epsilon = -\int \sigma^2 t d\epsilon, \\ \ln\bar{x} &= -\sigma^2 t\epsilon + C \\ \bar{x} &= e^C e^{-\sigma^2 t\epsilon}. \end{aligned}$$

Using the initial condition $\bar{x}(0) = x$ gives $\bar{x} = xe^{-\sigma^2 t\epsilon}$.
Lastly, we must have

$$\begin{aligned} \int \frac{d\bar{u}}{\bar{u}} &= \int \left(\ln\bar{x} + \left(r - \frac{1}{2}\sigma^2\right)\bar{t}\right) d\epsilon = \int \left(\ln x - \sigma^2 t\epsilon + \left(r - \frac{1}{2}\sigma^2\right)t\right) d\epsilon, \\ \ln\bar{u} &= \epsilon\ln x - \frac{1}{2}\sigma^2 t\epsilon^2 + \left(r - \frac{1}{2}\sigma^2\right)t\epsilon + C \\ \bar{u} &= e^C \exp\left(\ln x^\epsilon - \frac{1}{2}\sigma^2 t\epsilon^2 + \left(r - \frac{1}{2}\sigma^2\right)t\epsilon\right) \\ &= e^C x^\epsilon \exp\left(\left(-\frac{1}{2}\sigma^2\epsilon + r - \frac{1}{2}\sigma^2\right)t\epsilon\right). \end{aligned}$$

Again, setting $\bar{u}(0) = u$ gives $\bar{u} = x^\epsilon \exp\left(\left(-\frac{1}{2}\sigma^2\epsilon + r - \frac{1}{2}\sigma^2\right)t\epsilon\right)u(x,t)$. So the new solution $\bar{u}$ in terms of the new variables $\bar{x}$ and $\bar{t}$ is

$$\begin{aligned} \bar{u}(\bar{x},\bar{t}) &= (\bar{x}e^{\sigma^2\bar{t}\epsilon})^\epsilon \exp\left(\left(-\frac{1}{2}\sigma^2\epsilon + r - \frac{1}{2}\sigma^2\right)\bar{t}\epsilon\right)u(\bar{x}e^{\sigma^2\bar{t}\epsilon},\bar{t}) \\ &= \bar{x}^\epsilon e^{\sigma^2\bar{t}\epsilon^2} \exp\left(\left(-\frac{1}{2}\sigma^2\epsilon + r - \frac{1}{2}\sigma^2)\bar{t}\epsilon\right)\right)u(\bar{x}e^{\sigma^2\bar{t}\epsilon},\bar{t}) \\ &= \bar{x}^\epsilon \exp\left(\left(\frac{1}{2}\sigma^2\epsilon + r - \frac{1}{2}\sigma^2\right)\bar{t}\epsilon\right)u(\bar{x}e^{\sigma^2\bar{t}\epsilon},\bar{t}). \end{aligned}$$

For the sake of simplicity in notation, let us just write:

$$u(x,t;\epsilon) = x^{\epsilon} \exp\left(\left(\sigma^2(\epsilon-1)+2r\right)\frac{t\epsilon}{2}\right) u(xe^{\sigma^2 t\epsilon}, t). \tag{2.68}$$

It is then clear that using the notation in (2.38), the changes of variables are

$$a_1(x,t;\epsilon) = xe^{\sigma^2 t\epsilon}, \qquad a_2(x,t;\epsilon) = t$$

and the multiplier is $\sigma(x,t;\epsilon) = x^{\epsilon}\exp\left((\sigma^2(\epsilon-1)+2r))\frac{t\epsilon}{2}\right)$.
Observe now that a very simple stationary solution to the Black-Scholes PDE (2.65) is $u_0(x) = x$. Applying the symmetry (2.68) to this stationary solution we obtain the new solution

$$u(x,t;\epsilon) = x^{1+\epsilon}\exp\left(\left(\sigma^2(\epsilon+1)+2r\right)\frac{t\epsilon}{2}\right),$$

which has initial state

$$u(x,0;\epsilon) = x^{1+\epsilon}$$

and substituting into (2.41) we get

$$\int_0^{\infty} \underbrace{\sigma(y,0;\epsilon)}_{=y^{\epsilon}} \underbrace{u_0(a_1(y,0;\epsilon))}_{=y} p(t,x,y)dy = \underbrace{\sigma(x,t;\epsilon)u_0(a_1(x,t;\epsilon))}_{=x^{1+\epsilon}\exp\left(\left(\sigma^2(\epsilon+1)+2r\right)\frac{t\epsilon}{2}\right)},$$

so we are left with the integral equation

$$\int_0^{\infty} y^{1+\epsilon}p(t,x,y)dy = x^{1+\epsilon}\exp\left(\left(\sigma^2(\epsilon+1)+2r\right)\frac{t\epsilon}{2}\right). \tag{2.69}$$

We wish to identify the left-hand side of the above equation with a classical integral transform. To do so, we make the change $\epsilon \to s-2$ to obtain

$$\int_0^{\infty} y^{s-1}p(t,x,y)dy = x^{s-1}\exp\left(\left(\sigma^2(s-1)+2r\right)\frac{t(s-2)}{2}\right). \tag{2.70}$$

With this notation we easily recognise the left-hand side as the Mellin Transform (see Appendix A) of the fundamental solution $p(t,x,y)$ with respect to y, i.e.

$$\mathcal{M}\{p(t,x,y)\}(s) = x^{s-1}\exp\left(\left(\sigma^2(s-1)+2r\right)\frac{t(s-2)}{2}\right)$$

Hence we only need to perform a Mellin inversion from s to y to recover the fundamental solution. To do so, we use the relationship (A.6) obtained in Appendix A between the Mellin and Fourier transforms that allows us to convert the Inverse

Mellin transform into an appropriate Inverse Fourier Transform:

$$\begin{aligned}
p(t,x,y) &= \mathcal{M}^{-1}\left\{x^{s-1}\exp\left(\left(\sigma^2(s-1)+2r\right)\frac{t(s-2)}{2}\right)\right\}(y)\\
&= \frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}\left\{x^{-i\omega-1}\exp\left(\left(\sigma^2(-i\omega-1)+2r\right)\frac{t(-i\omega-2)}{2}\right)\right\}(-\ln y)\\
&= \frac{e^{-rt}}{\sigma y\sqrt{2\pi t}}\exp\left(-\frac{(\ln(\frac{y}{x})-(r-\frac{1}{2}\sigma^2)t)^2}{2\sigma^2 t}\right)
\end{aligned}$$

This final expression is precisely the transition probability density function for a *Geometrical Brownian Motion*. This result is well-known in classical Financial Mathematics.

We will mainly use this technique throughout this thesis, though other methods are also used in some sections. In particular, we use this approach in the next chapter to compute fundamental solutions for systems of PDEs.

2.2 Stochastic Processes and Stochastic Calculus: how to find transition density functions

Another fundamental topic on which our research is based is the theory of Stochastic Processes [38] and Stochastic Calculus [51]. We exploit a very interesting link between the study of fundamental solutions and the computation of transition densities for a given diffusion process. We will not present general results on Stochastic Calculus, since the reader can consult the extensive literature written on this subject. We only wish to remark some key concepts that relate our work in constructing fundamental solutions using symmetry methods with the study of diffusion processes. In particular, symmetry methods for the computation of fundamental solutions have proven to be very useful in the calculation of expectations for a wide range of diffusion processes (see [22, 19])

We will briefly give an idea of how these two concepts relate to each other in what follows. Suppose we have an Itô diffusion $X = \{X_t : t \geq 0\}$, which satisfies the Stochastic Differential Equation (SDE)

$$dX_t = b(X_t,t)dt + \sigma(X_t,t)dW_t \quad X_0 = x, \tag{2.71}$$

where $W = \{W_t : t \geq 0\}$ is a standard Wiener process.

The following result, which can be found with its proof in [51], tells us what conditions b and σ must satisfy so that the SDE (2.71) has a unique strong solution:

Theorem 2.2.1 (Existence and uniqueness theorem for SDEs [51]). *Let $T > 0$ and $b(x,t)$, $\sigma(x,t)$ be measurable functions with $b(\cdot,\cdot) : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$, $\sigma(\cdot,\cdot) : \mathbb{R}^n \times [0,T] \to \mathbb{R}^{n\times m}$ and $0 \le t \le T$ satisfying*

$$|b(x,t)| + |\sigma(x,t)| \le C(1+|x|); \quad x \in \mathbb{R}^n, \quad t \in [0,T] \tag{2.72}$$

for some constant C, where $|\sigma|^2 = \sum |\sigma_{ij}|^2$, and such that

$$|b(x,t) - b(y,t)| + |\sigma(x,t) - \sigma(y,t)| \le D|x-y|; \quad x,y \in \mathbb{R}^n, \quad t \in [0,T] \tag{2.73}$$

for some constant D. Let Z be a random variable which is independent of the σ-algebra $\mathcal{F}_\infty^{(m)}$ generated by $B_s(\cdot)$, $s \ge 0$ and such that

$$E[|Z|^2] < \infty.$$

Then the SDE $dX_t = b(X_t,t)dt + \sigma(X_t,t)dB_t$, with $0 \le t \le T$, $X_0 = Z$ has a unique t-continuous solution $X_t(\omega)$ with the property that $X_t(\omega)$ is adapted to the filtration $\mathcal{F}_t^Z$ generated by Z and $B_s(\cdot)$; $s \le t$ and

$$E\left[\int_0^T |X_t|^2 dt\right] < \infty$$

Now imagine that we have functions b and σ satisfying the conditions in Theorem 2.2.1, so that (2.71) has a unique strong solution, then the expectations

$$u(x,t) = E^x[\phi(X_t)] := E[\phi(X_t)|X_0 = x] \tag{2.74}$$

are solutions to a specific Cauchy problem given by the so called *Kolmogorov's Backward Equation*

Theorem 2.2.2 (Kolmogorov's Backward Equation). *Let $f \in C_0^2(\mathbb{R}^n)$*

(a) Define $u(x,t) = E^x[f(X_t)]$, then $u(\cdot,t) \in \mathcal{D}_A$ for each t and

$$\frac{\partial u}{\partial t} = Au, \quad t > 0, x \in \mathbb{R}^n \tag{2.75}$$

$$u(x,0) = f(x); \quad x \in \mathbb{R}^n \tag{2.76}$$

where $\mathcal{D}_A$ denotes the set of functions for which the generator A of X_t is defined for all $x \in \mathbb{R}^n$, and Au is interpreted as A applied to the function $x \to u(x,t)$

(b) Moreover, if $w(x,t) \in C^{2,1}(\mathbb{R}^n \times \mathbb{R})$ is a bounded function satisfying (2.75), (2.76) then $w(x,t) = u(x,t) = E^x[f(X_t)]$

We refer the reader to [51] for a proof of this result.

In the above theorem the infinitesimal generator A of the process X_t is mentioned. Recall (see for example Theorem 7.3.3 in [51]) the expression for the generator A of an Itô diffusion of the form

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t.$$

We know that if $f \in \mathcal{D}_A$ then

$$Af(x) = \sum_i b_i(x)\frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{i,j}(\sigma\sigma^T)_{i,j}(x)\frac{\partial f}{\partial x_i \partial x_j}. \tag{2.77}$$

It can be easily seen from the previous expression that for a diffusion given as the solution to the SDE (2.71) the generator A can be expressed as $A_t f(x) = b(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x)$.

The combination of the two previous results yields the Cauchy problem that u, defined as in (2.74), solves

$$u_t = b(x)u_x + \frac{1}{2}\sigma^2(x)u_{xx} \tag{2.78}$$

$$u(x,0) = \phi(x) \tag{2.79}$$

It is in this context where the study of fundamental solutions can be applied. Note that if $p(x,y,t)$ is an appropriate fundamental solution of the above Cauchy problem, then we can compute the expectations as an integral transform of such fundamental solution, i.e.

$$E^x[\phi(X_t)] = \int_\Omega \phi(y)p(x,y,t)dy \tag{2.80}$$

With this expression, the fundamental solution $p(x,y,t)$ is the probability transition density for the process. Note, however, that our Cauchy problem may in general have many different fundamental solutions, from which only one can be regarded as the probability transition density for the process. For instance, observe that in order for $p(x,y,t)$ to be a probability transition density, we need that $\int_\Omega p(x,y,t)dy = 1$

This tells us that finding fundamental solutions to the given Cauchy problem is in general not enough to obtain probability transition densities. This is precisely one of the main problems of using group-invariant solution methods or reduction to canonical form which, in a wide number of cases, produce fundamental solutions but fail to produce probability transition densities.

In [19], Craddock developed a method for finding fundamental solutions that are indeed probability transition densities. He observed that integral transform methods for finding transition densities could potentially produce two non equivalent fundamental solutions when applied to the Kolmogorov forward equation for a squared Bessel process of dimension $2n$ (see theorem 2.2.4). Interestingly, both fundamental solutions satisfied the conditions one would look for in a TDF a priori (i.e. they both integrated to 1, were positive functions, etc), but only one could indeed be the TDF, since the TDF for a squared Bessel process is known to be unique. He introduced the following result for a particular type of stochastic process that ensures uniqueness of solutions and thus guarantees that integral transform methods will produce a TDF:

Proposition 2.2.3. *Let $X = \{X_t : t \geq 0\}$ be an Itô diffusion which is the unique strong solution of*

$$X_t = X_0 + \int_0^t f(X_s)ds + \int_0^t \sqrt{2\sigma X_t}dW_t;$$

where $W = \{W_t : t \geq 0\}$ is a standard Wiener process. Suppose further that f is measurable and there exist constants $K > 0$; $a > 0$ such that $|f(x)| \leq Ke^{ax}$ for all x. Then there exists a $T > 0$ such that $u(x, t, \lambda) = \mathbf{E}^x[e^{-\lambda X_t}]$ is the unique solution of the first order PDE

$$\frac{\partial u}{\partial t} + \lambda^2\sigma\frac{\partial u}{\partial \lambda} + \lambda\mathbf{E}^x[f(X_t)e^{-\lambda X_t}] = 0$$

subject to $u(x, 0, \lambda) = e^{-\lambda x}$, for $0 \leq t < T$; $\lambda > a$.

A proof of this result can be found in [19]. Similar results can be derived for other kinds of stochastic processes.

Let us proceed with an example to illustrate the above theory. A similar study for this example can also be found in [19]

Example 2.2.1. We wish to obtain the transition density for the Cox- Ingersoll-Ross (CIR) process of interest rate modelling. Let $X = \{X_t : t \geq 0\}$ satisfying the SDE

$$dX_t = (a - bX_t)dt + \sqrt{2\sigma X_t}dW_t, \quad X_0 = x \tag{2.81}$$

Note that the Kolmogorov Backwards equation gives us

$$u_t = \sigma x u_{xx} + (a - bx)u_x \tag{2.82}$$

as the PDE we need to work with to find the TDF for a CIR process of the above type. We start working with the trivial solution $u(x, t) = 1$. Computation of the symmetries of the PDE (2.82) gives the following solution (see [19] to consult elements of the Lie algebra of this equation):

$$u_\epsilon(x,t) = \frac{b^{\frac{a}{\sigma}} \exp\left(\frac{ab}{\sigma}t\right)}{(\epsilon\sigma(e^{bt}-1)+be^{bt})^{\frac{a}{\sigma}}} \exp\left(-\frac{\epsilon bx}{\epsilon\sigma(e^{bt}-1)+be^{bt}}\right) \tag{2.83}$$

Observe that $u_\epsilon(x,0) = e^{-\epsilon x}$. Now take $U(x,t) = \int_0^\infty \phi(\epsilon)u_\epsilon(x,t)d\epsilon$, which is a solution according to Theorem 2.1.5. We can see how the initial condition for this new solution is precisely the Laplace transform of the function ϕ:

$$U(x,0) = \int_0^\infty \phi(\epsilon)u_\epsilon(x,0)d\epsilon = \int_0^\infty \phi(\epsilon)e^{-\epsilon x}d\epsilon = \Phi(x)$$

Here Φ denotes the Laplace transform of ϕ. Therefore if we can express the transformed solution $u_\epsilon(x,t)$ as the Laplace transform of some suitable function $p(x,y,t)$, it is easy to see that

$$\begin{aligned} U(x,t) &= \int_0^\infty \phi(\epsilon)u_\epsilon(x,t)d\epsilon = \int_0^\infty \phi(\epsilon)\left(\int_0^\infty p(x,y,t)e^{-\epsilon y}dy\right)d\epsilon \\ &= \int_0^\infty \int_0^\infty \phi(\epsilon)p(x,y,t)e^{-\epsilon y}d\epsilon dy = \int_0^\infty p(x,y,t)\int_0^\infty \phi(\epsilon)e^{-\epsilon y}d\epsilon dy \\ &= \int_0^\infty \Phi(y)p(x,y,t)dy \end{aligned}$$

So we have that $U(x,t) = \int_0^\infty \Phi(y)p(x,y,t)dy$ with $U(x,0) = \Phi(x)$. Hence, if p satisfies all the appropriate conditions (namely it is a positive function, it integrates to 1, etc.), it is potentially the transition density for the CIR process X_t satisfying (2.81).
The inverse Laplace transform of u_ϵ can be calculated to be the following

$$p(x,y,t) = \frac{b\exp\left(b(\frac{a}{\sigma}+1)t\right)}{\sigma(e^{bt}-1)}\left(\frac{y}{x}\right)^{\frac{\frac{a}{\sigma}-1}{2}} \exp\left(-\frac{b(x+e^{bt}y)}{\sigma(e^{bt}-1)}\right) I_{\frac{a}{\sigma}-1}\left(\frac{b\sqrt{xy}}{\sigma\sinh(\frac{bt}{2})}\right) \tag{2.84}$$

Observe that $\int_0^\infty p(x,y,t)dy = 1$. Proposition 2.2.3 guarantees that this is indeed the TDF for our process.

It is important to point out that, even though throughout this thesis we will always be obtaining transition densities for a process using the Kolmogorov backward equation, we could have decided to use the Kolmogorov forward equation instead. We have chosen the former because it suits our purposes better, but the latter could work equally well. The Kolmogorov forward equation theorem states the following:

Theorem 2.2.4 (Kolmogorov's Forward Equation). *Let X_t be an Itô diffusion in $\mathbb{R}^n$ with generator*

$$Af(y) = \sum_{i,j} a_{ij}(y) \frac{\partial^2 f}{\partial y_i \partial y_j} + \sum_i b_i(y) \frac{\partial f}{\partial y_i}; \quad f \in C_0^2 \tag{2.85}$$

where $a_{ij} \in C^2(\mathbb{R}^n)$ and $b_i \in C^1(\mathbb{R}^n)$ for all i, j and assume that the transition measure of X_t has a density $p_t(x, y)$, i.e. that

$$E^x[f(X_t)] = \int_{\mathbb{R}^n} f(y) p_t(x, y) dy; \quad f \in C_0^2. \tag{2.86}$$

Assume that $y \to p_t(x, y)$ is smooth for each t, x. Then $p_t(x, y)$ satisfies the Kolmogorov forward equation

$$\frac{d}{dt} p_t(x, y) = A_y^* p_t(x, y) \text{ for all } x, y, \tag{2.87}$$

where A_y^* operates on the variable y and is given by

$$A_y^* \phi(y) = \sum_{i,j} \frac{\partial^2}{\partial y_i \partial y_j}(a_{ij}\phi) - \sum_i \frac{\partial}{\partial y_i}(b_i \phi); \quad \phi \in C^2, \tag{2.88}$$

i.e. A_y^* is the adjoint of A_y.

Another very useful result and one we will extensively use is a generalization of the Kolmogorov backward equation theorem: The Feynman-Kac formula.

Theorem 2.2.5 (Feynman-Kac Formula [51]). *Let $f \in C_0^2(\mathbb{R}^n)$ and $q \in C(\mathbb{R}^n)$. Assume that q is lower bounded.*

(a) Put

$$v(x, t) = E^x \left[\exp\left(-\int_0^t q(X_s) ds \right) f(X_t) \right] \tag{2.89}$$

Then

$$\frac{\partial v}{\partial t} = Av - qv; \quad t > 0, x \in \mathbb{R}^n \tag{2.90}$$

$$v(x, 0) = f(x); \quad x \in \mathbb{R}^n \tag{2.91}$$

(b) Moreover, if $w(x, t) \in C^{2,1}(\mathbb{R}^n \times \mathbb{R})$ is bounded on $\mathbb{R}^n \times K$ for each compact $K \subset \mathbb{R}$ and solves (2.90), (2.91), then $w(x, t) = v(x, t)$, given by (2.89).

This theorem will allow us to compute the expectations of some functionals of a diffusion process. In particular, we will be using this theorem in Chapter 5 to compute functionals of the eigenvalues of a 2×2 Wishart process.

Chapter 3

Integral transform methods for the computation of fundamental solutions for a system of PDEs

In this chapter we explore the computation of fundamental solutions via integral transform methods as described in chapter one. However, instead of focusing in single PDEs we extend the scope of our study to linear parabolic systems of PDEs. In particular, we study one family of 2-dimensional systems. We include the computation of symmetries of some specific subfamilies as well as the computation of their fundamental solutions in each case.

Many authors have considered the problem of finding lie symmetries for particular systems of PDEs. For example, Olver computes the lie symmetries of the Euler equations in [52]; the Navier-Stokes equations are treated in [13] or [50], and these and some other examples are also considered by Ibragimov in [42]. There doesn't seem to be much literature available on the use of these symmetries for the computation of fundamental solutions. However, Ortner and Wagner use Fourier analysis to compute fundamental solutions for linear systems of PDEs with constant coefficients in [55]. Craddock provides an example of a computation of a fundamental matrix for a particular system in [18], which turns out to be incorrect. We will correct this here.

Let $u(x,t) : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$, $v(x,t) : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$. Consider a system of PDEs of the form

$$\begin{cases} u_t = u_{xx} + f(x)v \\ v_t = v_{xx} + g(x)u \end{cases} \tag{3.1}$$

The aim is to find functions $f(x)$ and $g(x)$ for which the system has non-trivial symmetries. This is not a simple question to address, and most of the times we will not be able to find all the possible classes of functions f and g for which we can find symmetries. In section 2.1 we have shown how to compute symmetries for a given system of differential equations. However, a more complicated step is to find a general class of functions for which we can find non-trivial symmetries of a system of PDEs with some undetermined coefficient functions. Some work along these lines has been done for a single PDE by Craddock and his co-authors in [25, 24], for example.

We will address this problem for this particular system (3.1) by first using Lie's method to find the symmetries of the system (3.1) for general functions f and g. This will produce a set of *determining equations* in terms of $f(x)$ and $g(x)$ for the symmetry group of this system. Looking at these equations we will then analyse what conditions our coefficient functions must satisfy and hence determine (if possible) the class of functions for which the system has non-trivial symmetries. Once these symmetries have been obtained, we will focus on each case separately to produce an expression for a fundamental matrix. Note that the problem of finding a fundamental solution described as in Definition 2.1.4 or Definition 2.1.5 for a single PDE extends in this case to the following:
Let $L = (L_{i,j})$, be the 2×2 matrix linear differential operator defined by

$$L = \begin{pmatrix} \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} & -f(x) \\ -g(x) & \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \end{pmatrix} \tag{3.2}$$

on an appropriate domain Ω. We are looking for a 2×2 matrix $P(t, x, y) = (p_{i,j})$ with the property that

$$LP = 0$$

and

$$\lim_{t \to 0} P(t, x, y) = \delta_y(x) I_2.$$

In the above expressions, 0 refers to the 2×2 zero matrix and I_2 is the 2-dimensional identity matrix. That is, we are looking for a matrix

$$P(t, x, y) = \begin{pmatrix} p_{1,1}(t, x, y) & p_{1,2}(t, x, y) \\ p_{2,1}(t, x, y) & p_{2,2}(t, x, y) \end{pmatrix}, \tag{3.3}$$

whose columns are solutions to the system

$$L \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and with the property that

$$\left[\int \begin{pmatrix} p_{1,1}(t,x,y) & p_{1,2}(t,x,y) \\ p_{2,1}(t,x,y) & p_{2,2}(t,x,y) \end{pmatrix} \begin{pmatrix} f_1(y) \\ f_2(y) \end{pmatrix} dy \right]_{t=0} = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} \tag{3.4}$$

Remark. Note that both the matrix P and the integral (3.4) must be defined over an appropriate domain.

3.1 Computation of symmetries

Let us look for a vector field of the form

$$\mathbf{w} = \xi \frac{\partial}{\partial x} + \tau \frac{\partial}{\partial t} + \phi \frac{\partial}{\partial u} + \eta \frac{\partial}{\partial v} \tag{3.5}$$

that will map any solution of our system (3.1) to another solution of such system.

We need to know how the derivatives of u and v behave under the action of $\mathbf{w}$, so we need to compute the second prolongation of $\mathbf{w}$ (see section 2.1 or [52]), which will act not only on the independent variables x, t and the dependent variables $u(x,t)$, $v(x,t)$ but also on the derivatives of u and v up to second order. Therefore, $\mathsf{pr}^{(2)}\mathbf{w}$ will be of the form

$$\begin{aligned} \mathsf{pr}^{(2)}\mathbf{w} = \mathbf{w} + \phi^x \frac{\partial}{\partial u_x} + \phi^t \frac{\partial}{\partial u_t} + \phi^{xx} \frac{\partial}{\partial u_{xx}} + \phi^{xt} \frac{\partial}{\partial u_{xt}} + \phi^{tt} \frac{\partial}{\partial u_{tt}} + \\ \eta^x \frac{\partial}{\partial v_x} + \eta^t \frac{\partial}{\partial v_t} + \eta^{xx} \frac{\partial}{\partial v_{xx}} + \eta^{xt} \frac{\partial}{\partial v_{xt}} + \eta^{tt} \frac{\partial}{\partial v_{tt}}, \end{aligned} \tag{3.6}$$

where the expressions for the coefficient functions will later be calculated using the formula (2.6). We want $\mathbf{w}$ to map solutions of the system to other solutions so, according to Lie's theorem (Theorem 2.1.2), we must have that

$$\mathsf{pr}^{(2)}\mathbf{w} \begin{bmatrix} u_t - u_{xx} - f(x)v \\ v_t - v_{xx} - g(x)u \end{bmatrix} = 0 \tag{3.7}$$

whenever

$$\begin{aligned} u_t - u_{xx} - f(x)v = 0 \\ v_t - v_{xx} - g(x)u = 0 \end{aligned} \tag{3.8}$$

Therefore, the following equations must be satisfied

$$\begin{cases} \phi^t = \phi^{xx} + \xi f'(x)v + f(x)\eta \\ \eta^t = \eta^{xx} + \xi g'(x)u + g(x)\phi \end{cases} \tag{3.9}$$

Recall now the formula for the coefficient functions (2.6), which can be found in [52], and which in our case translates to the following:

$$\begin{aligned} \phi^t &= D_t(\phi - \xi u_x - \tau u_t) + \xi u_{xt} + \tau u_{tt} \\ \phi^{xx} &= D_{xx}(\phi - \xi u_x - \tau u_t) + \xi u_{xxx} + \tau u_{xxt} \\ \eta^t &= D_t(\eta - \xi v_x - \tau v_t) + \xi v_{xt} + \tau v_{tt} \\ \eta^{xx} &= D_{xx}(\eta - \xi v_x - \tau v_t) + \xi v_{xxx} + \tau v_{xxt} \end{aligned}$$

Computation of the appropriate derivatives gives the following expression for these coefficient functions:

$$\begin{aligned} \phi^t &= \phi_t - \xi_t u_x + (\phi_u - \tau_t)u_t + \phi_v v_t - \xi_u u_x u_t - \xi_v u_x v_t - \tau_u u_t^2 - \tau_v u_t v_t \\ \phi^{xx} &= \phi_{xx} + (2\phi_{xu} - \xi_{xx})u_x - \tau_{xx}u_t + 2\phi_{xv}v_x + (\phi_{uu} - 2\xi_{xu})u_x^2 - 2\tau_{xu}u_t u_x \\ &\quad + (2\phi_{uv} - 2\xi_{xv})u_x v_x - 2\tau_{xv}u_t v_x + \phi_{vv}v_x^2 - \xi_{uu}u_x^3 - 2\xi_{uv}u_x^2 v_x - \tau_{uu}u_x^2 u_t \\ &\quad - 2\tau_{uv}u_x u_t v_x - \tau_{vv}u_t v_x^2 - \xi_{vv}u_x v_x^2 + (\phi_u - 2\xi_x)u_{xx} - 2\tau_x u_{xt} + \phi_v v_{xx} \\ &\quad - 3\xi_u u_x u_{xx} - \tau_u u_t u_{xx} - 2\tau_u u_x u_{xt} - 2\xi_v v_x u_{xx} - 2\tau_v v_x u_{xt} - \xi_v u_x v_{xx} \\ &\quad - \tau_v u_t v_{xx} \\ \eta^t &= \eta_t - \xi_t v_x + (\eta_v - \tau_t)v_t + \eta_u u_t - \xi_v v_x v_t - \xi_u v_x u_t - \tau_v v_t^2 - \tau_u v_t u_t \\ \eta^{xx} &= \eta_{xx} + (2\eta_{xv} - \xi_{xx})v_x - \tau_{xx}v_t + 2\eta_{xu}u_x + (\eta_{vv} - 2\xi_{xv})v_x^2 - 2\tau_{xv}v_x v_t \\ &\quad + (2\eta_{uv} - 2\xi_{xu})v_x u_x - 2\tau_{xu}v_t u_x + \eta_{uu}u_x^2 - \xi_{vv}v_x^3 - 2\xi_{uv}v_x^2 u_x - \tau_{vv}v_x^2 v_t \\ &\quad - 2\tau_{uv}v_x v_t u_x - \tau_{uu}v_t u_x^2 - \xi_{uu}v_x u_x^2 + (\eta_v - 2\xi_x)v_{xx} - 2\tau_x v_{xt} + \eta_u u_{xx} \\ &\quad - 3\xi_v v_x v_{xx} - 2\tau_v v_x v_{xt} - 2\xi_u u_x v_{xx} - \tau_v v_t v_{xx} - 2\tau_u u_x v_{xt} - \xi_u v_x u_{xx} \\ &\quad - \tau_u v_t u_{xx} \end{aligned} \tag{3.10}$$

The next step is to substitute the above expressions (3.10) into the system (3.9) and equate the coefficients of u, v and their respective derivatives.

The coefficients of u_{xt} and v_{xt} give:

$$\begin{cases} 0 = -2\tau_x - 2\tau_u u_x - 2\tau_v v_x \\ 0 = -2\tau_x - 2\tau_v v_x - 2\tau_u u_x, \end{cases} \tag{3.11}$$

which means that $\tau_x = \tau_v = \tau_u = 0$, so the coefficient function τ only depends on t: $\tau = \tau(t)$

Next, the coefficients of u_{xx} and v_{xx} produce:

$$\begin{cases} \phi_u - \tau_t - \xi_u u_x = \phi_u - 2\xi_x - 3\xi_u u_x - 2\xi_v v_x \\ \eta_v - \tau_t - \xi_v v_x = \eta_v - 2\xi_x - 3\xi_v v_x - 2\xi_u u_x, \end{cases} \tag{3.12}$$

and hence $\xi_u = \xi_v = 0$ so $\xi = \xi(x,t)$ and $\tau_t = 2\xi_x$, giving

$$\xi(x,t) = \frac{1}{2}\tau_t x + \sigma(t)$$

To continue, consider the coefficients of u_x^2 and v_x^2, from which we get $\phi_{uu} = \eta_{uu} = \phi_{vv} = \eta_{vv} = 0$. Note that this translates into both ϕ and η being of the form:

$$\begin{aligned} \phi(x,t,u,v) &= A(x,t) + B(x,t)u + C(x,t)v + D(x,t)uv \\ \eta(x,t,u,v) &= \alpha(x,t) + \beta(x,t)u + \gamma(x,t)v + \delta(x,t)uv \end{aligned}$$

We will now proceed to look at the coefficients of u_x and v_x, producing the following equations:

$$\begin{cases} \xi_t &= -2\phi_{xu} - 2\phi_{uv}v_x \\ 0 &= 2\eta_{xu} \\ \xi_t &= -2\eta_{xv} - 2\eta_{uv}u_x \\ 0 &= 2\phi_{xv}, \end{cases} \tag{3.13}$$

and so we must have $D(x,t) = \delta(x,t) = 0$ and $C_x = \beta_x = 0$ (meaning that $C = C(t)$, $\beta = \beta(t)$). Note also that

$$\phi_{xu} = B_x(x,t) = -\frac{1}{2}\xi_t = -\frac{1}{2}\left(\frac{1}{2}\tau_{tt}x + \sigma'(t)\right) = \gamma_x(x,t) = \eta_{xv},$$

which yields the following expression for $B(x,t)$ and $\gamma(x,t)$:

$$B(x,t) = -\frac{x^2}{8}\tau_{tt} - \frac{1}{2}x\sigma'(t) + K_1(t)$$

$$\gamma(x,t) = -\frac{x^2}{8}\tau_{tt} - \frac{1}{2}x\sigma'(t) + K_2(t)$$

So, at the moment we have the following information regarding the coefficient functions τ, ξ, ϕ and η:

$$\begin{cases} \tau = \tau(t) \\ \xi = \xi(x,t) = \frac{\tau_t}{2}x + \sigma(t) \\ \phi = \phi(x,t,u,v) = A(x,t) + \left(-\frac{\tau_{tt}}{8}x^2 - \frac{\sigma'(t)}{2}x + K_1(t)\right)u + C(t)v \\ \eta = \eta(x,t,u,v) = \alpha(x,t) + \left(-\frac{\tau_{tt}}{8}x^2 - \frac{\sigma'(t)}{2}x + K_2(t)\right)v + \beta(t)u \end{cases} \tag{3.14}$$

Finally, equating the remaining terms produces the following system:

$$\begin{cases} \phi_t + (\phi_u - \tau_t)f(x)v + \phi_v g(x)u = \phi_{xx} + \xi f'(x)v + f(x)\eta \\ \eta_t + (\eta_v - \tau_t)g(x)u + \eta_u f(x)v = \eta_{xx} + \xi g'(x)u + g(x)\phi. \end{cases} \tag{3.15}$$

The reader may check that substitution of the coefficient functions τ, ξ, ϕ and η and their respective derivatives, according to the forms in (3.14), yields the following system of equations:

$$\begin{aligned} &A_t(x,t) + u\left(-\frac{\tau_{ttt}}{8}x^2 - \frac{\sigma''(t)}{2}x + K_1'(t)\right) + C'(t)v + f(x)v\left(K_1(t) - \tau_t\right) + C(t)g(x)u \\ &= A_{xx}(x,t) - \frac{\tau_{tt}}{4}u + f'(x)v\left(\frac{\tau_t}{2}x + \sigma(t)\right) + f(x)\left(vK_2(t) + u\beta(t) + \alpha(x,t)\right) \end{aligned} \tag{3.16}$$

$$\begin{aligned} &\alpha_t(x,t) + v\left(-\frac{\tau_{ttt}}{8}x^2 - \frac{\sigma''(t)}{2}x + K_2'(t)\right) + \beta'(t)u + g(x)u\left(K_2(t) - \tau_t\right) + \beta(t)f(x)v \\ &= \alpha_{xx}(x,t) - \frac{\tau_{tt}}{4}v + g'(x)u\left(\frac{\tau_t}{2}x + \sigma(t)\right) + g(x)\left(uK_1(t) + vC(t) + A(x,t)\right) \end{aligned} \tag{3.17}$$

If we consider the terms in (3.16), (3.17) not involving neither u nor v, we realise that

$$\begin{pmatrix} A(x,t) \\ \alpha(x,t) \end{pmatrix}$$

must be a solution of the initial system (3.1), since (3.16) and (3.17) give

$$\begin{cases} A_t(x,t) - A_{xx}(x,t) - f(x)\alpha(x,t) = 0 \\ \alpha_t(x,t) - \alpha_{xx}(x,t) - g(x)A(x,t) = 0. \end{cases}$$

Also, by separately looking at the terms involving u and the terms involving v, we obtain the following system of equations

$$-\frac{\tau_{ttt}}{8}x^2 - \frac{\sigma''(t)}{2}x + \frac{\tau_{tt}}{4} + K_1'(t) + C(t)g(x) - f(x)\beta(t) = 0 \tag{3.18}$$

$$C'(t) + f(x)\left(K_1(t) - \tau_t - K_2(t)\right) - f'(x)\left(\frac{\tau_t}{2}x + \sigma(t)\right) = 0 \tag{3.19}$$

$$-\frac{\tau_{ttt}}{8}x^2 - \frac{\sigma''(t)}{2}x + \frac{\tau_{tt}}{4} + K_2'(t) + \beta(t)f(x) - g(x)C(t) = 0 \tag{3.20}$$

$$\beta'(t) + g(x)\left(K_2(t) - \tau_t - K_1(t)\right) - g'(x)\left(\frac{\tau_t}{2}x + \sigma(t)\right) = 0 \tag{3.21}$$

As it can be seen from the above system of equations, it is not easy to determine a general class of functions for $f(x)$ and $g(x)$ for which the initial system of PDEs (3.1) has non-trivial symmetries. For any system of PDEs with undetermined functions, this will be the case in general: we are left with a rather complicated system of equations for which it is not an easy task to find the most general solution.

However, the above system can be simplified by adding (3.18) and (3.20):

$$-\frac{\tau_{ttt}}{4}x^2 - \sigma_{tt}x + \frac{\tau_{tt}}{2} + K_1'(t) + K_2'(t) = 0, \tag{3.22}$$

which gives

$$\tau_{ttt} = 0, \quad \sigma_{tt} = 0, \quad \frac{\tau_{tt}}{2} + K_1'(t) + K_2'(t) = 0.$$

Hence

$$\tau = C_1t^2 + C_2t + C_3, \quad \sigma = C_4t + C_5, \quad K_1(t) + K_2(t) = -C_1t + C_6 \tag{3.23}$$

where C_i, $i = 1, \dots 6$ are arbitrary constants. Substitution into the remaining equations (3.19), (3.21) gives:

$$C'(t) + f(x)\left(2K_1(t) - C_1t - C_2 - C_6\right) - f'(x)\left(\frac{(2C_1t + C_2)}{2}x + C_4t + C_5\right) = 0 \tag{3.24}$$

$$\beta'(t) + g(x)\left(-2K_1(t) - 3C_1t + C_6 - C_2\right) - g'(x)\left(\frac{(2C_1t + C_2)}{2}x + C_4t + C_5\right) = 0 \tag{3.25}$$

Differentiation with respect to t twice yields

$$C'''(t) + 2f(x)K_1''(t) = 0 \tag{3.26}$$

$$\beta'''(t) - 2g(x)K_1''(t) = 0 \tag{3.27}$$

and further differentiation of the above equations with respect to x produces:

$$2f'(x)K_1''(t) = 0 \tag{3.28}$$
$$-2g'(x)K_1''(t) = 0 \tag{3.29}$$

The choice of $K_1'' = 0$ or $K_1'' \neq 0$ will then result in different cases of functions $f(x)$ and $g(x)$ for which it is possible to compute the symmetries of the system. However, since the main purpose here is to ultimately compute fundamental solutions, we will only include a few cases for which the Lie algebra contains enough symmetries to compute these fundamental matrices.

3.1.1 Case A: τ quadratic and $\sigma \equiv 0$

In this case, we have $\tau(t) = C_1t^2 + C_2t + C_3$ and equations(3.18)-(3.21) become

$$\frac{C_1}{2} + K_1'(t) + C(t)g(x) - f(x)\beta(t) = 0 \tag{3.30}$$
$$C'(t) + f(x)\left(K_1(t) - 2C_1t - C_2 - K_2(t)\right) - xf'(x)\left(C_1t + \frac{C_2}{2}\right) = 0 \tag{3.31}$$
$$\frac{C_1}{2} + K_2'(t) + \beta(t)f(x) - g(x)C(t) = 0 \tag{3.32}$$
$$\beta'(t) + g(x)\left(K_2(t) - 2C_1t - C_2 - K_1(t)\right) - xg'(x)\left(C_1t + \frac{C_2}{2}\right) = 0 \tag{3.33}$$

Note from (3.31) that we must have $kf(x) = xf'(x)$, so the function f must be $f(x) = \rho_1x^k$. This particular form of f transforms equation (3.31) into:

$$C'(t) + \rho_1x^k\left(K_1(t) - 2C_1t - C_2 - K_2(t)\right) - \rho_1kx^k\left(C_1t + \frac{C_2}{2}\right) = 0$$

Hence $C'(t) = 0$ and thus we can write $C(t) = \gamma_1$. Moreover, we must have

$$K_1(t) - K_2(t) = C_1t(2 + k) + C_2\left(1 + \frac{k}{2}\right) \tag{3.34}$$

Similarly, (3.33) gives $g(x) = \rho_2x^q$ and so it becomes

$$\beta'(t) + \rho_2x^q\left(K_2(t) - 2C_1t - C_2 - K_1(t)\right) - \rho_2qx^q\left(C_1t + \frac{C_2}{2}\right) = 0,$$

giving $\beta(t) = \gamma_2$ and

$$K_2(t) - K_1(t) = C_1t(2 + q) + C_2\left(1 + \frac{q}{2}\right) \tag{3.35}$$

The combination of equations (3.34) and (3.35) yields $q = -(4+k)$.

Next, substitution of the previously obtained forms for f and g into equations (3.30) and (3.32) produces the following equations:

$$\frac{C_1}{2} + K_1'(t) + \gamma_1\rho_2 x^{-(4+k)} - \gamma_2\rho_1 x^k = 0 \tag{3.36}$$

$$\frac{C_1}{2} + K_2'(t) + \gamma_2\rho_1 x^k - \gamma_1\rho_2 x^{-(4+k)} = 0 \tag{3.37}$$

One must distinguish now between 3 cases that will produce different sets of symmetries:

Case A.1: The case $k = -(4+k)$, i.e. $k = q = -2$

This case corresponds to the system

$$\begin{cases} u_t = u_{xx} + \frac{\rho_1}{x^2}v \\ v_t = v_{xx} + \frac{\rho_2}{x^2}u \end{cases} \tag{3.38}$$

For this particular case, the reader may check that the combination of equations (3.36), (3.37) and (3.35) yields

$$\begin{cases} K_1(t) = -\frac{C_1}{2}t + C_4 \\ K_2(t) = -\frac{C_1}{2}t + C_4 \\ \gamma_1\rho_2 - \gamma_2\rho_1 = 0, \end{cases} \tag{3.39}$$

and thus, the coefficient functions must be of the form:

$$\begin{cases} \tau(t) = C_1t^2 + C_2t + C_3 \\ \xi(x,t) = C_1tx + \frac{C_2}{2}x \\ \phi(x,t,u,v) = \left(-C_1\left(\frac{x^2}{4} + \frac{1}{2}t\right) + C_4\right)u + \gamma_1 v + A(x,t) \\ \eta(x,t,u,v) = \left(-C_1\left(\frac{x^2}{4} + \frac{1}{2}t\right) + C_4\right)v + \gamma_1\frac{\rho_2}{\rho_1}u + \alpha(x,t), \end{cases} \tag{3.40}$$

with the pair $(A(x,t), \alpha(x,t))$ any solution of the system (3.38).

Note. In this case, since f and g differ only by a constant, we have found similar coefficient functions ϕ and η and, therefore, these will produce similar transformations for u and v.

Substituting these coefficient functions obtained in (3.40) into the expression for our general infinitesimal symmetry

$$\mathbf{w} = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \phi\frac{\partial}{\partial u} + \eta\frac{\partial}{\partial v}$$

gives a general form for an element of the lie algebra of (3.38). Therefore, we can conclude that the lie algebra of our system (3.38) is spanned by the following infinitesimal generators

$$\begin{cases} \mathbf{w}_1 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} - u\left(\frac{x^2}{4} + \frac{t}{2}\right)\frac{\partial}{\partial u} - v\left(\frac{x^2}{4} + \frac{t}{2}\right)\frac{\partial}{\partial v} \\ \mathbf{w}_2 = t\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} \\ \mathbf{w}_3 = \frac{\partial}{\partial t} \\ \mathbf{w}_4 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} \\ \mathbf{w}_5 = v\frac{\partial}{\partial u} + u\frac{\rho_2}{\rho_1}\frac{\partial}{\partial v} \\ \mathbf{w}_{A\alpha} = A(x,t)\frac{\partial}{\partial u} + \alpha(x,t)\frac{\partial}{\partial v}, \end{cases} \tag{3.41}$$

where the pair (A, α) is a solution of (3.38).

It therefore follows that our system has the following symmetry groups:

$$\begin{cases} G_1 : \left(\frac{x}{1-\epsilon t}, \frac{t}{1-\epsilon t}, u(1-\epsilon t)^{1/2}\exp\left(\frac{-\epsilon x^2}{4(1-\epsilon t)}\right), v(1-\epsilon t)^{1/2}\exp\left(\frac{-\epsilon x^2}{4(1-\epsilon t)}\right)\right) \\ G_2 : (e^{\epsilon/2}x, e^{\epsilon}t, u, v) \\ G_3 : (x, t+\epsilon, u, v) \\ G_4 : (x, t, e^{\epsilon}u, e^{\epsilon}v) \\ G_5 : \left(x, t, u\cosh\left(\sqrt{\rho}\epsilon\right) + \frac{1}{\rho}v\sinh\left(\sqrt{\rho}\epsilon\right), v\cosh\left(\sqrt{\rho}\epsilon\right) + \rho u\sinh\left(\sqrt{\rho}\epsilon\right)\right) \\ G_{A\alpha} : (x, t, u+\epsilon A(x,t), v+\epsilon\alpha(x,t)), \end{cases} \tag{3.42}$$

where $\rho = \frac{\rho_2}{\rho_1}$ and where the pair (A, α) is a solution of (3.38).

The above symmetry groups have been obtained by simply solving the system of equations (2.2)-(2.3) for each of the vector fields in (3.41).

One may now recover the action of these symmetry groups to conclude the following:

Proposition 3.1.1. *Let $U = (u(x,t), v(x,t))$ be a solution of (3.38), then*

$$\begin{aligned}
U_1 &= (u\left(\frac{x}{1+\epsilon t}, \frac{t}{1+\epsilon t}\right)(1+\epsilon t)^{-1/2}\exp\left(\frac{-\epsilon x^2}{4(1+\epsilon t)}\right),\\
&v\left(\frac{x}{1+\epsilon t}, \frac{t}{1+\epsilon t}\right)(1+\epsilon t)^{-1/2}\exp\left(\frac{-\epsilon x^2}{4(1+\epsilon t)}\right))\\
U_2 &= (u(e^{-\epsilon/2}x, e^{-\epsilon}t), v(e^{-\epsilon/2}x, e^{-\epsilon}t))\\
U_3 &= (u(x, t-\epsilon), v(x, t-\epsilon))\\
U_4 &= (e^{\epsilon}u(x,t), e^{\epsilon}v(x,t))\\
U_5 &= (u(x,t)\cosh\left(\sqrt{\frac{\rho_2}{\rho_1}}\epsilon\right) + \frac{\rho_1}{\rho_2}v(x,t)\sinh\left(\sqrt{\frac{\rho_2}{\rho_1}}\epsilon\right),\\
&\quad v(x,t)\cosh\left(\sqrt{\frac{\rho_2}{\rho_1}}\epsilon\right) + \frac{\rho_2}{\rho_1}u(x,t)\sinh\left(\sqrt{\frac{\rho_2}{\rho_1}}\epsilon\right))\\
U_{A\alpha} &= (u(x,t) + \epsilon A(x,t), v(x,t) + \epsilon\alpha(x,t))
\end{aligned}$$

are also solutions of the given system. Here $(A(x,t), \alpha(x,t))$ is any arbitrary solution of (3.38).

Case A.2: The case $k = 0$ and $q = -4$ (or, equivalently, $q = 0$ and $k = -4$)
In this case the relevant system of PDEs is

$$\begin{cases} u_t = u_{xx} + \rho_1 v \\ v_t = v_{xx} + \frac{\rho_2}{x^4}u \end{cases} \tag{3.43}$$

Note that for this case, equation (3.36) gives

$$\frac{C_1}{2} + K_1'(t) + \gamma_1\rho_2 x^{-4} - \gamma_2\rho_1 = 0$$

Thus we must have $\gamma_1 = 0$ and $K_1'(t) = -\frac{C_1}{2} + \gamma_2\rho_1$. That is,

$$K_1(t) = \left(\gamma_2\rho_1 - \frac{C_1}{2}\right)t + C_4.$$

Then, equation (3.35) yields the expression for $K_2(t)$

$$K_2(t) = K_1(t) - 2C_1 t - C_2 = \left(\gamma_2\rho_1 - \frac{5}{2}C_1\right)t + C_4 - C_2$$

And substitution of $K_2'(t)$ into (3.37) gives

$$\frac{C_1}{2} + \gamma_2\rho_1 - \frac{5}{2}C_1 + \gamma_2\rho_1 = 2\gamma_2\rho_1 - 2C_1 = 0$$

or, equivalently, $\gamma_2 = \frac{C_1}{\rho_1}$. Hence the forms of our coefficient functions are

$$\begin{cases} \tau(t) = C_1 t^2 + C_2 t + C_3 \\ \xi(x,t) = C_1 tx + \frac{C_2}{2}x \\ \phi(x,t,u,v) = \left(C_1\left(-\frac{x^2}{4} + \frac{1}{2}t\right) + C_4\right)u + A(x,t) \\ \eta(x,t,u,v) = \left(-C_1\left(\frac{x^2}{4} + \frac{3}{2}t\right) + C_4 - C_2\right)v + \frac{C_1}{\rho_1}u + \alpha(x,t). \end{cases} \tag{3.44}$$

Therefore, a spanning set for the lie algebra of the system (3.43) is given by

$$\begin{cases} \mathbf{w}_1 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + u\left(-\frac{x^2}{4} + \frac{t}{2}\right)\frac{\partial}{\partial u} + \left(\frac{u}{\rho_1} - v\left(\frac{x^2}{4} + \frac{3}{2}t\right)\right)\frac{\partial}{\partial v} \\ \mathbf{w}_2 = t\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} - v\frac{\partial}{\partial v} \\ \mathbf{w}_3 = \frac{\partial}{\partial t} \\ \mathbf{w}_4 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} \\ \mathbf{w}_{A\alpha} = A(x,t)\frac{\partial}{\partial u} + \alpha(x,t)\frac{\partial}{\partial v}, \end{cases} \tag{3.45}$$

where the pair (A, α) is an arbitrary solution of the given system. In a similar way as in the previous case, one need only solve a relatively simple system of differential equations to obtain the following symmetry groups for the system (3.43):

$$\begin{cases} G_1 : \left(\frac{x}{1-\epsilon t}, \frac{t}{1-\epsilon t}, u\exp\left(\frac{-\epsilon x^2}{4(1-\epsilon t)}\right)\frac{1}{\sqrt{1-\epsilon t}}, \exp\left(\frac{-\epsilon x^2}{4(1-\epsilon t)}\right)\left(v(1-\epsilon t)^{3/2} + \frac{\epsilon}{\rho_1}u\sqrt{1-\epsilon t}\right)\right) \\ G_2 : (e^{\epsilon/2}x, e^{\epsilon}t, u, e^{-\epsilon}v) \\ G_3 : (x, t+\epsilon, u, v) \\ G_4 : (x, t, e^{\epsilon}u, e^{\epsilon}v) \\ G_{A\alpha} : (x, t, u + \epsilon A(x,t), v + \epsilon\alpha(x,t)). \end{cases} \tag{3.46}$$

Recovering the action of these groups to any particular solution $(u(x,t), v(x,t))$ one may obtain the following result:

Proposition 3.1.2. *Let $U = (u(x,t), v(x,t))$ be a solution of (3.43), then*

$$U_1 = (u\left(\frac{x}{1+\epsilon t}, \frac{t}{1+\epsilon t}\right)(1+\epsilon t)^{1/2} \exp\left(\frac{-\epsilon x^2}{4(1+\epsilon t)}\right),$$
$$\left(v\left(\frac{x}{1+\epsilon t}, \frac{t}{1+\epsilon t}\right)\frac{1}{(1+\epsilon t)^{3/2}} + \frac{\epsilon}{\rho_1}u\left(\frac{x}{1+\epsilon t}, \frac{t}{1+\epsilon t}\right)\frac{1}{\sqrt{1+\epsilon t}}\right)\exp\left(\frac{-\epsilon x^2}{4(1+\epsilon t)}\right))$$
$$U_2 = (u(e^{-\epsilon/2}x, e^{-\epsilon}t), e^{-\epsilon}v(e^{-\epsilon/2}x, e^{-\epsilon}t))$$
$$U_3 = (u(x, t-\epsilon), v(x, t-\epsilon))$$
$$U_4 = (e^{\epsilon}u(x,t), e^{\epsilon}v(x,t))$$
$$U_{A\alpha} = (u(x,t) + \epsilon A(x,t), v(x,t) + \epsilon\alpha(x,t))$$

are also solutions of the given system. Here (A, α) is any solution of (3.43).

Case A.3: The case $k \neq -(4+k)$ and $k, q \neq 0$ i.e. $k, q \neq -2, 0, -4$
In this case, we are dealing with a system of PDEs of the form:

$$\begin{cases} u_t = u_{xx} + \rho_1 x^k v \\ v_t = v_{xx} + \rho_2 x^{-(4+k)} u \end{cases} \tag{3.47}$$

Observe that according to (3.36) and (3.37) and because we want the functions $f(x)$ and $g(x)$ to be non-zero, we must have $\gamma_1 = \gamma_2 = 0$. Furthermore, equations (3.35)-(3.37) give $C_1 = 0$ and

$$\begin{cases} K_1(t) = C_4 \\ K_2(t) = C_4 - C_2\left(1 + \frac{k}{2}\right) \end{cases} \tag{3.48}$$

Putting all these conditions together we get that the coefficient functions for this case are of the form

$$\begin{cases} \tau(t) = C_2 t + C_3 \\ \xi(x,t) = \frac{C_2}{2}x \\ \phi(x,t,u,v) = A(x,t) + C_4 u \\ \eta(x,t,u,v) = \alpha(x,t) + \left(C_4 - C_2\left(1 + \frac{k}{2}\right)\right)v, \end{cases} \tag{3.49}$$

so the lie algebra of the system (3.47) is spanned by the following vector fields:

$$\begin{cases} \mathbf{w}_1 = t\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} - v(1+\frac{k}{2})\frac{\partial}{\partial v} \\ \mathbf{w}_2 = \frac{\partial}{\partial t} \\ \mathbf{w}_3 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} \\ \mathbf{w}_{A\alpha} = A(x,t)\frac{\partial}{\partial u} + \alpha(x,t)\frac{\partial}{\partial v} \end{cases} \tag{3.50}$$

Here, the pair (A, α) is again a solution of (3.47).

The reader may check that exponentiation of the above vector fields generates the following symmetry groups for our system (3.47), where the entries give the transformed point $(\tilde{x}, \tilde{t}, \tilde{u}, \tilde{v}) = exp(\epsilon \mathbf{w}_i)(x, t, u, v)$:

$$\begin{cases} G_1 : (e^{\epsilon/2}x, e^{\epsilon}t, u, e^{-\epsilon(1+\frac{k}{2})}v) \\ G_2 : (x, t+\epsilon, u, v) \\ G_3 : (x, t, e^{\epsilon}u, e^{\epsilon}v) \\ G_{A\alpha} : (x, t, u+\epsilon A(x,t), v+\epsilon\alpha(x,t)) \end{cases} \tag{3.51}$$

Once more, this result can be translated into the following:

Proposition 3.1.3. *Let $U = (u(x,t), v(x,t))$ be a solution of (3.47), then*

$$\begin{aligned} U_1 &= (u(e^{-\epsilon/2}x, e^{-\epsilon}t), e^{-\epsilon(1+\frac{k}{2})}v(e^{-\epsilon/2}x, e^{-\epsilon}t)) \\ U_2 &= (u(x, t-\epsilon), v(x, t-\epsilon)) \\ U_3 &= (e^{\epsilon}u(x,t), e^{\epsilon}v(x,t)) \\ U_{A\alpha} &= (u(x,t)+\epsilon A(x,t), v(x,t)+\epsilon\alpha(x,t)) \end{aligned}$$

are also solutions of the given system. The pair (A, α) denotes any solution of (3.47).

3.1.2 Case B: τ quadratic and σ linear

By assuming that τ is a quadratic function of t and σ a linear function of t, i.e.

$$\tau(t) = C_1t^2 + C_2t + C_3, \quad \sigma(t) = C_4t + C_5,$$

equations (3.18) - (3.21) simplify to the following:

$$\frac{C_1}{2} + K_1'(t) + C(t)g(x) - f(x)\beta(t) = 0 \tag{3.52}$$

$$C'(t) + f(x)\left(K_1(t) - 2C_1t - C_2 - K_2(t)\right) - f'(x)\left(\frac{2C_1t + C_2}{2}x + C_4t + C_5\right) = 0 \tag{3.53}$$

$$\frac{C_1}{2} + K_2'(t) + \beta(t)f(x) - g(x)C(t) = 0 \tag{3.54}$$

$$\beta'(t) + g(x)\left(K_2(t) - 2C_1t - C_2 - K_1(t)\right) - g'(x)\left(\frac{2C_1t + C_2}{2}x + C_4t + C_5\right) = 0 \tag{3.55}$$

Just as in the previous case, a thorough analysis of the above system of DEs allows us to determine the type of functions that f and g can be. For the above equations, one obtains that in order to satisfy conditions (3.52)-(3.55) we can only have constant functions f and g, say $f = \rho_1$, $g = \rho_2$. The system of PDES arising from this choice of f and g is

$$\begin{cases} u_t = u_{xx} + \rho_1 v \\ v_t = v_{xx} + \rho_2 u. \end{cases} \tag{3.56}$$

Substitution of these particular forms of f and g, into equations (3.53) and (3.55) yields

$$K_1(t) - K_2(t) = -\frac{C'(t)}{\rho_1} + 2C_1t + C_2 \tag{3.57}$$

$$K_2(t) - K_1(t) = -\frac{\beta'(t)}{\rho_2} + 2C_1t + C_2. \tag{3.58}$$

Therefore

$$\beta'(t) = -\frac{\rho_2}{\rho_1}C'(t) + 4C_1\rho_2 t + 2C_2\rho_2$$

or, integrating the above equation,

$$\beta(t) = -\frac{\rho_2}{\rho_1}C(t) + 2C_1\rho_2 t^2 + 2C_2\rho_2 t + C_6.$$

Next, equations (3.52) and (3.54) respectively give

$$\frac{C_1}{2} + K_1'(t) + 2\rho_2 C(t) - 2C_1\rho_1\rho_2 t^2 - 2C_2\rho_1\rho_2 t - \rho_1 C_6 = 0$$

$$\frac{C_1}{2} + K_2'(t) - 2\rho_2 C(t) + 2C_1\rho_1\rho_2 t^2 + 2C_2\rho_1\rho_2 t + \rho_1 C_6 = 0, \tag{3.59}$$

and hence $K_1'(t) + K_2'(t) = -C_1$. Therefore, we must have that $K_1(t) = -K_2(t) - C_1t + C_7$. Substituting this form for $K_1(t)$ into (3.57) gives

$$K_2(t) = \frac{C'(t)}{2\rho_1} - \frac{3}{2}C_1t - \frac{C_2}{2} + \frac{C_7}{2}$$

and differentiation of $K_2(t)$ and substitution into (3.59) yields the following ODE for the function $C(t)$:

$$\frac{C''(t)}{2\rho_1} - 2\rho_2 C(t) + 2C_1\rho_1\rho_2 t^2 + 2C_2\rho_1\rho_2 t + \rho_1 C_6 - C_1 = 0.$$

This ODE has solution

$$C(t) = \frac{\rho_1}{\rho_2}\frac{C_6}{2} + \rho_1 C_1 t^2 + \rho_1 C_2 t + C_8 e^{2\sqrt{\rho_1\rho_2}t} + C_9 e^{-2\sqrt{\rho_1\rho_2}t}$$

and therefore

$$\begin{aligned}
\beta(t) &= -\frac{\rho_2}{\rho_1}C(t) + 2C_1\rho_2 t^2 + 2C_2\rho_2 t + C_6 \\
&= \frac{C_6}{2} + \rho_2 C_1 t^2 + \rho_2 C_2 t - C_8\frac{\rho_2}{\rho_1}e^{2\sqrt{\rho_1\rho_2}t} - C_9\frac{\rho_2}{\rho_1}e^{-2\sqrt{\rho_1\rho_2}t} \\
K_2(t) &= \frac{C'(t)}{2\rho_1} - \frac{3}{2}C_1 t - \frac{C_2}{2} + \frac{C_7}{2} \\
&= C_8\sqrt{\frac{\rho_2}{\rho_1}}e^{2\sqrt{\rho_1\rho_2}t} - C_9\sqrt{\frac{\rho_2}{\rho_1}}e^{-2\sqrt{\rho_1\rho_2}t} - \frac{C_1}{2}t + \frac{C_7}{2} \\
K_1(t) &= -K_2(t) - C_1 t + C_7 \\
&= -C_8\sqrt{\frac{\rho_2}{\rho_1}}e^{2\sqrt{\rho_1\rho_2}t} + C_9\sqrt{\frac{\rho_2}{\rho_1}}e^{-2\sqrt{\rho_1\rho_2}t} - \frac{C_1}{2}t + \frac{C_7}{2}.
\end{aligned}$$

So our coefficient functions τ, ξ, ϕ and η will be of the form

$$\begin{cases}
\tau(t) = C_1 t^2 + C_2 t + C_3 \\
\xi(x,t) = C_1 tx + \frac{C_2}{2}x + C_4 t + C_5 \\
\phi(x,t,u,v) = A(x,t) + \left(\frac{\rho_1}{\rho_2}\frac{C_6}{2} + \rho_1 C_1 t^2 + \rho_1 C_2 t + C_8 e^{2\sqrt{\rho_1\rho_2}t} + C_9 e^{-2\sqrt{\rho_1\rho_2}t}\right)v \\
\qquad + \left(-\frac{C_1}{4}x^2 - \frac{C_4}{2}x - C_8\sqrt{\frac{\rho_2}{\rho_1}}e^{2\sqrt{\rho_1\rho_2}t} + C_9\sqrt{\frac{\rho_2}{\rho_1}}e^{-2\sqrt{\rho_1\rho_2}t} - \frac{C_1}{2}t + \frac{C_7}{2}\right)u \\
\eta(x,t,u,v) = \alpha(x,t) + \left(\frac{C_6}{2} + \rho_2 C_1 t^2 + \rho_2 C_2 t - C_8\frac{\rho_2}{\rho_1}e^{2\sqrt{\rho_1\rho_2}t} - C_9\frac{\rho_2}{\rho_1}e^{-2\sqrt{\rho_1\rho_2}t}\right)u \\
\qquad + \left(-\frac{C_1}{4}x^2 - \frac{C_4}{2}x + C_8\sqrt{\frac{\rho_2}{\rho_1}}e^{2\sqrt{\rho_1\rho_2}t} - C_9\sqrt{\frac{\rho_2}{\rho_1}}e^{-2\sqrt{\rho_1\rho_2}t} - \frac{C_1}{2}t + \frac{C_7}{2}\right)v,
\end{cases} \tag{3.60}$$

This gives the following infinitesimal generators as a spanning set of the lie algebra of (3.56):

$$\begin{cases}
\mathbf{w}_1 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + \left(\rho_1 t^2 v - u\left(\frac{t}{2} + \frac{x^2}{4}\right)\right)\frac{\partial}{\partial u} + \left(\rho_2 t^2 u - v\left(\frac{t}{2} + \frac{x^2}{4}\right)\right)\frac{\partial}{\partial v} \\
\mathbf{w}_2 = t\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} + \rho_1 tv\frac{\partial}{\partial u} + \rho_2 tu\frac{\partial}{\partial v} \\
\mathbf{w}_3 = \frac{\partial}{\partial t} \\
\mathbf{w}_4 = t\frac{\partial}{\partial x} - \frac{xu}{2}\frac{\partial}{\partial u} - \frac{xv}{2}\frac{\partial}{\partial v}
\end{cases}$$

$$\begin{cases} \mathbf{w}_5 = \frac{\partial}{\partial x} \\ \mathbf{w}_6 = \rho_1 v \frac{\partial}{\partial u} + \rho_2 u \frac{\partial}{\partial v} \\ \mathbf{w}_7 = u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v} \\ \mathbf{w}_8 = e^{2\sqrt{\rho_1\rho_2}t}\left(-\sqrt{\rho_1\rho_2}u + \rho_1 v\right)\frac{\partial}{\partial u} + e^{2\sqrt{\rho_1\rho_2}t}\left(\sqrt{\rho_1\rho_2}v - \rho_2 u\right)\frac{\partial}{\partial v} \\ \mathbf{w}_9 = e^{-2\sqrt{\rho_1\rho_2}t}\left(\sqrt{\rho_1\rho_2}u + \rho_1 v\right)\frac{\partial}{\partial u} + e^{-2\sqrt{\rho_1\rho_2}t}\left(-\sqrt{\rho_1\rho_2}v - \rho_2 u\right)\frac{\partial}{\partial v} \\ \mathbf{w}_{A\alpha} = A(x,t)\frac{\partial}{\partial u} + \alpha(x,t)\frac{\partial}{\partial v}, \end{cases} \tag{3.61}$$

where the pair (A, α) is a solution of the given system.

Exponentiation of the above vector fields produces the following symmetry groups (in the usual notation) for (3.56):

$$\begin{cases} G_1 : & \Bigg(\frac{x}{1-\epsilon t}, \frac{t}{1-\epsilon t}, \\ & \sqrt{1-\epsilon t}\left(u \cosh\left(\frac{\epsilon\sqrt{16\rho_1\rho_2 t^4 + x^4}}{4(1-\epsilon t)}\right) + \frac{(4\rho_1 t^2 v - u x^2)}{\sqrt{16\rho_1\rho_2 t^4 + x^4}} \sinh\left(\frac{\epsilon\sqrt{16\rho_1\rho_2 t^4 + x^4}}{4(1-\epsilon t)}\right)\right), \\ & \sqrt{1-\epsilon t}\left(v \cosh\left(\frac{\epsilon\sqrt{16\rho_1\rho_2 t^4 + x^4}}{4(1-\epsilon t)}\right) + \frac{(4\rho_2 t^2 u + v x^2)}{\sqrt{16\rho_1\rho_2 t^4 + x^4}} \sinh\left(\frac{\epsilon\sqrt{16\rho_1\rho_2 t^4 + x^4}}{4(1-\epsilon t)}\right)\right)\Bigg) \\ G_2 : & (e^{\epsilon/2}x, e^{\epsilon}t, u\cosh(\mu t(e^{\epsilon}-1)) + v\sqrt{\frac{\rho_1}{\rho_2}}\sinh(\mu t(e^{\epsilon}-1)), \\ & v\cosh(\mu t(e^{\epsilon}-1)) + u\sqrt{\frac{\rho_2}{\rho_1}}\sinh(\mu t(e^{\epsilon}-1))) \\ G_3 : & (x, t+\epsilon, u, v) \\ G_4 : & \left(x+\epsilon t, t, u\exp\left(-\frac{\epsilon}{2}\left(x+\frac{t\epsilon}{2}\right)\right), v\exp\left(-\frac{\epsilon}{2}\left(x+\frac{t\epsilon}{2}\right)\right)\right) \\ G_5 : & (x+\epsilon, t, u, v) \\ G_6 : & (x, t, u\cosh(\mu\epsilon) + \sqrt{\frac{\rho_1}{\rho_2}}v\sinh(\mu\epsilon), \sqrt{\frac{\rho_2}{\rho_1}}u\sinh(\mu\epsilon) + v\cosh(\mu\epsilon)) \\ G_7 : & (x, t, e^{\epsilon}u, e^{\epsilon}v) \\ G_8 : & (x, t, (1-\epsilon\mu e^{2\mu t})u + \epsilon\rho_1 e^{2\mu t}v, (1+\epsilon\mu e^{2\mu t})v - \epsilon\rho_2 e^{2\mu t}u) \\ G_9 : & (x, t, (1+\epsilon\mu e^{-2\mu t})u + \epsilon\rho_1 e^{-2\mu t}v, (1-\epsilon\mu e^{-2\mu t})v - \epsilon\rho_2 e^{-2\mu t}u) \\ G_{A\alpha} : & (x, t, u+\epsilon A(x,t), v+\epsilon\alpha(x,t)), \end{cases} \tag{3.62}$$

where $\mu = \sqrt{\rho_1\rho_2}$.
From the above result, the next proposition naturally follows by simply recovering the action of these symmetry groups on any given solution of the system (3.56):

Proposition 3.1.4. *Let* $U = (u(x,t), v(x,t))$ *be a solution of (3.56), then*

$$
\begin{aligned}
U_1 = \Bigg(& \frac{u(\frac{x}{1+\epsilon t}, \frac{t}{1+\epsilon t})}{\sqrt{1+\epsilon t}} \cosh\left(\frac{\epsilon\sqrt{16\rho_1\rho_2 t^4 + x^4}}{4(1+\epsilon t)}\right) \\
& + \frac{\left(4\rho_1 t^2 v(\frac{x}{1+\epsilon t}, \frac{t}{1+\epsilon t}) - x^2 u(\frac{x}{1+\epsilon t}, \frac{t}{1+\epsilon t})\right)}{\sqrt{(16\rho_1\rho_2 t^4 + x^4)(1+\epsilon t)}} \sinh\left(\frac{\epsilon\sqrt{16\rho_1\rho_2 t^4 + x^4}}{4(1+\epsilon t)}\right), \\
& \frac{v(\frac{x}{1+\epsilon t}, \frac{t}{1+\epsilon t})}{\sqrt{1+\epsilon t}} \cosh\left(\frac{\epsilon\sqrt{16\rho_1\rho_2 t^4 + x^4}}{4(1+\epsilon t)}\right) \\
& + \frac{\left(4\rho_2 t^2 u(\frac{x}{1+\epsilon t}, \frac{t}{1+\epsilon t}) + x^2 v(\frac{x}{1+\epsilon t}, \frac{t}{1+\epsilon t})\right)}{\sqrt{(16\rho_1\rho_2 t^4 + x^4)(1+\epsilon t)}} \sinh\left(\frac{\epsilon\sqrt{16\rho_1\rho_2 t^4 + x^4}}{4(1+\epsilon t)}\right)\Bigg) \\
U_2 = \Big(& u(xe^{-\epsilon/2}, te^{-\epsilon}) \cosh(\mu t(1-e^{-\epsilon})) + v(xe^{-\epsilon/2}, te^{-\epsilon})\sqrt{\frac{\rho_1}{\rho_2}} \sinh(\mu t(1-e^{-\epsilon})), \\
& v(xe^{-\epsilon/2}, te^{-\epsilon}) \cosh(\mu t(1-e^{-\epsilon})) + u(xe^{-\epsilon/2}, te^{-\epsilon})\sqrt{\frac{\rho_2}{\rho_1}} \sinh(\mu t(1-e^{-\epsilon}))\Big) \\
U_3 = & (u(x, t-\epsilon), v(x, t-\epsilon)) \\
U_4 = & \left(u(x-\epsilon t, t)\exp\left(-\frac{\epsilon}{2}\left(x - \frac{t\epsilon}{2}\right)\right), v(x-\epsilon t, t)\exp\left(-\frac{\epsilon}{2}\left(x - \frac{t\epsilon}{2}\right)\right)\right) \\
U_5 = & (u(x-\epsilon, t), v(x-\epsilon, t)) \\
U_6 = & \left(u(x,t)\cosh(\mu\epsilon) + \sqrt{\frac{\rho_1}{\rho_2}} v(x,t)\sinh(\mu\epsilon), \sqrt{\frac{\rho_2}{\rho_1}} u(x,t)\sinh(\mu\epsilon) + v(x,t)\cosh(\mu\epsilon)\right) \\
U_7 = & (e^{\epsilon} u(x,t), e^{\epsilon} v(x,t)) \\
U_8 = & ((1-\epsilon\mu e^{2\mu t})u(x,t) + \epsilon\rho_1 e^{2\mu t} v(x,t), (1+\epsilon\mu e^{2\mu t})v(x,t) - \epsilon\rho_2 e^{2\mu t} u(x,t)) \\
U_9 = & ((1+\epsilon\mu e^{-2\mu t})u(x,t) + \epsilon\rho_1 e^{-2\mu t} v(x,t), (1-\epsilon\mu e^{-2\mu t})v(x,t) - \epsilon\rho_2 e^{-2\mu t} u(x,t)) \\
U_{A\alpha} = & (u(x,t) + \epsilon A(x,t), v(x,t) + \epsilon\alpha(x,t))
\end{aligned}
$$

are also solutions of the given system. Here $\mu = \sqrt{\rho_1\rho_2}$ *and* (A, α) *is an arbitrary solution of (3.56).*

Note. The symmetry groups in (3.62) are given in the usual notation for the transformed points $(\tilde{x}, \tilde{t}, \tilde{u}, \tilde{v}) = exp(\epsilon \mathbf{w}_i)(x, t, u, v)$. These are obtained as usual by solving the system of equations (2.2)-(2.3) for each of the vector fields in (3.61). However, we do not include all the steps of these computations here, since they do get quite long and messy for some particular vector fields.

3.1.3 Case C: τ and σ constant functions

A similar analysis of the system of determining equations to the previous cases can be done for this case. The reader may check that these case will lead to a choice of functions $f(x) = a + bx$ and $g(x) = k(a + bx)$ for some constants a and b. With this, the system of determining equations can be solved to produce the following coefficient functions:

$$\begin{cases} \tau(t) = C_1 \\ \xi(x,t) = 2C_2 t + C_3 \\ \phi(x,t,u,v) = A(x,t) + (-C_2 x + C_4)\,u + \left(aC_2 t^2 + C_5 t + C_6\right) v \\ \eta(x,t,u,v) = \alpha(x,t) + (-C_2 x + C_4)\,v + k\left(aC_2 t^2 + C_5 t + C_6\right) u, \end{cases} \tag{3.63}$$

Hence, a spanning set for the Lie algebra of our system will be

$$\begin{cases} \mathbf{w}_1 = \frac{\partial}{\partial t} \\ \mathbf{w}_2 = 2t\frac{\partial}{\partial x} + (-xu + at^2 v)\frac{\partial}{\partial u} + (-xv + kat^2 u)\frac{\partial}{\partial v} \\ \mathbf{w}_3 = \frac{\partial}{\partial x} \\ \mathbf{w}_4 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} \\ \mathbf{w}_5 = tv\frac{\partial}{\partial u} + ktu\frac{\partial}{\partial v} \\ \mathbf{w}_6 = v\frac{\partial}{\partial u} + ku\frac{\partial}{\partial v} \\ \mathbf{w}_{A\alpha} = A(x,t)\frac{\partial}{\partial u} + \alpha(x,t)\frac{\partial}{\partial v}, \end{cases} \tag{3.64}$$

where the pair (A, α) is a solution of the given system. The usual exponentiation of the above vector fields produces the following symmetry groups for the system:

$$\begin{cases} G_1: & (x, t+\epsilon, u, v) \\ G_2: & (x + 2\epsilon t, t, e^{-\epsilon(t\epsilon + x)}(u\cosh(a\sqrt{k}t^2\epsilon) + \frac{1}{\sqrt{k}}v\sinh(a\sqrt{k}t^2\epsilon)), \\ & e^{-\epsilon(t\epsilon+x)}(\sqrt{k}u\sinh(a\sqrt{k}t^2\epsilon) + v\cosh(a\sqrt{k}t^2\epsilon))) \\ G_3: & (x+\epsilon, t, u, v) \\ G_4: & (x, t, e^{\epsilon}u, e^{\epsilon}v) \\ G_5: & (x, t, u\cosh(\sqrt{k}t\epsilon) + \frac{1}{\sqrt{k}}v\sinh(\sqrt{k}t\epsilon), \sqrt{k}u\sinh(\sqrt{k}t\epsilon) + v\cosh(\sqrt{k}t\epsilon)) \\ G_6: & (x, t, u\cosh(\sqrt{k}\epsilon) + \frac{1}{\sqrt{k}}v\sinh(\sqrt{k}\epsilon), \sqrt{k}u\sinh(\sqrt{k}\epsilon) + v\cosh(\sqrt{k}\epsilon)) \\ G_{A\alpha}: & (x, t, u + \epsilon A(x,t), v + \epsilon\alpha(x,t)), \end{cases}$$

(3.65)

Hence it follows that

Proposition 3.1.5. *For the system*

$$\begin{cases} u_t = u_{xx} + (ax+b)v \\ v_t = v_{xx} + k(ax+b)u, \end{cases} \tag{3.66}$$

if the pair $U = (u(x,t), v(x,t))$ *is a solution of (3.66), then*

$$\begin{aligned}
&U_1 = (u(x,t-\epsilon), v(x,t-\epsilon)) \\
&U_2 = (e^{-x\epsilon+t\epsilon^2}(u(x-2\epsilon t,t)\cosh(a\sqrt{k}t^2\epsilon) + \frac{1}{\sqrt{k}}v(x-2\epsilon t,t)\sinh(a\sqrt{k}t^2\epsilon)), \\
&e^{-x\epsilon+t\epsilon^2}\left(\sqrt{k}u(x-2\epsilon t,t)\sinh(a\sqrt{k}t^2\epsilon) + v(x-2\epsilon t,t)\cosh(a\sqrt{k}t^2\epsilon)\right)) \\
&U_3 = (u(x-\epsilon,t), v(x-\epsilon,t)) \\
&U_4 = (e^{\epsilon}u(x,t), e^{\epsilon}v(x,t)) \\
&U_5 = (u(x,t)\cosh(\sqrt{k}t\epsilon) + \frac{1}{\sqrt{k}}v(x,t)\sinh(\sqrt{k}t\epsilon), \\
&\sqrt{k}u(x,t)\sinh(\sqrt{k}t\epsilon) + v(x,t)\cosh(\sqrt{k}t\epsilon)) \\
&U_6 = (u(x,t)\cosh(\sqrt{k}\epsilon) + \frac{1}{\sqrt{k}}v(x,t)\sinh(\sqrt{k}\epsilon), \\
&\sqrt{k}u(x,t)\sinh(\sqrt{k}\epsilon) + v(x,t)\cosh(\sqrt{k}\epsilon)) \\
&U_{A\alpha} = (u(x,t) + \epsilon A(x,t), v(x,t) + \epsilon\alpha(x,t))
\end{aligned}$$

are also solutions of the given system. Here (A,α) *is an arbitrary solution of (3.66).*

3.2 Fundamental solutions

In this section we show how the symmetries found in Section 3.1 can be used to find fundamental solutions of the given systems of PDEs via the use of Integral Transform methods. We separate our study into the same cases we distinguished in the previous section. To the best of our knowledge, the results we present here are new.

3.2.1 Case A.1: The Laplace Transform

The first step is to look for stationary solutions of the system (3.38), which can be done solving the system:

$$\begin{cases} 0 = u_{xx} + \frac{\rho_1}{x^2}v \\ 0 = v_{xx} + \frac{\rho_2}{x^2}u. \end{cases}$$

This gives $v = -\frac{x^2}{\rho_1} u_{xx}$ and so $v_{xx} = -\frac{1}{\rho_1}(2u_{xx} + 4x u_{xxx} + x^2 u_{xxxx})$. Substitution of v_{xx} in the second equation yields the following differential equation for u:

$$x^4 u_{xxxx} + 4x^3 u_{xxx} + 2x^2 u_{xx} - \rho_1 \rho_2 u = 0. \tag{3.67}$$

This is an Euler-type equation and, therefore, we look for a solution of the type $u = x^\alpha$. We have $u_x = \alpha x^{\alpha-1}$, $u_{xx} = \alpha(\alpha-1)x^{\alpha-2}$, $u_{xxx} = \alpha(\alpha-1)(\alpha-2)x^{\alpha-3}$ and $u_{xxxx} = \alpha(\alpha-1)(\alpha-2)(\alpha-3)x^{\alpha-4}$. Substitution into equation (3.67) gives that α must satisfy

$$\alpha(\alpha-1)(\alpha-2)(\alpha-3) + 4\alpha(\alpha-1)(\alpha-2) + 2\alpha(\alpha-1) - \rho_1\rho_2 = 0. \tag{3.68}$$

Note that

$$\alpha_1 = \frac{1}{2} + \frac{\sqrt{1+4\sqrt{\rho_1\rho_2}}}{2} \quad \text{and} \quad \alpha_2 = \frac{1}{2} + \frac{\sqrt{1-4\sqrt{\rho_1\rho_2}}}{2}$$

are both solutions of (3.68) and let $\mu = \frac{\sqrt{1+4\sqrt{\rho_1\rho_2}}}{2}$, $\nu = \frac{\sqrt{1-4\sqrt{\rho_1\rho_2}}}{2}$ respectively. For the sake of simplicity, suppose that $\mu, \nu \in \mathbb{R}$, that is, suppose that either

- $\rho_1 = 0$,
- $\rho_1 < 0$ and $\frac{1}{16\rho_1} < \rho_2 \leq 0$, or
- $\rho_1 > 0$ and $0 \leq \rho_2 < \frac{1}{16\rho_1}$.

Separate analysis is needed for different values of ρ_1 and ρ_2.
Then $u_1 = x^{\frac{1}{2}+\mu}$ and $u_2 = x^{\frac{1}{2}+\nu}$ are solutions of (3.67), which produce the following two pairs of stationary solutions for our system (3.38):

$$\begin{cases} u_1 = x^{\mu+\frac{1}{2}} \\ v_1 = -\frac{x^2}{\rho_1}(\mu+\frac{1}{2})(\mu-\frac{1}{2})x^{\mu-\frac{3}{2}} = -\frac{\mu^2-\frac{1}{4}}{\rho_1}x^{\mu+\frac{1}{2}} = -\sqrt{\frac{\rho_2}{\rho_1}}x^{\mu+\frac{1}{2}} \end{cases} \tag{3.69}$$

and

$$\begin{cases} u_2 = x^{\nu+\frac{1}{2}} \\ v_2 = -\frac{x^2}{\rho_1}(\nu+\frac{1}{2})(\nu-\frac{1}{2})x^{\nu-\frac{3}{2}} = -\frac{\nu^2-\frac{1}{4}}{\rho_1}x^{\nu+\frac{1}{2}} = \sqrt{\frac{\rho_2}{\rho_1}}x^{\nu+\frac{1}{2}} \end{cases} \tag{3.70}$$

We know by Proposition 3.1.1 that if $(u(x,t), v(x,t))^\top$ is a solution of (3.38), so is

$$\tilde{U}_\epsilon(x,t) = \begin{pmatrix} \tilde{u}_\epsilon(x,t) \\ \tilde{v}_\epsilon(x,t) \end{pmatrix} = \begin{pmatrix} u\left(\frac{x}{1+\epsilon t}, \frac{t}{1+\epsilon t}\right)(1+\epsilon t)^{-1/2}\exp\left(\frac{-\epsilon x^2}{4(1+\epsilon t)}\right) \\ v\left(\frac{x}{1+\epsilon t}, \frac{t}{1+\epsilon t}\right)(1+\epsilon t)^{-1/2}\exp\left(\frac{-\epsilon x^2}{4(1+\epsilon t)}\right) \end{pmatrix}.$$

Applying this transformation to our stationary solutions $(u_1, v_1)^\top$ and $(u_2, v_2)^\top$, respectively, produces the time dependent solutions:

$$\tilde{U}_1(x,t,\epsilon) = \begin{pmatrix} \frac{x^{\mu+\frac{1}{2}}}{(1+\epsilon t)^{\mu+1}} \exp\left(\frac{-\epsilon x^2}{4(1+\epsilon t)}\right) \\ -\sqrt{\frac{\rho_2}{\rho_1}} \frac{x^{\mu+\frac{1}{2}}}{(1+\epsilon t)^{\mu+1}} \exp\left(\frac{-\epsilon x^2}{4(1+\epsilon t)}\right) \end{pmatrix},$$

$$\tilde{U}_2(x,t,\epsilon) = \begin{pmatrix} \frac{x^{\nu+\frac{1}{2}}}{(1+\epsilon t)^{\nu+1}} \exp\left(\frac{-\epsilon x^2}{4(1+\epsilon t)}\right) \\ \sqrt{\frac{\rho_2}{\rho_1}} \frac{x^{\nu+\frac{1}{2}}}{(1+\epsilon t)^{\nu+1}} \exp\left(\frac{-\epsilon x^2}{4(1+\epsilon t)}\right) \end{pmatrix}.$$

for our system (3.38).

Now, for convenience, let us make the change $\epsilon \to 4\epsilon$ in the expressions of $\tilde{U}_1$ and $\tilde{U}_2$, which is a valid change since ϵ is an arbitrary constant. Thus we obtain the new expressions:

$$\tilde{U}_1(x,t,\epsilon) = \begin{pmatrix} \frac{x^{\mu+\frac{1}{2}}}{(1+4\epsilon t)^{\mu+1}} \exp\left(\frac{-\epsilon x^2}{1+4\epsilon t}\right) \\ -\sqrt{\frac{\rho_2}{\rho_1}} \frac{x^{\mu+\frac{1}{2}}}{(1+4\epsilon t)^{\mu+1}} \exp\left(\frac{-\epsilon x^2}{1+4\epsilon t}\right) \end{pmatrix}$$

and

$$\tilde{U}_2(x,t,\epsilon) = \begin{pmatrix} \frac{x^{\nu+\frac{1}{2}}}{(1+4\epsilon t)^{\nu+1}} \exp\left(\frac{-\epsilon x^2}{1+4\epsilon t}\right) \\ \sqrt{\frac{\rho_2}{\rho_1}} \frac{x^{\nu+\frac{1}{2}}}{(1+4\epsilon t)^{\nu+1}} \exp\left(\frac{-\epsilon x^2}{1+4\epsilon t}\right) \end{pmatrix}.$$

These satisfy the initial conditions

$$\tilde{U}_1(x,0,\epsilon) = \begin{pmatrix} \tilde{u}_1(x,0,\epsilon) \\ \tilde{v}_1(x,0,\epsilon) \end{pmatrix} = \begin{pmatrix} x^{\mu+\frac{1}{2}} e^{-\epsilon x^2} \\ -\sqrt{\frac{\rho_2}{\rho_1}} x^{\mu+\frac{1}{2}} e^{-\epsilon x^2} \end{pmatrix}$$

and

$$\tilde{U}_2(x,0,\epsilon) = \begin{pmatrix} \tilde{u}_2(x,0,\epsilon) \\ \tilde{v}_2(x,0,\epsilon) \end{pmatrix} = \begin{pmatrix} x^{\nu+\frac{1}{2}} e^{-\epsilon x^2} \\ \sqrt{\frac{\rho_2}{\rho_1}} x^{\nu+\frac{1}{2}} e^{-\epsilon x^2} \end{pmatrix}.$$

The next step is to express $\tilde{u}_1$, $\tilde{v}_1$, $\tilde{u}_2$ and $\tilde{v}_2$ as Laplace transforms of certain functions. To do so, we rewrite $\tilde{u}_1$ as

$$\begin{aligned} \tilde{u}_1(x,t,\epsilon) &= \frac{x^{\mu+\frac{1}{2}}}{(4t)^{\mu+1}} \frac{1}{(\epsilon + \frac{1}{4t})^{\mu+1}} \exp\left(-\frac{(\epsilon + \frac{1}{4t} - \frac{1}{4t})x^2}{4t(\epsilon + \frac{1}{4t})}\right) \\ &= \frac{x^{\mu+\frac{1}{2}}}{(4t)^{\mu+1}} \exp\left(-\frac{x^2}{4t}\right) \frac{1}{(\epsilon + \frac{1}{4t})^{\mu+1}} \exp\left(\frac{(\frac{x}{4t})^2}{\epsilon + \frac{1}{4t}}\right), \end{aligned} \tag{3.71}$$

and, similarly, we express $\tilde{v}_1$ as

$$\tilde{v}_1(x,t,\epsilon) = -\sqrt{\frac{\rho_2}{\rho_1}} \frac{x^{\mu+\frac{1}{2}}}{(4t)^{\mu+1}} \exp\left(-\frac{x^2}{4t}\right) \frac{1}{(\epsilon + \frac{1}{4t})^{\mu+1}} \exp\left(\frac{(\frac{x}{4t})^2}{\epsilon + \frac{1}{4t}}\right). \tag{3.72}$$

Taking inverse Laplace transform of $\tilde{u}_1$ and $\tilde{v}_1$ with respect to ϵ yields

$$\begin{aligned}
\mathcal{L}^{-1}(\tilde{u}_1(x,t,\epsilon)) &= \mathcal{L}^{-1}\left(\frac{x^{\mu+\frac{1}{2}}}{(4t)^{\mu+1}}\exp\left(-\frac{x^2}{4t}\right)\frac{1}{(\epsilon+\frac{1}{4t})^{\mu+1}}\exp\left(\frac{(\frac{x}{4t})^2}{\epsilon+\frac{1}{4t}}\right)\right) \\
&= \frac{x^{\mu+\frac{1}{2}}}{(4t)^{\mu+1}}\exp\left(-\frac{x^2}{4t}\right)\mathcal{L}^{-1}\left(\frac{1}{(\epsilon+\frac{1}{4t})^{\mu+1}}\exp\left(\frac{(\frac{x}{4t})^2}{\epsilon+\frac{1}{4t}}\right)\right) \\
&= \frac{x^{\mu+\frac{1}{2}}}{(4t)^{\mu+1}}\exp\left(-\frac{x^2+z}{4t}\right)\mathcal{L}^{-1}\left(\frac{1}{\epsilon^{\mu+1}}\exp\left(\frac{(\frac{x}{4t})^2}{\epsilon}\right)\right) \\
&= \frac{x^{\mu+\frac{1}{2}}}{(4t)^{\mu+1}}\exp\left(-\frac{x^2+z}{4t}\right)\left(\frac{x}{4t}\right)^{-\mu} z^{\frac{\mu}{2}} I_\mu\left(\frac{x\sqrt{z}}{2t}\right) \\
&= \frac{x^{\frac{1}{2}}}{4t}\exp\left(-\frac{x^2+z}{4t}\right) z^{\frac{\mu}{2}} I_\mu\left(\frac{x\sqrt{z}}{2t}\right),
\end{aligned} \tag{3.73}$$

and, similarly, for $\tilde{v}_1$ we get

$$\begin{aligned}
\mathcal{L}^{-1}(\tilde{v}_1(x,t,\epsilon)) &= \mathcal{L}^{-1}\left(-\sqrt{\frac{\rho_2}{\rho_1}}\frac{x^{\mu+\frac{1}{2}}}{(4t)^{\mu+1}}\exp\left(-\frac{x^2}{4t}\right)\frac{1}{(\epsilon+\frac{1}{4t})^{\mu+1}}\exp\left(\frac{(\frac{x}{4t})^2}{\epsilon+\frac{1}{4t}}\right)\right) \\
&= -\sqrt{\frac{\rho_2}{\rho_1}}\mathcal{L}^{-1}\left(\frac{x^{\mu+\frac{1}{2}}}{(4t)^{\mu+1}}\exp\left(-\frac{x^2}{4t}\right)\frac{1}{(\epsilon+\frac{1}{4t})^{\mu+1}}\exp\left(\frac{(\frac{x}{4t})^2}{\epsilon+\frac{1}{4t}}\right)\right) \\
&= -\sqrt{\frac{\rho_2}{\rho_1}}\frac{x^{\frac{1}{2}}}{4t}\exp\left(-\frac{x^2+z}{4t}\right) z^{\frac{\mu}{2}} I_\mu\left(\frac{x\sqrt{z}}{2t}\right).
\end{aligned} \tag{3.74}$$

The reader may check that a similar computation for $\tilde{u}_2$ and $\tilde{v}_2$ yields:

$$\mathcal{L}^{-1}(\tilde{u}_2(x,t,\epsilon)) = \frac{x^{\frac{1}{2}}}{4t}\exp\left(-\frac{x^2+z}{4t}\right) z^{\frac{\nu}{2}} I_\nu\left(\frac{x\sqrt{z}}{2t}\right), \tag{3.75}$$

$$\mathcal{L}^{-1}(\tilde{v}_2(x,t,\epsilon)) = \sqrt{\frac{\rho_2}{\rho_1}}\frac{x^{\frac{1}{2}}}{4t}\exp\left(-\frac{x^2+z}{4t}\right) z^{\frac{\nu}{2}} I_\nu\left(\frac{x\sqrt{z}}{2t}\right). \tag{3.76}$$

These expressions allow us to write $\tilde{U}_1(x,t,\epsilon)$ and $\tilde{U}_2(x,t,\epsilon)$ as Laplace transforms:

$$\begin{aligned}
\tilde{U}_1(x,t,\epsilon) &= \begin{pmatrix} \tilde{u}_1(x,t,\epsilon) \\ \tilde{v}_1(x,t,\epsilon) \end{pmatrix} \\
&= \begin{pmatrix} \mathcal{L}\left(\frac{x^{\frac{1}{2}}}{4t}\exp\left(-\frac{x^2+z}{4t}\right) z^{\frac{\mu}{2}} I_\mu\left(\frac{x\sqrt{z}}{2t}\right)\right) \\ \mathcal{L}\left(-\sqrt{\frac{\rho_2}{\rho_1}}\frac{x^{\frac{1}{2}}}{4t}\exp\left(-\frac{x^2+z}{4t}\right) z^{\frac{\mu}{2}} I_\mu\left(\frac{x\sqrt{z}}{2t}\right)\right) \end{pmatrix}
\end{aligned}$$

$$= \int_0^\infty \begin{pmatrix} \frac{x^{\frac{1}{2}}}{4t} \exp\left(-\frac{x^2+z}{4t}\right) z^{\frac{\mu}{2}} I_\mu\left(\frac{x\sqrt{z}}{2t}\right) e^{-\epsilon z} \\ -\sqrt{\frac{\rho_2}{\rho_1}} \frac{x^{\frac{1}{2}}}{4t} \exp\left(-\frac{x^2+z}{4t}\right) z^{\frac{\mu}{2}} I_\mu\left(\frac{x\sqrt{z}}{2t}\right) e^{-\epsilon z} \end{pmatrix} dz. \tag{3.77}$$

and, similarly

$$\tilde{U}_2(x,t,\epsilon) = \begin{pmatrix} \tilde{u}_2(x,t,\epsilon) \\ \tilde{v}_2(x,t,\epsilon) \end{pmatrix}$$
$$= \int_0^\infty \begin{pmatrix} \frac{x^{\frac{1}{2}}}{4t} \exp\left(-\frac{x^2+z}{4t}\right) z^{\frac{\nu}{2}} I_\nu\left(\frac{x\sqrt{z}}{2t}\right) e^{-\epsilon z} \\ \sqrt{\frac{\rho_2}{\rho_1}} \frac{x^{\frac{1}{2}}}{4t} \exp\left(-\frac{x^2+z}{4t}\right) z^{\frac{\nu}{2}} I_\nu\left(\frac{x\sqrt{z}}{2t}\right) e^{-\epsilon z} \end{pmatrix} dz. \tag{3.78}$$

Observe now that linearity and Theorem 2.1.5 give that, for appropriate functions $\varphi(\epsilon)$ and $\psi(\epsilon)$ with sufficiently rapid decay, we can produce a new solution to the system (3.38) by taking

$$\begin{pmatrix} U(x,t) \\ V(x,t) \end{pmatrix} = \int_0^\infty \left(\varphi(\epsilon)\tilde{U}_1(x,t,\epsilon) + \psi(\epsilon)\tilde{U}_2(x,t,\epsilon)\right) d\epsilon$$
$$= \int_0^\infty \begin{pmatrix} \varphi(\epsilon)\tilde{u}_1(x,t,\epsilon) + \psi(\epsilon)\tilde{u}_2(x,t,\epsilon) \\ \varphi(\epsilon)\tilde{v}_1(x,t,\epsilon) + \psi(\epsilon)\tilde{v}_2(x,t,\epsilon) \end{pmatrix} d\epsilon.$$

This new solution will satisfy the initial condition

$$\begin{pmatrix} U(x,0) \\ V(x,0) \end{pmatrix} = \int_0^\infty \begin{pmatrix} \varphi(\epsilon)\tilde{u}_1(x,0,\epsilon) + \psi(\epsilon)\tilde{u}_2(x,0,\epsilon) \\ \varphi(\epsilon)\tilde{v}_1(x,0,\epsilon) + \psi(\epsilon)\tilde{v}_2(x,0,\epsilon) \end{pmatrix} d\epsilon$$
$$= \int_0^\infty \begin{pmatrix} \varphi(\epsilon)x^{\mu+\frac{1}{2}}e^{-\epsilon x^2} + \psi(\epsilon)x^{\nu+\frac{1}{2}}e^{-\epsilon x^2} \\ -\sqrt{\frac{\rho_2}{\rho_1}}\varphi(\epsilon)x^{\mu+\frac{1}{2}}e^{-\epsilon x^2} + \sqrt{\frac{\rho_2}{\rho_1}}\psi(\epsilon)x^{\nu+\frac{1}{2}}e^{-\epsilon x^2} \end{pmatrix} d\epsilon$$
$$= \begin{pmatrix} x^{\mu+\frac{1}{2}}\Phi(x^2) + x^{\nu+\frac{1}{2}}\Psi(x^2) \\ -\sqrt{\frac{\rho_2}{\rho_1}}x^{\mu+\frac{1}{2}}\Phi(x^2) + \sqrt{\frac{\rho_2}{\rho_1}}x^{\nu+\frac{1}{2}}\Psi(x^2) \end{pmatrix}$$
$$= \begin{pmatrix} x^{\mu+\frac{1}{2}} & x^{\nu+\frac{1}{2}} \\ -\sqrt{\frac{\rho_2}{\rho_1}}x^{\mu+\frac{1}{2}} & \sqrt{\frac{\rho_2}{\rho_1}}x^{\nu+\frac{1}{2}} \end{pmatrix} \begin{pmatrix} \Phi(x^2) \\ \Psi(x^2) \end{pmatrix}. \tag{3.79}$$

Note that putting

$$\begin{pmatrix} U(x,0) \\ V(x,0) \end{pmatrix} = \begin{pmatrix} x^{\mu+\frac{1}{2}}\Phi(x^2) + x^{\nu+\frac{1}{2}}\Psi(x^2) \\ -\sqrt{\frac{\rho_2}{\rho_1}}x^{\mu+\frac{1}{2}}\Phi(x^2) + \sqrt{\frac{\rho_2}{\rho_1}}x^{\nu+\frac{1}{2}}\Psi(x^2) \end{pmatrix} := \begin{pmatrix} f(x) \\ g(x) \end{pmatrix} \tag{3.80}$$

we can write

$$\underbrace{\begin{pmatrix} x^{\mu+\frac{1}{2}} & x^{\nu+\frac{1}{2}} \\ -\sqrt{\frac{\rho_2}{\rho_1}}x^{\mu+\frac{1}{2}} & \sqrt{\frac{\rho_2}{\rho_1}}x^{\nu+\frac{1}{2}} \end{pmatrix}}_{C(x)} \begin{pmatrix} \Phi(x^2) \\ \Psi(x^2) \end{pmatrix} = \begin{pmatrix} f(x) \\ g(x) \end{pmatrix},$$

which gives

$$\begin{pmatrix} \Phi(x^2) \\ \Psi(x^2) \end{pmatrix} = C^{-1}(x) \begin{pmatrix} f(x) \\ g(x) \end{pmatrix}. \tag{3.81}$$

Note. We can choose any pair of sufficiently well behaved functions $f(x)$, $g(x)$ for the initial state (3.80) due to the known smoothing properties of parabolic differential operators. This will be the case here and for the rest of examples presented in this work. However, we will not discuss this topic here, since the details can get quite technical.

The reader may check that the matrix $C^{-1}(x)$ can easily be calculated to be

$$\begin{aligned} C^{-1}(x) &= \frac{1}{2\sqrt{\frac{\rho_2}{\rho_1}}x^{\mu+\nu+1}} \begin{pmatrix} \sqrt{\frac{\rho_2}{\rho_1}}x^{\nu+\frac{1}{2}} & -x^{\nu+\frac{1}{2}} \\ \sqrt{\frac{\rho_2}{\rho_1}}x^{\mu+\frac{1}{2}} & x^{\mu+\frac{1}{2}} \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} x^{-\mu-\frac{1}{2}} & -\sqrt{\frac{\rho_1}{\rho_2}}x^{-\mu-\frac{1}{2}} \\ x^{-\nu-\frac{1}{2}} & \sqrt{\frac{\rho_1}{\rho_2}}x^{-\nu-\frac{1}{2}} \end{pmatrix}. \end{aligned}$$

We may now write the components U and V using the expressions obtained in (3.77) and (3.78) for $\tilde{u}_1(x,t,\epsilon)$, $\tilde{u}_2(x,t,\epsilon)$, $\tilde{v}_1(x,t,\epsilon)$ and $\tilde{v}_2(x,t,\epsilon)$:

$$\begin{aligned} U(x,t) &= \int_0^\infty \left(\varphi(\epsilon)\tilde{u}_1(x,t,\epsilon) + \psi(\epsilon)\tilde{u}_2(x,t,\epsilon)\right) d\epsilon \\ &= \int_0^\infty \varphi(\epsilon) \int_0^\infty \frac{\sqrt{x}}{4t} \exp\left(-\frac{x^2+z}{4t}\right) z^{\frac{\mu}{2}} I_\mu\left(\frac{x\sqrt{z}}{2t}\right) e^{-\epsilon z} dz d\epsilon \\ &+ \int_0^\infty \psi(\epsilon) \int_0^\infty \frac{\sqrt{x}}{4t} \exp\left(-\frac{x^2+z}{4t}\right) z^{\frac{\nu}{2}} I_\nu\left(\frac{x\sqrt{z}}{2t}\right) e^{-\epsilon z} dz d\epsilon \\ &= \int_0^\infty \int_0^\infty \varphi(\epsilon)\frac{\sqrt{x}}{4t} \exp\left(-\frac{x^2+z}{4t}\right) z^{\frac{\mu}{2}} I_\mu\left(\frac{x\sqrt{z}}{2t}\right) e^{-\epsilon z} d\epsilon dz \\ &+ \int_0^\infty \int_0^\infty \psi(\epsilon)\frac{\sqrt{x}}{4t} \exp\left(-\frac{x^2+z}{4t}\right) z^{\frac{\nu}{2}} I_\nu\left(\frac{x\sqrt{z}}{2t}\right) e^{-\epsilon z} d\epsilon dz \end{aligned}$$

$$= \int_0^\infty \frac{\sqrt{x}}{4t} \exp\left(-\frac{x^2+z}{4t}\right) z^{\frac{\mu}{2}} I_\mu\left(\frac{x\sqrt{z}}{2t}\right) \underbrace{\int_0^\infty \varphi(\epsilon) e^{-\epsilon z} d\epsilon}_{\Phi(z)} \, dz$$

$$+ \int_0^\infty \frac{\sqrt{x}}{4t} \exp\left(-\frac{x^2+z}{4t}\right) z^{\frac{\nu}{2}} I_\nu\left(\frac{x\sqrt{z}}{2t}\right) \underbrace{\int_0^\infty \psi(\epsilon) e^{-\epsilon z} d\epsilon}_{\Psi(z)} \, dz$$

$$= \int_0^\infty \frac{\sqrt{x}}{4t} \exp\left(-\frac{x^2+z}{4t}\right) \left(\Phi(z) z^{\frac{\mu}{2}} I_\mu\left(\frac{x\sqrt{z}}{2t}\right) + \Psi(z) z^{\frac{\nu}{2}} I_\nu\left(\frac{x\sqrt{z}}{2t}\right)\right) dz$$

Make the change of variables $z \to y^2$ to obtain

$$U(x,t) = \int_0^\infty \frac{\sqrt{x}}{2t} \exp\left(-\frac{x^2+y^2}{4t}\right) \left(\Phi(y^2) y^{\mu+1} I_\mu\left(\frac{xy}{2t}\right) + \Psi(y^2) y^{\nu+1} I_\nu\left(\frac{xy}{2t}\right)\right) dy$$

$$= \int_0^\infty \begin{pmatrix} a_{11}(x,y,t) & a_{12}(x,y,t) \end{pmatrix} \begin{pmatrix} \Phi(y^2) \\ \Psi(y^2) \end{pmatrix} dy, \tag{3.82}$$

where

$$a_{11}(x,y,t) := y^{\mu+\frac{1}{2}} \frac{\sqrt{xy}}{2t} \exp\left(-\frac{x^2+y^2}{4t}\right) I_\mu\left(\frac{xy}{2t}\right),$$

$$a_{12}(x,y,t) := y^{\nu+\frac{1}{2}} \frac{\sqrt{xy}}{2t} \exp\left(-\frac{x^2+y^2}{4t}\right) I_\nu\left(\frac{xy}{2t}\right).$$

Similarly, the expression for V becomes:

$$V(x,t) = \int_0^\infty \left(\varphi(\epsilon)\tilde{v}_1(x,t,\epsilon) + \psi(\epsilon)\tilde{v}_2(x,t,\epsilon)\right) d\epsilon$$

$$= \int_0^\infty \varphi(\epsilon) \int_0^\infty -\sqrt{\frac{\rho_2}{\rho_1}} \frac{\sqrt{x}}{4t} \exp\left(-\frac{x^2+z}{4t}\right) z^{\frac{\mu}{2}} I_\mu\left(\frac{x\sqrt{z}}{2t}\right) e^{-\epsilon z} dz d\epsilon$$

$$+ \int_0^\infty \psi(\epsilon) \int_0^\infty \sqrt{\frac{\rho_2}{\rho_1}} \frac{\sqrt{x}}{4t} \exp\left(-\frac{x^2+z}{4t}\right) z^{\frac{\nu}{2}} I_\nu\left(\frac{x\sqrt{z}}{2t}\right) e^{-\epsilon z} dz d\epsilon$$

$$= \int_0^\infty \int_0^\infty -\sqrt{\frac{\rho_2}{\rho_1}} \varphi(\epsilon) \frac{\sqrt{x}}{4t} \exp\left(-\frac{x^2+z}{4t}\right) z^{\frac{\mu}{2}} I_\mu\left(\frac{x\sqrt{z}}{2t}\right) e^{-\epsilon z} d\epsilon dz$$

$$+ \int_0^\infty \int_0^\infty \sqrt{\frac{\rho_2}{\rho_1}} \psi(\epsilon) \frac{\sqrt{x}}{4t} \exp\left(-\frac{x^2+z}{4t}\right) z^{\frac{\nu}{2}} I_\nu\left(\frac{x\sqrt{z}}{2t}\right) e^{-\epsilon z} d\epsilon dz$$

$$= \int_0^\infty -\sqrt{\frac{\rho_2}{\rho_1}} \frac{\sqrt{x}}{4t} \exp\left(-\frac{x^2+z}{4t}\right) z^{\frac{\mu}{2}} I_\mu\left(\frac{x\sqrt{z}}{2t}\right) \underbrace{\int_0^\infty \varphi(\epsilon) e^{-\epsilon z} d\epsilon}_{\Phi(z)} \, dz$$

$$+ \int_0^\infty \sqrt{\frac{\rho_2}{\rho_1}} \frac{\sqrt{x}}{4t} \exp\left(-\frac{x^2+z}{4t}\right) z^{\frac{\nu}{2}} I_\nu \left(\frac{x\sqrt{z}}{2t}\right) \underbrace{\int_0^\infty \psi(\epsilon) e^{-\epsilon z} d\epsilon}_{\Psi(z)} dz$$

$$= \int_0^\infty -\sqrt{\frac{\rho_2}{\rho_1}} \Phi(z) \frac{\sqrt{x}}{4t} \exp\left(-\frac{x^2+z}{4t}\right) z^{\frac{\mu}{2}} I_\mu \left(\frac{x\sqrt{z}}{2t}\right) dz$$

$$+ \int_0^\infty \sqrt{\frac{\rho_2}{\rho_1}} \Psi(z) \frac{\sqrt{x}}{4t} \exp\left(-\frac{x^2+z}{4t}\right) z^{\frac{\nu}{2}} I_\nu \left(\frac{x\sqrt{z}}{2t}\right) dz, \tag{3.83}$$

and making the change $z \to y^2$ yields

$$\begin{aligned} V(x,t) &= \int_0^\infty -\sqrt{\frac{\rho_2}{\rho_1}} \Phi(y^2) \frac{\sqrt{x}}{2t} \exp\left(-\frac{x^2+y^2}{4t}\right) y^{\mu+1} I_\mu \left(\frac{xy}{2t}\right) dy \\ &= \int_0^\infty \sqrt{\frac{\rho_2}{\rho_1}} \Psi(y^2) \frac{\sqrt{x}}{2t} \exp\left(-\frac{x^2+y^2}{4t}\right) y^{\nu+1} I_\nu \left(\frac{xy}{2t}\right) dy \\ &= \int_0^\infty \begin{pmatrix} a_{21}(x,y,t) & a_{22}(x,y,t) \end{pmatrix} \begin{pmatrix} \Phi(y^2) \\ \Psi(y^2) \end{pmatrix} dy, \end{aligned} \tag{3.84}$$

where

$$a_{21}(x,y,t) := -\sqrt{\frac{\rho_2}{\rho_1}} y^{\mu+\frac{1}{2}} \frac{\sqrt{xy}}{2t} \exp\left(-\frac{x^2+y^2}{4t}\right) I_\mu \left(\frac{xy}{2t}\right),$$

$$a_{22}(x,y,t) := \sqrt{\frac{\rho_2}{\rho_1}} y^{\nu+\frac{1}{2}} \frac{\sqrt{xy}}{2t} \exp\left(-\frac{x^2+y^2}{4t}\right) I_\nu \left(\frac{xy}{2t}\right).$$

Observe that we have written

$$\begin{pmatrix} U(x,t) \\ V(x,t) \end{pmatrix} = \int_0^\infty \begin{pmatrix} a_{11}(x,y,t) & a_{12}(x,y,t) \\ a_{21}(x,y,t) & a_{22}(x,y,t) \end{pmatrix} \begin{pmatrix} \Phi(y^2) \\ \Psi(y^2) \end{pmatrix} dy,$$

and replace $\begin{pmatrix} \Phi(y^2) & \Psi(y^2) \end{pmatrix}^\top$ by the expression obtained in (3.81). This yields the following expression for $(U \quad V)^\top$

$$\begin{pmatrix} U(x,t) \\ V(x,t) \end{pmatrix} = \int_0^\infty \underbrace{\begin{pmatrix} a_{11}(x,y,t) & a_{12}(x,y,t) \\ a_{21}(x,y,t) & a_{22}(x,y,t) \end{pmatrix}}_{A(x,y,t)} C^{-1}(y) \begin{pmatrix} f(y) \\ g(y) \end{pmatrix} dy. \tag{3.85}$$

Recall that according to (3.80) the initial condition was

$$\begin{pmatrix} U(x,0) \\ V(x,0) \end{pmatrix} = \begin{pmatrix} f(x) \\ g(x) \end{pmatrix},$$

so we can conclude that the matrix $P(t,x,y) = (p_{ij}(t,x,y))$ defined by the product

$$\begin{aligned} P(t,x,y) &= \begin{pmatrix} p_{11}(t,x,y) & p_{12}(t,x,y) \\ p_{21}(t,x,y) & p_{22}(t,x,y) \end{pmatrix} := A(x,y,t)C^{-1}(y) \\ &= \frac{1}{2}\begin{pmatrix} a_{11}(x,y,t) & a_{12}(x,y,t) \\ a_{21}(x,y,t) & a_{22}(x,y,t) \end{pmatrix} \begin{pmatrix} y^{-\mu-\frac{1}{2}} & -\sqrt{\frac{\rho_1}{\rho_2}}y^{-\mu-\frac{1}{2}} \\ y^{-\nu-\frac{1}{2}} & \sqrt{\frac{\rho_1}{\rho_2}}y^{-\nu-\frac{1}{2}} \end{pmatrix} \end{aligned} \tag{3.86}$$

is a fundamental solution of (3.38).
The components $p_{ij}(t,x,y)$ can be calculated to be:

$$\begin{aligned} p_{11}(t,x,y) &:= \frac{1}{2}\left(y^{-\mu-\frac{1}{2}}a_{11}(x,y,t) + y^{-\nu-\frac{1}{2}}a_{12}(x,y,t)\right) \\ &= \frac{\sqrt{xy}}{4t}\exp\left(-\frac{x^2+y^2}{4t}\right)\left(I_\mu\left(\frac{xy}{2t}\right) + I_\nu\left(\frac{xy}{2t}\right)\right) \\ p_{12}(t,x,y) &:= \frac{1}{2}\left(-\sqrt{\frac{\rho_1}{\rho_2}}y^{-\mu-\frac{1}{2}}a_{11}(x,y,t) + \sqrt{\frac{\rho_1}{\rho_2}}y^{-\nu-\frac{1}{2}}a_{12}(x,y,t)\right) \\ &= \sqrt{\frac{\rho_1}{\rho_2}}\frac{\sqrt{xy}}{4t}\exp\left(-\frac{x^2+y^2}{4t}\right)\left(-I_\mu\left(\frac{xy}{2t}\right) + I_\nu\left(\frac{xy}{2t}\right)\right) \\ p_{21}(t,x,y) &:= \frac{1}{2}\left(y^{-\mu-\frac{1}{2}}a_{21}(x,y,t) + y^{-\nu-\frac{1}{2}}a_{22}(x,y,t)\right) \\ &= \sqrt{\frac{\rho_2}{\rho_1}}\frac{\sqrt{xy}}{4t}\exp\left(-\frac{x^2+y^2}{4t}\right)\left(-I_\mu\left(\frac{xy}{2t}\right) + I_\nu\left(\frac{xy}{2t}\right)\right) \\ p_{22}(t,x,y) &:= \frac{1}{2}\left(-\sqrt{\frac{\rho_1}{\rho_2}}y^{-\mu-\frac{1}{2}}a_{21}(x,y,t) + \sqrt{\frac{\rho_1}{\rho_2}}y^{-\nu-\frac{1}{2}}a_{22}(x,y,t)\right) \\ &= \frac{\sqrt{xy}}{4t}\exp\left(-\frac{x^2+y^2}{4t}\right)\left(I_\mu\left(\frac{xy}{2t}\right) + I_\nu\left(\frac{xy}{2t}\right)\right), \end{aligned}$$

thus obtaining the following expression for $P(t,x,y)$:

$$P(t,x,y) = \frac{\sqrt{xy}}{4t}e^{-\frac{x^2+y^2}{4t}}\begin{pmatrix} I_\mu\left(\frac{xy}{2t}\right) + I_\nu\left(\frac{xy}{2t}\right) & \sqrt{\frac{\rho_1}{\rho_2}}\left(I_\nu\left(\frac{xy}{2t}\right) - I_\mu\left(\frac{xy}{2t}\right)\right) \\ \sqrt{\frac{\rho_2}{\rho_1}}\left(I_\nu\left(\frac{xy}{2t}\right) - I_\mu\left(\frac{xy}{2t}\right)\right) & I_\mu\left(\frac{xy}{2t}\right) + I_\nu\left(\frac{xy}{2t}\right) \end{pmatrix}. \tag{3.87}$$

Remark. The more general problem

$$\begin{cases} u_t = u_{xx} + (\eta_1 + \frac{\rho_1}{x^2})v \\ v_t = v_{xx} + (\eta_2 + \frac{\rho_2}{x^2})u. \end{cases}$$

does also possess enough symmetries to be able to compute fundamental solutions in principle. However, we cannot find the necessary stationary solutions to compute these fundamental matrices.

3.2.2 Case A.2: A more complex case with the Laplace Transform

Similarly to the previous case, we need to start by looking for stationary solutions of (3.43), i.e. we must solve:

$$\begin{cases} 0 = u_{xx} + \rho_1 v \\ 0 = v_{xx} + \frac{\rho_2}{x^4} u. \end{cases}$$

It is clear we must have $v = -\frac{u_{xx}}{\rho_1}$ and hence, substituting $v_{xx} = -\frac{u_{xxxx}}{\rho_1}$ into the second equation above, we observe that u must solve:

$$x^4 u_{xxxx} - \rho_1 \rho_2 u = 0, \tag{3.88}$$

which again is an Euler-type equation. Therefore, solutions will be of the form $u = x^\alpha$. This will turn equation (3.88) into

$$\alpha(\alpha-1)(\alpha-2)(\alpha-3)x^\alpha - \rho_1\rho_2 x^\alpha = 0, \tag{3.89}$$

so we need only find roots of the polynomial $\alpha(\alpha-1)(\alpha-2)(\alpha-3) - \rho_1\rho_2$. The reader may check that

$$\alpha_1 = \frac{3}{2} + \frac{\sqrt{5 + 4\sqrt{1+\rho_1\rho_2}}}{2} \quad \text{and} \quad \alpha_2 = \frac{3}{2} + \frac{\sqrt{5 - 4\sqrt{1+\rho_1\rho_2}}}{2}$$

are two of such roots. For simplicity in the notation, let us define $\mu = \frac{\sqrt{5+4\sqrt{1+\rho_1\rho_2}}}{2}$ and $\nu = \frac{\sqrt{5-4\sqrt{1+\rho_1\rho_2}}}{2}$. Again, for convenience, we assume that both $\mu, \nu \in \mathbb{R}$, i.e. either

- $\rho_1 = 0$,
- $\rho_1 < 0$ and $\frac{9}{16\rho_1} < \rho_2 \leq -\frac{1}{\rho_1}$, or
- $\rho_1 > 0$ and $-\frac{1}{\rho_1} \leq \rho_2 < \frac{9}{16\rho_1}$.

Other choices of ρ_1 and ρ_2 can be considered separately in a similar study.

It is clear that the pairs

$$\begin{cases} u_1 = x^{\frac{3}{2}+\mu} \\ v_1 = -\frac{u_{xx}}{\rho_1} = -\frac{(\mu+\frac{3}{2})(\mu+\frac{1}{2})}{\rho_1} x^{\mu-\frac{1}{2}} \end{cases} \tag{3.90}$$

$$\begin{cases} u_2 = x^{\frac{3}{2}+\nu} \\ v_2 = -\frac{u_{xx}}{\rho_1} = -\frac{(\nu+\frac{3}{2})(\nu+\frac{1}{2})}{\rho_1} x^{\nu-\frac{1}{2}} \end{cases} \tag{3.91}$$

are two stationary solutions of (3.43). Therefore, using the symmetry U_1 in Proposition 3.1.2 and making the change $\epsilon \to 4\epsilon$ (just as in the previous case), we know that

$$\begin{aligned} \tilde{U}_1(x,t,\epsilon) &= \begin{pmatrix} \tilde{u}_1(x,t,\epsilon) \\ \tilde{v}_1(x,t,\epsilon) \end{pmatrix} \\ &= \begin{pmatrix} u_1\left(\frac{x}{1+4\epsilon t}\right)(1+4\epsilon t)^{1/2}\exp\left(\frac{-\epsilon x^2}{1+4\epsilon t}\right) \\ \left(v_1\left(\frac{x}{1+4\epsilon t}\right)(1+4\epsilon t)^{-3/2} + \frac{4\epsilon}{\rho_1}u_1\left(\frac{x}{1+4\epsilon t}\right)(1+4\epsilon t)^{-1/2}\right)\exp\left(\frac{-\epsilon x^2}{1+4\epsilon t}\right) \end{pmatrix} \\ &= \begin{pmatrix} \frac{x^{\mu+\frac{3}{2}}}{(1+4\epsilon t)^{\mu+1}}\exp\left(\frac{-\epsilon x^2}{1+4\epsilon t}\right) \\ \left(-\frac{(\mu+\frac{3}{2})(\mu+\frac{1}{2})}{\rho_1}\frac{x^{\mu-\frac{1}{2}}}{(1+4\epsilon t)^{\mu+1}} + \frac{4\epsilon}{\rho_1}\frac{x^{\mu+\frac{3}{2}}}{(1+4\epsilon t)^{\mu+2}}\right)\exp\left(\frac{-\epsilon x^2}{1+4\epsilon t}\right) \end{pmatrix} \end{aligned}$$

and similarly

$$\begin{aligned} \tilde{U}_2(x,t,\epsilon) &= \begin{pmatrix} \tilde{u}_2(x,t,\epsilon) \\ \tilde{v}_2(x,t,\epsilon) \end{pmatrix} \\ &= \begin{pmatrix} u_2\left(\frac{x}{1+4\epsilon t}\right)(1+4\epsilon t)^{1/2}\exp\left(\frac{-\epsilon x^2}{1+4\epsilon t}\right) \\ \left(v_2\left(\frac{x}{1+4\epsilon t}\right)(1+4\epsilon t)^{-3/2} + \frac{4\epsilon}{\rho_1}u_2\left(\frac{x}{1+4\epsilon t}\right)(1+4\epsilon t)^{-1/2}\right)\exp\left(\frac{-\epsilon x^2}{1+4\epsilon t}\right) \end{pmatrix} \\ &= \begin{pmatrix} \frac{x^{\nu+\frac{3}{2}}}{(1+4\epsilon t)^{\nu+1}}\exp\left(\frac{-\epsilon x^2}{1+4\epsilon t}\right) \\ \left(-\frac{(\nu+\frac{3}{2})(\nu+\frac{1}{2})}{\rho_1}\frac{x^{\nu-\frac{1}{2}}}{(1+4\epsilon t)^{\nu+1}} + \frac{4\epsilon}{\rho_1}\frac{x^{\nu+\frac{3}{2}}}{(1+4\epsilon t)^{\nu+2}}\right)\exp\left(\frac{-\epsilon x^2}{1+4\epsilon t}\right) \end{pmatrix} \end{aligned}$$

are also solutions, satisfying the initial conditions:

$$\tilde{U}_1(x,0,\epsilon) = \begin{pmatrix} x^{\mu+\frac{3}{2}}e^{-\epsilon x^2} \\ \left(-\frac{(\mu+\frac{3}{2})(\mu+\frac{1}{2})}{\rho_1}x^{\mu-\frac{1}{2}} + \frac{4\epsilon}{\rho_1}x^{\mu+\frac{3}{2}}\right)e^{-\epsilon x^2} \end{pmatrix} \tag{3.92}$$

and

$$\tilde{U}_2(x,0,\epsilon) = \begin{pmatrix} x^{\nu+\frac{3}{2}} e^{-\epsilon x^2} \\ \left(-\frac{(\nu+\frac{3}{2})(\nu+\frac{1}{2})}{\rho_1} x^{\nu-\frac{1}{2}} + \frac{4\epsilon}{\rho_1} x^{\nu+\frac{3}{2}}\right) e^{-\epsilon x^2} \end{pmatrix} \tag{3.93}$$

respectively. Recall that by linearity and Theorem 2.1.5, for suitable functions $\phi(\epsilon)$ an $\psi(\epsilon)$, we can produce yet another solution given by

$$\begin{aligned} \tilde{U}(x,t) &= \int_0^\infty \left(\phi(\epsilon)\tilde{U}_1(x,t,\epsilon) + \psi(\epsilon)\tilde{U}_2(x,t,\epsilon)\right) d\epsilon \\ &= \int_0^\infty \begin{pmatrix} \phi(\epsilon) \frac{x^{\mu+\frac{3}{2}}}{(1+4\epsilon t)^{\mu+1}} \exp\left(\frac{-\epsilon x^2}{1+4\epsilon t}\right) \\ \phi(\epsilon)\left(-\frac{(\mu+\frac{3}{2})(\mu+\frac{1}{2})}{\rho_1}\frac{x^{\mu-\frac{1}{2}}}{(1+4\epsilon t)^{\mu+1}} + \frac{4\epsilon}{\rho_1}\frac{x^{\mu+\frac{3}{2}}}{(1+4\epsilon t)^{\mu+2}}\right)\exp\left(\frac{-\epsilon x^2}{1+4\epsilon t}\right) \end{pmatrix} d\epsilon \\ &+ \int_0^\infty \begin{pmatrix} \psi(\epsilon) \frac{x^{\nu+\frac{3}{2}}}{(1+4\epsilon t)^{\nu+1}} \exp\left(\frac{-\epsilon x^2}{1+4\epsilon t}\right) \\ \psi(\epsilon)\left(-\frac{(\nu+\frac{3}{2})(\nu+\frac{1}{2})}{\rho_1}\frac{x^{\nu-\frac{1}{2}}}{(1+4\epsilon t)^{\nu+1}} + \frac{4\epsilon}{\rho_1}\frac{x^{\nu+\frac{3}{2}}}{(1+4\epsilon t)^{\nu+2}}\right)\exp\left(\frac{-\epsilon x^2}{1+4\epsilon t}\right) \end{pmatrix} d\epsilon, \end{aligned}$$

which has initial condition

$$\begin{aligned} \begin{pmatrix} f(x) \\ g(x) \end{pmatrix} := \tilde{U}(x,0) &= \int_0^\infty \begin{pmatrix} \phi(\epsilon) x^{\mu+\frac{3}{2}} e^{-\epsilon x^2} \\ \phi(\epsilon)\left(-\frac{(\mu+\frac{3}{2})(\mu+\frac{1}{2})}{\rho_1} x^{\mu-\frac{1}{2}} + \frac{4\epsilon}{\rho_1} x^{\mu+\frac{3}{2}}\right) e^{-\epsilon x^2} \end{pmatrix} d\epsilon \\ &+ \int_0^\infty \begin{pmatrix} \psi(\epsilon) x^{\nu+\frac{3}{2}} e^{-\epsilon x^2} \\ \psi(\epsilon)\left(-\frac{(\nu+\frac{3}{2})(\nu+\frac{1}{2})}{\rho_1} x^{\nu-\frac{1}{2}} + \frac{4\epsilon}{\rho_1} x^{\nu+\frac{3}{2}}\right) e^{-\epsilon x^2} \end{pmatrix} d\epsilon \end{aligned}$$

The reader may check that the first component of the above expression gives

$$x^{\mu+\frac{3}{2}} \int_0^\infty \phi(\epsilon) e^{-\epsilon x^2} d\epsilon + x^{\nu+\frac{3}{2}} \int_0^\infty \psi(\epsilon) e^{-\epsilon x^2} d\epsilon = f(x)$$

or, equivalently,

$$x^{\mu+\frac{3}{2}}\Phi(x^2) + x^{\nu+\frac{3}{2}}\Psi(x^2) = f(x), \tag{3.94}$$

where Φ and Ψ denote the Laplace transforms of ϕ and ψ respectively. Similarly, the second component can be written as

$$\begin{aligned} &-\frac{\left(\mu+\frac{3}{2}\right)\left(\mu+\frac{1}{2}\right)}{\rho_1} x^{\mu-\frac{1}{2}} \int_0^\infty \phi(\epsilon) e^{-\epsilon x^2} d\epsilon - \frac{2}{\rho_1} x^{\mu+\frac{1}{2}} \int_0^\infty -2x\epsilon\phi(\epsilon) e^{-\epsilon x^2} d\epsilon \\ &-\frac{\left(\nu+\frac{3}{2}\right)\left(\nu+\frac{1}{2}\right)}{\rho_1} x^{\nu-\frac{1}{2}} \int_0^\infty \psi(\epsilon) e^{-\epsilon x^2} d\epsilon - \frac{2}{\rho_1} x^{\nu+\frac{1}{2}} \int_0^\infty -2x\epsilon\psi(\epsilon) e^{-\epsilon x^2} d\epsilon = g(x), \end{aligned}$$

thus yielding the following differential equation:

$$\left(\mu+\frac{3}{2}\right)\left(\mu+\frac{1}{2}\right)x^{\mu-\frac{1}{2}}\Phi(x^2)+\left(\nu+\frac{3}{2}\right)\left(\nu+\frac{1}{2}\right)x^{\nu-\frac{1}{2}}\Psi(x^2)$$
$$+2x^{\mu+\frac{1}{2}}\frac{d}{dx}\Phi(x^2)+2x^{\nu+\frac{1}{2}}\frac{d}{dx}\Psi(x^2)=-\rho_1 g(x).$$

Therefore we need to solve the system:

$$\begin{cases} x^{\mu+\frac{3}{2}}\Phi(x^2)+x^{\nu+\frac{3}{2}}\Psi(x^2)=f(x) \\ \left(\mu+\frac{3}{2}\right)\left(\mu+\frac{1}{2}\right)x^{\mu-\frac{1}{2}}\Phi(x^2)+\left(\nu+\frac{3}{2}\right)\left(\nu+\frac{1}{2}\right)x^{\nu-\frac{1}{2}}\Psi(x^2) \\ \quad +2x^{\mu+\frac{1}{2}}\frac{d}{dx}\Phi(x^2)+2x^{\nu+\frac{1}{2}}\frac{d}{dx}\Psi(x^2)=-\rho_1 g(x) \end{cases} \tag{3.95}$$

The first equation in this system gives

$$\Phi(x^2)=x^{-(\mu+\frac{3}{2})}f(x)-x^{\nu-\mu}\Psi(x^2) \tag{3.96}$$

Differentiation with respect to x produces the following expression for $\frac{d}{dx}\Phi(x^2)$:

$$\frac{d}{dx}\Phi(x^2)=-(\mu+\frac{3}{2})x^{-(\mu+\frac{5}{2})}f(x)+x^{-(\mu+\frac{3}{2})}f'(x)-(\nu-\mu)x^{\nu-\mu-1}\Psi(x^2)-x^{\nu-\mu}\frac{d}{dx}\Psi(x^2) \tag{3.97}$$

Substitution of expressions (3.96) and (3.97) into the second equation in (3.95) results in the following expression for $\Psi(x^2)$:

$$\Psi(x^2)=\frac{1}{\mu^2-\nu^2}\left(\rho_1 x^{-(\nu-\frac{1}{2})}g(x)+2x^{-(\nu+\frac{1}{2})}f'(x)+\left(\mu^2-\frac{9}{4}\right)x^{-(\nu+\frac{3}{2})}f(x)\right),$$

and hence

$$\Phi(x^2)=-\frac{1}{\mu^2-\nu^2}\left(\rho_1 x^{-(\mu-\frac{1}{2})}g(x)+2x^{-(\mu+\frac{1}{2})}f'(x)+\left(\nu^2-\frac{9}{4}\right)x^{-(\mu+\frac{3}{2})}f(x)\right).$$

That is, we can write

$$\begin{pmatrix}\Phi(x^2)\\ \Psi(x^2)\end{pmatrix}=\underbrace{\frac{1}{\eta}\begin{pmatrix}-\left(\nu^2-\frac{9}{4}\right)x^{-(\mu+\frac{3}{2})}-2x^{-(\mu+\frac{1}{2})}\frac{d}{dx} & -\rho_1 x^{-(\mu-\frac{1}{2})}\\ \left(\mu^2-\frac{9}{4}\right)x^{-(\nu+\frac{3}{2})}+2x^{-(\nu+\frac{1}{2})}\frac{d}{dx} & \rho_1 x^{-(\nu-\frac{1}{2})}\end{pmatrix}}_{C(x)}\begin{pmatrix}f(x)\\ g(x)\end{pmatrix},$$

where $\eta=\mu^2-\nu^2$. (3.98)

Next, let us write the functions $\tilde{u}_1(x,t,\epsilon)$, $\tilde{v}_1(x,t,\epsilon)$, $\tilde{u}_2(x,t,\epsilon)$ and $\tilde{v}_2(x,t,\epsilon)$ as Laplace transforms of the following functions:

$$\tilde{u}_1(x,t,\epsilon) = \frac{x^{\mu+\frac{3}{2}}}{(1+4\epsilon t)^{\mu+1}} \exp\left(\frac{-\epsilon x^2}{1+4\epsilon t}\right)$$

$$= \mathcal{L}\left(\frac{x^{3/2}}{4t} z^{\mu/2} \exp\left(-\frac{x^2+z}{4t}\right) I_\mu\left(\frac{x\sqrt{z}}{2t}\right)\right)$$

$$\tilde{v}_1(x,t,\epsilon) = \left(-\frac{\left(\mu+\frac{3}{2}\right)\left(\mu+\frac{1}{2}\right)}{\rho_1}\frac{x^{\mu-\frac{1}{2}}}{(1+4\epsilon t)^{\mu+1}} + \frac{4\epsilon}{\rho_1}\frac{x^{\mu+\frac{3}{2}}}{(1+4\epsilon t)^{\mu+2}}\right)\exp\left(\frac{-\epsilon x^2}{1+4\epsilon t}\right)$$

$$= \mathcal{L}\left(z^{\mu/2}\exp\left(-\frac{x^2+z}{4t}\right)\left(\frac{x^{3/2}}{4\rho_1 t^2} - \frac{\left(\mu+\frac{3}{2}\right)\left(\mu+\frac{1}{2}\right)x^{-1/2}}{4\rho_1 t}\right) I_\mu\left(\frac{x\sqrt{z}}{2t}\right)\right.$$

$$\left. -\frac{\sqrt{x}}{4\rho_1 t^2} z^{\frac{\mu+1}{2}} \exp\left(-\frac{x^2+z}{4t}\right) I_{\mu+1}\left(\frac{x\sqrt{z}}{2t}\right)\right)$$

$$\tilde{u}_2(x,t,\epsilon) = \frac{x^{\nu+\frac{3}{2}}}{(1+4\epsilon t)^{\nu+1}} \exp\left(\frac{-\epsilon x^2}{1+4\epsilon t}\right)$$

$$= \mathcal{L}\left(\frac{x^{3/2}}{4t} z^{\nu/2} \exp\left(-\frac{x^2+z}{4t}\right) I_\nu\left(\frac{x\sqrt{z}}{2t}\right)\right)$$

$$\tilde{v}_2(x,t,\epsilon) = \left(-\frac{\left(\nu+\frac{3}{2}\right)\left(\nu+\frac{1}{2}\right)}{\rho_1}\frac{x^{\nu-\frac{1}{2}}}{(1+4\epsilon t)^{\nu+1}} + \frac{4\epsilon}{\rho_1}\frac{x^{\nu+\frac{3}{2}}}{(1+4\epsilon t)^{\nu+2}}\right)\exp\left(\frac{-\epsilon x^2}{1+4\epsilon t}\right)$$

$$= \mathcal{L}\left(z^{\nu/2}\exp\left(-\frac{x^2+z}{4t}\right)\left(\frac{x^{3/2}}{4\rho_1 t^2} - \frac{\left(\nu+\frac{3}{2}\right)\left(\nu+\frac{1}{2}\right)x^{-1/2}}{4\rho_1 t}\right) I_\nu\left(\frac{x\sqrt{z}}{2t}\right)\right.$$

$$\left. -\frac{\sqrt{x}}{4\rho_1 t^2} z^{\frac{\nu+1}{2}} \exp\left(-\frac{x^2+z}{4t}\right) I_{\nu+1}\left(\frac{x\sqrt{z}}{2t}\right)\right).$$

Let us write

$$m_1(x,z,t) := \frac{x^{3/2}}{4t} z^{\mu/2} \exp\left(-\frac{x^2+z}{4t}\right) I_\mu\left(\frac{x\sqrt{z}}{2t}\right)$$

$$n_1(x,z,t) := z^{\mu/2}\exp\left(-\frac{x^2+z}{4t}\right)\left(\frac{x^{3/2}}{4\rho_1 t^2} - \frac{\left(\mu+\frac{3}{2}\right)\left(\mu+\frac{1}{2}\right)x^{-1/2}}{4\rho_1 t}\right)I_\mu\left(\frac{x\sqrt{z}}{2t}\right)$$
$$-\frac{\sqrt{x}}{4\rho_1 t^2}z^{\frac{\mu+1}{2}}\exp\left(-\frac{x^2+z}{4t}\right)I_{\mu+1}\left(\frac{x\sqrt{z}}{2t}\right)$$
$$m_2(x,z,t) := \frac{x^{3/2}}{4t}z^{\nu/2}\exp\left(-\frac{x^2+z}{4t}\right)I_\nu\left(\frac{x\sqrt{z}}{2t}\right)$$
$$n_2(x,z,t) := z^{\nu/2}\exp\left(-\frac{x^2+z}{4t}\right)\left(\frac{x^{3/2}}{4\rho_1 t^2} - \frac{\left(\nu+\frac{3}{2}\right)\left(\nu+\frac{1}{2}\right)x^{-1/2}}{4\rho_1 t}\right)I_\nu\left(\frac{x\sqrt{z}}{2t}\right)$$
$$-\frac{\sqrt{x}}{4\rho_1 t^2}z^{\frac{\nu+1}{2}}\exp\left(-\frac{x^2+z}{4t}\right)I_{\nu+1}\left(\frac{x\sqrt{z}}{2t}\right)$$

Then the solution $\tilde{U}(x,t)$ can be writen as

$$\tilde{U}(x,t) = \begin{pmatrix} \int_0^\infty \left(\phi(\epsilon)\int_0^\infty m_1(x,z,t)e^{-\epsilon z}dz + \psi(\epsilon)\int_0^\infty m_2(x,z,t)e^{-\epsilon z}dz\right)d\epsilon \\ \int_0^\infty \left(\phi(\epsilon)\int_0^\infty n_1(x,z,t)e^{-\epsilon z}dz + \psi(\epsilon)\int_0^\infty n_2(x,z,t)e^{-\epsilon z}dz\right)d\epsilon \end{pmatrix}$$
$$= \begin{pmatrix} \int_0^\infty \left(\left(\int_0^\infty \phi(\epsilon)e^{-\epsilon z}d\epsilon\right)m_1(x,z,t) + \left(\int_0^\infty \psi(\epsilon)e^{-\epsilon z}d\epsilon\right)m_2(x,z,t)\right)dz \\ \int_0^\infty \left(\left(\int_0^\infty \phi(\epsilon)e^{-\epsilon z}d\epsilon\right)n_1(x,z,t) + \left(\int_0^\infty \psi(\epsilon)e^{-\epsilon z}d\epsilon\right)n_2(x,z,t)\right)dz \end{pmatrix}$$
$$= \int_0^\infty \begin{pmatrix} \Phi(z)m_1(x,z,t) + \Psi(z)m_2(x,z,t) \\ \Phi(z)n_1(x,z,t) + \Psi(z)n_2(x,z,t) \end{pmatrix} dz$$
$$= \int_0^\infty \begin{pmatrix} m_1(x,z,t) & m_2(x,z,t) \\ n_1(x,z,t) & n_2(x,z,t) \end{pmatrix}\begin{pmatrix} \Phi(z) \\ \Psi(z) \end{pmatrix} dz$$

Make the change of variables $z = y^2$ so that $dz = 2ydy$ and our solution becomes

$$\tilde{U}(x,t) = \int_0^\infty 2y \begin{pmatrix} m_1(x,y^2,t) & m_2(x,y^2,t) \\ n_1(x,y^2,t) & n_2(x,y^2,t) \end{pmatrix}\begin{pmatrix} \Phi(y^2) \\ \Psi(y^2) \end{pmatrix} dy.$$

Let us substitute $(\Phi(y^2) \quad \Psi(y^2))^\top$ by the expression found in (3.98) to obtain

$$\tilde{U}(x,t) = \int_0^\infty 2y \begin{pmatrix} m_1(x,y^2,t) & m_2(x,y^2,t) \\ n_1(x,y^2,t) & n_2(x,y^2,t) \end{pmatrix} C(y) \begin{pmatrix} f(y) \\ g(y) \end{pmatrix} dy.$$

Therefore, a fundamental matrix for this system is given by:

$$P(x,t,y) = \begin{pmatrix} p_{11}(x,t,y) & p_{12}(x,t,y) \\ p_{21}(x,t,y) & p_{22}(x,t,y) \end{pmatrix}$$
$$:= 2y \begin{pmatrix} m_1(x,y^2,t) & m_2(x,y^2,t) \\ n_1(x,y^2,t) & n_2(x,y^2,t) \end{pmatrix} C(y).$$

The reader may check that the expressions one obtains for each of the components $p_{ij}(x,t,y)$ of the fundamental matrix are respectively:

$$\begin{aligned} p_{11}(x,t,y) &= \frac{xe^{-\frac{x^2+y^2}{4t}}}{2t(\mu^2-\nu^2)}\sqrt{\frac{x}{y}}\left(\left(\mu^2-\frac{9}{4}\right)I_\nu\left(\frac{xy}{2t}\right)-\left(\nu^2-\frac{9}{4}\right)I_\mu\left(\frac{xy}{2t}\right)\right) \\ &+\frac{x\sqrt{xy}e^{-\frac{x^2+y^2}{4t}}}{t(\mu^2-\nu^2)}\left(I_\nu\left(\frac{xy}{2t}\right)+I_\mu\left(\frac{xy}{2t}\right)\right)\frac{d}{dy} \\ p_{12}(x,t,y) &= \frac{\rho_1(xy)^{\frac{3}{2}}e^{-\frac{x^2+y^2}{4t}}}{2t(\mu^2-\nu^2)}\left(I_\nu\left(\frac{xy}{2t}\right)-I_\mu\left(\frac{xy}{2t}\right)\right) \\ p_{21}(x,t,y) &= -\frac{e^{-\frac{x^2+y^2}{4t}}}{2\rho_1 t}\sqrt{\frac{x}{y}}\left(\left(\frac{xy}{t}-AB\frac{y}{x}\right)I_\mu\left(\frac{xy}{2t}\right)-\frac{y^2}{t}I_{\mu+1}\left(\frac{xy}{2t}\right)\right) \\ &\times\left(\frac{\left(\nu^2-\frac{9}{4}\right)}{y}-2\frac{d}{dy}\right) \\ &+\frac{e^{-\frac{x^2+y^2}{4t}}}{2\rho_1 t}\sqrt{\frac{x}{y}}\left(\left(\frac{xy}{t}-CD\frac{y}{x}\right)I_\nu\left(\frac{xy}{2t}\right)-\frac{y^2}{t}I_{\nu+1}\left(\frac{xy}{2t}\right)\right) \\ &\times\left(\frac{\left(\mu^2-\frac{9}{4}\right)}{y}+2\frac{d}{dy}\right) \end{aligned}$$

$$\begin{aligned} p_{22}(x,t,y) &= -\frac{\sqrt{xy}}{2t}e^{-\frac{x^2+y^2}{4t}}\left(\left(\frac{xy}{t}-AB\frac{y}{x}\right)I_\mu\left(\frac{xy}{2t}\right)-\frac{y^2}{t}I_{\mu+1}\left(\frac{xy}{2t}\right)\right) \\ &+\frac{\sqrt{xy}}{2t}e^{-\frac{x^2+y^2}{4t}}\left(\left(\frac{xy}{t}-CD\frac{y}{x}\right)I_\nu\left(\frac{xy}{2t}\right)-\frac{y^2}{t}I_{\nu+1}\left(\frac{xy}{2t}\right)\right), \end{aligned}$$

where $A=\mu+\frac{3}{2}$, $B=\mu+\frac{1}{2}$, $C=\nu+\frac{3}{2}$ and $D=\nu+\frac{1}{2}$.

Note. The fundamental matrix obtained in this case is not a matrix of scalar functions as in the previous example, but a matrix of differential operators. This will be the case whenever the expressions of the appropriate integral transforms of ϕ

and ψ are given in terms of not only the initial conditions $f(x)$ and $g(x)$, but also their derivatives. This is the case here, as it can be seen in (3.98). Further, we will see in other examples that these fundamental matrices sometimes include integral operators as well as differential operators.

3.2.3 Case A.3: Not enough symmetries

Similarly to the previous cases, a stationary solution for this case can be found by solving the system:

$$\begin{cases} 0 = u_{xx} + \rho_1 x^k v \\ 0 = v_{xx} + \rho_2 x^{-(4+k)} u \end{cases} \tag{3.99}$$

which will give an Euler type equation for the function u if we take $v = -\frac{u_{xx}}{\rho_1 x^k}$, calculate v_{xx} an substitute it into the second equation in the above system. However, if we look at the set of symmetries obtained for this particular case, it is clear that we cannot obtain time-dependent solutions from the application of a symmetry to a stationary solution. There are no symmetries in the Lie algebra for this case that introduce the time variable "t" from a solution that is time independent. So we cannot use our usual methodology to obtain fundamental solutions through the action of a symmetry on a stationary solution and the inversion of a classic integral transform. For these cases, a scaling symmetry often suffices, provided that we can find a time-dependent solution to the given system. These solutions may be found by inspection in some cases. However, in most cases it is not easy to find such solutions by inspection. This is the case for this example, where we have not yet found a suitable time-dependent solution to which we can apply the scaling symmetry in the lie algebra for this system of PDEs.

In later chapters we will provide a set of tools that can be used when one is aiming to find a transition density for a given stochastic process as a fundamental solution for the associated Kolmogorov Backwards equation but the symmetries for this equation are not "complex" enough to provide such fundamental solution. It turns out that these methods, while not producing the transition density function, can be extremely useful to calculate all sorts of expected values for the considered stochastic processes, as well as for functionals of these. Such tools rely heavily on the use of symmetries and the classical integral transforms of fundamental solutions.

3.2.4 Case B: The Fourier Transform

Recall that, for this case, the system of PDEs we are dealing with is of the form:

$$\begin{cases} u_t = u_{xx} + Av \\ v_t = v_{xx} + Bu, \end{cases} \tag{3.100}$$

and recall that the symmetries obtained for this type of system are of the form:

Proposition 3.2.1. *Let $U = (u(x,t), v(x,t))$ be a solution of (3.100), then*

$$U_1 = \left(\frac{\left(4At^2 v(\frac{x}{1+\epsilon t}, \frac{t}{1+\epsilon t}) - x^2 u(\frac{x}{1+\epsilon t}, \frac{t}{1+\epsilon t})\right) \sinh\left(\sigma(x,t,\epsilon)\right)}{\sqrt{(16ABt^4 + x^4)(1+\epsilon t)}} \right.$$
$$+ \frac{u(\frac{x}{1+\epsilon t}, \frac{t}{1+\epsilon t}) \cosh\left(\sigma(x,t,\epsilon)\right)}{\sqrt{1+\epsilon t}}, \frac{v(\frac{x}{1+\epsilon t}, \frac{t}{1+\epsilon t}) \cosh\left(\sigma(x,t,\epsilon)\right)}{\sqrt{1+\epsilon t}}$$
$$\left. + \frac{\left(4Bt^2 u(\frac{x}{1+\epsilon t}, \frac{t}{1+\epsilon t}) + x^2 v(\frac{x}{1+\epsilon t}, \frac{t}{1+\epsilon t})\right) \sinh\left(\sigma(x,t,\epsilon)\right)}{\sqrt{(16ABt^4 + x^4)(1+\epsilon t)}} \right)$$

$$U_2 = (u(xe^{-\epsilon/2}, te^{-\epsilon}) \cosh(\mu t(1-e^{-\epsilon})) + v(xe^{-\epsilon/2}, te^{-\epsilon}) \sqrt{\frac{A}{B}} \sinh(\mu t(1-e^{-\epsilon})),$$
$$v(xe^{-\epsilon/2}, te^{-\epsilon}) \cosh(\mu t(1-e^{-\epsilon})) + u(xe^{-\epsilon/2}, te^{-\epsilon}) \sqrt{\frac{B}{A}} \sinh(\mu t(1-e^{-\epsilon})))$$

$$U_3 = (u(x, t-\epsilon), v(x, t-\epsilon))$$

$$U_4 = (u(x-\epsilon t, t) \exp\left(-\frac{\epsilon}{2}\left(x - \frac{t\epsilon}{2}\right)\right), v(x-\epsilon t, t) \exp\left(-\frac{\epsilon}{2}\left(x - \frac{t\epsilon}{2}\right)\right))$$

$$U_5 = (u(x-\epsilon, t), v(x-\epsilon, t))$$

$$U_6 = (u(x,t)\cosh(\mu\epsilon) + \sqrt{\frac{A}{B}} v(x,t)\sinh(\mu\epsilon), \sqrt{\frac{B}{A}} u(x,t)\sinh(\mu\epsilon) + v(x,t)\cosh(\mu\epsilon))$$

$$U_7 = (e^\epsilon u(x,t), e^\epsilon v(x,t))$$

$$U_8 = ((1 - \epsilon\mu e^{2\mu t})u(x,t) + \epsilon A e^{2\mu t} v(x,t), (1 + \epsilon\mu e^{2\mu t})v(x,t) - \epsilon B e^{2\mu t} u(x,t))$$

$$U_9 = ((1 + \epsilon\mu e^{-2\mu t})u(x,t) + \epsilon A e^{-2\mu t} v(x,t), (1 - \epsilon\mu e^{-2\mu t})v(x,t) - \epsilon B e^{-2\mu t} u(x,t))$$

$$U_{CD} = (u(x,t) + \epsilon C(x,t), v(x,t) + \epsilon D(x,t))$$

are also solutions of the given system. Here $\mu = \sqrt{AB}$, $\sigma(x,t,\epsilon) = \frac{\epsilon\sqrt{16ABt^4+x^4}}{4(1+\epsilon t)}$ and (C,D) is an arbitrary solution of (3.56).

With this, we proceed to look for a stationary solution for (3.100), i.e. we set $u_t = v_t = 0$ and solve the remaining system for u and v. That is, we need to solve

$$\begin{cases} 0 = u_{xx} + Av \\ 0 = v_{xx} + Bu, \end{cases}$$

which gives $v = -\frac{u_{xx}}{A}$ and so $v_{xx} = -\frac{u_{xxxx}}{A}$. Substitution of v_{xx} in the second equation yields the following differential equation for u:

$$u_{xxxx} - ABu = 0 \tag{3.101}$$

Note that $u_1 = e^{\sqrt[4]{AB}x}$ and $u_2 = e^{i\sqrt[4]{AB}x}$ are both solutions of (3.101), which respectively yield the pairs

$$\begin{cases} u_1 = e^{\sqrt[4]{AB}x} \\ v_1 = -\frac{\sqrt{AB}}{A} e^{\sqrt[4]{AB}x} = -\sqrt{\frac{B}{A}} e^{\sqrt[4]{AB}x} \end{cases} \tag{3.102}$$

and

$$\begin{cases} u_2 = e^{i\sqrt[4]{AB}x} \\ v_2 = \frac{\sqrt{AB}}{A} e^{i\sqrt[4]{AB}x} = \sqrt{\frac{B}{A}} e^{i\sqrt[4]{AB}x} \end{cases} \tag{3.103}$$

as stationary solutions of (3.100).

We know by Proposition 3.2.1 that if $(u(x,t) \quad v(x,t))^\top$ is a solution of (3.100), so is

$$\tilde{U}_\epsilon(x,t) = \begin{pmatrix} \tilde{u}_\epsilon(x,t) \\ \tilde{v}_\epsilon(x,t) \end{pmatrix} = \begin{pmatrix} u(x-\epsilon t, t)\exp\left(-\frac{\epsilon}{2}\left(x - \frac{t\epsilon}{2}\right)\right) \\ v(x-\epsilon t, t)\exp\left(-\frac{\epsilon}{2}\left(x - \frac{t\epsilon}{2}\right)\right) \end{pmatrix}. \tag{3.104}$$

Applying this transformation to the stationary solutions (3.102) and (3.103) and letting $\mu = \sqrt[4]{AB}$ produces the following time-dependent pairs of solutions of (3.100):

$$\tilde{U}_1(x,t,\epsilon) = \begin{pmatrix} \tilde{u}_1(x,t,\epsilon) \\ \tilde{v}_1(x,t,\epsilon) \end{pmatrix} = \begin{pmatrix} e^{\mu x} e^{-\epsilon(\mu t + \frac{x}{2}) + \epsilon^2 \frac{t}{4}} \\ -\sqrt{\frac{B}{A}} e^{\mu x} e^{-\epsilon(\mu t + \frac{x}{2}) + \epsilon^2 \frac{t}{4}} \end{pmatrix}$$

$$\tilde{U}_2(x,t,\epsilon) = \begin{pmatrix} \tilde{u}_2(x,t,\epsilon) \\ \tilde{v}_2(x,t,\epsilon) \end{pmatrix} = \begin{pmatrix} e^{i\mu x} e^{-\epsilon(i\mu t + \frac{x}{2}) + \epsilon^2 \frac{t}{4}} \\ \sqrt{\frac{B}{A}} e^{i\mu x} e^{-\epsilon(i\mu t + \frac{x}{2}) + \epsilon^2 \frac{t}{4}} \end{pmatrix}$$

Let us now substitute the constant ϵ by $i\epsilon$. This substitution yields

$$\tilde{U}_1(x,t,\epsilon) = \begin{pmatrix} e^{\mu x} e^{-i\epsilon(\mu t + \frac{x}{2}) - \epsilon^2 \frac{t}{4}} \\ -\sqrt{\frac{B}{A}} e^{\mu x} e^{-i\epsilon(\mu t + \frac{x}{2}) - \epsilon^2 \frac{t}{4}} \end{pmatrix}$$

$$\tilde{U}_2(x,t,\epsilon) = \begin{pmatrix} e^{i\mu x} e^{-i\epsilon(i\mu t + \frac{x}{2}) - \epsilon^2 \frac{t}{4}} \\ \sqrt{\frac{B}{A}} e^{i\mu x} e^{-i\epsilon(i\mu t + \frac{x}{2}) - \epsilon^2 \frac{t}{4}} \end{pmatrix}.$$

The reader may check that the inverse Fourier transforms of these solutions with respect to the variable ϵ are given respectively by:

$$\mathcal{F}^{-1}\left(\tilde{U}_1(x,t,\epsilon)\right) = \begin{pmatrix} \sqrt{\frac{2}{t}}e^{-\left(\frac{(\mu t+y)^2}{t}+\frac{x^2}{4t}+\frac{xy}{t}\right)} \\ -\sqrt{\frac{B}{A}}\sqrt{\frac{2}{t}}e^{-\left(\frac{(\mu t+y)^2}{t}+\frac{x^2}{4t}+\frac{xy}{t}\right)} \end{pmatrix} \tag{3.105}$$

$$\mathcal{F}^{-1}\left(\tilde{U}_2(x,t,\epsilon)\right) = \begin{pmatrix} \sqrt{\frac{2}{t}}e^{\mu^2 t-\frac{(x+2y)^2}{4t}-2i\mu y} \\ \sqrt{\frac{B}{A}}\sqrt{\frac{2}{t}}e^{\mu^2 t-\frac{(x+2y)^2}{4t}-2i\mu y} \end{pmatrix}, \tag{3.106}$$

where we understand the Fourier transform $\mathcal{F}$ to be defined in the classical form as

$$\mathcal{F}(g(t))(\omega) = G(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(t)e^{i\omega t}dt. \tag{3.107}$$

With this definition, the inverse Fourier transform is given by:

$$\mathcal{F}^{-1}(G(\omega))(t) = g(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} G(\omega)e^{-i\omega t}d\omega. \tag{3.108}$$

Next, recall that by linearity and Theorem 2.1.5, for appropriate functions φ and ψ with sufficiently rapid decay, we have that

$$\begin{aligned} \begin{pmatrix} U(x,t) \\ V(x,t) \end{pmatrix} &= \int_{-\infty}^{\infty}\left(\varphi(\epsilon)\tilde{U}_1(x,t,\epsilon)+\psi(\epsilon)\tilde{U}_2(x,t,\epsilon)\right)d\epsilon \\ &= \int_{-\infty}^{\infty}\begin{pmatrix} \varphi(\epsilon)\tilde{u}_1(x,t,\epsilon)+\psi(\epsilon)\tilde{u}_2(x,t,\epsilon) \\ \varphi(\epsilon)\tilde{v}_1(x,t,\epsilon)+\psi(\epsilon)\tilde{v}_2(x,t,\epsilon) \end{pmatrix} d\epsilon \end{aligned} \tag{3.109}$$

will be a new solution of the system (3.100), with initial condition

$$\begin{aligned} \begin{pmatrix} U(x,0) \\ V(x,0) \end{pmatrix} &= \int_{-\infty}^{\infty}\begin{pmatrix} \varphi(\epsilon)\tilde{u}_1(x,0,\epsilon)+\psi(\epsilon)\tilde{u}_2(x,0,\epsilon) \\ \varphi(\epsilon)\tilde{v}_1(x,0,\epsilon)+\psi(\epsilon)\tilde{v}_2(x,0,\epsilon) \end{pmatrix} d\epsilon \\ &= \int_{-\infty}^{\infty}\begin{pmatrix} \varphi(\epsilon)e^{\mu x}e^{-\frac{i\epsilon x}{2}}+\psi(\epsilon)e^{i\mu x}e^{-\frac{i\epsilon x}{2}} \\ -\varphi(\epsilon)\sqrt{\frac{B}{A}}e^{\mu x}e^{-\frac{i\epsilon x}{2}}+\psi(\epsilon)\sqrt{\frac{B}{A}}e^{i\mu x}e^{-\frac{i\epsilon x}{2}} \end{pmatrix} d\epsilon \\ &= \begin{pmatrix} e^{\mu x}\int_{-\infty}^{\infty}\varphi(\epsilon)e^{-\frac{i\epsilon x}{2}}d\epsilon + e^{i\mu x}\int_{-\infty}^{\infty}\psi(\epsilon)e^{-\frac{i\epsilon x}{2}}d\epsilon \\ -\sqrt{\frac{B}{A}}e^{\mu x}\int_{-\infty}^{\infty}\varphi(\epsilon)e^{-\frac{i\epsilon x}{2}}d\epsilon + \sqrt{\frac{B}{A}}e^{i\mu x}\int_{-\infty}^{\infty}\psi(\epsilon)e^{-\frac{i\epsilon x}{2}}d\epsilon \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{2\pi}e^{\mu x}\Phi\left(-\frac{x}{2}\right)+\sqrt{2\pi}e^{i\mu x}\Psi\left(-\frac{x}{2}\right) \\ -\sqrt{2\pi}\sqrt{\frac{B}{A}}e^{\mu x}\Phi\left(-\frac{x}{2}\right)+\sqrt{2\pi}\sqrt{\frac{B}{A}}e^{i\mu x}\Psi\left(-\frac{x}{2}\right) \end{pmatrix} \end{aligned}$$

$$= \underbrace{\begin{pmatrix} \sqrt{2\pi}e^{\mu x} & \sqrt{2\pi}e^{i\mu x} \\ -\sqrt{2\pi}\sqrt{\frac{B}{A}}e^{\mu x} & \sqrt{2\pi}\sqrt{\frac{B}{A}}e^{i\mu x} \end{pmatrix}}_{C(x)} \begin{pmatrix} \Phi\left(-\frac{x}{2}\right) \\ \Psi\left(-\frac{x}{2}\right) \end{pmatrix},$$

where $\Phi = \mathcal{F}(\varphi)$ and $\Psi = \mathcal{F}(\psi)$.

Putting

$$\begin{pmatrix} U(x,0) \\ V(x,0) \end{pmatrix} = C(x) \begin{pmatrix} \Phi\left(-\frac{x}{2}\right) \\ \Psi\left(-\frac{x}{2}\right) \end{pmatrix} := \begin{pmatrix} f(x) \\ g(x) \end{pmatrix}, \tag{3.110}$$

we obtain the expression

$$\begin{pmatrix} \Phi\left(-\frac{x}{2}\right) \\ \Psi\left(-\frac{x}{2}\right) \end{pmatrix} = C^{-1}(x) \begin{pmatrix} f(x) \\ g(x) \end{pmatrix}. \tag{3.111}$$

The reader may check that C is a non-singular matrix and that the corresponding expression for $C^{-1}(x)$ is

$$C^{-1}(x) = \frac{1}{4\pi} \begin{pmatrix} e^{-\mu x} & -\sqrt{\frac{A}{B}}e^{-\mu x} \\ e^{-i\mu x} & \sqrt{\frac{A}{B}}e^{-i\mu x} \end{pmatrix}. \tag{3.112}$$

Let us now substitute $\tilde{u}_1$, $\tilde{v}_1$, $\tilde{u}_2$ and $\tilde{v}_2$ in the expression (3.109) of our solution $(U(x,t) \quad V(x,t))^\top$ by the Fourier transforms of the expressions obtained in (3.105) and (3.106). This yields:

$$\begin{aligned}
\begin{pmatrix} U(x,t) \\ V(x,t) \end{pmatrix} &= \begin{pmatrix} \int_{-\infty}^{\infty} \varphi(\epsilon)\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{t}} e^{-\frac{(\mu t+y)^2}{t} - \frac{x^2}{4t} - \frac{xy}{t}} e^{i\epsilon y} dy d\epsilon \\ -\int_{-\infty}^{\infty} \varphi(\epsilon)\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{B}{A}}\sqrt{\frac{2}{t}} e^{-\frac{(\mu t+y)^2}{t} - \frac{x^2}{4t} - \frac{xy}{t}} e^{i\epsilon y} dy d\epsilon \end{pmatrix} \\
&+ \begin{pmatrix} \int_{-\infty}^{\infty} \psi(\epsilon)\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{t}} e^{\mu^2 t - \frac{(x+2y)^2}{4t} - 2i\mu y} e^{i\epsilon y} dy d\epsilon \\ \int_{-\infty}^{\infty} \psi(\epsilon)\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{B}{A}}\sqrt{\frac{2}{t}} e^{\mu^2 t - \frac{(x+2y)^2}{4t} - 2i\mu y} e^{i\epsilon y} dy d\epsilon \end{pmatrix} \\
&= \begin{pmatrix} \int_{-\infty}^{\infty} \sqrt{\frac{2}{t}} e^{-\frac{(\mu t+y)^2}{t} - \frac{x^2}{4t} - \frac{xy}{t}} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(\epsilon) e^{i\epsilon y} d\epsilon\right) dy \\ -\int_{-\infty}^{\infty} \sqrt{\frac{B}{A}}\sqrt{\frac{2}{t}} e^{-\frac{(\mu t+y)^2}{t} - \frac{x^2}{4t} - \frac{xy}{t}} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(\epsilon) e^{i\epsilon y} d\epsilon\right) dy \end{pmatrix} \\
&+ \begin{pmatrix} \int_{-\infty}^{\infty} \sqrt{\frac{2}{t}} e^{\mu^2 t - \frac{(x+2y)^2}{4t} - 2i\mu y} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(\epsilon) e^{i\epsilon y} d\epsilon\right) dy \\ \int_{-\infty}^{\infty} \sqrt{\frac{B}{A}}\sqrt{\frac{2}{t}} e^{\mu^2 t - \frac{(x+2y)^2}{4t} - 2i\mu y} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(\epsilon) e^{i\epsilon y} d\epsilon\right) dy \end{pmatrix} \\
&= \begin{pmatrix} \int_{-\infty}^{\infty} \sqrt{\frac{2}{t}} \left(e^{-\frac{(\mu t+y)^2}{t} - \frac{x^2}{4t} - \frac{xy}{t}} \Phi(y) + e^{\mu^2 t - \frac{(x+2y)^2}{4t} - 2i\mu y} \Psi(y) \right) dy \\ -\int_{-\infty}^{\infty} \sqrt{\frac{B}{A}}\sqrt{\frac{2}{t}} \left(e^{-\frac{(\mu t+y)^2}{t} - \frac{x^2}{4t} - \frac{xy}{t}} \Phi(y) - e^{\mu^2 t - \frac{(x+2y)^2}{4t} - 2i\mu y} \Psi(y) \right) dy \end{pmatrix}
\end{aligned}$$

Make the change of variables $y = -\frac{z}{2}$ to obtain

$$\begin{pmatrix} U(x,t) \\ V(x,t) \end{pmatrix} = \begin{pmatrix} -\int_{-\infty}^{\infty} \frac{e^{-\frac{(x-z)^2}{4t}}}{\sqrt{2t}} \left(e^{-\mu^2 t-\mu z}\Phi(-\frac{z}{2}) + e^{\mu^2 t+i\mu z}\Psi(-\frac{z}{2})\right) dz \\ \int_{-\infty}^{\infty} \sqrt{\frac{B}{A}} \frac{e^{-\frac{(x-z)^2}{4t}}}{\sqrt{2t}} \left(e^{-\mu^2 t-\mu z}\Phi(-\frac{z}{2}) - e^{\mu^2 t+i\mu z}\Psi(-\frac{z}{2})\right) dz \end{pmatrix}$$

$$= \int_{-\infty}^{\infty} \begin{pmatrix} -\frac{e^{-\frac{(x-z)^2}{4t}}}{\sqrt{2t}} \left(e^{-\mu^2 t-\mu z}\Phi(-\frac{z}{2}) + e^{\mu^2 t+i\mu z}\Psi(-\frac{z}{2})\right) \\ \sqrt{\frac{B}{A}} \frac{e^{-\frac{(x-z)^2}{4t}}}{\sqrt{2t}} \left(e^{-\mu^2 t-\mu z}\Phi(-\frac{z}{2}) - e^{\mu^2 t+i\mu z}\Psi(-\frac{z}{2})\right) \end{pmatrix} dz$$

$$= \int_{-\infty}^{\infty} \underbrace{\begin{pmatrix} -\frac{e^{-\frac{(x-z)^2}{4t}}}{\sqrt{2t}} e^{-\mu^2 t-\mu z} & -\frac{e^{-\frac{(x-z)^2}{4t}}}{\sqrt{2t}} e^{\mu^2 t+i\mu z} \\ \sqrt{\frac{B}{A}} \frac{e^{-\frac{(x-z)^2}{4t}}}{\sqrt{2t}} e^{-\mu^2 t-\mu z} & -\sqrt{\frac{B}{A}} \frac{e^{-\frac{(x-z)^2}{4t}}}{\sqrt{2t}} e^{\mu^2 t+i\mu z} \end{pmatrix}}_{A(x,t,z)} \begin{pmatrix} \Phi(-\frac{z}{2}) \\ \Psi(-\frac{z}{2}) \end{pmatrix} dz$$

Use (3.124) to write

$$\begin{pmatrix} U(x,t) \\ V(x,t) \end{pmatrix} = \int_{-\infty}^{\infty} A(x,t,z)C^{-1}(z) \begin{pmatrix} f(z) \\ g(z) \end{pmatrix} dz.$$

Therefore, a fundamental matrix $P(t,x,z) = (p_{ij}(t,x,z))$ for the system (3.100) is given by the product:

$$\begin{aligned} P(x,t,z) &:= A(x,t,z)C^{-1}(z) \\ &= \frac{e^{-\frac{(x-z)^2}{4t}}}{\sqrt{2t}} \begin{pmatrix} -e^{-\mu^2 t-\mu z} & -e^{\mu^2 t+i\mu z} \\ \sqrt{\frac{B}{A}} e^{-\mu^2 t-\mu z} & -\sqrt{\frac{B}{A}} e^{\mu^2 t+i\mu z} \end{pmatrix} \frac{1}{4\pi} \begin{pmatrix} e^{-\mu z} & -\sqrt{\frac{A}{B}} e^{-\mu z} \\ e^{-i\mu z} & \sqrt{\frac{A}{B}} e^{-i\mu z} \end{pmatrix} \\ &= -\frac{e^{-\frac{(x-z)^2}{4t}+\mu^2 t}}{4\pi\sqrt{2t}} \begin{pmatrix} 1+e^{-2\mu^2 t-2\mu z} & \sqrt{\frac{A}{B}}(1-e^{-2\mu^2 t-2\mu z}) \\ \sqrt{\frac{B}{A}}(1-e^{-2\mu^2 t-2\mu z}) & (1+e^{-2\mu^2 t-2\mu z}) \end{pmatrix} \end{aligned} \tag{3.113}$$

Note. If $AB \neq 0$, it would have also been possible to find fundamental solutions for the system (3.100) by first transforming the system, via a simple scaling, to the same system with $A, B = \pm 1$ and later decoupling it to

$$U_t = U_{xx}, \quad V_t = V_{xx}, \tag{3.114}$$

by using either

$$\begin{aligned} u &= \pm(U-V)\sinh t + (U+V)\cosh t \\ v &= (U-V)\cosh t \mp (U+V)\sinh t \end{aligned} \tag{3.115}$$

where the positive case is used when $A = B = 1$ and the negative case when $A = B = -1$ or

$$u = \pm(U - V)\sin t + (U + V)\cos t$$
$$v = (U - V)\cos t \mp (U + V)\sin t \tag{3.116}$$

where the positive case is used when $A = -B = 1$ and the negative case when $-A = B = 1$.

3.2.5 Case C: A more complex case with the Fourier Transform

As in the previous cases, we start by looking for a stationary solution for (3.66), that is, we look for u and v satisfying:

$$\begin{cases} 0 = u_{xx} + (ax + b)v \\ 0 = v_{xx} + k(ax + b)u. \end{cases}$$

which gives $v = -\frac{u_{xx}}{ax+b}$. Computation of v_{xx} and substitution into the second equation yields the following differential equation for u:

$$-\frac{2a^2u''(x)}{(ax+b)^3} - \frac{u^{(4)}(x)}{ax+b} + \frac{2au^{(3)}(x)}{(ax+b)^2} = -ku(x)(ax+b) \tag{3.117}$$

Without loss of generality take $b = 0$, since the system with $b \neq 0$ can be easily transformed to one with $b = 0$ via a change of variables. Assume $a, x, k > 0$ for the sake of simplicity in the calculations and note that

$$u_1 = \sqrt{ax}\left(J_{\frac{1}{3}}\left(\frac{2}{3}\sqrt[4]{k}x\sqrt{ax}\right) + I_{\frac{1}{3}}\left(\frac{2}{3}\sqrt[4]{k}x\sqrt{ax}\right)\right)$$

and

$$u_2 = \sqrt{ax}\left(I_{\frac{1}{3}}\left(\frac{2}{3}\sqrt[4]{k}x\sqrt{ax}\right) - J_{\frac{1}{3}}\left(\frac{2}{3}\sqrt[4]{k}x\sqrt{ax}\right)\right)$$

are both solutions of (3.117), which respectively yield the pairs

$$\begin{cases} u_1 = \sqrt{ax}\left(J_{\frac{1}{3}}\left(\frac{2}{3}\sqrt[4]{k}x\sqrt{ax}\right) + I_{\frac{1}{3}}\left(\frac{2}{3}\sqrt[4]{k}x\sqrt{ax}\right)\right) \\ v_1 = \sqrt{akx}\left(J_{\frac{1}{3}}\left(\frac{2}{3}\sqrt[4]{k}\sqrt{ax^3}\right) - I_{\frac{1}{3}}\left(\frac{2}{3}\sqrt[4]{k}\sqrt{ax^3}\right)\right) \end{cases} \tag{3.118}$$

and

$$\begin{cases} u_2 = \sqrt{ax}\left(I_{\frac{1}{3}}\left(\frac{2}{3}\sqrt[4]{k}x\sqrt{ax}\right) - J_{\frac{1}{3}}\left(\frac{2}{3}\sqrt[4]{k}x\sqrt{ax}\right)\right) \\ v_2 = -\sqrt{akx}\left(J_{\frac{1}{3}}\left(\frac{2}{3}\sqrt[4]{k}\sqrt{ax^3}\right) + I_{\frac{1}{3}}\left(\frac{2}{3}\sqrt[4]{k}\sqrt{ax^3}\right)\right) \end{cases} \tag{3.119}$$

as stationary solutions of the system

$$\begin{cases} u_t = u_{xx} + (ax+b)v \\ v_t = v_{xx} + k(ax+b)u. \end{cases} \tag{3.120}$$

We know by Proposition 3.1.5 that if $(u(x,t) \quad v(x,t))^\top$ is a solution of (3.120), so is

$$\begin{aligned} \tilde{U}_\epsilon(x,t) &= \begin{pmatrix} \tilde{u}_\epsilon(x,t) \\ \tilde{v}_\epsilon(x,t) \end{pmatrix} \\ &= \begin{pmatrix} e^{-x\epsilon+t\epsilon^2}(u(x-2\epsilon t,t)\cosh(a\sqrt{k}t^2\epsilon) + \frac{1}{\sqrt{k}}v(x-2\epsilon t,t)\sinh(a\sqrt{k}t^2\epsilon)) \\ e^{-x\epsilon+t\epsilon^2}\left(\sqrt{k}u(x-2\epsilon t,t)\sinh(a\sqrt{k}t^2\epsilon) + v(x-2\epsilon t,t)\cosh(a\sqrt{k}t^2\epsilon)\right) \end{pmatrix} \end{aligned} \tag{3.121}$$

Applying this transformation to the stationary solutions (3.118) and (3.119) and letting $k = \kappa^2$ produces the time-dependent pairs of solutions of (3.120):

$$\tilde{U}_1(x,t,\epsilon) = \begin{pmatrix} e^{-x\epsilon+t\epsilon^2}\tilde{u}_1(x,t,\epsilon) \\ e^{-x\epsilon+t\epsilon^2}\tilde{v}_1(x,t,\epsilon) \end{pmatrix}, \quad \tilde{U}_2(x,t,\epsilon) = \begin{pmatrix} e^{-x\epsilon+t\epsilon^2}\tilde{u}_2(x,t,\epsilon) \\ e^{-x\epsilon+t\epsilon^2}\tilde{v}_2(x,t,\epsilon) \end{pmatrix},$$

where

$$\begin{aligned} \tilde{u}_1(x,t,\epsilon) &= \sqrt{a(x-2t\epsilon)}J_{\frac{1}{3}}\left(\frac{2\sqrt{a\kappa(x-2t\epsilon)^3}}{3}\right)(\sinh(a\kappa t^2\epsilon) + \cosh(a\kappa t^2\epsilon)) \\ &\quad + \sqrt{a(x-2t\epsilon)}I_{\frac{1}{3}}\left(\frac{2\sqrt{a\kappa(x-2t\epsilon)^3}}{3}\right)(\cosh(a\kappa t^2\epsilon) - \sinh(a\kappa t^2\epsilon)) \\ \tilde{v}_1(x,t,\epsilon) &= \kappa\sqrt{a(x-2t\epsilon)}J_{\frac{1}{3}}\left(\frac{2\sqrt{a\kappa(x-2t\epsilon)^3}}{3}\right)(\sinh(a\kappa t^2\epsilon) + \cosh(a\kappa t^2\epsilon)) \\ &\quad + \kappa\sqrt{a(x-2t\epsilon)}I_{\frac{1}{3}}\left(\frac{2\sqrt{a\kappa(x-2t\epsilon)^3}}{3}\right)(\sinh(a\kappa t^2\epsilon) - \cosh(a\kappa t^2\epsilon)) \\ \tilde{u}_2(x,t,\epsilon) &= \sqrt{a(x-2t\epsilon)}I_{\frac{1}{3}}\left(\frac{2\sqrt{a\kappa(x-2t\epsilon)^3}}{3}\right)(\cosh(a\kappa t^2\epsilon) - \sinh(a\kappa t^2\epsilon)) \\ &\quad - \sqrt{a(x-2t\epsilon)}J_{\frac{1}{3}}\left(\frac{2\sqrt{a\kappa(x-2t\epsilon)^3}}{3}\right)(\sinh(a\kappa t^2\epsilon) + \cosh(a\kappa t^2\epsilon)) \\ \tilde{v}_2(x,t,\epsilon) &= \kappa\sqrt{a(x-2t\epsilon)}I_{\frac{1}{3}}\left(\frac{2\sqrt{a\kappa(x-2t\epsilon)^3}}{3}\right)(\sinh(a\kappa t^2\epsilon) - \cosh(a\kappa t^2\epsilon)) \\ &\quad - \kappa\sqrt{a(x-2t\epsilon)}J_{\frac{1}{3}}\left(\frac{2\sqrt{a\kappa(x-2t\epsilon)^3}}{3}\right)(\sinh(a\kappa t^2\epsilon) + \cosh(a\kappa t^2\epsilon)) \end{aligned}$$

Substitution of the constant ϵ by $i\epsilon$ yields

$$\tilde{U}_1(x,t,\epsilon) = \begin{pmatrix} e^{-(xi\epsilon+t\epsilon^2)}\tilde{u}_1(x,t,i\epsilon) \\ e^{-(xi\epsilon+t\epsilon^2)}\tilde{v}_1(x,t,i\epsilon) \end{pmatrix}, \quad \tilde{U}_2(x,t,\epsilon) = \begin{pmatrix} e^{-(xi\epsilon+t\epsilon^2)}\tilde{u}_2(x,t,i\epsilon) \\ e^{-(xi\epsilon+t\epsilon^2)}\tilde{v}_2(x,t,i\epsilon) \end{pmatrix}.$$

Construct now the new solution:

$$\begin{aligned} \begin{pmatrix} U(x,t) \\ V(x,t) \end{pmatrix} &= \int_{-\infty}^{\infty} \left(\varphi(\epsilon)\tilde{U}_1(x,t,\epsilon) + \psi(\epsilon)\tilde{U}_2(x,t,\epsilon)\right) d\epsilon \\ &= \int_{-\infty}^{\infty} \begin{pmatrix} \varphi(\epsilon)e^{-(xi\epsilon+t\epsilon^2)}\tilde{u}_1(x,t,i\epsilon) + \psi(\epsilon)e^{-(xi\epsilon+t\epsilon^2)}\tilde{u}_2(x,t,i\epsilon) \\ \varphi(\epsilon)e^{-(xi\epsilon+t\epsilon^2)}\tilde{v}_1(x,t,i\epsilon) + \psi(\epsilon)e^{-(xi\epsilon+t\epsilon^2)}\tilde{v}_2(x,t,i\epsilon) \end{pmatrix} d\epsilon, \end{aligned} \tag{3.122}$$

for appropriate functions φ and ψ with sufficiently rapid decay. This solution has initial condition

$$\begin{aligned} \begin{pmatrix} U(x,0) \\ V(x,0) \end{pmatrix} &= \int_{-\infty}^{\infty} \begin{pmatrix} \varphi(\epsilon)e^{-xi\epsilon}\tilde{u}_1(x,0,i\epsilon) + \psi(\epsilon)e^{-xi\epsilon}\tilde{u}_2(x,0,i\epsilon) \\ \varphi(\epsilon)e^{-xi\epsilon}\tilde{v}_1(x,0,i\epsilon) + \psi(\epsilon)e^{-xi\epsilon}\tilde{v}_2(x,0,i\epsilon) \end{pmatrix} d\epsilon \\ &= \int_{-\infty}^{\infty} \begin{pmatrix} \sqrt{ax}e^{-ix\epsilon}\varphi(\epsilon)\left(I_{\frac{1}{3}}\left(\frac{2\sqrt{a\kappa x^3}}{3}\right) - J_{\frac{1}{3}}\left(\frac{2}{3}\sqrt{a\kappa x^3}\right)\right) \\ \kappa\sqrt{ax}e^{-ix\epsilon}\varphi(\epsilon)\left(J_{\frac{1}{3}}\left(\frac{2\sqrt{a\kappa x^3}}{3}\right) - I_{\frac{1}{3}}\left(\frac{2\sqrt{a\kappa x^3}}{3}\right)\right) \end{pmatrix} d\epsilon \\ &+ \int_{-\infty}^{\infty} \begin{pmatrix} \sqrt{ax}e^{-ix\epsilon}\psi(\epsilon)\left(I_{\frac{1}{3}}\left(\frac{2\sqrt{a\kappa x^3}}{3}\right) - J_{\frac{1}{3}}\left(\frac{2}{3}\sqrt{a\kappa x^3}\right)\right) \\ \kappa\sqrt{ax}e^{-ix\epsilon}\psi(\epsilon)\left(-J_{\frac{1}{3}}\left(\frac{2\sqrt{a\kappa x^3}}{3}\right) - I_{\frac{1}{3}}\left(\frac{2\sqrt{a\kappa x^3}}{3}\right)\right) \end{pmatrix} d\epsilon \\ &= C(x)\begin{pmatrix} \Phi(-x) \\ \Psi(-x) \end{pmatrix}, \end{aligned}$$

where

$$C(x) := \sqrt{2\pi ax}\begin{pmatrix} I_{\frac{1}{3}}\left(\frac{2\sqrt{a\kappa x^3}}{3}\right) - J_{\frac{1}{3}}\left(\frac{2}{3}\sqrt{a\kappa x^3}\right) & I_{\frac{1}{3}}\left(\frac{2\sqrt{a\kappa x^3}}{3}\right) - J_{\frac{1}{3}}\left(\frac{2}{3}\sqrt{a\kappa x^3}\right) \\ \kappa(J_{\frac{1}{3}}\left(\frac{2\sqrt{a\kappa x^3}}{3}\right) - I_{\frac{1}{3}}\left(\frac{2\sqrt{a\kappa x^3}}{3}\right)) & -\kappa(J_{\frac{1}{3}}\left(\frac{2\sqrt{a\kappa x^3}}{3}\right) + I_{\frac{1}{3}}\left(\frac{2\sqrt{a\kappa x^3}}{3}\right)) \end{pmatrix}$$

and $\Phi = \mathcal{F}(\varphi)$, $\Psi = \mathcal{F}(\psi)$.

Let us now write

$$\begin{pmatrix} U(x,0) \\ V(x,0) \end{pmatrix} = C(x) \begin{pmatrix} \Phi(-x) \\ \Psi(-x) \end{pmatrix} := \begin{pmatrix} f(x) \\ g(x) \end{pmatrix}. \tag{3.123}$$

This gives

$$\begin{pmatrix} \Phi(-x) \\ \Psi(-x) \end{pmatrix} = C^{-1}(x) \begin{pmatrix} f(x) \\ g(x) \end{pmatrix}, \tag{3.124}$$

where the matrix $C^{-1}(x)$ can be computed using Cramer's formula.

Consider the functions

$$\begin{aligned} U_1(x,t,z) &:= \mathcal{F}^{-1}\left(e^{-(xi\epsilon+t\epsilon^2)}\tilde{u}_1(x,t,i\epsilon)\right) \\ U_2(x,t,z) &:= \mathcal{F}^{-1}\left(e^{-(xi\epsilon+t\epsilon^2)}\tilde{u}_2(x,t,i\epsilon)\right) \\ V_1(x,t,z) &:= \mathcal{F}^{-1}\left(e^{-(xi\epsilon+t\epsilon^2)}\tilde{v}_1(x,t,i\epsilon)\right) \\ V_2(x,t,z) &:= \mathcal{F}^{-1}\left(e^{-(xi\epsilon+t\epsilon^2)}\tilde{v}_2(x,t,i\epsilon)\right) \end{aligned}$$

and hence write

$$e^{-(xi\epsilon+t\epsilon^2)}\tilde{u}_j(x,t,i\epsilon) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} U_j(x,t,z)e^{i\epsilon z}dz$$

and

$$e^{-(xi\epsilon+t\epsilon^2)}\tilde{v}_j(x,t,i\epsilon) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} V_j(x,t,z)e^{i\epsilon z}dz.$$

This gives

$$\begin{aligned} \begin{pmatrix} U(x,t) \\ V(x,t) \end{pmatrix} &= \int_{-\infty}^{\infty} \begin{pmatrix} \varphi(\epsilon)e^{-(xi\epsilon+t\epsilon^2)}\tilde{u}_1(x,t,i\epsilon) + \psi(\epsilon)e^{-(xi\epsilon+t\epsilon^2)}\tilde{u}_2(x,t,i\epsilon) \\ \varphi(\epsilon)e^{-(xi\epsilon+t\epsilon^2)}\tilde{v}_1(x,t,i\epsilon) + \psi(\epsilon)e^{-(xi\epsilon+t\epsilon^2)}\tilde{v}_2(x,t,i\epsilon) \end{pmatrix} d\epsilon \\ &= \begin{pmatrix} \int_{-\infty}^{\infty}\varphi(\epsilon)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} U_1(x,t,z)e^{i\epsilon z}dzd\epsilon + \int_{-\infty}^{\infty}\psi(\epsilon)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} U_2(x,t,z)e^{i\epsilon z}dzd\epsilon \\ \int_{-\infty}^{\infty}\varphi(\epsilon)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} V_1(x,t,z)e^{i\epsilon z}dzd\epsilon + \int_{-\infty}^{\infty}\psi(\epsilon)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} V_2(x,t,z)e^{i\epsilon z}dzd\epsilon \end{pmatrix} \\ &= \begin{pmatrix} \int_{-\infty}^{\infty} U_1(x,t,z)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\varphi(\epsilon)e^{i\epsilon z}d\epsilon dz + \int_{-\infty}^{\infty} U_2(x,t,z)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi(\epsilon)e^{i\epsilon z}d\epsilon dz \\ \int_{-\infty}^{\infty} V_1(x,t,z)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\varphi(\epsilon)e^{i\epsilon z}d\epsilon dz + \int_{-\infty}^{\infty} V_2(x,t,z)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi(\epsilon)e^{i\epsilon z}d\epsilon dz \end{pmatrix} \\ &= \begin{pmatrix} \int_{-\infty}^{\infty} U_1(x,t,z)\Phi(z)dz + \int_{-\infty}^{\infty} U_2(x,t,z)\Psi(z)dz \\ \int_{-\infty}^{\infty} V_1(x,t,z)\Phi(z)dz + \int_{-\infty}^{\infty} V_2(x,t,z)\Psi(z)dz \end{pmatrix} \end{aligned}$$

The change of variables $y = -z$ then gives:

$$\begin{aligned}\begin{pmatrix} U(x,t) \\ V(x,t) \end{pmatrix} &= -\int_{-\infty}^{\infty} \begin{pmatrix} U_1(x,t,-y)\Phi(-y) + U_2(x,t,-y)\Psi(-y) \\ V_1(x,t,-y)\Phi(-y) + V_2(x,t,-y)\Psi(-y) \end{pmatrix} dy \\ &= -\int_{-\infty}^{\infty} \begin{pmatrix} U_1(x,t,-y) & U_2(x,t,-y) \\ V_1(x,t,-y) & V_2(x,t,-y) \end{pmatrix} \begin{pmatrix} \Phi(-y) \\ \Psi(-y) \end{pmatrix} dy \\ &= -\int_{-\infty}^{\infty} \begin{pmatrix} U_1(x,t,-y) & U_2(x,t,-y) \\ V_1(x,t,-y) & V_2(x,t,-y) \end{pmatrix} C^{-1}(y) \begin{pmatrix} f(y) \\ g(y) \end{pmatrix} dy\end{aligned}$$

Therefore, a fundamental matrix for the system (3.120) is given by

$$P(x,t,y) := -\begin{pmatrix} U_1(x,t,-y) & U_2(x,t,-y) \\ V_1(x,t,-y) & V_2(x,t,-y) \end{pmatrix} C^{-1}(y).$$

Remark. A closed-form expression for the inverse Fourier transforms $U_j(x,t,z)$, $V_j(x,t,z)$ does not seem easy to obtain. However, these can be approximated numerically.

Chapter 4

Systems of PDEs involving real functions arising from single PDEs for a complex-valued function

In this Chapter we further explore Lie Symmetry methods for systems of PDEs. In this case, these systems arise from separating the real and complex components of a single PDE involving a complex valued function u. We use known results for that particular PDE to determine a fundamental solution for the system arising from the real and imaginary components respectively.

Consider the following Theorem from [23]:

Theorem 4.0.1. *Suppose that $\gamma \neq 2$ and for a given g, $h(x) = x^{1-\gamma} f(x)$ is a solution of the Riccati equation*

$$\sigma x h' - \sigma h + \frac{1}{2} h^2 + 2\sigma x^{2-\gamma} g(x) = 2\sigma A x^{2-\gamma} + B. \tag{4.1}$$

Then the PDE

$$u_t = \sigma x^{\gamma} u_{xx} + f(x) u_x - g(x) u, \quad x \geq 0 \tag{4.2}$$

has a symmetry of the form

$$\bar{U}_\epsilon(t,x) = \frac{1}{(1+4\epsilon t)^{\frac{1-\gamma}{2-\gamma}}} \exp\left\{ \frac{-4\epsilon(x^{2-\gamma} + A\sigma(2-\gamma)^2 t^2)}{\sigma(2-\gamma)^2(1+4\epsilon t)} \right\} \times$$
$$\exp\left\{ \frac{1}{2\sigma}\left(F\left(\frac{x}{(1+4\epsilon t)^{\frac{2}{2-\gamma}}} \right) - F(x) \right) \right\} u\left(\frac{t}{1+4\epsilon t}, \frac{x}{(1+4\epsilon t)^{\frac{2}{2-\gamma}}} \right),$$

where $F'(x) = \frac{f(x)}{x^\gamma}$ and u is a solution of (4.2). That is, for ϵ sufficiently small, $\bar{U}_\epsilon$ is a solution of (4.2) whenever u is. If $u(t,x) = u_0(x)$ with u_0 an analytic, stationary solution, *then there is a fundamental solution $p(t,x,y)$ of (4.2) such that*

$$\int_0^\infty e^{-\lambda y^{2-\gamma}} u_0(y) p(t,x,y) dy = U_\lambda(t,x). \tag{4.3}$$

Here $U_\lambda(t,x) = \bar{U}_{\frac{1}{4}\sigma(2-\gamma)^2\lambda}(t,x)$. Further, if $g = 0$, then we may take $u_0 = 1$, and the fundamental solution arising from this choice satisfies $\int_0^\infty p(t,x,y)dy = 1$.

This theorem allows us to formulate the following result:

Theorem 4.0.2. *Let $\alpha(x)$ and $\beta(x)$ be real valued functions satisfying:*

$$\begin{cases} \sigma x\alpha'(x) - \sigma\alpha(x) + \frac{1}{2}(\alpha(x)^2 - \beta(x)^2) = 2\sigma A_1 x + B_1 \\ \sigma x\beta'(x) - \sigma\beta(x) + \alpha(x)\beta(x) = 2\sigma A_2 x + B_2, \end{cases} \tag{4.4}$$

for some $\sigma \in \mathbb{R}$ and $A_i, B_i \in \mathbb{C}$, $i = 1,2$. Then the system

$$\begin{cases} v_t &= \sigma x v_{xx} + \alpha(x)v_x - \beta(x)w_x \\ w_t &= \sigma x w_{xx} + \beta(x)v_x + \alpha(x)w_x, \end{cases}, \quad x > 0, \quad t > 0 \tag{4.5}$$

has a fundamental matrix of the form:

$$P(x,t,z) = \begin{pmatrix} p_{11}(x,t,z) & p_{12}(x,t,z) \\ p_{21}(x,t,z) & p_{22}(x,t,z) \end{pmatrix}, \tag{4.6}$$

with

$$\begin{aligned}
p_{11}(x,t,z) &= \frac{1}{\sigma}\tilde{v}\left(x,t,\frac{z}{\sigma}\right) \\
p_{21}(x,t,z) &= \frac{1}{\sigma}\tilde{w}\left(x,t,\frac{z}{\sigma}\right) \\
p_{12}(x,t,z) &= \tilde{v}\left(x,t,\frac{z}{\sigma}\right)\left(z\frac{\beta'(z)}{\beta(z)^2} - \frac{\alpha(z)}{\sigma\beta(z)} - \frac{z}{\beta(z)}\frac{d}{dz}\right) + \frac{1}{\sigma}\tilde{v}_1\left(x,t,\frac{z}{\sigma}\right) I_\beta \\
p_{22}(x,t,z) &= \tilde{w}\left(x,t,\frac{z}{\sigma}\right)\left(z\frac{\beta'(z)}{\beta(z)^2} - \frac{\alpha(z)}{\sigma\beta(z)} - \frac{z}{\beta(z)}\frac{d}{dz}\right) + \frac{1}{\sigma}\tilde{w}_1\left(x,t,\frac{z}{\sigma}\right) I_\beta,
\end{aligned}$$

where I_β denotes the integral operator defined as

$$I_\beta f(x) := \int_{x_0}^x \frac{f(s)}{\beta(s)} ds.$$

Here, the functions $\tilde{v}$, $\tilde{w}$, $\tilde{v}_1$ and $\tilde{w}_1$ are defined as follows:

$$\begin{aligned}\tilde{v}(x,t,y) &= \int_0^\infty v(x,t,\epsilon)e^{-\epsilon y}d\epsilon \qquad \tilde{v}_1(x,t,y) &&= \int_0^\infty v_1(x,t,\epsilon)e^{-\epsilon y}d\epsilon\\ \tilde{w}(x,t,y) &= \int_0^\infty w(x,t,\epsilon)e^{-\epsilon y}d\epsilon \qquad \tilde{w}_1(x,t,y) &&= \int_0^\infty w_1(x,t,\epsilon)e^{-\epsilon y}d\epsilon, \end{aligned} \tag{4.7}$$

where

$$\begin{aligned} v(x,t,\epsilon) &= \exp(r(x,t,\epsilon))\cos(s(x,t,\epsilon))\\ w(x,t,\epsilon) &= \exp(r(x,t,\epsilon))\sin(s(x,t,\epsilon))\\ v_1(x,t,\epsilon) &= e^{r(x,t,\epsilon)}\cos\big(s(x,t,\epsilon)\big)\left(-\frac{2\epsilon t A_1}{1+\epsilon t}+\frac{\epsilon^2(x+A_1\sigma t^2)}{\sigma(1+\epsilon t)^2}-\frac{\epsilon\alpha(\frac{x}{(1+\epsilon t)^2})}{\sigma(1+\epsilon t)}\right)\\ &\quad - e^{r(x,t,\epsilon)}\sin\big(s(x,t,\epsilon)\big)\left(-\frac{2\epsilon t A_2}{1+\epsilon t}+\frac{\epsilon^2 A_2 t^2}{(1+\epsilon t)^2}-\frac{\epsilon\beta(\frac{x}{(1+\epsilon t)^2})}{\sigma(1+\epsilon t)}\right)\\ w_1(x,t,\epsilon) &= e^{r(x,t,\epsilon)}\sin\big(s(x,t,\epsilon)\big)\left(-\frac{2\epsilon t A_1}{1+\epsilon t}+\frac{\epsilon^2(x+A_1\sigma t^2)}{\sigma(1+\epsilon t)^2}-\frac{\epsilon\alpha(\frac{x}{(1+\epsilon t)^2})}{\sigma(1+\epsilon t)}\right)\\ &\quad + e^{r(x,t,\epsilon)}\cos(s(x,t,\epsilon))\left(-\frac{2\epsilon t A_2}{1+\epsilon t}+\frac{\epsilon^2 A_2 t^2}{(1+\epsilon t)^2}-\frac{\epsilon\beta(\frac{x}{(1+\epsilon t)^2})}{\sigma(1+\epsilon t)}\right), \end{aligned}$$

with

$$\begin{cases} r(x,t,\epsilon) &= -\frac{\epsilon(x+A_1\sigma t^2)}{\sigma(1+\epsilon t)}+\frac{1}{2\sigma}\left(K\left(\frac{x}{(1+\epsilon t)^2}\right)-K(x)\right)\\ s(x,t,\epsilon) &= -\frac{\epsilon A_2 t^2}{1+\epsilon t}+\frac{1}{2\sigma}\left(L\left(\frac{x}{(1+\epsilon t)^2}\right)-L(x)\right)\\ K'(x) &= \frac{\alpha(x)}{x}\\ L'(x) &= \frac{\beta(x)}{x} \end{cases}$$

That is, a solution $(\bar{V}(x,t) \quad \bar{W}(x,t))^\top$ for the system (4.5), with initial condition

$$\begin{pmatrix}\bar{V}(x,0)\\ \bar{W}(x,0)\end{pmatrix} = \begin{pmatrix} m(x)\\ n(x)\end{pmatrix}$$

can be written as

$$\begin{pmatrix}\bar{V}(x,t)\\ \bar{W}(x,t)\end{pmatrix} = \int_0^\infty \underbrace{\begin{pmatrix} p_{11}(x,t,z) & p_{12}(x,t,z)\\ p_{21}(x,t,z) & p_{22}(x,t,z)\end{pmatrix}}_{P(x,t,z)} \begin{pmatrix} m(z)\\ n(z)\end{pmatrix} dz,$$

where the components $p_{ij}(x,t,z)$ of the fundamental matrix $P(x,t,z)$ are defined as above.

Proof. Choose $\gamma = 1$ and $g(x) \equiv 0$ in Craddock's theorem (4.0.1). We have that if f satisfies the following Riccati equation with $A, B \in \mathbb{C}$

$$\sigma x f'(x) - \sigma f(x) + \frac{1}{2} f(x)^2 = 2\sigma A x + B, \tag{4.8}$$

then the PDE

$$u_t = \sigma x u_{xx} + f(x) u_x, \quad x \geq 0 \tag{4.9}$$

has a symmetry of the form

$$\bar{U}_\epsilon(x,t) = \exp\left(\frac{-4\epsilon(x + A\sigma t^2)}{\sigma(1+4\epsilon t)} + \frac{F\left(\frac{x}{(1+4\epsilon t)^2}\right) - F(x)}{2\sigma} \right) u\left(\frac{x}{(1+4\epsilon t)^2}, \frac{t}{1+4\epsilon t} \right), \tag{4.10}$$

where $F'(x) = \frac{f(x)}{x}$.

Since $u_0 = 1$ is a solution to (4.9) we have that

$$u(x,t,\epsilon) = \exp\left(\frac{-4\epsilon(x + A\sigma t^2)}{\sigma(1+4\epsilon t)} \right) \exp\left(\frac{1}{2\sigma} \left(F\left(\frac{x}{(1+4\epsilon t)^2} \right) - F(x) \right) \right) \tag{4.11}$$

is also a solution. Let us simplify this expression by taking $\epsilon \to \frac{\epsilon}{4}$ to get the solution

$$u(x,t,\epsilon) = \exp\left(\frac{-\epsilon(x + A\sigma t^2)}{\sigma(1+\epsilon t)} \right) \exp\left(\frac{1}{2\sigma} \left(F\left(\frac{x}{(1+\epsilon t)^2} \right) - F(x) \right) \right). \tag{4.12}$$

Now, let $f(x) = \alpha(x) + i\beta(x)$ and write $u(x,t,\epsilon) = v(x,t,\epsilon) + iw(x,t,\epsilon)$. With this, equation (4.9) becomes

$$v_t + iw_t = \sigma x(v_{xx} + iw_{xx}) + (\alpha(x) + i\beta(x))(v_x + iw_x), \qquad x \geq 0 \tag{4.13}$$

which translates into the system:

$$\begin{cases} v_t &= \sigma x v_{xx} + \alpha(x) v_x - \beta(x) w_x \\ w_t &= \sigma x w_{xx} + \beta(x) v_x + \alpha(x) w_x \end{cases}, \qquad x \geq 0 \tag{4.14}$$

Observe also that the Riccati equation (4.8) becomes:

$$\sigma x(\alpha'(x) + i\beta'(x)) - \sigma(\alpha(x) + i\beta(x)) + \frac{1}{2}(\alpha(x) + i\beta(x))^2 = 2\sigma(A_1 + iA_2)x + B_1 + iB_2, \tag{4.15}$$

which is equivalent to

$$\begin{cases} \sigma x\alpha'(x) - \sigma\alpha(x) + \frac{1}{2}(\alpha(x)^2 - \beta(x)^2) = 2\sigma A_1 x + B_1 \\ \sigma x\beta'(x) - \sigma\beta(x) + \alpha(x)\beta(x) = 2\sigma A_2 x + B_2. \end{cases} \tag{4.16}$$

Note that the functions $v(x,t,\epsilon)$ and $w(x,t,\epsilon)$ corresponding to the real and imaginary parts of the function $u(x,t,\epsilon)$ in (4.12) can be written respectively as:

$$\begin{aligned} v(x,t,\epsilon) &= \exp\left(r(x,t,\epsilon)\right)\cos\left(s(x,t,\epsilon)\right) \\ w(x,t,\epsilon) &= \exp\left(r(x,t,\epsilon)\right)\sin\left(s(x,t,\epsilon)\right), \end{aligned} \tag{4.17}$$

where

$$\begin{cases} r(x,t,\epsilon) & = \frac{-\epsilon(x+A_1\sigma t^2)}{\sigma(1+\epsilon t)} + \frac{1}{2\sigma}\left(K\left(\frac{x}{(1+\epsilon t)^2}\right) - K(x)\right) \\ s(x,t,\epsilon) & = -\frac{\epsilon A_2 t^2}{1+\epsilon t} + \frac{1}{2\sigma}\left(L\left(\frac{x}{(1+\epsilon t)^2}\right) - L(x)\right) \\ K'(x) & = \frac{\alpha(x)}{x} \\ L'(x) & = \frac{\beta(x)}{x}. \end{cases} \tag{4.18}$$

Consider the pair $v_1(x,t,\epsilon)$, $w_1(x,t,\epsilon)$ given by

$$\begin{cases} v_1(x,t,\epsilon) = \frac{\partial}{\partial t}v(x,t,\epsilon) \\ w_1(x,t,\epsilon) = \frac{\partial}{\partial t}w(x,t,\epsilon) \end{cases} \tag{4.19}$$

and note that this pair is also a solution to our system.
The reader may check that the explicit expressions for v_1 and w_1 are given, respectively, by:

$$\begin{cases} v_1(x,t,\epsilon) & = e^{r(x,t,\epsilon)}\cos\left(s(x,t,\epsilon)\right)\left(-\frac{2\epsilon t A_1}{1+\epsilon t} + \frac{\epsilon^2 x}{\sigma(1+\epsilon t)^2} + \frac{\epsilon^2 A_1 t^2}{(1+\epsilon t)^2} - \frac{\epsilon\alpha(\frac{x}{(1+\epsilon t)^2})}{\sigma(1+\epsilon t)}\right) \\ & \quad -e^{r(x,t,\epsilon)}\sin\left(s(x,t,\epsilon)\right)\left(-\frac{2\epsilon t A_2}{1+\epsilon t} + \frac{\epsilon^2 A_2 t^2}{(1+\epsilon t)^2} - \frac{\epsilon\beta(\frac{x}{(1+\epsilon t)^2})}{\sigma(1+\epsilon t)}\right) \\ w_1(x,t,\epsilon) & = e^{r(x,t,\epsilon)}\sin\left(s(x,t,\epsilon)\right)\left(-\frac{2\epsilon t A_1}{1+\epsilon t} + \frac{\epsilon^2 x}{\sigma(1+\epsilon t)^2} + \frac{\epsilon^2 A_1 t^2}{(1+\epsilon t)^2} - \frac{\epsilon\alpha(\frac{x}{(1+\epsilon t)^2})}{\sigma(1+\epsilon t)}\right) \\ & \quad +e^{r(x,t,\epsilon)}\cos(s(x,t,\epsilon))\left(-\frac{2\epsilon t A_2}{1+\epsilon t} + \frac{\epsilon^2 A_2 t^2}{(1+\epsilon t)^2} - \frac{\epsilon\beta(\frac{x}{(1+\epsilon t)^2})}{\sigma(1+\epsilon t)}\right), \end{cases} \tag{4.20}$$

where the functions $r(x,t,\epsilon)$ and $s(x,t,\epsilon)$ are defined as in (4.18).
Next, for suitable functions ϕ and ψ, with sufficiently rapid decay, define the new

solution

$$\begin{cases} \bar{V}(x,t) := \int_0^\infty (\phi(\epsilon)v(x,t,\epsilon) + \psi(\epsilon)v_1(x,t,\epsilon))d\epsilon \\ \bar{W}(x,t) := \int_0^\infty (\phi(\epsilon)w(x,t,\epsilon) + \psi(\epsilon)w_1(x,t,\epsilon))d\epsilon, \end{cases} \tag{4.21}$$

which has initial condition

$$\begin{cases} \bar{V}(x,0) & := \int_0^\infty (\phi(\epsilon)v(x,0,\epsilon) + \psi(\epsilon)v_1(x,0,\epsilon))d\epsilon \\ & = \int_0^\infty (\exp\left(-\frac{\epsilon x}{\sigma}\right)\left(\phi(\epsilon) - \frac{\epsilon}{\sigma}\psi(\epsilon)(\alpha(x) - \epsilon x)\right))d\epsilon \\ \bar{W}(x,0) & := \int_0^\infty (\phi(\epsilon)w(x,0,\epsilon) + \psi(\epsilon)w_1(x,0,\epsilon))d\epsilon \\ & = \int_0^\infty (\exp\left(-\frac{\epsilon x}{\sigma}\right)\left(-\frac{\epsilon}{\sigma}\psi(\epsilon)\beta(x)\right))d\epsilon. \end{cases}$$

Write the initial condition for each component as

$$\bar{V}(x,0) = \int_0^\infty \exp\left(-\frac{\epsilon x}{\sigma}\right)\left(\phi(\epsilon) - \frac{\epsilon}{\sigma}\psi(\epsilon)(\alpha(x) - \epsilon x)\right) d\epsilon := m(x) \tag{4.22}$$

and

$$\bar{W}(x,0) = \int_0^\infty \exp\left(-\frac{\epsilon x}{\sigma}\right)\left(-\frac{\epsilon}{\sigma}\psi(\epsilon)\beta(x)\right) d\epsilon := n(x) \tag{4.23}$$

respectively.

The reader may check that expressions (4.22) and (4.23) yield the following pair of ordinary differential equations for the Laplace transforms of the functions ϕ and ψ:

$$\int_0^\infty \phi(\epsilon)e^{-\frac{\epsilon x}{\sigma}} d\epsilon + \alpha(x)\int_0^\infty -\frac{\epsilon}{\sigma}\psi(\epsilon)e^{-\frac{\epsilon x}{\sigma}} d\epsilon + \sigma x \int_0^\infty \left(\frac{\epsilon}{\sigma}\right)^2 \psi(\epsilon)e^{-\frac{\epsilon x}{\sigma}} d\epsilon = m(x)$$
$$\iff \Phi\left(\frac{x}{\sigma}\right) + \alpha(x)\frac{d}{dx}\Psi\left(\frac{x}{\sigma}\right) + \sigma x\frac{d^2}{dx^2}\Psi\left(\frac{x}{\sigma}\right) = m(x),$$

and

$$\beta(x)\int_0^\infty -\frac{\epsilon}{\sigma}\psi(\epsilon)e^{-\frac{\epsilon x}{\sigma}} d\epsilon = n(x)$$
$$\iff \beta(x)\frac{d}{dx}\Psi\left(\frac{x}{\sigma}\right) = n(x).$$

Here Φ and Ψ denote the Laplace transforms of ϕ and ψ respectively.

Solving the above system of ODEs, one may write the following expression for the Laplace transforms Φ and Ψ in terms of the initial conditions $m(x)$ and $n(x)$:

$$\begin{pmatrix} \Phi\left(\frac{x}{\sigma}\right) \\ \Psi\left(\frac{x}{\sigma}\right) \end{pmatrix} = \begin{pmatrix} 1 & \sigma x\frac{\beta'(x)}{\beta(x)^2} - \frac{\alpha(x)}{\beta(x)} - \frac{\sigma x}{\beta(x)}\frac{d}{dx} \\ 0 & I_\beta \end{pmatrix} \begin{pmatrix} m(x) \\ n(x) \end{pmatrix}, \tag{4.24}$$

where I_β denotes the integral operator defined by:

$$I_\beta f(x) := \int_{x_0}^{x} \frac{f(s)}{\beta(s)} ds. \tag{4.25}$$

Suppose we can write v, $v1$, w and w_1 as Laplace transforms of some suitable functions, i.e.

$$\begin{aligned} v(x,t,\epsilon) &= \mathcal{L}(\tilde{v}(x,t,y)) = \int_0^\infty \tilde{v}(x,t,y)e^{-\epsilon y}dy, \\ v_1(x,t,\epsilon) &= \mathcal{L}(\tilde{v}_1(x,t,y)) = \int_0^\infty \tilde{v}_1(x,t,y)e^{-\epsilon y}dy, \\ w(x,t,\epsilon) &= \mathcal{L}(\tilde{w}(x,t,y)) = \int_0^\infty \tilde{w}(x,t,y)e^{-\epsilon y}dy \text{ and} \\ w_1(x,t,\epsilon) &= \mathcal{L}(\tilde{w}_1(x,t,y)) = \int_0^\infty \tilde{w}_1(x,t,y)e^{-\epsilon y}dy \end{aligned} \tag{4.26}$$

respectively. Then, using the above expressions, we can write

$$\begin{aligned} \begin{pmatrix} \bar{V}(x,t) \\ \bar{W}(x,t) \end{pmatrix} &= \begin{pmatrix} \int_0^\infty \phi(\epsilon)v(x,t,\epsilon)d\epsilon + \int_0^\infty \psi(\epsilon)v_1(x,t,\epsilon)d\epsilon \\ \int_0^\infty \phi(\epsilon)w(x,t,\epsilon)d\epsilon + \int_0^\infty \psi(\epsilon)w_1(x,t,\epsilon)d\epsilon \end{pmatrix} \\ &= \begin{pmatrix} \int_0^\infty \phi(\epsilon)\left(\int_0^\infty \tilde{v}(x,t,y)e^{-\epsilon y}dy\right)d\epsilon + \int_0^\infty \psi(\epsilon)\left(\int_0^\infty \tilde{v}_1(x,t,y)e^{-\epsilon y}dy\right)d\epsilon \\ \int_0^\infty \phi(\epsilon)\left(\int_0^\infty \tilde{w}(x,t,y)e^{-\epsilon y}dy\right)d\epsilon + \int_0^\infty \psi(\epsilon)\left(\int_0^\infty \tilde{w}_1(x,t,y)e^{-\epsilon y}dy\right)d\epsilon \end{pmatrix} \\ &= \int_0^\infty \begin{pmatrix} \left(\int_0^\infty \phi(\epsilon)e^{-\epsilon y}d\epsilon\right)\tilde{v}(x,t,y) + \left(\int_0^\infty \psi(\epsilon)e^{-\epsilon y}d\epsilon\right)\tilde{v}_1(x,t,y) \\ \left(\int_0^\infty \phi(\epsilon)e^{-\epsilon y}d\epsilon\right)\tilde{w}(x,t,y) + \left(\int_0^\infty \psi(\epsilon)e^{-\epsilon y}d\epsilon\right)\tilde{w}_1(x,t,y) \end{pmatrix} dy \\ &= \int_0^\infty \begin{pmatrix} \tilde{v}(x,t,y) & \tilde{v}_1(x,t,y) \\ \tilde{w}(x,t,y) & \tilde{w}_1(x,t,y) \end{pmatrix} \begin{pmatrix} \Phi(y) \\ \Psi(y) \end{pmatrix} dy \end{aligned}$$

The change of variables $z = \sigma y$ and the expression (4.24) for the Laplace transforms Φ and Ψ give:

$$\begin{aligned} \begin{pmatrix} \bar{V}(x,t) \\ \bar{W}(x,t) \end{pmatrix} &= \int_0^\infty \begin{pmatrix} \tilde{v}(x,t,\frac{z}{\sigma}) & \tilde{v}_1(x,t,\frac{z}{\sigma}) \\ \tilde{w}(x,t,\frac{z}{\sigma}) & \tilde{w}_1(x,t,\frac{z}{\sigma}) \end{pmatrix} \begin{pmatrix} \Phi(\frac{z}{\sigma}) \\ \Psi(\frac{z}{\sigma}) \end{pmatrix} \frac{dz}{\sigma} \\ &= \int_0^\infty \begin{pmatrix} \tilde{v}(x,t,\frac{z}{\sigma}) & \tilde{v}_1(x,t,\frac{z}{\sigma}) \\ \tilde{w}(x,t,\frac{z}{\sigma}) & \tilde{w}_1(x,t,\frac{z}{\sigma}) \end{pmatrix} \begin{pmatrix} 1 & \sigma z\frac{\beta'(z)}{\beta(z)^2} - \frac{\alpha(z)}{\beta(z)} - \frac{\sigma z}{\beta(z)}\frac{d}{dz} \\ 0 & I_\beta \end{pmatrix} \begin{pmatrix} m(z) \\ n(z) \end{pmatrix} \frac{dz}{\sigma} \end{aligned}$$

That is, the solution can be expressed as

$$\begin{pmatrix} \bar{V}(x,t) \\ \bar{W}(x,t) \end{pmatrix} = \int_0^\infty \underbrace{\begin{pmatrix} p_{11}(x,t,z) & p_{12}(x,t,z) \\ p_{21}(x,t,z) & p_{22}(x,t,z) \end{pmatrix}}_{P(x,t,z)} \begin{pmatrix} m(z) \\ n(z) \end{pmatrix} dz,$$

with

$$\begin{pmatrix} \bar{V}(x,0) \\ \bar{W}(x,0) \end{pmatrix} = \begin{pmatrix} m(x) \\ n(n) \end{pmatrix},$$

where the components $p_{ij}(x,t,z)$ of the fundamental matrix $P(x,t,z)$ are given by

$$\begin{aligned}
p_{11}(x,t,z) &= \frac{1}{\sigma}\tilde{v}\left(x,t,\frac{z}{\sigma}\right) \\
p_{21}(x,t,z) &= \frac{1}{\sigma}\tilde{w}\left(x,t,\frac{z}{\sigma}\right) \\
p_{12}(x,t,z) &= \tilde{v}\left(x,t,\frac{z}{\sigma}\right)\left(z\frac{\beta'(z)}{\beta(z)^2} - \frac{\alpha(z)}{\sigma\beta(z)} - \frac{z}{\beta(z)}\frac{d}{dz}\right) + \frac{1}{\sigma}\tilde{v}_1\left(x,t,\frac{z}{\sigma}\right) I_\beta \\
p_{22}(x,t,z) &= \tilde{w}\left(x,t,\frac{z}{\sigma}\right)\left(z\frac{\beta'(z)}{\beta(z)^2} - \frac{\alpha(z)}{\sigma\beta(z)} - \frac{z}{\beta(z)}\frac{d}{dz}\right) + \frac{1}{\sigma}\tilde{w}_1\left(x,t,\frac{z}{\sigma}\right) I_\beta,
\end{aligned}$$

and where I_β denotes the integral operator defined before as

$$I_\beta f(x) := \int_{x_0}^x \frac{f(s)}{\beta(s)} ds.$$

Hence $P(x,t,z)$ defined as above is a fundamental matrix for the system (4.14). □

Note. Observe that the above case could not have been handled through reduction to the heat equation. Lie proved that a linear parabolic PDE in one dimension can be mapped to the heat equation if and only if its Lie symmetry algebra is six dimensional. Craddock and his coauthors have shown that this happens for exactly one choice of the constant B in equation (4.1). Thus although some special cases might be handled by reducing to the heat equation, the general case that we study cannot be handled in this manner.

Let us now present two examples of systems of PDEs for which we can explicitly calculate a fundamental matrix $P(x,t,z)$ as defined in theorem 4.0.2.

Example 4.0.1. Choice of constant functions $\alpha(x)$ and $\beta(x)$
The following result follows naturally from Theorem 4.0.2:

Corollary. *A solution* $(\bar{V}(x,t) \quad \bar{W}(x,t))^\top$ *for the system:*

$$\begin{cases} v_t &= \sigma x v_{xx} + \lambda_1 v_x - \lambda_2 w_x \\ w_t &= \sigma x w_{xx} + \lambda_2 v_x + \lambda_1 w_x, \end{cases} \quad x,t>0 \tag{4.27}$$

with initial condition

$$\begin{pmatrix} \bar{V}(x,0) \\ \bar{W}(x,0) \end{pmatrix} = \begin{pmatrix} m(x) \\ n(x) \end{pmatrix}$$

can be written as

$$\begin{pmatrix} \bar{V}(x,t) \\ \bar{W}(x,t) \end{pmatrix} = \int_0^\infty \underbrace{\begin{pmatrix} p_{11}(x,t,z) & p_{12}(x,t,z) \\ p_{21}(x,t,z) & p_{22}(x,t,z) \end{pmatrix}}_{P(x,t,z)} \begin{pmatrix} m(z) \\ n(z) \end{pmatrix} dz,$$

where the components $p_{ij}(x,t,z)$ *of the fundamental matrix* $P(x,t,z)$ *are given by*

$$p_{11}(x,t,z) = \frac{1}{\sigma}\tilde{v}\left(x,t,\frac{z}{\sigma}\right)$$
$$p_{21}(x,t,z) = \frac{1}{\sigma}\tilde{w}\left(x,t,\frac{z}{\sigma}\right)$$
$$p_{12}(x,t,z) = -\tilde{v}\left(x,t,\frac{z}{\sigma}\right)\left(\frac{\lambda_1}{\sigma\lambda_2} + \frac{z}{\lambda_2}\frac{d}{dz}\right) + \frac{1}{\sigma}\tilde{v}_1\left(x,t,\frac{z}{\sigma}\right) I_{\lambda_2}$$
$$p_{22}(x,t,z) = -\tilde{w}\left(x,t,\frac{z}{\sigma}\right)\left(\frac{\lambda_1}{\sigma\lambda_2} + \frac{z}{\lambda_2}\frac{d}{dz}\right) + \frac{1}{\sigma}\tilde{w}_1\left(x,t,\frac{z}{\sigma}\right) I_{\lambda_2},$$

and

$$\tilde{v}\left(x,t,\frac{z}{\sigma}\right) = \frac{e^{-\frac{x+z}{\sigma t}}}{2t}\left(\left(\frac{x}{z}\right)^{\frac{\sigma-\bar{\lambda}}{2\sigma}} I_{\frac{\bar{\lambda}}{\sigma}-1}\left(\frac{2\sqrt{xz}}{\sigma t}\right) + \left(\frac{x}{z}\right)^{\frac{\sigma-\lambda}{2\sigma}} I_{\frac{\lambda}{\sigma}-1}\left(\frac{2\sqrt{xz}}{\sigma t}\right)\right)$$

$$\tilde{w}\left(x,t,\frac{z}{\sigma}\right) = \frac{ie^{-\frac{x+z}{\sigma t}}}{2t}\left(\left(\frac{x}{z}\right)^{\frac{\sigma-\bar{\lambda}}{2\sigma}} I_{\frac{\bar{\lambda}}{\sigma}-1}\left(\frac{2\sqrt{xz}}{\sigma t}\right) - \left(\frac{x}{z}\right)^{\frac{\sigma-\lambda}{2\sigma}} I_{\frac{\lambda}{\sigma}-1}\left(\frac{2\sqrt{xz}}{\sigma t}\right)\right)$$

$$\tilde{v}_1\left(x,t,\frac{z}{\sigma}\right) = \frac{e^{-\frac{x+z}{\sigma t}}}{2\sigma t^3}\left(\left(\frac{x}{z}\right)^{-\frac{\bar{\lambda}}{2\sigma}}\left(\frac{\bar{\lambda}t}{z}(x+z-\bar{\lambda}t) - 2x\right) I_{\frac{\bar{\lambda}}{\sigma}}\left(\frac{2\sqrt{xz}}{\sigma t}\right)\right.$$
$$+ \left(\frac{x}{z}\right)^{\frac{\sigma-\lambda}{2\sigma}}(x+z-\lambda t) I_{\frac{\sigma+\lambda}{\sigma}}\left(\frac{2\sqrt{xz}}{\sigma t}\right) + \left(\frac{x}{z}\right)^{\frac{\sigma-\bar{\lambda}}{2\sigma}}(x+z-\bar{\lambda}t) I_{\frac{\sigma+\bar{\lambda}}{\sigma}}\left(\frac{2\sqrt{xz}}{\sigma t}\right)$$
$$\left. + \left(\frac{x}{z}\right)^{-\frac{\lambda}{2\sigma}}\left(\frac{\lambda t}{z}(x+z-\lambda t) - 2x\right) I_{\frac{\lambda}{\sigma}}\left(\frac{2\sqrt{xz}}{\sigma t}\right)\right)$$

$$\tilde{w}_1\left(x,t,\frac{z}{\sigma}\right)=\frac{ie^{-\frac{x+z}{\sigma t}}}{2\sigma t^3}\left(z\left(\frac{x}{z}\right)^{\frac{1}{2}-\frac{\bar{\lambda}}{2\sigma}}I_{\frac{\sigma+\bar{\lambda}}{\sigma}}\left(\frac{2\sqrt{xz}}{\sigma t}\right)-z\left(\frac{x}{z}\right)^{\frac{1}{2}-\frac{\lambda}{2\sigma}}I_{\frac{\sigma+\lambda}{\sigma}}\left(\frac{2\sqrt{xz}}{\sigma t}\right)\right.$$
$$+\left(\bar{\lambda}t-2x\right)\left(\frac{x}{z}\right)^{-\frac{\bar{\lambda}}{2\sigma}}I_{\frac{\bar{\lambda}}{\sigma}}\left(\frac{2\sqrt{xz}}{\sigma t}\right)+\left(2x-\lambda t\right)\left(\frac{x}{z}\right)^{-\frac{\lambda}{2\sigma}}I_{\frac{\lambda}{\sigma}}\left(\frac{2\sqrt{xz}}{\sigma t}\right)$$
$$\left.+\left(\lambda t-x\right)\left(\frac{x}{z}\right)^{\frac{1}{2}-\frac{\lambda}{2\sigma}}I_{\frac{\lambda-\sigma}{\sigma}}\left(\frac{2\sqrt{xz}}{\sigma t}\right)+\left(x-\bar{\lambda}t\right)\left(\frac{x}{z}\right)^{\frac{1}{2}-\frac{\bar{\lambda}}{2\sigma}}I_{\frac{\bar{\lambda}-\sigma}{\sigma}}\left(\frac{2\sqrt{xz}}{\sigma t}\right)\right),$$

where $\lambda=\lambda_1+i\lambda_2$ *and* $\bar{\lambda}=\lambda_1-i\lambda_2$.
Hence, the matrix $P(x,t,z)$ *defined as above is a fundamental matrix for the system* (4.27)

Proof. Let $\alpha(x)=\lambda_1$ and $\beta(x)=\lambda_2$. It is straightforward to check that this choice of functions α and β satisfies all the necessary conditions for theorem 4.0.2 to apply. Observe that the system (4.4) is in this case:

$$\begin{cases}-\sigma\lambda_1+\frac{1}{2}(\lambda_1^2-\lambda_2^2)=2\sigma A_1x+B_1\\ -\sigma\lambda_2+\lambda_1\lambda_2=2\sigma A_2x+B_2,\end{cases}\tag{4.28}$$

Therefore α and β are solutions of (4.4) for $A_1=A_2=0$, $B_1=\frac{1}{2}(\lambda_1^2-\lambda_2^2)-\sigma\lambda_1$ and $B_2=\lambda_1\lambda_2-\sigma\lambda_2$. Hence, theorem 4.0.2 holds for this example. That is, a fundamental solution for the system (4.27) will be of the form given in theorem 4.0.2.

One need only calculate the explicit forms of the functions $K(x)$ and $L(x)$ appearing in (4.17) to later be able to obtain explicit forms for the functions $\tilde{v}$, $\tilde{w}$, $\tilde{v}_1$ and $\tilde{w}_1$. Observe that he have

$$K'(x)=\frac{\lambda_1}{x}\implies K(x)=\lambda_1\log x,$$

$$L'(x)=\frac{\lambda_2}{x}\implies L(x)=\lambda_2\log x$$

Therefore the function $\tilde{v}=\mathcal{L}^{-1}(v)$ from (4.49) can be calculated to be:

$$\tilde{v}(x,t,y)=\mathcal{L}^{-1}(v(x,t,\epsilon))$$
$$=\mathcal{L}^{-1}\left(e^{\frac{-\epsilon(x+A_1\sigma t^2)}{\sigma(1+\epsilon t)}+\frac{K\left(\frac{x}{(1+\epsilon t)^2}\right)-K(x)}{2\sigma}}\cos\left(-\frac{\epsilon A_2t^2}{1+\epsilon t}+\frac{L\left(\frac{x}{(1+\epsilon t)^2}\right)-L(x)}{2\sigma}\right)\right)$$
$$=\mathcal{L}^{-1}\left(e^{\frac{-\epsilon x}{\sigma(1+\epsilon t)}+\frac{\lambda_1\log\left(\frac{x}{(1+\epsilon t)^2}\right)-\lambda_1\log x}{2\sigma}}\cos\left(\frac{\lambda_2\log\left(\frac{x}{(1+\epsilon t)^2}\right)-\lambda_2\log x}{2\sigma}\right)\right)$$

$$= \mathcal{L}^{-1}\left(\exp\left(\frac{-\epsilon x}{\sigma(1+\epsilon t)} + \frac{\lambda_1}{2\sigma}\log\left(\frac{1}{(1+\epsilon t)^2}\right)\right)\cos\left(\frac{\lambda_2}{2\sigma}\log\left(\frac{1}{(1+\epsilon t)^2}\right)\right)\right)$$

$$= \mathcal{L}^{-1}\left(\exp\left(\frac{-\epsilon x}{\sigma(1+\epsilon t)}\right)(1+\epsilon t)^{-\frac{\lambda_1}{\sigma}}\frac{1}{2}\left(e^{i\frac{\lambda_2}{2\sigma}\log\left(\frac{1}{(1+\epsilon t)^2}\right)} + e^{-i\frac{\lambda_2}{2\sigma}\log\left(\frac{1}{(1+\epsilon t)^2}\right)}\right)\right)$$

$$= \mathcal{L}^{-1}\left(\exp\left(\frac{-\epsilon x}{\sigma(1+\epsilon t)}\right)(1+\epsilon t)^{-\frac{\lambda_1}{\sigma}}\frac{1}{2}\left((1+\epsilon t)^{-i\frac{\lambda_2}{\sigma}} + (1+\epsilon t)^{i\frac{\lambda_2}{\sigma}}\right)\right)$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left(\exp\left(\frac{-\epsilon x}{\sigma(1+\epsilon t)}\right)\left((1+\epsilon t)^{-\frac{\lambda_1+i\lambda_2}{\sigma}} + (1+\epsilon t)^{\frac{i\lambda_2-\lambda_1}{\sigma}}\right)\right)$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left(e^{\frac{-\left(\epsilon+\frac{1}{t}-\frac{1}{t}\right)x}{\sigma t\left(\epsilon+\frac{1}{t}\right)}}\left(t^{-\frac{\lambda_1+i\lambda_2}{\sigma}}\left(\epsilon+\frac{1}{t}\right)^{-\frac{\lambda_1+i\lambda_2}{\sigma}} + t^{\frac{i\lambda_2-\lambda_1}{\sigma}}\left(\epsilon+\frac{1}{t}\right)^{\frac{i\lambda_2-\lambda_1}{\sigma}}\right)\right)$$

$$= \frac{e^{-\frac{x}{\sigma t}}}{2}\mathcal{L}^{-1}\left(e^{\frac{x}{\sigma t^2\left(\epsilon+\frac{1}{t}\right)}}\left(t^{-\frac{\lambda_1+i\lambda_2}{\sigma}}\left(\epsilon+\frac{1}{t}\right)^{-\frac{\lambda_1+i\lambda_2}{\sigma}} + t^{\frac{i\lambda_2-\lambda_1}{\sigma}}\left(\epsilon+\frac{1}{t}\right)^{\frac{i\lambda_2-\lambda_1}{\sigma}}\right)\right)$$

$$= \frac{e^{-\frac{x}{\sigma t}-\frac{y}{t}}}{2}\mathcal{L}^{-1}\left(\exp\left(\frac{x}{\sigma t^2\epsilon}\right)\left(t^{-\frac{\lambda_1+i\lambda_2}{\sigma}}\epsilon^{-\frac{\lambda_1+i\lambda_2}{\sigma}} + t^{\frac{i\lambda_2-\lambda_1}{\sigma}}\epsilon^{\frac{i\lambda_2-\lambda_1}{\sigma}}\right)\right)$$

$$= \frac{e^{-\frac{x}{\sigma t}-\frac{y}{t}}}{2}\left(t^{-\frac{\lambda_1+i\lambda_2}{\sigma}}\mathcal{L}^{-1}\left(e^{\frac{x}{\sigma t^2\epsilon}}\epsilon^{-\frac{\lambda_1+i\lambda_2}{\sigma}}\right) + t^{\frac{i\lambda_2-\lambda_1}{\sigma}}\mathcal{L}^{-1}\left(e^{\frac{x}{\sigma t^2\epsilon}}\epsilon^{\frac{i\lambda_2-\lambda_1}{\sigma}}\right)\right)$$

$$= \frac{e^{-\frac{x}{\sigma t}-\frac{y}{t}}}{2t}\left(\frac{x}{\sigma y}\right)^{\frac{-\lambda_1+i\lambda_2+\sigma}{2\sigma}} I_{\frac{\lambda_1-i\lambda_2}{\sigma}-1}\left(2\sqrt{\frac{xy}{t^2\sigma}}\right)$$

$$+ \frac{e^{-\frac{x}{\sigma t}-\frac{y}{t}}}{2t}\left(\frac{x}{\sigma y}\right)^{\frac{-\lambda_1-i\lambda_2+\sigma}{2\sigma}} I_{\frac{\lambda_1+i\lambda_2}{\sigma}-1}\left(2\sqrt{\frac{xy}{t^2\sigma}}\right)$$

Using a similar argument one obtains the expression for $\tilde{w} = \mathcal{L}^{-1}(w)$:

$$\tilde{w}(x,t,y) = \mathcal{L}^{-1}(w(x,t,\epsilon))$$

$$= \mathcal{L}^{-1}\left(e^{\frac{-\epsilon(x+A_1\sigma t^2)}{\sigma(1+\epsilon t)} + \frac{K\left(\frac{x}{(1+\epsilon t)^2}\right)-K(x)}{2\sigma}}\sin\left(-\frac{\epsilon A_2 t^2}{1+\epsilon t} + \frac{L\left(\frac{x}{(1+\epsilon t)^2}\right)-L(x)}{2\sigma}\right)\right)$$

$$= \mathcal{L}^{-1}\left(-\frac{i}{2}\exp\left(-\frac{x\epsilon}{\sigma(1+\epsilon t)}\right)\left((1+\epsilon t)^{-\frac{\lambda_1+i\lambda_2}{\sigma}} - (1+\epsilon t)^{\frac{-\lambda_1+i\lambda_2}{\sigma}}\right)\right)$$

$$= -\frac{ie^{-\frac{x}{\sigma t}}}{2}\mathcal{L}^{-1}\left(e^{\frac{x}{\sigma t^2\left(\frac{1}{t}+\epsilon\right)}}\left(t^{-\frac{\lambda_1+i\lambda_2}{\sigma}}\left(\frac{1}{t}+\epsilon\right)^{-\frac{\lambda_1+i\lambda_2}{\sigma}} - t^{\frac{-\lambda_1+i\lambda_2}{\sigma}}\left(\frac{1}{t}+\epsilon\right)^{\frac{-\lambda_1+i\lambda_2}{\sigma}}\right)\right)$$

$$= -\frac{i}{2}e^{-\frac{x}{\sigma t}-\frac{y}{t}}\mathcal{L}^{-1}\left(\exp\left(\frac{x}{\sigma t^2\epsilon}\right)\left(t^{-\frac{\lambda_1+i\lambda_2}{\sigma}}\epsilon^{-\frac{\lambda_1+i\lambda_2}{\sigma}} - t^{\frac{-\lambda_1+i\lambda_2}{\sigma}}\epsilon^{\frac{-\lambda_1+i\lambda_2}{\sigma}}\right)\right)$$
$$= -\frac{ie^{-\frac{x}{\sigma t}-\frac{y}{t}}}{2t}\left(\frac{x}{\sigma y}\right)^{\frac{-\lambda_1-i\lambda_2+\sigma}{2\sigma}} I_{\frac{\lambda_1+i\lambda_2}{\sigma}-1}\left(2\sqrt{\frac{xy}{t^2\sigma}}\right)$$
$$+\frac{ie^{-\frac{x}{\sigma t}-\frac{y}{t}}}{2t}\left(\frac{x}{\sigma y}\right)^{\frac{-\lambda_1+i\lambda_2+\sigma}{2\sigma}} I_{\frac{\lambda_1-i\lambda_2}{\sigma}-1}\left(2\sqrt{\frac{xy}{t^2\sigma}}\right)$$

Finally, the inverse Laplace transforms $\tilde{v_1} = \mathcal{L}^{-1}(v_1)$ and $\tilde{w_1} = \mathcal{L}^{-1}(w_1)$ can be calculated to be the following (note that here we omit most of the algebra for these two calculations but the arguments used are very similar to those used for $\tilde{v}$ and $\tilde{w}$):

$$\tilde{v_1}(x,t,y) = \mathcal{L}^{-1}(v_1(x,t,\epsilon)) = \mathcal{L}^{-1}\left(\frac{\partial}{\partial t}v(x,t,\epsilon)\right)$$
$$= \mathcal{L}^{-1}\left(e^{-\frac{x\epsilon}{\sigma(1+\epsilon t)}}\frac{\epsilon(1+\epsilon t)^{-\frac{\lambda_1+i\lambda_2+2\sigma}{\sigma}}}{2\sigma}\left(x\epsilon-(\lambda_1-i\lambda_2)(1+\epsilon t)\right)(1+\epsilon t)^{\frac{2i\lambda_2}{\sigma}}\right.$$
$$\left.+\, e^{-\frac{x\epsilon}{\sigma(1+\epsilon t)}}\frac{\epsilon(1+\epsilon t)^{-\frac{\lambda_1+i\lambda_2+2\sigma}{\sigma}}}{2\sigma}(x\epsilon-(\lambda_1+i\lambda_2)(1+\epsilon t))\right)$$
$$= \frac{e^{-\frac{x+\sigma y}{\sigma t}}}{2\sigma t^3}\left(\left(\frac{x}{\sigma y}\right)^{\frac{1}{2}-\frac{\lambda_1+i\lambda_2}{2\sigma}}\left(x+\sigma y-(\lambda_1+i\lambda_2)t\right)I_{\frac{\sigma+\lambda_1+i\lambda_2}{\sigma}}\left(2\sqrt{\frac{xy}{t^2\sigma}}\right)\right.$$
$$+\left(\frac{x}{\sigma y}\right)^{-\frac{\lambda_1+i\lambda_2}{2\sigma}}\left(\frac{(\lambda_1+i\lambda_2)t}{\sigma y}\left(x+\sigma y-(\lambda_1+i\lambda_2)t\right)-2x\right)I_{\frac{\lambda_1+i\lambda_2}{\sigma}}\left(2\sqrt{\frac{xy}{t^2\sigma}}\right)$$
$$+\left(\frac{x}{\sigma y}\right)^{-\frac{\lambda_1-i\lambda_2}{2\sigma}}\left(\frac{(\lambda_1-i\lambda_2)t}{\sigma y}\left(x+\sigma y-(\lambda_1-i\lambda_2)t\right)-2x\right)I_{\frac{\lambda_1-i\lambda_2}{\sigma}}\left(2\sqrt{\frac{xy}{t^2\sigma}}\right)$$
$$\left.+\left(\frac{x}{\sigma y}\right)^{\frac{1}{2}-\frac{\lambda_1-i\lambda_2}{2\sigma}}\left(x+\sigma y-(\lambda_1-i\lambda_2)t\right)I_{\frac{\sigma+\lambda_1-i\lambda_2}{\sigma}}\left(2\sqrt{\frac{xy}{t^2\sigma}}\right)\right)$$

$$\tilde{w_1}(x,t,y) = \mathcal{L}^{-1}(w_1(x,t,\epsilon)) = \mathcal{L}^{-1}\left(\frac{\partial}{\partial t}w(x,t,\epsilon)\right)$$
$$= \mathcal{L}^{-1}\left(-\frac{i\epsilon}{2\sigma}e^{-\frac{x\epsilon}{\sigma(1+\epsilon t)}}(1+\epsilon t)^{-\frac{\lambda_1+i\lambda_2+2\sigma}{\sigma}}\left((\lambda_1-i\lambda_2)(1+\epsilon t)-x\epsilon\right)(1+\epsilon t)^{\frac{2i\lambda_2}{\sigma}}\right.$$
$$\left.+\frac{i\epsilon}{2\sigma}e^{-\frac{x\epsilon}{\sigma(1+\epsilon t)}}(1+\epsilon t)^{-\frac{\lambda_1+i\lambda_2+2\sigma}{\sigma}}((\lambda_1+i\lambda_2)(1+\epsilon t)-x\epsilon)\right)$$
$$= -\frac{ie^{-\frac{x}{\sigma t}-\frac{y}{t}}}{2\sigma t^3}\left((-2x+(\lambda_1+i\lambda_2)t)\left(\frac{x}{\sigma y}\right)^{-\frac{\lambda_1+i\lambda_2}{2\sigma}}I_{\frac{\lambda_1+i\lambda_2}{\sigma}}\left(2\sqrt{\frac{xy}{t^2\sigma}}\right)\right.$$

$$+\left(2x-(\lambda_1-i\lambda_2)\,t\right)\left(\frac{x}{\sigma y}\right)^{-\frac{\lambda_1-i\lambda_2}{2\sigma}} I_{\frac{\lambda_1-i\lambda_2}{\sigma}}\left(2\sqrt{\frac{xy}{t^2\sigma}}\right)$$

$$+\left(x-(\lambda_1+i\lambda_2)\,t\right)\left(\frac{x}{\sigma y}\right)^{\frac{1}{2}-\frac{\lambda_1+i\lambda_2}{2\sigma}} I_{\frac{\lambda_1+i\lambda_2}{\sigma}-1}\left(2\sqrt{\frac{xy}{t^2\sigma}}\right)$$

$$+\left(-x+(\lambda_1-i\lambda_2)\,t\right)\left(\frac{x}{\sigma y}\right)^{\frac{1}{2}-\frac{\lambda_1-i\lambda_2}{2\sigma}} I_{\frac{\lambda_1-i\lambda_2}{\sigma}-1}\left(2\sqrt{\frac{xy}{t^2\sigma}}\right)$$

$$+\sigma y\left(\frac{x}{\sigma y}\right)^{\frac{1}{2}-\frac{\lambda_1+i\lambda_2}{2\sigma}}\left(I_{\frac{\sigma+\lambda_1+i\lambda_2}{\sigma}}\left(2\sqrt{\frac{xy}{t^2\sigma}}\right)-\left(\frac{x}{\sigma y}\right)^{\frac{i\lambda_2}{\sigma}} I_{\frac{\sigma+\lambda_1-i\lambda_2}{\sigma}}\left(2\sqrt{\frac{xy}{t^2\sigma}}\right)\right)\Bigg)$$

Substitution of these expressions into the general form for the fundamental solution given in theorem 4.0.2 yields the desired result. □

Example 4.0.2. Choice of $\alpha(x)=\frac{2\sigma x(c_1+x)}{(c_1+x)^2+c_2^2}$ and $\beta(x)=-\frac{2\sigma x c_2}{(c_1+x)^2+c_2^2}$

The following result follows naturally from Theorem 4.0.2:

Corollary. *Consider the system*

$$\begin{cases} v_t &= \sigma x v_{xx}+\frac{2\sigma x(c_1+x)}{(c_1+x)^2+c_2^2}v_x+\frac{2\sigma x c_2}{(c_1+x)^2+c_2^2}w_x \\ w_t &= \sigma x w_{xx}-\frac{2\sigma x c_2}{(c_1+x)^2+c_2^2}v_x+\frac{2\sigma x(c_1+x)}{(c_1+x)^2+c_2^2}w_x, \end{cases}, \qquad x,t>0 \tag{4.29}$$

where both $v(x,t)$ and $w(x,t)$ are real-valued functions and where $\sigma>0$, $c_1,c_2\in\mathbb{R}$. Then a fundamental matrix for this system is

$$P(x,t,z)=\begin{pmatrix} p_{11}(x,t,z) & p_{12}(x,t,z) \\ p_{21}(x,t,z) & p_{22}(x,t,z)\end{pmatrix}, \tag{4.30}$$

where

$$p_{11}(x,t,z)=e^{-\frac{x+z}{\sigma t}}\left(\frac{(c_1+x)(c_1+z)+c_2^2}{\sigma t\left((c_1+x)^2+c_2^2\right)}\sqrt{\frac{x}{z}}I_1\left(2\frac{\sqrt{xz}}{t\sigma}\right)+\frac{c_1(c_1+x)+c_2^2}{(c_1+x)^2+c_2^2}\delta(z)\right)$$

$$p_{21}(x,t,z)=\frac{c_2 e^{-\frac{x+z}{\sigma t}}}{(c_1+x)^2+c_2^2}\left(\frac{(x-z)}{\sigma t}\sqrt{\frac{x}{z}}I_1\left(2\frac{\sqrt{xz}}{t\sigma}\right)+x\delta(z)\right)$$

$$p_{12}(x,t,z)=\tilde{v}\left(x,t,\frac{z}{\sigma}\right)\left(-\frac{(c_1+z)\,(c_1-3z)+c_2^2}{2c_2\sigma z}+\frac{(c_1+z)^2+c_2^2}{2c_2\sigma}\frac{d}{dz}\right)$$

$$+\frac{1}{\sigma}\tilde{v}_1\left(x,t,\frac{z}{\sigma}\right)I_\beta$$

$$p_{22}(x,t,z) = \tilde{w}\left(x,t,\frac{z}{\sigma}\right)\left(-\frac{(c_1+z)(c_1-3z)+c_2^2}{2c_2\sigma z} + \frac{(c_1+z)^2+c_2^2}{2c_2\sigma}\frac{d}{dz}\right) + \frac{1}{\sigma}\tilde{w}_1\left(x,t,\frac{z}{\sigma}\right)I_\beta,$$

and where I_β denotes the integral operator defined as

$$I_\beta f(x) := -\int_{x_0}^{x}\frac{(c_1+s)^2+c_2^2}{2\sigma s c_2}f(s)ds.$$

In the above expressions,

$$\begin{aligned}
\tilde{v}(x,t,y) &= \mathcal{L}^{-1}(v(x,t,\epsilon))\\
&= e^{-\frac{x}{\sigma t}-\frac{y}{t}}\left(\frac{(c_1+x)(c_1+\sigma y)+c_2^2}{y\left((c_1+x)^2+c_2^2\right)}\sqrt{\frac{xy}{\sigma t^2}}I_1\left(2\sqrt{\frac{xy}{t^2\sigma}}\right) + \frac{c_1(c_1+x)+c_2^2}{(c_1+x)^2+c_2^2}\delta(y)\right)\\
\tilde{w}(x,t,y) &= \mathcal{L}^{-1}(w(x,t,\epsilon))\\
&= \frac{e^{-\frac{x}{\sigma t}-\frac{y}{t}}}{(c_1+x)^2+c_2^2}\left(\frac{(c_2(x-\sigma y))}{y}\sqrt{\frac{xy}{\sigma t^2}}I_1\left(2\sqrt{\frac{xy}{t^2\sigma}}\right)+c_2 x\delta(y)\right)\\
\tilde{v}_1(x,t,y) &= \mathcal{L}^{-1}(v_1(x,t,\epsilon))\\
&= e^{-\frac{x}{\sigma t}-\frac{y}{t}}\left(\frac{xt\left(c_1(c_1+x)+c_2^2\right)}{\sigma t^3((c_1+x)^2+c_2^2)}\delta(y) - \frac{2x\left(c_1(c_1+x)+c_2^2\right)}{\sigma t^3((c_1+x)^2+c_2^2)}I_0\left(2\sqrt{\frac{xy}{t^2\sigma}}\right)\right.\\
&+ \frac{\left(c_1(c_1+x)+c_2^2\right)(x+\sigma y)+\sigma y(c_1+x)(x-2\sigma t)}{\sigma t^3((c_1+x)^2+c_2^2)}\sqrt{\frac{x}{\sigma y}}I_1\left(2\sqrt{\frac{xy}{t^2\sigma}}\right)\\
&+ \left.\frac{\sigma y(c_1+x)\sqrt{x\sigma y}}{\sigma t^3((c_1+x)^2+c_2^2)}I_3\left(2\sqrt{\frac{xy}{t^2\sigma}}\right) - \frac{2\sigma y(c_1+x)(x-\sigma t)}{\sigma t^3((c_1+x)^2+c_2^2)}I_2\left(2\sqrt{\frac{xy}{t^2\sigma}}\right)\right)\\
\tilde{w}_1(x,t,y) &= \mathcal{L}^{-1}(w_1(x,t,\epsilon))\\
&= e^{-\frac{x}{\sigma t}-\frac{y}{t}}\left(\frac{\sqrt{\sigma y x}c_2}{((c_1+x)^2+c_2^2)\sigma t^3}\left(\left(2\sigma t+\frac{x^2}{\sigma y}\right)I_1\left(2\sqrt{\frac{xy}{t^2\sigma}}\right)-\sigma y I_3\left(2\sqrt{\frac{xy}{t^2\sigma}}\right)\right)\right.\\
&+\frac{x^2c_2}{((c_1+x)^2+c_2^2)\sigma t^3}\left(t\delta(y)-2I_0\left(2\sqrt{\frac{xy}{t^2\sigma}}\right)\right)\\
&+\left.\frac{c_2(2\sigma y(x-\sigma t))}{((c_1+x)^2+c_2^2)\sigma t^3}I_2\left(2\sqrt{\frac{xy}{t^2\sigma}}\right)\right)
\end{aligned}$$

That is, a solution $(\bar{V}(x,t) \quad \bar{W}(x,t))^\top$ *for the system* (4.29) *with initial condition*

$$\begin{pmatrix} \bar{V}(x,0) \\ \bar{W}(x,0) \end{pmatrix} = \begin{pmatrix} m(x) \\ n(x) \end{pmatrix}$$

can be written as

$$\begin{pmatrix} \bar{V}(x,t) \\ \bar{W}(x,t) \end{pmatrix} = \int_0^\infty \underbrace{\begin{pmatrix} p_{11}(x,t,z) & p_{12}(x,t,z) \\ p_{21}(x,t,z) & p_{22}(x,t,z) \end{pmatrix}}_{P(x,t,z)} \begin{pmatrix} m(z) \\ n(z) \end{pmatrix} dz,$$

with the matrix $P(x,t,z)$ *defined as above.*

Proof. Let $\alpha(x) = \frac{2\sigma x(c_1+x)}{(c_1+x)^2+c_2^2}$ and $\beta(x) = -\frac{2\sigma x c_2}{(c_1+x)^2+c_2^2}$. The reader may check that this pair of functions solves the system (4.4) for $A_1 = A_2 = B_1 = B_2 = 0$, that is, α and β are solutions of:

$$\begin{cases} \sigma x\alpha'(x) - \sigma\alpha(x) + \frac{1}{2}(\alpha(x)^2 - \beta(x)^2) = 0 \\ \sigma x\beta'(x) - \sigma\beta(x) + \alpha(x)\beta(x) = 0. \end{cases} \tag{4.31}$$

Hence a fundamental solution for the system (4.29) will be of the form given in theorem 4.0.2.

Computation of the functions K and L appearing in (4.17) can be done as follows:

$$K'(x) = \frac{\alpha(x)}{x} = \frac{\sigma(2c_1+2x)}{c_1^2+c_2^2+x^2+2c_1x} \implies K(x) = \sigma\log((c_1+x)^2+c_2^2),$$

$$L'(x) = \frac{\beta(x)}{x} = -\frac{2\sigma c_2}{(c_1+x)^2+c_2^2} \implies L(x) = -2\sigma\tan^{-1}\left(\frac{c_1+x}{c_2}\right).$$

Substitution of these expressions into the general for of the functions $\tilde{v}$, $\tilde{w}$, $\tilde{v}_1$ and $\tilde{w}_2$ gives

$$\begin{aligned} \tilde{v}(x,t,y) &= \mathcal{L}^{-1}(v(x,t,\epsilon)) \\ &= \mathcal{L}^{-1}\left(e^{\frac{-\epsilon(x+A_1\sigma t^2)}{\sigma(1+\epsilon t)} + \frac{K\left(\frac{x}{(1+\epsilon t)^2}\right)-K(x)}{2\sigma}} \cos\left(-\frac{\epsilon A_2 t^2}{1+\epsilon t} + \frac{L\left(\frac{x}{(1+\epsilon t)^2}\right)-L(x)}{2\sigma}\right)\right) \\ &= \mathcal{L}^{-1}\left(\exp\left(\frac{-\epsilon x}{\sigma(1+\epsilon t)} + \frac{1}{2}\log\left(\frac{(c_1+\frac{x}{(1+\epsilon t)^2})^2+c_2^2}{(c_1+x)^2+c_2^2}\right)\right)\right. \\ &\quad \left.\times\cos\left(\tan^{-1}\left(\frac{c_1+x}{c_2}\right) - \tan^{-1}\left(\frac{c_1+\frac{x}{(1+\epsilon t)^2}}{c_2}\right)\right)\right) \end{aligned}$$

$$
\begin{aligned}
&= \mathcal{L}^{-1}\left(\exp\left(\frac{-\epsilon x}{\sigma(1+\epsilon t)}\right)\sqrt{\frac{(c_1+\frac{x}{(1+\epsilon t)^2})^2+c_2^2}{(c_1+x)^2+c_2^2}}\right.\\
&\left.\times\cos\left(\tan^{-1}\left(\frac{c_1+x}{c_2}\right)-\tan^{-1}\left(\frac{c_1+\frac{x}{(1+\epsilon t)^2}}{c_2}\right)\right)\right)\\
&= \mathcal{L}^{-1}\left(\exp\left(\frac{-\epsilon x}{\sigma(1+\epsilon t)}\right)\frac{c_1x(2+\epsilon t(2+\epsilon t))+(c_1^2+c_2^2)(1+\epsilon t)^2+x^2}{(t\epsilon+1)^2\left((c_1+x)^2+c_2^2\right)}\right)\\
&= \frac{1}{\left((c_1+x)^2+c_2^2\right)}\mathcal{L}^{-1}\left(e^{\frac{-\epsilon x}{\sigma(1+\epsilon t)}}\left(\frac{c_1xt^2\epsilon^2}{(1+\epsilon t)^2}+\frac{2c_1x}{(1+\epsilon t)}+c_1^2+c_2^2+\frac{x^2}{(1+\epsilon t)^2}\right)\right)\\
&= \frac{e^{-\frac{x}{\sigma t}}}{\left((c_1+x)^2+c_2^2\right)}\mathcal{L}^{-1}\left(e^{\frac{x}{\sigma t^2(\epsilon+\frac{1}{t})}}\left(c_1^2+c_2^2+\frac{x^2}{t^2(\epsilon+\frac{1}{t})^2}+c_1x+\frac{c_1x}{t^2(\epsilon+\frac{1}{t})^2}\right)\right)\\
&= \frac{e^{-\frac{x}{\sigma t}-\frac{y}{t}}}{\left((c_1+x)^2+c_2^2\right)}\mathcal{L}^{-1}\left(e^{\frac{x}{\sigma t^2\epsilon}}\left(c_1^2+c_2^2+c_1x+\frac{x(c_1+x)}{t^2\epsilon^2}\right)\right)\\
&= e^{-\frac{x}{\sigma t}-\frac{y}{t}}\left(\frac{(c_1+x)(c_1+\sigma y)+c_2^2}{y\left((c_1+x)^2+c_2^2\right)}\sqrt{\frac{xy}{\sigma t^2}}I_1\left(2\sqrt{\frac{xy}{t^2\sigma}}\right)+\frac{c_1(c_1+x)+c_2^2}{(c_1+x)^2+c_2^2}\delta(y)\right).
\end{aligned}
$$

Similarly, the reader may check that we obtain

$$
\begin{aligned}
&\tilde{w}(x,t,y) = \mathcal{L}^{-1}(w(x,t,\epsilon))\\
&\quad= \mathcal{L}^{-1}\left(e^{\frac{-\epsilon(x+A_1\sigma t^2)}{\sigma(1+\epsilon t)}+\frac{K\left(\frac{x}{(1+\epsilon t)^2}\right)-K(x)}{2\sigma}}\sin\left(-\frac{\epsilon A_2t^2}{1+\epsilon t}+\frac{L\left(\frac{x}{(1+\epsilon t)^2}\right)-L(x)}{2\sigma}\right)\right)\\
&\quad= \mathcal{L}^{-1}\left(\exp\left(\frac{-\epsilon x}{\sigma(1+\epsilon t)}\right)\frac{c_2tx\epsilon(2+\epsilon t)}{(1+\epsilon t)^2\left((c_1+x)^2+c_2^2\right)}\right)\\
&\quad= \frac{e^{-\frac{x}{\sigma t}}}{(c_1+x)^2+c_2^2}\mathcal{L}^{-1}\left(e^{\frac{x}{\sigma t^2(\epsilon+\frac{1}{t})}}\left(c_2x-\frac{c_2x}{t^2(\epsilon+\frac{1}{t})^2}\right)\right)\\
&\quad= \frac{e^{-\frac{x}{\sigma t}-\frac{y}{t}}}{(c_1+x)^2+c_2^2}\mathcal{L}^{-1}\left(e^{\frac{x}{\sigma t^2\epsilon}}\left(c_2x-\frac{c_2x}{t^2\epsilon^2}\right)\right)\\
&\quad= \frac{e^{-\frac{x}{\sigma t}-\frac{y}{t}}}{(c_1+x)^2+c_2^2}\left(\frac{(c_2(x-\sigma y))}{y}\sqrt{\frac{xy}{\sigma t^2}}I_1\left(2\sqrt{\frac{xy}{t^2\sigma}}\right)+c_2x\delta(y)\right),
\end{aligned}
$$

and that some basic algebraic manipulation (similar to that used for $\tilde{v}$ and $\tilde{w}$) produces the following expressions for $\tilde{v}_1$ and $\tilde{w}_1$:

$$
\begin{aligned}
\tilde{v}_1(x,t,y) &= \mathcal{L}^{-1}(v_1(x,t,\epsilon)) \\
&= \frac{e^{-\frac{x}{\sigma t}-\frac{y}{t}}}{(c_1+x)^2+c_2^2}\mathcal{L}^{-1}\left(e^{\frac{x}{\sigma t^2\epsilon}}\left(\frac{x\left(c_1^2+c_2^2\right)+2c_1x^2+x^3}{\sigma t^4\epsilon^2}-\frac{2\left(x\left(c_1^2+c_2^2\right)+c_1x^2\right)}{\sigma t^3\epsilon}\right)\right. \\
&+ e^{\frac{x}{\sigma t^2\epsilon}}\left(\frac{x\left(c_1^2+c_2^2\right)+c_1x^2}{\sigma t^2}+\frac{x^2(c_1+x)}{\sigma t^6\epsilon^4}-\frac{2x^2(c_1+x)}{\sigma t^5\epsilon^3}\right) \\
&+ \left. e^{\frac{x}{\sigma t^2\epsilon}}\left(\frac{2x(c_1+x)}{t^4\epsilon^3}-\frac{2x(c_1+x)}{t^3\epsilon^2}\right)\right) \\
&= e^{-\frac{x}{\sigma t}-\frac{y}{t}}\left(\frac{xt\left(c_1(c_1+x)+c_2^2\right)}{\sigma t^3((c_1+x)^2+c_2^2)}\delta(y)-\frac{2x\left(c_1(c_1+x)+c_2^2\right)}{\sigma t^3((c_1+x)^2+c_2^2)}I_0\left(2\sqrt{\frac{xy}{t^2\sigma}}\right)\right. \\
&+ \frac{\left(c_1(c_1+x)+c_2^2\right)(x+\sigma y)+\sigma y(c_1+x)(x-2\sigma t)}{\sigma t^3((c_1+x)^2+c_2^2)}\sqrt{\frac{x}{\sigma y}}I_1\left(2\sqrt{\frac{xy}{t^2\sigma}}\right) \\
&+ \left.\frac{\sigma y(c_1+x)\sqrt{x\sigma y}}{\sigma t^3((c_1+x)^2+c_2^2)}I_3\left(2\sqrt{\frac{xy}{t^2\sigma}}\right)-\frac{2\sigma y(c_1+x)(x-\sigma t)}{\sigma t^3((c_1+x)^2+c_2^2)}I_2\left(2\sqrt{\frac{xy}{t^2\sigma}}\right)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{w}_1(x,t,y) &= \mathcal{L}^{-1}(w_1(x,t,\epsilon)) \\
&= \frac{xc_2e^{-\frac{x}{\sigma t}-\frac{y}{t}}}{(c_1+x)^2+c_2^2}\mathcal{L}^{-1}\left(e^{\frac{x}{\sigma t^2\epsilon}}\left(-\frac{2x}{\sigma t^3\epsilon}+\frac{x}{\sigma t^2}-\frac{x}{\sigma t^6\epsilon^4}+\frac{2x}{\sigma t^5\epsilon^3}-\frac{2}{t^4\epsilon^3}+\frac{2}{t^3\epsilon^2}\right)\right) \\
&= e^{-\frac{x}{\sigma t}-\frac{y}{t}}\left(\frac{\sqrt{\sigma yx}c_2}{((c_1+x)^2+c_2^2)\sigma t^3}\left(\left(2\sigma t+\frac{x^2}{\sigma y}\right)I_1\left(2\sqrt{\frac{xy}{t^2\sigma}}\right)-\sigma yI_3\left(2\sqrt{\frac{xy}{t^2\sigma}}\right)\right)\right. \\
&+ \frac{x^2c_2}{((c_1+x)^2+c_2^2)\sigma t^3}\left(t\delta(y)-2I_0\left(2\sqrt{\frac{xy}{t^2\sigma}}\right)\right) \\
&+ \left.\frac{c_2(2\sigma y(x-\sigma t))}{((c_1+x)^2+c_2^2)\sigma t^3}I_2\left(2\sqrt{\frac{xy}{t^2\sigma}}\right)\right)
\end{aligned}
$$

Again, substitution of these expressions in the general form for the matrix $P(x,t,z)$ given in theorem 4.0.2 yields the desired result. □

4.1 A more general result

The results obtained so far in this chapter can actually be generalised using the original form of Theorem 4.0.1. We will distinguish between two different cases depending on the choice of g in Theorem 4.0.1:

(I) $g(x) \equiv 0$. For this choice of the function g, equation (4.2) has $u_0 = 1$ as a stationary solution, so we might develop our techniques from this starting point. Note that in this case the aim will be to obtain a fundamental matrix for the system of PDEs

$$\begin{cases} v_t = \sigma x^\gamma v_{xx} + f_1(x)v_x - f_2(x)w_x \\ w_t = \sigma x^\gamma w_{xx} + f_2(x)v_x + f_1(x)w_x, \end{cases} \quad x \geq 0 \tag{4.32}$$

provided that the functions f_1 and f_2 satisfy

$$\begin{cases} -\gamma\sigma x^{1-\gamma} f_1(x) + \sigma x^{2-\gamma} f_1'(x) + \frac{1}{2}x^{2(1-\gamma)}(f_1(x)^2 - f_2(x)^2) = 2\sigma A_1 x^{2-\gamma} + B_1 \\ -\gamma\sigma x^{1-\gamma} f_2(x) + \sigma x^{2-\gamma} f_2'(x) + x^{2(1-\gamma)} f_1(x) f_2(x) = 2\sigma A_2 x^{2-\gamma} + B_2 \end{cases} \tag{4.33}$$

(II) $g \neq 0$. In this case, since $u = 1$ is not a solution, we will first need to find an appropriate stationary solution of (4.2) to be able to apply our methodology. The outcome here will be a method to compute, for appropriate choices of the function g, a fundamental matrix for the system

$$\begin{cases} v_t = \sigma x^\gamma v_{xx} + f_1(x)v_x - f_2(x)w_x - g_1(x)v + g_2(x)w \\ w_t = \sigma x^\gamma w_{xx} + f_2(x)v_x + f_1(x)w_x - g_2(x)v - g_1(x)w, \end{cases} \quad x \geq 0 \tag{4.34}$$

for any choice of functions f_1 and f_2 satisfying

$$\begin{cases} \sigma x^{2-\gamma}(f_1'(x) + 2g_1(x)) - \gamma\sigma x^{1-\gamma} f_1(x) + \frac{1}{2}x^{2(1-\gamma)}(f_1(x)^2 - f_2(x)^2) = 2\sigma A_1 x^{2-\gamma} + B_1 \\ \sigma x^{2-\gamma}(f_2'(x) + 2g_2(x)) - \gamma\sigma x^{1-\gamma} f_2(x) + x^{2(1-\gamma)} f_1(x) f_2(x) = 2\sigma A_2 x^{2-\gamma} + B_2 \end{cases} \tag{4.35}$$

4.1.1 Case (I): Starting from the stationary solution $u_0 = 1$

The most general form of theorem 4.0.1 allows us to state the following theorem:

Theorem 4.1.1. *Let $A_i, B_i \in \mathbb{R}$ for $i = 1, 2$ and let $\sigma, \gamma \in \mathbb{R}$ with $\gamma \neq 2$.*
Suppose further that $v(t,x)$ and $w(t,x)$ are real-valued functions and that $f_1(x)$, $f_2(x)$ are

also real-valued functions, satisfying

$$\begin{cases} -\gamma\sigma x^{1-\gamma} f_1(x) + \sigma x^{2-\gamma} f_1'(x) + \frac{1}{2} x^{2(1-\gamma)}(f_1(x)^2 - f_2(x)^2) = 2\sigma A_1 x^{2-\gamma} + B_1 \\ -\gamma\sigma x^{1-\gamma} f_2(x) + \sigma x^{2-\gamma} f_2'(x) + x^{2(1-\gamma)} f_1(x) f_2(x) = 2\sigma A_2 x^{2-\gamma} + B_2. \end{cases} \tag{4.36}$$

Then the system of PDEs

$$\begin{cases} v_t = \sigma x^\gamma v_{xx} + f_1(x) v_x - f_2(x) w_x \\ w_t = \sigma x^\gamma w_{xx} + f_2(x) v_x + f_1(x) w_x, \end{cases} \qquad x, t > 0 \tag{4.37}$$

has a fundamental matrix $P(t,x,y)$ given by:

$$\underbrace{\frac{y^{1-\gamma}}{\sigma(2-\gamma)} \begin{pmatrix} \tilde{v}(t,x,\frac{y^{2-\gamma}}{(2-\gamma)^2\sigma}) & \tilde{v}_1(t,x,\frac{y^{2-\gamma}}{(2-\gamma)^2\sigma}) \\ \tilde{w}(t,x,\frac{y^{2-\gamma}}{(2-\gamma)^2\sigma}) & \tilde{w}_1(t,x,\frac{y^{2-\gamma}}{(2-\gamma)^2\sigma}) \end{pmatrix} \begin{pmatrix} 1 & \sigma y^\gamma \frac{f_2'(y)}{f_2(y)^2} - \frac{f_1(y)}{f_2(y)} - \frac{\sigma y^\gamma}{f_2(y)} \frac{d}{dy} \\ 0 & I_{f_2} \end{pmatrix}}_{P(t,x,y)}, \tag{4.38}$$

where

$$I_{f_2} g(y) := \int_{y_0}^{y} \frac{g(s)}{f_2(s)} ds, \tag{4.39}$$

and where the functions $\tilde{v}$, $\tilde{w}$, $\tilde{v_1}$ and $\tilde{w_1}$ are given by

$$\tilde{v}(t,x,z) = \mathcal{L}^{-1}(v(t,x,\epsilon)), \qquad \tilde{w}(t,x,z) = \mathcal{L}^{-1}(w(t,x,\epsilon))$$
$$\tilde{v_1}(t,x,z) = \mathcal{L}^{-1}(v_1(t,x,\epsilon)), \qquad \tilde{w_1}(t,x,z) = \mathcal{L}^{-1}(w_1(t,x,\epsilon))$$

for

$$\begin{cases} v(t,x,\epsilon) &= (1+\epsilon t)^{-\frac{1-\gamma}{2-\gamma}} \cos\left(-\frac{\epsilon A_2 t^2}{1+\epsilon t} + \frac{1}{2\sigma}\left(F_2\left(\frac{x}{(1+\epsilon t)^{\frac{2}{2-\gamma}}}\right) - F_2(x)\right)\right) \\ & \times \exp\left(\frac{-\epsilon(x^{2-\gamma} + A_1\sigma(2-\gamma)^2 t^2)}{\sigma(2-\gamma)^2(1+\epsilon t)} + \frac{1}{2\sigma}\left(F_1\left(\frac{x}{(1+\epsilon t)^{\frac{2}{2-\gamma}}}\right) - F_1(x)\right)\right) \\ w(t,x,\epsilon) &= (1+\epsilon t)^{-\frac{1-\gamma}{2-\gamma}} \sin\left(-\frac{\epsilon A_2 t^2}{1+\epsilon t} + \frac{1}{2\sigma}\left(F_2\left(\frac{x}{(1+\epsilon t)^{\frac{2}{2-\gamma}}}\right) - F_2(x)\right)\right) \\ & \times \exp\left(\frac{-\epsilon(x^{2-\gamma} + A_1\sigma(2-\gamma)^2 t^2)}{\sigma(2-\gamma)^2(1+\epsilon t)} + \frac{1}{2\sigma}\left(F_1\left(\frac{x}{(1+\epsilon t)^{\frac{2}{2-\gamma}}}\right) - F_1(x)\right)\right) \end{cases}$$

where $F_1'(x) = \frac{f_1(x)}{x^\gamma}$, $F_2'(x) = \frac{f_2(x)}{x^\gamma}$, *and*

$$\begin{cases} v_1(t,x,\epsilon) = \frac{\partial}{\partial t} v(t,x,\epsilon) \\ w_1(t,x,\epsilon) = \frac{\partial}{\partial t} w(t,x,\epsilon). \end{cases} \tag{4.40}$$

That is, a solution $(\bar{V}(x,t) \quad \bar{W}(x,t))^\top$ *for the system (4.37) with initial condition*

$$\begin{pmatrix} \bar{V}(x,0) \\ \bar{W}(x,0) \end{pmatrix} = \begin{pmatrix} m(x) \\ n(x) \end{pmatrix}$$

can be written as

$$\begin{pmatrix} \bar{V}(x,t) \\ \bar{W}(x,t) \end{pmatrix} = \int_0^\infty \underbrace{\begin{pmatrix} p_{11}(x,t,z) & p_{12}(x,t,z) \\ p_{21}(x,t,z) & p_{22}(x,t,z) \end{pmatrix}}_{P(x,t,z)} \begin{pmatrix} m(z) \\ n(z) \end{pmatrix} dz,$$

with the matrix $P(x,t,z)$ *defined as above.*

Proof. Consider Craddock's theorem 4.0.1. Suppose $\sigma \in \mathbb{R}$ and $\gamma \neq 2$. Write $A = A_1 + iA_2$ and $B = B_1 + iB_2$.
Let $u(t,x) = v(t,x) + iw(t,x)$ and $f(x) = f_1(x) + if_2(x)$. Then for $g(x) \equiv 0$ we have that equation (4.2) and (4.1) in Theorem 4.0.1 read as:

$$v_t + iw_t = \sigma x^\gamma (v_{xx} + iw_{xx}) + (f_1(x) + if_2(x))(v_x + iw_x), \quad x \geq 0 \tag{4.41}$$

and

$$\sigma x (x^{1-\gamma}(f_1 + if_2))' - \sigma (x^{1-\gamma}(f_1 + if_2)) + \frac{1}{2}(x^{1-\gamma}(f_1 + if_2))^2 = 2\sigma A x^{2-\gamma} + B. \tag{4.42}$$

This is equivalent to considering the system of PDEs:

$$\begin{cases} v_t = \sigma x^\gamma v_{xx} + f_1(x) v_x - f_2(x) w_x \\ w_t = \sigma x^\gamma w_{xx} + f_2(x) v_x + f_1(x) w_x, \end{cases} \quad x \geq 0 \tag{4.43}$$

subject to the condition that f_1 and f_2 satisfy

$$\begin{cases} -\gamma\sigma x^{1-\gamma} f_1(x) + \sigma x^{2-\gamma} f_1'(x) + \frac{1}{2} x^{2(1-\gamma)} (f_1(x)^2 - f_2(x)^2) = 2\sigma A_1 x^{2-\gamma} + B_1 \\ -\gamma\sigma x^{1-\gamma} f_2(x) + \sigma x^{2-\gamma} f_2'(x) + x^{2(1-\gamma)} f_1(x) f_2(x) = 2\sigma A_2 x^{2-\gamma} + B_2 \end{cases} \tag{4.44}$$

Clearly $u_0 = 1$, i.e. $v_0 = 1$, $w_0 = 0$, is a stationary solution of (4.43).
The symmetry $\bar{U}_\epsilon$ in Theorem 4.0.1 applied to this stationary solution u_0 yields the new solution:

$$\bar{U}_\epsilon(t,x) = \frac{1}{(1+4\epsilon t)^{\frac{1-\gamma}{2-\gamma}}} \exp\left\{\frac{-4\epsilon(x^{2-\gamma} + (A_1 + iA_2)\sigma(2-\gamma)^2 t^2)}{\sigma(2-\gamma)^2(1+4\epsilon t)}\right\} \times$$
$$\exp\left\{\frac{1}{2\sigma}\left(F_1\left(\frac{x}{(1+4\epsilon t)^{\frac{2}{2-\gamma}}}\right) + iF_2\left(\frac{x}{(1+4\epsilon t)^{\frac{2}{2-\gamma}}}\right) - F_1(x) - iF_2(x)\right)\right\},$$

where $F_1'(x) = \frac{f_1(x)}{x^\gamma}$, $F_2'(x) = \frac{f_2(x)}{x^\gamma}$. Or, alternatively, making the change $\epsilon \to \epsilon/4$ we get

$$U_\epsilon(t,x) = \frac{1}{(1+\epsilon t)^{\frac{1-\gamma}{2-\gamma}}} \exp\left\{\frac{-\epsilon(x^{2-\gamma} + (A_1 + iA_2)\sigma(2-\gamma)^2 t^2)}{\sigma(2-\gamma)^2(1+\epsilon t)}\right\} \times$$
$$\exp\left\{\frac{1}{2\sigma}\left(F_1\left(\frac{x}{(1+\epsilon t)^{\frac{2}{2-\gamma}}}\right) + iF_2\left(\frac{x}{(1+\epsilon t)^{\frac{2}{2-\gamma}}}\right) - F_1(x) - iF_2(x)\right)\right\}.$$

That is, the pair

$$\begin{cases} v(t,x,\epsilon) & = (1+\epsilon t)^{-\frac{1-\gamma}{2-\gamma}} \cos\left(-\frac{\epsilon A_2 t^2}{1+\epsilon t} + \frac{1}{2\sigma}\left(F_2\left(\frac{x}{(1+\epsilon t)^{\frac{2}{2-\gamma}}}\right) - F_2(x)\right)\right) \\ & \times \exp\left(\frac{-\epsilon(x^{2-\gamma}+A_1\sigma(2-\gamma)^2 t^2)}{\sigma(2-\gamma)^2(1+\epsilon t)} + \frac{1}{2\sigma}\left(F_1\left(\frac{x}{(1+\epsilon t)^{\frac{2}{2-\gamma}}}\right) - F_1(x)\right)\right) \\ w(t,x,\epsilon) & = (1+\epsilon t)^{-\frac{1-\gamma}{2-\gamma}} \sin\left(-\frac{\epsilon A_2 t^2}{1+\epsilon t} + \frac{1}{2\sigma}\left(F_2\left(\frac{x}{(1+\epsilon t)^{\frac{2}{2-\gamma}}}\right) - F_2(x)\right)\right) \\ & \times \exp\left(\frac{-\epsilon(x^{2-\gamma}+A_1\sigma(2-\gamma)^2 t^2)}{\sigma(2-\gamma)^2(1+\epsilon t)} + \frac{1}{2\sigma}\left(F_1\left(\frac{x}{(1+\epsilon t)^{\frac{2}{2-\gamma}}}\right) - F_1(x)\right)\right) \end{cases}$$

is a solution of (4.43). Clearly, the pair $(v_1(t,x,\epsilon) \quad w_1(t,x,\epsilon))^\top$ defined as

$$\begin{pmatrix} v_1(t,x,\epsilon) \\ w_1(t,x,\epsilon) \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial t} v(t,x,\epsilon) \\ \frac{\partial}{\partial t} w(t,x,\epsilon) \end{pmatrix} \tag{4.45}$$

is also a solution to our system, where the explicit expressions for the components v_1 and w_1 are given by:

$$\begin{cases} v_1(t,x,\epsilon) = -\frac{e^{a(t,x,\epsilon)}\sin\big(b(t,x,\epsilon)\big)}{(1+\epsilon t)^{3-\frac{1}{2-\gamma}}}\left(A_2 t\epsilon(2+\epsilon t) - \frac{\epsilon x^{1-\gamma} f_2\left(\frac{x}{(1+\epsilon t)^{\frac{2}{2-\gamma}}}\right)}{(\gamma-2)\sigma(1+\epsilon t)^{-\frac{\gamma}{2-\gamma}}}\right) \\ -\frac{e^{a(t,x,\epsilon)}\cos\big(b(t,x,\epsilon)\big)}{(1+\epsilon t)^{3-\frac{1}{2-\gamma}}}\left(A_1 t\epsilon(2+\epsilon t) + \frac{\epsilon x^{1-\gamma} f_1\left(\frac{x}{(1+\epsilon t)^{\frac{2}{2-\gamma}}}\right)}{(2-\gamma)\sigma(t\epsilon+1)^{-\frac{\gamma}{2-\gamma}}} + \frac{\epsilon(1-\gamma)(1+\epsilon t)}{2-\gamma} - \frac{\epsilon^2 x^{2-\gamma}}{(\gamma-2)^2\sigma}\right) \\ w_1(t,x,\epsilon) = -\frac{e^{a(t,x,\epsilon)}\cos\big(b(t,x,\epsilon)\big)}{(1+\epsilon t)^{3-\frac{1}{2-\gamma}}}\left(A_2 t\epsilon(2+\epsilon t) - \frac{\epsilon x^{1-\gamma} f_2\left(\frac{x}{(1+\epsilon t)^{\frac{2}{2-\gamma}}}\right)}{(\gamma-2)\sigma(1+\epsilon t)^{-\frac{\gamma}{2-\gamma}}}\right) \\ +\frac{e^{a(t,x,\epsilon)}\sin\big(b(t,x,\epsilon)\big)}{(1+\epsilon t)^{3-\frac{1}{2-\gamma}}}\left(A_1 t\epsilon(2+\epsilon t) + \frac{\epsilon x^{1-\gamma} f_1\left(\frac{x}{(1+\epsilon t)^{\frac{2}{2-\gamma}}}\right)}{(2-\gamma)\sigma(1+\epsilon t)^{-\frac{\gamma}{2-\gamma}}} - \frac{\epsilon(\gamma-1)(1+\epsilon t)}{2-\gamma} - \frac{\epsilon^2 x^{2-\gamma}}{(\gamma-2)^2\sigma}\right) \end{cases}$$

with

$$\begin{cases} a(t,x,\epsilon) & := -\frac{\epsilon\left(A_1(\gamma-2)^2\sigma t^2 + x^{2-\gamma}\right)}{(\gamma-2)^2\sigma(1+\epsilon t)} - \frac{1}{2\sigma}\left(F_1(x) - F_1\Big(\frac{x}{(1+\epsilon t)^{\frac{2}{2-\gamma}}}\Big)\right) \\ b(t,x,\epsilon) & := \frac{A_2 t^2\epsilon}{1+\epsilon t} + \frac{1}{2\sigma}\left(F_2(x) - F_2\Big(\frac{x}{(1+\epsilon t)^{\frac{2}{2-\gamma}}}\Big)\right). \end{cases}$$

Consider, for suitable functions ϕ and ψ with sufficiently rapid decay, the usual construction of a new solution given by

$$\begin{pmatrix} V(t,x) \\ W(t,x) \end{pmatrix} = \int_0^\infty \begin{pmatrix} \phi(\epsilon)v(t,x,\epsilon) + \psi(\epsilon)v_1(t,x,\epsilon) \\ \phi(\epsilon)w(t,x,\epsilon) + \psi(\epsilon)w_1(t,x,\epsilon) \end{pmatrix} d\epsilon, \tag{4.46}$$

which, as the reader may easily check, has initial condition

$$\begin{pmatrix} V(0,x) \\ W(0,x) \end{pmatrix} = \begin{pmatrix} \int_0^\infty e^{-\frac{\epsilon x^{2-\gamma}}{(2-\gamma)^2\sigma}}\left(\phi(\epsilon) - \frac{\epsilon\psi(\epsilon)}{2-\gamma}\left((1-\gamma) + \frac{f_1(x)x^{1-\gamma}}{\sigma} - \frac{\epsilon x^{2-\gamma}}{(2-\gamma)\sigma}\right)\right) d\epsilon \\ \int_0^\infty -e^{-\frac{\epsilon x^{2-\gamma}}{(2-\gamma)^2\sigma}}\frac{\epsilon f_2(x)x^{1-\gamma}\psi(\epsilon)}{(2-\gamma)\sigma} d\epsilon \end{pmatrix}$$
$$:= \begin{pmatrix} m(x) \\ n(x) \end{pmatrix}.$$

Such initial condition may be written individually for each component as

$$\int_0^\infty \phi(\epsilon)e^{-\frac{\epsilon x^{2-\gamma}}{(2-\gamma)^2\sigma}}d\epsilon + f_1(x)\int_0^\infty \left(-\frac{\epsilon x^{1-\gamma}}{\sigma(2-\gamma)}\right)\psi(\epsilon)e^{-\frac{\epsilon x^{2-\gamma}}{(2-\gamma)^2\sigma}}d\epsilon$$
$$+\frac{\sigma}{x^{-\gamma}}\int_0^\infty \left(\frac{\epsilon^2 x^{2-2\gamma}}{\sigma^2(2-\gamma)^2} - \frac{\epsilon(1-\gamma)x^{-\gamma}}{\sigma(2-\gamma)}\right)\psi(\epsilon)e^{-\frac{\epsilon x^{2-\gamma}}{(2-\gamma)^2\sigma}}d\epsilon = m(x)$$
$$\iff \Phi\left(\frac{x^{2-\gamma}}{(2-\gamma)^2\sigma}\right) + f_1(x)\frac{d}{dx}\Psi\left(\frac{x^{2-\gamma}}{(2-\gamma)^2\sigma}\right) + \sigma x^\gamma \frac{d^2}{dx^2}\Psi\left(\frac{x^{2-\gamma}}{(2-\gamma)^2\sigma}\right) = m(x)$$

and

$$f_2(x)\int_0^\infty -\frac{\epsilon x^{1-\gamma}}{\sigma(2-\gamma)}\psi(\epsilon)e^{-\frac{\epsilon x^{2-\gamma}}{(2-\gamma)^2\sigma}}d\epsilon = n(x)$$
$$\iff f_2(x)\frac{d}{dx}\Psi\left(\frac{x^{2-\gamma}}{(2-\gamma)^2\sigma}\right) = n(x)$$

respectively. Hence we may write :

$$\begin{pmatrix} \Phi\left(\frac{x^{2-\gamma}}{(2-\gamma)^2\sigma}\right) \\ \Psi\left(\frac{x^{2-\gamma}}{(2-\gamma)^2\sigma}\right) \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & \sigma x^\gamma \frac{f_2'(x)}{f_2(x)^2} - \frac{f_1(x)}{f_2(x)} - \frac{\sigma x^\gamma}{f_2(x)}\frac{d}{dx} \\ 0 & I_{f_2} \end{pmatrix}}_{C(x)} \begin{pmatrix} m(x) \\ n(x) \end{pmatrix}, \tag{4.47}$$

where I_{f_2} denotes the integral operator defined by:

$$I_{f_2}g(x) := \int_{x_0}^x \frac{g(s)}{f_2(s)}ds. \tag{4.48}$$

Then, as usual, suppose we can write

$$\begin{aligned} v(t,x,\epsilon) &= \mathcal{L}(\tilde{v}(t,x,z)) = \int_0^\infty \tilde{v}(t,x,z)e^{-\epsilon z}dz, \\ v_1(t,x,\epsilon) &= \mathcal{L}(\tilde{v}_1(t,x,z)) = \int_0^\infty \tilde{v}_1(t,x,z)e^{-\epsilon z}dz, \\ w(t,x,\epsilon) &= \mathcal{L}(\tilde{w}(t,x,z)) = \int_0^\infty \tilde{w}(t,x,z)e^{-\epsilon z}dz \text{ and} \\ w_1(t,x,\epsilon) &= \mathcal{L}(\tilde{w}_1(t,x,z)) = \int_0^\infty \tilde{w}_1(t,x,z)e^{-\epsilon z}dz \end{aligned} \tag{4.49}$$

so that

$$\begin{pmatrix} V(t,x) \\ W(t,x) \end{pmatrix} = \int_0^\infty \begin{pmatrix} \tilde{v}(t,x,z) & \tilde{v}_1(t,x,z) \\ \tilde{w}(t,x,z) & \tilde{w}_1(t,x,z) \end{pmatrix} \begin{pmatrix} \Phi(z) \\ \Psi(z) \end{pmatrix} dz$$

Then the change of variables $z = \frac{y^{2-\gamma}}{(2-\gamma)^2\sigma}$, together with expression(4.47), produce

$$\begin{pmatrix} V(t,x) \\ W(t,x) \end{pmatrix} = \int_0^\infty \underbrace{\begin{pmatrix} \tilde{v}(t,x,\frac{y^{2-\gamma}}{(2-\gamma)^2\sigma}) & \tilde{v}_1(t,x,\frac{y^{2-\gamma}}{(2-\gamma)^2\sigma}) \\ \tilde{w}(t,x,\frac{y^{2-\gamma}}{(2-\gamma)^2\sigma}) & \tilde{w}_1(t,x,\frac{y^{2-\gamma}}{(2-\gamma)^2\sigma}) \end{pmatrix}}_{L\left(t,x,\frac{y^{2-\gamma}}{\sigma(2-\gamma)^2}\right)} \begin{pmatrix} \Phi(\frac{y^{2-\gamma}}{(2-\gamma)^2\sigma}) \\ \Psi(\frac{y^{2-\gamma}}{(2-\gamma)^2\sigma}) \end{pmatrix} \frac{y^{1-\gamma}}{\sigma(2-\gamma)} dy$$

$$= \int_0^\infty L\left(t,x,\frac{y^{2-\gamma}}{\sigma(2-\gamma)^2}\right) C(y) \begin{pmatrix} m(y) \\ n(y) \end{pmatrix} \frac{y^{1-\gamma}}{\sigma(2-\gamma)} dy,$$

which gives the desired result. □

4.1.2 Case (II): Starting from any other stationary solution

Suppose $\sigma \in \mathbb{R}$ and $\gamma \neq 2$. Write $A = A_1 + iA_2$ and $B = B_1 + iB_2$.
Let $u(t,x) = v(t,x) + iw(t,x)$ and $f(x) = f_1(x) + if_2(x)$.
Then for $g(x) = g_1(x) + ig(x) \neq 0$ we have that equation (4.2) and (4.1) in Theorem 4.0.1 read as:

$$v_t + iw_t = \sigma x^\gamma (v_{xx} + iw_{xx}) + (f_1(x) + if_2(x))(v_x + iw_x) - (g_1(x) + ig_2(x))(v + iw), \quad x \geq 0 \tag{4.50}$$

and

$$\sigma x(x^{1-\gamma}(f_1 + if_2))' - \sigma(x^{1-\gamma}(f_1 + if_2)) + \frac{1}{2}(x^{1-\gamma}(f_1 + if_2))^2 + 2\sigma x^{2-\gamma}(g_1 + ig_2) = 2\sigma(A_1 + iA_2)x^{2-\gamma} + (B_1 + iB_2).$$

This case is equivalent to considering the system of PDEs:

$$\begin{cases} v_t = \sigma x^\gamma v_{xx} + f_1(x)v_x - f_2(x)w_x - g_1(x)v + g_2(x)w \\ w_t = \sigma x^\gamma w_{xx} + f_2(x)v_x + f_1(x)w_x - g_2(x)v - g_1(x)w, \end{cases} \quad x \geq 0 \tag{4.51}$$

subject to the condition that f_1 and f_2 satisfy

$$\begin{cases} \sigma x^{2-\gamma}(f_1'(x) + 2g_1(x)) - \gamma\sigma x^{1-\gamma} f_1(x) + \frac{1}{2}x^{2(1-\gamma)}(f_1(x)^2 - f_2(x)^2) = 2\sigma A_1 x^{2-\gamma} + B_1 \\ \sigma x^{2-\gamma}(f_2'(x) + 2g_2(x)) - \gamma\sigma x^{1-\gamma} f_2(x) + x^{2(1-\gamma)} f_1(x) f_2(x) = 2\sigma A_2 x^{2-\gamma} + B_2 \end{cases} \tag{4.52}$$

In this case, $v = 1$, $w = 0$ is not a solution, so we cannot proceed as in the previous case. Here, if we wish to obtain a similar result to Theorem 4.1.1, we need to find (for our particular choice of g) an analytic stationary solution $u_0(x)$ of the PDE

$$u_t = \sigma x^\gamma u_{xx} + f(x)u_x - g(x)u, \quad x \geq 0 \tag{4.53}$$

and then transform it according to the symmetry in Theorem 4.0.1, that is, construct the solution

$$U(t,x,\epsilon) = \frac{\exp\left(\frac{-4\epsilon(x^{2-\gamma}+A\sigma(2-\gamma)^2t^2)}{\sigma(2-\gamma)^2(1+4\epsilon t)} + \frac{F\left(x/(1+4\epsilon t)^{\frac{2}{2-\gamma}}\right)-F(x)}{2\sigma}\right)}{(1+4\epsilon t)^{\frac{1-\gamma}{2-\gamma}}} u_0\left(\frac{x}{(1+4\epsilon t)^{\frac{2}{2-\gamma}}}\right), \tag{4.54}$$

where $F'(x) = \frac{f(x)}{x^\gamma}$.

With this, by separating the real and imaginary parts in (4.54), we can follow the argument in the previous case to obtain a fundamental matrix for the system (4.51) in a similar way as we did in Section 4.1.1 for the system (4.37). However, we have not yet obtained a stationary solution to this problem with a general function g. This line of study is open for future work. We would like to consider particular choices of g for which a stationary solution can be easily found.

4.2 Extension to more complicated cases

The results in the previous section can be extended to the computation of fundamental matrices for systems of the type (4.32) and (4.34) for a wider class of functions f_1 and f_2. We do not write down any results here for these cases. We only wish to do the preliminaries and we will leave these extensions for future work. The details for the fundamental matrices for these other cases become extremely complicated and so we have not pursued this in this work. However, we wish to indicate what kind of results can be obtained as an extension of those obtained in the previous section. To do so, we need the following two results from Craddock. [23]

Theorem 4.2.1. *[[23]] Suppose that $\gamma \neq 2$ and that for a given g, the drift f in the PDE (4.2) is such that $h(x) = x^{1-\gamma}f(x)$ satisfies the Riccati equation*

$$\sigma x h' - \sigma h + \frac{1}{2}h^2 + 2\sigma x^{2-\gamma}g(x) = \frac{A}{2(2-\gamma)^2}x^{4-2\gamma} + \frac{B}{2-\gamma}x^{2-\gamma} + C, \tag{4.55}$$

where $A > 0$, B and C are arbitraty constants. Let u_0 be an analytic, stationary solution of (4.2). Then for ϵ sufficiently small (4.2) has a solution

$$\bar{U}_\epsilon(t,x) = (1+2\epsilon^2(\cosh(\sqrt{A}t)-1)+2\epsilon\sinh(\sqrt{A}t))^{-c}$$
$$\times\left|\frac{\cosh(\frac{\sqrt{A}t}{2})+(1+2\epsilon)\sinh(\frac{\sqrt{A}t}{2})}{\cosh(\frac{\sqrt{A}t}{2})-(1-2\epsilon)\sinh(\frac{\sqrt{A}t}{2})}\right|^{\frac{B}{2\sigma\sqrt{A}(2-\gamma)}} e^{-\frac{1}{2\sigma}F(x)-\frac{Bt}{\sigma(2-\gamma)}}$$
$$\times\exp\left\{\frac{-\sqrt{A}\epsilon x^{2-\gamma}(\cosh(\sqrt{A}t)+\epsilon\sinh(\sqrt{A}t))}{\sigma(2-\gamma)^2(1+2\epsilon^2(\cosh(\sqrt{A}t)-1)+2\epsilon\sinh(\sqrt{A}t))}\right\}$$
$$\times\exp\left\{\frac{1}{2\sigma}F\left(\frac{x}{(1+2\epsilon^2(\cosh(\sqrt{A}t)-1)+2\epsilon\sinh(\sqrt{A}t))^{\frac{1}{2-\gamma}}}\right)\right\}$$
$$\times u_0\left(\frac{x}{(1+2\epsilon^2(\cosh(\sqrt{A}t)-1)+2\epsilon\sinh(\sqrt{A}t))^{\frac{1}{2-\gamma}}}\right),$$

where $F'(x) = \frac{f(x)}{x^\gamma}$ *and* $c = \frac{1-\gamma}{2-\gamma}$. *Furthermore, there exists a fundamental solution* $p(t,x,y)$ *of (4.2) such that*

$$\int_0^\infty e^{-\lambda y^{2-\gamma}} u_0(y)p(t,x,y)dy = U_\lambda(t,x). \tag{4.56}$$

in which $U_\lambda(t,x) = \bar{U}_{\frac{\sigma(2-\gamma)^2\lambda}{\sqrt{A}}}(t,x)$. *If* $g=0$, *then we may take* $u_0 = 1$, *and the fundamental solution satisfies* $\int_0^\infty p(t,x,y)dy = 1$.

Theorem 4.2.2. *[[23]] Suppose that* $\gamma \neq 2$ *and that for a given* g, $h(x) = x^{1-\gamma}f(x)$ *is a solution of the Riccati equation*

$$\sigma x h' - \sigma h + \frac{1}{2}h^2 + 2\sigma x^{2-\gamma}g(x) = \frac{Ax^{4-2\gamma}}{2(2-\gamma)^2} + \frac{Bx^{3-\frac{3}{2}\gamma}}{3-\frac{3}{2}\gamma} + \frac{Cx^{2-\gamma}}{2-\gamma} - \kappa, \tag{4.57}$$

where $\kappa = \frac{\gamma}{8}(\gamma-4)\sigma^2$, $\gamma \neq 2$ *and* $A > 0$. *Let* u_0 *be an analytic stationary solution of the PDE (4.2). Define the following constants:* $a = \frac{C}{2\sigma(2-\gamma)}$, $b = \frac{(1-\gamma)\sqrt{A}}{2(2-\gamma)}$, $k = \frac{2(2-\gamma)B}{3\sqrt{A}}$, $d = \frac{B^2}{9A\sigma}$, $l = \frac{B\gamma}{3Ak}$ *and* $s = \frac{a+d}{\sqrt{A}} - \frac{\sqrt{A}k^2}{2\sigma(2-\gamma)^2}$. *Let* ϵ *be sufficiently small and*

$$X(\epsilon,x,t) = \left(\frac{x^{1-\frac{\gamma}{2}}+k}{\sqrt{1+2\epsilon^2(\cosh(\sqrt{A}t)-1)+2\epsilon\sinh(\sqrt{A}t)}} - k\right)^{\frac{2}{2-\gamma}},$$

and $F'(x) = \frac{f(x)}{x^\gamma}$. *Then equation (4.2) has a solution*

$$\bar{U}_\epsilon(t,x) = \frac{x^l(1+2\epsilon^2(\cosh(\sqrt{A}t)-1)+2\epsilon\sinh(\sqrt{A}t))^{-\frac{2b}{\sqrt{A}}}}{(k+kx^{\frac{\gamma}{2}}(1-\sqrt{1+2\epsilon^2(\cosh(\sqrt{A}t)-1)+2\epsilon\sinh(\sqrt{A}t)}))^l}$$
$$\times \left|\frac{\cosh(\frac{\sqrt{A}t}{2})+(1+2\epsilon)\sinh(\frac{\sqrt{A}t}{2})}{\cosh(\frac{\sqrt{A}t}{2})-(1-2\epsilon)\sinh(\frac{\sqrt{A}t}{2})}\right|^s e^{\frac{\sqrt{A}k^2}{\sigma(2-\gamma)^2}-2s\sqrt{A}t}$$
$$\times \exp\left\{\frac{-\sqrt{A}\epsilon(x^{1-\frac{\gamma}{2}}+k)^2(\cosh(\sqrt{A}t)+\epsilon\sinh(\sqrt{A}t))}{\sigma(2-\gamma)^2(1+2\epsilon^2(\cosh(\sqrt{A}t)-1)+2\epsilon\sinh(\sqrt{A}t))}\right\}$$
$$\times \exp\left\{\frac{1}{2\sigma}(F(X(\epsilon,x,t))-F(x))\right\}u_0(X(\epsilon,x,t)).$$

Further, (4.2) has a fundamental solution $p(t,x,y)$ such that

$$\int_0^\infty e^{-\lambda(y^{2-\gamma}+2ky^{1-\frac{\gamma}{2}})}u_0(y)p(t,x,y)dy = U_\lambda(t,x), \tag{4.58}$$

in which $U_\lambda(t,x) = \bar{U}_{\frac{\sigma(2-\gamma)^2\lambda}{\sqrt{A}}}(t,x)$. If $g=0$, then we may take $u_0=1$, and $\int_0^\infty p(t,x,y)dy = 1$ for the fundamental solution arising from this choice.

In this work we have focused on systems arising from separating real and imaginary parts of a single PDE involving a function f satisfying the Riccati equation (4.1) given in Theorem 4.0.1. However, using a similar argument to that we used in the previous section with Theorem 4.0.1 and applying it to either Theorem 4.2.1 or Theorem 4.2.1 will broaden the class of functions f_1 and f_2 for which we can compute fundamental solutions of the systems:

$$\begin{cases} v_t = \sigma x^\gamma v_{xx} + f_1(x)v_x - f_2(x)w_x \\ w_t = \sigma x^\gamma w_{xx} + f_2(x)v_x + f_1(x)w_x, \end{cases} \quad x \geq 0 \tag{4.59}$$

and

$$\begin{cases} v_t = \sigma x^\gamma v_{xx} + f_1(x)v_x - f_2(x)w_x - g_1(x)v + g_2(x)w \\ w_t = \sigma x^\gamma w_{xx} + f_2(x)v_x + f_1(x)w_x - g_2(x)v - g_1(x)w, \end{cases} \quad x \geq 0. \tag{4.60}$$

Observe that for the choice $g(x)=0$, the methodology described in the previous section applied to Theorem 4.2.1 will produce a fundamental matrix for the system (4.59) with functions f_1 and f_2 satisfying

$$\begin{cases} -\gamma\sigma x^{1-\gamma}f_1(x) + \sigma x^{2-\gamma}f_1'(x) + \frac{1}{2}x^{2(1-\gamma)}(f_1(x)^2 - f_2(x)^2) = \frac{A_1x^{4-2\gamma}}{2(2-\gamma)^2} + \frac{B_1x^{2-\gamma}}{2-\gamma} + C_1 \\ -\gamma\sigma x^{1-\gamma}f_2(x) + \sigma x^{2-\gamma}f_2'(x) + x^{2(1-\gamma)}f_1(x)f_2(x) = \frac{A_2x^{4-2\gamma}}{2(2-\gamma)^2} + \frac{B_2x^{2-\gamma}}{2-\gamma} + C_2, \end{cases}$$

while, if applied to Theorem 4.2.2, will produce a fundamental solution for the same system but with functions f_1 and f_2 satisfying

$$\begin{cases} \sigma x^{2-\gamma} f_1'(x) - \gamma\sigma x^{1-\gamma} f_1(x) + \frac{1}{2}x^{2(1-\gamma)}(f_1(x)^2 - f_2(x)^2) \\ \qquad\qquad = \frac{A_1 x^{4-2\gamma}}{2(2-\gamma)^2} + \frac{B_1 x^{3-\frac{3}{2}\gamma}}{3-\frac{3}{2}\gamma} + \frac{C_1 x^{2-\gamma}}{2-\gamma} - \kappa \\ \sigma x^{2-\gamma} f_2'(x) - \gamma\sigma x^{1-\gamma} f_2(x) + x^{2(1-\gamma)} f_1(x) f_2(x) = \frac{A_2 x^{4-2\gamma}}{2(2-\gamma)^2} + \frac{B_2 x^{3-\frac{3}{2}\gamma}}{3-\frac{3}{2}\gamma} + \frac{C_2 x^{2-\gamma}}{2-\gamma}. \end{cases}$$

In both these cases, $u_0 = 1$ will be a valid stationary solution that we can transform through the action of the symmetries described in Theorem 4.2.1 and Theorem 4.2.2 to then separate into real and imaginary parts.

Similarly, the choice $g(x) \neq 0$ will yield fundamental matrices for the system (4.60) with f_1 and f_2 satisfying either

$$\begin{cases} \sigma x^{2-\gamma}(f_1'(x) + 2g_1(x)) - \gamma\sigma x^{1-\gamma} f_1(x) + \frac{1}{2}x^{2(1-\gamma)}(f_1(x)^2 - f_2(x)^2) \\ \qquad\qquad = \frac{A_1 x^{4-2\gamma}}{2(2-\gamma)^2} + \frac{B_1 x^{2-\gamma}}{2-\gamma} + C_1 \\ \sigma x^{2-\gamma}(f_2'(x) + 2g_2(x)) - \gamma\sigma x^{1-\gamma} f_2(x) + x^{2(1-\gamma)} f_1(x) f_2(x) \\ \qquad\qquad = \frac{A_2 x^{4-2\gamma}}{2(2-\gamma)^2} + \frac{B_2 x^{2-\gamma}}{2-\gamma} + C_2, \end{cases} \tag{4.61}$$

when using Theorem 4.2.1, or

$$\begin{cases} \sigma x^{2-\gamma}(f_1'(x) + 2g_1(x)) - \gamma\sigma x^{1-\gamma} f_1(x) + \frac{1}{2}x^{2(1-\gamma)}(f_1(x)^2 - f_2(x)^2) \\ \qquad\qquad = \frac{A_1 x^{4-2\gamma}}{2(2-\gamma)^2} + \frac{B_1 x^{3-\frac{3}{2}\gamma}}{3-\frac{3}{2}\gamma} + \frac{C_1 x^{2-\gamma}}{2-\gamma} - \kappa \\ \sigma x^{2-\gamma}(f_2'(x) + 2g_2(x)) - \gamma\sigma x^{1-\gamma} f_2(x) + x^{2(1-\gamma)} f_1(x) f_2(x) \\ \qquad\qquad = \frac{A_2 x^{4-2\gamma}}{2(2-\gamma)^2} + \frac{B_2 x^{3-\frac{3}{2}\gamma}}{3-\frac{3}{2}\gamma} + \frac{C_2 x^{2-\gamma}}{2-\gamma} \end{cases} \tag{4.62}$$

if using Theorem 4.2.2 instead. In both these cases we must first find an analytic stationary solution $u_0(x)$ to transform according to the symmetry in Theorem 4.2.1 and Theorem 4.2.1.

Computation of fundamental matrices using the results in these two theorems is a line of future research we would like to explore.

Note. Since both Theorem 4.2.1 and Theorem 4.2.2 contain expressions in terms of the $\sqrt{A}$, it is convenient to write $\sqrt{A} = \tilde{A}_1 + \tilde{A}_2 i$ and then substitute $A = (\tilde{A}_1^2 - \tilde{A}_2^2) + 2\tilde{A}_1\tilde{A}_2 i$ when splitting the solution into real and imaginary parts.

Note. Observe that just like in the examples presented in the first section of this chapter, these cases cannot be handled by reducing to the heat equation in general. Craddock and his coauthors have shown that the Lie symmetry algebra of (4.2) is six dimensional (and hence the equation can be mapped to the heat equation) for exactly one choice of the constant C in equation (4.55).

Chapter 5

Wishart Processes and their Eigenvalues

In this chapter we formally introduce a particular type of stochastic matrix process, the Wishart process, focusing in particular on its eigenvalues. We show that while the usual Lie symmetry methods fail to produce a transition density function for the eigenvalues, the techniques we have developed can produce expressions for the expected value of a wide range of functionals of these eigenvalues. These techniques still rely heavily on the use of Lie symmetries and integral transforms, but they combine those with all sorts of results in the area of Mathematical Analysis.
If one tries to simulate the behaviour of the eigenvalues of a Wishart process using, for example, Monte-Carlo simulation techniques, these expectations we can obtain using our methodology can be used as our control variables. This is only an example of how Lie symmetry methods can be extremely useful even when the symmetries of the PDE are not enough to produce a transition density.

We will start this chapter introducing the theoretical notion of a Wishart process. We will then show how the usual symmetry analysis of the Kolmogorov Backwards equation associated to the Wishart process itself produces a set of symmetries that does not suffice to find a transition density function for the process.
We will then move on to study the stochastic process of the eigenvalues of a Wishart process. We will see how the methodology that has been used so far can be applied to study the Kolmogorov Backwards equation associated to this process but how some issues arise with boundary conditions and with solving a particular Sturm-Liouville problem for which we do not know the solution.
It is precisely in this context where we introduce a set of tools and methods that will allow us to compute the expected value of these eigenvalues, as well as all sorts of functionals of these. As pointed out before, knowledge of all these expected values gives us a good idea on how these processes behave, and provides us at the same time with a range of potential control variables for a Monte-Carlo simulation of these eigenvalues.

5.1 Introducing Wishart processes

Wishart processes were first introduced by Marie France Bru (see [12] or [11]) in the field of Biology as a tool to study the perturbation of experimental data but, since then, they have been studied theoretically by many authors such as Yor et al. [27] or Pfaffel [54]. Recently, great attention has been paid to these processes for their applications in Finance. It turns out that these processes are a great tool to model stochastic volatility. There are a great number of derivative pricing models that include Wishart processes as the stochastic volatility matrix. For example, Gourieroux and Sufana (see [35] or [36]) use this approach to create a multiasset analogue of the well known Cox-Ingersoll-Ross model. They provide in [36] an example of a multiasset extension of the Heston stochastic volatility model as well as an extension of the Merton model for credit risk analysis to a framework with stochastic firm liability, stochastic volatility, and several firms. In [30], Grasselli, Da Fonseca and Tebaldi further explore the use of a Wishart (multifactor) affine process to model the volatility of a single risky asset. In [31] they present the Wishart Affine Stochastic Correlation model (WASC) as the first analytically tractable model that is consistent with the apparent effects in typical market pricing of plain vanilla option prices while contemplating non-trivial stochastic volatility of asset returns and stochastic correlation of cross-sectional asset returns. Later on, in [29], Fonseca et al. discuss an estimation strategy for this WASC model.
In [2], Asai et al. provide an extensive review of the literature on Multivariate Stochastic Volatility (MSV) models, in most of which one can see that Wishart processes play an essential role. Other work that relies heavily on the use of such processes includes for example that by Leung et al. in [48].

It is then clear that this particular type of matrix processes seem to be a very convenient while still rather realistic way of modelling stochastic volatility in modern financial models. Hence the importance of understanding their behaviour as well as their main properties.

One may typically define a matrix Wishart process as follows:

Definition 5.1.1. Let $n, p \in \mathbb{Z}^+$ (not including 0). Consider the $n \times p$ matrix W_t, whose elements are independent scalar valued Brownian motions and whose initial state is $W_0 = C$. A Wishart process $S = \{S_t, t \geq 0\}$ of dimension p, index n and initial state S_0 (denoted $S_t \sim WIS(n, p, S_0)$) is defined as

$$S_t = W_t^\top W_t, \text{ with } S_0 = C^\top C$$

Moreover, it can be seen that such process S_t satisfies the SDE

$$dS_t = n\ I\ dt + dW_t^\top \sqrt{S_t} + \sqrt{S_t} dW_t \tag{5.1}$$

Using the common tools of Itô calculus one can compute the infinitesimal generator for such process to obtain the following:

Proposition 5.1.1. *[11] Let $X_t \sim WIS(n, p, X_0)$. The infinitesimal generator of this process is*

$$A_X = Tr[nD + 2XD^2],\ \textit{with}\ D = (D_{ij}) = (\partial/\partial X_{ij}), \quad X \in \mathcal{S}^p, \tag{5.2}$$

where $\mathcal{S}^p$ denotes the set of all symmetric $p \times p$ matrices.

Wishart processes possess the following *additivity property*:

Proposition 5.1.2. *Let (X_t) and (Y_t) be two independent Wishart processes $X_t \sim WIS(n, p, X_0)$ and $Y_t \sim WIS(m, p, Y_0)$ respectively. Then the process $(Z_t) := (X_t + Y_t)$ is also a Wishart process with $Z_t \sim WIS(n + m, p, X_0 + Y_0)$.*

Proof. Let $X_t = W_t^\top W_t$ and $Y_t = V_t^\top V_t$ with W_t and V_t independent Brownian motions of dimension $n \times p$ and $m \times p$ respectively. Then, it clearly follows that

$$U_t = \begin{pmatrix} W_t \\ V_t \end{pmatrix}$$

is an $(n + m) \times p$ matrix of independent Brownian motions, and

$$Z_t = X_t + Y_t = W_t^\top W_t + V_t^\top V_t = U_t^\top U_t$$

□

It is well known that Wishart processes can be generalized to processes with a non-integer index α and they naturally conserve the above additivity property. However, we will only consider Wishart processes of integer index throughout this work.

Just like for any type of matrix process, the behaviour of the eigenvalues will have a major impact on the overall properties of the Wishart process. There are some interesting results by Bru ([11]) on these eigenvalues and their characteristics. Amongst the most important of those properties is the non-colliding property of the eigenvalues:

Theorem 5.1.3. *[11] Let $S_t \sim WIS(n, p, S_0)$, with $n \geq p$. If at time $t = 0$ the p eigenvalues of $S_0 = C^\top C$ are distinct, labeled*

$$\lambda_1(0) > \cdots > \lambda_p(0) \geq 0$$

then for all $t \geq 0$, the p eigenvalues of S_t are distinct

$$\lambda_1(t) > \cdots > \lambda_p(t) \geq 0 \quad a.s$$

the process $(\lambda_1(t), \dots, \lambda_p(t))$ is a diffusion, solution of the stochastic differential system

$$d\lambda_i = 2\sqrt{\lambda_i} dB_i + \left(n + \sum_{j \neq i} \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j} \right) dt, \quad i = 1, \dots, p, \tag{5.3}$$

where $B_1(t), \dots, B_p(t)$ are p independent Brownian motions, adapted to the natural filtration $(\mathcal{F}_t)_{t \geq 0}$ associated to the process (S_t).

In the rest of this chapter we will try to obtain information about these processes and their eigenvalues by drawing upon the Lie symmetries of the Kolmogorov Backwards equation linked to the Itô diffusions defined by (5.1) and (5.3).

5.2 Limitations of the existing techniques

Let us first focus on the study of the associated Kolmogorov backwards equation for a 2×2 Wishart process for the sake of simplicity. Let $W_t = (W_{ij})_t \sim WIS(n, 2, W_0)$, with $i, j \in \{1, 2\}$. Then, the infinitesimal generator of this process can easily be calculated to be

$$\begin{aligned} \mathcal{A}_W = n \left(\frac{\partial}{\partial W_{11}} + \frac{\partial}{\partial W_{22}} \right) + 2W_{11} \left(\frac{\partial^2}{\partial W_{11}^2} + \frac{\partial^2}{\partial W_{12}^2} \right) \\ + 4W_{12} \left(\frac{\partial^2}{\partial W_{11} \partial W_{12}} + \frac{\partial^2}{\partial W_{12} \partial W_{22}} \right) + 2W_{22} \left(\frac{\partial^2}{\partial W_{12}^2} + \frac{\partial^2}{\partial W_{22}^2} \right) \end{aligned} \tag{5.4}$$

Note that we have expressed this generator in terms of W_{11}, W_{12} and W_{22} only, since we know that our matrix W is symmetric and so $W_{12} = W_{21}$. This means that, effectively, we are only dealing with 3 variables instead of 4. Let us rename these variables $(W_{11}(t), W_{12}(t), W_{22}(t)) \mapsto (X_t, Y_t, Z_t)$ to simplify notation. Then, if we let $u(x, y, z, t) = E^{x,y,z}[f(X_t, Y_t, Z_t)]$, the Kolmogorov Backwards equation theorem (Theorem 2.2.2) yields the following Cauchy problem for u:

$$u_t = n(u_x + u_z) + 2xu_{xx} + 2zu_{zz} + (2x + 2z)u_{yy} + 4y(u_{xy} + u_{yz}), \tag{5.5}$$

$$u(x,y,z,0) = f(x,y,z)$$

The computation of the symmetries of this PDE is a difficult task. This particular case yields a rather complex system of determining equations[1] that produce only a very trivial set of symmetries that is not enough to work with if we wish to find fundamental solutions of (5.5) via integral transform methods. These trivial symmetries we obtain are only the following:

Proposition 5.2.1. *Let $u(x,y,z,t)$ be a solution of the PDE (5.5). Then*

$$\begin{cases} u_1(x,y,z,t) = u(x,y,z,t-\epsilon) \\ u_2(x,y,z,t) = u(e^{-\epsilon}x, e^{-\epsilon}y, e^{-\epsilon}z, e^{-\epsilon}t) \\ u_v(x,y,z,t) = u(x,y,z,t) + \epsilon v(x,y,z,t), \end{cases}$$

where $v(x,y,z,t)$ is any solution of (5.5), are also solutions of such equation.

The reader can check that reduction of (5.5) under the symmetry u_2 leads to a 3-dimensional PDE that, however, we cannot solve.

Therefore, since we come across the obstacle that we cannot obtain enough symmetries to find the desired fundamental solution, one might try to turn to the study of the eigenvalues instead. The symmetry analysis of the Kolmogorov backwards equation resulting from the generator of the SDE (5.3) will potentially provide us with a better understanding of how these matrix processes behave. It is well known that the eigenvalues of a matrix are one of their most characteristic features. They are inherent to the matrix and invariant under changes of basis.

Let $\mu_t = (\mu_1(t), \dots, \mu_p(t))$ be the vector of the eigenvalues of a $p \times p$ Wishart process with index $n \geq p$, such that all the eigenvalues μ_i are distinct and ordered from largest to smallest. Then the SDE (5.3) reads as

$$\underbrace{\begin{pmatrix} d\mu_1 \\ \vdots \\ d\mu_p \end{pmatrix}}_{d\mu_t} = 2 \underbrace{\begin{pmatrix} \sqrt{\mu_1} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sqrt{\mu_p} \end{pmatrix}}_{\sigma(\mu_t)} \begin{pmatrix} dB_1 \\ \vdots \\ dB_p \end{pmatrix} + \underbrace{\begin{pmatrix} n + \sum_{j \neq 1} \frac{\mu_1 + \mu_j}{\mu_1 - \mu_j} \\ \vdots \\ n + \sum_{j \neq p} \frac{\mu_p + \mu_j}{\mu_p - \mu_j} \end{pmatrix}}_{b(\mu_t)} dt, \tag{5.6}$$

[1] This system of determining equations can actually be simplified by introducing the new variables

$$t = 4T, \quad x = X^2, \quad y = Y^2, \quad z = Z^2, \quad u = U$$

hence the so called drift function is

$$b(\mu_t) = \begin{pmatrix} n + \sum_{j\neq 1} \frac{\mu_1+\mu_j}{\mu_1-\mu_j} \\ \vdots \\ n + \sum_{j\neq p} \frac{\mu_p+\mu_j}{\mu_p-\mu_j} \end{pmatrix}$$

and the diffusion function is

$$(\sigma\sigma^T)(\mu_t) = \begin{pmatrix} 2\sqrt{\mu_1} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 2\sqrt{\mu_p} \end{pmatrix} \begin{pmatrix} 2\sqrt{\mu_1} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 2\sqrt{\mu_p} \end{pmatrix}$$

$$= \begin{pmatrix} 4\mu_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 4\mu_p \end{pmatrix}$$

These expressions for the drift and diffusion functions can be used to compute the generator for the process μ_t via the formula (2.77). Then, taking $u(\lambda, t) = E^\lambda[\phi(\mu_t)] := E[\phi(\mu_t)|\mu_0 = \lambda]$, Kolmogorov's backward equation (2.2.2) for our diffusion process yields the following Cauchy problem:

$$u_t(\lambda, t) = \sum_i \left(n + \sum_{j\neq i} \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j} \right) u_{\lambda_i}(\lambda, t) + \sum_i 2\lambda_i u_{\lambda_i\lambda_i}(\lambda, t) \tag{5.7}$$

$$u(\lambda, 0) = \phi(\lambda); \quad \lambda \in \mathbb{R}^p$$

Our aim this time will be to try to find a fundamental solution to the above equation. In particular, we are interested in finding the transition density for the eigenvalues of a Wishart process.

5.2.1 Classical integral transform and Lie symmetry methods: An infinite series expansion for the transition densities of the eigenvalues of a $p \times p$ Wishart process

In this section, we show how the existing results relying on the symmetry analysis of the Cauchy problem (5.7) potentially produce an infinite series expression for the transition densities of the eigenvalues of a Wishart process in terms of a set of eigenvalues and eigenfunctions of a particular Sturm-Liouville problem. We show that, however, some issues arise with the boundary conditions. Moreover, the Sturm-Liouville problem that can be obtained through these methods is a rather complicated one and one for which we fail to obtain an exact solution. We first consider the case of a 2-dimensional Wishart process and then move on to the generalisation to a process of dimension p. In both cases, however, we need to first transform the Cauchy problem in (5.7) to one of a more convenient form through a series of changes of variables.

Let us start by writing $\lambda_i = \frac{\bar{\lambda}_i^2}{2}$ for $1 \leq i \leq p$. With this transformation, (5.7) becomes

$$u_t = \sum_i \left(n + \sum_{j \neq i} \frac{2\bar{\lambda}_j^2}{\bar{\lambda}_i^2 - \bar{\lambda}_j^2} \right) \frac{u_{\bar{\lambda}_i}}{\bar{\lambda}_i} + \underbrace{\sum_i u_{\bar{\lambda}_i \bar{\lambda}_i}}_{\Delta u} \tag{5.8}$$

$$u(\bar{\lambda}, 0) = \phi\left(\frac{\bar{\lambda}^2}{2}\right); \quad \bar{\lambda} \in \mathbb{R}^n$$

Next, define the function

$$\rho(\bar{\lambda}_1, \ldots, \bar{\lambda}_p) = -\frac{1}{2}\left[(n - 2(p-1)) \sum_{i=1}^{p} \log(\bar{\lambda}_i) + \sum_{i=1}^{p-1} \sum_{j>i} \log(\bar{\lambda}_i^2 - \bar{\lambda}_j^2)\right] \tag{5.9}$$

and let $u(\bar{\lambda}_1, \ldots, \bar{\lambda}_p, t) = e^{\rho(\bar{\lambda}_1, \ldots, \bar{\lambda}_p)} v(\bar{\lambda}_1, \ldots, \bar{\lambda}_p, t)$. Then the condition that u satisfies (5.8) is equivalent to the condition that the function v satisfies the following:

$$v_t = \Delta v - \frac{1}{4}\left[C \sum_{i=1}^{p} \frac{1}{\bar{\lambda}_i^2} - 2\left(\sum_{i=1}^{p-1} \sum_{j>i} \left\{ \frac{1}{(\bar{\lambda}_i + \bar{\lambda}_j)^2} + \frac{1}{(\bar{\lambda}_i - \bar{\lambda}_j)^2} \right\}\right)\right] v, \tag{5.10}$$

$$v(\bar{\lambda}, 0) = e^{-\rho(\bar{\lambda})} \phi\left(\frac{\bar{\lambda}^2}{2}\right) := \psi(\bar{\lambda}); \quad \bar{\lambda} \in \mathbb{R}^n$$

where $C = (n - 2p)(n - 2(p-1))$.

Finally, for the sake of simplicity, let us go back to our initial notation and say that the problem we need to solve is

$$u_t = \Delta u - \frac{1}{4}\left[C \sum_{i=1}^{p} \frac{1}{\lambda_i^2} - 2\left(\sum_{i=1}^{p-1} \sum_{j>i} \left\{ \frac{1}{(\lambda_i + \lambda_j)^2} + \frac{1}{(\lambda_i - \lambda_j)^2} \right\}\right)\right] u, \tag{5.11}$$

$$u(\lambda, 0) = \psi(\lambda); \quad \lambda \in \mathbb{R}^n$$

with $C = (n-2p)(n-2(p-1))$. One need only revert the changes of variables to get the solution to the initial problem.

Observe that the above equation can be written as

$$u_t = \Delta u + \frac{1}{\lambda_1^2} K\left(\frac{\lambda_2}{\lambda_1}, \dots, \frac{\lambda_p}{\lambda_1}\right) u, \tag{5.12}$$

with

$$K(\xi_1, \dots, \xi_{p-1}) = -\frac{C}{4}\left(1 + \sum_{i=1}^{p-1} \frac{1}{\xi_i^2}\right) + \frac{1}{2}\left(\sum_{i=1}^{p-1}\left\{\frac{1}{(1+\xi_i)^2} + \frac{1}{(1-\xi_i)^2}\right\} + \underbrace{\sum_{i=1}^{p-2}\sum_{j>i}\left\{\frac{1}{(\xi_i+\xi_j)^2} + \frac{1}{(\xi_i-\xi_j)^2}\right\}}_{(*)}\right),$$

$C = (n-2p)(n-2(p-1))$, and where the term $(*)$ vanishes for $p = 2$. From this form of our problem, we will proceed to discuss the 2×2 case and the general case for a p-dimensional Wishart process separately.

5.2.1.1 The 2×2 case

Let $W_t \sim WIS(n, 2, W_0)$ with eigenvalues $\mu_1(t) > \mu_2(t) \geq 0$ and with index $n \geq 2$. Let

$$\begin{aligned}\tilde{u}((\lambda_1, \lambda_2), t) &= E^{(\lambda_1,\lambda_2)}[\phi((\mu_1(t), \mu_2(t)))] \\ &:= E[\phi((\mu_1(t), \mu_2(t)))|(\mu_1(0), \mu_2(0)) = (\lambda_1, \lambda_2)],\end{aligned}$$

Then recall that the Cauchy problem that such function $\tilde{u}$ must satisfy can be recast as follows, according to (5.12):

$$u_t = \Delta u + \frac{1}{\lambda_1^2} K\left(\frac{\lambda_2}{\lambda_1}\right) u, \tag{5.13}$$

$$u((\lambda_1, \lambda_2), 0) = \psi((\lambda_1, \lambda_2)),$$

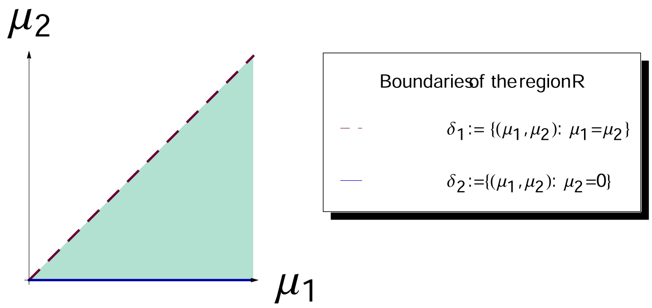
where

$$\begin{cases} K\left(\frac{\lambda_2}{\lambda_1}\right) & := -\frac{1}{4}\left[(n-4)(n-2)\left(1+\frac{1}{\left(\frac{\lambda_2}{\lambda_1}\right)^2}\right) - 2\left(\frac{1}{\left(1+\frac{\lambda_2}{\lambda_1}\right)^2} + \frac{1}{\left(1-\frac{\lambda_2}{\lambda_1}\right)^2}\right)\right] \\ \psi((\lambda_1,\lambda_2)) & := e^{-\rho((\lambda_1,\lambda_2))}\phi\left(\frac{(\lambda_1,\lambda_2)^2}{2}\right) \\ \rho((\lambda_1,\lambda_2)) & := -\frac{1}{2}\left[(n-2)(\log(\lambda_1)+\log(\lambda_2)) + \log(\lambda_1^2-\lambda_2^2)\right]. \end{cases}$$

Transform this problem to its equivalent in the ordinary polar coordinates by letting

$$\begin{cases} \lambda_1 & = r\cos\theta \\ \lambda_2 & = r\sin\theta, \end{cases}$$

with $r > 0$ and $\theta \in [0, \frac{\pi}{4})$, since we must have that $\mu_1(t) > \mu_2(t) \geq 0$. We are essentially considering our equation in the following region for the "spatial" variables:



The reader may check that letting $v(r,\theta,t) = u(r\cos\theta, r\sin\theta, t) = u(\lambda_1,\lambda_2,t)$, our equation can be rewritten as

$$v_t(r,\theta,t) = v_{rr}(r,\theta,t) + \frac{1}{r}v_r(r,\theta,t) + \frac{1}{r^2}v_{\theta\theta}(r,\theta,t) + \frac{1}{r^2}K(\theta)\,v(r,\theta,t), \qquad (5.14)$$

with $K(\theta) := -\frac{1}{4}\left(\frac{(n-4)(n-2)}{\sin^2\theta\cos^2\theta} - \frac{4}{(2\cos^2\theta-1)^2}\right)$, $r > 0$, $\theta \in [0, \frac{\pi}{4})$. Similarly, the initial condition is

$$v(r,\theta,0) = \psi(r\cos\theta, r\sin\theta) := \delta(r,\theta).$$

It turns out that Craddock and Lennox [21] managed to compute an infinite series expansion for a fundamental solution of an equation of the type (5.14) using Lie symmetry methods. They propose the following theorem:

Theorem 5.2.2. *[21] Suppose that K is continuous and that the Sturm–Liouville problem*

$$L''(\theta) + (K(\theta) + \lambda)L(\theta) = 0 \tag{5.15a}$$

$$\alpha_1 L(a) + \alpha_2 L'(a) = 0 \tag{5.15b}$$

$$\beta_1 L(b) + \beta_2 L'(b) = 0, \tag{5.15c}$$

has a complete set of eigenfunctions and eigenvalues, and that the eigenvalues are all positive. Consider the initial and boundary value problem

$$u_t = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + \frac{K(\theta)}{r^2}u, \tag{5.16}$$

$$r > 0, \quad a \leq \theta \leq b, \quad a, b \in [0, 2\pi], \tag{5.17}$$

$$u(r, \theta, 0) = f(r, \theta), \quad f \in \mathcal{D}(\Omega), \tag{5.18}$$

$$\alpha_1 u(r, a, t) + \alpha_2 u_\theta(r, a, t) = 0 \tag{5.19}$$

$$\beta_1 u(r, b, t) + \beta_2 u_\theta(r, b, t) = 0. \tag{5.20}$$

Here $\Omega = [0, \infty) \times [a, b]$ in polar coordinates. Then there is a solution of the form

$$u(r, \theta, t) = \int_0^\infty \int_a^b f(\rho, \phi) p(t, r, \theta, \rho, \phi) \rho d\phi d\rho, \tag{5.21}$$

where

$$p(t, r, \theta, \rho, \phi) = \frac{1}{2t} \exp\left(-\frac{r^2 + \rho^2}{4t}\right) \sum_n \overline{L_n(\phi)} L_n(\theta) I_{\sqrt{\lambda_n}}\left(\frac{r\rho}{2t}\right), \tag{5.22}$$

in which $L_n(\theta)$, λ_n, $n = 1, 2, 3, \ldots$ are the normalized eigenfunctions and corresponding eigenvalues for the given Sturm-Liouville problem.

Hence, if we set some boundary conditions of the form (5.19)-(5.20) for our problem, i.e.

$$\begin{cases} \alpha_1 v(r, 0, t) + \alpha_2 v_\theta(r, 0, t) = 0 \\ \beta_1 v(r, \frac{\pi}{4}, t) + \beta_2 v_\theta(r, \frac{\pi}{4}, t) = 0 \end{cases} \quad \alpha_i, \beta_i \quad \text{constants for} \quad i \in \{1, 2\}, \tag{5.23}$$

we need to consider the Sturm-Liouville problem given by

$$L"(\theta) + (K(\theta) + \nu)L(\theta) = 0 \tag{5.24}$$

$$\alpha_1 L(0) + \alpha_2 L'(0) = 0 \tag{5.25}$$

$$\beta_1 L(\pi/4) + \beta_2 L'(\pi/4) = 0. \tag{5.26}$$

If one can prove that the above problem satisfies the conditions in Theorem 5.2.2 (i.e. that the S-L problem has a complete set of eigenfunctions and eigenvalues, and the eigenvalues are all positive), then this theorem gives that there exists a solution of (5.14) in the considered region and with the chosen boundary conditions (5.23) of the form

$$v(r,\theta,t) = \int_0^\infty \int_0^{\frac{\pi}{4}} \delta(\rho,\phi) p(t,r,\theta,\rho,\phi)\rho d\phi d\rho, \tag{5.27}$$

where

$$p(t,r,\theta,\rho,\phi) = \frac{1}{2t}\exp\left(-\frac{r^2+\rho^2}{4t}\right)\sum_k \overline{L_k(\phi)} L_k(\theta) I_{\sqrt{\nu_k}}\left(\frac{r\rho}{2t}\right). \tag{5.28}$$

In the expression (5.28) of the fundamental solution $p(t,r,\theta,\rho,\phi)$, $L_k(\theta)$, ν_k, $k = 1,2,3,\ldots$ are the normalized eigenfunctions and corresponding eigenvalues for the Sturm Liouville problem (5.24).

That is, one can potentially obtain fundamental solutions to our initial problem by solving (5.24) and later reversing the changes of variables applied to the initial PDE. However, there are a few questions that need to be addressed in order to be able to apply Craddock and Lennox's theorem:

- The function $K(\theta)$ is continuous everywhere in the interior of our region R and asymptotically approaches $\pm\infty$ at the boundaries $\delta_1 = \{(r,\theta) : \theta = \pi/4\}$ and $\delta_2 = \{(r,\theta) : \theta = 0\}$ respectively. $\longrightarrow$ This problem could be addressed by considering the region $\bar{R} := \{(r,\theta) : r > 0, \epsilon \le \theta \le \frac{\pi}{4} - \epsilon\}$ instead, and then trying to extrapolate the results to the region R by letting $\epsilon \to 0$. This might, of course, lead to potential divergence problems when taking limits, so one would need to pay special attention to this.

- To determine the boundary conditions (5.23) one must have some knowledge of the behaviour of the eigenvalues of a Wishart process and transform the conditions from the initial variables in (5.7) to those in the transformed problem (5.13). Setting particular conditions to those eigenvalues might lead to complicated boundary conditions for the problem in polar coordinates, so more research needs to be done in regard to what kind of conditions we can impose if we wish to be able to solve the problem.

We have not yet been able to solve the Sturm-Liouville problem (5.24) analytically, but potentially the eigenfunctions $L_k(\theta)$ and the eigenvalues ν_k in (5.28) can be approximated numerically. All these considerations are potential future work on this topic. More research in this direction is needed to try to extend the scope of this result.

Note that Craddock and Lennox obtain the results stated in Theorem 5.2.2 [21] trough the use of Lie symmetry methods. Although these methods provide a theoretical result if one can overcome the difficulties highlighted above, it does no seem to be a very effective approach for practical purposes in our particular case.

5.2.1.2 The general $p \times p$ case

An extension of the work presented above to the general p-dimensional case is introduced in this section through a generalisation to higher dimensions of Theorem 5.2.2 by Craddock and Lennox. This higher dimensional version of Theorem 5.2.2 is due to the same authors and can be found in [21]:

Theorem 5.2.3. *Consider the equation*

$$u_t = u_{rr} + \frac{n-1}{r}u_r + \frac{1}{r^2}(\Delta_{S^{n-1}} + G(\Theta))u, \tag{5.29}$$

$$u(r,\Theta,0) = f(r,\Theta), \tag{5.30}$$

and $\alpha(\Theta)\Psi(\Theta) + (1-\alpha(\Theta))\partial\Psi/\partial n = 0$*, with* α *a continuous function and* $\partial\Psi/\partial n$ *the normal derivative on the surface of the unit sphere* S^{n-1} *of dimension* $n-1$*. Here* Ψ *is* u *restricted to* S^{n-1}*,* $\Delta_{S^{n-1}}$ *is the Laplace–Beltrami operator on the sphere and* $f \in \mathcal{D}(\mathbb{R}^n)$*. Let* $\Theta = (\theta, \phi_1, \ldots, \phi_{m-2})$*. Then there is a solution of the form*

$$U(r,\Theta,t) = \int_0^\infty \int_{S^{n-1}} f(\rho,\xi)p(t,r,\Theta,\rho,\xi)\rho d\xi d\rho, \tag{5.31}$$

where for $n \geq 2$*,*

$$p(t,r,\Theta,\rho,\xi) = \frac{1}{2t}\left(\frac{\rho}{r}\right)^{\frac{n}{2}-1} \exp\left(-\frac{r^2+\rho^2}{4t}\right)\sum_{\lambda_m} L_m(\Theta)\overline{L_m(\xi)}I_{\mu(m)}\left(\frac{r\rho}{2t}\right), \tag{5.32}$$

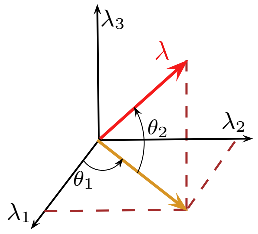
where $\mu(m) = \frac{1}{2}\sqrt{4\lambda_m + (n-2)^2}$ *and* $L_m(\Theta)$ *are normalized eigenfunctions of the problem* $\Delta_S L + (\lambda + G)L = 0$ *and* λ_m *are the eigenvalues.*

According to the above theorem, and similarly to the 2×2 case, we can find a series expression for the fundamental solution of an equation of the type (5.29) in terms of the eigenvalues and eigenfunctions of an appropriate Sturm-Liouville problem. In what follows, we show how equation (5.12) can be recast into one of the form considered in the above theorem, and we discuss what difficulties we come across in the process.

Consider equation (5.12) in the general p-dimensional case and let

$$\begin{cases} \lambda_1 &= r\prod_{k=1}^{p-1}\cos(\theta_k) \\ \lambda_i &= r\sin(\theta_{i-1})\prod_{k=i}^{p-1}\cos(\theta_k), \quad i = 2,\dots,p-1 \\ \lambda_p &= r\sin(\theta_{p-1}) \end{cases}$$

Note that, with this notation, $r > 0$ is the radius and the angle θ_i, for $i = 1,\dots,p-2$, is the angle from the projection of λ onto the hyperplane in $\mathbb{R}^{i+1}$ generated by $\{\lambda_1,\dots,\lambda_i\}$ to the projection of λ onto the hyperplane in $\mathbb{R}^{i+2}$ generated by $\{\lambda_1,\dots,\lambda_i+1\}$. Similarly, the angle θ_{p-1} is the angle from the projection of λ onto the hyperplane in $\mathbb{R}^p$ generated by $\{\lambda_1,\dots,\lambda_{p-1}\}$ to the vector λ itself. To illustrate, see figure [5.2.1.2] for the 3×3 case.



Note that this definition for the 3×3 case does not correspond to the usual spherical coordinates definition.

The reader may check that the following conditions must be imposed on the angles

$$\begin{cases} \theta_1 &\in (0,\pi/4) \\ \theta_i &\in (0,\arctan(\sin(\theta_{i-1})), \quad i = 2,\dots,p-2 \\ \theta_{p-1} &\in [0,\arctan(\sin(\theta_{p-2})), \end{cases}$$

since we must have $\mu_1(t) > \cdots > \mu_p(t) \geq 0$.

Putting

$$\begin{aligned} v(r,\theta_1,\dots,\theta_{p-1},t) &= u(\lambda_1(r,\theta_1,\dots,\theta_{p-1}),\dots,\lambda_p(r,\theta_1,\dots,\theta_{p-1}),t) \\ &= u(r\prod_{k=1}^{p-1}\cos(\theta_k),\dots,r\sin(\theta_{p-1}),t) \end{aligned}$$

transforms the problem (5.11) into the following problem for v:

$$v_t = v_{rr} + \frac{p-1}{r} v_r + \frac{1}{r^2}\sum_{k=1}^{p-1}\left(\prod_{j=k+1}^{p-1}\sec^2(\theta_j)\right) v_{\theta_k\theta_k}$$

$$-\frac{1}{r^2}\sum_{k=1}^{p-1}\left((k-1)\tan(\theta_k)\prod_{j=k+1}^{p-1}\sec^2(\theta_j)\right) v_{\theta_k} + \frac{1}{r^2}K\left(\theta_1,\dots,\theta_{p-1}\right)v, \tag{5.33}$$

$$= v_{rr} + \frac{p-1}{r}v_r + \frac{1}{r^2}(L+K(\Theta))v, \tag{5.34}$$

where L is the differential operator defined by

$$L = \sum_{k=1}^{p-1}\left(\prod_{j=k+1}^{p-1}\sec^2(\theta_j)\right)\left(\frac{\partial^2}{\partial\theta_k^2} - (k-1)\tan(\theta_k)\frac{\partial}{\partial\theta_k}\right)$$

and

$$K(\Theta) := K\left(\theta_1,\dots,\theta_{p-1}\right) := -\frac{C}{4}\sum_{k=1}^{p}\left(\csc^2(\theta_{k-1})\prod_{j=k}^{p-1}\sec^2(\theta_j)\right)$$

$$+\sum_{k=1}^{p-1}\left[\prod_{l=k+1}^{p-1}\sec^2(\theta_l)\left(\frac{\sin^2(\theta_k)+\prod_{j=1}^{k}\cos^2(\theta_j)}{\left(\sin^2(\theta_k)-\prod_{j=1}^{k}\cos^2(\theta_j)\right)^2}\right)\right]$$

$$+\sum_{k=1}^{p-1}\left[\prod_{l=k+1}^{p-1}\sec^2(\theta_l)\left(\sum_{j<k}\frac{\sin^2(\theta_k)+\sin^2(\theta_j)\prod_{i=j+1}^{k}\cos^2(\theta_i)}{\left(\sin^2(\theta_k)-\sin^2(\theta_j)\prod_{i=j+1}^{k}\cos^2(\theta_i)\right)^2}\right)\right],$$

with $C = (n-2p)(n-2(p-1))$, $r > 0$ and

$$\begin{cases} \theta_1 & \in (0,\pi/4) \\ \theta_i & \in (0,\arctan(\sin(\theta_{i-1}))), \quad i = 2,\dots,p-2 \\ \theta_{p-1} & \in [0,\arctan(\sin(\theta_{p-2})) \end{cases}$$

Moreover, the initial condition in (5.11) can be written in terms of v as:

$$v(r,\theta_1,\dots,\theta_{p-1},0) = \psi(r\prod_{k=1}^{p-1}\cos(\theta_k),\dots,r\sin(\theta_{p-1})) := \delta(r,\theta_1,\dots,\theta_{p-1}).$$

In a similar way as for the 2-dimensional case in the previous section, this problem can be compared to that considered by Craddock and Lennox in theorem 5.2.3.

That is, ideally, one could obtain a series expansion for the fundamental solution of PDE (5.33) in terms of the eigenvalues and eigenfunctions of a particular Sturm-Liouville problem. However, just as in the 2×2 case, the boundary conditions that should be imposed for this problem are not yet clear and we haven't been able to analytically solve the associated Sturm-Liouville problem. Further course of study could be to try to solve this problem numerically.

5.3 An alternative approach to the study of the eigenvalues

In this section we focus mainly in the 2-dimensional case for the sake of simplicity, although we do provide some results for general p-dimensional Wishart processes.

We first present the results of the lie symmetry study of the Kolmogorov backwards equation associated to the eigenvalues of a Wishart process of dimension 2 and then we introduce our approach to the study of these eigenvalue processes.

Let $W_t \sim WIS(n, 2, W_0)$ with $n \geq 2$ and with eigenvalues $\mu_1(t) > \mu_2(t) \geq 0$. Let

$$u(\lambda_1, \lambda_2, t) = E^{\lambda_1,\lambda_2}[\phi(\mu_1(t), \mu_2(t))] := E[\phi((\mu_1(t), \mu_2(t))|\mu_1(0) = \lambda_1, \mu_2(0) = \lambda_2]$$

and recall that Kolmogorov's backwards equation theorem yields the following Cauchy problem for the function u:

$$u_t = \left(n + \frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2}\right) u_{\lambda_1} + \left(n - \frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2}\right) u_{\lambda_2} + 2\lambda_1 u_{\lambda_1\lambda_1} + 2\lambda_2 u_{\lambda_2\lambda_2} \tag{5.35}$$

$$u(\lambda, 0) = \phi(\lambda); \quad \lambda \in \mathbb{R}^2$$

The reader may check that Lie's method for the systematic computation of symmetries of the above PDE (described in Chapter 2) yields the following result:

Proposition 5.3.1. *Let $u(\lambda_1, \lambda_2, t)$ be a solution of the PDE (5.35). Then*

$$\begin{cases} u_1(\lambda_1, \lambda_2, t) = u(\lambda_1, \lambda_2, t - \epsilon) \\ u_2(\lambda_1, \lambda_2, t) = u(e^{-\epsilon}\lambda_1, e^{-\epsilon}\lambda_2, e^{-\epsilon}t) \\ u_3(\lambda_1, \lambda_2, t) = u\left(\frac{\lambda_1}{(1+\epsilon t)^2}, \frac{\lambda_2}{(1+\epsilon t)^2}, \frac{t}{(1+\epsilon t)}\right) \exp\left(-\frac{\epsilon}{2}\frac{(\lambda_1+\lambda_2)}{(1+\epsilon t)}\right)(1+\epsilon t)^{-n} \\ u_4(\lambda_1, \lambda_2, t) = e^{\epsilon} u(\lambda_1, \lambda_2, t) \\ u_B(\lambda_1, \lambda_2, t) = u(\lambda_1, \lambda_2, t) + \epsilon B(\lambda_1, \lambda_2, t), \end{cases}$$

are also solutions of such equation. Here $B(\lambda_1, \lambda_2, t)$ is any solution of (5.35).

Note. This result follows from the exponentiation of the following vector fields

$$\begin{cases} \mathbf{v}_1 = \frac{\partial}{\partial t} \\ \mathbf{v}_2 = t\frac{\partial}{\partial t} + \lambda_1\frac{\partial}{\partial \lambda_1} + \lambda_2\frac{\partial}{\partial \lambda_2} \\ \mathbf{v}_3 = t^2\frac{\partial}{\partial t} + 2\lambda_1 t\frac{\partial}{\partial \lambda_1} + 2\lambda_2 t\frac{\partial}{\partial \lambda_2} + u\left(-\frac{\lambda_1}{2} - \frac{\lambda_2}{2} - nt\right)\frac{\partial}{\partial u} \\ \mathbf{v}_4 = u\frac{\partial}{\partial u} \\ \mathbf{v}_B = B(\lambda_1, \lambda_2, t)\frac{\partial}{\partial u}, \end{cases} \tag{5.36}$$

which are a spanning set for the lie algebra of (5.35).

It turns out that the application of the integral transform methods introduced in Chapter 2 fails to produce the transition density for the eigenvalues $\mu_1(t)$ and $\mu_2(t)$. These methods do, however, successfully produce the transition density for the sum $\mu_1(t) + \mu_2(t)$, i.e. the trace of the Wishart process. This transition density is not a new result, since the distribution of the trace of a Wishart process is known to be that of a squared-Bessel process, but it does provide an alternative proof for this well-known result.

Recall the following theorem for the probability density function for a squared-Bessel distribution:

Theorem 5.3.2. *Let $\rho^2 \sim BESQ^\delta(x)$. The probability density function for ρ^2 is*

$$f_t^\delta(x, y) = \frac{1}{2t}\left(\frac{y}{x}\right)^{\frac{\nu}{2}} e^{-\frac{(x+y)}{2t}} I_\nu\left(\frac{\sqrt{xy}}{t}\right) 1_{\{y>0\}}, \quad \nu = \frac{\delta}{2} - 1$$

Let us provide a new proof, based on lie symmetries and integral transforms, for the following well-known result:

Theorem 5.3.3. *The sum of the eigenvalues of a Wishart process is a squared-Bessel process with parameter $\nu = n - 1$ (or, equivalently $\delta = 2n$)*

Proof. Observe that $u(\lambda_1, \lambda_2, t) = 1$ is a solution of (5.35). The symmetry u_3 in Proposition 5.3.1 gives that

$$\begin{aligned} u_\epsilon(\lambda_1, \lambda_2, t) &= u\left(\frac{\lambda_1}{(1+\epsilon t)^2}, \frac{\lambda_2}{(1+\epsilon t)^2}, \frac{t}{(1+\epsilon t)}\right) \exp\left(-\frac{\epsilon}{2}\frac{(\lambda_1+\lambda_2)}{(1+\epsilon t)}\right)(1+\epsilon t)^{-n} \\ &= \exp\left(-\frac{\epsilon}{2}\frac{(\lambda_1+\lambda_2)}{(1+\epsilon t)}\right)(1+\epsilon t)^{-n} \end{aligned}$$

is also a solution. Make the change $\epsilon \to 2\epsilon$ so that u_ϵ becomes

$$u_\epsilon(\lambda_1, \lambda_2, t) = \exp\left(\frac{-\epsilon(\lambda_1+\lambda_2)}{(1+2\epsilon t)}\right)(1+2\epsilon t)^{-n}$$

and observe that

$$u_\epsilon(\lambda_1, \lambda_2, 0) = \exp\left(-\epsilon(\lambda_1+\lambda_2)\right)$$

Use (2.34) with an appropriate function φ to construct a new solution of (5.35) given by

$$\begin{aligned} U(\lambda_1,\lambda_2,t) &= \int_0^\infty \varphi(\epsilon)u_\epsilon(\lambda_1,\lambda_2,t)d\epsilon \\ &= \int_0^\infty \varphi(\epsilon)\exp\left(\frac{-\epsilon(\lambda_1+\lambda_2)}{(1+2\epsilon t)}\right)(1+2\epsilon t)^{-n}d\epsilon, \end{aligned} \tag{5.37}$$

which has initial condition

$$\begin{aligned} U(\lambda_1,\lambda_2,0) &= \int_0^\infty \varphi(\epsilon)u_\epsilon(\lambda_1,\lambda_2,0)d\epsilon \\ &= \int_0^\infty \varphi(\epsilon)\exp\left(-\epsilon(\lambda_1+\lambda_2)\right)d\epsilon \\ &= \Phi(\lambda_1+\lambda_2). \end{aligned}$$

Here Φ denotes the Laplace transform of φ.

We wish to identify $u_\epsilon(\lambda_1,\lambda_2,t)$ as the Laplace transform of some function, so we rewrite u_ϵ as

$$\begin{aligned} u_\epsilon(\lambda_1,\lambda_2,t) &= \exp\left(\frac{-\epsilon(\lambda_1+\lambda_2)}{(1+2\epsilon t)}\right)\frac{1}{(1+2\epsilon t)^n} \\ &= \exp\left(\frac{-(\lambda_1+\lambda_2)\left(\epsilon+\frac{1}{2t}-\frac{1}{2t}\right)}{2t(\epsilon+\frac{1}{2t})}\right)\frac{1}{(2t)^n}\frac{1}{(\epsilon+\frac{1}{2t})^n} \\ &= \exp\left(\frac{-(\lambda_1+\lambda_2)}{2t}\right)\exp\left(\frac{(\lambda_1+\lambda_2)}{(2t)^2}\frac{1}{(\epsilon+\frac{1}{2t})}\right)\frac{1}{(2t)^n}\frac{1}{(\epsilon+\frac{1}{2t})^n} \end{aligned}$$

and compute the inverse Laplace transform with respect to ϵ:

$$\begin{aligned} &\mathcal{L}^{-1}\left(\frac{1}{(2t)^n}\frac{1}{(\epsilon+\frac{1}{2t})^n}\exp\left(\frac{-(\lambda_1+\lambda_2)}{2t}\right)\exp\left(\frac{(\lambda_1+\lambda_2)}{(2t)^2}\frac{1}{(\epsilon+\frac{1}{2t})}\right)\right) \\ &= \frac{1}{(2t)^n}\exp\left(\frac{-(\lambda_1+\lambda_2)}{2t}\right)\mathcal{L}^{-1}\left(\frac{1}{(\epsilon+\frac{1}{2t})^n}\exp\left(\frac{(\lambda_1+\lambda_2)}{(2t)^2}\frac{1}{(\epsilon+\frac{1}{2t})}\right)\right) \\ &= \frac{1}{(2t)^n}\exp\left(\frac{-(\lambda_1+\lambda_2)}{2t}\right)\exp\left(-\frac{z}{2t}\right)\mathcal{L}^{-1}\left(\frac{1}{\epsilon^n}\exp\left(\frac{(\lambda_1+\lambda_2)}{(2t)^2}\frac{1}{\epsilon}\right)\right) \\ &= \frac{1}{(2t)^n}\exp\left(\frac{-(\lambda_1+\lambda_2)}{2t}\right)\exp\left(-\frac{z}{2t}\right)\left(\frac{(\lambda_1+\lambda_2)}{z(2t)^2}\right)^{\frac{1-n}{2}} I_{n-1}\left(\frac{\sqrt{z(\lambda_1+\lambda_2)}}{t}\right) \end{aligned}$$

$$= \frac{1}{2t} \exp\left(\frac{-(\lambda_1+\lambda_2+z)}{2t}\right)\left(\frac{(\lambda_1+\lambda_2)}{z}\right)^{\frac{1-n}{2}} I_{n-1}\left(\frac{\sqrt{z(\lambda_1+\lambda_2)}}{t}\right)$$

Hence,

$$\begin{aligned} u_\epsilon(\lambda_1,\lambda_2,t) &= \mathcal{L}\left(\frac{1}{2t}\exp\left(\frac{-(\lambda_1+\lambda_2+z)}{2t}\right)\left(\frac{\lambda_1+\lambda_2}{z}\right)^{\frac{1-n}{2}} I_{n-1}\left(\frac{\sqrt{z(\lambda_1+\lambda_2)}}{t}\right)\right) \\ &= \int_0^\infty \exp\left(\frac{-(\lambda_1+\lambda_2+z)}{2t}\right)\frac{(\lambda_1+\lambda_2)^{\frac{1-n}{2}}}{2t} z^{\frac{n-1}{2}} I_{n-1}\left(\frac{\sqrt{z(\lambda_1+\lambda_2)}}{t}\right) e^{-\epsilon z} dz. \end{aligned}$$

Substitution into (5.37) gives

$$\begin{aligned} U(\lambda_1,\lambda_2,t) &= \int_0^\infty \varphi(\epsilon) u_\epsilon(\lambda_1,\lambda_2,t) d\epsilon \\ &= \int_0^\infty \varphi(\epsilon) \int_0^\infty e^{\frac{-(\lambda_1+\lambda_2+z)}{2t}} \frac{(\lambda_1+\lambda_2)^{\frac{1-n}{2}}}{2t} z^{\frac{n-1}{2}} I_{n-1}\left(\frac{\sqrt{z(\lambda_1+\lambda_2)}}{t}\right) e^{-\epsilon z} dz d\epsilon \\ &= \int_0^\infty \int_0^\infty \varphi(\epsilon) e^{\frac{-(\lambda_1+\lambda_2+z)}{2t}} \frac{(\lambda_1+\lambda_2)^{\frac{1-n}{2}}}{2t} z^{\frac{n-1}{2}} I_{n-1}\left(\frac{\sqrt{z(\lambda_1+\lambda_2)}}{t}\right) e^{-\epsilon z} dz d\epsilon \\ &= \int_0^\infty \int_0^\infty \varphi(\epsilon) e^{\frac{-(\lambda_1+\lambda_2+z)}{2t}} \frac{(\lambda_1+\lambda_2)^{\frac{1-n}{2}}}{2t} z^{\frac{n-1}{2}} I_{n-1}\left(\frac{\sqrt{z(\lambda_1+\lambda_2)}}{t}\right) e^{-\epsilon z} d\epsilon dz \\ &= \int_0^\infty e^{\frac{-(\lambda_1+\lambda_2+z)}{2t}} \frac{(\lambda_1+\lambda_2)^{\frac{1-n}{2}}}{2t} z^{\frac{n-1}{2}} I_{n-1}\left(\frac{\sqrt{z(\lambda_1+\lambda_2)}}{t}\right)\left(\int_0^\infty \varphi(\epsilon) e^{-\epsilon z} d\epsilon\right) dz \\ &= \int_0^\infty e^{\frac{-(\lambda_1+\lambda_2+z)}{2t}} \frac{(\lambda_1+\lambda_2)^{\frac{1-n}{2}}}{2t} z^{\frac{n-1}{2}} I_{n-1}\left(\frac{\sqrt{z(\lambda_1+\lambda_2)}}{t}\right) \Phi(z) dz \end{aligned}$$

Furthermore, $U(\lambda_1,\lambda_2,t)$ can be written as a function of $\lambda_1+\lambda_2$ and t only:

$$\begin{aligned} U(\lambda_1,\lambda_2,t) &= U(\lambda_1+\lambda_2,t) \\ &= \int_0^\infty \underbrace{\Phi(z)}_{f(z)} \underbrace{e^{\frac{-((\lambda_1+\lambda_2)+z)}{2t}} \frac{(\lambda_1+\lambda_2)^{\frac{1-n}{2}}}{2t} z^{\frac{n-1}{2}} I_{n-1}\left(\frac{\sqrt{z(\lambda_1+\lambda_2)}}{t}\right)}_{p(t,\lambda_1+\lambda_2,z)} dz, \end{aligned}$$

with $$U(\lambda_1+\lambda_2,0) = \Phi(\lambda_1+\lambda_2) = f(\lambda_1+\lambda_2).$$

Let $Z(t) := \mu_1(t)+\mu_2(t)$. Then $Z(0) = \mu_1(0)+\mu_2(0) = \lambda_1+\lambda_2 := z_0$. Therefore we can write

$$U(z_0,t) = E^{z_0}[f(Z(t))] := E[f(Z(t))|Z(0)=z_0]$$
$$= \int_0^\infty \underbrace{\Phi(z)}_{f(z)} \underbrace{e^{\frac{-(z_0+z)}{2t}} \frac{z_0^{\frac{1-n}{2}}}{2t} z^{\frac{n-1}{2}} I_{n-1}\left(\frac{\sqrt{zz_0}}{t}\right)}_{p(t,z_0,z)} dz$$

Note that indeed

$$\int_0^\infty p(t,z_0,z)dz = 1,$$

and that $p(t,z_0,z)1_{\{z>0\}}$ satisfies all the necessary conditions to be the transition density for the process $Z(t) = \mu_1(t) + \mu_2(t)$ starting at $z_0 = \lambda_1 + \lambda_2$. Compare the expression for $p(t,z_0,z)1_{\{z>0\}}$ with that in theorem 5.3.2 to conclude that the process $Z(t) = \mu_1(t)+\mu_2(t)$ is distributed according to a squared-Bessel distribution with parameter $\nu = n-1$. □

Having the transition density function of the sum $\mu_1(t)+\mu_2(t)$ allows us to compute expected values of the form $E^{\lambda_1,\lambda_2}[f(\mu_1(t)+\mu_2(t))]$ via a simple integration:

$$E^{\lambda_1,\lambda_2}[f(\mu_1(t)+\mu_2(t))] = \int_0^\infty f(z)\frac{e^{\frac{-(\lambda_1+\lambda_2+z)}{2t}}}{2t}\left(\frac{\lambda_1+\lambda_2}{z}\right)^{\frac{1-n}{2}} I_{n-1}\left(\frac{\sqrt{z(\lambda_1+\lambda_2)}}{t}\right) dz,$$

but this provides us with very limited information about the overall behaviour of the eigenvalues $\mu_1(t)$ and $\mu_2(t)$. This motivates the following question: can we extend the existing knowledge about the behaviour of these eigenvalues by extending the classes of functions for which we can obtain expected values?

Ideally, we would like to be able to compute

$$E^{\lambda_1,\lambda_2}\left[f(\mu_1(t),\mu_2(t))e^{-\int_0^t g(\mu_1(s),\mu_2(s))ds}\right] \tag{5.38}$$

for any function f and g. Knowledge of the transition density function for the eigenvalues $\mu_1(t),\mu_2(t)$ would allow us to compute $E^{\lambda_1,\lambda_2}[f(\mu_1(t),\mu_2(t))]$ for an arbitrary f, but as far as we know, this transition density is still unknown. We have seen that the usual lie symmetry methods for the obtention of transition densities also fail to produce this result. This naturally leads to the question: for which types of functions f and g are these expectations computable?

In what follows, we answer this question by providing techniques for the computation of expected values of the above type (5.38) for a wide range of functions. We start with $g = 0$ and, in the last section, we provide some results for different choices of g. Note that if $E^{\lambda_1,\lambda_2}[f_i(\mu_1(t),\mu_2(t))]$ is computable for $i \in I$, then $E^{\lambda_1,\lambda_2}\left[\sum_{i\in I_j} f_i(\mu_1(t),\mu_2(t))\right]$, $I_j \subset I$ is also computable.

5.3.1 Expectations of any symmetric polynomial in the eigenvalues of a Wishart process

Here we present a result that allows us to generate an expression for

$$E^{\lambda_1,\lambda_2}[p(\mu_1(t),\mu_2(t))]$$

for any symmetric polynomial $p(\mu_1,\mu_2)$, i.e. any polynomial p such that $p(\mu_1,\mu_2) = p(\mu_2,\mu_1)$:

Theorem 5.3.4 (Recursive method for generating the expectations of any symmetric polynomial in the eigenvalues of a 2×2 Wishart process).
Let $W_t \sim WIS(n,2,W_0)$ with $n \geq 2$ and with eigenvalues $\mu_1(t) > \mu_2(t) \geq 0$. Let L be the differential operator defined as

$$L = \left(n + \frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}\right)\frac{\partial}{\partial\lambda_1} + \left(n - \frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}\right)\frac{\partial}{\partial\lambda_2} + 2\lambda_1\frac{\partial^2}{\partial\lambda_1^2} + 2\lambda_2\frac{\partial^2}{\partial\lambda_2^2},$$

and let $p(x_1,x_2)$ be a symmetric polynomial of degree k. That is, let p be a polynomial such that $p(x_1,x_2) = p(x_2,x_1)$ with $k = deg(p(x_1,x_2))$.
Take

$$u_0(\lambda_1,\lambda_2,t) = p(\lambda_1,\lambda_2)$$

and

$$u_s(\lambda_1,\lambda_2,t) = \int_0^t Lu_{s-1}(\lambda_1,\lambda_2,r)dr + p(\lambda_1,\lambda_2), \quad \textit{for } s = 1,\ldots,k.$$

Then

$$E^{\lambda_1,\lambda_2}[p(\mu_1(t),\mu_2(t))] = u_k(\lambda_1,\lambda_2,t).$$

Proof. It is clear from the definition of u_k that it satisfies $u_k(\lambda_1,\lambda_2,0) = p(\lambda_1,\lambda_2)$. On the other hand, we need to show that u_k solves $u_t = Lu$, which can be verified by looking at the recursive construction of u_k:

$$\begin{aligned}
u_0(\lambda_1,\lambda_2,t) &= p(\lambda_1,\lambda_2)\\
u_1(\lambda_1,\lambda_2,t) &= \int_0^t Lp(\lambda_1,\lambda_2)dr + p(\lambda_1,\lambda_2) = tLp(\lambda_1,\lambda_2) + p(\lambda_1,\lambda_2)\\
u_2(\lambda_1,\lambda_2,t) &= \int_0^t L(rLp(\lambda_1,\lambda_2) + p(\lambda_1,\lambda_2))dr + p(\lambda_1,\lambda_2)\\
&= \int_0^t (rL^2p(\lambda_1,\lambda_2) + Lp(\lambda_1,\lambda_2))dr + p(\lambda_1,\lambda_2)\\
&= \frac{t^2}{2}L^2p(\lambda_1,\lambda_2) + tLp(\lambda_1,\lambda_2) + p(\lambda_1,\lambda_2)
\end{aligned}$$

$$\vdots$$

$$\begin{aligned} u_k(\lambda_1, \lambda_2, t) &= \frac{t^k}{k!} L^k p(\lambda_1, \lambda_2) + \frac{t^{k-1}}{(k-1)!} L^{k-1} p(\lambda_1, \lambda_2) + \cdots + tLp(\lambda_1, \lambda_2) + p(\lambda_1, \lambda_2) \\ &= \sum_{i=0}^{k} \frac{(tL)^i}{i!} p(\lambda_1, \lambda_2) \end{aligned}$$

Hence

$$\begin{aligned} \frac{\partial}{\partial t} u_k(\lambda_1, \lambda_2, t) &= \sum_{i=1}^{k} \frac{t^{i-1} L^i}{(i-1)!} p(\lambda_1, \lambda_2) \\ &= \frac{t^{k-1}}{(k-1)!} L^k p(\lambda_1, \lambda_2) + \frac{t^{k-2}}{(k-2)!} L^{k-1} p(\lambda_1, \lambda_2) + \cdots + Lp(\lambda_1, \lambda_2) \\ &= Lu_{k-1}(\lambda_1, \lambda_2, t) = \sum_{i=0}^{k-1} \frac{t^i L^{i+1}}{i!} p(\lambda_1, \lambda_2), \end{aligned}$$

while on the other hand

$$Lu_k(\lambda_1, \lambda_2, t) = L \sum_{i=0}^{k} \frac{(tL)^i}{i!} p(\lambda_1, \lambda_2) = \sum_{i=0}^{k} \frac{t^i L^{i+1}}{i!} p(\lambda_1, \lambda_2). \tag{5.39}$$

But the last term $\frac{t^k L^{k+1}}{k!} p(\lambda_1, \lambda_2)$ vanishes due to the symmetry and the order of the polynomial p. Hence $(u_k)_t = Lu_{k-1} = Lu_k$ as claimed.

Therefore, by part (b) of Kolmogorov's Backward Theorem (2.2.2), we must have $u_k(\lambda_1, \lambda_2, t) = E^{\lambda_1, \lambda_2}[p(\mu_1(t), \mu_2(t))]$. □

The above method, while indeed effective for any symmetric polynomial p, becomes quite time-consuming for polynomials of higher degrees. This is due to the fact that we need to calculate $k+1$ iterates in each case. Alternatively, we can calculate L^i for $i = 1, \ldots, k$ and use the series expansion definition

$$u_k(\lambda_1, \lambda_2, t) = \sum_{i=0}^{k} \frac{(tL)^i}{i!} p(\lambda_1, \lambda_2)$$

obtained in the proof of the above theorem. This is, however, equally inconvenient for higher values of k.

It turns out that the lie symmetries of equation (5.35) give us an alternative to this method:

Theorem 5.3.5. *Let $W_t \sim WIS(n, 2, W_0)$ with $n \geq 2$ and with eigenvalues $\mu_1(t) > \mu_2(t) \geq 0$. Let L be the differential operator defined as*

$$L = \left(n + \frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2}\right)\frac{\partial}{\partial \lambda_1} + \left(n - \frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2}\right)\frac{\partial}{\partial \lambda_2} + 2\lambda_1 \frac{\partial^2}{\partial \lambda_1^2} + 2\lambda_2 \frac{\partial^2}{\partial \lambda_2^2}.$$

Suppose $u(\lambda_1, \lambda_2, t) \in C^{2,1}(\mathbb{R}^2 \times \mathbb{R})$ is a solution of $u_t = Lu$ with $u(\lambda_1, \lambda_2, 0) = f(\lambda_1, \lambda_2)$ and u is bounded for $t \in K$, for each compact $K \subset \mathbb{R}$, i.e. suppose that $u(\lambda_1, \lambda_2, t) = E^{\lambda_1, \lambda_2}[f(\mu_1(t), \mu_2(t))]$.
Consider

$$v(\lambda_1, \lambda_2, t) = \left[\frac{d^i}{d\epsilon^i} u\left(\frac{\lambda_1}{(1+\epsilon t)^2}, \frac{\lambda_2}{(1+\epsilon t)^2}, \frac{t}{(1+\epsilon t)}\right) \frac{\exp\left(-\frac{\epsilon}{2}\frac{(\lambda_1+\lambda_2)}{(1+\epsilon t)}\right)}{(1+\epsilon t)^n}\right]_{\epsilon=0} \tag{5.40}$$

for any $i \in \mathbb{N} \cup \{0\}$, and with $v(\lambda_1, \lambda_2, 0) = g(\lambda_1, \lambda_2)$. If $v(\lambda_1, \lambda_2, t) \in C^{2,1}(\mathbb{R}^2 \times \mathbb{R})$, and is bounded for $t \in K$, for each compact $K \subset \mathbb{R}$, then

$$v(\lambda_1, \lambda_2, t) = E^{\lambda_1, \lambda_2}[g(\mu_1(t), \mu_2(t))].$$

Proof. Let $u(\lambda_1, \lambda_2, t) = E^{\lambda_1, \lambda_2}[f(\mu_1(t), \mu_2(t))]$. Note that Proposition 5.3.1 gives that

$$u\left(\frac{\lambda_1}{(1+\epsilon t)^2}, \frac{\lambda_2}{(1+\epsilon t)^2}, \frac{t}{(1+\epsilon t)}\right) \exp\left(-\frac{\epsilon}{2}\frac{(\lambda_1+\lambda_2)}{(1+\epsilon t)}\right)(1+\epsilon t)^{-n}$$

is also a solution of $u_t = Lu$. Further, by Theorem 2.1.5, differentiation of this symmetry with respect to ϵ any number of times produces other solutions of the equation $u_t = Lu$. Hence,

$$v(\lambda_1, \lambda_2, t) = \left[\frac{d^i}{d\epsilon^i} u\left(\frac{\lambda_1}{(1+\epsilon t)^2}, \frac{\lambda_2}{(1+\epsilon t)^2}, \frac{t}{(1+\epsilon t)}\right) \frac{\exp\left(-\frac{\epsilon}{2}\frac{(\lambda_1+\lambda_2)}{(1+\epsilon t)}\right)}{(1+\epsilon t)^n}\right]_{\epsilon=0}$$

satisfies $v_t = Lv$. Moreover, if $v(\lambda_1, \lambda_2, t) \in C^{2,1}(\mathbb{R}^2 \times \mathbb{R})$ and is bounded for $t \in K$, for each compact $K \subset \mathbb{R}$ and $v(\lambda_1, \lambda_2, 0) = g(\lambda_1, \lambda_2)$, then part (b) of Kolmogorov Backwards equation theorem yields the desired result. □

Note. The above result allows us to compute $E^{\lambda_1, \lambda_2}[g(\mu_1(t), \mu_2(t))]$ for any function g that can be obtained as the initial condition of a solution u of $u_t = Lu$ that can be written as a derivative with respect to ϵ of any known solution $\tilde{u}$ transformed through the action of any of the symmetries of $u_t = Lu$.

In what follows we illustrate how to use this result in the computation of expected values of symmetric polynomials in the eigenvalues μ_1 and μ_2. Roughly, the idea is that we start with a particular solution u_1 of equation (5.35) with initial condition $u_1(\lambda_1, \lambda_2, 0) = p_1(\lambda_1, \lambda_2)$, where p_1 is a symmetric polynomial. Then, via the transformation through symmetries of this first solution, we obtain another solution u_2 with different initial condition $u_2(\lambda_1, \lambda_2, 0) = p_2(\lambda_1, \lambda_2)$, where p_2 is again a symmetric polynomial. Thus, we obtain the expectations $E^{\lambda_1,\lambda_2}[p_2(\mu_1(t), \mu_2(t))]$ from the manipulation through symmetry of $E^{\lambda_1,\lambda_2}[p_1(\mu_1(t), \mu_2(t))]$.

Example 5.3.1. Consider the polynomial $\mu_1^2(t)\mu_2(t) + \mu_2^2(t)\mu_1(t)$, where $\mu_1(t)$ and $\mu_2(t)$ are the eigenvalues of a paticular Wishart process $W_t \sim WIS(n, 2, W_0)$ and where $\mu_1(t) > \mu_2(t) \geq 0$. Our goal here is to calculate the expectations

$$E^{\lambda_1,\lambda_2}[\mu_1^2(t)\mu_2(t) + \mu_2^2(t)\mu_1(t)]$$

<u>Method 1</u>*: Using the recursive definition of u described in Theorem 5.3.4:*
Our polynomial $p(x, y) = x^2y+y^2x$ is of degree 3, therefore we will have to compute u_0, u_1, u_2 and u_3.
According to the above theorem, we define

$$\begin{aligned}
u_0(\lambda_1, \lambda_2, t) &= p(\lambda_1, \lambda_2) = \lambda_1^2\lambda_2 + \lambda_2^2\lambda_1,\\
u_1(\lambda_1, \lambda_2, t) &= \int_0^t Lu_0(\lambda_1, \lambda_2, r)dr + p(\lambda_1, \lambda_2)\\
&= \int_0^t ((n-1)(\lambda_1^2 + \lambda_2^2) + 2(3+2n)\lambda_1\lambda_2)dr + \lambda_1^2\lambda_2 + \lambda_2^2\lambda_1\\
&= ((n-1)(\lambda_1^2 + \lambda_2^2) + 2(3+2n)\lambda_1\lambda_2)t + \lambda_1^2\lambda_2 + \lambda_2^2\lambda_1,\\
u_2(\lambda_1, \lambda_2, t) &= \int_0^t Lu_1(\lambda_1, \lambda_2, r)dr + p(\lambda_1, \lambda_2)\\
&= \int_0^t ((n-1)(\lambda_1^2 + \lambda_2^2) + 2(3+2n)\lambda_1\lambda_2 + 6r(n-1)(n+2)(\lambda_1 + \lambda_2))dr\\
&+ \lambda_1^2\lambda_2 + \lambda_2^2\lambda_1\\
&= ((n-1)(\lambda_1^2 + \lambda_2^2) + 2(3+2n)\lambda_1\lambda_2)t + 3t^2(n-1)(n+2)(\lambda_1 + \lambda_2)\\
&+ \lambda_1^2\lambda_2 + \lambda_2^2\lambda_1\\
u_3(\lambda_1, \lambda_2, t) &= \int_0^t Lu_2(\lambda_1, \lambda_2, r)dr + p(\lambda_1, \lambda_2)\\
&= \int_0^t (6n(n-1)(n+2)r^2 + (n-1)(\lambda_1^2 + \lambda_2^2) + 2(3+2n)\lambda_1\lambda_2)dr\\
&+ \int_0^t (6r(n-1)(n+2)(\lambda_1 + \lambda_2))dr + \lambda_1^2\lambda_2 + \lambda_2^2\lambda_1\\
&= 2n(n-1)(n+2)t^3 + ((n-1)(\lambda_1^2 + \lambda_2^2) + 2(3+2n)\lambda_1\lambda_2)t\\
&+ 3t^2(n-1)(n+2)(\lambda_1 + \lambda_2) + \lambda_1^2\lambda_2 + \lambda_2^2\lambda_1
\end{aligned}$$

So the expectations we are looking for are given by

$$\begin{aligned}E^{\lambda_1,\lambda_2}[\mu_1^2(t)\mu_2(t)+\mu_2^2(t)\mu_1(t)] &= 2n(n-1)(n+2)t^3+3t^2(n-1)(n+2)(\lambda_1+\lambda_2)\\ &+((n-1)(\lambda_1^2+\lambda_2^2)+2(3+2n)\lambda_1\lambda_2)t+\lambda_1^2\lambda_2+\lambda_2^2\lambda_1.\end{aligned}$$

Note that an alternative for this method would have been to calculate the first three powers of L, i.e. L,L^2 and L^3 and then compute the desired expectations as

$$E^{\lambda_1,\lambda_2}[\mu_1^2(t)\mu_2(t)+\mu_2^2(t)\mu_1(t)] = \sum_{i=0}^{3}\frac{(tL)^i}{i!}(\lambda_1^2\lambda_2+\lambda_2^2\lambda_1).$$

<u>Method 2</u>: *Using the lie symmetries of equation (5.35) as in Theorem 5.3.5:*
Let

$$\begin{aligned}u_1(\lambda_1,\lambda_2,t) &= E^{\lambda_1,\lambda_2}[\mu_1^2(t)+\mu_2^2(t)] = 2n(n+3)t^2+2(n+3)(\lambda_1+\lambda_2)t+\lambda_1^2+\lambda_2^2\\ u_2(\lambda_1,\lambda_2,t) &= E^{\lambda_1,\lambda_2}[\mu_1^3(t)+\mu_2^3(t)] = 2n(n+2)(n+7)t^3\\ &+3(n+2)(n+7)(\lambda_1+\lambda_2)t^2+\Big(3(n+5)(\lambda_1^2+\lambda_2^2)+6\lambda_1\lambda_2\Big)t+\lambda_1^3+\lambda_2^3\end{aligned}$$

Observe that $u_1(\lambda_1,\lambda_2,0)=\lambda_1^3+\lambda_2^3$ and $u_2(\lambda_1,\lambda_2,0)=\lambda_1^2+\lambda_2^2$ respectively. Define

$$v_k(\lambda_1,\lambda_2,t) = \left[\frac{d}{d\epsilon}u_k\left(\frac{\lambda_1}{(1+\epsilon t)^2},\frac{\lambda_2}{(1+\epsilon t)^2},\frac{t}{(1+\epsilon t)}\right)\frac{\exp\left(-\frac{\epsilon}{2}\frac{(\lambda_1+\lambda_2)}{(1+\epsilon t)}\right)}{(1+\epsilon t)^n}\right]_{\epsilon=0}, \tag{5.41}$$

for any u_k solution of (5.35). We know by theorem 5.3.5 that v_k solves (5.35). The reader may check that the explicit expression for v_k is

$$\begin{aligned}v_k(\lambda_1,\lambda_2,t) &= \left[\frac{d}{d\epsilon}u_k\left(\frac{\lambda_1}{(1+\epsilon t)^2},\frac{\lambda_2}{(1+\epsilon t)^2},\frac{t}{(1+\epsilon t)}\right)\frac{\exp\left(-\frac{\epsilon}{2}\frac{(\lambda_1+\lambda_2)}{(1+\epsilon t)}\right)}{(1+\epsilon t)^n}\right]_{\epsilon=0}\\ &= \Bigg[-\exp\left(-\frac{\epsilon}{2}\frac{(\lambda_1+\lambda_2)}{(1+\epsilon t)}\right)\Bigg(\frac{t^2}{(1+\epsilon t)^{n+2}}\partial_3 u_k\left(\frac{\lambda_1}{(1+\epsilon t)^2},\frac{\lambda_2}{(1+\epsilon t)^2},\frac{t}{(1+\epsilon t)}\right)\\ &+\left(\frac{\lambda_1+\lambda_2}{2(1+\epsilon t)^{n+2}}+\frac{nt}{(1+\epsilon t)^{n+1}}\right)u_k\left(\frac{\lambda_1}{(1+\epsilon t)^2},\frac{\lambda_2}{(1+\epsilon t)^2},\frac{t}{(1+\epsilon t)}\right)\\ &+\frac{2\lambda_1 t}{(1+\epsilon t)^{n+3}}\partial_1 u_k\left(\frac{\lambda_1}{(1+\epsilon t)^2},\frac{\lambda_2}{(1+\epsilon t)^2},\frac{t}{(1+\epsilon t)}\right)\\ &+\frac{2\lambda_2 t}{(1+\epsilon t)^{n+3}}\partial_2 u_k\left(\frac{\lambda_1}{(1+\epsilon t)^2},\frac{\lambda_2}{(1+\epsilon t)^2},\frac{t}{(1+\epsilon t)}\right)\Bigg)\Bigg]_{\epsilon=0}\end{aligned}$$

$$= -2\lambda_1 t\partial_1 u_k(\lambda_1,\lambda_2,t) - 2\lambda_2 t\partial_2 u_k(\lambda_1,\lambda_2,t) - t^2\partial_3 u_k(\lambda_1,\lambda_2,t) - \left(\frac{\lambda_1+\lambda_2}{2} + nt\right) u_k(\lambda_1,\lambda_2,t)$$

Hence $v_k(\lambda_1,\lambda_2,0) = -\frac{\lambda_1+\lambda_2}{2}u_k(\lambda_1,\lambda_2,0)$. That is,

$$v_1(\lambda_1,\lambda_2,0) = -\frac{\lambda_1+\lambda_2}{2}u_1(\lambda_1,\lambda_2,0) = -\frac{\lambda_1+\lambda_2}{2}\lambda_1^2+\lambda_2^2 = -\frac{\lambda_1^3+\lambda_1^2\lambda_2+\lambda_1\lambda_2^2+\lambda_2^3}{2}$$

Therefore $v_1(\lambda_1,\lambda_2,t) = E^{\lambda_1,\lambda_2}\left[-\frac{\mu_1(t)^3+\mu_1(t)^2\mu_2(t)+\mu_1(t)\mu_2(t)^2+\mu_2(t)^3}{2}\right]$, thus the linear combination $v(\lambda_1,\lambda_2,t) = -2v_1(\lambda_1,\lambda_2,t) - u_2(\lambda_1,\lambda_2,t)$ will give us the desired expected value. The explicit computation of v gives

$$\begin{aligned}
v(\lambda_1,\lambda_2,t) &= -2v_1(\lambda_1,\lambda_2,t) - u_2(\lambda_1,\lambda_2,t)\\
&= \lambda_1^3+\lambda_1^2\lambda_2+\lambda_1\lambda_2^2+\lambda_2^3 + 2t((n+3)(\lambda_1+\lambda_2)^2+(n+4)(\lambda_1^2+\lambda_2^2))\\
&\quad + 6t^2(n+2)(n+3)(\lambda_1+\lambda_2) + 4n(n+2)(n+3)t^3\\
&\quad - 2n(n+2)(n+7)t^3 - 3(n+2)(n+7)(\lambda_1+\lambda_2)t^2\\
&\quad - \left(3(n+5)(\lambda_1^2+\lambda_2^2)+6\lambda_1\lambda_2\right)t - \lambda_1^3 - \lambda_2^3\\
&=2n(n-1)(n+2)t^3 + 3t^2(n-1)(n+2)(\lambda_1+\lambda_2)\\
&\quad + ((n-1)(\lambda_1^2+\lambda_2^2) + 2(3+2n)\lambda_1\lambda_2)t + \lambda_1^2\lambda_2 + \lambda_2^2\lambda_1.
\end{aligned}$$

Clearly v solves (5.35) since $v_t = -2(v_1)_t - (u_2)_t = -2Lv_1 - Lu_2 = L(-2v_1-u_2) = Lv$ and $v(\lambda_1,\lambda_2,0) = \lambda_1^2\lambda_2 + \lambda_2^2\lambda_1$. Therefore

$$v(\lambda_1,\lambda_2,t) = E^{\lambda_1,\lambda_2}[\mu_1(t)^2\mu_2(t) + \mu_2(t)^2\mu_1(t)].$$

Remark. We have seen in this example that transforming any solution u_k of (5.35) through the symmetry given by v_k produces another solution with initial condition $v_k(\lambda_1,\lambda_2,0) = -\frac{\lambda_1+\lambda_2}{2}u_k(\lambda_1,\lambda_2,0)$. Hence, if $u_k(\lambda_1,\lambda_2,0)$ is a symmetric polynomial of degree m, $v_k(\lambda_1,\lambda_2,0)$ will be a symmetric polynomial of degree $m+1$.

This suggests that, ideally, we would like to be able to compute the expectations $E^{\lambda_1,\lambda_2}[p(\mu_1(t),\mu_2(t))]$ for any symmetric polynomial p of degree k in terms of linear combinations of $E^{\lambda_1,\lambda_2}[q_i(\mu_1(t),\mu_2(t))]$ and the action of lie symmetries on $E^{\lambda_1,\lambda_2}[q_i(\mu_1(t),\mu_2(t))]$ for some symmetric polynomials q_i of degree $k' < k$. However, is this really possible? We answer this question with the following result:

Proposition 5.3.6. *Let $W_t \sim WIS(n,2,W_0)$ with $n \geq 2$ and with eigenvalues $\mu_1(t) > \mu_2(t) \geq 0$. Let $u_i(\lambda_1,\lambda_2,t) = E^{\lambda_1,\lambda_2}[(\mu_1(t)\mu_2(t))^i]$, for $i \in \mathbb{N}\cup\{0\}$ and let*

$$v_i^j(\lambda_1,\lambda_2,t) = \left[\frac{d^j}{d\epsilon^j}u_i\left(\frac{\lambda_1}{(1+\epsilon t)^2},\frac{\lambda_2}{(1+\epsilon t)^2},\frac{t}{(1+\epsilon t)}\right)\frac{\exp\left(-\frac{\epsilon}{2}\frac{(\lambda_1+\lambda_2)}{(1+\epsilon t)}\right)}{(1+\epsilon t)^n}\right]_{\epsilon=0} \tag{5.42}$$

for $i,j \in \mathbb{N}\cup\{0\}$. Then, for any polynomial $p(x,y)$ with $\deg(p(x,y)) = k$ such that $p(x,y) = p(y,x)$,

$$E^{\lambda_1,\lambda_2}[p(\mu_1(t),\mu_2(t))] = \sum_{i\in I,j\in J} C_{ij}v_i^j(\lambda_1,\lambda_2,t),\ \ I\subset\left\{0,1,\ldots,\left\lfloor\frac{k}{2}\right\rfloor\right\},\ \ J\subset\{0,1,\ldots,k\} \tag{5.43}$$

for some constants C_{ij}.

Proof. We will prove this result by induction. First consider the following observations:

1. Let P_S^k denote the set of all symmetric polynomials of degree k. To show that the result holds for any $q \in P_S^k$, it is enough to show that it holds for each element of a basis of P_S^k. That is, let $B^k = \{b_l^k\}$, $l \in L$ be a basis of P_S^k. Then it is enough to show that $E^{\lambda_1,\lambda_2}[b_l^k(\mu_1(t),\mu_2(t))] = \sum_{i\in I,j\in J} C_{ij}^l v_i^j(\lambda_1,\lambda_2,t)$ for each $l \in L$, for some constants C_{ij}^l. From this, the result trivially follows for any $q \in P_S^k$, since q can be written as $q = \sum_l \alpha_l b_l^k$, for some constants α_l and hence

$$\begin{aligned}
E^{\lambda_1,\lambda_2}[q(\mu_1(t),\mu_2(t))] &= E^{\lambda_1,\lambda_2}[\sum_{l\in L}\alpha_l b_l^k(\mu_1(t),\mu_2(t))] \\
&= \sum_{l\in L}\alpha_l E^{\lambda_1,\lambda_2}[b_l^k(\mu_1(t),\mu_2(t))] \\
&= \sum_{l\in L}\alpha_l \sum_{i\in I,j\in J} C_{ij}^l v_i^j(\lambda_1,\lambda_2,t) \\
&= \sum_{i\in I,j\in J}\left(\sum_{l\in L}\alpha_l C_{ij}^l\right) v_i^j(\lambda_1,\lambda_2,t) \\
&= \sum_{i\in I,j\in J} A_{ij} v_i^j(\lambda_1,\lambda_2,t),
\end{aligned}$$

for some constants $A_{ij} = \sum_{l\in L}\alpha_l C_{ij}^l$.

2. Observe that if $u_i(\lambda_1, \lambda_2, t) = E[p(\mu_1(t), \mu_2(t))]$, i.e. $u_i(\lambda_1, \lambda_2, 0) = p(\lambda_1, \lambda_2)$ then $v_i^j(\lambda_1, \lambda_2, t) = E\left[\left(\frac{\mu_1(t)+\mu_2(t)}{-2}\right)^j p(\mu_1(t), \mu_2(t))\right]$, since

$$v_i^j(\lambda_1, \lambda_2, 0) = \left(\frac{\lambda_1+\lambda_2}{-2}\right)^j u_i(\lambda_1, \lambda_2, 0) = \left(\frac{\lambda_1+\lambda_2}{-2}\right)^j p(\lambda_1, \lambda_2).$$

   So it follows that if $v_i^j(\lambda_1, \lambda_2, t) = E\left[q(\mu_1(t), \mu_2(t))\right]$ then $-2v_i^{j+1}(\lambda_1, \lambda_2, t) = E\left[(\mu_1(t)+\mu_2(t))q(\mu_1(t), \mu_2(t))\right]$.

3. A basis B^k for P_S^k will consist of $\left\lceil \frac{k+1}{2} \right\rceil$ elements. That is $|B^k| = \left\lceil \frac{k+1}{2} \right\rceil$. Hence, if k is even $|B^k| = \left\lceil \frac{k+1}{2} \right\rceil = \frac{k}{2} + 1$. However, if k is odd then $|B^k| = \left\lceil \frac{k+1}{2} \right\rceil = \frac{k+1}{2}$. For practical purposes, we will define the polynomials in the basis slightly different for even and odd k.

   - **Even degrees:** A basis $B^k = \{b_1^k, \ldots, b_{\frac{k}{2}+1}^k\}$ is
   $$\begin{cases} b_i^k = x^{k-i+1}y^{i-1} + x^{i-1}y^{k-i+1}, & i = 1, \ldots, \frac{k}{2}, \\ b_{\frac{k}{2}+1}^k = x^{\frac{k}{2}}y^{\frac{k}{2}}. \end{cases}$$
   - **Odd degrees:** A basis $B^k = \{b_1^k, \ldots, b_{\frac{k+1}{2}}^k\}$ is
   $$\begin{cases} b_i^k = x^{k-i+1}y^{i-1} + x^{i-1}y^{k-i+1}, & i = 1, \ldots, \frac{k+1}{2}. \end{cases}$$

We will now prove the result for $k = 1$ and $k = 2$ and show that, if the result holds for $k = n-1$, then it must hold for $k = n$. Then, by induction, the result can be said to be true for all k. To ease the notation, let us refer to $(\mu_1(t), \mu_2(t))$ as (X_t, Y_t) so that the initial condition $(\mu_1(0), \mu_2(0)) = (\lambda_1, \lambda_2)$ becomes $(X_0, Y_0) = (x, y)$.

- $k = 1$: $B^1 = \{b_1^1\}$,
  $b_1^1 = x + y$. Clearly, $E[b_1^1(X_t, Y_t)] = -2v_0^1(x, y, t)$

- $k = 2$: $B^2 = \{b_1^2, b_2^2\}$
  First, for $b_2^2 = xy$ we have that clearly $E[b_2^2(X_t, Y_t)] = v_1^0(x, y, t)$.
  Then $b_1^2 = x^2 + y^2 = (x+y)(x+y) - 2xy = (x+y)b_1^1 - 2b_2^2$,
  so $E[b_1^2(X_t, Y_t)] = E[(X_t + Y_t)b_1^1(X_t, Y_t) - 2b_2^2(X_t, Y_t)] = -2(-2v_0^2(x, y, t)) - 2v_1^0(x, y, t) = 4v_0^2(x, y, t) - 2v_1^0(x, y, t)$

- **Induction argument**: Suppose the result holds for $k = n-1$. Consider the case $k = n$:
  If n is even: $B^n = \{b_1^n, \ldots, b_{\frac{n}{2}+1}^n\}$ and $B^{n-1} = \{b_1^{n-1}, \ldots, b_{\frac{n}{2}}^{n-1}\}$,
  with $E[b_l^{n-1}(X_t, Y_t)] = \sum_{i,j} \gamma_{ij}^l v_i^j$ for each $l = 1, \ldots, \frac{n}{2}$
  (*) $b_{\frac{n}{2}+1}^n = x^{\frac{n}{2}}y^{\frac{n}{2}}$, for which clearly $E[b_{\frac{n}{2}+1}^n(X_t, Y_t)] = v_{\frac{n}{2}}^0(x, y, t)$.

$$(*) \quad b^n_{\frac{n}{2}} = x^{\frac{n}{2}+1}y^{\frac{n}{2}-1} + x^{\frac{n}{2}-1}y^{\frac{n}{2}+1}$$
$$= (x+y)(x^{\frac{n}{2}}y^{\frac{n}{2}-1} + x^{\frac{n}{2}-1}y^{\frac{n}{2}}) - 2x^{\frac{n}{2}}y^{\frac{n}{2}}$$
$$= (x+y)b^{n-1}_{\frac{n}{2}} - 2b^n_{\frac{n}{2}+1},$$

but $E[b^{n-1}_{\frac{n}{2}}(X_t,Y_t)] = \sum_{i,j}\gamma^{\frac{n}{2}}_{ij}v^j_i(x,y,t)$, and $E[b^n_{\frac{n}{2}+1}(X_t,Y_t)] = v^0_{\frac{n}{2}}(x,y,t)$ so
$E[b^n_{\frac{n}{2}}(X_t,Y_t)] = -2\left(\sum_{i,j}\gamma^{\frac{n}{2}}_{ij}v^{j+1}_i(x,y,t)\right) - 2v^0_{\frac{n}{2}}(x,y,t) := \sum_{i,j}\beta^{\frac{n}{2}}_{ij}v^j_i(x,y,t).$

$$(*) \quad b^n_{\frac{n}{2}-1} = x^{\frac{n}{2}+2}y^{\frac{n}{2}-2} + x^{\frac{n}{2}-2}y^{\frac{n}{2}+2}$$
$$= (x+y)(x^{\frac{n}{2}+1}y^{\frac{n}{2}-2} + x^{\frac{n}{2}-2}y^{\frac{n}{2}+1}) - (x^{\frac{n}{2}+1}y^{\frac{n}{2}-1} + x^{\frac{n}{2}-1}y^{\frac{n}{2}+1})$$
$$= (x+y)b^{n-1}_{\frac{n}{2}-1} - b^n_{\frac{n}{2}},$$

but $E[b^{n-1}_{\frac{n}{2}-1}(X_t,Y_t)] = \sum_{i,j}\gamma^{\frac{n}{2}-1}_{ij}v^j_i(x,y,t)$, $E[b^n_{\frac{n}{2}}(X_t,Y_t)] = \sum_{i,j}\beta^{\frac{n}{2}}_{ij}v^j_i(x,y,t)$ so
$E[b^n_{\frac{n}{2}-1}(X_t,Y_t)] = -2\left(\sum_{i,j}\gamma^{\frac{n}{2}-1}_{ij}v^{j+1}_i(x,y,t)\right) - \sum_{i,j}\beta^{\frac{n}{2}}_{ij}v^j_i(x,y,t)$
$:= \sum_{i,j}\beta^{\frac{n}{2}-1}_{ij}v^j_i(x,y,t)$

$\vdots$

(*) $b^n_1 = x^n + y^n = (x+y)(x^{n-1}+y^{n-1}) - (x^{n-1}y + xy^{n-1}) = (x+y)b^{n-1}_1 - b^n_2$,
but $E[b^{n-1}_1(X_t,Y_t)] = \sum_{i,j}\gamma^1_{ij}v^j_i(x,y,t)$, and $E[b^n_2(X_t,Y_t)] = \sum_{i,j}\beta^2_{ij}v^j_i(x,y,t)$ hence
$E[b^n_1(X_t,Y_t)] = -2\left(\sum_{i,j}\gamma^1_{ij}v^{j+1}_i(x,y,t)\right) - \sum_{i,j}\beta^2_{ij}v^j_i(x,y,t)$
$:= \sum_{i,j}\beta^1_{ij}v^j_i(x,y,t)$

Note that all the polynomials b^n_s, $s = \frac{n}{2}-2,\dots,2$, which have not been written here can be written as $b^n_s = (x+y)b^{n-1}_s - b^n_{s+1}$ and hence, $E[b^n_s(X_t,Y_t)] = -2\left(\sum_{i,j}\gamma^s_{ij}v^{j+1}_i(x,y,t)\right) - \sum_{i,j}\beta^{s+1}_{ij}v^j_i(x,y,t)$, where $\sum_{i,j}\beta^{s+1}_{ij}v^j_i(x,y,t) = E[b^n_{s+1}(X_t,Y_t)]$

If n is odd: $B^n = \{b^n_1,\dots,b^n_{\frac{n+1}{2}}\}$ and $B^{n-1} = \{b^{n-1}_1,\dots,b^{n-1}_{\frac{n+1}{2}}\}$,
with $E[b^{n-1}_l(X_t,Y_t)] = \sum_{i,j}\gamma^l_{ij}v^j_i$ for each $l = 1,\dots,\frac{n+1}{2}$

$$(*) \quad b^n_{\frac{n+1}{2}} = x^{\frac{n-1}{2}+1}y^{\frac{n-1}{2}} + x^{\frac{n-1}{2}}y^{\frac{n-1}{2}+1}$$
$$= (x+y)x^{\frac{n-1}{2}}y^{\frac{n-1}{2}}$$
$$= (x+y)b^{n-1}_{\frac{n+1}{2}}$$

but $E[b^{n-1}_{\frac{n+1}{2}}(X_t,Y_t)] = \sum_{i,j}\gamma^{\frac{n+1}{2}}_{ij}v^j_i$, so
$E[b^n_{\frac{n+1}{2}}(X_t,Y_t)] = -2\sum_{i,j}\gamma^{\frac{n+1}{2}}_{ij}v^{j+1}_i := \sum_{i,j}\beta^{\frac{n+1}{2}}_{ij}v^j_i$

(*) $b^n_{\frac{n+1}{2}-1} = x^{\frac{n-1}{2}+2}y^{\frac{n-1}{2}-1} + x^{\frac{n-1}{2}-1}y^{\frac{n-1}{2}+2}$
$= (x+y)(x^{\frac{n-1}{2}+1}y^{\frac{n-1}{2}-1} + x^{\frac{n-1}{2}-1}y^{\frac{n-1}{2}+1})$
$- (x^{\frac{n-1}{2}+1}y^{\frac{n-1}{2}} + x^{\frac{n-1}{2}}y^{\frac{n-1}{2}+1})$
$= (x+y)b^{n-1}_{\frac{n+1}{2}-1} - b^n_{\frac{n+1}{2}}$

but $E[b^{n-1}_{\frac{n+1}{2}-1}(X_t, Y_t)] = \sum_{i,j} \gamma_{ij}^{\frac{n+1}{2}-1} v_i^j$, $E[b^n_{\frac{n+1}{2}}(X_t, Y_t)] = \sum_{i,j} \beta_{ij}^{\frac{n+1}{2}} v_i^j$ so
$E[b^n_{\frac{n+1}{2}}(X_t, Y_t)] = -2\sum_{i,j} \gamma_{ij}^{\frac{n+1}{2}-1} v_i^{j+1} - \sum_{i,j} \beta_{ij}^{\frac{n+1}{2}} v_i^j := \sum_{i,j} \beta_{ij}^{\frac{n+1}{2}-1} v_i^j$
⋮
(*) $b^n_1 = x^n + y^n = (x+y)(x^{n-1} + y^{n-1}) - (x^{n-1}y + xy^{n-1}) = (x+y)b^{n-1}_1 - b^n_2$,
but $E[b^{n-1}_1(X_t, Y_t)] = \sum_{i,j} \gamma^1_{ij} v_i^j(x,y,t)$, and $E[b^n_2(X_t, Y_t)] = \sum_{i,j} \beta^2_{ij} v_i^j(x,y,t)$
hence
$E[b^n_1(X_t, Y_t)] = -2\left(\sum_{i,j} \gamma^1_{ij} v_i^{j+1}(x,y,t)\right) - \sum_{i,j} \beta^2_{ij} v_i^j(x,y,t)$
$:= \sum_{i,j} \beta^1_{ij} v_i^j(x,y,t)$
Note that, again, all the polynomials b^n_s, $s = \frac{n+1}{2} - 2, \ldots, 2$, which have not been written here can be written as $b^n_s = (x+y)b^{n-1}_s - b^n_{s+1}$ and hence,
$E[b^n_s(X_t, Y_t)] = -2\left(\sum_{i,j} \gamma^s_{ij} v_i^{j+1}(x,y,t)\right) - \sum_{i,j} \beta^{s+1}_{ij} v_i^j(x,y,t)$,
where $\sum_{i,j} \beta^{s+1}_{ij} v_i^j(x,y,t) = E[b^n_{s+1}(X_t, Y_t)]$

□

Remark. The above result implies that one need only compute

$$u_i(\lambda_1, \lambda_2, t) = E^{\lambda_1,\lambda_2}[(\mu_1(t)\mu_2(t))^i], \quad \text{for} \quad i = 1, \ldots, \left\lfloor \frac{k}{2} \right\rfloor$$

using Theorem 5.3.4 to be able to compute $E^{\lambda_1,\lambda_2}[p(\mu_1(t), \mu_2(t))]$ for any symmetric polynomial p of degree k through symmetry transformation as described in Theorem 5.3.5.

The following proposition presents only a sample of the results that can be obtained combining Theorem 5.3.4 and Theorem 5.3.5:

Proposition 5.3.7. *Let $W_t \sim WIS(n, 2, W_0)$ with $n \geq 2$ and with eigenvalues $\mu_1(t) > \mu_2(t) \geq 0$. Then*

$$\begin{aligned}
&E^{\lambda_1,\lambda_2}[\mu_1(t)\mu_2(t)] = n(n-1)t^2 + (n-1)t(\lambda_1+\lambda_2) + \lambda_1\lambda_2\\
&E^{\lambda_1,\lambda_2}[\mu_1^2(t)\mu_2^2(t)] = (n-1)n(n+1)(n+2)t^4 + 2(n-1)(n+1)(n+2)(\lambda_1+\lambda_2)t^3\\
&\quad + \Big((n-1)(n+1)(\lambda_1+\lambda_2)^2 + 2(n+1)(n+4)\lambda_1\lambda_2\Big)\, t^2\\
&\quad +2(n+1)(\lambda_1+\lambda_2)\lambda_1\lambda_2 t + \lambda_1^2\lambda_2^2\\
&E^{\lambda_1,\lambda_2}[\mu_1(t)+\mu_2(t)] = 2nt + \lambda_1 + \lambda_2,\\
&E^{\lambda_1,\lambda_2}[\mu_1^2(t)+\mu_2^2(t)] = 2n(n+3)t^2 + 2(n+3)(\lambda_1+\lambda_2)t + \lambda_1^2 + \lambda_2^2,\\
&E^{\lambda_1,\lambda_2}[\mu_1^3(t)+\mu_2^3(t)] = 2n(n+2)(n+7)t^3 + 3(n+2)(n+7)(\lambda_1+\lambda_2)t^2\\
&\quad + \Big(3(n+5)(\lambda_1^2+\lambda_2^2) + 6\lambda_1\lambda_2\Big)\, t + \lambda_1^3 + \lambda_2^3,\\
&E^{\lambda_1,\lambda_2}[\mu_1^4(t)+\mu_2^4(t)] = 2n(n+2)(n^2+16n+47)t^4\\
&\quad +4(n+2)(n^2+16n+47)(\lambda_1+\lambda_2)t^3\\
&\quad +2((n+4)(3n+19)(\lambda_1^2+\lambda_2^2) + (7n+27)(\lambda_1+\lambda_2)^2)t^2\\
&\quad +4((n+7)(\lambda_1^3+\lambda_2^3) + 2\lambda_1\lambda_2(\lambda_1+\lambda_2))t + \lambda_1^4 + \lambda_2^4,\\
&E^{\lambda_1,\lambda_2}[\mu_1^5(t)+\mu_2^5(t)] = 2n(n+2)(n+4)(n^2+24n+103)t^5\\
&\quad +5(n+2)(n+4)(n^2+24n+103)(\lambda_1+\lambda_2)t^4\\
&\quad +10((n+4)(n+6)(n+9)(\lambda_1^2+\lambda_2^2) + 2(n+4)(2n+11)(\lambda_1+\lambda_2)^2)t^3\\
&\quad +5((2n^2+33n+125))(\lambda_1^3+\lambda_2^3) + 3(3n+17)\lambda_1\lambda_2(\lambda_1+\lambda_2))t^2\\
&\quad +5((n+9)(\lambda_1^4+\lambda_2^4) + 2\lambda_1\lambda_2(\lambda_1+\lambda_2)^2 - 2\lambda_1^2\lambda_2^2)t + \lambda_1^5 + \lambda_2^5,\\
&E^{\lambda_1,\lambda_2}[\mu_1^6(t)+\mu_2^6(t)] = 2n(n+2)(n+4)(n+7)(n^2+32n+159)t^6\\
&\quad +6(n+2)(n+4)(n+7)(n^2+32n+159)(\lambda_1+\lambda_2)t^5\\
&\quad +3((n+4)(n+6)(5n^2+100n+479)(\lambda_1^2+\lambda_2^2))t^4\\
&\quad +6(n+4)(3n+19)(5n+33)(\lambda_1+\lambda_2)^2t^4\\
&\quad +2(3(n+7)(25n+153)\lambda_1\lambda_2(\lambda_1+\lambda_2))t^3\\
&\quad +2(10n^3+285n^2+2486n+6819)(\lambda_1^3+\lambda_2^3))t^3\\
&\quad +3((5n^2+102n+493)(\lambda_1^4+\lambda_2^4) + 2(11n+83)\lambda_1\lambda_2(\lambda_1+\lambda_2)^2)t^2\\
&\quad -3(11n+85)\lambda_1^2\lambda_2^2t^2\\
&\quad +6(2\lambda_1\lambda_2(\lambda_1+\lambda_2)^3 - 4(\lambda_1+\lambda_2)\lambda_1^2\lambda_2^2 + (n+11)(\lambda_1^5+\lambda_2^5))t + \lambda_1^6 + \lambda_2^6.
\end{aligned}$$

The scope of these techniques extends to any symmetric polynomial in the eigenvalues, i.e. any polynomial that remains invariant under permutation of the variables. The symmetry property is essential since it is one of the key aspects that ensures in the proof of Theorem 5.3.4 that the last term in the series expansion of

Lu_k (5.39) vanishes.

However, if one is interested in a large number of expected values of the above type, while computing these is possible with the described techniques, one may ask if a general explicit formula may be found for some particular cases. In general, this can become quite complex, but there are some examples for which this is possible. We have obtained a closed expression for the expected value of any power of the trace of a 2-dimensional Wishart process, i.e. we have obtained an explicit expression for $E^{\lambda_1,\lambda_2}[(\mu_1(t)+\mu_2(t))^k]$ for any $k \in \mathbb{N}$:

Proposition 5.3.8. *Let $W_t \sim WIS(n,2,W_0)$ with $n \geq 2$ and with eigenvalues $\mu_1(t) > \mu_2(t) \geq 0$, and let $k \in \mathbb{N}$. Then*

$$E^{\lambda_1,\lambda_2}[(\mu_1(t)+\mu_2(t))^k] = (\lambda_1+\lambda_2)^k + \sum_{i=1}^{k} \frac{(2t)^i}{i!} \left(\prod_{j=1}^{i} (k-j+1)(n+k-j) \right) (\lambda_1+\lambda_2)^{k-i}.$$

Proof. To prove this result one need only observe that

$$u(\lambda_1,\lambda_2,t) := (\lambda_1+\lambda_2)^k + \sum_{i=1}^{k} \frac{(2t)^i}{i!} \left(\prod_{j=1}^{i} (k-j+1)(n+k-j) \right) (\lambda_1+\lambda_2)^{k-i}$$

satisfies the initial condition $u(\lambda_1,\lambda_2,0) = (\lambda_1+\lambda_2)^k$ and that it is a solution to the Kolmogorv Backwards equation for the proccess of the eigenvalues $(\mu_1(t),\mu_2(t))$, i.e. u solves

$$u_t = 2\lambda_1 u_{\lambda_1\lambda_1} + 2\lambda_2 u_{\lambda_2\lambda_2} + \left(n + \frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}\right) u_{\lambda_1} + \left(n - \frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}\right) u_{\lambda_2}$$

Observe that

$$\begin{aligned} u_t &= \sum_{i=1}^{k} \frac{2^i t^{i-1}}{(i-1)!} \left(\prod_{j=1}^{i} (k-j+1)(n+k-j) \right) (\lambda_1+\lambda_2)^{k-i} \\ u_{\lambda_1} &= k(\lambda_1+\lambda_2)^{k-1} + \sum_{i=1}^{k-1} \frac{(2t)^i}{i!} \left(\prod_{j=1}^{i} (k-j+1)(n+k-j) \right) (k-i)(\lambda_1+\lambda_2)^{k-i-1} \\ &= u_{\lambda_2} \\ u_{\lambda_1\lambda_1} &= u_{\lambda_2\lambda_2} = k(k-1)(\lambda_1+\lambda_2)^{k-2} \\ &\quad + \sum_{i=1}^{k-2} \frac{(2t)^i}{i!} \left(\prod_{j=1}^{i} (k-j+1)(n+k-j) \right) (k-i)(k-i-1)(\lambda_1+\lambda_2)^{k-i-2} \end{aligned}$$

Substitution into the right-hand side (RHS) of the above PDE gives:

$$
\begin{aligned}
RHS &= 2k(k-1)(\lambda_1+\lambda_2)^{k-1} \\
&+ \sum_{i=1}^{k-2} \frac{2^{i+1}t^i}{i!} \left(\prod_{j=1}^{i} (k-j+1)(n+k-j) \right) (k-i)(k-i-1)(\lambda_1+\lambda_2)^{k-i-1} \\
&+ 2nk(\lambda_1+\lambda_2)^{k-1} + \sum_{i=1}^{k-1} \frac{2^{i+1}t^i}{i!} \left(\prod_{j=1}^{i} (k-j+1)(n+k-j) \right) (k-i)n(\lambda_1+\lambda_2)^{k-i-1} \\
&= 2k(n+k-1)(\lambda_1+\lambda_2)^{k-1} + \frac{2^k t^{k-1}}{(k-1)!} \left(\prod_{j=1}^{k-1} (k-j+1)(n+k-j) \right) n \\
&+ \sum_{i=1}^{k-2} \frac{2^{i+1}t^i}{i!} \left(\prod_{j=1}^{i} (k-j+1)(n+k-j) \right) (k-i)(n+k-i-1)(\lambda_1+\lambda_2)^{k-i-1} \\
&= 2k(n+k-1)(\lambda_1+\lambda_2)^{k-1} + \frac{2^k t^{k-1}}{(k-1)!} \left(\prod_{j=1}^{k-1} (k-j+1)(n+k-j) \right) n \\
&+ \sum_{i=2}^{k-1} \frac{2^{i}t^{i-1}}{(i-1)!} \left(\prod_{j=1}^{i-1} (k-j+1)(n+k-j) \right) (k-i+1)(n+k-i)(\lambda_1+\lambda_2)^{k-i} \\
&= \sum_{i=1}^{k} \frac{2^{i}t^{i-1}}{(i-1)!} \left(\prod_{j=1}^{i-1} (k-j+1)(n+k-j) \right) (k-i+1)(n+k-i)(\lambda_1+\lambda_2)^{k-i} \\
&= \sum_{i=1}^{k} \frac{2^{i}t^{i-1}}{(i-1)!} \left(\prod_{j=1}^{i} (k-j+1)(n+k-j) \right) (\lambda_1+\lambda_2)^{k-i} = u_t = LHS
\end{aligned}
$$

Therefore u is a solution to the Kolmogorov Backwards equation for the process of the eigenvalues and it has initial condition $u(\lambda_1, \lambda_2, 0) = (\lambda_1 + \lambda_2)^k$, so by part (b) in Theorem 2.2.2

$$E^{\lambda_1,\lambda_2}[(\mu_1(t) + \mu_2(t))^k] = u(\lambda_1, \lambda_2, t)$$

□

5.3.1.1 Extension to p-dimensional Wishart processes

The methods used in the previous section can be easily proven to be extensible to any dimension p. For example, Theorem 5.3.4 naturally extends to the following for a p-dimensional Wishart process.

Theorem 5.3.9. *Let $W_t \sim WIS(n, p, W_0)$ with $n \geq p$ and with eigenvalues $\mu_1(t) > \cdots > \mu_p(t) \geq 0$. Let L_p be the differential operator defined as*

$$L_p = \sum_{i=1}^{p} 2\lambda_i \frac{\partial^2}{\partial \lambda_i^2} + \sum_i \left(n + \sum_{j \neq i} \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j} \right) \frac{\partial}{\partial \lambda_i},$$

and let $p(x_1, \dots, x_p)$ be a symmetric polynomial of degree k. That is, let p be a polynomial with $k = deg(p(x_1, \dots, x_p))$ and such that $p(x_1, \dots, x_p) = (p \circ \sigma)(x_1, \dots, x_p)$, for any permutation $\sigma : \{x_1, \dots, x_p\} \to \{x_1, \dots, x_p\}$.
Take

$$u_0(\lambda_1, \dots, \lambda_p, t) = p(\lambda_1, \dots, \lambda_p)$$

and

$$u_s(\lambda_1, \dots, \lambda_p, t) = \int_0^t L_p u_{s-1}(\lambda_1, \dots, \lambda_p, r) dr + p(\lambda_1, \dots, \lambda_p), \quad \text{for } s = 1, \dots, k.$$

Then

$$E^{\lambda_1, \dots, \lambda_p}[p(\mu_1(t), \dots, \mu_p(t))] = u_k(\lambda_1, \dots, \lambda_p, t).$$

Proof. The proof of this result is similar to that of the 2-dimensional version (Theorem 5.3.4). □

Similarly, Theorem 5.3.5 extends to

Theorem 5.3.10. *Let $W_t \sim WIS(n, p, W_0)$ with $n \geq p$ and with eigenvalues $\mu_1(t) > \cdots > \mu_p(t) \geq 0$. Let L_p be the differential operator defined as*

$$L_p = \sum_{i=1}^{p} 2\lambda_i \frac{\partial^2}{\partial \lambda_i^2} + \sum_i \left(n + \sum_{j \neq i} \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j} \right) \frac{\partial}{\partial \lambda_i},$$

Suppose $u(\lambda_1, \dots, \lambda_p, t) \in C^{2,1}(\mathbb{R}^p \times \mathbb{R})$ is a solution of $u_t = L_p u$, with $u(\lambda_1, \dots, \lambda_p, 0) = f(\lambda_1, \dots, \lambda_p)$ and u is bounded for $t \in K$, for each compact $K \subset \mathbb{R}$, i.e. suppose that $u(\lambda_1, \dots, \lambda_p, t) = E^{\lambda_1, \dots, \lambda_p}[f(\mu_1(t), \dots, \mu_p(t))]$.
Consider

$$v(\lambda_1, \dots, \lambda_p, t) = \left[\frac{d^i}{d\epsilon^i} \tilde{u}_\epsilon(\lambda_1, \dots, \lambda_p, t) \right]_{\epsilon=0} \tag{5.44}$$

for any $i \in \mathbb{N} \cup \{0\}$, where

$$\tilde{u}_\epsilon(\lambda_1, \dots, \lambda_p, t) = \sigma(\lambda_1, \dots, \lambda_p, t, \epsilon) u \left(a_1(\lambda_1, \dots, \lambda_p, t, \epsilon), \dots, a_{p+1}(\lambda_1, \dots, \lambda_p, t, \epsilon) \right)$$

denotes any symmetry of the equation $u_t = L_p u$.
Suppose that $v(\lambda_1, \dots, \lambda_p, 0) = g(\lambda_1, \dots, \lambda_p)$. If $v(\lambda_1, \dots, \lambda_p, t) \in C^{2,1}(\mathbb{R}^p \times \mathbb{R})$, and is

bounded for $t \in K$, for each compact $K \subset \mathbb{R}$, then

$$v(\lambda_1, \dots, \lambda_p, t) = E^{\lambda_1, \dots, \lambda_p}[g(\mu_1(t), \dots, \mu_p(t))].$$

Proof. Again, the proof for this result is similar to that of Theorem 5.3.5. □

Through a similar argument, an extension of Proposition 5.3.6 can also be formulated for general dimension p. Therefore we can compute the expected value for any symmetric polynomial in the eigenvalues of a $p \times p$ Wishart process.

Remark. It is important to remark that what we refer to as a symmetric polynomial in the eigenvalues of a $p \times p$ Wishart process is any polynomial q such that $q(\mu_1, \mu_2, \dots, \mu_p) = (q \circ \sigma)(\mu_1, \mu_2, \dots, \mu_p)$, for any permutation $\sigma : \{\mu_1, \dots, \mu_p\} \to \{\mu_1, \dots, \mu_p\}$. Observe that this does not include polynomials like $\mu_1^2\mu_2 + \mu_1\mu_2^2$ if we are regarding a Wishart process of dimension $p > 2$, while this polynomial does indeed fit the definition if we are working with a $2-$dimensional Wishart process.

Even though the above results give us all the necessary tools to compute the expected value for any symmetric polynomial in the eigenvalues $\mu_1, \dots, \mu_p$ of a $p \times p$ Wishart process, one may observe that these tools involve either calculating powers of the linear operator

$$L_p = \sum_{i=1}^{p} 2\lambda_i \frac{\partial^2}{\partial \lambda_i^2} + \sum_i \left(n + \sum_{j \neq i} \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j} \right) \frac{\partial}{\partial \lambda_i}, \tag{5.45}$$

computing a large number of iterations involving integrals of L_p applied to some functions or, alternatively, computing the symmetries of $u_t = L_p u$. As a general rule, for a fixed p, the higher the degree of the considered polynomial, the more computationally demanding these tasks become. Moreover, if one wishes to study the general case p, these methods can become quite messy, since we need to work with the general form of the operator L_p as well as the general form of the relevant symmetric polynomial.

It turns out that it is possible to obtain a closed form expression in terms of the dimension p for the expectations of some symmetric polynomials. This gives a much more convenient way of evaluating these expected values for these particular polynomials. We provide a sample of these results in the following propositions:

Proposition 5.3.11. *Let $W_t \sim WIS(n, p, W_0)$ with $n \geq p$, and with eigenvalues $\mu_1(t) > \dots > \mu_p(t) \geq 0$, the expected values for the sums $\sum_{i=1}^{p} \mu_i^k(t)$, $k = 1, 2, 3, 4$ are given respectively by the following solutions of the associated Kolmogorov Backward Equation :*

(a) $k = 1$:

$$E^{\lambda_1,\ldots,\lambda_p}\left[\sum_{i=1}^{p}\mu_i(t)\right] : = u_1(\lambda_1,\ldots,\lambda_p,t) := \sum_{i=1}^{p}\lambda_i + pnt$$

(b) $k = 2$:

$$E^{\lambda_1,\ldots,\lambda_p}\left[\sum_{i=1}^{p}\mu_i^2(t)\right] := u_2(\lambda_1,\ldots,\lambda_p,t)$$
$$:= pn^2t^2 + \sum_{i=1}^{p}\lambda_i^2 + 2(p+1)t\sum_{i=1}^{p}\lambda_i + nt\left(p(p+1)t + 2\sum_{i=1}^{p}\lambda_i\right) \tag{5.46}$$

(c) $k = 3$:

$$E^{\lambda_1,\ldots,\lambda_p}\left[\sum_{i=1}^{p}\mu_i^3(t)\right] := u_3(\lambda_1,\ldots,\lambda_p,t)$$
$$:= pn^3t^3 + \sum_{i=1}^{p}\lambda_i^3 + 3((p+1)(p+2)+2)t^2\sum_{i=1}^{p}\lambda_i$$
$$+ 3n^2t^2\left(p(p+1)t + \sum_{i=1}^{p}\lambda_i\right) + 3t\left((p+3)\sum_{i=1}^{p}\lambda_i^2 + 2\sum_{j\neq i}\lambda_i\lambda_j\right)$$
$$+ nt\left[(p(p+1)(p+2)+2p)t^2 + 9(p+1)t\sum_{i=1}^{p}\lambda_i + 3\sum_{i=1}^{p}\lambda_i^2\right] \tag{5.47}$$

(d) $k = 4$:

$$E^{\lambda_1,\ldots,\lambda_p}\left[\sum_{i=1}^{p}\mu_i^4(t)\right] := u_4(\lambda_1,\ldots,\lambda_p,t)$$
$$:= A\left(npt^4 + 4t^3\sum_{i=1}^{p}\lambda_i\right) + 2Bt^2\sum_{i=1}^{p}\lambda_i^2 + 4Ct^2\sum_{j\neq i}\lambda_i\lambda_j$$
$$+ 4Dt\sum_{i=1}^{p}\lambda_i^3 + 8t\sum_{i=1}^{p}\lambda_i^2\left(\sum_{j\neq i}\lambda_j\right) + \sum_{i=1}^{p}\lambda_i^4, \tag{5.48}$$

where

$A := n^3+6(p+1)n^2+n[6(p+1)(p+2)-(p-9)]+[(p+1)(p+2)(p+3)+10(p+1)+4]$,
$B := 3n^2 + 2n(4p+11)[3(p+3)(p+4)+(p+11)]$,
$C := 7n + (7p+13)$ *and*
$D := n + (p+5)$

Proof. We need to show that for each $k = 1, 2, 3, 4$, $u_k(\lambda_1, \ldots, \lambda_p, t)$ has initial condition $u_k(\lambda_1, \ldots, \lambda_p, 0) = \sum_{i=1}^{p} \lambda_i^k$ and solves the PDE $u_t = L_p u$, where the linear operator L_p is defined as in (5.45).

It is straightforward that for each $k \in \{1, 2, 3, 4\}$, u_k satisfies the required initial condition: one need only substitute $t = 0$ in the given expressions and the result trivially follows. Therefore it only remains to show that u_k is a solution of the mentioned PDE in each case:

(a) k=1: On the one hand, we have $(u_1)_t = np$ and, on the other hand

$$\begin{aligned}
L_p u_1 &= \sum_{i=1}^{p} \left(n + \sum_{j \neq i} \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j} \right) (u_1)_{\lambda_i} + \sum_{i=1}^{p} 2\lambda_i (u_1)_{\lambda_i \lambda_i} \\
&= \sum_{i=1}^{p} \left(n + \sum_{j \neq i} \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j} \right) \times 1 + \sum_{i=1}^{p} 2\lambda_i \times 0 \\
&= \sum_{i=1}^{p} n + \sum_{i=1}^{p} \left(\sum_{j \neq i} \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j} \right) \\
&= np,
\end{aligned}$$

Therefore the PDE $(u_1)_t = L_p u_1$ is indeed satisfied.
Note that the sum $\sum_{i=1}^{p} \left(\sum_{j \neq i} \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j} \right) = 0$ since $\frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j} = -\frac{\lambda_j + \lambda_i}{\lambda_j - \lambda_i}, \forall i, j$

(b) k=2: The left-hand side of our PDE is

$$\begin{aligned}
(u_2)_t &= 2pn^2 t + 2(p+1) \sum_{i=1}^{p} \lambda_i + 2np(p+1)t + 2n \sum_{i=1}^{p} \lambda_i \\
&= 2pnt(n + p + 1) + 2(p + 1 + n) \sum_{i=1}^{p} \lambda_i \\
&= 2(p + 1 + n) \left(pnt + \sum_{i=1}^{p} \lambda_i \right)
\end{aligned}$$

On the other hand, we have that the right-hand side is

$$\begin{aligned}
L_p u_2 &= \sum_{i=1}^{p} \left(n + \sum_{j \neq i} \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j} \right) (u_2)_{\lambda_i} + \sum_{i=1}^{p} 2\lambda_i (u_2)_{\lambda_i \lambda_i} \\
&= \sum_{i=1}^{p} \left(n + \sum_{j \neq i} \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j} \right) (2\lambda_i + 2(p+1)t + 2nt) + \sum_{i=1}^{p} 2\lambda_i \times 2
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{p} n\left(2\lambda_i + 2(p+1)t + 2nt\right) + 4\sum_{i=1}^{p}\lambda_i \\
&+ \sum_{i=1}^{p}\left(\sum_{j\neq i}\frac{\lambda_i+\lambda_j}{\lambda_i-\lambda_j}\right)\left(2\lambda_i + 2(p+1)t + 2nt\right) \\
&=(2n+4)\sum_{i=1}^{p}\lambda_i + 2np(p+1)t + 2n^2pt + 2\sum_{i=1}^{p}\left(\sum_{j\neq i}\frac{\lambda_i+\lambda_j}{\lambda_i-\lambda_j}\right)\lambda_i \\
&=(2n+4)\sum_{i=1}^{p}\lambda_i + 2ntp(p+1+n) + 2(p-1)\sum_{i=1}^{p}\lambda_i \\
&=(2n+4+2p-2)\sum_{i=1}^{p}\lambda_i + 2ntp(p+1+n) \\
&=2(n+1+p)\sum_{i=1}^{p}\lambda_i + 2ntp(p+1+n) \\
&=2(p+1+n)\left(pnt + \sum_{i=1}^{p}\lambda_i\right).
\end{aligned}
$$

So we have verified that u_2 satisfies $(u_2)_t = L_p u_2$ as required.

(c) k=3: In this case, on the one hand we have:

$$
\begin{aligned}
(u_3)_t = {} & 3pn^3t^2 + 6((p+1)(p+2)+2)t\sum_{i=1}^{p}\lambda_i \\
& + 9n^2p(p+1)t^2 + 6n^2t\sum_{i=1}^{p}\lambda_i + 3\left((p+3)\sum_{i=1}^{p}\lambda_i^2 + 2\sum_{j\neq i}\lambda_i\lambda_j\right) \\
& + 3nt^2(p(p+1)(p+2)+2p) + 18nt(p+1)\sum_{i=1}^{p}\lambda_i + 3n\sum_{i=1}^{p}\lambda_i^2 \\
= {} & t^2\left[3np^3 + 9n(n+1)p^2 + 3np(n^2+3n+4)\right] \\
& + t\sum_{i=1}^{p}\lambda_i\left[6p^2 + 18(n+1)p + 6(n^2+3n+4)\right] \\
& + \sum_{i=1}^{p}\lambda_i^2(3p+3(n+3)) + 6\sum_{j\neq i}\lambda_i\lambda_j,
\end{aligned}
$$

while on the other hand, the RHS of the PDE is

$$
\begin{aligned}
L_p u_3 =& \sum_{i=1}^{p}\left(n+\sum_{j\neq i}\frac{\lambda_i+\lambda_j}{\lambda_i-\lambda_j}\right)(u_3)_{\lambda_i}+\sum_{i=1}^{p}2\lambda_i(u_3)_{\lambda_i\lambda_i}\\
=&\sum_{i=1}^{p}2\lambda_i\underbrace{\left[6\lambda_i+6t(p+(n+3))\right]}_{(u_3)_{\lambda_i\lambda_i}}+\sum_{i=1}^{p}\left(n+\sum_{j\neq i}\frac{\lambda_i+\lambda_j}{\lambda_i-\lambda_j}\right)\\
&\underbrace{\times\left(3\lambda_i^2+6t\lambda_i(p+(n+3))+6t\sum_{j\neq i}\lambda_j+3t^2\left(p^2+3p(n+1)+n^2+3n+4\right)\right)}_{(u_3)_{\lambda_i}}\\
=&12\sum_{i=1}^{p}\lambda_i^2+12t(p+(n+3))\sum_{i=1}^{p}\lambda_i+3n\sum_{i=1}^{p}\lambda_i^2+6tn(p+(n+3))\sum_{i=1}^{p}\lambda_i\\
&+6tn\sum_{i=1}^{p}\sum_{j\neq i}\lambda_j+3t^2np\left(p^2+3p(n+1)+n^2+3n+4\right)+\sum_{i=1}^{p}\left(\sum_{j\neq i}\frac{\lambda_i+\lambda_j}{\lambda_i-\lambda_j}\right)\\
&\times\left(3\lambda_i^2+6t\lambda_i(p+(n+3))+6t\sum_{j\neq i}\lambda_j+3t^2\left(p^2+3p(n+1)+n^2+3n+4\right)\right)\\
=&(12+3n)\sum_{i=1}^{p}\lambda_i^2+6t((n+2)p+(n^2+5n+6))\sum_{i=1}^{p}\lambda_i+6tn(p-1)\sum_{i=1}^{p}\lambda_i\\
&+t^2\left(3np^3+9np^2(n+1)+3np(n^2+3n+4)\right)+\sum_{i=1}^{p}\left(\sum_{j\neq i}\frac{\lambda_i+\lambda_j}{\lambda_i-\lambda_j}\right)\\
&\times\left(3\lambda_i^2+6t\lambda_i(p+(n+3))+6t\sum_{j\neq i}\lambda_j\right)\\
=&(12+3n)\sum_{i=1}^{p}\lambda_i^2+6t(2(n+1)p+(n^2+4n+6))\sum_{i=1}^{p}\lambda_i+\\
&+t^2\left(3np^3+9np^2(n+1)+3np(n^2+3n+4)\right)+3\sum_{i=1}^{p}\left(\sum_{j\neq i}\frac{\lambda_i+\lambda_j}{\lambda_i-\lambda_j}\right)\lambda_i^2\\
&+6t(p+(n+3))\sum_{i=1}^{p}\left(\sum_{j\neq i}\frac{\lambda_i+\lambda_j}{\lambda_i-\lambda_j}\right)\lambda_i+6t\sum_{i=1}^{p}\left(\sum_{j\neq i}\frac{\lambda_i+\lambda_j}{\lambda_i-\lambda_j}\right)\left(\sum_{j\neq i}\lambda_j\right)\\
=&(12+3n)\sum_{i=1}^{p}\lambda_i^2+6t(2(n+1)p+(n^2+4n+6))\sum_{i=1}^{p}\lambda_i+\\
&+t^2\left(3np^3+9np^2(n+1)+3np(n^2+3n+4)\right)+3(p-1)\sum_{i=1}^{p}\lambda_i^2\\
&+6\sum_{j\neq i}\lambda_i\lambda_j+6t(p+(n+3))(p-1)\sum_{i=1}^{p}\lambda_i-6t(p-1)\sum_{i=1}^{p}\lambda_i
\end{aligned}
$$

$$
\begin{aligned}
&= t^2 \left(3np^3 + 9np^2(n+1) + 3np(n^2+3n+4)\right) \\
&+ 6t \sum_{i=1}^{p} \lambda_i (p^2 + 3(n+1)p + (n^2+3n+4)) \\
&+ \sum_{i=1}^{p} \lambda_i^2 (3p + 3(n+3)) + 6 \sum_{j \neq i} \lambda_i \lambda_j
\end{aligned}
$$

Again, it is verified that $(u_3)_t = L_p u_3$ as claimed.

(d) k=4: Let us first compute the appropriate partial derivatives of u_4:

$$
\begin{cases}
(u_4)_t & = 4Anpt^3 + 12At^2 \sum_{i=1}^{p} \lambda_i + 4Bt \sum_{i=1}^{p} \lambda_i^2 + 8Ct \sum_{j \neq i} \lambda_i \lambda_j \\
& \quad + 4D \sum_{i=1}^{p} \lambda_i^3 + 8 \sum_{i=1}^{p} \lambda_i^2 \left(\sum_{j \neq i} \lambda_j \right) \\
(u_4)_{\lambda_i} & = 4At^3 + 4Bt^2 \lambda_i + 4Ct^2 \sum_{j \neq i} \lambda_j + 12Dt\lambda_i^2 + 16t\lambda_i \sum_{j \neq i} \lambda_j \\
& \quad + 8t \sum_{j \neq i} \lambda_j^2 + 4\lambda_i^3 \\
(u_4)_{\lambda_i \lambda_i} & = 4Bt^2 + 24Dt\lambda_i + 16t \sum_{j \neq i} \lambda_j + 12\lambda_i^2,
\end{cases}
$$

where A, B, C and D are defined as in the statement of the proposition. Then, the RHS of our PDE becomes:

$$
\begin{aligned}
L_p u_4 &= \sum_{i=1}^{p} \left(n + \sum_{j \neq i} \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j} \right) (u_4)_{\lambda_i} + \sum_{i=1}^{p} 2\lambda_i (u_4)_{\lambda_i \lambda_i} \\
&= \sum_{i=1}^{p} n4At^3 + 4Bt^2 \sum_{i=1}^{p} n\lambda_i + 4Ct^2 \sum_{i=1}^{p} n \sum_{j \neq i} \lambda_j + 12Dt \sum_{i=1}^{p} n\lambda_i^2 \\
&+ 16t \sum_{i=1}^{p} n\lambda_i \sum_{j \neq i} \lambda_j + 8t \sum_{i=1}^{p} n \sum_{j \neq i} \lambda_j^2 + 4 \sum_{i=1}^{p} n\lambda_i^3 \\
&+ 4At^3 \sum_{i=1}^{p} \left(\sum_{j \neq i} \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j} \right) + 4Bt^2 \sum_{i=1}^{p} \left(\sum_{j \neq i} \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j} \right) \lambda_i \\
&+ 4Ct^2 \sum_{i=1}^{p} \left(\sum_{j \neq i} \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j} \right) \sum_{j \neq i} \lambda_j + 12Dt \sum_{i=1}^{p} \left(\sum_{j \neq i} \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j} \right) \lambda_i^2 \\
&+ 16t \sum_{i=1}^{p} \left(\sum_{j \neq i} \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j} \right) \lambda_i \sum_{j \neq i} \lambda_j + 8t \sum_{i=1}^{p} \left(\sum_{j \neq i} \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j} \right) \sum_{j \neq i} \lambda_j^2 \\
&+ 4 \sum_{i=1}^{p} \left(\sum_{j \neq i} \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j} \right) \lambda_i^3 + 4Bt^2 \sum_{i=1}^{p} 2\lambda_i + 24Dt \sum_{i=1}^{p} 2\lambda_i \lambda_i
\end{aligned}
$$

$$
\begin{aligned}
&+ 16t\sum_{i=1}^{p} 2\lambda_i \sum_{j\neq i}\lambda_j + 12\sum_{i=1}^{p} 2\lambda_i\lambda_i^2 \\
&= 4npAt^3 + 4Bnt^2\sum_{i=1}^{p}\lambda_i + 4Cnt^2 \underbrace{\sum_{i=1}^{p}\sum_{j\neq i}\lambda_j}_{(p-1)\sum_{i=1}^{p}\lambda_i} + 12nDt\sum_{i=1}^{p}\lambda_i^2 \\
&+ 16nt\underbrace{\sum_{i=1}^{p}\lambda_i\sum_{j\neq i}\lambda_j}_{2\sum_{j\neq i}\lambda_i\lambda_j} + 8nt \underbrace{\sum_{i=1}^{p}\sum_{j\neq i}\lambda_j^2}_{(p-1)\sum_{i=1}^{p}\lambda_i^2} + 4n\sum_{i=1}^{p}\lambda_i^3 \\
&+ 4At^3\underbrace{\sum_{i=1}^{p}\left(\sum_{j\neq i}\frac{\lambda_i+\lambda_j}{\lambda_i-\lambda_j}\right)}_{=0} + 4Bt^2\underbrace{\sum_{i=1}^{p}\left(\sum_{j\neq i}\frac{\lambda_i+\lambda_j}{\lambda_i-\lambda_j}\right)\lambda_i}_{(p-1)\sum_{i=1}^{p}\lambda_i} \\
&+ 4Ct^2\underbrace{\sum_{i=1}^{p}\left(\sum_{j\neq i}\frac{\lambda_i+\lambda_j}{\lambda_i-\lambda_j}\right)\sum_{j\neq i}\lambda_j}_{-(p-1)\sum_{i=1}^{p}\lambda_i} + 12Dt\underbrace{\sum_{i=1}^{p}\left(\sum_{j\neq i}\frac{\lambda_i+\lambda_j}{\lambda_i-\lambda_j}\right)\lambda_i^2}_{(p-1)\sum_{i=1}^{p}\lambda_i^2 + 2\sum_{j\neq i}\lambda_i\lambda_j} \\
&+ 16t\underbrace{\sum_{i=1}^{p}\left(\sum_{j\neq i}\frac{\lambda_i+\lambda_j}{\lambda_i-\lambda_j}\right)\lambda_i\sum_{j\neq i}\lambda_j}_{2(p-2)\sum_{j\neq i}\lambda_i\lambda_j} + 8t\underbrace{\sum_{i=1}^{p}\left(\sum_{j\neq i}\frac{\lambda_i+\lambda_j}{\lambda_i-\lambda_j}\right)\sum_{j\neq i}\lambda_j^2}_{-(p-1)\sum_{i=1}^{p}\lambda_i^2 - 2\sum_{j\neq i}\lambda_i\lambda_j} \\
&+ 4\underbrace{\sum_{i=1}^{p}\left(\sum_{j\neq i}\frac{\lambda_i+\lambda_j}{\lambda_i-\lambda_j}\right)\lambda_i^3}_{(p-1)\sum_{i=1}^{p}\lambda_i^3 + 2\sum_{i=1}^{p}\lambda_i^2\left(\sum_{j\neq i}\lambda_j\right)} + 8Bt^2\sum_{i=1}^{p}\lambda_i + 48Dt\sum_{i=1}^{p}\lambda_i^2 \\
&+ 32t\underbrace{\sum_{i=1}^{p}\lambda_i\sum_{j\neq i}\lambda_j}_{2\sum_{j\neq i}\lambda_i\lambda_j} + 24\sum_{i=1}^{p}\lambda_i^3 \\
&= t^2\sum_{i=1}^{p}\lambda_i\left[4Bn + 4B(p-1) + 8B + 4Cn(p-1) - 4C(p-1)\right] \\
&+ t\sum_{i=1}^{p}\lambda_i^2\left[12nD + 48D + 12D(p-1) + 8n(p-1) - 8(p-1)\right] \\
&+ t\sum_{j\neq i}\lambda_i\lambda_j\left[32n + 32(p-2) - 16 + 64 + 24D\right] \\
&+ \sum_{i=1}^{p}\lambda_i^3\left[4n + 4(p-1) + 24\right] + 8\sum_{i=1}^{p}\lambda_i^2\left(\sum_{j\neq i}\lambda_j\right) + 4npAt^3
\end{aligned}
$$

$$
\begin{aligned}
&= 4t^2 \sum_{i=1}^{p} \lambda_i \left[B(n+p+1) + C(n-1)(p-1)\right] \\
&+ 4t \sum_{i=1}^{p} \lambda_i^2 \left[D(3n+3p+9) + 2(n-1)(p-1)\right] \\
&+ 8t \sum_{j \neq i} \lambda_i \lambda_j \left[4n + 4p - 2 + 3D\right] + 4 \sum_{i=1}^{p} \lambda_i^3 \left[n+p+5\right] \\
&+ 8 \sum_{i=1}^{p} \lambda_i^2 \left(\sum_{j \neq i} \lambda_j \right) + 4npAt^3 \\
&= (u_4)_t,
\end{aligned}
$$

since it can be easily shown that:

$$
\begin{cases}
12A &= 4\left[B(n+p+1) + C(n-1)(p-1)\right] \\
4B &= 4\left[D(3n+3p+9) + 2(n-1)(p-1)\right] \\
8C &= 8\left[4n + 4p - 2 + 3D\right]
\end{cases}
$$

Hence the PDE $(u_4)_t = L_p u_4$ is satisfied.

So for each case, we have verified that $(u_k)_t = L_p u_k$, $k \in \{1, 2, 3, 4\}$ and $u_k(\lambda_1, \ldots, \lambda_p, 0) = \sum_{i=1}^{p} \lambda_i^k$. Hence, by part (b) of Theorem 2.2.2, we have that for each k

$$
u_k(\lambda_1, \ldots, \lambda_p, t) = E^{\lambda_1, \ldots, \lambda_p} \left[\sum_{i=1}^{p} \mu_i^k(t) \right]
$$

as claimed. □

Note. The case $k = 1$ in the above proposition gives the expected value of the trace of W_t. That is, for $W_t \sim WIS(n, p, W_0)$ with $n \geq p$ and eigenvalues $\mu_1(t) > \cdots > \mu_p(t) \geq 0$ with $\mu_i(0) = \lambda_i$, $i = 1, \ldots, p$ we have

$$
E\left[Tr(W_t)\right] = \sum_{i=1}^{p} \lambda_i + pnt
$$

We now provide an explicit expression for the expected value of the determinant of a p-dimensional Wishart process:

Proposition 5.3.12. *Let $J = \{1, \ldots, p\}$ and let $W_t \sim WIS(n, p, W_0)$ with $n \geq p$ and with eigenvalues $\mu_1(t) > \cdots > \mu_p(t) \geq 0$. Then, the expected value for the product of the eigenvalues (that is, the expected value of the determinant of the process W_t) is given by the* *following solution of the Kolmogorov's Backward Equation:*

$$E^{\lambda_1,\ldots,\lambda_p}\left[\prod_{i=1}^{p}\mu_i(t)\right] = E^{\lambda}\left[\det(W_t)\right] = u(\lambda_1,\ldots,\lambda_p,t)$$
$$= \prod_{i=1}^{p}\lambda_i + \sum_{i=1}^{p} t^i \left(\prod_{k=1}^{i}(n-p+k)\right)\left(\sum_{\substack{J_m\subset J,\\ |J_m|=p-i}}\prod_{r\in J_m}\lambda_r\right) \tag{5.49}$$

Proof. As in the previous proposition, we must only check that the given expression u is indeed a solution of $u_t = L_p u$ with initial condition $u(\lambda_1,\ldots,\lambda_p,0) = \prod_{i=1}^{p}\lambda_i$. Again, it is trivial that the initial condition is satisfied since:

$$u(\lambda_1,\ldots,\lambda_p,0) = \prod_{i=1}^{p}\lambda_i + \sum_{i=1}^{p} 0^i \left(\prod_{k=1}^{i}(n-p+k)\right)\left(\sum_{\substack{J_m\subset J,\\ |J_m|=p-i}}\prod_{r\in J_m}\lambda_r\right) = \prod_{i=1}^{p}\lambda_i.$$

Now, to check that u is indeed a solution to the required PDE, we have :

$$u_t = \sum_{i=1}^{p} i t^{i-1}\left(\prod_{k=1}^{i}(n-p+k)\right)\left(\sum_{\substack{J_m\subset J,\\ |J_m|=p-i}}\prod_{r\in J_m}\lambda_r\right)$$
$$= \underbrace{(n-p+1)\sum_{\substack{J_m\subset J,\\ |J_m|=p-1}}\prod_{r\in J_m}\lambda_r}_{I} + \underbrace{\sum_{i=2}^{p} i t^{i-1}\left(\prod_{k=1}^{i}(n-p+k)\right)\left(\sum_{\substack{J_m\subset J,\\ |J_m|=p-i}}\prod_{r\in J_m}\lambda_r\right)}_{II}$$

and

$$L_p u = \sum_{i=1}^{p}\left(n + \sum_{j\neq i}\frac{\lambda_i+\lambda_j}{\lambda_i-\lambda_j}\right)u_{\lambda_i} + \sum_{i=1}^{p} 2\lambda_i \underbrace{u_{\lambda_i\lambda_i}}_{=0}$$
$$= \sum_{i=1}^{p}\left(n + \sum_{j\neq i}\frac{\lambda_i+\lambda_j}{\lambda_i-\lambda_j}\right)\left(\prod_{\substack{j=1\\ j\neq i}}^{p}\lambda_j + \sum_{j=1}^{p-1} t^j \left(\prod_{k=1}^{j}(n-p+k)\right)\left(\sum_{\substack{J_m\subset J\setminus\{i\},\\ |J_m|=p-j-1}}\prod_{r\in J_m}\lambda_r\right)\right)$$

$$= \underbrace{\sum_{i=1}^{p} n \prod_{\substack{j=1\\ j\neq i}}^{p} \lambda_j}_{A} + \underbrace{\sum_{i=1}^{p} \left(\sum_{j\neq i} \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j} \right) \left(\sum_{j=1}^{p-1} t^j \left(\prod_{k=1}^{j} (n-p+k) \right) \left(\sum_{\substack{J_m \subset J\setminus\{i\},\\ |J_m|=p-j-1}} \prod_{r\in J_m} \lambda_r \right) \right)}_{B}$$

$$+ \underbrace{\sum_{i=1}^{p} \left(\sum_{j\neq i} \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j} \right) \prod_{\substack{j=1\\ j\neq i}}^{p} \lambda_j}_{C} + \underbrace{\sum_{i=1}^{p} n \left(\sum_{j=1}^{p-1} t^j \left(\prod_{k=1}^{j} (n-p+k) \right) \left(\sum_{\substack{J_m \subset J\setminus\{i\},\\ |J_m|=p-j-1}} \prod_{r\in J_m} \lambda_r \right) \right)}_{D}$$

Observe that the individual terms A, B, C and D can be expressed respectively as follows:

$$A = n \sum_{\substack{J_m \subset J,\\ |J_m|=p-1}} \prod_{r\in J_m} \lambda_r,$$

$$C = -(p-1) \sum_{\substack{J_m \subset J,\\ |J_m|=p-1}} \prod_{r\in J_m} \lambda_r,$$

$$B = -\sum_{j=1}^{p-1} (p-(j+1))(j+1) t^j \left(\prod_{k=1}^{j} (n-p+k) \right) \left(\sum_{\substack{J_m \subset J,\\ |J_m|=p-j-1}} \prod_{r\in J_m} \lambda_r \right)$$

$$= -\sum_{j=2}^{p} (p-j) j t^{j-1} \left(\prod_{k=1}^{j-1} (n-p+k) \right) \left(\sum_{\substack{J_m \subset J,\\ |J_m|=p-j}} \prod_{r\in J_m} \lambda_r \right),$$

$$D = \sum_{j=1}^{p-1} n(j+1) t^j \left(\prod_{k=1}^{j} (n-p+k) \right) \left(\sum_{\substack{J_m \subset J,\\ |J_m|=p-j-1}} \prod_{r\in J_m} \lambda_r \right)$$

$$= \sum_{j=2}^{p} n j t^{j-1} \left(\prod_{k=1}^{j-1} (n-p+k) \right) \left(\sum_{\substack{J_m \subset J,\\ |J_m|=p-j}} \prod_{r\in J_m} \lambda_r \right)$$

Substitution back into L_p gives:

$$
\begin{aligned}
L_p u =& n \sum_{\substack{J_m \subset J, \\ |J_m|=p-1}} \prod_{r \in J_m} \lambda_r - (p-1) \sum_{\substack{J_m \subset J, \\ |J_m|=p-1}} \prod_{r \in J_m} \lambda_r \\
&+ \sum_{j=2}^{p} n j t^{j-1} \left(\prod_{k=1}^{j-1} (n-p+k) \right) \left(\sum_{\substack{J_m \subset J, \\ |J_m|=p-j}} \prod_{r \in J_m} \lambda_r \right) \\
&- \sum_{j=2}^{p} (p-j) j t^{j-1} \left(\prod_{k=1}^{j-1} (n-p+k) \right) \left(\sum_{\substack{J_m \subset J, \\ |J_m|=p-j}} \prod_{r \in J_m} \lambda_r \right) \\
&= (n-(p-1)) \sum_{\substack{J_m \subset J, \\ |J_m|=p-1}} \prod_{r \in J_m} \lambda_r \\
&+ \sum_{j=2}^{p} (n-(p-j)) j t^{j-1} \left(\prod_{k=1}^{j-1} (n-p+k) \right) \left(\sum_{\substack{J_m \subset J, \\ |J_m|=p-j}} \prod_{r \in J_m} \lambda_r \right) \\
&= (n-p+1) \sum_{\substack{J_m \subset J, \\ |J_m|=p-1}} \prod_{r \in J_m} \lambda_r + \sum_{j=2}^{p} j t^{j-1} \left(\prod_{k=1}^{j} (n-p+k) \right) \left(\sum_{\substack{J_m \subset J, \\ |J_m|=p-j}} \prod_{r \in J_m} \lambda_r \right),
\end{aligned}
$$

which is exactly the same as expressions I and II in u_t.

Therefore, we have that the function u defined as in (5.49) is a solution of $u_t = L_p u$ with initial condition $u(\lambda_1, \ldots, \lambda_p, 0) = \prod_{i=1}^{p} \lambda_i$ and so, by part (b) of the Kolmogorov Backward equation theorem (Theorem 2.2.2),

$$
u(\lambda_1, \ldots, \lambda_p, t) = E^{\lambda_1, \ldots, \lambda_p} \left[\prod_{i=1}^{p} \mu_i(t) \right]
$$

as claimed. □

It can be easily appreciated that the general expressions for the expectations in the $p-$dimensional case get substantially more complicated as we increase the degree of the polynomials. Thus, we have not included any more cases in this work. Theorems 5.3.9 and 5.3.10 will produce the expected value for any particular symmetric polynomial we wish to consider.

5.3.2 A method to extend the computations of expected values to a wider class of functions in the eigenvalues of a Wishart process

In this section we present some results that allow us to extend the range of functions f for which we can compute $E[f(\mu_1(t), \mu_2(t))]$. Note that Theorem 5.3.5 in the previous section does precisely this through the use of lie symmetries. This particular theorem is not only valid to obtain expected values of symmetric polynomials, but for a much wider range of functions.

Similarly to the above mentioned theorem, the results in this section extend the scope of our results to a wider class of functions f but, in this case, by making use of a combination of the basic tools of Itô calculus and the results obtained in the previous section. The results presented here are for the 2-dimensional case for simplicity.

Remark. From this point on, we will typically refer to the eigenvalues of a Wishart process as X_t and Y_t to ease the notation. Even though this new notation looks somewhat less natural for eigenvalues, it will make all the upcoming expressions a lot simpler. In this and the upcoming sections, we will consider $W_t \sim WIS(n, 2, W_0)$, with $n \geq 2$ and with eigenvalues $X(t) > Y(t) \geq 0$, that we will usually denote by $X_t := X(t)$, $Y_t := Y(t)$.

Theorem 5.3.13. *Let $W_t \sim WIS(n, 2, W_0)$ with index $n \geq 2$ and with eigenvalues $X(t) > Y(t) \geq 0$. Let P denote the set of all polynomials in two variables with coefficients in $\mathbb{R}$ and let $P_S = \{p(x, y) \in P | p(x, y) = p(y, x)\}$, i.e. P_S denotes the set of all symmetric polynomials in 2 variables with real coefficients. Let L be the linear operator defined as:*

$$L = 2x\frac{\partial^2}{\partial x^2} + 2y\frac{\partial^2}{\partial y^2} + \left(n + \frac{x+y}{x-y}\right)\frac{\partial}{\partial x} + \left(n - \frac{x+y}{x-y}\right)\frac{\partial}{\partial y} \tag{5.50}$$

Then, for any function f such that

$$Lf = g, \quad g \in P_S$$

we have that

$$E[f(X_t, Y_t)] = f(x, y) + \int_0^t E[g(X_s, Y_s)]ds,$$

where the integrand in the above expression can be calculated using the methodology described in Theorems 5.3.4 and 5.3.5.

Proof. Let $g \in P_S$. Theorems 5.3.4 and 5.3.5 have been proven to produce an expression for $E[g(X_t, Y_t)]$ for all $g \in P_S$.

Now let f be a function satisfying $Lf = g$. Then, by Itô's formula, we have that

$$f(X_t, Y_t) = f(x, y) + \int_0^t (Lf)(X_s, Y_s)ds + M_t, \tag{5.51}$$

where the term M_t is a martingale. Therefore, taking expectations yields

$$\begin{aligned} E[f(X_t, Y_t)] &= f(x, y) + \int_0^t E[g(X_s, Y_s)]ds + E[M_t] \\ &= f(x, y) + \int_0^t E[g(X_s, Y_s)]ds \end{aligned}$$

as claimed. Furthermore the integrand in the above expression is known since $g \in P_S$. □

Corollary. *Let $W_t \sim WIS(n, 2, W_0)$ with index $n \geq 2$ and with eigenvalues $X(t) > Y(t) \geq 0$. Let L be the linear operator defined as:*

$$L = 2x\frac{\partial^2}{\partial x^2} + 2y\frac{\partial^2}{\partial y^2} + \left(n + \frac{x+y}{x-y}\right)\frac{\partial}{\partial x} + \left(n - \frac{x+y}{x-y}\right)\frac{\partial}{\partial y} \tag{5.52}$$

Then, for any function f such that

$$L^k f = g, \quad g \in P_S$$

we have that

$$E[f(X_t, Y_t)] = \sum_{n=0}^{k-1} \frac{t^n}{n!} L^n f(x, y) + \int_0^t \int_0^{t_{k-1}} \cdots \int_0^{t_1} E[g(X_s, Y_s)]ds dt_1 \ldots dt_{k-1},$$

where the integrand in the above expression can be calculated using the methodology described in Theorems 5.3.4 and 5.3.5.

Proof. To prove this Corollary one need only use Theorem 5.3.14 iteratively k times. Note that for $k = 1$ we have that $Lf = g$ and we get exactly the statement of Theorem 5.3.14:

$$E[f(X_t, Y_t)] = f(x, y) + \int_0^t E[g(X_s, Y_s)]ds,$$

which has already been proved.
For $k = 2$, we have that $L^2 f = g$. Let us denote $Lf := f_1$. We then have $L^2 f = L(Lf) = Lf_1 = g$. Similarly to the proof of Theorem 5.3.14, Itô's formula yields:

$$f(X_t, Y_t) = f(x, y) + \int_0^t (Lf)(X_s, Y_s)ds + M_t,$$
$$= f(x, y) + \int_0^t f_1(X_s, Y_s)ds + M_t,$$

where M_t is a martingale. Hence, taking expectations:

$$E[f(X_t, Y_t)] = f(x, y) + \int_0^t E[f_1(X_s, Y_s)]ds + E[M_t],$$
$$= f(x, y) + \int_0^t \left(f_1(x, y) + \int_0^{t_1} E[g(X_s, Y_s)]ds \right) dt_1$$
$$= f(x, y) + t f_1(x, y) \int_0^t \int_0^{t_1} E[g(X_s, Y_s)]dsdt_1,$$

as claimed. Observe that the expression used for $E[f_1(X_s, Y_s)]$ above comes directly from Theorem 5.3.14, or form the case $k = 1$ in this Corollary.

We need only show that if the result is true for $k = n - 1$, i.e. if for any h such that $L^{n-1}h = g$, $g \in P_S$ it is satisfied that

$$E[h(X_t, Y_t)] = \sum_{i=0}^{n-2} \frac{t^i}{i!} L^i h(x, y) + \int_0^t \int_0^{t_{n-2}} \cdots \int_0^{t_1} E[g(X_s, Y_s)]dsdt_1 \dots dt_{n-2},$$

then the result also holds for $k = n$. Hence, by induction, the result will be proven to be true for all k. Observe that for $k = n$ we have $L^n f = g$. If we denote $Lf := f_{n-1}$ we have that $L^n f = L^{n-1}(Lf) = L^{n-1}(f_{n-1}) = g$. Then, as in the previous cases, Itô's formula gives

$$f(X_t, Y_t) = f(x, y) + \int_0^t (Lf)(X_s, Y_s)ds + M_t,$$
$$= f(x, y) + \int_0^t f_{n-1}(X_s, Y_s)ds + M_t,$$

where M_t is a martingale. And again, computing the expectations yields

$$E[f(X_t, Y_t)] = f(x, y) + \int_0^t E[f_{n-1}(X_s, Y_s)]ds + E[M_t],$$
$$= f(x, y) + \int_0^t \sum_{i=0}^{n-2} \frac{t_{n-1}^i}{i!} L^i f_{n-1}(x, y)dt_{n-1}$$
$$+ \int_0^t \int_0^{t_{n-1}} \int_0^{t_{n-2}} \cdots \int_0^{t_1} E[g(X_s, Y_s)]dsdt_1 \dots dt_{n-2}dt_{n-1}$$

But note that

$$f(x,y)+\int_0^t\sum_{i=0}^{n-2}\frac{t_{n-1}^i}{i!}L^i f_{n-1}(x,y)dt_{n-1}=f(x,y)+\int_0^t\sum_{i=0}^{n-2}\frac{t_{n-1}^i}{i!}L^i(Lf)(x,y)dt_{n-1}$$
$$=f(x,y)+\sum_{i=0}^{n-2}\frac{t^{i+1}}{(i+1)!}L^{i+1}f(x,y)=f(x,y)+\sum_{i=1}^{n-1}\frac{t^i}{i!}L^i f(x,y)$$
$$=\sum_{i=0}^{n-1}\frac{t^i}{i!}L^i f(x,y)$$

So we have that

$$E[f(X_t,Y_t)]=\sum_{i=0}^{n-1}\frac{t^i}{i!}L^i f(x,y)+\int_0^t\int_0^{t_{n-1}}\cdots\int_0^{t_1}E[g(X_s,Y_s)]dsdt_1\dots dt_{n-1},$$

as required. Hence, by induction, we have that for all $k\in\mathbb{N}$, if $L^k f=g,\quad g\in P_S$ then

$$E[f(X_t,Y_t)]=\sum_{n=0}^{k-1}\frac{t^n}{n!}L^n f(x,y)+\int_0^t\int_0^{t_{k-1}}\cdots\int_0^{t_1}E[g(X_s,Y_s)]dsdt_1\dots dt_{k-1},$$

where the integrand $E[g(X_s,Y_s)]$ can be calculated according to Proposition 5.3.4.
□

We will illustrate this methodology by presenting an example in what follows:

Example 5.3.2. Consider the PDE:

$$Lf(x,y)=x+y, \tag{5.53}$$

where the differential operator L is defined as in (5.58).
To solve this PDE let $z=x+y$. Then equation (5.53) becomes:

$$2xf_{xx}+2yf_{yy}+\left(n+\frac{x+y}{x-y}\right)f_x+\left(n-\frac{x+y}{x-y}\right)f_y=x+y$$
$$\Leftrightarrow 2zf_{zz}+2nf_z=z \tag{5.54}$$

Note that equation (5.54) has solution:

$$f(z)=C_1+C_2\frac{z^{1-n}}{1-n}+\frac{z^2}{4(n+1)}, \tag{5.55}$$

which can be expressed in terms of x and y as:

$$f(x,y) = C_1 + C_2\frac{(x+y)^{1-n}}{1-n} + \frac{(x+y)^2}{4(n+1)}. \tag{5.56}$$

Therefore using Theorem 5.3.14 we have that

$$\begin{aligned} E\left[C_1 + C_2\frac{(X_t+Y_t)^{1-n}}{1-n} + \frac{(X_t+Y_t)^2}{4(n+1)}\right] &= C_1 + C_2\frac{(x+y)^{1-n}}{1-n} + \frac{(x+y)^2}{4(n+1)} \\ &+ \int_0^t E[X_s+Y_s]ds \end{aligned}$$

But we know from the previous sections that $E[X_s+Y_s] = 2ns + x + y$ so we have

$$\begin{aligned} E\left[C_1 + C_2\frac{(X_t+Y_t)^{1-n}}{1-n} + \frac{(X_t+Y_t)^2}{4(n+1)}\right] &= C_1 + C_2\frac{(x+y)^{1-n}}{1-n} + \frac{(x+y)^2}{4(n+1)} \\ &+ \int_0^t (2ns+x+y)ds \\ &= C_1 + C_2\frac{(x+y)^{1-n}}{1-n} + \frac{(x+y)^2}{4(n+1)} \\ &+ nt^2 + (x+y)t \end{aligned}$$

This allows us to calculate the expectations $E\left[(X_t+Y_t)^{1-n}\right]$ in terms of $E\left[(X_t+Y_t)^2\right]$ as follows:

$$E[(X_t+Y_t)^{1-n}] = (x+y)^{1-n} + (1-n)\left(\frac{(x+y)^2 - E[(X_t+Y_t)^2]}{4(n+1)} + nt^2 + (x+y)t\right)$$

Note that $(x+y)^{1-n}$ with $n \geq 2$ is not a polynomial, so this was not a function of the eigenvalues for which we could calculate the expectations using the methodology presented in previous sections. However, $E[(X_t+Y_t)^2]$ is indeed something we already know how to calculate. Therefore, using this method we are extending the class of functions of the eigenvalues of a Wishart process for which we can calculate the expectations. Note that in this case:

$$E^{x,y}[(X_t+Y_t)^2] = 4n(n+1)t^2 + 4(n+1)t(x+y) + (x+y)^2 \tag{5.57}$$

Hence

$$\begin{aligned} E[(X_t+Y_t)^{1-n}] &= (x+y)^{1-n} + (1-n)(nt^2 + (x+y)t) \\ &+ (1-n)\frac{(x+y)^2 - 4n(n+1)t^2 - 4(n+1)t(x+y) - (x+y)^2}{4(n+1)} \\ &= (x+y)^{1-n} + (1-n)(nt^2 + (x+y)t) - (1-n)(nt^2 + t(x+y)) \\ &= (x+y)^{1-n} \end{aligned}$$

Remark. This particular example deals with the expectations of a function that can be expressed in terms of the sum $X_t + Y_t$, for which we know the density function. So this particular expected value could have been obtained by simply integrating against the transition density function of the process $X_t + Y_t$. However, we use this example here to illustrate how it can be done using symmetries.

An even wider range of possibilities can be explored by the combination of these results with those we can obtain using Theorem 5.3.5. In the previous section, the mentioned theorem produced expected values of some symmetric polynomials through the transformation by symmetry of the expected value of a different symmetric polynomial but, when applied to different kinds of functions, it can produce expected values for other classes of functions.

Let us show how theorem 5.3.5 can, for instance, be used in Example 5.3.2 to obtain the expectations for $(X_t + Y_t)^{-k}$, $k \in \mathbb{N}$ for every $k \in [0, n-2]$:

Example 5.3.3. In the previous example we obtained that for a Wishart process $W_t \sim WIS(n, 2, W_0)$ with index $n \geq 2$ and with eigenvalues $X(t) > Y(t) \geq 0$

$$E[(X_t + Y_t)^{1-n}] = (x+y)^{1-n}$$

Let $u(x, y, t) := (x+y)^{1-n}$, which is a solution of $u_t = Lu$, where L the generator of the eigenvalue process (X_t, Y_t), i.e. L is defined as in (5.58). Define

$$v(x,y,t) = \left[\frac{d}{d\epsilon} u\left(\frac{x}{(1+\epsilon t)^2}, \frac{y}{(1+\epsilon t)^2}, \frac{t}{(1+\epsilon t)} \right) \frac{\exp\left(-\frac{\epsilon}{2}\frac{(x+y)}{(1+\epsilon t)}\right)}{(1+\epsilon t)^n} \right]_{\epsilon=0}$$

as in theorem 5.3.5. Observe that the new solution is

$$v(x,y,t) = -\frac{(x+y)^{2-n}}{2} - (2-n)t(x+y)^{1-n},$$

which has initial condition $v(x, y, 0) = -\frac{(x+y)^{2-n}}{2}$. Thus, multiplying by -2 we get the new solution $\tilde{v}(x, y, t) = (x+y)^{2-n} + (2-n)2t(x+y)^{1-n}$, which, according to Theorem 5.3.5 satisfies

$$E[(X_t + Y_t)^{2-n}] = \tilde{v}(x, y, t).$$

We could use derivatives of higher orders too, for example:

$$\begin{aligned} w(x,y,t) &= \left[\frac{d^2}{d\epsilon^2} u\left(\frac{x}{(1+\epsilon t)^2}, \frac{y}{(1+\epsilon t)^2}, \frac{t}{(1+\epsilon t)} \right) \frac{\exp\left(-\frac{\epsilon}{2}\frac{(x+y)}{(1+\epsilon t)}\right)}{(1+\epsilon t)^n} \right]_{\epsilon=0} \\ &= \frac{1}{4}(x+y)^{3-n} + t(3-n)(x+y)^{2-n} + t^2(2-n)(3-n)(x+y)^{1-n}, \end{aligned}$$

which is a solution with initial condition $w(x,y,0) = \frac{(x+y)^{3-n}}{4}$. Hence $\tilde{w}(x,y,t) = 4w(x,y,t) = (x+y)^{3-n} + 4t(3-n)(x+y)^{2-n} + 4t^2(2-n)(3-n)(x+y)^{1-n}$ will satisfy, according to Theorem 5.3.5,

$$E[(X_t + Y_t)^{3-n}] = \tilde{w}(x,y,t)$$

We can repeat this for higher orders of the derivative to obtain the explicit expression of $E[(X_t + Y_t)^{-k}]$ for each $k \in \mathbb{N}$ such that $k \in [0, n-2]$.

We must remark that Theorem 5.3.14 and its associated Corollary could have been formulated in the general form:

Theorem 5.3.14. *Let $W_t \sim WIS(n, 2, W_0)$ with index $n \geq 2$ and with eigenvalues $X(t) > Y(t) \geq 0$. Let L be the linear operator defined as:*

$$L = 2x\frac{\partial^2}{\partial x^2} + 2y\frac{\partial^2}{\partial y^2} + \left(n + \frac{x+y}{x-y}\right)\frac{\partial}{\partial x} + \left(n - \frac{x+y}{x-y}\right)\frac{\partial}{\partial y} \tag{5.58}$$

Suppose $E[g(X_t, Y_t)]$ is a known integrable function. Then, for any function f such that $Lf = g$, we have that

$$E[f(X_t, Y_t)] = f(x,y) + \int_0^t E[g(X_s, Y_s)]ds.$$

Furthermore, if $E[g(X_s, Y_s)]$ is k times integrable with respect to the time variable, then for any h such that $L^k h = g$, we have

$$E[h(X_t, Y_t)] = \sum_{n=0}^{k-1} \frac{t^n}{n!} L^n h(x,y) + \int_0^t \int_0^{t_{k-1}} \cdots \int_0^{t_1} E[g(X_s, Y_s)]ds dt_1 \ldots dt_{k-1}.$$

5.3.3 Integral transform methods for the computation of the expectations of the eigenvalues of a 2×2 Wishart process and a bound for their variance

In previous sections we have presented a proof of the well known result that $Z_t = X_t + Y_t \sim BESQ^{2n}(z)$, so the probability density function for the process Z_t is a known function. Hence, if $p_t(z, \xi)$ is the transition density function for a Bessel-squared distribution, any expected value of the form $E[f(X_t + Y_t)]$ may be calculated as

$$E[f(X_t + Y_t)] = \int_0^\infty f(\xi) p_t(x+y, \xi) d\xi.$$

But, as preciously remarked, this is very limited if one wishes to understand the behaviour of the eigenvalues of a Wishart process.

We have already seen in previous sections how there are other types of functions of the eigenvalues for which it is possible to obtain the expected value. In this section, we produce an expression for $E[X_t]$ and $E[Y_t]$ via the use of integral transform methods. We obtain the Fourier cosine transform of the transition density of the difference $X_t - Y_t$ and hence obtain an integral expression for such transition density function. Moreover, we give some bounds for the variances of X_t and Y_t.

We will refer to the following well known definitions and results throughout this section:

Definition 5.3.1. The *Schwartz space* or, alternatively, the *space of rapidly decreasing functions on* $\mathbb{R}^n$ is the function space

$$\mathcal{S}(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) : \|f\|_{\alpha,\beta} < \infty \quad \forall \alpha, \beta \in \mathbb{N}^n\}, \tag{5.59}$$

where α, β are multi-indices, $C^\infty(\mathbb{R}^n)$ is the set of smooth functions $f : \mathbb{R}^n \to \mathbb{C}$, and

$$\|f\|_{\alpha,\beta} = \sup_{x \in \mathbb{R}^n} \left| x^\alpha D^\beta f(x) \right|.$$

Note that $\mathcal{S}(\mathbb{R}^n)$ is a subspace of the function space $C^\infty(\mathbb{R}^n)$.
We say that a function $f : \mathbb{R} \to \mathbb{C}$ is a Schwartz function if it is an infinitely differentiable function such that $\forall n \in \mathbb{N} \cup \{0\}$, $f^{(n)}(x) \xrightarrow[x \to \pm\infty]{} 0$ faster than $\frac{1}{x^k}$ $\forall k$. That is, f must be a C^∞ function such that for all $n \in \mathbb{N} \cup \{0\}$

$$\lim_{|x| \to \infty} \frac{f^{(n)}(x)}{x^{-k}} = 0, \quad \forall k.$$

Next we present a well known result regarding the classical Fourier transform of the Schwartz class:

Theorem 5.3.15. *The Fourier transform $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$ are homeomorphisms of $\mathcal{S}(\mathbb{R}^n)$ onto itself.*

With this results, we are ready to present the following theorem:

Theorem 5.3.16. *Let $W_t \sim WIS(n, 2, W_0)$, with $n \geq 2$ and with eigenvalues $X_t > Y_t \geq 0$. Let $S_t = X_t - Y_t$ and let $q_t(\xi, s)$ be the transition density function of the process S_t. Then*

$$\hat{q}_t(\lambda, s) = \sqrt{\frac{2}{\pi}} \sum_{k=0}^{\infty} (-1)^k \frac{\lambda^{2k}}{(2k)!} E[(S_t)^{2k}],$$

where $\hat{q}$ denotes the classical Fourier cosine transform of the function q. Further, the expected values $E[(S_t)^{2k}]$ appearing in the above expression are computable for all k, since S_t^{2k} is a symmetric polynomial in the eigenvalues X_t, Y_t for each k.

Proof. Let us start defining $Z_t = X_t + Y_t$ and let $p_t(z, \xi)$ be the transition density function for a Bessel-squared distribution. Then, clearly,

$$\begin{aligned} E[\cos(\lambda Z_t)] &= \int_0^\infty \cos(\lambda\xi) p_t(z,\xi) d\xi \\ &= \int_0^\infty \frac{e^{\lambda\xi i} + e^{-\lambda\xi i}}{2} \frac{e^{\frac{-(z+\xi)}{2t}}}{2t} \left(\frac{z}{\xi}\right)^{\frac{1-n}{2}} I_{n-1}\left(\frac{\sqrt{\xi z}}{t}\right) d\xi \\ &= \frac{z^{\frac{1-n}{2}} e^{-\frac{z}{2t}}}{4t} \int_0^\infty (e^{\xi(\lambda i - \frac{1}{2t})} + e^{-\xi(\lambda i + \frac{1}{2t})}) \xi^{\frac{n-1}{2}} I_{n-1}\left(\frac{\sqrt{\xi z}}{t}\right) d\xi \end{aligned}$$

Making the change of variables $\xi = y^2$ we have that $d\xi = 2ydy$, so our integral becomes

$$\begin{aligned} E[\cos(\lambda Z_t)] &= \frac{z^{\frac{1-n}{2}} e^{-\frac{z}{2t}}}{2t} \int_0^\infty (e^{y^2(\lambda i - \frac{1}{2t})} + e^{-y^2(\lambda i + \frac{1}{2t})}) y^n I_{n-1}\left(\frac{y\sqrt{z}}{t}\right) dy \\ &= \frac{z^{\frac{1-n}{2}} e^{-\frac{z}{2t}}}{2t} \left(\frac{z}{t^2}\right)^{\frac{n-1}{2}} \left(\left(\frac{1}{t} + 2i\lambda\right)^{-n} e^{\frac{iz}{-4\lambda t^2 + 2it}} + \left(\frac{1}{t} - 2i\lambda\right)^{-n} e^{\frac{iz}{4\lambda t^2 + 2it}}\right) \\ &= \frac{e^{-\frac{z}{2t}}}{2} (1 + 4\lambda^2 t^2)^{-n} \left((1 - 2i\lambda t)^n e^{\frac{iz}{-4\lambda t^2 + 2it}} + (1 + 2i\lambda t)^n e^{\frac{iz}{4\lambda t^2 + 2it}}\right) \end{aligned} \quad (5.60)$$

Expand this result as a series around $\lambda = 0$ to get

$$\begin{aligned} E[\cos(\lambda Z_t)] \sim 1 &- \frac{1}{2}\lambda^2 \left(4(n+1)nt^2 + 4(n+1)tz + z^2\right) \\ &+ \frac{1}{24}\lambda^4 \Big(16(n+1)(n+2)(n+3)nt^4 + 32(n+1)(n+2)(n+3)t^3 z \\ &\qquad + 24(n+2)(n+3)t^2 z^2 + 8(n+3)tz^3 + z^4\Big) \\ &- \frac{1}{720}\lambda^6 \Big(64(n+1)(n+2)(n+3)(n+4)(n+5)nt^6 \\ &\qquad + 192(n+1)(n+2)(n+3)(n+4)(n+5)t^5 z \\ &\qquad + 240(n+2)(n+3)(n+4)(n+5)t^4 z^2 \\ &\qquad + 160(n+3)(n+4)(n+5)t^3 z^3 + 60(n+4)(n+5)t^2 z^4 \\ &\qquad + 12(n+5)tz^5 + z^6\Big) + \dots \\ &= 1 - \frac{\lambda^2}{2} E[Z_t^2] + \frac{\lambda^4}{24} E[Z_t^4] - \frac{\lambda^6}{720} E[Z_t^6] + \dots = \sum_{n=0}^\infty (-1)^n \frac{\lambda^{2n}}{(2n)!} E[Z_t^{2n}], \end{aligned}$$

where $E[Z_t^{2k}] = z^{2k} + \sum_{i=1}^{2k} \frac{(2t)^i}{i!} \left(\prod_{j=1}^{i} (2k - j + 1)(n + 2k - j)\right) z^{2k-i}$,

as computed in previous sections. But observe also that

$$E[\cos(\lambda Z_t)] = \int_0^\infty \cos(\lambda\xi) p_t(z,\xi) d\xi = \sqrt{\frac{\pi}{2}} \hat{p}_t(z,\lambda),$$

where $\hat{p}$ denotes the classical Fourier cosine transform defined as

$$\hat{f}(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos(\omega t) dt.$$

Therefore, by the Paley-Wiener theorem we have that $E[\cos(\lambda Z_t)]$ is an analytic function and hence it is equal to its Taylor series expansion. So we have that

$$E[\cos(\lambda Z_t)] = E\left[\sum_{n=0}^\infty (-1)^n \frac{\lambda^{2n}}{(2n)!} Z_t^{2n}\right] = \sum_{n=0}^\infty (-1)^n \frac{\lambda^{2n}}{(2n)!} E[Z_t^{2n}]$$

Now observe that

$$\left|\cos(\lambda(X_t - Y_t)) - \cos(\lambda(X_t + Y_t))\right| \leq 2$$

and hence

$$-2 + \cos(\lambda(X_t + Y_t)) \leq \cos(\lambda(X_t - Y_t)) \leq 2 + \cos(\lambda(X_t + Y_t))$$

Taking expectations one gets:

$$-2 + E[\cos(\lambda(X_t + Y_t))] \leq E[\cos(\lambda(X_t - Y_t))] \leq 2 + E[\cos(\lambda(X_t + Y_t))]$$

Therefore the expectations $E[\cos(\lambda(X_t - Y_t))]$ exist since we have shown before that $E[\cos(\lambda(X_t + Y_t))]$ exist. Let $g(z) = \cos(\lambda z)$ and observe that we can expand $g(X_t + Y_t)$ and $g(X_t - Y_t)$ as Taylor series as:

$$\cos(\lambda(X_t + Y_t)) = \sum_{k=0}^N (-1)^k \frac{\lambda^{2k}}{(2k)!} (X_t + Y_t)^{2k} + R_N^+,$$
$$\cos(\lambda(X_t - Y_t)) = \sum_{k=0}^N (-1)^k \frac{\lambda^{2k}}{(2k)!} (X_t - Y_t)^{2k} + R_N^-,$$

where the error terms are

$$R_N^+ = \frac{g^{(2N+1)}(\xi)}{(2N+1)!} (X_t + Y_t)^{2N+1}, \quad \xi \in (0, X_t + Y_t)$$
$$R_N^- = \frac{g^{(2N+1)}(\eta)}{(2N+1)!} (X_t - Y_t)^{2N+1}, \quad \eta \in (0, X_t - Y_t)$$

and where $R_N^+, R_N^- \to 0$ as $N \to \infty$.
The expectations for the $\cos(\lambda(X_t - Y_t))$ are thus given by:

$$\begin{aligned} E[\cos(\lambda(X_t - Y_t))] &= E[\sum_{k=0}^{N}(-1)^k \frac{\lambda^{2k}}{(2k)!}(X_t - Y_t)^{2k} + R_N^-] \\ &= \sum_{k=0}^{N}(-1)^k \frac{\lambda^{2k}}{(2k)!}E[(X_t - Y_t)^{2k}] + E[R_N^-] \end{aligned}$$

by linearity. Take limits as $N \to \infty$ to obtain

$$E[\cos(\lambda(X_t - Y_t))] = \lim_{N\to\infty}\sum_{k=0}^{N}(-1)^k \frac{\lambda^{2k}}{(2k)!}E[(X_t - Y_t)^{2k}] + \lim_{N\to\infty} E[R_N] \quad (5.61)$$

$$= \sum_{k=0}^{\infty}(-1)^k \frac{\lambda^{2k}}{(2k)!}E[(X_t - Y_t)^{2k}] + E[\lim_{N\to\infty} R_N] \quad (5.62)$$

$$= \sum_{k=0}^{\infty}(-1)^k \frac{\lambda^{2k}}{(2k)!}E[(X_t - Y_t)^{2k}].$$

The reader may have observed that the expectations and the limit have been swapped in the error term from (5.61) to (5.62). Observe that this is possible because

$$\left|\frac{R_N^-}{R_N^+}\right| = \left|\frac{g^{(2N+1)}(\eta)(X_t - Y_t)^{2N+1}}{g^{(2N+1)}(\xi)(X_t + Y_t)^{2N+1}}\right| \leq \frac{1}{|g^{(2N+1)}(\xi)|}\left|\frac{(X_t - Y_t)^{2N+1}}{(X_t + Y_t)^{2N+1}}\right| \leq \frac{1}{|g^{(2N+1)}(\xi)|},$$

hence

$$|R_N^-| \leq K|R_N^+|, \quad \text{with } K = \frac{1}{|g^{(2N+1)}(\xi)|}.$$

But we have seen before that $\lim_{N\to\infty} E[R_N^+] = 0 < \infty$, so by the Dominated Convergence Theorem we have that

$$\lim_{N\to\infty} E[R_N] = E[\lim_{N\to\infty} R_N] = 0.$$

Therefore, we have proved that

$$K_t(\lambda, x - y) := E[\cos(\lambda(X_t - Y_t))] = \sum_{k=0}^{\infty}(-1)^k \frac{\lambda^{2k}}{(2k)!}E[(X_t - Y_t)^{2k}]$$

But

$$K_t(\lambda, x - y) = E[\cos(\lambda(X_t - Y_t))] = \int_0^{\infty} \cos(\lambda\xi)q_t(\xi, x - y)d\xi = \sqrt{\frac{\pi}{2}}\hat{q}_t(\lambda, x - y),$$

where the function q is the transition density function of the process $X_t - Y_t$ and $\hat{q}$ denotes the Fourier cosine transform of q. Hence, writing $\hat{q}_t(\lambda, x-y) = \sqrt{\frac{2}{\pi}}K_t(\lambda, x-y)$ and putting $S_t = X_t - Y_t$ gives the desired result. Note that $(x - y)^{2k}$ is a symmetric polynomial for every $k \in \mathbb{N}$, so $E[(X_t - Y_t)^{2k}]$ is a computable function for each $k \in \mathbb{N}$. □

Note. In a similar way to how we have proved that

$$E[\cos(\lambda(X_t - Y_t))] = \sum_{k=0}^{\infty}(-1)^k \frac{\lambda^{2k}}{(2k)!} E[(X_t - Y_t)^{2k}],$$

one may obtain $E[\cos(\lambda p(X_t, Y_t))] = \sum_{k=0}^{\infty}(-1)^k \frac{\lambda^{2k}}{(2k)!} E[p(X_t, Y_t)^{2k}]$ for any polynomial p such that $p(X_t, Y_t)^{2k}$ is a symmetric polynomial in X_t, Y_t. This includes expectations such as $E[\cos(\lambda(X_t^2 + Y_t^2))]$ among many others.

Corollary. *Let $W_t \sim WIS(n, 2, W_0)$, with $n \geq 2$ and with eigenvalues $X_t > Y_t \geq 0$. Let $S_t = X_t - Y_t$ and let $q_t(\xi, s)$ be the transition density function of the process S_t. Then*

$$q_t(\xi, s) = \frac{2}{\pi} \int_0^{\infty} \sum_{k=0}^{\infty}(-1)^k \frac{\lambda^{2k}}{(2k)!} E[(S_t)^{2k}] \cos(\lambda\xi) d\lambda. \tag{5.63}$$

Proof. This is just an exercise of inverting the Fourier cosine transform obtained in the proof of the previous theorem, i.e.

$$\begin{aligned} q_t(\xi, x - y) &= \frac{2}{\pi} \int_0^{\infty} K_t(\lambda, x - y) \cos(\lambda\xi) d\lambda \\ &= \frac{2}{\pi} \int_0^{\infty} \sum_{k=0}^{\infty}(-1)^k \frac{\lambda^{2k}}{(2k)!} E[(X_t - Y_t)^{2k}] \cos(\lambda\xi) d\lambda. \end{aligned}$$

□

Remark. The integral in (5.63) cannot be calculated term by term but it can be dealt with as:

$$q_t(\xi, x - y) = \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \lim_{\epsilon \to 0} \int_0^{\infty} e^{-\epsilon\lambda^2} \lambda^{2k} E[(X_t - Y_t)^{2k}] \cos(\lambda\xi) d\lambda. \tag{5.64}$$

Hence the density function q can be approximated by calculating as many terms as desired in the above expression. The amount of terms needed for a particular accuracy is something we would like to investigate in future work.

We proceed now to present a result that provides us with an expression for $E[X_t]$ and $E[Y_t]$ respectively.

Theorem 5.3.17. *Let $W_t \sim WIS(n, 2, W_0)$, with $n \geq 2$ and with eigenvalues $X_t > Y_t \geq 0$. Then*

$$E[X_t] = nt + \frac{x+y}{2} - \frac{1}{\pi}\int_0^\infty \frac{\left(\sum_{k=0}^\infty (-1)^k \frac{\lambda^{2k}}{(2k)!} E[(X_t - Y_t)^{2k}]\right) - 1}{\lambda^2} d\lambda \tag{5.65}$$

$$E[Y_t] = nt + \frac{x+y}{2} + \frac{1}{\pi}\int_0^\infty \frac{\left(\sum_{k=0}^\infty (-1)^k \frac{\lambda^{2k}}{(2k)!} E[(X_t - Y_t)^{2k}]\right) - 1}{\lambda^2} d\lambda, \tag{5.66}$$

where the expected values $E[(X_t - Y_t)^{2k}]$ are computable for all k.

Proof. We will obtain $E[X_t]$ and $E[Y_t]$ as $E[X_t] = \frac{1}{2}(E[X_t + Y_t] + E[X_t - Y_t])$ and $E[Y_t] = \frac{1}{2}(E[X_t + Y_t] - E[X_t - Y_t])$ respectively. We know that $E[X_t + Y_t] = 2nt + x + y$, so we need only compute $E[X_t - Y_t]$.

Let q_t be the transition density function for the difference $X_t - Y_t$, then

$$\begin{aligned} E[X_t - Y_t] &= \int_0^\infty z q_t(z, x-y) dz \\ &= \int_0^\infty z \frac{2}{\pi} \int_0^\infty K_t(\lambda, x-y) \cos(\lambda z) d\lambda dz \\ &= \int_0^\infty K_t(\lambda, x-y) \sqrt{\frac{2}{\pi}} \underbrace{\left(\sqrt{\frac{2}{\pi}} \int_0^\infty z \cos(\lambda z) dz \right)}_{\mathcal{F}_c(z)} d\lambda, \end{aligned}$$

where $\mathcal{F}_c(z)$ denotes the Fourier cosine transform of z, and where the function $K_t(\lambda, x-y)$ is defined as in the previous proofs. Note that this Fourier cosine transform does not exist in the usual sense, since the integral is not convergent. However, it can be defined as a distribution. Let the pseudo-function $Pf\left(\frac{1}{\lambda^2}\right)$ be defined by

$$\int_0^\infty Pf\left(\frac{1}{\lambda^2}\right) \phi(\lambda) d\lambda = -\int_0^\infty \frac{\phi(\lambda) - \phi(0) - \phi'(0)\lambda}{\lambda^2} d\lambda \tag{5.67}$$

and take $\phi(\lambda) = \cos(\lambda z)$. Note that

$$\begin{aligned} \mathcal{F}_c^{-1}\left(Pf\left(\frac{1}{\lambda^2}\right)\right) &= \sqrt{\frac{2}{\pi}} \int_0^\infty Pf\left(\frac{1}{\lambda^2}\right) \cos(\lambda z) d\lambda = -\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\cos(\lambda z) - 1}{\lambda^2} d\lambda \\ &= \sqrt{\frac{\pi}{2}} z \end{aligned}$$

So we have that $\mathcal{F}_c(z) = \sqrt{\frac{2}{\pi}} Pf\left(\frac{1}{\lambda^2}\right)$. Using this, and the definition (5.67) of $Pf\left(\frac{1}{\lambda^2}\right)$ in the expresion for $E[X_t - Y_t]$ one gets:

$$\begin{aligned} E[X_t - Y_t] &= \int_0^\infty K_t(\lambda, x-y)\sqrt{\frac{2}{\pi}}\sqrt{\frac{2}{\pi}} Pf\left(\frac{1}{\lambda^2}\right) d\lambda \\ &= -\frac{2}{\pi}\int_0^\infty \frac{K_t(\lambda, x-y) - 1}{\lambda^2} d\lambda, \end{aligned}$$

since $K_t(0, x-y) = 1$ and $(K_t)_\lambda(0, x-y) = 0$.

Therefore $E[X_t - Y_t]$ is given by

$$E[X_t - Y_t] = -\frac{2}{\pi}\int_0^\infty \frac{K_t(\lambda, x-y) - 1}{\lambda^2} d\lambda \tag{5.68}$$

$$= -\frac{2}{\pi}\int_0^\infty \frac{\left(\sum_{k=0}^\infty (-1)^k \frac{\lambda^{2k}}{(2k)!} E[(X_t - Y_t)^{2k}]\right) - 1}{\lambda^2} d\lambda \tag{5.69}$$

With this expression one can obtain $E[X_t]$ and $E[Y_t]$ as

$$\begin{aligned} E[X_t] &= \frac{1}{2}(E[X_t + Y_t] + E[X_t - Y_t]) \\ &= \frac{1}{2}\left(2nt + x + y - \frac{2}{\pi}\int_0^\infty \frac{\left(\sum_{k=0}^\infty (-1)^k \frac{\lambda^{2k}}{(2k)!} E[(X_t - Y_t)^{2k}]\right) - 1}{\lambda^2} d\lambda\right) \end{aligned} \tag{5.70}$$

$$\begin{aligned} E[Y_t] &= \frac{1}{2}(E[X_t + Y_t] - E[X_t - Y_t]) \\ &= \frac{1}{2}\left(2nt + x + y + \frac{2}{\pi}\int_0^\infty \frac{\left(\sum_{k=0}^\infty (-1)^k \frac{\lambda^{2k}}{(2k)!} E[(X_t - Y_t)^{2k}]\right) - 1}{\lambda^2} d\lambda\right), \end{aligned} \tag{5.71}$$

as claimed. □

Remark. The evaluation problem for the integrals appearing in (5.70) and (5.71) can be treated in a similar way as to how expression (5.64) is used to evaluate (5.63).

Recall that $E[(X_t - Y_t)^{2k}$ can be calculated using the results presented in previous sections for symmetric polynomials in the eigenvalues. As an example, we provide a few of these for lower degrees:

$$\begin{aligned}
E[(X_t - Y_t)^2 &= 8nt^2 + 8t(x+y) + (x-y)^2 \\
E[(X_t - Y_t)^4 &= 128n(n+2)t^4 + 256t^3(n+2)(x+y) \\
&+ 32t^2\left((n+4)(x-y)^2 + 4(x+y)^2\right) \\
&+ 32t(x-y)^2(x+y) + (x-y)^4 \\
E[(X_t - Y_t)^6 &= 3072n(n+2)(n+4)t^6 + 9216(n+2)(n+4)t^5(x+y) \\
&+ 1152(n+4)t^4\left((n+6)(x-y)^2 + 8(x+y)^2\right) \\
&+ 768t^3(x+y)\left(3(n+6)(x-y)^2 + 4(x+y)^2\right) \\
&+ 72t^2(x-y)^2\left((n+8)(x-y)^2 + 16(x+y)^2\right) + 72t(x-y)^4(x+y) \\
&+ (x-y)^6 \\
E[(X_t - Y_t)^8 &= 98304n(n+2)(n+4)(n+6)t^8 \\
&+ 393216(n+2)(n+4)(n+6)t^7(x+y) \\
&+ 49152(n+4)(n+6)t^6\left((n+8)(x-y)^2 + 12(x+y)^2\right) \\
&+ 49152(n+6)t^5(x+y)\left(3(n+8)(x-y)^2 + 8(x+y)^2\right) \\
&+ 1536t^4\left(3(n+8)(n+10)(x-y)^4\right) \\
&+ 1536t^4\left(96(n+8)(x+y)^2(x-y)^2 + 64(x+y)^4\right) \\
&+ 3072t^3(x-y)^2(x+y)\left(3(n+10)(x-y)^2 + 16(x+y)^2\right) \\
&+ 128t^2(x-y)^4\left((n+12)(x-y)^2 + 36(x+y)^2\right) \\
&+ 128t(x-y)^6(x+y) + (x-y)^8
\end{aligned}$$

An alternative expression can be obtained for $E[X_t]$ and $E[Y_t]$:

Theorem 5.3.18. *Let $W_t \sim WIS(n, 2, W_0)$, with $n \geq 2$ and with eigenvalues $X_t > Y_t \geq 0$. Then*

$$E[X_t] = 2nt + x + y + \frac{1}{\pi}\int_0^\infty \frac{\sum_{k=0}^\infty (-1)^k \frac{\lambda^{2k}}{(2k)!} E[(X_t+Y_t)^{2k} - (X_t - Y_t)^{2k}]}{\lambda^2} d\lambda \tag{5.72}$$

$$E[Y_t] = -\frac{1}{\pi}\int_0^\infty \frac{\sum_{k=0}^\infty (-1)^k \frac{\lambda^{2k}}{(2k)!} E[(X_t+Y_t)^{2k} - (X_t - Y_t)^{2k}]}{\lambda^2} d\lambda, \tag{5.73}$$

where the expected values $E[(X_t + Y_t)^{2k} - (X_t - Y_t)^{2k}]$ are computable for all k.

Proof. This proof relies on the following identity for $a, b \in \mathbb{R}$:

$$\int_0^\infty \frac{\cos(\lambda a) - \cos(\lambda b)}{\lambda^2} d\lambda = \frac{\pi}{2}(|b| - |a|). \tag{5.74}$$

Note that taking $a = X_t + Y_t$ and $b = X_t - Y_t$ we have that

$$\int_0^\infty \frac{\cos(\lambda(X_t + Y_t)) - \cos(\lambda(X_t - Y_t))}{\lambda^2} d\lambda = \frac{\pi}{2}((X_t - Y_t) - (X_t + Y_t)) = -\pi Y_t \tag{5.75}$$

Hence

$$E[Y_t] = -\frac{1}{\pi} E\left[\int_0^\infty \frac{\cos(\lambda(X_t + Y_t)) - \cos(\lambda(X_t - Y_t))}{\lambda^2} d\lambda\right] \tag{5.76}$$

Observe that

$$\begin{aligned}
I &:= \int_0^\infty \frac{\cos(\lambda(X_t + Y_t)) - \cos(\lambda(X_t - Y_t))}{\lambda^2} d\lambda \\
&\leq \int_0^\infty \left| \frac{\cos(\lambda(X_t + Y_t)) - \cos(\lambda(X_t - Y_t))}{\lambda^2} \right| d\lambda \\
&= \int_0^1 \left| \frac{\cos(\lambda(X_t + Y_t)) - \cos(\lambda(X_t - Y_t))}{\lambda^2} \right| d\lambda \\
&+ \int_1^\infty \left| \frac{\cos(\lambda(X_t + Y_t)) - \cos(\lambda(X_t - Y_t))}{\lambda^2} \right| d\lambda \\
&\leq \int_0^1 \left| -2X_tY_t + \frac{\lambda^2 X_t Y_t (X_t^2 + Y_t^2)}{3} - \frac{\lambda^4((X_t + Y_t)^6 - (X_t - Y_t)^6)}{6!} + O(\lambda^6) \right| d\lambda \\
&+ \int_1^\infty \frac{2}{\lambda^2} d\lambda \\
&< \infty
\end{aligned}$$

So we can write

$$\begin{aligned}
E[Y_t] &= -\frac{1}{\pi} \int_0^\infty \frac{E[\cos(\lambda(X_t + Y_t))] - E[\cos(\lambda(X_t - Y_t))]}{\lambda^2} d\lambda \\
&= -\frac{1}{\pi} \int_0^\infty \frac{\sum_{k=0}^\infty (-1)^k \frac{\lambda^{2k}}{(2k)!} E[(X_t + Y_t)^{2k}] - \sum_{k=0}^\infty (-1)^k \frac{\lambda^{2k}}{(2k)!} E[(X_t - Y_t)^{2k}]}{\lambda^2} d\lambda \\
&= -\frac{1}{\pi} \int_0^\infty \frac{\sum_{k=0}^\infty (-1)^k \frac{\lambda^{2k}}{(2k)!} (E[(X_t + Y_t)^{2k}] - E[(X_t - Y_t)^{2k}])}{\lambda^2} d\lambda \\
&= -\frac{1}{\pi} \int_0^\infty \frac{\sum_{k=0}^\infty (-1)^k \frac{\lambda^{2k}}{(2k)!} E[(X_t + Y_t)^{2k} - (X_t - Y_t)^{2k}]}{\lambda^2} d\lambda,
\end{aligned}$$

and, therefore,

$$E[X_t] = E[X_t + Y_t] - E[Y_t]$$
$$= 2nt + x + y + \frac{1}{\pi}\int_0^\infty \frac{\sum_{k=0}^{\infty}(-1)^k \frac{\lambda^{2k}}{(2k)!}E[(X_t+Y_t)^{2k} - (X_t - Y_t)^{2k}]}{\lambda^2}d\lambda.$$

The expectations $E[(X_t+Y_t)^{2k} - (X_t-Y_t)^{2k}]$ are clearly computable for each k since $(X_t+Y_t)^{2k} - (X_t-Y_t)^{2k}$ is a symmetric polynomial in X_t, Y_t. Note that $E[(X_t+Y_t)^{2k} - (X_t-Y_t)^{2k}] = E[4X_tY_t + 8X_tY_t(X_t^2+Y_t^2) + \dots]$ so the $k=0$ term in the above expressions for $E[Y_t]$, $E[X_t]$ is zero, and hence the integral has no singularity at 0. □

The integrals expressions (5.72) and (5.73) can once again be treated similarly to those in (5.64) and (5.69).

Note. The eigenvalues X_t and Y_t are related to each other by their initial values. The expected values for X_t and Y_t are by definition

$$E[X_t] = \int_\Omega \bar{x} p_t(\bar{x}, x, \bar{y}, y)d\bar{x}d\bar{y} \quad E[X_t] = \int_\Omega \bar{y} p_t(\bar{x}, x, \bar{y}, y)d\bar{x}d\bar{y}, \tag{5.77}$$

for a suitable region Ω, and where the function $p_t(\bar{x}, x, \bar{y}, y)$ is the transition density for the eigenvalue process (X_t, Y_t). Therefore, the expected values $E[X_t]$ and $E[Y_t]$ naturally depend on both the initial values $X(0) = x$ and $Y(0) = y$, as can be seen in the expressions in both Theorems 5.3.17 and 5.3.18.

Having an expression for $E[X_t]$ and $E[Y_t]$, the next natural question to ask is: can we obtain expressions for $Var[X_t]$ and $Var[Y_t]$? We have not yet been able to produce such expressions, but we have some bounds for these variances:

Theorem 5.3.19. *Let $W_t \sim WIS(n, 2, W_0)$, with $n \geq 2$ and with eigenvalues $X_t > Y_t \geq 0$. Then some upper and lower bounds for the variance of X_t are*

$$Var[X_t] \leq n(n+3)t^2 + (n+3)(x+y)t + \frac{x^2+y^2}{2}$$
$$+ \sqrt{\left(n(n+1)t^2 + (n+1)(x+y)t + \frac{(x+y)^2}{4}\right)(8nt^2 + 8(x+y)t + (x-y)^2)}$$
$$- \frac{1}{4}\left(2nt + x + y - \frac{2}{\pi}\int_0^\infty \frac{\left(\sum_{k=0}^{\infty}(-1)^k \frac{\lambda^{2k}}{(2k)!}E[(X_t-Y_t)^{2k}]\right) - 1}{\lambda^2}d\lambda\right)^2$$

$$Var[X_t] \geq n(n+3)t^2 + (n+3)(x+y)t + \frac{x^2+y^2}{2}$$
$$- \frac{1}{4}\left(2nt + x + y - \frac{2}{\pi}\int_0^\infty \frac{\left(\sum_{k=0}^{\infty}(-1)^k \frac{\lambda^{2k}}{(2k)!}E[(X_t-Y_t)^{2k}]\right) - 1}{\lambda^2}d\lambda\right)^2$$

and, similarly, some upper and lower bounds for the variance of Y_t are

$$Var[Y_t] \leq n(n+3)t^2 + (n+3)(x+y)t + \frac{x^2+y^2}{2} - \frac{1}{4}\left(2nt + x + y + \frac{2}{\pi}\int_0^\infty \frac{\left(\sum_{k=0}^\infty (-1)^k \frac{\lambda^{2k}}{(2k)!} E[(X_t - Y_t)^{2k}]\right) - 1}{\lambda^2} d\lambda\right)^2$$

$$Var[Y_t] \geq n(n+3)t^2 + (n+3)(x+y)t + \frac{x^2+y^2}{2} - \sqrt{\left(n(n+1)t^2 + (n+1)(x+y)t + \frac{(x+y)^2}{4}\right)(8nt^2 + 8(x+y)t + (x-y)^2)} - \frac{1}{4}\left(2nt + x + y + \frac{2}{\pi}\int_0^\infty \frac{\left(\sum_{k=0}^\infty (-1)^k \frac{\lambda^{2k}}{(2k)!} E[(X_t - Y_t)^{2k}]\right) - 1}{\lambda^2} d\lambda\right)^2,$$

where the expected values $E[(X_t - Y_t)^{2k}]$ are computable for all k.

Proof. In order to complete this proof, we will need the expected values of some symmetric polynomials. We compute those using Theorems 5.3.4 and 5.3.5 to obtain:

$$\begin{aligned}
E[X_t^2 + Y_t^2] &= 2n(n+3)t^2 + 2(n+3)(x+y)t + x^2 + y^2 \\
E[(X_t + Y_t)^2] &= 4n(n+1)t^2 + 4(n+1)(x+y)t + (x+y)^2 \\
E[(X_t - Y_t)^2] &= E[X_t^2 + Y_t^2 - 2X_tY_t] = E[X_t^2 + Y_t^2] - 2E[X_tY_t] \\
&= 2n(n+3)t^2 + 2(n+3)(x+y)t + x^2 + y^2 \\
&\quad - 2(n(n-1)t^2 + (n-1)t(x+y) + xy) \\
&= 8nt^2 + 8(x+y)t + (x-y)^2
\end{aligned}$$

Now observe that Hölder's inequality gives:

$$E[X_t^2 - Y_t^2] = E[(X_t + Y_t)(X_t - Y_t)] \leq E[(X_t + Y_t)^2]^{1/2} E[(X_t - Y_t)^2]^{1/2} \tag{5.78}$$

Hence

$$\begin{aligned}
E[X_t^2] &= \frac{1}{2}(E[X_t^2 + Y_t^2] + E[X_t^2 - Y_t^2]) \\
&\leq \frac{1}{2}(E[X_t^2 + Y_t^2] + E[(X_t + Y_t)^2]^{1/2} E[(X_t - Y_t)^2]^{1/2}) \\
&= \frac{1}{2}(2n(n+3)t^2 + 2(n+3)(x+y)t + x^2 + y^2 \\
&\quad + \sqrt{(4n(n+1)t^2 + 4(n+1)(x+y)t + (x+y)^2)(8nt^2 + 8(x+y)t + (x-y)^2)})
\end{aligned}$$

$$= n(n+3)t^2 + (n+3)(x+y)t + \frac{x^2+y^2}{2}$$
$$+ \sqrt{\left(n(n+1)t^2 + (n+1)(x+y)t + \frac{(x+y)^2}{4}\right)(8nt^2 + 8(x+y)t + (x-y)^2)}$$

On the other hand, since $E[X_t^2 - Y_t^2] \geq 0$ we have that

$$n(n+3)t^2 + (n+3)(x+y)t + \frac{x^2+y^2}{2} = \frac{1}{2}E[X_t^2 + Y_t^2] \leq E[X_t^2]$$

This gives us the following bound for the variance of X_t

$$\begin{aligned}
Var[X_t] &= E[X_t^2] - E[X_t]^2 \\
&\leq n(n+3)t^2 + (n+3)(x+y)t + \frac{x^2+y^2}{2} \\
&+ \sqrt{\left(n(n+1)t^2 + (n+1)(x+y)t + \frac{(x+y)^2}{4}\right)(8nt^2 + 8(x+y)t + (x-y)^2)} \\
&- \frac{1}{4}\left(2nt + x + y - \frac{2}{\pi}\int_0^\infty \frac{\left(\sum_{k=0}^\infty (-1)^k \frac{\lambda^{2k}}{(2k)!} E[(X_t - Y_t)^{2k}]\right) - 1}{\lambda^2} d\lambda\right)^2
\end{aligned}$$

And

$$\begin{aligned}
Var[X_t] &\geq n(n+3)t^2 + (n+3)(x+y)t + \frac{x^2+y^2}{2} \\
&- \frac{1}{4}\left(2nt + x + y - \frac{2}{\pi}\int_0^\infty \frac{\left(\sum_{k=0}^\infty (-1)^k \frac{\lambda^{2k}}{(2k)!} E[(X_t - Y_t)^{2k}]\right) - 1}{\lambda^2} d\lambda\right)^2
\end{aligned}$$

Similarly, since $E[(X_t^2 - Y_t^2)] > 0$, an upper bound for $E[Y_t^2]$ can be:

$$\begin{aligned}
E[Y_t^2] &= \frac{1}{2}(E[X_t^2 + Y_t^2] - E[X_t^2 - Y_t^2]) \\
&\leq \frac{1}{2}E[X_t^2 + Y_t^2] \\
&= n(n+3)t^2 + (n+3)(x+y)t + \frac{x^2+y^2}{2}
\end{aligned}$$

and using (5.78) we can also establish a lower bound given by

$$\begin{aligned}
E[Y_t^2] &= \frac{1}{2}(E[X_t^2 + Y_t^2] - E[X_t^2 - Y_t^2]) \\
&\geq \frac{1}{2}(E[X_t^2 + Y_t^2] - E[(X_t + Y_t)^2]^{1/2} E[(X_t - Y_t)^2]^{1/2})
\end{aligned}$$

$$= n(n+3)t^2 + (n+3)(x+y)t + \frac{x^2+y^2}{2}$$
$$- \sqrt{\left(n(n+1)t^2 + (n+1)(x+y)t + \frac{(x+y)^2}{4}\right)(8nt^2 + 8(x+y)t + (x-y)^2)}$$

This gives the following upper and lower bounds for the variance:

$$\begin{aligned} Var[Y_t] &= E[Y_t^2] - E[Y_t]^2 \\ &\leq n(n+3)t^2 + (n+3)(x+y)t + \frac{x^2+y^2}{2} \\ &\quad - \frac{1}{4}\left(2nt + x + y + \frac{2}{\pi}\int_0^\infty \frac{\left(\sum_{k=0}^\infty (-1)^k \frac{\lambda^{2k}}{(2k)!} E[(X_t - Y_t)^{2k}]\right) - 1}{\lambda^2} d\lambda\right)^2 \end{aligned}$$

and

$$\begin{aligned} Var[Y_t] \geq\; & n(n+3)t^2 + (n+3)(x+y)t + \frac{x^2+y^2}{2} \\ & - \sqrt{\left(n(n+1)t^2 + (n+1)(x+y)t + \frac{(x+y)^2}{4}\right)(8nt^2 + 8(x+y)t + (x-y)^2)} \\ & - \frac{1}{4}\left(2nt + x + y + \frac{2}{\pi}\int_0^\infty \frac{\left(\sum_{k=0}^\infty (-1)^k \frac{\lambda^{2k}}{(2k)!} E[(X_t - Y_t)^{2k}]\right) - 1}{\lambda^2} d\lambda\right)^2, \end{aligned}$$

as claimed. Again, the expectations $E[(X_t+Y_t)^{2k} - (X_t-Y_t)^{2k}]$ are clearly computable for each k since $(X_t+Y_t)^{2k} - (X_t-Y_t)^{2k}$ is a symmetric polynomial in X_t, Y_t. □

Remark. In the upper and lower bounds for the variances of X_t and Y_t in the above Theorem, we have used the expressions for $E[X_t]$ and $E[Y_t]$ given by (5.70) and (5.71) respectively. However, those could have been replaced by the alternative integral expressions (5.72) and (5.73).

In this section, we have obtained the expected value of $\cos(\lambda(X_t+Y_t))$ and $\cos(\lambda(X_t-Y_t))$. This naturally provides us with $E[f(X_t,Y_t)]$ for any function f that can be written as $f(X_t,Y_t) = A\cos(\lambda(X_t+Y_t))] + B\cos(\lambda(X_t-Y_t))$ for some constants A and B:

Example 5.3.4.

$$
\begin{aligned}
E[\cos(\lambda X_t)\cos(\lambda Y_t)] &= \frac{1}{2}(E[\cos(\lambda(X_t - Y_t))] + E[\cos(\lambda(X_t + Y_t))]) \\
&= \frac{1}{2}\left(\sum_{k=0}^{\infty}(-1)^k\frac{\lambda^{2k}}{(2k)!}E[(X_t - Y_t)^{2k}] + \sum_{k=0}^{\infty}(-1)^k\frac{\lambda^{2k}}{(2k)!}E[(X_t + Y_t)^{2k}]\right) \\
&= \frac{1}{2}\left(\sum_{k=0}^{\infty}(-1)^k\frac{\lambda^{2k}}{(2k)!}E[(X_t - Y_t)^{2k} + (X_t + Y_t)^{2k}]\right)
\end{aligned} \tag{5.79}
$$

$$
\begin{aligned}
E[\sin(\lambda X_t)\sin(\lambda Y_t)] &= \frac{1}{2}(E[\cos(\lambda(X_t - Y_t))] - E[\cos(\lambda(X_t + Y_t))]) \\
&= \frac{1}{2}\left(\sum_{k=0}^{\infty}(-1)^k\frac{\lambda^{2k}}{(2k)!}E[(X_t - Y_t)^{2k}] - \sum_{k=0}^{\infty}(-1)^k\frac{\lambda^{2k}}{(2k)!}E[(X_t + Y_t)^{2k}]\right) \\
&= \frac{1}{2}\left(\sum_{k=0}^{\infty}(-1)^k\frac{\lambda^{2k}}{(2k)!}E[(X_t - Y_t)^{2k} - (X_t + Y_t)^{2k}]\right)
\end{aligned} \tag{5.80}
$$

The expected values appearing in the above expressions are all computable using the techniques provides thus far. Moreover, the expression for $E[\cos(\lambda(X_t + Y_t))]$ used in (5.79) and (5.80) can be substituted by the expression (5.60) obtained via direct integration of the $\cos(\lambda z)$ against the transition density for the sum $Z_t = X_t + Y_t$.

In addition, Theorem 5.3.16 and its subsequent Corollary allow us to formulate the following result:

Proposition 5.3.20. *Let $W_t \sim WIS(n, 2, W_0)$, with $n \geq 2$ and with eigenvalues $X_t > Y_t \geq 0$. Let $f \in \mathcal{S}(\mathbb{R}^+)$, i.e. let f be any function in the Schwartz class. Then*

$$
E[f(X_t - Y_t)] = \sqrt{\frac{2}{\pi}}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}E[(X_t - Y_t)^{2k}]\int_0^{\infty}\hat{f}(\lambda)\lambda^{2k}d\lambda, \tag{5.81}
$$

where $\hat{f}$ denotes the classical Fourier cosine transform of the function f. Further, the expected values $E[(X_t - Y_t)^{2k}]$ appearing in the above expression are computable for all k, since $(X_t - Y_t)^{2k}$ is a symmetric polynomial in the eigenvalues X_t, Y_t for each k.

Proof. Let $q_t(\xi, x-y)$ be the transition density function of the process $X_t - Y_t$. Then by (5.63)

$$\begin{aligned}
E[f(X_t-Y_t)] &= \int_0^\infty f(\xi) q_t(\xi, x-y) d\xi \\
&= \int_0^\infty f(\xi) \frac{2}{\pi} \int_0^\infty \sum_{k=0}^{\infty} (-1)^k \frac{\lambda^{2k}}{(2k)!} E[(X_t-Y_t)^{2k}] \cos(\lambda\xi) d\lambda d\xi \\
&= \sqrt{\frac{2}{\pi}} \int_0^\infty \int_0^\infty \sqrt{\frac{2}{\pi}} f(\xi) \sum_{k=0}^{\infty} (-1)^k \frac{\lambda^{2k}}{(2k)!} E[(X_t-Y_t)^{2k}] \cos(\lambda\xi) d\lambda d\xi \\
&= \sqrt{\frac{2}{\pi}} \int_0^\infty \sum_{k=0}^{\infty} (-1)^k \frac{\lambda^{2k}}{(2k)!} E[(X_t-Y_t)^{2k}] \left(\sqrt{\frac{2}{\pi}} \int_0^\infty f(\xi) \cos(\lambda\xi) d\xi \right) d\lambda \\
&= \sqrt{\frac{2}{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} E[(X_t-Y_t)^{2k}] \int_0^\infty \hat{f}(\lambda) \lambda^{2k} d\lambda,
\end{aligned}$$

where $\hat{f}$ is Fourier cosine transform of the function f, as claimed. □

Example 5.3.5. Consider the above Proposition with $f(z) = e^{-\mu z^2}$, $\mu > 0$. Note that $f \in \mathcal{S}(\mathbb{R}^+)$ and that

$$\hat{f}(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(z) \cos(\lambda z) dz = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\mu z^2} \cos(\lambda z) dz = \frac{e^{-\frac{\lambda^2}{4\mu}}}{\sqrt{2\mu}}$$

Hence

$$\begin{aligned}
E[e^{-\mu(X_t-Y_t)^2}] &= \sqrt{\frac{2}{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} E[(X_t-Y_t)^{2k}] \int_0^\infty \hat{f}(\lambda) \lambda^{2k} d\lambda \\
&= \frac{1}{\sqrt{\mu\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} E[(X_t-Y_t)^{2k}] \int_0^\infty e^{-\frac{\lambda^2}{4\mu}} \lambda^{2k} d\lambda \\
&= \frac{1}{\sqrt{\mu\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} E[(X_t-Y_t)^{2k}] 4^k \mu^{k+\frac{1}{2}} \Gamma\left(k+\frac{1}{2}\right) \\
&= \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-4\mu)^k}{(2k)!} E[(X_t-Y_t)^{2k}] \Gamma\left(k+\frac{1}{2}\right) \\
&= \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-4\mu)^k}{(2k)!} E[(X_t-Y_t)^{2k}] \frac{(2k)!}{4^k k!} \sqrt{\pi} \\
&= \sum_{k=0}^{\infty} \frac{(-\mu)^k}{k!} E[(X_t-Y_t)^{2k}],
\end{aligned}$$

and $E[(X_t-Y_t)^{2k}$ are computable for all k.

Moreover, for other functions $g(X_t - Y_t)$ that are not in $\mathcal{S}(\mathbb{R}^+)$, the expected value $E[g(X_t - Y_t)]$ may still be computable through the appropriate manipulation of $E[f(X_t - Y_t)]$ for some $f \in \mathcal{S}(\mathbb{R}^+)$.

Example 5.3.6. Claim: Let $g(z) = \frac{1}{z^2}$, which is not in $\mathcal{S}(\mathbb{R}^+)$. We can obtain $E[g(X_t - Y_t)]$ through the integration with respect to μ of $E[f(X_t - Y_t)]$ for $f(z) = e^{-\mu z^2}$, $\mu > 0$, which is clearly in $\mathcal{S}(\mathbb{R}^+)$. That is, we can obtain $E\left[\frac{1}{(X_t - Y_t)^2}\right]$ as

$$E\left[\frac{1}{(X_t - Y_t)^2}\right] = \int_0^\infty \sum_{k=0}^\infty \frac{(-\mu)^k}{k!} E[(X_t - Y_t)^{2k}] d\mu.$$

To show this, let $r_t(z, x - y)$ be the transition density function for the difference of the eigenvalues. Then:

$$\begin{aligned}
\int_0^\infty E\left[e^{-\mu(X_t - Y_t)^2}\right] d\mu &= \int_0^\infty \int_0^\infty e^{-\mu z^2} r_t(z, x - y) dz d\mu \\
&= \int_0^\infty \left(\int_0^\infty e^{-\mu z^2} d\mu\right) r_t(z, x - y) dz \\
&= \int_0^\infty \left[\frac{e^{-\mu z^2}}{-z^2}\right]_0^\infty r_t(z, x - y) dz \\
&= \int_0^\infty \frac{1}{z^2} r_t(z, x - y) dz \\
&= E\left[\frac{1}{(X_t - Y_t)^2}\right]
\end{aligned}$$

Therefore

$$E\left[\frac{1}{(X_t - Y_t)^2}\right] = \int_0^\infty \sum_{k=0}^\infty \frac{(-\mu)^k}{k!} E[(X_t - Y_t)^{2k}] d\mu,$$

as claimed. Once more, we cannot integrate the previous expression term by term, so if we wish to be able to obtain an approximation for this expected value, we must first express it as

$$E\left[\frac{1}{(X_t - Y_t)^2}\right] = \sum_{k=0}^\infty \frac{(-1)^k}{k!} \lim_{\epsilon \to 0} \int_0^\infty e^{-\epsilon \mu^2} \mu^k E[(X_t - Y_t)^{2k}] d\mu \tag{5.82}$$

and consider as many terms as needed in the above expression. The amount of terms needed for a desired accuracy is yet to be studied.

5.3.4 The Feynman-Kac formula to compute expected values for some functionals of the eigenvalues of a 2×2 Wishart process

In the previous sections, we have been concerned with finding expected values of the type

$$E^{x,y}\left[f(X(t),Y(t))e^{-\int_0^t g(X(s),Y(s))ds}\right], \tag{5.83}$$

for some cases with $g = 0$. In this section, we present some results for $g \neq 0$ that can be obtained through the so called Feynman-Kac formula (see Theorem 2.2.5). The results presented in what follows have been obtained as group-invariant solutions for the Cauchy problem for $u(x, y, t)$:

$$u_t = Lu - q(x,y)u, \quad t > 0 \tag{5.84}$$

$$u(x,y,0) = f(x,y), \tag{5.85}$$

where the differential operator L is the generator of the eigenvalue process of a Wishart process $W_t \sim WIS(n, 2, W_0)$ with $n \geq 2$:

$$L = 2x\frac{\partial^2}{\partial x^2} + 2y\frac{\partial^2}{\partial y^2} + \left(n + \frac{x+y}{x-y}\right)\frac{\partial}{\partial x} + \left(n - \frac{x+y}{x-y}\right)\frac{\partial}{\partial y} \tag{5.86}$$

Theorem 5.3.21. *Let $W_t \sim WIS(n, 2, W_0)$ with eigenvalues $X(t) > Y(t) \geq 0$ and with index $n \geq 2$. Then*

$$E^{(x,y)}\left[\exp\left(-\int_0^t \left(\lambda + \frac{1}{X_s}f\left(\frac{Y_s}{X_s}\right)\right)ds\right)\exp\left(-\frac{X_t+Y_t}{2} - \lambda\right)\right.$$
$$\left.\times\left(1-\frac{Y_t}{X_t}\right)^{-1/2}\left(\frac{Y_t}{X_t}\right)^{\frac{1-n}{4}}\left(1+\frac{Y_t}{X_t}\right)^{\frac{n-2}{2}}\alpha\left(\frac{Y_t}{X_t}\right)\right] \tag{5.87}$$
$$= (t+1)^{-n}\exp\left(-\frac{x+y}{2(t+1)} - \lambda(t+1)\right)\left(1-\frac{y}{x}\right)^{-1/2}\left(\frac{y}{x}\right)^{\frac{1-n}{4}}\left(1+\frac{y}{x}\right)^{\frac{n-2}{2}}\alpha\left(\frac{y}{x}\right) \tag{5.88}$$

where $\alpha(z)$ satisfies the ODE:

$$\alpha''(z) + \phi(z)\alpha(z) = 0, \tag{5.89}$$

and

$$\phi(z) = -\frac{n^2(z-1)^4 - 2n(z-1)^2(3+z(2+3z))}{16(z^2-1)^2} - \frac{(1+z)^2(5+z(-14+5z)) + 8(z-1)^2 z(1+z)f(z)}{16(z^2-1)^2}.$$

Furthermore, for the following choices of $f(z)$, we can explicitly write these expectations with the given function α in each case.

1.

$$f(z) = -\frac{n^2(-1+z)^4 - 2n(z-1)^2(3+z(2+3z)) + (1+z)^2(5+z(-14+5z)}{8(z-1)^2 z(z+1)},$$

which yields $\alpha(z) = C_1 + C_2 z$.

2.

$$f(z) = \frac{-5}{8(z-1)^2 z(z+1)} + \frac{-n^2(z-1)^2 + 2n(3+z(2+3z))}{8z(z+1)} + \frac{4+z(18+(4-5z)z+16a(-1+z^2)^2)}{8(z-1)^2(z+1)}, \quad \text{with } a \in \mathbb{R}.$$

This choice of f produces $\alpha(z) = C_1 \exp(\sqrt{a}z) + C_2 \exp(-\sqrt{a}z)$.

3.

$$f(z) = \frac{1}{8}\left(-5+6n-n^2 + \frac{8}{(z-1)^2} + \frac{4}{(z-1)} - \frac{(n-5)(n-1)}{z}\right) + 2z(b+(a+b)z+az^2) + \frac{n(n-2)}{2(z+1)}, \quad \text{with } a, b \in \mathbb{R}.$$

In this case we have $\alpha(z) = C_1 Ai\left(\frac{b+az}{a^{2/3}}\right) + C_2 Bi\left(\frac{b+az}{a^{2/3}}\right)$, where Ai and Bi are the traditional Airy functions, that is

$$Ai(z) = \frac{1}{3^{2/3}\Gamma\left(\frac{2}{3}\right)}\,{}_0F_1\left(;\frac{2}{3};\frac{1}{9}z^3\right) - \frac{z}{3^{1/3}\Gamma\left(\frac{1}{3}\right)}\,{}_0F_1\left(;\frac{4}{3};\frac{1}{9}z^3\right)$$

$$Bi(z) = \frac{1}{3^{1/6}\Gamma\left(\frac{2}{3}\right)}\,{}_0F_1\left(;\frac{2}{3};\frac{1}{9}z^3\right) + \frac{3^{1/6}z}{\Gamma\left(\frac{1}{3}\right)}\,{}_0F_1\left(;\frac{4}{3};\frac{1}{9}z^3\right)$$

where ${}_0F_1(;a;z) \equiv \lim_{q\to\infty} {}_1F_1(q;a;z/q)$ or, as a series expansion ${}_0F_1(;a;z) = \sum_{n=0}^{\infty} \frac{z^n}{(a)_n n!}$

4.

$$f(z) = \frac{-5 - n^2(z-1)^4 + 2n(z-1)^2(3+z(2+3z))}{8(z-1)^2 z(z+1)} + \frac{4 + z(18 + (4-5z)z + 16a(z^2-1)^2(1+az^2))}{8(z-1)^2(z+1)}, \quad \text{with } a \in \mathbb{R}.$$

For this choice of f we must have $\alpha(z) = C_1 D_{-1}(\sqrt{2}\sqrt{a}z) + C_2 D_0(i\sqrt{2}\sqrt{a}z)$. Here, $D_\nu(z)$ refers to the traditional Parabolic Cylinder function, i.e.

$$D_\nu(z) = \frac{2^{\nu/2} e^{-\frac{z^2}{4}}}{\sqrt{\pi}} \cos\left(\frac{\pi\nu}{2}\right) \Gamma\left(\frac{\nu+1}{2}\right) {}_1F_1\left(-\frac{\nu}{2}; \frac{1}{2}; \frac{z^2}{2}\right) + \frac{2^{(\nu+1)/2} e^{-\frac{z^2}{4}} z}{\sqrt{\pi}} \sin\left(\frac{\pi\nu}{2}\right) \Gamma\left(\frac{\nu}{2}+1\right) {}_1F_1\left(\frac{1}{2} - \frac{\nu}{2}; \frac{3}{2}; \frac{z^2}{2}\right)$$

5.

$$f(z) = \frac{-n^2(z-1)^2 + n(6+4z+6z^2) + 16z^2(z+1)^2(b+az^2)}{(z+1)8z} + \frac{-5 + z(9 + (9-5z)z)}{(z-1)^2 8z}, \quad \text{with } a, b \in \mathbb{R},$$

which produces $\alpha(z) = C_1 D_{-\frac{b+\sqrt{a}}{2\sqrt{a}}}(\sqrt{2}a^{1/4}z) + C_2 D_{\frac{b-\sqrt{a}}{2\sqrt{a}}}(i\sqrt{2}a^{1/4}z)$, where $D_\nu(z)$ is again the traditional Parabolic Cylinder function.

6.

$$f(z) = \frac{-5 - n^2(z-1)^4 + 2n(z-1)^2(3+z(2+3z))}{8(z-1)^2 z(z+1)} + \frac{4 + z(18 + z(4 - 5z - 16a^3(az-2)(z^2-1)^2))}{8(z-1)^2(z+1)}, \quad \text{with } a \in \mathbb{R}.$$

This particular $f(z)$ leads to

$$\alpha(z) = C_1 D_{-\frac{1}{2}(1+i)}((-1+i)(-1+az)) + C_2 D_{-\frac{1}{2}(1-i)}((1+i)(-1+az)),$$

with $D_\nu(z)$ the Parabolic Cylinder function.

7.

$$f(z) = \frac{-n^2(z-1)^2 + n(6+4z+6z^2) + 16z^2(z+1)^2(c+z(b+az))}{(z+1)8z} + \frac{-5 + z(9 + z(9-5z))}{(z-1)^2 8z}, \quad \text{with } a, b, c \in \mathbb{R}.$$

Choosing such an $f(z)$ yields the following expression for $\alpha(z)$:

$$\alpha(z) = C_1 D_{\frac{b^2-4a(\sqrt{a}+c)}{8a^{3/2}}}\left(\frac{b+2az}{\sqrt{2}a^{3/4}}\right) + C_2 D_{\frac{-b^2-4a(\sqrt{a}-c)}{8a^{3/2}}}\left(i\frac{b+2az}{\sqrt{2}a^{3/4}}\right).$$

In the above expression $D_\nu(z)$ denotes the traditional Parabolic Cylinder function.

8.

$$f(z) = \frac{-5 - n^2(z-1)^4 + 2n(z-1)^2(3+z(2+3z))}{8(z-1)^2 z(z+1)} + \frac{4 + z(18 + (4-5z)z + 16az^q(z^2-1)^2)}{8(z-1)^2(z+1)}, \quad \textit{with } a \in \mathbb{R}, \quad q \in \mathbb{Z},$$

which gives

$$\alpha(z) = \sqrt{z} C_1 I_{-\frac{1}{2+q}}\left(\frac{2\sqrt{a}z^{1+\frac{q}{2}}}{2+q}\right)\Gamma\left(\frac{1+q}{2+q}\right) + \sqrt{z} C_2 (-1)^{\frac{1}{2+q}} I_{\frac{1}{2+q}}\left(\frac{2\sqrt{a}z^{1+\frac{q}{2}}}{2+q}\right)\Gamma\left(1+\frac{1}{2+q}\right)$$

where $I_n(z)$ is the modified Bessel function of the first kind and $\Gamma(z)$ is the Euler gamma function. That is

$$I_n(z) = \sum_{m=0}^{\infty} \frac{1}{m!\,\Gamma(m+n+1)}\left(\frac{z}{2}\right)^{2m+n}$$

$$\Gamma(z) = \frac{1}{z}\prod_{n=1}^{\infty}\frac{\left(1+\frac{1}{n}\right)^z}{1+\frac{z}{n}}, \quad z \in \mathbb{C} \setminus \{0, \mathbb{Z}^-\}$$

9.

$$f(z) = \frac{z+1}{8z}\left(-\frac{n^2(z-1)^2}{(z+1)^2} + \frac{-5+(14-5z)z}{(z-1)^2} + 2az^q(z+1)(q+az^{1+q})\right) + \frac{n(3+z(2+3z))}{4z(z+1)} \quad \textit{with } a \in \mathbb{R}, \quad q \in \mathbb{Z}.$$

In this case, a particular solution for the function $\alpha(z)$ is $\alpha(z) = C_1 \exp(\frac{a}{q+1} z^{q+1})$

10.

$$f(z) = \frac{1+z}{8z}\left(-\frac{n^2(z-1)^2}{(z+1)^2} + \frac{-5+z(14-5z)}{(z-1)^2}\right.$$
$$\left. + 2az^{q-1}(z+1)(1+q+az^q) + \frac{n(3+z(2+3z))}{4z(z+1)}\right) \quad \textit{with } a \in \mathbb{R}, \quad q \in \mathbb{Z}.$$

Similarly to the previous case, a particular solution for the function $\alpha(z)$ *is* $\alpha(z) = C_1 z \exp(\frac{az^q}{q})$.

11.

$$f(z) = \frac{4+z(18+z(4-5z))+16bz^q(z^2-1)^2+16az^{1+2q}(z^2-1)^2}{8(z-1)^2(z+1)}$$
$$+ \frac{-5-n^2(z-1)^4+2n(z-1)^2(3+z(2+3z))}{8(z-1)^2z(z+1)} \quad \textit{with } a,b \in \mathbb{R}, \quad q \in \mathbb{Z}.$$

For this last choice of $f(z)$ *the expression for* $\alpha(z)$ *is*

$$\alpha(z) = C_1 e^{\left(\frac{i\sqrt{a}z^{1+q}}{|1+q|}\right)} U\left(\frac{q}{2(1+q)} - i\frac{b}{2\sqrt{a}|1+q|}, \frac{q}{1+q}, -\frac{(2i\sqrt{a}z^{1+q}}{|1+q|}\right)$$
$$+ C_2 e^{\left(\frac{i\sqrt{a}z^{1+q}}{|1+q|}\right)} L^{-\frac{1}{1+q}}_{-\frac{q}{2(1+q)}+i\frac{b}{2\sqrt{a}|1+q|}}\left(-\frac{2i\sqrt{a}z^{1+q}}{|1+q|}\right).$$

In this expression for $\alpha(z)$*, the function* $U(a,b,z)$ *is the confluent hypergeometric function and* $L_n^a(x)$ *is the generalized Laguerre polynomial.*

Proof. Let

$$u(x,y,t) = (t+1)^{-n} e^{-\frac{x+y}{2(t+1)} - \lambda(t+1)} \left(1-\frac{y}{x}\right)^{-1/2} \left(\frac{y}{x}\right)^{\frac{1-n}{4}} \left(1+\frac{y}{x}\right)^{\frac{n-2}{2}} \alpha\left(\frac{y}{x}\right)$$

where $\alpha(z)$ satisfies the ODE:

$$\alpha''(z) + \phi(z)\alpha(z) = 0, \tag{5.90}$$

and

$$\phi(z) = -\frac{n^2(z-1)^4 - 2n(z-1)^2(3+z(2+3z))}{16(z^2-1)^2}$$
$$- \frac{(1+z)^2(5+z(-14+5z)) + 8(z-1)^2z(1+z)f(z)}{16(z^2-1)^2}.$$

We first need to check that

$$u(x,y,0) = \exp\left(-\frac{x+y}{2} - \lambda\right)\left(1 - \frac{y}{x}\right)^{-1/2}\left(\frac{y}{x}\right)^{\frac{1-n}{4}}\left(1+\frac{y}{x}\right)^{\frac{n-2}{2}}\alpha\left(\frac{y}{x}\right),$$

as required by Theorem 2.2.5, but this is trivial if we substitute $t = 0$ in the expression of u. Next we need to check that u satisfies the PDE (5.84), with $q(x,y) = \lambda + \frac{1}{x} f\left(\frac{y}{x}\right)$. That is, we must check that u solves

$$\begin{aligned} u_t &= Lu - \left(\lambda + \frac{1}{x} f\left(\frac{y}{x}\right)\right) u \\ &= 2xu_{xx} + 2yu_{yy} + \left(n + \frac{x+y}{x-y}\right) u_x + \left(n - \frac{x+y}{x-y}\right) u_y - \left(\lambda + \frac{1}{x} f\left(\frac{y}{x}\right)\right) u \end{aligned} \tag{5.91}$$

Observe that on the one hand we have

$$\begin{aligned} u_t &= (t+1)^{-n} e^{-\frac{x+y}{2(t+1)} - \lambda(t+1)} \left(1 - \frac{y}{x}\right)^{-1/2}\left(\frac{y}{x}\right)^{\frac{1-n}{4}}\left(1+\frac{y}{x}\right)^{\frac{n-2}{2}}\alpha\left(\frac{y}{x}\right) \\ &\times \left(\frac{x+y}{2}(t+1)^{-2} - \lambda - n(t+1)^{-1}\right) \\ &= u \times \left(\frac{x+y}{2}(t+1)^{-2} - \lambda - n(t+1)^{-1}\right) \end{aligned}$$

Similarly, we can write u_x, u_y, u_{xx} and u_{yy} in terms of u as follows:

$$\begin{aligned} u_x &= -(t+1)^{-n} e^{-\frac{x+y}{2(t+1)} - \lambda(t+1)} \left(1 - \frac{y}{x}\right)^{-1/2}\left(\frac{y}{x}\right)^{\frac{1-n}{4}}\left(1+\frac{y}{x}\right)^{\frac{n-2}{2}}\alpha\left(\frac{y}{x}\right) \\ &\times \left(\frac{(1-n)(x-2y)x + (5-n)y^2}{4x(x-y)(x+y)} + \frac{1}{2(t+1)} + \frac{y}{x^2}\frac{\alpha'\left(\frac{y}{x}\right)}{\alpha\left(\frac{y}{x}\right)}\right) \\ &= -u \times \left(\frac{(1-n)(x-2y)x + (5-n)y^2}{4x(x-y)(x+y)} + \frac{1}{2(t+1)} + \frac{y}{x^2}\frac{\alpha'\left(\frac{y}{x}\right)}{\alpha\left(\frac{y}{x}\right)}\right) \end{aligned}$$

$$\begin{aligned} u_y &= -(t+1)^{-n} e^{-\frac{x+y}{2(t+1)} - \lambda(t+1)} \left(1 - \frac{y}{x}\right)^{-1/2}\left(\frac{y}{x}\right)^{\frac{1-n}{4}}\left(1+\frac{y}{x}\right)^{\frac{n-2}{2}}\alpha\left(\frac{y}{x}\right) \\ &\times \left(\frac{(n-1)\left(x^2 - 2xy\right) + (n-5)y^2}{4y(x-y)(x+y)} + \frac{1}{2(t+1)} - \frac{1}{x}\frac{\alpha'\left(\frac{y}{x}\right)}{\alpha\left(\frac{y}{x}\right)}\right) \\ &= -u \times \left(\frac{(n-1)\left(x^2 - 2xy\right) + (n-5)y^2}{4y(x-y)(x+y)} + \frac{1}{2(t+1)} - \frac{1}{x}\frac{\alpha'\left(\frac{y}{x}\right)}{\alpha\left(\frac{y}{x}\right)}\right) \end{aligned}$$

$$u_{xx} = \underbrace{(t+1)^{-n} e^{-\frac{x+y}{2(t+1)} - \lambda(t+1)} \left(1 - \frac{y}{x}\right)^{-1/2} \left(\frac{y}{x}\right)^{\frac{1-n}{4}} \left(1 + \frac{y}{x}\right)^{\frac{n-2}{2}} \alpha\left(\frac{y}{x}\right)}_{u}$$
$$\times \left(\frac{6((n-6)n+13)x^2y^2 + (n-5)(n-1)\left(x^4 - 4xy\left(x^2+y^2\right) + y^4\right)}{16x^2(x-y)^2(x+y)^2} \right.$$
$$+ \frac{1}{4(t+1)^2} - \frac{(n-1)x(x-2y) + (n-5)y^2}{4(t+1)x(x+y)(x-y)}$$
$$\left. - \left(\frac{(n-5)x^2y + (n-1)y^2(y-2x)}{2x^3(x+y)(x-y)} - \frac{y}{(t+1)x^2} \right) \frac{\alpha'\left(\frac{y}{x}\right)}{\alpha\left(\frac{y}{x}\right)} + \frac{y^2}{x^4} \frac{\alpha''\left(\frac{y}{x}\right)}{\alpha\left(\frac{y}{x}\right)} \right)$$

$$u_{yy} = \underbrace{(t+1)^{-n} e^{-\frac{x+y}{2(t+1)} - \lambda(t+1)} \left(1 - \frac{y}{x}\right)^{-1/2} \left(\frac{y}{x}\right)^{\frac{1-n}{4}} \left(1 + \frac{y}{x}\right)^{\frac{n-2}{2}} \alpha\left(\frac{y}{x}\right)}_{u}$$
$$\times \left(\frac{(n-1)x(x-2y) + (n-5)y^2}{4(t+1)y(x+y)(x-y)} + \frac{2(3(n-6)n+23)x^2y^2 + (n-9)(n-5)y^4}{16y^2(x+y)^2(x-y)^2} \right.$$
$$+ \frac{(n-1)\left((n+3)x^4 - 4(n-1)x^3y - 4(n-9)xy^3\right)}{16y^2(x+y)^2(x-y)^2} + \frac{1}{4(t+1)^2}$$
$$\left. - \left(\frac{(n-1)x(x-2y) + (n-5)y^2}{2xy(x+y)(x-y)} + \frac{2y}{2(t+1)xy} \right) \frac{\alpha'\left(\frac{y}{x}\right)}{\alpha\left(\frac{y}{x}\right)} + \frac{1}{x^2} \frac{\alpha''\left(\frac{y}{x}\right)}{\alpha\left(\frac{y}{x}\right)} \right)$$

It can be checked that substituting these expressions of the derivatives into (5.91) produces the following differential equation for the function α:

$$0 = -\frac{1}{x} f\left(\frac{y}{x}\right) + \frac{2y(x+y)}{x^3} \frac{\alpha''\left(\frac{y}{x}\right)}{\alpha\left(\frac{y}{x}\right)}$$
$$+ \frac{2(n(2-3n)+9)x^2y^2 + (n-1)\left(4(n-1)xy\left(x^2+y^2\right) - (n-5)\left(x^4+y^4\right)\right)}{8xy(x-y)^2(x+y)},$$

which, after setting $z = \frac{y}{x}$ is equivalent to

$$0 = -\left(\frac{8z(z+1)(z-1)^2 f(z) + n^2(z-1)^4 - 2n(z(3z+2)+3)(z-1)^2}{16z^2\left(z^2-1\right)^2} \right.$$
$$\left. + \frac{(z+1)^2(z(5z-14)+5)}{16z^2\left(z^2-1\right)^2} \right) \alpha(z) + \alpha''(z),$$

as claimed. Hence, if α is indeed a solution of the above DE, then u is in fact a solution of

$$u_t = Lu - \left(\lambda + \frac{1}{x} f\left(\frac{y}{x}\right)\right) u$$

with

$$u(x, y, 0) = \exp\left(-\frac{x+y}{2} - \lambda\right)\left(1 - \frac{y}{x}\right)^{-1/2}\left(\frac{y}{x}\right)^{\frac{1-n}{4}}\left(1 + \frac{y}{x}\right)^{\frac{n-2}{2}} \alpha\left(\frac{y}{x}\right),$$

and so, by part (b) of the Feyman-Kac theorem, we have that

$$u(x, y, t) = E^{(x,y)}\left[\exp\left(-\int_0^t \left(\lambda + \frac{1}{X_s} f\left(\frac{Y_s}{X_s}\right)\right) ds\right) \exp\left(-\frac{X_t + Y_t}{2} - \lambda\right) \right.$$
$$\left. \times \left(1 - \frac{Y_t}{X_t}\right)^{-1/2}\left(\frac{Y_t}{X_t}\right)^{\frac{1-n}{4}}\left(1 + \frac{Y_t}{X_t}\right)^{\frac{n-2}{2}} \alpha\left(\frac{Y_t}{X_t}\right)\right]$$

For the proof of the second part of the theorem where particular choices for f are given, one need only check that substituting each particular choice of f into the coefficient function $\phi(z)$ appearing in the DE

$$\alpha'(z) + \phi(z)\alpha(z) = 0$$

will yield the given function $\alpha(z)$ as a solution to the DE in each case. We do not include these computations here, but they can easily be checked using any basic computer algebra system. □

Another result involving the expectations of functionals of the eigenvalues of a 2×2 Wishart Process that can be obtained using the Feynman-Kac theorem (Theorem 2.2.5) is the following:

Theorem 5.3.22. *Let $W_t \sim WIS(n, 2, W_0)$ with eigenvalues $X(t) > Y(t) \geq 0$ and with index $n \geq 2$. Then*

$$E^{(x,y)}\left[e^{-\lambda t} \omega\left(\frac{X_t}{Y_t}\right)\right] = e^{-\lambda t} \omega\left(\frac{x}{y}\right), \tag{5.92}$$

where

$$\omega(z) = C_1 - C_2 z^{\frac{3-n}{2}} \left(F_1\left(\frac{3-n}{2}, -n, 1, \frac{5-n}{2}, -z, z\right)\right.$$
$$\left. + {}_2F_1\left(1-n, \frac{3-n}{2}, \frac{5-n}{2}, -z\right) + 2{}_2F_1\left(2-n, \frac{3-n}{2}, \frac{5-n}{2}, -z\right)\right), \tag{5.93}$$

where $F_1(a; b_1, b_2; c; x, y)$ is the Appell hypergeometric function of two variables and ${}_2F_1(a, b; c; z)$ is the traditional hypergeometric function ${}_2F_1$.

Proof. Let $u(x, y, t) = e^{-\lambda t}\omega\left(\frac{x}{y}\right)$ with $\omega(z)$ defined as in (5.93). Note that $u(x, y, 0) = \omega\left(\frac{x}{y}\right)$. We need to check next that u satisfies the PDE:

$$u_t = Lu - \lambda u \tag{5.94}$$

Note that

$$\begin{aligned}
u_t &= -\lambda e^{-\lambda t}\omega\left(\frac{x}{y}\right) \\
u_x &= \frac{e^{-\lambda t}}{y}\omega'\left(\frac{x}{y}\right) \\
u_y &= -\frac{xe^{-\lambda t}}{y^2}\omega'\left(\frac{x}{y}\right) \\
u_{xx} &= \frac{e^{-\lambda t}}{y^2}\omega''\left(\frac{x}{y}\right) \\
u_{yy} &= e^{-\lambda t}\left(\frac{x^2}{y^4}\omega''\left(\frac{x}{y}\right) + \frac{2x}{y^3}\omega'\left(\frac{x}{y}\right)\right)
\end{aligned}$$

So equation (5.94) becomes:

$$\begin{aligned}
-\lambda e^{-\lambda t}\omega\left(\frac{x}{y}\right) = 2x\frac{e^{-\lambda t}}{y^2}\omega''\left(\frac{x}{y}\right) + 2ye^{-\lambda t}\left(\frac{x^2}{y^4}\omega''\left(\frac{x}{y}\right) + \frac{2x}{y^3}\omega'\left(\frac{x}{y}\right)\right) \\
+ \left(n + \frac{x+y}{x-y}\right)\frac{e^{-\lambda t}}{y}\omega'\left(\frac{x}{y}\right) - \left(n - \frac{x+y}{x-y}\right)\frac{xe^{-\lambda t}}{y^2}\omega'\left(\frac{x}{y}\right) \\
- \lambda e^{-\lambda t}\omega\left(\frac{x}{y}\right)
\end{aligned}$$

Note that the above ODE is equivalent (after multiplying by $ye^{\lambda t}$) to:

$$0 = \left(2\frac{x}{y} + 2\frac{x^2}{y^2}\right)\omega''\left(\frac{x}{y}\right) + \left(4\frac{x}{y} + n\left(1 - \frac{x}{y}\right) + \frac{x+y}{x-y}\left(1 + \frac{x}{y}\right)\right)\omega'\left(\frac{x}{y}\right)$$

Now let $z = \frac{x}{y}$. The above ODE for ω becomes the following in terms of z:

$$0 = 2z(1+z)\omega''(z) + \left(4z + n(1-z) + \frac{z+1}{z-1}(1+z)\right)\omega'(z), \tag{5.95}$$

So we only need to make sure that $w(z)$ defined as in (5.93) is a solution to this ODE. This can easily be checked with any basic computer algebra system, so we do not include the computations here. We have thus proved that u satisfies (5.94) with initial condition $u(x, y, 0) = \omega\left(\frac{x}{y}\right)$ and hence, by part (b) of the Feynman-Kac Theorem, we have

$$u(x, y, t) = E^{(x,y)}\left[e^{-\lambda t}\omega\left(\frac{X_t}{Y_t}\right)\right] \tag{5.96}$$

□

Chapter 6

Summary

6.1 Systems of PDEs

In this thesis, we have shown how lie symmetries and integral transforms may be used to obtain fundamental matrices for some classes of systems of PDEs. In Chapter 3 we have computed the lie algebra for systems of the type

$$\begin{cases} u_t = u_{xx} + g_1(x)v \\ v_t = v_{xx} + g_2(x)u \end{cases}$$

for the choices:

$$\begin{aligned}
&(1) \quad g_1(x) = \frac{C_1}{x^2}, \quad g_2(x) = \frac{C_2}{x^2},\\
&(2) \quad g_1(x) = C_1, \quad g_2(x) = \frac{C_2}{x^4},\\
&(3) \quad g_1(x) = C_1 x^k, \quad g_2(x) = C_2 x^{-(4+k)}, k \neq 0, -2,\\
&(4) \quad g_1(x) = C_1 + C_2 x, \quad g_2(x) = k(C_1 + C_2 x),\\
&(5) \quad g_1(x) = C_1, \quad g_2(x) = C_2.
\end{aligned}$$

We have obtained fundamental matrices for cases (1), (2),(4) and (5). We have shown how the simplest cases, (1),(4) and (5), produce fundamental matrices of scalar functions, while the more complex cases, such as (2), produce matrices of differential operators.

In Chapter 4, we have obtained fundamental matrices for general systems of the type

$$\begin{cases} v_t = \sigma x^\gamma v_{xx} + f_1(x)v_x - f_2(x)w_x \\ w_t = \sigma x^\gamma w_{xx} + f_2(x)v_x + f_1(x)w_x, \end{cases} \qquad x, t > 0,$$

where $A_i, B_i \in \mathbb{R}$ for $i = 1, 2$, $\sigma, \gamma \in \mathbb{R}$, $\gamma \neq 2$ and where $f_1(x)$, $f_2(x)$ are real-valued functions satisfying

$$\begin{cases} -\gamma\sigma x^{1-\gamma} f_1(x) + \sigma x^{2-\gamma} f_1'(x) + \frac{1}{2} x^{2(1-\gamma)}(f_1(x)^2 - f_2(x)^2) = 2\sigma A_1 x^{2-\gamma} + B_1 \\ -\gamma\sigma x^{1-\gamma} f_2(x) + \sigma x^{2-\gamma} f_2'(x) + x^{2(1-\gamma)} f_1(x) f_2(x) = 2\sigma A_2 x^{2-\gamma} + B_2. \end{cases}$$

We have obtained a general formula for a fundamental matrix for any system of this particular type and we have obtained some explicit formulas for some examples. These typically contain both integral and differential operators. We have also prepared the ground to obtain similar results for a wider class of functions f_1 and f_2.

6.1.1 Future work

We would like to extend our work on this topic in different aspects:

- First, we would like to find a non-stationary solution for the the above mentioned case (3) to be able to compute a fundamental matrix for this case too. For this choice of functions g_1 and g_2, the system possesses a scaling symmetry, and we believe this is enough to produce a fundamental matrix.
- Second, we would like to further explore the methods used in Chapter 4 to extend the scope of the results to a wider class of functions f_1 and f_2, as indicated at the end of that chapter.
- Third, we would like to explore other ways of using Lie symmetries to compute fundamental matrices for a given system of PDEs and try to determine if it is possible to obtain simpler fundamental matrices in the cases where we currently have things that are not in terms of scalar functions only.
- Finally, we would like to explore these methods for higher dimensional systems, systems of higher order and, ideally, elliptic systems as well.

6.2 Wishart Processes and their eigenvalues

We have studied Wishart processes in Chapter 5. In particular, we have focused on their eigenvalues. We have shown how the usual lie symmetry methods fail to produce a transition density function for the eigenvalues of a Wishart process. However, we have produced a set of tools that allow us to compute a wide range of expected values of functions of these eigenvalues. We have focused mainly on 2-dimensional Wishart processes but we have also provided some results for general dimension p. In particular we have obtained expected values of the form:

$$E^{x,y}\left[f(X(t),Y(t))\right], \tag{6.1}$$

for a wide range of functions f including symmetric polynomials and some trigonometric functions such as $\cos(\lambda(X_t - Y_t))$ or $\sin(\lambda X_t)\sin(\lambda Y_t)$, or for any function $f(X_t - Y_t)$ of the Schwartz class. Further, we have given some methods that allow us to compute new expected values of these type from known ones.

In addition, we have obtained the cosine transform of the transition density function of $X_t - Y_t$ and an integral expression for the expected values of X_t and Y_t. We have also given some bounds for the variances of X_t and Y_t.

Lastly, we have computed some expectations of the type

$$E^{x,y}\left[f(X(t),Y(t))e^{-\int_0^t g(X(s),Y(s))ds}\right], \tag{6.2}$$

through the Feynman-Kac formula.

6.2.1 Future work

- We would like to pursue the idea of solving the Sturm-Liouville problem obtained in section 5.2.1.1, which would produce an expression for the transition density of the eigenvalues.
- We would like to further explore the evaluation problem for the integral expressions appearing in the last section. We would like to determine the amount of terms needed for a particular accuracy.

6.3 Conclusion

Lie symmetries have been proven to be useful for the computation of fundamental matrices for systems of PDEs. So far, we have only studied a small number of parabolic systems but we would like to explore these techniques for higher dimensional systems and for systems of higher orders. Ideally, we would also like to extend this to elliptic systems of PDEs, so it is clear that this is still an area with much room for further study.

Additionally, we have seen how lie symmetries can be employed to study diffusion processes even when they fail to produce the transition density for the process. They provide a good amount of information about the diffusion and they allow us to obtain the expected values for a large number of functions of this diffusion.

Appendix A

Integral Transforms

A.1 Some classical integral transforms

Throughout our work, we make extensive use of classical integral transforms. Here is a recollection of the transforms we use and their definitions, as well as some useful properties.
For a suitable function f in each case, let us define the following transforms:

(1) **Laplace Transform**

$$\mathcal{L}\{f(t)\}(s) = \int_0^\infty f(t)e^{-st}dt \tag{A.1}$$

(2) **Fourier Transform**

$$\mathcal{F}\{f(x)\}(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(x)e^{-iyx}dx \tag{A.2}$$

(3) **Fourier Cosine Transform**

$$\mathcal{F}_c\{f(t)\}(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t)\cos(\omega t)dt \tag{A.3}$$

(4) **Mellin Transform**

$$\mathcal{M}\{f(x)\}(s) = \int_0^\infty x^{s-1} f(x)dx \tag{A.4}$$

(5) **Hankel Transform of order** n

$$\mathcal{H}_n\{f(t)\}(r) = \int_0^\infty t f(t) J_n(rt)dt, \tag{A.5}$$

where $J_n(x)$ is a Bessel function of the first kind.

A useful property of the Laplace transform is the *shift property*. We use this property a few times throughout this thesis to invert some Laplace transforms. Let

$\mathcal{L}\{f(t)\}(s) = F(s)$, then

$$\mathcal{L}\{e^{-at}f(t)\}(s) = F(s+a).$$

This implies that

$$\mathcal{L}^{-1}\{F(s+a)\}(t) = e^{-at}f(t)$$

Let us now derive a relationship between the Mellin transform and the Fourier transform defined in (A.2):

$$\begin{aligned}
\mathcal{M}\{f(x)\}(s) &= \int_0^\infty x^{s-1}f(x)dx \quad \text{(make the change of variables } x = e^{-u}\text{)} \\
&= -\int_\infty^{-\infty} e^{-u(s-1)}f(e^{-u})e^{-u}du \\
&= \int_{-\infty}^\infty e^{-us}f(e^{-u})du \\
&= \sqrt{2\pi}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-i(-is)u}f(e^{-u})du \\
&= \sqrt{2\pi}\mathcal{F}\{f(e^{-x})\}(-is) \qquad \text{(A.6)}
\end{aligned}$$

Hence we can invert any Mellin transform by converting it to the corresponding Inverse Fourier transform.

A.2 The distributional Laplace transform

In this section we give a brief explanation on how to extend the notion of the classical Laplace transform defined in (A.1) to one that is capable of dealing with the broader concept of generalised functions or distributions. We also deal with what is commonly referred to as *pseudo-functions*. These pseudo-functions will be our main tool to deal with integrands that have singularities at a finite point. The approach we use to deal with such singularities is due to the French mathematician Jacques Hadamard. An explanation of this approach can be found in [37].

We will first consider the case of probably one of the most famous generalised functions: the *Dirac Delta function*, $\delta(t)$.
It is clear, due to the very special properties of this particular function, that extending the classical notion of the Laplace transform (A.1) to generalised functions should produce the following result for δ:

$$\mathcal{L}\{\delta(t)\}(s) = \int_0^\infty \delta(t)e^{-st}dt = 1 \qquad \text{(A.7)}$$

We can also define the Laplace transform for any derivatives of the Delta function in a similar manner:

$$\begin{aligned}\mathcal{L}\{\delta'(t)\}(s) &= \int_0^\infty \delta'(t)e^{-st}dt = -\int_0^\infty (-s)\delta(t)e^{-st}dt = s\int_0^\infty \delta(t)e^{-st}dt = s\\ \mathcal{L}\{\delta''(t)\}(s) &= \int_0^\infty \delta''(t)e^{-st}dt = -\int_0^\infty (-s)\delta'(t)e^{-st}dt = s\int_0^\infty \delta'(t)e^{-st}dt = s^2\\ &\vdots\end{aligned}$$

One can therefore easily obtain the following general formula recursively:

$$\mathcal{L}\{\delta^{(n)}(t)\}(s) = s^n \tag{A.8}$$

This seems rather intuitive and, although it requires some knowledge on the manipulation of distributions, it does not apparently require any new definitions of the Laplace transform. In the literature, the Laplace transform of the Dirac Delta function (A.7) commonly appears denoted as

$$\mathcal{L}_-\{\delta(t)\}(s) = \int_{0^-}^\infty \delta(t)e^{-st}dt := \lim_{\epsilon\to 0}\int_{-\epsilon}^\infty \delta(t)e^{-st}dt = 1. \tag{A.9}$$

This notation is used to emphasize that any point mass located at the origin is entirely captured by the Laplace transform.
However, one often needs to deal with functions of the type $f(t) = t^\lambda$ with $\lambda \in \mathbb{R}$, $\lambda \leq -1$. For such functions it is easy to see that the Laplace integral

$$\mathcal{L}\{t^\lambda\}(s) = \int_0^\infty t^\lambda e^{-st}dt \tag{A.10}$$

diverges near the origin. It is in this context that the so called pseudo-functions arise. To deal with this type of functions one needs to redefine the notion of a Laplace transform.
There are a number of authors that have worked on the definition of distributional integral transforms (and their inversions) using different approaches. Amongst them, we would like to remark the work by Zemanian in [58], [61], [57], [60],[59] or by Pathak in [53].
Both Pathak (in [53]) and Zemanian (in [57]) first introduce the testing function space $\mathscr{L}_a$ and its dual space $\mathscr{L}'_a$ for which the ordinary right-sided Laplace transform can be generalised in a natural way and then define the generalized Laplace transform for any function $f \in \mathscr{L}'_a$ as

$$\mathcal{L}\{f\}(s) = \langle f(t), e^{-st}\rangle, \quad s \in \Lambda, \tag{A.11}$$

for a suitable region $\Lambda \subset \mathbb{C}$.
We provide in what follows a very brief explanation on how these spaces and transform are defined. We do not provide all the proofs and we refer the reader to the above literature for a more thorough explanation on the concept.

Let $I = (0, \infty)$ and let ϕ be a $\mathcal{C}^\infty$ complex-valued function on I. Then $\phi \in \mathscr{L}_a(I)$ for $a \in \mathbb{R}$ if

$$\gamma_k(\phi) := \gamma_{a,k}(\phi) := \sup_{0<t<\infty} |e^{at} D^k \phi(t)| < \infty, \quad \forall k \in \mathbb{N}. \tag{A.12}$$

With this definition it can easily be seen that $\mathscr{L}_a(I)$ is a linear space and that for $a < b$ we have that $\mathscr{L}_b(I) \subset \mathscr{L}_a(I)$. One can also easily check that for each fixed s, the exponential $e^{-st} \in \mathscr{L}_a(I)$ if and only if $a \leq Re[s]$.
We now need to define the appropriate space of generalized functions, i.e. the dual of $\mathscr{L}_a(I)$, $\mathscr{L}'_a(I)$:
Let f be a locally integrable function such that $e^{-at} f(t)$ is absolutely integrable on $(0, \infty)$. Then f generates a regular generalized function in $\mathscr{L}'_a(I)$ through

$$\langle f, \phi \rangle := \int_0^\infty f(t)\phi(t)dt, \quad \phi \in \mathscr{L}_a(I). \tag{A.13}$$

We say that f is a *Laplace transformable generalized function* if there exists $a \in \mathbb{R}$ such that $f \in \mathscr{L}'_a(I)$. It is then clear that $f \in \mathscr{L}'_b(I)$ for every $b > a$; hence there exists a real number σ_f (possibly $-\infty$) such that $f \in \mathscr{L}'_a(I)$ for every $a > \sigma_f$ and $f \notin \mathscr{L}'_a(I)$ for every $a < \sigma_f$.
The last step is to define the generalized Laplace transform of any function $f \in \mathscr{L}'_a(I)$. One expects to be using the test function e^{-st} to define such transform but, as mentioned before, certain conditions must be satisfied for such exponential to be in $\mathscr{L}_a(I)$. So we need to produce one last definition before we can properly define our distributional Laplace transform.
Let

$$\Omega_f := \{s \in \mathbb{C} : Re[s] > \sigma_f, s \neq 0, |arg(s)| < \pi\}. \tag{A.14}$$

Note that for negative σ_f this region Ω_f is a cut half-plane obtained by deleting all non-positive values of s.

Definition A.2.1. Let $s \in \Omega_f$. We define the **generalized Laplace transform** of $f \in \mathscr{L}'_a(I)$ by

$$\mathcal{L}\{f\}(s) := F(s) := \langle f(t), e^{-st} \rangle. \tag{A.15}$$

With this definition, the real number σ_f is called the *abscissa of definition* and the region Ω_f is referred to as the *region of definition* of the generalized Laplace transform.

It is important to remark that the above definition makes sense as the application of a certain function $f \in \mathscr{L}'_a(I)$ to $e^{-st} \in \mathscr{L}_a(I)$ for $a \in \mathbb{R}$ with $\sigma_f < a < Re[s]$. It turns out that with such a definition this construction possesses all the expected nice qualities such as analyticity in Ω_f. The following theorem can be found with its proof in [53]:

Theorem A.2.1. *Let $F(s) = \mathcal{L}\{f\}(s)$ for $s \in \Omega_f$. Then $F(s)$ is an analytic function on Ω_f and*

$$D^{(k)}F(s) = \langle f(t), (-t)^k e^{-st} \rangle, \quad k \in \mathbb{N} \tag{A.16}$$

This result is however not surprising since it is in some way the analogous of the differentiation property of the ordinary Laplace transform.
Other properties of the distributional Laplace transform that can be intuitively related to those of the ordinary Laplace transform. Here is a summary (without proofs) of some of the most interesting properties of the distributional Laplace transform:

(i) Differentiation of the Laplace transform:

$$D^\alpha \mathcal{L}\{f(t)\}(s) = \mathcal{L}\{(-t)^\alpha f(t)\}(s), \quad s \in \Omega_f \tag{A.17}$$

(ii) The Laplace transform of a derivative:

$$\mathcal{L}\{D^\alpha f(t)\}(s) = s^\alpha \mathcal{L}\{f(t)\}(s), \quad s \in \Omega_f \tag{A.18}$$

(iii) The translation of a Laplace transform:

$$\mathcal{L}\{f(t)e^{-at}\}(s) = \mathcal{L}\{f(t)\}(s+a), \quad a > 0, \quad s+a \in \Omega_f \tag{A.19}$$

(iv) The Laplace transform of a translation:

$$\mathcal{L}\{H(t-t_0)f(t-t_0)\}(s) = e^{-st_0}\mathcal{L}\{f(t)\}(s), \quad s \in \Omega_f \tag{A.20}$$

(v) The Laplace transform under a change of scale:

$$\mathcal{L}\{f(at)\}(s) = \frac{1}{a}\mathcal{L}\{f(t)\}(\frac{s}{a}), \quad a > 0, \quad \frac{s}{a} \in \Omega_f \tag{A.21}$$

Furthermore, just as with the classical Laplace transform, an inversion formula can be obtained and the convolution of two elements in $\mathscr{L}'_a(I)$ can be defined in such a way that its distributional Laplace transform will be the product of the individual Laplace transform of the elements involved in the convolution.

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