CLOSED FORMS FOR 10 FAMILIES OF FINITE SUMS OF FRACTIONAL GENERALIZED FIBONACCI PRODUCTS

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Abstract. In this paper, we present closed forms for 10 families of finite sums where the denominator of the summand is a product of generalized Fibonacci numbers. In each of these 10 families, the product in the denominator of the summand is arbitrarily long.

1. Introduction

For all integers n, define the sequence $\{W_n\}$ by

$$W_n(a,b,p) = W_n = pW_{n-1} + W_{n-2},\ W_0 = a,\ W_1 = b, \tag{1.1}$$

in which $a \geq 0$, $b \geq 1$, and $p \geq 1$ are integers, and $(a,b) \neq (0,0)$. These restrictions on the parameters a, b, and p, ensure that $\{W_n\}$ is an integer sequence with $W_n \geq 1$ for $n \geq 1$. When $(a,b,p) = (0,1,1)$, $\{W_n\}$ becomes $\{F_n\}$, the sequence of Fibonacci numbers, so we refer to $\{W_n\}$ as a sequence of generalized Fibonacci numbers.

Define also

$$\overline{W}_n(a,b,p) = \overline{W}_n = W_{n-1} + W_{n+1},$$

which has the same relationship with $\{W_n\}$ as the Lucas sequence $\{L_n\}$ has with the Fibonacci sequence. With $\Delta = p^2 + 4$, it follows that

$$\overline{\overline{W}}_n = \Delta W_n. \tag{1.2}$$

We remark that identity (1.2) is required, for instance, if we take $W_n = L_n = \overline{F}_n$ in S_4 or S_5 (see Section 2).

Setting $(a,b) = (0,1)$, and allowing p to remain arbitrary, we write $\{W_n(p)\} = \{U_n\}$, and $\{\overline{W}_n(p)\} = \{V_n\}$. The sequence $\{U_n\}$ generalizes $\{F_n\}$, and $\{V_n\}$ generalizes $\{L_n\}$.

Let α and β denote the two distinct real roots of $x^2 - px - 1 = 0$. Set $A = b - a\beta$ and $B = b - a\alpha$. Then, the Binet forms for $\{W_n\}$ and $\{\overline{W}_n\}$ are, respectively,

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}, \tag{1.3}$$

$$\overline{W}_n = A\alpha^n + B\beta^n. \tag{1.4}$$

In this paper, we present closed forms for 10 families of finite reciprocal sums of products that involve the sequence $\{W_n\}$. Each of the 10 (families of finite reciprocal) sums that we define is parametrized by integers $j \geq 0$ and $k \geq 1$, so we describe our sums as "families". In each of the sums that we define, the number of factors in the denominator of the summand is arbitrarily large, and it is this feature that distinguishes the main results in the present paper from results on reciprocal summation that we have cited in the literature. In this regard, we refer the interested reader to [2], which is the only reference that presents closed forms for families of finite reciprocal sums of arbitrarily long products of generalized Fibonacci numbers.

The following is an example of the kinds of sums that arise from the main results in the present paper. For $n \geq 1$, we have

$$\sum_{i=1}^{n} \frac{F_{i+2j+1}}{F_i F_{i+1} \cdots F_{i+4j+2}} = \frac{1}{L_{2j+1}} \left(\frac{1}{F_1 \cdots F_{4j+2}} - \frac{1}{F_{n+1} \cdots F_{n+4j+2}} \right), \tag{1.5}$$

in which $j \geq 0$ is an integer. In (1.5), which is a special case of (3.1), the product in the denominator of the summand has $4j+3$ Fibonacci factors, and so can be arbitrarily long. For $j = 1$, (1.5) becomes

$$\sum_{i=1}^{n} \frac{1}{F_i F_{i+1} F_{i+2} F_{i+4} F_{i+5} F_{i+6}} = \frac{1}{4} \left(\frac{1}{240} - \frac{1}{F_{n+1} F_{n+2} \cdots F_{n+6}} \right).$$

In Section 2, we define the 10 sums that are the topic of this paper. In Section 3, we state our main results, which are the closed forms for the 10 sums defined in Section 2. In Section 4, we give a sample proof, and in Section 5, we give several special cases of our main results.

2. The Finite Sums

We now define the 10 sums whose closed forms we give in the next section. In each of these 10 sums, the upper limit of summation is a positive integer $n \geq 1$. Furthermore, each sum that we define is parametrized by integers $j \geq 0$ and $k \geq 1$. The first three sums are

$$\begin{aligned}
S_1(n,k,j) &= \sum_{i=1}^{n} \frac{(-1)^{(k+1)i} W_{k(i+2j+1)}}{W_{ki} W_{k(i+1)} \cdots W_{k(i+4j+2)}}, \\
S_2(n,k,j) &= \sum_{i=1}^{n} \frac{(-1)^{(k+1)i} W_{k(2i+2j+1)}}{W_{2ki} \cdots W_{2k(i+2j+1)}}, \\
S_3(n,k,j) &= \sum_{i=1}^{n} \frac{(-1)^{i} W_{k(i+2j+2)}}{W_{ki} W_{k(i+1)} \cdots W_{k(i+4j+4)}}.
\end{aligned}$$

The next three sums, which can be viewed as counterparts to S_1, S_2, and S_3 are

$$\begin{aligned}
S_4(n,k,j) &= \sum_{i=1}^{n} \frac{(-1)^{ki} \overline{W}_{k(i+2j+1)}}{W_{ki} W_{k(i+1)} \cdots W_{k(i+4j+2)}}, \\
S_5(n,k,j) &= \sum_{i=1}^{n} \frac{(-1)^{ki} \overline{W}_{k(2i+2j+1)}}{W_{2ki} \cdots W_{2k(i+2j+1)}}, \\
S_6(n,k,j) &= \sum_{i=1}^{n} \frac{\overline{W}_{k(i+2j+2)}}{W_{ki} W_{k(i+1)} \cdots W_{k(i+4j+4)}}.
\end{aligned}$$

Each of the next four sums that we define involves a run of squared terms in the denominator of the summand. Define

$$S_7(n,k,j) = \sum_{i=1}^{n} \frac{W_{k(i+j+1)}\overline{W}_{k(i+j+1)}}{W_{ki}^2 W_{k(i+1)}^2 \cdots W_{k(i+2j+2)}^2},$$
$$S_8(n,k,j) = \sum_{i=1}^{n} \frac{W_{k(2i+2j+1)}\overline{W}_{k(2i+2j+1)}}{W_{2ki}^2 \cdots W_{2k(i+2j+1)}^2}.$$

Finally, the two sums that follow involve the sequences $\{U_n\}$ and $\{V_n\}$.

$$S_9(n,k,j) = \sum_{i=1}^{n} \frac{(-1)^{ki} U_{k(2i+2j+1)}}{U_{ki}^2 \cdots U_{k(i+2j+1)}^2},$$
$$S_{10}(n,k,j) = \sum_{i=1}^{n} \frac{(-1)^{ki} U_{k(2i+2j+1)}}{V_{ki}^2 \cdots V_{k(i+2j+1)}^2}.$$

In the next section, we state our main results, which are the closed forms for the 10 sums defined above.

3. The Closed Forms

To succinctly present the formulas in this section, we employ some familiar notation. In all that follows, we take i as the dummy variable, so that, for instance, $[W_{ki}]_m^n$ means $W_{kn} - W_{km}$.

The four theorems in this section give the closed forms for the sums defined in Section 2. The first theorem gives the closed forms for S_1, S_2, and S_3.

Theorem 3.1. *Suppose $j \geq 0$ and $k \geq 1$ are integers. Then, the closed forms for S_1, S_2, and S_3 are*

$$S_1(n,k,j) = \frac{(-1)^k}{V_{k(2j+1)}} \left[\frac{(-1)^{(k+1)i}}{W_{k(i+1)} \cdots W_{k(i+4j+2)}} \right]_0^n, \tag{3.1}$$
$$S_2(n,k,j) = \frac{(-1)^k}{V_{k(2j+1)}} \left[\frac{(-1)^{(k+1)i}}{W_{2k(i+1)} \cdots W_{2k(i+2j+1)}} \right]_0^n, \tag{3.2}$$
$$S_3(n,k,j) = \frac{1}{V_{k(2j+2)}} \left[\frac{(-1)^{i}}{W_{k(i+1)} \cdots W_{k(i+4j+4)}} \right]_0^n. \tag{3.3}$$

Theorem 3.2 gives the closed forms for S_4, S_5, and S_6.

Theorem 3.2. *Suppose $j \geq 0$ and $k \geq 1$ are integers. Then, the closed forms for S_4, S_5, and S_6 are*

$$S_4(n,k,j) = \frac{(-1)^{k+1}}{U_{k(2j+1)}} \left[\frac{(-1)^{ki}}{W_{k(i+1)} \cdots W_{k(i+4j+2)}} \right]_0^n, \tag{3.4}$$
$$S_5(n,k,j) = \frac{(-1)^{k+1}}{U_{k(2j+1)}} \left[\frac{(-1)^{ki}}{W_{2k(i+1)} \cdots W_{2k(i+2j+1)}} \right]_0^n, \tag{3.5}$$
$$S_6(n,k,j) = \frac{-1}{U_{k(2j+2)}} \left[\frac{1}{W_{k(i+1)} \cdots W_{k(i+4j+4)}} \right]_0^n. \tag{3.6}$$

Our next theorem gives the closed forms for S_7 and S_8.

Theorem 3.3. *Suppose $j \geq 0$ and $k \geq 1$ are integers. Then, the closed forms for S_7 and S_8 are*

$$S_7(n,k,j) = \frac{-1}{U_{2k(j+1)}} \left[\frac{1}{W^2_{k(i+1)} \cdots W^2_{k(i+2j+2)}} \right]_0^n, \tag{3.7}$$

$$S_8(n,k,j) = \frac{-1}{U_{2k(2j+1)}} \left[\frac{1}{W^2_{2k(i+1)} \cdots W^2_{2k(i+2j+1)}} \right]_0^n. \tag{3.8}$$

For a sum similar in spirit to the two sums given in Theorem 3.3, see (6.3) in [2]. Our final theorem gives the closed forms for S_9 and S_{10}.

Theorem 3.4. *Suppose $j \geq 0$ and $k \geq 1$ are integers. Then, the closed forms for S_9 and S_{10} are*

$$S_9(n,k,j) = \frac{(-1)^{k+1}}{U_{k(2j+1)}} \left[\frac{(-1)^{ki}}{U^2_{k(i+1)} \cdots U^2_{k(i+2j+1)}} \right]_0^n, \tag{3.9}$$

$$S_{10}(n,k,j) = \frac{(-1)^{k+1}}{\Delta U_{k(2j+1)}} \left[\frac{(-1)^{ki}}{V^2_{k(i+1)} \cdots V^2_{k(i+2j+1)}} \right]_0^n. \tag{3.10}$$

We remark that, by virtue of the parameter k, (3.9) and (3.10) generalize the two results contained in the array (6.4) in [2].

4. A Sample Proof

We require formulas of (72)–(75) in [1], which in the notation of the present paper are

$$\begin{aligned} W_{n+k} + W_{n-k} &= W_n V_k, \quad k \text{ even}, \\ W_{n+k} - W_{n-k} &= \overline{W}_n U_k, \quad k \text{ even}, \\ W_{n+k} + W_{n-k} &= \overline{W}_n U_k, \quad k \text{ odd}, \\ W_{n+k} - W_{n-k} &= W_n V_k, \quad k \text{ odd}. \end{aligned} \tag{4.1}$$

What follows is a proof of (3.7), and this method of proof applies to each of (3.1)–(3.10).

Proof. For $n \geq 1$, denote the right side of (3.7) by $r(n,j,k)$. Then the difference $r(n+1,k,j) - r(n,k,j)$ is

$$\begin{aligned} &\frac{W^2_{k(n+2j+3)} - W^2_{k(n+1)}}{U_{2k(j+1)} W^2_{k(n+1)} \cdots W^2_{k(n+2j+3)}} \\ &= \frac{\left(W_{k(n+2j+3)} + W_{k(n+1)}\right)\left(W_{k(n+2j+3)} - W_{k(n+1)}\right)}{U_{2k(j+1)} W^2_{k(n+1)} \cdots W^2_{k(n+2j+3)}} \\ &= \frac{U_{k(j+1)} V_{k(j+1)} W_{k(n+j+2)} \overline{W}_{k(n+j+2)}}{U_{2k(j+1)} W^2_{k(n+1)} \cdots W^2_{k(n+2j+3)}}, \quad \text{from the results in (4.1)} \\ &= S_7(n+1,k,j) - S_7(n,k,j), \end{aligned} \tag{4.2}$$

since $U_{2n} = U_n V_n$ for all n.

Next, after performing manipulations similar to those in (4.2), we see that

$$\begin{aligned} r(1,k,j) &= \frac{\left(W_{k(2j+3)} + W_k\right)\left(W_{k(2j+3)} - W_k\right)}{U_{2k(j+1)} W_k^2 \cdots W_{k(2j+3)}^2} \\ &= \frac{U_{k(j+1)} V_{k(j+1)} W_{k(j+2)} \overline{W}_{k(j+2)}}{U_{2k(j+1)} W_k^2 \cdots W_{k(2j+3)}^2} \\ &= S_7(1,k,j). \end{aligned} \tag{4.3}$$

This, together with $r(n+1,k,j) - r(n,k,j) = S_7(n+1,k,j) - S_7(n,k,j)$, proves (3.7). □

We remark that the proofs of (3.9) and (3.10) proceed along similar lines. In this regard however, the identities in (4.1) are of no use. Instead, one requires the identities

$$\begin{aligned} U_{k(n+2j+1)}^2 + (-1)^{k+1} U_{kn}^2 &= U_{k(2j+1)} U_{k(2n+2j+1)}, \\ V_{k(n+2j+1)}^2 + (-1)^{k+1} V_{kn}^2 &= \Delta U_{k(2j+1)} U_{k(2n+2j+1)}, \end{aligned} \tag{4.4}$$

which are valid for all integers j, k, and n. The identities in (4.4) can be proved with the use of the Binet forms, a task that we leave to the interested reader.

5. Some Special Cases of Our Main Results

In this section, for the purpose of illustration, we give a selection of special cases of our main results. We begin with (3.4) and (3.5), where we set $W_n = F_n$ and take $(j,k) = (0,1)$. Then (3.4) and (3.5) become, respectively,

$$\sum_{i=1}^{n} \frac{(-1)^i L_{i+1}}{F_i F_{i+1} F_{i+2}} = \frac{(-1)^n}{F_{n+1} F_{n+2}} - 1,$$

$$\sum_{i=1}^{n} \frac{(-1)^i L_{2i+1}}{F_{2i} F_{2i+2}} = \frac{(-1)^n}{F_{2n+2}} - 1.$$

Staying with (3.4) and (3.5), setting $W_n = L_n$, and taking $(j,k) = (1,1)$, we see that (3.4) and (3.5) become, respectively,

$$\sum_{i=1}^{n} \frac{(-1)^i F_{i+3}}{L_i L_{i+1} L_{i+2} \cdots L_{i+6}} = \frac{1}{10}\left(\frac{(-1)^n}{L_{n+1} L_{n+2} \cdots L_{n+6}} - \frac{1}{16632}\right),$$

$$\sum_{i=1}^{n} \frac{(-1)^i F_{2i+3}}{L_{2i} L_{2i+2} L_{2i+4} L_{2i+6}} = \frac{1}{10}\left(\frac{(-1)^n}{L_{2n+2} L_{2n+4} L_{2n+6}} - \frac{1}{378}\right).$$

Next, consider (3.7) and (3.8), set $W_n = F_n$, and take $(j,k) = (1,1)$. Then (3.7) and (3.8) become, respectively,

$$\sum_{i=1}^{n} \frac{L_{i+2}}{F_i^2 F_{i+1}^2 F_{i+2} F_{i+3}^2 F_{i+4}^2} = \frac{1}{3}\left(\frac{1}{36} - \frac{1}{F_{n+1}^2 F_{n+2}^2 F_{n+3}^2 F_{n+4}^2}\right),$$

$$\sum_{i=1}^{n} \frac{F_{4i+6}}{F_{2i}^2 F_{2i+2}^2 F_{2i+4}^2 F_{2i+6}^2} = \frac{1}{8}\left(\frac{1}{576} - \frac{1}{F_{2n+2}^2 F_{2n+4}^2 F_{2n+6}^2}\right).$$

Finally, in (3.9) and (3.10), set $U_n = F_n$ so that $V_n = L_n$. Then, with $(j,k) = (1,3)$, (3.9) and (3.10) become, respectively,

$$\sum_{i=1}^{n} \frac{(-1)^i F_{6i+9}}{F_{3i}^2 F_{3i+3}^2 F_{3i+6}^2 F_{3i+9}^2} = \frac{1}{34}\left(\frac{(-1)^n}{F_{3n+3}^2 F_{3n+6}^2 F_{3n+9}^2} - \frac{1}{295936}\right),$$

$$\sum_{i=1}^{n} \frac{(-1)^i F_{6i+9}}{L_{3i}^2 L_{3i+3}^2 L_{3i+6}^2 L_{3i+9}^2} = \frac{1}{170}\left(\frac{(-1)^n}{L_{3n+3}^2 L_{3n+6}^2 L_{3n+9}^2} - \frac{1}{29942784}\right).$$

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