CLOSED FORMULAS FOR FINITE SUMS OF WEIGHTED FRACTIONAL GENERALIZED FIBONACCI PRODUCTS

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Abstract. In this paper, we present closed formulas for finite sums of fractions involving weighted products of generalized Fibonacci numbers. There are two significant features that characterize the main results. First, the product in the denominator of each summand can be arbitrarily long. Second, in each summand that we consider, there is a so-called weight term. The weight term occurs either in the numerator or the denominator of the summand.

1. Introduction

We begin by defining the sequences that we employ throughout this paper. Since the topic is reciprocal summation, throughout our presentation we place restrictions on the parameters to avoid zero denominators.

Let $a \geq 0$, $b \geq 0$, and $p \geq 1$ be integers with $(a,b) \neq (0,0)$. Define the integer sequence $\{W_n\}$ by

$$W_n(a,b,p) = W_n = pW_{n-1} + W_{n-2},\ W_0 = a,\ W_1 = b. \tag{1.1}$$

Our restrictions on a, b, and p ensure that $\{W_n\}$ is an integer sequence in which $W_n \geq 0$ for $n \geq 0$, thus maintaining the analogy with the Fibonacci sequence.

Define also

$$\overline{W}_n(a,b,p) = \overline{W}_n = W_{n-1} + W_{n+1},$$

which bears the same relationship with $\{W_n\}$ as the Lucas sequence bears with the Fibonacci sequence. With $\Delta = p^2 + 4$, it follows that

$$\overline{\overline{W}}_n = \Delta W_n. \tag{1.2}$$

We remark that identity (1.2) is required, for instance, if we take $H_n = L_n = \overline{F}_n$ in S_4 or S_5 (see Section 2).

In the itemized list that follows, we set the notation for the special cases of $\{W_n\}$ that we consider throughout this paper.

- For $(a,b,p) = (0,1,1)$, we have $\{W_n\} = \{F_n\}$, and $\{\overline{W}_n\} = \{L_n\}$, which are the Fibonacci and Lucas sequences, respectively;
- For $(a,b,p) = (0,1,2)$, we have $\{W_n\} = \{P_n\}$, and $\{\overline{W}_n\} = \{Q_n\}$, which are the Pell and Pell-Lucas sequences, respectively;
- For $p = 1$, we write $\{W_n(a,b,1)\} = \{H_n(a,b)\} = \{H_n\}$;
- For $(a,b) = (0,1)$, we write $\{W_n(p)\} = \{U_n\}$, and $\{\overline{W}_n(p)\} = \{V_n\}$.

The sequences $\{H_n\}$ and $\{\overline{H}_n\}$ satisfy the same recurrence as $\{F_n\}$, and are generalizations of $\{F_n\}$ and $\{L_n\}$, respectively. The sequence $\{U_n\}$ generalizes both $\{F_n\}$ and $\{P_n\}$, and $\{V_n\}$ generalizes both $\{L_n\}$ and $\{Q_n\}$.

Let α and β denote the two distinct real roots of $x^2 - px - 1 = 0$. Set $A = b - a\beta$ and $B = b - a\alpha$. Then the Binet forms for $\{W_n\}$ and $\{\overline{W}_n\}$ are, respectively,

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}, \tag{1.3}$$

$$\overline{W}_n = A\alpha^n + B\beta^n. \tag{1.4}$$

The Binet forms of each sequence defined in the list above can be obtained from (1.3) or (1.4).

Readers of this journal who are interested in summations that involve reciprocals of Fibonacci numbers will be familiar with the two seminal papers of Brousseau, [1] and [2]. In these two papers, Brousseau's focus is on summing the reciprocals of certain products of Fibonacci numbers. Although Brousseau's focus is on infinite summation, his methods (for the most part) also yield the corresponding finite sums.

In [5], [6], [7], [8], and [9], we continue the theme of reciprocal summation, and consider finite reciprocal sums of products that involve generalized Fibonacci numbers. Indeed, we give closed forms, in terms of rational numbers, for these sums. In Section 2 of [5], while not attempting to give a detailed history of the broad topic of reciprocal summation involving (generalized) Fibonacci numbers, we cite several references that propelled us along the path in question.

In the present paper, the denominator of the summand of each finite sum that we consider contains a product of generalized Fibonacci numbers. However, there are two significant points of difference from the results in the papers referenced in the two paragraphs immediately above. Regarding our main results (Section 3) in the present paper,

- the product in the denominator of each summand can be arbitrarily long;
- there is a so-called *weight* term located in the numerator or the denominator of the summand.

To motivate our presentation, we now give examples of sums that illustrate these two points. These examples are instances of our main results, which we give in Section 3. For integers $n \geq 2$ and $j \geq 1$, we have

$$\sum_{i=1}^{n-1} \frac{L_{i-1}L_{j+1}^i}{F_i \cdots F_{i+j}} = \frac{1}{F_j}\left(\frac{L_{j+1}^n}{F_n \cdots F_{n+j-1}} - \frac{L_{j+1}}{F_1 \cdots F_j}\right), \tag{1.5}$$

$$\sum_{i=1}^{n-1} \frac{L_{i-1}7^i}{F_iF_{i+1}F_{i+2}F_{i+3}} = \frac{7}{2}\left(\frac{7^{n-1}}{F_nF_{n+1}F_{n+2}} - \frac{1}{2}\right). \tag{1.6}$$

In (1.5), which we obtain from (3.7) by putting $H_n = F_n$ and $k = 1$, the product in the denominator of the summand has $j+1$ Fibonacci factors, and so can be arbitrarily long. The weight term is L_{j+1}^i. For $j = 3$, (1.5) reduces to (1.6), and the weight term is 7^i.

To succinctly present our next example, which contains a run of squared terms in the denominator of the summand, we employ some familiar notations. Throughout this paper, we take i as the dummy variable, so that, for instance, $[F_{ki}]_m^n$ means $F_{kn} - F_{km}$.

For integers $n \geq 2$ and $j \geq 1$, we have

$$\sum_{i=1}^{n-1} \frac{P_{4i+2j}}{P_{2i}^2 \cdots P_{2(i+j)}^2} = -\frac{1}{P_{2j}}\left[\frac{1}{P_{2i}^2 \cdots P_{2(i+j-1)}^2}\right]_1^n, \tag{1.7}$$

$$\sum_{i=1}^{n-1} \frac{Q_{2i+2}}{P_{2i}^2P_{2i+2}P_{2i+4}^2} = \frac{1}{12}\left(\frac{1}{576} - \frac{1}{P_{2n}^2P_{2n+2}^2}\right). \tag{1.8}$$

We obtain (1.7) from (6.3) by setting $W_n = P_n$ and $k = 2$. The denominator of the summand in (1.7) has $j+1$ squared factors, and thus can be arbitrarily long. For $j = 2$, (1.7) reduces to (1.8). The weight term in (1.7) and (1.8) is unity. We include several more instances of our main results in Section 5 and Section 6.

When conducting the research for this paper, we began by searching for results that hold for the sequences $\{H_n\}$ and $\{\overline{H}_n\}$. From these results, we then determined those that also hold for the more general sequences $\{W_n\}$ and $\{\overline{W}_n\}$. Accordingly, we present each of our main results in terms of the most general sequence(s) for which it is valid.

In Section 2, we define the 14 finite sums that are the focus of this paper. In Section 3, we state our main results, which are the closed forms for the 14 sums defined in Section 2. In Section 4, we give a sample proof, and in Section 5, we give several special cases of our main results. We conclude with Section 6, where we present the closed forms for a limited number of sums in which the denominator of the summand contains squared factors.

2. The Finite Sums

In each of the finite sums that we define below, the upper limit of summation is a positive integer $n \geq 2$. Furthermore, $j \geq 1$ and $k \geq 1$ are assumed to be integers. We now define the 14 finite sums whose closed forms we give in the next section. The first three finite sums involve sequences generated by the recurrence given in (1.1). These finite sums are

$$S_1(n,j,k) = \sum_{i=1}^{n-1} \frac{W_{ki+1}U_{jk-1}^i}{W_{ki}\cdots W_{k(i+j)}}, \quad jk \neq 1,$$
$$S_2(n,j,k) = \sum_{i=1}^{n-1} \frac{W_{ki-1}U_{jk+1}^i}{W_{ki}\cdots W_{k(i+j)}}, \quad jk \neq -1,$$
$$S_3(n,j,k) = \sum_{i=1}^{n-1} \frac{W_{k(i-j)}V_{jk}^i}{W_{ki}\cdots W_{k(i+j)}}.$$

Next, we define four finite sums that involve sequences generated by the recurrence given in (1.1) in which $p = 1$. These finite sums are

$$S_4(n,j,k) = \sum_{i=1}^{n-1} \frac{\overline{H}_{ki-1}L_{jk+1}^i}{H_{ki}\cdots H_{k(i+j)}},$$
$$S_5(n,j,k) = \sum_{i=1}^{n-1} \frac{(-1)^i\overline{H}_{ki+1}L_{jk-1}^i}{H_{ki}\cdots H_{k(i+j)}},$$
$$S_6(n,j,k) = \sum_{i=1}^{n-1} \frac{H_{ki-2}F_{jk+2}^i}{H_{ki}\cdots H_{k(i+j)}}, \quad jk \neq -2,$$
$$S_7(n,j,k) = \sum_{i=1}^{n-1} \frac{(-1)^iH_{ki+2}F_{jk-2}^i}{H_{ki}\cdots H_{k(i+j)}}, \quad jk \neq 2.$$

In each of the next two groups of finite sums that we define, the weight term is located in the denominator of the summand. The three finite sums that follow involve sequences generated by the recurrence given in (1.1). These finite sums are

$$S_8(n,j,k) = \sum_{i=1}^{n-1} \frac{(-1)^{jki} W_{k(i+j)-1}}{U_{jk-1}^i W_{ki} \cdots W_{k(i+j)}}, \quad jk \neq 1,$$
$$S_9(n,j,k) = \sum_{i=1}^{n-1} \frac{(-1)^{jki} W_{k(i+j)+1}}{U_{jk+1}^i W_{ki} \cdots W_{k(i+j)}}, \quad jk \neq -1,$$
$$S_{10}(n,j,k) = \sum_{i=1}^{n-1} \frac{(-1)^{jki} W_{k(i+2j)}}{V_{jk}^i W_{ki} \cdots W_{k(i+j)}}.$$

Finally, the four finite sums that follow involve sequences generated by the recurrence given in (1.1) in which $p = 1$.

$$S_{11}(n,j,k) = \sum_{i=1}^{n-1} \frac{(-1)^{(jk+1)i} \overline{H}_{k(i+j)-1}}{L_{jk-1}^i H_{ki} \cdots H_{k(i+j)}},$$
$$S_{12}(n,j,k) = \sum_{i=1}^{n-1} \frac{(-1)^{jki} \overline{H}_{k(i+j)+1}}{L_{jk+1}^i H_{ki} \cdots H_{k(i+j)}},$$
$$S_{13}(n,j,k) = \sum_{i=1}^{n-1} \frac{(-1)^{(jk+1)i} H_{k(i+j)-2}}{F_{jk-2}^i H_{ki} \cdots H_{k(i+j)}}, \quad jk \neq 2,$$
$$S_{14}(n,j,k) = \sum_{i=1}^{n-1} \frac{(-1)^{jki} H_{k(i+j)+2}}{F_{jk+2}^i H_{ki} \cdots H_{k(i+j)}}, \quad jk \neq -2.$$

In the next section, we state our main results, which are the closed forms for the 14 sums defined above.

3. The Closed Forms

We begin this section with Lemma 3.1, and follow this with Theorem 3.2, which gives the closed forms for S_1, S_2, and S_3.

Lemma 3.1. *Let n, j, and k be integers. Then*

$$W_{k(n+j)} - U_{jk-1} W_{kn} = U_{jk} W_{kn+1}, \tag{3.1}$$
$$W_{k(n+j)} - U_{jk+1} W_{kn} = U_{jk} W_{kn-1}, \tag{3.2}$$
$$W_{k(n+j)} - V_{jk} W_{kn} = (-1)^{jk+1} W_{k(n-j)}. \tag{3.3}$$

For a method of proof that applies to each result in Lemma 3.1, see Section 4.

Theorem 3.2. *Suppose $j \geq 1$ and $k \geq 1$ are integers. Then, with the constraints on j and k in the definitions of S_1, S_2, and S_3, we have*

$$S_1(n,j,k) = -\frac{1}{U_{jk}} \left[\frac{U_{jk-1}^i}{W_{ki} \cdots W_{k(i+j-1)}} \right]_1^n, \tag{3.4}$$

$$S_2(n,j,k) = -\frac{1}{U_{jk}} \left[\frac{U_{jk+1}^i}{W_{ki} \cdots W_{k(i+j-1)}} \right]_1^n, \tag{3.5}$$

$$S_3(n,j,k) = (-1)^{jk} \left[\frac{V_{jk}^i}{W_{ki} \cdots W_{k(i+j-1)}} \right]_1^n. \tag{3.6}$$

Lemma 3.1 contains key identities that we require for the proof of Theorem 3.2. More precisely, we require (3.1), (3.2), and (3.3) for the proofs of (3.4), (3.5), and (3.6), respectively.

Before proceeding, we make some remarks regarding identities (3.1), (3.2), and (3.3). Each is a so-called *Product Difference Fibonacci Identity* (PDFI) of order 2, a description introduced by Fairgrieve and Gould [3]. To see the motivation for this terminology, write (3.1) as

$$U_{jk}W_{kn+1} - (-U_{jk-1}W_{kn}) = W_{k(n+j)}.$$

Here we have a difference of products, involving terms from generalized Fibonacci sequences, that yields a succinct right side. Perhaps the most celebrated PDFI of order 2 is Simson's identity

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n.$$

For a comprehensive commentary on PDFIs, we refer the interested reader to [4].

We now state Lemma 3.3, which contains (in order) the four PDFIs that we require for the proof of the four results in Theorem 3.4.

Lemma 3.3. *Let n, j, and k be integers. Then*

$$\begin{aligned} H_{k(n+j)} - L_{jk+1}H_{kn} &= -F_{jk}\overline{H}_{kn-1}, \\ H_{k(n+j)} + L_{jk-1}H_{kn} &= F_{jk}\overline{H}_{kn+1}, \\ H_{k(n+j)} - F_{jk+2}H_{kn} &= -F_{jk}H_{kn-2}, \\ H_{k(n+j)} + F_{jk-2}H_{kn} &= F_{jk}H_{kn+2}. \end{aligned}$$

Theorem 3.4. *Suppose $j \geq 1$ and $k \geq 1$ are integers. Then, with the constraints on j and k in the definitions of S_4, S_5, S_6, and S_7, we have*

$$S_4(n,j,k) = \frac{1}{F_{jk}} \left[\frac{L_{jk+1}^i}{H_{ki} \cdots H_{k(i+j-1)}} \right]_1^n, \tag{3.7}$$

$$S_5(n,j,k) = \frac{1}{F_{jk}} \left[\frac{(-1)^{i-1}L_{jk-1}^i}{H_{ki} \cdots H_{k(i+j-1)}} \right]_1^n, \tag{3.8}$$

$$S_6(n,j,k) = \frac{1}{F_{jk}} \left[\frac{F_{jk+2}^i}{H_{ki} \cdots H_{k(i+j-1)}} \right]_1^n, \tag{3.9}$$

$$S_7(n,j,k) = \frac{1}{F_{jk}} \left[\frac{(-1)^{i-1}F_{jk-2}^i}{H_{ki} \cdots H_{k(i+j-1)}} \right]_1^n. \tag{3.10}$$

Next, we state Lemma 3.5, which contains the PDFIs that we require for the proof of the results in Theorem 3.6.

Lemma 3.5. *Let n, j, and k be integers. Then*

$$U_{jk-1}W_{k(n+j)} + (-1)^{jk+1}W_{kn} = U_{jk}W_{k(n+j)-1},$$
$$U_{jk+1}W_{k(n+j)} + (-1)^{jk+1}W_{kn} = U_{jk}W_{k(n+j)+1},$$
$$V_{jk}W_{k(n+j)} + (-1)^{jk+1}W_{kn} = W_{k(n+2j)}.$$

Theorem 3.6. *Suppose $j \geq 1$ and $k \geq 1$ are integers. Then, with the constraints on j and k in the definitions of S_8, S_9, and S_{10}, we have*

$$S_8(n,j,k) = \frac{1}{U_{jk}}\left[\frac{(-1)^{jki+1}}{U_{jk-1}^{i-1}W_{ki}\cdots W_{k(i+j-1)}}\right]_1^n, \tag{3.11}$$

$$S_9(n,j,k) = \frac{1}{U_{jk}}\left[\frac{(-1)^{jki+1}}{U_{jk+1}^{i-1}W_{ki}\cdots W_{k(i+j-1)}}\right]_1^n, \tag{3.12}$$

$$S_{10}(n,j,k) = \left[\frac{(-1)^{jki+1}}{V_{jk}^{i-1}W_{ki}\cdots W_{k(i+j-1)}}\right]_1^n. \tag{3.13}$$

Finally, for this section, we state Lemma 3.7, which contains the PDFIs that we require for the proof of the results in Theorem 3.8.

Lemma 3.7. *Let n, j, and k be integers. Then*

$$L_{jk-1}H_{k(n+j)} + (-1)^{jk}H_{kn} = F_{jk}\overline{H}_{k(n+j)-1},$$
$$L_{jk+1}H_{k(n+j)} + (-1)^{jk+1}H_{kn} = F_{jk}\overline{H}_{k(n+j)+1},$$
$$F_{jk-2}H_{k(n+j)} + (-1)^{jk}H_{kn} = F_{jk}H_{k(n+j)-2},$$
$$F_{jk+2}H_{k(n+j)} + (-1)^{jk+1}H_{kn} = F_{jk}H_{k(n+j)+2}.$$

Theorem 3.8. *Suppose $j \geq 1$ and $k \geq 1$ are integers. Then, with the constraints on j and k in the definitions of S_{11}, S_{12}, S_{13}, and S_{14}, we have*

$$S_{11}(n,j,k) = \frac{1}{F_{jk}}\left[\frac{(-1)^{(jk+1)i+1}}{L_{jk-1}^{i-1}H_{ki}\cdots H_{k(i+j-1)}}\right]_1^n, \tag{3.14}$$

$$S_{12}(n,j,k) = \frac{1}{F_{jk}}\left[\frac{(-1)^{jki+1}}{L_{jk+1}^{i-1}H_{ki}\cdots H_{k(i+j-1)}}\right]_1^n, \tag{3.15}$$

$$S_{13}(n,j,k) = \frac{1}{F_{jk}}\left[\frac{(-1)^{(jk+1)i+1}}{F_{jk-2}^{i-1}H_{ki}\cdots H_{k(i+j-1)}}\right]_1^n, \tag{3.16}$$

$$S_{14}(n,j,k) = \frac{1}{F_{jk}}\left[\frac{(-1)^{jki+1}}{F_{jk+2}^{i-1}H_{ki}\cdots H_{k(i+j-1)}}\right]_1^n. \tag{3.17}$$

4. A Sample Proof

To illustrate a method of proof that applies to each of (3.4)–(3.17), we prove (3.12) in which $n \geq 2$, $j \geq 1$, $k \geq 1$, and $jk \neq -1$. We begin by proving the PDFI that we require for the proof of (3.12).

Proof. In Lemma 3.5, the relevant PDFI that we require for the proof of (3.12) is

$$U_{jk+1}W_{k(n+j)} + (-1)^{jk+1}W_{kn} = U_{jk}W_{k(n+j)+1}. \tag{4.1}$$

Transposing the product on the right side of (4.1) to the left side, substituting the Binet forms throughout, then expanding and factoring, we obtain

$$(1+\alpha\beta)(\alpha\beta)^{jk}W_{kn}. \tag{4.2}$$

Since $\alpha\beta = -1$, (4.1) follows from (4.2).

Denote the right side of (3.12) by $r(n,j,k)$. Then after some elementary algebra, we see that

$$\begin{aligned} r(n+1,j,k) - r(n,j,k) &= \frac{(-1)^{jkn}\left(U_{jk+1}W_{k(n+j)} + (-1)^{jk+1}W_{kn}\right)}{U_{jk}U_{jk+1}^{n}W_{kn}\cdots W_{k(n+j)}} \\ &= \frac{(-1)^{jkn}W_{k(n+j)+1}}{U_{jk+1}^{n}W_{kn}\cdots W_{k(n+j)}}, \text{by } (4.1) \\ &= S_9(n+1,j,k) - S_9(n,j,k). \end{aligned} \tag{4.3}$$

Next, after performing manipulations similar to those immediately above, we see that

$$\begin{aligned} r(2,j,k) &= \frac{1}{U_{jk}}\left(\frac{(-1)^{jk}U_{jk+1}W_{k(1+j)} - W_k}{U_{jk+1}W_k\cdots W_{k(1+j)}}\right) \\ &= \frac{(-1)^{jk}}{U_{jk}}\left(\frac{U_{jk+1}W_{k(1+j)} + (-1)^{jk+1}W_k}{U_{jk+1}W_k\cdots W_{k(1+j)}}\right) \\ &= \frac{(-1)^{jk}W_{k(1+j)+1}}{U_{jk+1}W_k\cdots W_{k(1+j)}}, \text{by } (4.1) \\ &= S_9(2,j,k). \end{aligned} \tag{4.4}$$

Together, (4.3) and (4.4) prove (3.12). □

5. Some Special Cases of Our Main Results

In this section, we take a selection of our main results from Section 3, and write down special cases of them. In (3.4) and (3.6), let $k=1$, and take $W_n = P_n$. Then (3.4) and (3.6) become, respectively,

$$\sum_{i=1}^{n-1}\frac{P_{i+1}P_{j-1}^{i}}{P_i\cdots P_{i+j}} = -\frac{1}{P_j}\left[\frac{P_{j-1}^{i}}{P_i\cdots P_{i+j-1}}\right]_1^n, \tag{5.1}$$

$$\sum_{i=1}^{n-1}\frac{P_{i-j}Q_{j}^{i}}{P_i\cdots P_{i+j}} = (-1)^j\left[\frac{Q_{j}^{i}}{P_i\cdots P_{i+j-1}}\right]_1^n. \tag{5.2}$$

With $j=2$, (5.1) and (5.2) yield, respectively,

$$\sum_{i=1}^{n-1}\frac{1}{P_iP_{i+2}} = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{P_nP_{n+1}}\right),$$

$$\sum_{i=1}^{n-1}\frac{P_{i-2}6^{i}}{P_iP_{i+1}P_{i+2}} = \frac{6^n}{P_nP_{n+1}} - 3.$$

Next, consider (3.7) and (3.8) with $H_n = F_n$, and take $k = 2$. Then (3.7) and (3.8) become, respectively,

$$\sum_{i=1}^{n-1} \frac{L_{2i-1}L_{2j+1}^i}{F_{2i}\cdots F_{2(i+j)}} = \frac{1}{F_{2j}}\left[\frac{L_{2j+1}^i}{F_{2i}\cdots F_{2(i+j-1)}}\right]_1^n, \tag{5.3}$$

$$\sum_{i=1}^{n-1} \frac{(-1)^i L_{2i+1}L_{2j-1}^i}{F_{2i}\cdots F_{2(i+j)}} = \frac{1}{F_{2j}}\left[\frac{(-1)^{i-1}L_{2j-1}^i}{F_{2i}\cdots F_{2(i+j-1)}}\right]_1^n. \tag{5.4}$$

With $j = 1$, (5.3) and (5.4) yield, respectively,

$$\sum_{i=1}^{n-1} \frac{L_{2i-1}2^{2i}}{F_{2i}F_{2(i+1)}} = 4\left(\frac{2^{2n-2}}{F_{2n}} - 1\right),$$

$$\sum_{i=1}^{n-1} \frac{(-1)^i L_{2i+1}}{F_{2i}F_{2(i+1)}} = \frac{(-1)^{n-1}}{F_{2n}} - 1.$$

Finally for this section, consider (3.13), and take $H_n = F_n$. With $k = 3$, (3.13) then becomes

$$\sum_{i=1}^{n-1} \frac{F_{3(i+2j)}}{L_{3j}^i F_{3i}\cdots F_{3(i+j)}} = \left[\frac{-1}{L_{3j}^{i-1}F_{3i}\cdots F_{3(i+j-1)}}\right]_1^n, \; j \text{ even}, \tag{5.5}$$

$$\sum_{i=1}^{n-1} \frac{(-1)^i F_{3(i+2j)}}{L_{3j}^i F_{3i}\cdots F_{3(i+j)}} = \left[\frac{(-1)^{i+1}}{L_{3j}^{i-1}F_{3i}\cdots F_{3(i+j-1)}}\right]_1^n, \; j \text{ odd}. \tag{5.6}$$

With $j = 2$ in (5.5), and with $j = 3$ in (5.6), we have, respectively,

$$\sum_{i=1}^{n-1} \frac{F_{3(i+4)}}{18^i F_{3i}F_{3(i+1)}F_{3(i+2)}} = \frac{1}{16} - \frac{1}{18^{n-1}F_{3n}F_{3(n+1)}},$$

$$\sum_{i=1}^{n-1} \frac{(-1)^i F_{3(i+6)}}{76^i F_{3i}F_{3(i+1)}F_{3(i+2)}F_{3(i+3)}} = \frac{(-1)^{n-1}}{76^{n-1}F_{3n}F_{3(n+1)}F_{3(n+2)}} - \frac{1}{544}.$$

6. Where There Are Squared Factors in the Denominator of the Summand

Results analogous to those presented above, where the denominator of the summand contains squared factors, seem to be rare. We have discovered nine such results, which we present below. The interested reader may wish to supply the proofs for some (or all) of these results. For each of the six results in (6.1) and (6.2), the denominator of the summand is bounded in length. However, for each of the three results in (6.3) and (6.4), the summand contains arbitrarily long products in its denominator.

The three results in (6.1) hold for the special sequence $\{H_n\}$.

$$\begin{aligned}
\sum_{i=1}^{n-1} \frac{H_{i-1}H_{i+2}}{H_i^2 H_{i+1}^2} &= -\left[\frac{1}{H_i^2}\right]_1^n,\\
\sum_{i=1}^{n-1} \frac{2^{2i}H_{i-2}\overline{H}_{i+1}}{H_i^2 H_{i+1}^2} &= \left[\frac{2^{2i}}{H_i^2}\right]_1^n,\\
\sum_{i=1}^{n-1} \frac{H_{i+3}\overline{H}_i}{2^{2i}H_i^2 H_{i+1}^2} &= -\left[\frac{1}{2^{2i-2}H_i^2}\right]_1^n.
\end{aligned} \tag{6.1}$$

The three results in (6.2) also hold for the special sequence $\{H_n\}$.

$$\begin{aligned}\sum_{i=1}^{n-1}\frac{1}{H_i^2H_{i+1}H_{i+2}H_{i+3}^2}&=-\frac{1}{4}\left[\frac{1}{H_i^2H_{i+1}^2H_{i+2}^2}\right]_1^n,\\ \sum_{i=1}^{n-1}\frac{3^{2i}H_{i-1}\overline{H}_{i+1}}{H_i^2H_{i+1}^2H_{i+2}^2H_{i+3}^2}&=-\frac{1}{4}\left[\frac{3^{2i}}{H_i^2H_{i+1}^2H_{i+2}^2}\right]_1^n,\\ \sum_{i=1}^{n-1}\frac{H_{i+4}\overline{H}_{i+2}}{3^{2i}H_i^2H_{i+1}^2H_{i+2}^2H_{i+3}^2}&=-\frac{1}{4}\left[\frac{1}{3^{2i-2}H_i^2H_{i+1}^2H_{i+2}^2}\right]_1^n.\end{aligned}\tag{6.2}$$

The setting for the next sum is the more general sequence $\{W_n\}$, where jk is even.

$$\sum_{i=1}^{n-1}\frac{W_{k(2i+j)/2}\overline{W}_{k(2i+j)/2}}{W_{ki}^2\cdots W_{k(i+j)}^2}=-\frac{1}{U_{jk}}\left[\frac{1}{W_{ki}^2\cdots W_{k(i+j-1)}^2}\right]_1^n,\ jk\text{ even.}\tag{6.3}$$

In (1.1), take $(a,b,p)=(0,1,2)$, and let $(j,k)=(2,1)$. Then (6.3) becomes

$$\sum_{i=1}^{n-1}\frac{Q_{i+1}}{P_i^2P_{i+1}P_{i+2}^2}=\frac{1}{2}\left(\frac{1}{4}-\frac{1}{P_n^2P_{n+1}^2}\right).$$

The two sums that follow are alternating, and contain only the parameter j. The most general setting for these sums are the sequences $\{U_n\}$ and $\{V_n\}$.

$$\begin{aligned}\sum_{i=1}^{n-1}\frac{(-1)^iU_{2i+2j-1}}{U_i^2\cdots U_{i+2j-1}^2}&=\frac{1}{U_{2j-1}}\left[\frac{(-1)^{i-1}}{U_i^2\cdots U_{i+2j-2}^2}\right]_1^n,\\ \sum_{i=1}^{n-1}\frac{(-1)^iU_{2i+2j-1}}{V_i^2\cdots V_{i+2j-1}^2}&=\frac{1}{\Delta U_{2j-1}}\left[\frac{(-1)^{i-1}}{V_i^2\cdots V_{i+2j-2}^2}\right]_1^n.\end{aligned}\tag{6.4}$$

Staying with the Pell and Pell-Lucas sequences, and taking $j=2$, we see that the two sums in (6.4) become, respectively,

$$\begin{aligned}\sum_{i=1}^{n-1}\frac{(-1)^iP_{2i+3}}{P_i^2P_{i+1}^2P_{i+2}^2P_{i+3}^2}&=\frac{1}{5}\left(\frac{(-1)^{n-1}}{P_n^2P_{n+1}^2P_{n+2}^2}-\frac{1}{100}\right),\\ \sum_{i=1}^{n-1}\frac{(-1)^iP_{2i+3}}{Q_i^2Q_{i+1}^2Q_{i+2}^2Q_{i+3}^2}&=\frac{1}{40}\left(\frac{(-1)^{n-1}}{Q_n^2Q_{n+1}^2Q_{n+2}^2}-\frac{1}{28224}\right).\end{aligned}$$

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