SUMS OF RECIPROCALS OF WEIGHTED PRODUCTS OF THE SINE AND COSINE FUNCTIONS

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Abstract. In this paper, we define 12 families of finite sums that involve the sine/cosine functions. Four of these families are parametrized by j, and the remaining eight families are parametrized by j and k. In each of the aforementioned 12 families, the denominator of the summand contains a product of sine or cosine functions, and the length of this product is governed by the parameter j. As such, the length of the product in question can be made as large as we please.

In each of the 12 families of finite sums that we consider, there is a so-called *weight term* in the summand. For instance, in S_4 (defined in Section 2), the weight term is $(\frac{1}{2\cos j})^i$.

1. Introduction

In this paper, we give closed forms for 12 families of finite sums that involve the sine/cosine functions. Four of the families of finite sums that we consider contain only the integer parameter $j \geq 1$. In addition to the parameter j, the remaining eight families of finite sums that we consider contain the parameter $k \neq 0$, which is a rational number. In each of the 12 finite sums in question, the parameter j can be arbitrarily large, and governs the length of the product that defines the denominator of the summand. In each case, the *denominator* of the summand contains a run of sine or cosine functions, and so we call the sums that we consider *reciprocal* sums. We refer to each of the finite sums that we consider as *weighted*.

We illustrate the concepts discussed in the previous paragraph by referring the reader to $S_5(n, j, k)$, defined in Section 2. In $S_5(n, j, k)$, the parameters are j and k, with $jk \neq -1$. The weight term is $\left(\frac{\sin 1}{\sin(jk+1)}\right)^i$. Excluding the weight term, the length of the product in the denominator of the summand is $j+1$, and this quantity can be arbitrarily large. In $S_4(n, j, k)$, also defined in Section 2, the weight term is $(\frac{1}{2\cos j})^i$.

In Section 2, we define the 12 families of finite sums that we consider in this paper. In Section 3, we give the closed form for each of the 12 families of finite sums in question. Furthermore, we list certain key identities that are used in the proofs, and give a selection of specific examples of our sums. In Section 4, we demonstrate a sample proof. In Section 5, we make some concluding comments.

2. The Finite Sums

Throughout this paper, the upper limit of summation is an integer $n-1$, with $n \geq 2$. Furthermore, the parameter $j \geq 1$ is taken to be an integer, and $k \neq 0$ is a rational number. We now define the 12 families of finite sums whose closed forms we give in the next section.

The four families of finite sums S_1, S_2, S_3, and S_4 are defined below, and contain only the parameter j.

$$S_1(n,j) = \sum_{i=1}^{n-1} \frac{(2\cos j)^i \sin(i-j)}{\sin i \cdots \sin(i+j)},$$
$$S_2(n,j) = \sum_{i=1}^{n-1} \frac{(2\cos j)^i \cos(i-j)}{\cos i \cdots \cos(i+j)},$$
$$S_3(n,j) = \sum_{i=1}^{n-1} \frac{\sin(i+2j)}{(2\cos j)^i \sin i \cdots \sin(i+j)},$$
$$S_4(n,j) = \sum_{i=1}^{n-1} \frac{\cos(i+2j)}{(2\cos j)^i \cos i \cdots \cos(i+j)}.$$

We next define two groups of four families of finite sums, where each family is parametrized by j and k. The content of each group of four families is based on the shape of the weight term. We also recall the conditions on j and k stated at the beginning of this section. The sums S_5, S_6, S_7, and S_8 are defined as

$$S_5(n,j,k) = \sum_{i=1}^{n-1} \left(\frac{\sin 1}{\sin(jk+1)}\right)^i \frac{\sin(k(i+j)+1)}{\sin(ki)\cdots\sin(k(i+j))}, \quad jk \neq -1,$$
$$S_6(n,j,k) = \sum_{i=1}^{n-1} \left(\frac{\sin 1}{\sin(jk+1)}\right)^i \frac{\cos(k(i+j)+1)}{\cos(ki)\cdots\cos(k(i+j))}, \quad jk \neq -1,$$
$$S_7(n,j,k) = \sum_{i=1}^{n-1} \left(\frac{-\sin 1}{\sin(jk-1)}\right)^i \frac{\sin(k(i+j)-1)}{\sin(ki)\cdots\sin(k(i+j))}, \quad jk \neq 1,$$
$$S_8(n,j,k) = \sum_{i=1}^{n-1} \left(\frac{-\sin 1}{\sin(jk-1)}\right)^i \frac{\cos(k(i+j)-1)}{\cos(ki)\cdots\cos(k(i+j))}, \quad jk \neq 1.$$

Finally for this section, we define

$$S_9(n,j,k) = \sum_{i=1}^{n-1} \left(\frac{\sin(jk+1)}{\sin 1}\right)^i \frac{\sin(ki-1)}{\sin(ki)\cdots\sin(k(i+j))}, \quad jk \neq -1,$$
$$S_{10}(n,j,k) = \sum_{i=1}^{n-1} \left(\frac{\sin(jk+1)}{\sin 1}\right)^i \frac{\cos(ki-1)}{\cos(ki)\cdots\cos(k(i+j))}, \quad jk \neq -1,$$
$$S_{11}(n,j,k) = \sum_{i=1}^{n-1} \left(\frac{-\sin(jk-1)}{\sin 1}\right)^i \frac{\sin(ki+1)}{\sin(ki)\cdots\sin(k(i+j))}, \quad jk \neq 1,$$
$$S_{12}(n,j,k) = \sum_{i=1}^{n-1} \left(\frac{-\sin(jk-1)}{\sin 1}\right)^i \frac{\cos(ki+1)}{\cos(ki)\cdots\cos(k(i+j))}, \quad jk \neq 1.$$

Before proceeding, we remark that there are two significant points of difference between the sums that we present in this paper, and the sums that we present in [1] and [2]. In the present paper,

- the product in the denominator of each summand can be arbitrarily long;

- there is a weight term in the summand.

We have found no results in the literature that are analogous to the results that we present here, or in the papers [1] and [2].

3. The Closed Forms

In this section, we give the closed form for each of the 12 families of finite sums defined in Section 2. We present these closed forms in three theorems. Throughout, we take the running variable i to be the dummy variable, so that, for instance, $[\sin(ki+1)]_1^n$ is taken to mean $\sin(kn+1) - \sin(k+1)$. Our first theorem follows.

Theorem 3.1. *Let $j \geq 1$ be an integer. Then,*

$$S_1(n,j) = \left[\frac{(2\cos j)^i}{\sin i \cdots \sin(i+j-1)}\right]_1^n, \tag{3.1}$$

$$S_2(n,j) = \left[\frac{(2\cos j)^i}{\cos i \cdots \cos(i+j-1)}\right]_1^n, \tag{3.2}$$

$$S_3(n,j) = -\left[\frac{1}{(2\cos j)^{i-1}\sin i \cdots \sin(i+j-1)}\right]_1^n, \tag{3.3}$$

$$S_4(n,j) = -\left[\frac{1}{(2\cos j)^{i-1}\cos i \cdots \cos(i+j-1)}\right]_1^n. \tag{3.4}$$

For the proof of each of (3.1)–(3.4), we require a key identity. We demonstrate how each key identity is used via a sample proof that we outline in the next section. For the proof of each of (3.1)–(3.4), the key identities are, respectively,

$$2\cos j \sin n - \sin(n+j) = \sin(n-j),$$
$$2\cos j \cos n - \cos(n+j) = \cos(n-j),$$
$$2\cos j \sin(n+j) - \sin n = \sin(n+2j),$$
$$2\cos j \cos(n+j) - \cos n = \cos(n+2j).$$

Now let $j = 1$. Then (3.3) and (3.4) become, respectively,

$$\sum_{i=1}^{n-1} \frac{\sin(i+2)}{(2\cos 1)^i \sin i \sin(i+1)} = \frac{1}{\sin 1} - \frac{1}{(2\cos 1)^{n-1}\sin n},$$
$$\sum_{i=1}^{n-1} \frac{\cos(i+2)}{(2\cos 1)^i \cos i \cos(i+1)} = \frac{1}{\cos 1} - \frac{1}{(2\cos 1)^{n-1}\cos n}.$$

Our second theorem is

Theorem 3.2. *With the restrictions imposed on j and k in the definitions of S_5, S_6, S_7, and S_8, we have*

$$S_5(n,j,k) = -\frac{\sin(jk+1)}{\sin(jk)}\left[\frac{\left(\frac{\sin 1}{\sin(jk+1)}\right)^i}{\sin(ki)\cdots\sin(k(i+j-1))}\right]_1^n, \tag{3.5}$$

$$S_6(n,j,k) = -\frac{\sin(jk+1)}{\sin(jk)}\left[\frac{\left(\frac{\sin 1}{\sin(jk+1)}\right)^i}{\cos(ki)\cdots\cos(k(i+j-1))}\right]_1^n, \tag{3.6}$$

$$S_7(n,j,k) = -\frac{\sin(jk-1)}{\sin(jk)}\left[\frac{\left(\frac{-\sin 1}{\sin(jk-1)}\right)^i}{\sin(ki)\cdots\sin(k(i+j-1))}\right]_1^n, \tag{3.7}$$

$$S_8(n,j,k) = -\frac{\sin(jk-1)}{\sin(jk)}\left[\frac{\left(\frac{-\sin 1}{\sin(jk-1)}\right)^i}{\cos(ki)\cdots\cos(k(i+j-1))}\right]_1^n. \tag{3.8}$$

The four key identities required for the proof of the four results in Theorem 3.2 are, respectively,

$$\sin(jk+1)\sin(k(n+j)) - \sin 1\sin(kn) = \sin(jk)\sin(k(n+j)+1),$$
$$\sin(jk+1)\cos(k(n+j)) - \sin 1\cos(kn) = \sin(jk)\cos(k(n+j)+1),$$
$$\sin(jk-1)\sin(k(n+j)) + \sin 1\sin(kn) = \sin(jk)\sin(k(n+j)-1),$$
$$\sin(jk-1)\cos(k(n+j)) + \sin 1\cos(kn) = \sin(jk)\cos(k(n+j)-1).$$

Now let $k = 1$. Then (3.7) and (3.8) become, respectively,

$$\sum_{i=1}^{n-1}\left(\frac{-\sin 1}{\sin(j-1)}\right)^i \frac{\sin(i+j-1)}{\sin i\cdots\sin(i+j)} = -\frac{\sin(j-1)}{\sin j}\left[\frac{\left(\frac{-\sin 1}{\sin(j-1)}\right)^i}{\sin i\cdots\sin(i+j-1)}\right]_1^n, \tag{3.9}$$

$$\sum_{i=1}^{n-1}\left(\frac{-\sin 1}{\sin(j-1)}\right)^i \frac{\cos(i+j-1)}{\cos i\cdots\cos(i+j)} = -\frac{\sin(j-1)}{\sin j}\left[\frac{\left(\frac{-\sin 1}{\sin(j-1)}\right)^i}{\cos i\cdots\cos(i+j-1)}\right]_1^n. \tag{3.10}$$

Let $j = 2$. After some simplification, (3.9) and (3.10) become, respectively,

$$\sum_{i=1}^{n-1}\frac{(-1)^i}{\sin i\sin(i+2)} = \frac{1}{2\cos 1}\left[\frac{(-1)^{i-1}}{\sin i\sin(i+1)}\right]_1^n, \tag{3.11}$$

$$\sum_{i=1}^{n-1}\frac{(-1)^i}{\cos i\cos(i+2)} = \frac{1}{2\cos 1}\left[\frac{(-1)^{i-1}}{\cos i\cos(i+1)}\right]_1^n. \tag{3.12}$$

For our final theorem, we have

Theorem 3.3. *With the restrictions imposed on j and k in the definitions of S_9, S_{10}, S_{11}, and S_{12}, we have*

$$S_9(n,j,k) = \frac{\sin 1}{\sin(jk)} \left[\frac{\left(\frac{\sin(jk+1)}{\sin 1}\right)^i}{\sin(ki)\cdots\sin(k(i+j-1))} \right]_1^n, \tag{3.13}$$

$$S_{10}(n,j,k) = \frac{\sin 1}{\sin(jk)} \left[\frac{\left(\frac{\sin(jk+1)}{\sin 1}\right)^i}{\cos(ki)\cdots\cos(k(i+j-1))} \right]_1^n, \tag{3.14}$$

$$S_{11}(n,j,k) = -\frac{\sin 1}{\sin(jk)} \left[\frac{\left(\frac{-\sin(jk-1)}{\sin 1}\right)^i}{\sin(ki)\cdots\sin(k(i+j-1))} \right]_1^n, \tag{3.15}$$

$$S_{12}(n,j,k) = -\frac{\sin 1}{\sin(jk)} \left[\frac{\left(\frac{-\sin(jk-1)}{\sin 1}\right)^i}{\cos(ki)\cdots\cos(k(i+j-1))} \right]_1^n. \tag{3.16}$$

The four key identities required for the proof of the four results in Theorem 3.3 are, respectively,

$$\begin{aligned}
\sin(jk+1)\sin(kn) - \sin 1 \sin(k(n+j)) &= \sin(jk)\sin(kn-1),\\
\sin(jk+1)\cos(kn) - \sin 1 \cos(k(n+j)) &= \sin(jk)\cos(kn-1),\\
\sin(jk-1)\sin(kn) + \sin 1 \sin(k(n+j)) &= \sin(jk)\sin(kn+1),\\
\sin(jk-1)\cos(kn) + \sin 1 \cos(k(n+j)) &= \sin(jk)\cos(kn+1).
\end{aligned}$$

Now let $k = 1$. Then (3.13) and (3.14) become, respectively,

$$\sum_{i=1}^{n-1} \left(\frac{\sin(j+1)}{\sin 1}\right)^i \frac{\sin(i-1)}{\sin i \cdots \sin(i+j)} = \frac{\sin 1}{\sin j} \left[\frac{\left(\frac{\sin(j+1)}{\sin 1}\right)^i}{\sin i \cdots \sin(i+j-1)} \right]_1^n, \tag{3.17}$$

$$\sum_{i=1}^{n-1} \left(\frac{\sin(j+1)}{\sin 1}\right)^i \frac{\cos(i-1)}{\cos i \cdots \cos(i+j)} = \frac{\sin 1}{\sin j} \left[\frac{\left(\frac{\sin(j+1)}{\sin 1}\right)^i}{\cos i \cdots \cos(i+j-1)} \right]_1^n. \tag{3.18}$$

Finally, for this section, let $j = 3$. Then (3.17) and (3.18) become, respectively,

$$\sum_{i=1}^{n-1} \left(\frac{\sin 4}{\sin 1}\right)^i \frac{\sin(i-1)}{\sin i \sin(i+1)\cdots\sin(i+3)} = \frac{\sin 1}{\sin 3} \left[\frac{\left(\frac{\sin 4}{\sin 1}\right)^i}{\sin i \sin(i+1)\sin(i+2)} \right]_1^n,$$

$$\sum_{i=1}^{n-1} \left(\frac{\sin 4}{\sin 1}\right)^i \frac{\cos(i-1)}{\cos i \cos(i+1)\cdots\cos(i+3)} = \frac{\sin 1}{\sin 3} \left[\frac{\left(\frac{\sin 4}{\sin 1}\right)^i}{\cos i \cos(i+1)\cos(i+2)} \right]_1^n.$$

4. A Sample Proof

Each of the 12 results stated in Theorem 3.1, Theorem 3.2, and Theorem 3.3 can be proved in the same manner. To illustrate the method, we now give a proof of (3.15). To proceed, we require two identities from elementary trigonometry. These identities are

$$\sin\alpha\sin\beta = \frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2}, \tag{4.1}$$

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right). \tag{4.2}$$

The key identity that we require for the proof of (3.15) is

$$\sin(jk-1)\sin(kn) + \sin 1\sin(k(n+j)) = \sin(jk)\sin(kn+1), \tag{4.3}$$

which is true for all real numbers j, k, and n. To prove (4.3), we apply (4.1) to each product on the left of (4.3), and then apply (4.2) to the result. We leave the details to the interested reader.

Denote the right side of (3.15) by $r(n,j,k)$. Also, for convenience, denote the left side of (4.3) by $d(n,j,k)$. Then after the obvious (lengthy) manipulations, we see that

$$\begin{aligned} r(n+1,j,k) - r(n,j,k) &= \frac{(-1)^n d(n,j,k)\sin^n(jk-1)}{\sin(jk)\sin^n 1\sin(kn)\cdots\sin(k(n+j))} \\ &= \frac{(-1)^n\sin(kn+1)\sin^n(jk-1)}{\sin^n 1\sin(kn)\cdots\sin(k(n+j))}, \quad \text{by (4.3)} \\ &= S_{11}(n+1,j,k) - S_{11}(n,j,k) \end{aligned} \tag{4.4}$$

Next, after performing manipulations similar to those immediately above, we obtain

$$\begin{aligned} r(2,j,k) &= \frac{-d(1,j,k)\sin(jk-1)}{\sin(jk)\sin 1\sin k\cdots\sin(k(1+j))} \\ &= \frac{-\sin(k+1)\sin(jk-1)}{\sin 1\sin k\cdots\sin(k(1+j))}, \quad \text{by (4.3)} \\ &= S_{11}(2,j,k). \end{aligned} \tag{4.5}$$

Together, (4.4) and (4.5) prove (3.15).

5. Concluding Comments

We remark that results analogous to those that we present in this paper, where the denominator of the summand contains a run of squared terms, seem to be extremely rare. Indeed, for $j \geq 1$ an integer, we have managed to find only

$$\sum_{i=1}^{n-1} \frac{\sin(2(i+j))}{\sin^2 i\cdots\sin^2(i+2j)} = -\frac{1}{\sin(2j)}\left[\frac{1}{\sin^2 i\cdots\sin^2(i+2j-1)}\right]_1^n,$$

$$\sum_{i=1}^{n-1} \frac{\sin(2(i+j))}{\cos^2 i\cdots\cos^2(i+2j)} = \frac{1}{\sin(2j)}\left[\frac{1}{\cos^2 i\cdots\cos^2(i+2j-1)}\right]_1^n,$$

both of which we believe to be new. We leave the proofs of these two results to the interested reader.

To present this paper succinctly, we have chosen to present all our results in an abbreviated manner. We now indicate how our results can be expressed in their most general form. Let θ be any real number that is not a rational multiple of π. This condition on θ eliminates the possibility of vanishing denominators. Then this *entire* paper can be generalized in the following manner: take *every* occurrence of sin and cos, and multiply the argument by θ. For instance, the generalized forms of the sums (3.11) and (3.12) are, respectively,

$$\sum_{i=1}^{n-1} \frac{(-1)^i}{\sin(i\theta)\sin((i+2)\theta)} = \frac{1}{2\cos\theta}\left[\frac{(-1)^{i-1}}{\sin(i\theta)\sin((i+1)\theta)}\right]_1^n,$$
$$\sum_{i=1}^{n-1} \frac{(-1)^i}{\cos(i\theta)\cos((i+2)\theta)} = \frac{1}{2\cos\theta}\left[\frac{(-1)^{i-1}}{\cos(i\theta)\cos((i+1)\theta)}\right]_1^n.$$

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References

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