Financial Markets with Multidimensional Uncertainty

Nihad Aliyev

A dissertation submitted in partial fulfilment of the requirements for the degree
of

Doctor of Philosophy

in

Finance Discipline Group

UTS Business School

University of Technology Sydney

April 4, 2019

Statement of Originality

I certify that this dissertation titled 'Financial Markets with Multidimensional Uncertainty' has not previously been submitted for a degree nor has it been submitted as part of requirement for a degree except as fully acknowledged within the text.

I also certify that the dissertation has been written by me. Any help that I have received in my research and the preparation of the dissertation itself has been acknowledged. In addition, I certify that all information sources and literature used are indicated in the dissertation.

Production Note:
Signature: Signature removed prior to publication.

Date: 4 April 2019

Working Papers

1. Aliyev, N. and He, X. (2018a), 'Ambiguous price formation', *Working paper, UTS Business School* is based on Chapter 2 of this dissertation and has been presented in the following conferences;

   - 2017 American Finance Association Annual Meeting, Chicago, USA,

   - 2017 Investment Management Research, Sydney, Australia,

   - 2017 Computing in Economics and Finance, New York, USA,

   - 2017 Quantitative Methods in Finance, Sydney, Australia.

2. Aliyev, N. and He, X. (2018b), 'The pricing of composition uncertainty', *Working paper, UTS Business School* is based on Chapter 3 of this dissertation and has been presented in the following conferences;

   - 2018 Financial Management Association Meeting, San Diego, USA,

   - 2018 FIRN Ph.D. Symposium, Brisbane, Australia,

   - 31st PhD Conference in Economics and Business, Sydney, Australia,

   - 2018 Quantitative Methods in Finance, Sydney, Australia,

   - 2019 Computing in Economics and Finance, Ottawa, Canada.

3. Aliyev, N., He, X. and Putnins, T. (2018), 'Learning about toxicity: Why order imbalance can destabilize markets', *Working paper, UTS Business School* is based on Chapter 4 of this dissertation and has been presented in the following conferences/seminars;

   - 2018 Financial Management Association Meeting, San Diego, USA,

   - 2018 Market Microstructure Meeting, Sydney, Australia,

   - 2018 Computing in Economics and Finance, Milan, Italy,

   - 2018 Multinational Finance Society Conference, Budapest, Hungary,

   - 2018 Seminar series in University of Sydney, Sydney, Australia,

   - 2019 Seminar series in Macquarie University, Sydney, Australia.

4. Aliyev, N. and He, X. (2018c), 'Toward a general model of financial markets', *Working paper, UTS Business School* is based on Chapter 5 of this dissertation and has been presented in the following conferences;

- 2016 Australasian Finance and Banking Conference, Sydney, Australia,

- 2016 Quantitative Methods in Finance, Sydney, Australia,

- 2016 UTS Business School Ph.D. Conference, Sydney, Australia.

To my family.

Acknowledgements

Abstract

The stability in financial markets is important to promote economic growth. Due to its fundamental importance, the causes of market instability are of broad interest. The purpose of this dissertation is to propose plausible explanations of financial market phenomena related to market stability such as prices, liquidity, volatility, information value, welfare, and market efficiency. The premise in the analysis is that uncertainty in financial markets is multidimensional and information structure is complex. To be more precise, in modern financial markets, forming consistent beliefs about the fundamental values of securities, the composition of market participants, and other market characteristics are complex and uncertain. On the basis of this premise, this dissertation investigates the trading decisions, order sizes, liquidity, security prices, information value, welfare, and market efficiency to shed light on the causes of financial market instability (fragility) and makes a number of empirical predictions some of which provide explanations for results that have been reported in the empirical market microstructure literature and others are yet to be tested. The dissertation also identifies conditions under which markets are vulnerable to instability and thus also has important policy implications.

The first phenomenon investigated in this dissertation is sudden liquidity deteriorations and improvements in financial markets. Chapter 2 presents a security price formation model with ambiguous liquidity provision. The model provides a unified and parsimonious framework to explain the empirically documented features that market liquidity can suddenly deteriorate during market crashes and improve during trading reforms. Consequently, ambiguity in liquidity provision can increase the value of information and social welfare. The ambiguous price formation model helps to understand (i) the dynamics of ambiguity, (ii) the determinants of time-varying ambiguity aversion of liquidity providers, (iii) the price and liquidity dynamics during various order flow patterns, and (iv) the effect of trade size on security prices during ambiguous market episodes.

Chapter 3 develops a model in which traders face uncertainty about the composition of informed and uninformed traders (composition uncertainty) to investigate the "crowded-trade" problem (not being able to know how many others are taking

the same position) in financial markets. This chapter characterizes the equilibrium in the information market where both types of traders are affected by composition uncertainty and in the financial market where only uninformed traders are affected, leading the uninformed traders to be disadvantaged in the face of composition uncertainty. This composition uncertainty distorts traders' information acquisition, demands, and perceived equity premium, resulting in undervalued (resp. overvalued) stock when traders are sufficiently (resp. insufficiently) uncertainty averse. The model helps to understand a linkage between liquidity and asset prices, proposes plausible explanations for large price swings, and demonstrates how regulations to enhance market efficiency may not work when the composition of traders is uncertain.

Chapter 4 shows that when market participants learn about the level of adverse selection from order flow, a large order imbalance can be destabilizing, causing sharp price movements and evaporation of liquidity, as it signals high "toxicity" (adverse selection). While such effect is consistent with the practitioner view that order flow is informative about toxicity, it contrasts with standard microstructure models in which the level of adverse selection is assumed to be known and thus order imbalance improves liquidity by revealing private information. The model helps to understand when markets are most susceptible to imbalance-induced instability and the dynamic process of how markets digest order imbalance.

Chapter 5 examines the implications of the true complexity of real-world information on market efficiency. Using the literature of decision theories and information sciences, Chapter 5 discusses how accounting different attributes of information can unify two controversial views, efficient markets hypothesis and behavioral finance. The main thesis advanced is that the roots of behavioral anomalies are the imprecision and reliability of information. By exemplifying different decision scenarios, Chapter 5 argues that the decision making is rational with precise and reliable information, whereas becomes more behavioral in nature as the information becomes more imprecise and unreliable.

Overall, the results of this dissertation suggest that multiple dimensions of uncertainty formalized in different languages can illuminate on various aspects of market stability that we otherwise label as anomalies and offer a promising middle ground between efficient markets hypothesis and behavioral finance.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

An educated mind is satisfied with the degree of precision that the nature of the subject admits and does not seek exactness where only approximation is possible.

Aristotle

1.1 Introduction

Information is fundamental to our understanding of financial markets. The fundamental importance of information in finance has resulted in an extensive literature on asset pricing and market microstructure investigating its role in determining security prices. Yet the theoretical market microstructure and asset pricing literature are dominated by standard models with simple information structures (e.g., Grossman and Stiglitz (1980), Kyle (1985), Glosten and Milgrom (1985)). These models are popular because of their clear economic intuition and mathematical simplicity. While these models have provided a number of important insights into numerous financial market phenomena, they lack to capture the complex information structures in modern financial markets.

Very often the information available to the majority of market participants is imprecise and partially reliable. Liquidity-providing market participants often lack the ability to form consistent beliefs about the fundamental values of securities. In extremely uncertain market episodes, forming beliefs can even become hardly possible. When investors trade with each other, it is often the case that they do not know who they are trading with. The composition of financial market

participants and the quality of their information have never been more complex and uncertain.[1]

Similarly, the stability in financial markets is fundamentally important to promote economic growth. Despite its fundamental importance, however, it is surprising that we know little about *what* causes market instability, *why* markets are vulnerable to instability and *when* they are most vulnerable. Paradoxically, the standard models with simple information structures (e.g., Kyle (1985), Glosten and Milgrom (1985)) can predict counterintuitive results about the stability in financial markets. This thesis proposes plausible explanations to financial market phenomena related to market stability.

The premise in the analysis is that uncertainty in financial markets is multidimensional and information structure is complex. To be more precise, in modern financial markets, forming consistent beliefs about the fundamental values of financial securities, the composition of market participants, and other market characteristics are complex and uncertain (e.g., Romer (1993), Banerjee and Green (2015)). On the basis of this premise, this thesis sheds light on the causes of financial market instability and makes a number of empirical predictions about what causes market instability and when markets are most vulnerable to instability. The thesis identifies conditions under which markets are vulnerable to instability and thus also sets forth *how* to cope with instability by suggesting important policy implications. Understanding *what*, *why*, *when*, and *how* questions of the market stability and instability is the main objective of this thesis. The thesis proposes to take the multiple dimensions of information uncertainty (either defined probabilistically or in a more general way such as ambiguity and fuzziness) in financial markets into account to investigate the trading decisions, prices, liquidity, volatility, value of information, welfare, and efficiency in financial markets.

This chapter briefly discusses the main focus of each chapter of the thesis and outline the modeling approaches and the main results to show the bigger picture before going into the details of each chapter. For the same purpose, this chapter also discusses where this thesis fits in the general financial economics literature and leaves the direct comparison of each chapter with existing subliterature as a section in the corresponding chapter.

---

[1]See, for instance, "When Silicon Valley came to Wall Street" (Financial Times, October 28, 2017) and "The big changes in US markets since Black Monday" (Financial Times, October 19, 2017). Even some of the data analytics firms entered into a hedge fund business (e.g., Cargometrics).

1.1.1 Chapter 2: Ambiguous price formation

Financial crisis and market crashes are often associated with elevated volatility, evaporation of liquidity, and extreme price inefficiency (e.g., Easley, López de Prado and O'Hara (2012), Kirilenko et al. (2017)). For example, recently, during the infamous May 2010 Flash Crash, the spread of the June 2010 E-mini (S&P 500 futures) contracts widened to 11 ticks (2.75 index points) and declined to about 1 tick (0.25) in a matter of minutes (CFTC - SEC (2010a, 2010b)). On the contrary, historically, financial markets also experienced liquidity improvements following the trading reforms in financial markets (e.g., Jones (2002)). For example, the spreads experienced a sudden decline after NYSE reduced the minimum tick size from eighths ($0.125) to sixteenths ($0.0625) of a dollar in 1997 and from sixteenths to one cent ($0.01) per share in 2001 (e.g., SEC (2012)).

Such scenarios of sudden liquidity deteriorations and improvements are hard to reconcile with the standard price formation models of market microstructure theory. What is causing such peculiar market behavior? Is there a unified approach that can explain both the liquidity deteriorations and improvements in financial markets? How does the behavior of market participants change during such periods? What are the implications for the value of information and total welfare of the market participants? What are the impacts on the price-order size relationship?

To answer these questions, building on the sequential trading model of Glosten and Milgrom (1985), Chapter 2 proposes an ambiguous price formation model in which liquidity providers are subject to ambiguity. Unlike the standard sequential trading model, in this model the market makers have ambiguous (uncertain) beliefs about the security payoffs. To model uncertainty in market makers' beliefs about the fundamental values, we adopt the Choquet Expected Utility (CEU) framework of Schmeidler (1989) with non-additive probabilities to capture the uncertainty and uncertainty aversion. We use non-extreme-outcome-additive (neo-additive) capacities proposed by Chateauneuf, Eichberger and Grant (2007) which is the convex combination of a probability and a constant parameter, measuring the degree of uncertainty aversion. Instead of Bayesian updating, we use generalized Bayesian updating (GBU) of the beliefs (e.g., Walley (1991)). Consistent with the CEU framework, we take Choquet expectation with respect to the market maker's beliefs about the payoffs.

The ambiguous price formation model in Chapter 2 provides a unified explanation for the sudden liquidity changes in financial markets and investigates its implications for value of information, welfare, and price-quantity relationship thoroughly. In this model, a liquidity distortion relative to the standard probabilistic model arises because ambiguity and ambiguity aversion of the market maker influence the perceived adverse selection risk (i.e., perceived proportion of informed traders). To be more precise, in the presence of ambiguity, when the market maker is insufficiently (resp. sufficiently) ambiguity averse, she perceives the number of informed traders less (resp. more) than the actual number of informed traders, resulting in a liquidity improvement (resp. deterioration). Chapter 2 characterizes the necessary and sufficient condition of ambiguity aversion for the market maker to be sufficiently ambiguity averse.

A liquidity distortion (a deterioration or improvement) relative to the standard model has an immediate consequence for the value of information to the financial market participants. Chapter 2 shows that ambiguity can make private information more or less valuable in financial markets. Thus it can distort information acquisition decisions of traders and ultimately impact the informativeness of prices in financial markets. This chapter also introduces trading motives to uninformed traders and investigates the total welfare of market participants. Consequently, by comparing the total welfare of market participants in the presence and absence of ambiguity, Chapter 2 shows that when the market maker is insufficiently (resp. sufficiently) ambiguity averse, ambiguity can contribute to a welfare gain (resp. loss) to society.

To investigate the relationship between the security prices and order sizes, Chapter 2 also distinguishes between the separating equilibrium, in which informed traders trade only large quantities, and the pooling equilibrium, in which informed traders trade either small or large quantities with positive probabilities. The presence of ambiguity makes the separating equilibrium to be prevalent in financial markets. In extremely ambiguous market episodes, the separating equilibrium becomes the only equilibrium. This implies large orders are more likely to be placed by informed traders during ambiguous market episodes, consistent with the empirical market microstructure literature.

1.1.2 Chapter 3: Pricing of composition uncertainty

Based on the intuition of Chapter 2, Chapter 3 develops a rational expectations equilibrium (REE) model to explicitly investigate the impacts of uncertainty about the number of informed traders in financial markets. Indeed, when investors trade with each other, it is often the case that they do not know whether or not they trade against better-informed counterparties. The "quant meltdown" of August 2007 and subsequent unfolding of the global financial crisis also highlighted the importance of this problem. In his presidential address, Stein (2009) emphasizes this as a "crowded-trade" problem (i.e., not being able to know how many others are taking the same position). The fundamental of the crowded-trade problem is that the actual number of informed and uninformed traders is hard to observe. Chapter 3 investigates how uncertainty about the composition of market participants (i.e., composition uncertainty) affects the traders' trading decisions, equity premium, stock prices, and the value of information.

Chapter 3 constructs an REE in the presence of composition uncertainty for a formal treatment of the crowded-traded problem. The model extends the standard CARA-normal REE model where market prices perfectly communicate information (e.g., Grossman (1976), Radner (1979)) along two dimensions. First, it introduces private investment opportunities only available to informed traders with a return correlated to idiosyncratic noise to provide motivation for informed trading as in Easley, O'Hara and Yang (2014) and Wang (1994). Second, to reflect the practical challenges of the crowded-trade problem, the model introduces composition uncertainty so that traders do not precisely know the number of informed and uninformed traders in the market. This uncertainty naturally generates deviations from the "fair" benchmark price without such uncertainty. The informational inefficiency of asset prices stemming from composition uncertainty helps to understand a linkage between liquidity and asset prices, proposes plausible explanations for large price movements, and demonstrates how regulations to enhance market efficiency may not work during uncertain composition of traders.

Chapter 3 first models the traders' preferences with the maxmin model of Gilboa and Schmeidler (1989) to investigate the impacts of composition uncertainty on the market and then extends the analysis to the $\alpha-$maxmin model of Marinacci (2002) and Ghirardato, Maccheroni and Marinacci (2004) to investigate the impacts of traders' uncertainty aversion. In this model, traders first decide whether they want

to be informed or uninformed as in the standard model and then trade. During the trading stage, the uninformed traders with maxmin preferences reduce their risky stockholdings relative to the informationally efficient benchmark and demand a composition uncertainty premium to be compensated. This occurs because composition uncertainty affects the uninformed more than the informed, leading them to be disadvantaged in the face of the composition uncertainty. Consequently, the perceived equity premium is higher and thus the stock is undervalued relative to the benchmark. An extension to the $\alpha-$maxmin model shows an undervaluation is robust as long as the traders are sufficiently uncertainty averse and an overvaluation can occur when they are not sufficiently uncertainty averse.

Chapter 3 characterizes the cost range for the unique information market equilibrium with maxmin preferences and shows that multiple equilibria can arise when traders have $\alpha-$maxmin preferences, offering predictions under both unique and multiple information market equilibria. When a unique information market equilibrium exists, Chapter 3 decomposes the benefit of informed trading into a standard and an uncertain "Knightian" component. The standard component satisfies all the standard results and the "Knightian" component reduces the value of information. The overall value of information is monotonically decreasing in the number of informed traders (i.e., strategic substitutability in information acquisition) as in the standard models.

In the extended $\alpha-$maxmin model, Chapter 3 derives the comparative statics with respect to composition uncertainty as well as the traders' uncertainty aversion. This model shows that the results of the maxmin model are the same (resp. reversed) when traders are sufficiently (resp. insufficiently) uncertainty averse. Chapter 3 also shows that the sufficient uncertainty aversion condition decreases in the number of informed traders. That means traders with a given uncertainty aversion can be sufficiently uncertainty averse when the proportion of informed traders is high but not as sufficient when it is low. Consequently, the value of becoming informed can be decreasing (resp. increasing) in the number of informed traders when traders are sufficiently (resp. insufficiently) uncertainty averse during the high (resp. low) informed market, leading to complementarity in information acquisition and multiple information market equilibria. Strategic complementarities and multiple information market equilibria propose an explanation for sharp price movements in financial markets and demonstrate the importance of exchange-specific uncertainty for regulations to enhance market efficiency.

1.1.3 Chapter 4: Learning about toxicity

Market microstructure theory is composed of two standard reference frameworks for modeling price formation process (Kyle (1985) and Glosten and Milgrom (1985)). In these frameworks, an imbalance between buyer- and seller-initiated trades (order imbalance) stabilizes markets ex post. Order imbalance reveals private information, moves prices towards the fundamental values and decreases uncertainty, thereby reduces adverse selection risks and increases liquidity in the form of lower price impacts in the framework of Kyle and narrower bid-ask spreads in the framework of Glosten-Milgrom. We should, therefore, expect calmer, more liquid, and more informative markets following periods of large order imbalances. Yet, in practice, order imbalance can often be highly destabilizing for markets. In the extreme, order imbalance can trigger 'flash crashes' —episodes of extreme price movements accompanied by evaporation of liquidity and elevated volatility (e.g., Easley et al. (2012), Kirilenko et al. (2017)). Despite the fundamental importance of market stability in promoting economic growth, it is surprising that the standard models offer little about *why* order imbalance can destabilize markets and *when* markets are most vulnerable to destabilizing order imbalance.

To answer both of these questions, Chapter 4 proposes to model the process by which market participants learn about adverse selection risk ('toxicity') from order flow, in particular order imbalance, and study the implications of this learning process. To an otherwise standard sequential trade model, Chapter 4 adds uncertainty about the proportion of informed traders (composition uncertainty) and/or the quality of their signals (signal quality uncertainty), resulting in uncertainty about the level of adverse selection. Consequently, liquidity providers must learn about toxicity, rather than knowing the probability of informed trading and the quality of informed traders' information. This learning occurs from order flow. Intuitively, because informed trading tends to result in order imbalance, observing an episode of highly unbalanced order flow acts as a signal that there is likely to be a high proportion of informed traders or that informed traders have very precise information. This upward revision in perceived adverse selection risk can cause liquidity providers to set wider spreads to protect themselves from higher toxicity, as well as sharp price adjustments as the information contained in past order flow is reassessed. Such effects, which all follow from learning about adverse selection, oppose the standard stabilizing effect of order imbalance (learning about

fundamental value). The tension between these stabilizing and destabilizing effects is what allows the model to illustrate why order imbalance can sometimes be destabilizing and offer insights about when the destabilizing effects are likely to dominate the stabilizing effects.

Using the model with uncertain level of adverse selection, Chapter 4 also explores how markets respond to various order flow patterns. Unlike the standard models with no change in liquidity in the face of balanced order flow, liquidity improves when the market maker with uncertain adverse selection risk receives balanced orders. This occurs because the market maker receiving balanced order flow maintains her initial belief about the security value, but revises her belief about the high adverse selection risk (high probability of informed trading or probability of high-quality information) downward, leading the bid-ask spread to be narrower than the initial bid-ask spread.

The market maker receiving sequences of sell (resp. buy) orders revises her belief about the security value downward (resp. upward), but revises her belief about the high adverse selection risk upward. This means, unlike the standard models, order imbalance can be destabilizing. Chapter 4 shows that order imbalance destabilizes the market when the initial belief about the toxicity (adverse selection risk) is sufficiently low, implying that financial markets are more susceptible to instability in response to order imbalance in low perceived toxicity periods.

Lastly, an order in the direction of the sequence is less informative than the reversal in the order flow due to a property we term "repricing history". During consecutive buy (resp. sell) orders, an additional buy (resp. sell) order contributes less than an additional sell (resp. buy) order in revising the market maker's belief about the security value. The reason for this is that in the presence of the consecutive buy (resp. sell) orders, the market maker receiving an additional buy (resp. sell) order confirms what she already knew, whereas an additional sell (resp. buy) order makes her to realize that the past order flow may not have been as informed, and hence, a reassessment of the previous learning. The large price movements and liquidity dry-ups similar to 'flash crashes' therefore naturally arise in this model, offering an explanation for the prevalence of flash crashes in modern financial markets. All these results follow from the fact that order flow in the presence of the composition and the signal quality uncertainty, in addition to revealing some information about the fundamental value, reveals information about the level of adverse selection risk of the market participants.

1.1.4   Chapter 5: General model of financial markets

The models in the first three parts of this dissertation (i.e., ambiguous price formation, ambiguous composition of traders, and learning about toxicity) provide useful characterizations of different dimensions of uncertainty in financial markets. Nevertheless, they require some informational structure to obtain closed-form solutions. This motivates us to ask "how complex the real-world information can get?" and "what are the implications of such information complexity for financial markets?" in the last part of this dissertation.

With increasingly complex information in financial markets, it is now more important to understand different roles played by different dimensions of information uncertainty in determining market outcomes. However, in doing so, we should also be cognizant about the true complexity of the real-world information and financial decision-making process with such information. To address the last issue, in Chapter 5, we incorporate information science, decision theory and financial economics literatures together, and discuss information in the broadest possible way that lends itself to possible quantitative scrutiny. In our discussion, we use Zadeh (2011) classification of information - numerical, interval-valued, second-order uncertain, fuzzy and Z information - based on its generality. We argue that individuals are subjectively rational if they apply *"correct"* decision technique to each class of information separately. We exemplify candidate decision theories in each information class and show that financial decision making becomes more behavioral in nature as information becomes imprecise and unreliable. We present a general approximation for subjective rationality in decision making and suggest a general framework of financial markets. We argue that efficient markets hypothesis and behavioral finance pioneered by Eugene Fama and Robert Shiller (e.g., Fama (1965), Shiller (1981)) become special cases of this framework with the imprecision and reliability of information approximately connecting them. That is, imprecise and partially reliable information generates "anomalies", whereas precise facts lead to efficient markets.

In summary, the main argument we develop in this chapter is that uncertainty is multidimensional and the real-world information to cope with multidimensional uncertainty is ambiguous (vague or imprecise), and unreliable. Accounting these attributes of information proposes a middle ground between two extreme views in finance literature and has a potential to unify these views.

1.2 Literature Review

In this section, we review the general literature on uncertainty and its impact on financial markets, and leave the comparisons of each chapter with related papers as a section in the corresponding chapter. The purpose is to provide the idea of an uncertainty (risk, ambiguity, fuzziness) as a broad concept and show how the focus of this thesis fits in the recent applications of uncertainty to financial markets. The thesis is related to four main strands of literature.

1.2.1 Decision making under uncertainty

The idea of ambiguity or Knightian uncertainty dates back to Knight (1921) and Keynes (1921), where they distinguish between risk (when relative odds of the events are known) and uncertainty (when the degree of knowledge only allows the decision maker to work with estimates). However, the argument that *"the decision makers behave as if they assign (subjective) odds even when they do not know the odds"* prevailed in much of the literature following Ramsey (1931) and Savage (1954). The experimental evidence of Ellsberg (1961) on the ambiguity aversion of decision makers revived the tentative ideas of Knight and Keynes. The behavior of ambiguity aversion documented by Ellsberg (1961) has been first axiomatized in the decision making context by Choquet expected utility of Schmeidler (1989) and maxmin expected utility of Gilboa and Schmeidler (1989). Since then, different approaches such as unanimity preferences of Bewley (2002), smooth preferences of Klibanoff, Marinacci and Mukerji (2005), variational preferences of Maccheroni, Marinacci and Rustichini (2006), $\alpha-$maxmin model of Ghirardato et al. (2004) have been taken to model information ambiguity. We refer to Gilboa and Marinacci (2013), Epstein and Schneider (2010), and Machina and Siniscalchi (2014) for extensive surveys of this literature.

1.2.2 Uncertainty and financial markets

This thesis is closely related to recent papers studying ambiguity of the financial market participants and its effects on market characteristics by modeling the preferences of traders with the above decision theories (e.g., Epstein and Schneider (2008), Caskey (2009), Easley and O'Hara (2010*a*, 2010*b*), Routledge and Zin

(2009), Easley et al. (2014), Condie and Ganguli (2017), Ozsoylev and Werner (2011), Mele and Sangiorgi (2015)). Chapters 2 and 3 contribute to this literature by modeling market participants with Choquet expected utility, maxmin, and $\alpha-$maxmin models to show that information ambiguity in different dimensions such as beliefs about the fundamental values and the composition of traders, in addition to the risk associated with the fundamental values, can explain a number of empirical regularities in financial markets. More precisely, Chapter 2 contributes to this literature by showing the ambiguity about the payoffs as a unifying mechanism to explain sudden liquidity deteriorations and improvements in financial markets, and its consequences for value of information and welfare to market participants. Chapter 2 also investigates the price-order size analysis in the presence of ambiguity. In addition, unlike the most of the existing studies with ambiguity about the fundamental values, Chapter 3 models an ambiguous composition of traders as a novel source of ambiguity and investigates its impacts on asset prices and value of information.

1.2.3 Multiple dimensions of uncertainty

Another closely related strand of literature to this thesis is multidimensional uncertainty quantified by probability distributions (e.g., uncertain risk aversion, uncertain wealth of traders, uncertain composition of traders, uncertain quality of informed traders' information) other than the fundamental values of securities and its effects on financial markets (e.g., Easley and O'Hara (1992), Leach and Madhavan (1992, 1993), Romer (1993), Gervais (1997), Avery and Zemsky (1998), Gao, Song and Wang (2013), Banerjee and Green (2015), Goldstein and Yang (2015), Banerjee, Davis and Gondhi (2018)). Chapter 4 contributes to this literature by modeling uncertain composition of traders and uncertain quality of informed traders' information in a sequential trading model to reconcile financial practice and empirical evidence associated with order imbalances with market microstructure theory. Unlike the existing studies, Chapter 4 shows how uncertainty about the adverse selection risk (either in the composition of traders, the quality of traders' information or both) combined with uncertainty about the fundamental value and why learning about different dimensions of uncertainty from the order flow can lead to market instability (liquidity evaporations, large price swings, and elevated volatility) in the face of large order imbalances.

1.2.4 Market efficiency

Lastly, the thesis is also related to the controversial debate among the two schools of thoughts, efficient markets hypothesis and behavioral finance, which shaped much of the modern finance literature (e.g., Osborne (1959), Samuelson (1965), Fama (1965, 1970, 2014), Rubinstein (2001), Thaler (1980), De Long et al. (1990), Shiller (1981, 2003, 2014)). Chapter 5 contributes to this debate by proposing a middle ground between efficient markets hypothesis and behavioral finance on the basis of above decision theories under ambiguity as well as fuzziness (e.g., Aliev and Huseynov (2014)) and information science literature (e.g., Zadeh (1965, 2011), Klir (2005)).

Chapter 2

Ambiguous Price Formation

The action which follows upon an opinion depends as much upon the amount of confidence in that opinion as it does upon the favorableness of the opinion itself.

Knight (1921) "Risk, Uncertainty and Profit" [p. 227].

2.1 Introduction

Liquidity provision in modern financial markets is a complex process. Modern liquidity providers (e.g., algorithmic market makers) are subject to multiple dimensions of uncertainty. The complexity in modern liquidity provision (e.g., due to the proliferation of HFT and demise of the designated market makers) has a first-order impact in fluctuations in the security prices and market liquidity. Given the fundamental importance of these fluctuations in promoting market stability and social welfare, understanding the mechanics of such fluctuations is of broader interest.

In particular, market crashes are often associated with an increase in volatility, evaporation of liquidity and extreme price inefficiency. For example, during 1997-1998 financial crisis, the spreads over treasuries widened on U.S. AAA bonds, AAA commercial mortgage pools, credit instruments and swap contracts (e.g., Scholes (2000)). Similar features are associated with the global financial crisis, in which for some assets virtually all markets have been characterized by large spreads and lack of trading. On the contrary, financial markets also experience liquidity improvements following landmark reforms in trading rules. For example,

the spreads fell dramatically after the SEC implemented new trading rules (e.g., permit the public to submit binding limit orders, displaying the superior quotes placed by dealers in private trading venues) in Nasdaq on January 20, 1997 (e.g., Barclay et al. (1999)).

Such scenarios of sudden liquidity deteriorations and improvements are hard to reconcile with the standard price formation models of the market microstructure theory (e.g., Kyle (1985), Glosten and Milgrom (1985)). In this chapter, we present a security price formation model to provide an explanation for such peculiar market behavior. Our premise is that the liquidity provision in the standard market microstructure models is too simple to capture the underlying complexity in the modern liquidity provision. The standard price formation models are designed for well-defined gambles, in which a single probability distribution captures the uncertainty of liquidity providers. We argue that this is in contrast to the real-life price formation, where ambiguity plays an important role in liquidity provision.

We propose a sequential trading model, in which market makers providing liquidity to the market have ambiguous beliefs about the final security payoff. In modeling ambiguity, we adopt the Choquet Expected Utility (CEU) framework of Schmeidler (1989). Specifically, we use non-extreme-outcome-additive (neo-additive) capacities proposed by Chateauneuf et al. (2007) to capture the ambiguity and ambiguity aversion of the market makers. A neo-additive capacity is the convex combination of a probability and a constant parameter, measuring the degree of ambiguity aversion. In addition, instead of Bayesian updating, we use generalized Bayesian updating (GBU) of the beliefs. Consistent with the CEU framework, we take Choquet expectation with respect to the market maker's beliefs about the final payoff of the security. The fundamental strength of our modeling approach is that it is consistent with a coherent axiomatic theory. Additionally, it parsimoniously allows for the separation between ambiguity and ambiguity attitude of the market makers.

In this setting, we first investigate the impacts of ambiguity and ambiguity aversion of the market maker on the quotes and bid-ask spread at a given point in time. In the presence of ambiguity, the bid-ask spread can be wider or narrower than the standard bid-ask spread depending on the combination of ambiguity and ambiguity aversion. More precisely, we introduce a concept of "bid-ask spread neutrality", in which the equilibrium bid-ask spread with ambiguity is the same as the one without ambiguity. Consequently, we show that the equilibrium spread can

be decomposed into the standard spread and an "ambiguity premium/discount" component characterizing the ambiguity and ambiguity aversion of the market maker. For the sufficiently ambiguity-averse market maker, the resulting ambiguity premium on the spread provides a potential explanation to drying liquidity and price inefficiency during the periods of extreme market stress. On the other hand, the ambiguity discount prevails when the market maker is optimistic (or not as sufficiently ambiguity averse) about the ambiguity. The liquidity distortion (either in the form of ambiguity premium or discount) relative to the standard model arises in our model because ambiguity and ambiguity aversion influence the market maker's perceived adverse selection risk.

We use our model to explore the market dynamics during continuous sell and balanced orders. When the ambiguity-averse market maker receives continuous sell orders, her perceived adverse selection risk (adverse selection risk after taking into account ambiguity) increases, which tends to make the market less liquid. This is consistent with the experience of the U.S. financial markets during the infamous May 2010 Flash Crash, in which the spread of the June 2010 E-mini (S&P 500 futures) contracts widened to 11 ticks (2.75 index points) in the face of selling pressure. This basic effect intrinsic to market crashes is in contrast to standard microstructure models in which liquidity improves (lower price impacts in the Kyle framework and narrower bid-ask spread in the Glosten-Milgrom framework) in the face of selling pressure due to the resolution of uncertainty about the payoff. Our model also shows that the market maker's ambiguity aversion increases in the face of selling pressure. Even if the market maker is initially not ambiguity averse, she can become ambiguity averse during the selling pressure, resulting a switch from an ambiguity discount to an ambiguity premium on the standard spread.

The prediction of our model is also different from the standard model during balanced orders. In the standard model, balanced order flow reveals no new information and thus has no effect on prices or liquidity. In our model, balanced orders can be stabilizing and destabilizing depending on the ambiguity attitude of the market maker. We show that balanced orders improve liquidity when the market maker is optimistic about her belief assessments, whereas deteriorate liquidity when she is pessimistic. In contrast to continuous sell orders, balanced orders have no effect on the ambiguity aversion of the market maker.

Next, we investigate the consequences of liquidity distortions due to the market maker's ambiguity on the value of information to informed traders and the total welfare of all market participants. The ambiguity in liquidity provision can make private information more or less valuable in financial markets. Intuitively, information becomes more (resp. less) valuable when it leads to more (resp. less) profits. To verify this intuition we decompose the value of information in our model into the standard and ambiguous components due to the market maker's ambiguity. Consequently, we show that the ambiguity premium (resp. discount) on the standard bid-ask spread is associated with a value discount (resp. premium) on the standard value of information. To investigate the total welfare of market participants we introduce trading motives to uninformed trading based on the distribution of private valuation of the security. When the private valuation of an uninformed trader arriving at the market lies inside the bid and ask quotes, he chooses not to trade, resulting in a welfare loss to society. By comparing the total welfare of market participants in the presence and absence of ambiguity, we show that when the market maker is insufficiently (resp. sufficiently) ambiguity averse, ambiguity contributes to a welfare gain (resp. loss) to society.

We also consider the possibility of different trade size and examine the behavior of liquidity providers and informed traders (e.g., Easley and O'Hara (1987)). To do this, we distinguish between the separating equilibrium, in which informed traders trade only large quantities and the pooling equilibrium, in which informed traders trade either small or large quantities. For the sufficiently ambiguity-averse market maker, the order size ratio (i.e., the ratio of large order size to small order size) for the separating equilibrium to exist reduces with ambiguity in liquidity provision. In extreme ambiguity, the separating equilibrium becomes the only equilibrium. This arises because the ambiguity-averse market maker behaves according to her worst case scenario and the informed traders with the knowledge of the market maker's pricing rule trade only large quantities to maximize their profits.

During the market crashes the liquidity-providing market participants often withdraw their quotes (e.g., Chordia, Roll and Subrahmanyam (2002), Anand and Venkataraman (2016)). The withdrawal of liquidity by some electronic market makers and enhanced uncertainty about the actions of the market participants exacerbate the magnitude of the crash. This is often associated with the "toxicity" (adverse selection risk) in the order flow (e.g., Easley et al. (2012)). In our model,

the ambiguity-averse market maker regards the order flow as "toxic" since her perceived adverse selection risk increases with ambiguity. Our model also captures the flip side of the story in which the ambiguity leads to liquidity improvements when the liquidity providers are sufficiently optimistic. The dual role of ambiguity and ambiguity aversion, in turn, generates the seemingly peculiar market behavior (liquidity deteriorations for the events that the market makers are pessimistic and liquidity improvements for the events that they are optimistic).

The next section relates this chapter to the existing literature. In Section 2.3, we present the ambiguous price formation model. In Section 2.4, we investigate the effects of ambiguity in liquidity provision on the properties of the market. In Section 2.5, we examine the consequences of ambiguous price formation on the value of information to informed traders and the total welfare to society. In Section 2.6, we incorporate different order sizes to our baseline model to explore the joint impact of order sizes and ambiguity in liquidity provision on the behavior of market participants. Section 2.7 examines the empirical implications of our results and concludes. Proofs and other extensions to relax the assumptions of our baseline model are relegated to the appendices.

2.2   Related Literature

This chapter contributes to various strands of literature. Market liquidity is the focus of market microstructure. Classic market microstructure models involve a dealer who provides liquidity at a cost that arises from the risk of holding inventory (e.g., Stoll (1978)) or trading against a better-informed trader (e.g., Kyle (1985), Glosten and Milgrom (1985)). Theoretically, there are several factors (with possibly overlapping reasons) that can result in liquidity crashes.

(i) Brunnermeier and Pedersen (2008) provide a model in which traders' liquidity provision depends on the availability of funding and vice versa. They show that the market liquidity and funding liquidity are mutually reinforcing and can lead to liquidity spirals.

(ii) Huang and Wang (2009) extends Grossman and Miller (1988) framework to link the cost of maintaining continuous market presence with market crashes. Thus the need for liquidity and market crashes emerge endogenously when

selling pressure overwhelms the insufficient risk-bearing capacity of market makers.

(iii) Duffie (2010) extends the standard dynamic general equilibrium model of Stapleton and Subrahmanyam (1978) to show that prices respond to supply or demand shocks with a sharp reaction and liquidity dries up due to the presence of inattentive investors.

(iv) Cespa and Foucault (2014) extend the standard CARA-Normal REE model to allow learning about the fundamental value of one asset from another asset. This cross-asset learning generates a self-reinforcing positive relationship between the price informativeness and liquidity, leading to liquidity spillovers and crashes.

(v) Aliyev, He and Putnins (2018) extend the standard sequential trading model of Glosten and Milgrom (1985) to show that large buying or selling pressure can lead to sharp price adjustments and evaporation of liquidity due to learning about the level of toxicity (adverse selection) in the market.

This chapter contributes to this literature in various ways. First, we explore ambiguity and ambiguity aversion of liquidity providers during uncertain market events to capture both liquidity deteriorations during financial crisis and improvements during trading reforms. Second, we investigate the implications of such ambiguity and ambiguity aversion on information value and welfare. Third, we examine the price-order size-ambiguity relationship when traders can trade different order sizes. More importantly, we provide different set of predictions about the impacts of ambiguity and ambiguity aversion of liquidity providers on the market outcomes.

This chapter is related to a subset of the market microstructure literature that investigates the impacts of ambiguity on the market liquidity. Routledge and Zin (2009) focus on a monopolist market maker for a derivative security. Ozsoylev and Werner (2011) develop a static rational expectations equilibrium model following maxmin expected utility of Gilboa and Schmeidler (1989). Similarly, in a static CARA-normal framework, Easley and O'Hara (2010a) use incomplete preferences of Bewley (2002) to show that no trade region naturally arises for a certain price range, resulting in an ambiguity spread. We model trading as a sequential trading process with a competitive market making and adopt CEU framework with neo-additive capacities. This allows us to investigate the impacts of ambiguity as

well as the ambiguity aversion of liquidity providers on both the static prices and liquidity as well as their dynamics.

There are three ways to explicitly allow for this separation: the *α-maxmin expected utility* (α-MEU) model of Marinacci (2002), the *smooth ambiguity* model of Klibanoff et al. (2005) and *Choquet expected utility* model of Schmeidler (1989) combined with neo-additive capacities. Both α-MEU and smooth ambiguity models do not have an axiomatization in terms of preference over Savage acts. In the smooth model, the ambiguity attitude is determined by a second-order preference over second-order acts. Our approach of separating ambiguity and ambiguity attitude is much simpler (tractable), axiomatically coherent and intuitive.

Another related branch of literature explores the impacts of ambiguity on the value of information and welfare (e.g., Mele and Sangiorgi (2015), Aliyev and He (2018b), Easley et al. (2014)). In a rational expectations framework, Mele and Sangiorgi (2015) show that traders' ambiguity about the asset fundamentals can make the value of information to increase in the number of informed traders (i.e., strategic complementarities) and drive large price swings. Easley et al. (2014) develop a static model with ambiguity-averse mutual funds facing ambiguity about the effective risk tolerance of hedge funds to show that this ambiguity decreases the welfare. In our model, market makers are subject to ambiguity as opposed to traders. This is natural because in today's fast and fragmented market, effectively monitoring market conditions in real time is challenging and liquidity providers are vulnerable to the presence of ambiguity. Our analysis contributes to this literature by showing that ambiguity can increase the value of information to informed traders and the welfare to society.

The chapter is also related to the market microstructure literature investigating the price-quantity relationship (e.g., Easley and O'Hara (1987), Ozsoylev and Takayama (2010)). Our contribution here is to characterize the conditions on ambiguity under which order size does separate informed and uninformed traders in equilibrium and show the price-quantity-ambiguity relationship. This is the first research, to our knowledge, to show the association between the order size and ambiguity. Finally, this chapter is related to recent papers studying the ambiguity and financial markets (e.g., Caskey (2009), Easley and O'Hara (2010b), Condie and Ganguli (2017)).

2.3   Ambiguous Price Formation

This section presents the model of ambiguous price formation to investigate the role of market making and price formation during extreme market events. The assumption that the uncertainty around this time can be explained by a single probability distribution is neither natural nor realistic. Taking this into account allows us to develop the underlying intuition of price formation during extreme market events. The next subsection introduces necessary concepts.

2.3.1   Preliminaries

We assume that the uncertainty can be described by a non-empty set of finite states, denoted by $\mathcal{S}$. A non-additive probability is a real-valued set function defined on the set of events $\mathcal{E}$ that is normalized ($v(\varnothing) = 0$, $v(\mathcal{S}) = 1$) and monotonic (for all A, B in $\mathcal{E}$, $A \subseteq B \Rightarrow v(A) \leq v(B)$). We specifically use neo-additive capacities of Chateauneuf et al. (2007) to capture the ambiguity and ambiguity aversion during extreme market events. Given an additive probability π on $\mathcal{E}$, a neo-additive capacity is defined by $v(A) = (1-\delta)\cdot\pi(A)+\delta\cdot\alpha$ for $\varnothing \subsetneq A \subsetneq \mathcal{S}$ and $\alpha, \delta \in [0, 1]$. The weight $(1 - \delta)$ given to $\pi(A)$ is a *degree of confidence* which the individual holds in her probabilistic belief $\pi(A)$. The parameter δ is thus a measure of ambiguity. When $\delta = 0$, the standard probabilistic analysis with no ambiguity applies, whereas when $\delta = 1$, the individual has no confidence (full ambiguity) in her probability assessment.

The parameter α measures the individual's attitude toward ambiguity. When $\alpha = 0$, the individual exhibits pure pessimism (ambiguity aversion) since $v(A) \leq \pi(A)$ and when $\alpha = 1$, she exhibits pure optimism (ambiguity seeking) since $v(A) \geq \pi(A)$. The belief (v) is revised by the generalized Bayesian updating, i.e.,

$$v(A|B) = \frac{v(A \cap B)}{v(A \cap B) + 1 - v(A \cup B^c)}. \tag{2.1}$$

It is straightforward to check that Eq. (2.1) reduces to Bayes' rule when v is additive; that is, $1 - v(A \cup B^c) = v(A^c \cap B)$ and $v(A \cap B) + v(A^c \cap B) = v(B)$, and hence Bayes' rule. Lastly, we introduce the Choquet expectation with respect to the beliefs. Without loss of generality, rank a non-negative function f on $\mathcal{S}$ as $f(s_1) \geq f(s_2) \geq ... \geq f(s_n)$ and $f(s_{n+1}) = 0$. The Choquet expectation

(i.e., integral) of the non-negative function f on $\mathcal{S}$ with respect to a non-additive probability v follows from Choquet (1955) as

$$E_v[f] := \int_{\mathcal{S}} f \cdot dv = \sum_{k=1}^{n} \big(f(s_k) - f(s_{k+1})\big) \cdot v\big(\{s_1, s_2, ..., s_k\}\big). \qquad (2.2)$$

To provide an intuition, we consider the following example.

Example 2.1. Consider an asset (a) that pays either \$2 in low state (l) or \$5 in high state (h) with non-additive probabilities of $v_l = 0.2$ and $v_h = 0.4$ respectively. The (Choquet) expected payoff of buying a unit of this asset is then given by $E_v[a_b] = 0.6 \cdot 2 + 0.4 \cdot 5 = 3.2$. The (Choquet) expected payoff of selling a unit of this asset is given by $E_v[a_s] = 0.2 \cdot (-2) + 0.8 \cdot (-5) = -4.4$.

In Example 2.1, the expected payoffs of buying and selling the asset follow from Eq. (2.2). The expectation is equivalent to adding the probability gap $(1 - v_l - v_h = 0.4)$ to the beliefs about the worst case scenarios. When buying the asset the worst case scenario is the low state (l), whereas when selling the asset the worst case scenario is the high state (h).

2.3.2 Setup

We consider a Glosten-Milgrom framework with one risky security and three different groups of traders; informed traders, uninformed traders and market makers (traders within each group are identical). Trade takes place in $t = 1, ..., T$ periods and the risky security pays off in period $T + 1$. The final payoff of the security is represented by a random variable $\hat{V}$ which either takes low value (V_l) with an initial prior probability of π_l or high value (V_h) with a probability of $\pi_h = 1 - \pi_l$.[2]

Consistent with the standard Glosten-Milgrom framework, in each period, the risk-neutral and competitive market maker posts bid $(B_{\alpha,\delta})$ and ask $(A_{\alpha,\delta})$ quotes for a fixed amount (normalized to one unit) to earn zero expected profit. Additionally, in our model, the ambiguity-averse market maker's beliefs about the final payoff are ambiguous. Her initial beliefs about the low and high outcomes of the payoff,

---

[2]The two-point distribution of the security payoff is certainly unrealistic. In Appendix 2.2, we extend the two-point distribution of the security payoff to the trinomial distribution to show that our results are robust to this distributional assumption.

respectively, are

$$v_l = (1 - \delta) \cdot \pi_l + \delta \cdot \alpha \quad \text{and} \quad v_h = (1 - \delta) \cdot (1 - \pi_l) + \delta \cdot \alpha, \qquad (2.3)$$

where δ is the amount of ambiguity and α is the market maker's ambiguity attitude. The amount of probability "lost" by the presence of ambiguity is $1 - v_l - v_h = \delta \cdot (1 - 2 \cdot \alpha)$, representing the confidence in the market maker's probability assessment. It measures the deviation of the belief $v = \{v_l, v_h\}$ from the additive probabilistic belief $\pi = \{\pi_l, \pi_h\}$. To capture the ambiguity aversion of the market maker, we assume $0 \leq \alpha \leq 0.5$. When $\alpha = 0$, the market maker is fully ambiguity averse and when $\alpha = 0.5$, she is ambiguity neutral.

At each trading round, a trader arrives at the market to buy or sell a fixed amount (normalized to one unit) of the security at the ask or bid quote. With a probability of μ the trader arriving at the market is informed and with a probability of $1 - \mu$ is uninformed. The informed traders are risk neutral and maximize their expected profits by trading on a perfect signal $\Theta = \{H, L\}$ about the payoff of the risky security.[3] They buy one unit of the security when $\Theta = H$ and sell one unit when $\Theta = L$. The uninformed traders trade according to their liquidity needs or hedging purposes, which are exogenous to the model.[4] For convenience, we assume that, with equal probabilities, they buy and sell with perfectly inelastic demand. The structure of the model is common knowledge to all market participants.

2.3.3   Equilibrium

The standard Bertrand competition argument that the competitive market maker expects a zero profit implies that the market maker's bid and ask quotes are the expected final security payoff conditional on receiving a sell (s) or buy (b) order respectively. She, therefore, sets ask price as $E_v[V|b]$ and bid price as $E_v[V|s]$ by revising her beliefs (v) about the final payoff with the generalized Bayesian

---

[3]In Appendix 2.3, we show that our results hold as long the informed traders' signal is sufficiently informative. We also show that the sufficiency condition for the information quality increases with ambiguity and ambiguity aversion of the market maker.

[4]Our results are robust to exogenous uninformed trading assumption. We show this in Section 2.5.2 when we model endogenous uninformed trading to investigate the welfare implications of ambiguity. Also assuming ambiguity about other market features (such as the trader composition) do not qualitatively change our results. Moreover, imposing ambiguity to other market participants is not binding in our baseline model since, in this model, the informed traders do not use the price function to extract information about the security payoff and the uninformed traders are assumed to trade exogenously.

updating.[5] This implies that she applies Eq. (2.1) to update her beliefs given in Eq. (2.3). Let π_b denote the probability of a buy order. The next lemma formally derives the updating of the market maker's belief about the low outcome conditional on a buy order, $v(\hat{V} = V_l|b) = v_l^b$. The beliefs v_h^b, v_l^s and v_h^s can be defined and calculated similarly.

Lemma 2.2. Suppose $\pi_b > 0$. The market maker's revised belief about the low outcome conditional on a buy order is given by

$$v_l^b = (1 - \delta^b) \cdot \pi_l^b + \delta^b \cdot \alpha, \tag{2.4}$$

where $\delta^b = \frac{\delta}{(1-\delta)\cdot\pi_b+\delta}$ and π_l^b is a Bayesian update of π_l conditional on a buy order. In addition, the updated ambiguity, δ^b, increases with the ambiguity, δ, and decreases with the probability of a buy order.

Several features of Lemma 2.2 deserve comments. First, note that the revised belief reduces to the Bayesian updated probability when there is no ambiguity, $\delta = 0$. Second, the market maker's revised belief, v_l^b, is also a convex combination of the posterior probability, π_l^b, and the ambiguity aversion, α. This means the market maker's ambiguity attitude stays the same, the probability is revised by Bayes' theorem, while the market maker's posterior belief is the weighted average of the two. Third, conditional on the buy order the ambiguity itself, however, is updated upward (i.e., $\delta^b \geq \delta$). The enhanced ambiguity irrespective of the type of trade is natural, especially during the crisis and trading reform periods. This is so because, during these periods, the market makers become less confident about their information processing with additional information. The updated ambiguity δ^b increases with the prior belief about the ambiguity δ. This means that for the given probability of a buy order, the updated ambiguity will be higher if the prior ambiguity is higher. Moreover, the updated ambiguity decreases with the probability of a buy order. This means that the more likely it was for the buy order to occur probabilistically, the updated ambiguity conditional on a buy order

---

[5]The intuition of the quotes in the presence of ambiguity follows the same intuition as in the standard model. When someone buys from the market maker the expected profit of the market maker is $A_{\alpha,\delta} - E_v[V|b]$ and when someone sells it is $E_v[V|s] - B_{\alpha,\delta}$. The competition drives down the profits to zero, leading to $A_{\alpha,\delta} = E_v[V|b]$ and $B_{\alpha,\delta} = E_v[V|s]$. In our analysis, we analytically characterize the initial quotes and spread and numerically investigate their evolutions by evaluating $E_v[V|b]$ and $E_v[V|s]$ iteratively using the generalized Bayesian updating in Eq. (2.1). In Appendix 2.4, we also investigate other (optimistic updating of Gilboa and Schmeidler (1993) and pessimistic updating of Dempster (1968) and Shafer (1976)) ways of updating Choquet beliefs.

will be lower. In extreme, when $\pi_b = 1$, the market maker's updated ambiguity and the prior ambiguity are equal. This type of probability tilting, though less structured, is characterized as a behavioral bias in behavioral finance (e.g., Gervais and Odean (2001)). Next, we define the equilibrium.

Definition 2.3. An equilibrium consists of the market maker's prices, informed traders' trading strategies, and posterior beliefs such that:

(i) The bid and ask prices satisfy the zero-expected-profit condition, given the market maker's posterior beliefs;

(ii) The informed traders maximize their expected profits given the signal $\Theta = \{H, L\}$;

(iii) The market maker's belief satisfies the generalized Bayesian updating.

2.4 Ambiguity, Ambiguity Aversion and Market Properties

Our model allows us to analyze the effects of ambiguity on the market maker's behavior and the properties of the market in two different ways. First, we can vary the size of the market maker's ambiguity and ambiguity aversion to get an idea of their impact on the quotes and bid-ask spread (i.e., comparative statics). Second, we can fix the initial ambiguity δ and ambiguity aversion α and determine the effects of different trading histories on the evolution of quotes and spread in the presence of ambiguity.

2.4.1 Effects of the ambiguity on the initial quotes/spread

Given the conditional expectations, the market maker can set initial bid and ask quotes. The bid is the expected value of the final payoff conditional on a sell order,

$$
\begin{aligned}
B_{\alpha,\delta} =& E_v[\hat{V}|s] = \left[v_l^s + \delta^s \cdot (1 - 2 \cdot \alpha) \right] \cdot V_l + v_h^s \cdot V_h \\
=& V_l + v_h^s \cdot (V_h - V_l).
\end{aligned}
\tag{2.5}
$$

Similarly, the ask is the expected value of the final payoff conditional on a buy order,

$$
\begin{aligned}
A_{\alpha,\delta} &= E_v[\hat{V}|b] = v_l^b \cdot V_l + \left[v_h^b + \delta^b \cdot (1 - 2 \cdot \alpha)\right] \cdot V_h \\
&= V_h - v_l^b \cdot (V_h - V_l).
\end{aligned}
\tag{2.6}
$$

Eqs. (2.5) and (2.6) directly follow from the definition of Choquet expectation given in Eq. (2.2). The intuition follows from Example 2.1. Since $V_h > V_l$ for buying and $(-V_l) > (-V_h)$ for selling, the minimum probability when evaluating an expectation is the probability that puts the most possible weight on V_l and $(-V_h)$ respectively. Therefore, the probability gap, $\delta^s \cdot (1 - 2 \cdot \alpha)$ and $\delta^b \cdot (1 - 2 \cdot \alpha)$ are added to v_l^s and v_h^b in the calculation of $B_{\alpha,\delta}$ and $A_{\alpha,\delta}$ respectively. The standard probabilistic bid and ask quotes are given, respectively, as

$$
B = V_l + \pi_h^s \cdot (V_h - V_l),
\tag{2.7}
$$

$$
A = V_h - \pi_l^b \cdot (V_h - V_l),
\tag{2.8}
$$

where $\pi_h^s = \frac{(1-\mu)\cdot\pi_h}{(1+\mu)\cdot\pi_l+(1-\mu)\cdot\pi_h}$ and $\pi_l^b = \frac{(1-\mu)\cdot\pi_l}{(1-\mu)\cdot\pi_l+(1+\mu)\cdot\pi_h}$ follow from Bayes' rule. Having described the structure of the model and determined the equilibrium quotes in the presence and absence of ambiguity, we now combine these quotes in the following proposition.

Proposition 2.4. The equilibrium bid/ask in the presence of ambiguity is a convex combination of the standard probabilistic bid/ask and α-weighted payoff,

$$
B_{\alpha,\delta} = \left(1 - \delta^s\right) \cdot B + \delta^s \cdot \left(\alpha \cdot V_h + (1 - \alpha) \cdot V_l\right),
\tag{2.9}
$$

$$
A_{\alpha,\delta} = \left(1 - \delta^b\right) \cdot A + \delta^b \cdot \left((1 - \alpha) \cdot V_h + \alpha \cdot V_l\right),
\tag{2.10}
$$

and the bid-ask spread takes the form of

$$
S_{\alpha,\delta} = S + \left(\delta^b \cdot \left(\pi_l^b - \alpha\right) + \delta^s \cdot \left(\pi_h^s - \alpha\right)\right) \cdot \left(V_h - V_l\right),
\tag{2.11}
$$

where $S_{\alpha,\delta}$ and S denote the bid-ask spread in the presence and absence of ambiguity respectively.

The quotes of the ambiguity-averse market maker in Proposition 2.4 diverge from the standard quotes as ambiguity increases, leading the bid-ask spread to reflect more the ambiguity and ambiguity aversion of the market maker. In our ambiguous price formation model, there is an ambiguity premium effect on the standard bid-ask spread (i.e., $S_{\alpha,\delta} - S > 0$) when $\alpha \leq \min\{\pi_h^s, \pi_l^b\}$. Thus an ambiguity premium effect always prevails when the market maker is fully ambiguity averse ($\alpha = 0$). This is because, for the non-zero conditional probabilities, the bid $B_{\alpha,\delta}$ is lower than the standard bid B and the ask $A_{\alpha,\delta}$ is higher than the standard ask A, widening the bid-ask spread. However, there is also an ambiguity discount effect (i.e., $S_{\alpha,\delta} - S < 0$), in which the bid-ask spread with ambiguity is less than the bid-ask spread without ambiguity when $\alpha \geq \max\{\pi_h^s, \pi_l^b\}$. In this scenario, the bid is higher than the standard bid and the ask is lower than the standard ask quote, narrowing the spread.

In Figure 2.1, we illustrate the equilibrium bid/ask quotes and spreads in the presence and absence of ambiguity against the amount of ambiguity δ and ambiguity aversion α of the market maker. The bid/ask quotes and spread in the absence of ambiguity (fixed black layers) obviously do not respond to changes in ambiguity and ambiguity aversion of the market maker. The quotes and spread in the presence of ambiguity (curved layers), however, have two distinct areas, in which ambiguity premium and ambiguity discount on the standard bid and ask quotes and therefore on the standard spread prevail. An ambiguity premium adds premium over the standard spread, while an ambiguity discount reduces it due to the optimistic behavior of the market maker.

Panels (A)-(C) show that when there is no ambiguity ($\delta = 0$), the bid/ask quotes and spread in our model reduces to the standard quotes and spread. Additionally, Panel (A) shows that when exposed to full ambiguity ($\delta = 1$), the fully ambiguity-averse market maker ($\alpha = 0$) sets the highest ask, whereas the ambiguity-neutral ($\alpha = 0.5$) sets the lowest. In contrast, Panel (B) shows that when $\delta = 1$, the fully ambiguity-averse market maker sets the lowest bid, whereas the ambiguity-neutral sets the highest. Thus as illustrated in Panel (C), the highest and lowest bid-ask spreads correspond to the fully ambiguity-averse ($\alpha = 0$) and ambiguity-neutral ($\alpha = 0.5$) cases when the market maker has full ambiguity in her beliefs about the final payoff.
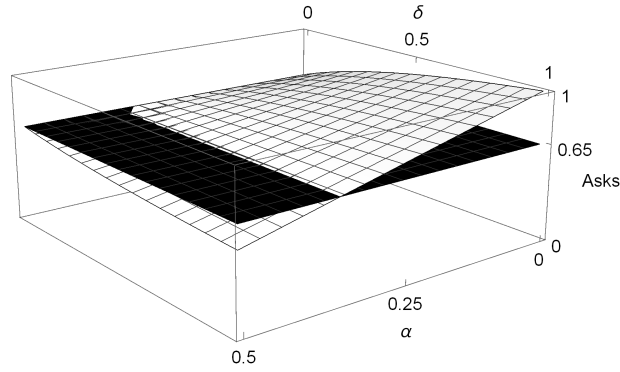
(A) Asks in the presence $A_{\alpha,\delta}$ (curved layer) and absence A (fixed layer) of ambiguity
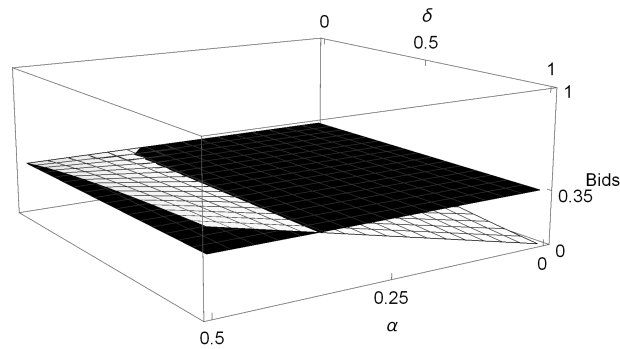
(B) Bids in the presence $B_{\alpha,\delta}$ (curved layer) and absence B (fixed layer) of ambiguity
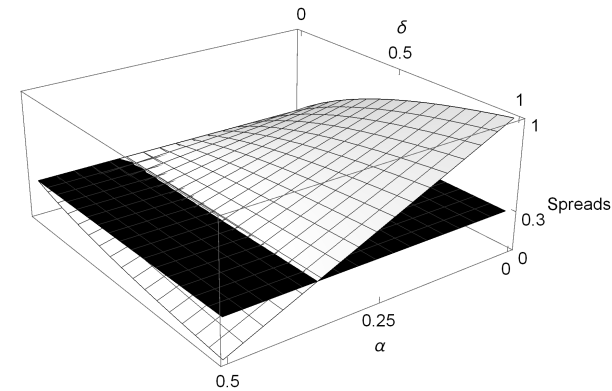
(C) Spreads in the presence $S_{\alpha,\delta}$ (curved layer) and absence S (fixed layer) of ambiguity

Figure 2.1: The relation between the bid/ask prices and spread in the presence (curved layers) and absence of ambiguity (fixed black layers) with respect to ambiguity, δ, and ambiguity aversion, α, of the market maker. Panel (A) plots bids, (B) plots asks and (C) plots the resulting bid-ask spreads. The parameter values are $V_l = 0, V_h = 1, \pi_l = \pi_h = 0.5$ and $\mu = 0.3$.

Panel (C) of the figure also shows that there is a combination of ambiguity δ^* and ambiguity aversion α^* that generates a bid-ask spread equal to the standard bid-ask spread (i.e., $S_{\alpha,\delta} = S$). We define this combination as a "bid-ask spread neutrality condition" and formally derive the condition in the following corollary. The condition sets an upper (resp. lower) bound for α that yields an ambiguity premium (resp. ambiguity discount) over the standard bid-ask spread. Put differently, the condition obtains the necessary and sufficient level of ambiguity aversion for the market maker to widen the bid-ask spread relative to the standard spread.

Corollary 2.5. In equilibrium, there is an ambiguity aversion α^* defined by

$$\alpha^* = w \cdot \pi_l^b + (1 - w) \cdot \pi_h^s, \quad w = \frac{\delta^b}{\delta^b + \delta^s}, \tag{2.12}$$

which equalizes the bid-ask spreads with and without ambiguity (i.e., $S_{\alpha,\delta} = S$) and divides the bid-ask spread with ambiguity into ambiguity premium, $\alpha < \alpha^*$, and ambiguity discount, $\alpha > \alpha^*$, areas.

Corollary 2.5 is very intuitive. Suppose the bid-ask spreads in the presence $S_{\alpha,\delta}$ and absence S of ambiguity given in Eq. (2.11) are, respectively, characterized by

$$S_{\alpha,\delta} = \beta \cdot \mu_{\alpha,\delta} \cdot (V_h - V_l) \quad \text{and} \quad S = \beta \cdot \mu \cdot (V_h - V_l), \tag{2.13}$$

where $\mu_{\alpha,\delta}$ is the market maker's perceived probability of informed trading after taking into account the ambiguity and $\beta = \frac{1-(1-2\cdot\pi_h)^2\cdot\mu}{1-(1-2\cdot\pi_h)^2\cdot\mu^2}$ is a parameter that follows from writing the conditional probabilities explicitly in the bid and ask (Eqs. (2.7) and (2.8)) and finding their difference. Combining Eqs. (2.11) and (2.13) obtains the perceived probability of informed trading as

$$\mu_{\alpha,\delta} = \mu + \frac{\delta^b \cdot \left(\pi_l^b - \alpha\right) - \delta^s \cdot \left(\alpha - \pi_h^s\right)}{\beta}. \tag{2.14}$$

Condition (2.12) is the necessary and sufficient condition that equalizes the perceived probability of informed trading $\mu_{\alpha,\delta}$ (probability of informed trading after accounting for ambiguity) and the actual probability of informed trading μ. The intuition is more clear by re-expressing Eq. (2.14) as

$$\mu_{\alpha,\delta} = \mu + \frac{\delta^b + \delta^s}{\beta} \cdot (\alpha^* - \alpha). \tag{2.15}$$

When $\alpha < \alpha^*$, the perceived probability of informed trading is greater than the probability of informed trading, leading to the ambiguity premium on the bid-ask spread. When $\alpha > \alpha^*$, however, the opposite prevails, leading to the ambiguity discount on the bid-ask spread.[6]

To see the intuition more clearly assume that the market maker initially has naive priors (i.e., $\pi_l = \pi_h = 0.5$) and the security payoffs are normalized (i.e., $V_l = 0$ and $V_h = 1$). It is well-known that in such a scenario, the bid $B = \pi_h^s = \left(\frac{1-\mu}{2}\right)$ and the ask $A = \pi_h^b = \left(\frac{1+\mu}{2}\right)$ lead to the bid-ask spread of the standard model given by the probability of informed trading, $S = \mu$ (this immediately follows from Eq. (2.13) since $\beta = 1$ when $\pi_l = \pi_h = 0.5$). This is because the bid-ask spread stems from the probability of informed trading due to the adverse selection risk of the market maker. In our model, the same argument holds. However, it is the market maker's perceived adverse selection risk, $\mu_{\alpha,\delta}$, that characterizes the spread. Substituting $\pi_l^b = \pi_h^s = \frac{1-\mu}{2}$ into Eq. (2.14) yields

$$\mu_{\alpha,\delta} = \mu + \delta^b \cdot (1 - 2\alpha - \mu) \tag{2.16}$$

where $\delta^b = \delta^s = \frac{2\cdot\delta}{1+\delta}$. Let $\phi \in [0, 1]$ be the normalized degree of ambiguity aversion given by $\phi = (1 - 2\cdot\alpha)$, where $\phi = 0$ is the case of no ambiguity aversion (ambiguity neutrality) and $\phi = 1$ is full ambiguity aversion. It follows from Eq. (2.13) that the spread is given by the market maker's perceived probability of informed trading, i.e.,

$$S_{\alpha,\delta} = \mu_{\alpha,\delta} = \mu + \delta^b \cdot (\phi - \mu). \tag{2.17}$$

When the market maker has no ambiguity ($\delta = \delta^b = 0$), the perceived probability of informed trading, $\mu_{\alpha,\delta}$, corresponds to the actual probability of informed trading, μ. When the information about the final payoff is fully ambiguous ($\delta = \delta^b = 1$), the bid-ask spread of the ambiguity-averse market maker is fully characterized by her degree of ambiguity aversion, ϕ. The spread in the non-extreme scenarios

---

[6]Note that the bid-ask spread neutrality condition only equalizes the bid-ask spreads with and without ambiguity, not the bid and ask quotes. In that sense, we can differentiate between the "strict form neutrality", where the bid, ask quotes with ambiguity (naturally the spread) are equal to the probabilistic bid, ask quotes and the "weak form neutrality", where only the bid-ask spreads are equal. The strict form bid-ask spread neutrality for the ambiguity-averse market maker is only attained when $\pi_l = \pi_h = 0.5$. The market maker can be a "weak form neutral" by increasing or decreasing the bid/ask mid price. The necessary and sufficient condition to be a "weak form neutral" by increasing (decreasing) the mid price is $\pi_l > 0.5$ ($\pi_l < 0.5$). That means, by changing the mid price, the ambiguity-averse market maker maintains the bid-ask spread with ambiguity the same as the probabilistic bid-ask spread even though the probabilistic beliefs suggest otherwise.

is determined by the interplay between the degree of ambiguity aversion of the market maker and the actual probability of informed trading. When the degree of ambiguity aversion exceeds the probability of informed trading, $\phi > \mu$, the perceived probability of informed trading is greater than the probability of informed trading, $\mu_{\alpha,\delta} > \mu$, and hence a wider bid-ask spread. There is, however, a flip side of the story, in which the market maker sets the spread lower than the standard spread when her degree of ambiguity aversion is less than the probabilistic belief about the informed trading. This occurs because she is optimistic (not as pessimistic) about the level of informed trading in the presence of ambiguity.

The current application of information ambiguity to financial markets mainly focuses on the ambiguity premium through non-participation of traders (e.g., Easley and O'Hara (2010a, 2010b)). While this approach has provided a number of important insights into financial market phenomena, our model shows that ambiguity premium effect is not the full picture and non-participation of traders is not the only economic channel. Our results suggest a link between the ambiguity, ambiguity aversion, and perceived adverse selection risk of liquidity providers. This link can explain the observed ambiguity premium (mainly during crash periods when liquidity providers are sufficiently pessimistic) and ambiguity discount (mainly during trading reforms when liquidity providers are sufficiently optimistic).

In Figure 2.2, we plot two extreme scenarios of informed trading (i.e., $\mu = 0$ in Panel (A) and $\mu = 1$ in Panel (B)) to emphasize the type of price inefficiency and liquidity distortion during extreme market events. In contrast to the standard models, Panel (A) shows that the perceived probability of informed trading can be very high due to liquidity providers' ambiguity even when the actual probability of informed trading is at its minimum ($\mu = 0$), which is useful in understanding the ambiguity premium effect during financial crashes. Panel (B) shows that the perceived probability of informed trading can be very low even when the actual probability of informed trading is at its maximum ($\mu = 1$), which is useful in understanding the ambiguity discount effect during reforms in trading rules.

We only observe the ambiguity premium in Panel (A) since $\phi \geq \mu = 0$ and ambiguity discount in Panel (B) since $\phi \leq \mu = 1$. An increase in the ambiguity δ of the market maker leads to an increase in the perceived probability of informed trading $\mu_{\alpha,\delta}$ in Panel (A) and a decrease in Panel (B). That is, when $\phi > \mu$, ambiguity increases the level of ambiguity premium, whereas when $\phi < \mu$, ambiguity increases the level of ambiguity discount. In extreme ambiguity, $\delta = 1$,

(A) $\mu = 0$ **(B)** $\mu = 1$
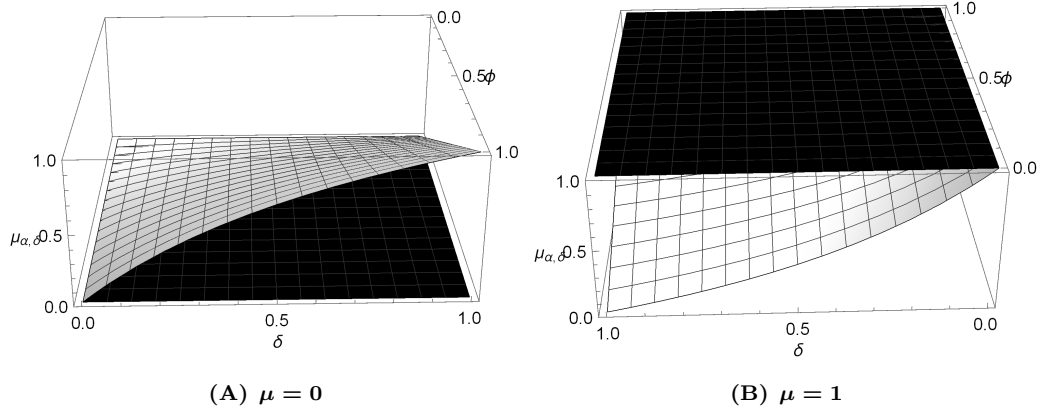
Figure 2.2: The relation between the perceived probability of informed trading (curved layer), $\mu_{\alpha,\delta}$, and the probability of informed trading (fixed black layer), μ, with respect to ambiguity, δ, and ambiguity aversion, ϕ, of the market maker. The parameter values are $V_h - V_l = 1, \pi_l = \pi_h = 0.5$.

which is almost always the case with all the financial crashes and controversial trading reforms, the liquidity is determined by the ambiguity aversion of the market maker, ϕ. Panels (A)-(B) also show that for the given level of ambiguity the spread is wider when the market maker is more ambiguity averse. Therefore, the liquidity dries up in crash episodes during which the liquidity providers face the extreme level of ambiguity and display extreme ambiguity-aversion, whereas the liquidity improves during trading reforms in which they also face ambiguity, but are optimistic toward this ambiguity. Formally we have the following corollaries.

Corollary 2.6. If the market maker is sufficiently ambiguity averse (i.e., $\alpha \leq \min\{\pi_h^s, \pi_l^b\}$);

   (i) there is an ambiguity premium on the bid-ask spread,

  (ii) incremental ambiguity increases the ambiguity premium,

 (iii) the magnitude of increase in the ambiguity premium is decreasing with ambiguity.

If the market maker is not sufficiently ambiguity averse $\alpha \geq \max\{\pi_h^s, \pi_l^b\}$;

   (i) there is an ambiguity discount on the bid-ask spread,

  (ii) incremental ambiguity increases the ambiguity discount,

 (iii) the magnitude of increase in the ambiguity discount is decreasing with ambiguity.

Corollary 2.6 provides an intuition about the evaporation of liquidity during the historical flight-to-liquidity episodes such as shorting ban, transaction taxes or in extreme, market crashes and liquidity improvements during the historical trading reforms such as permitting the public to compete directly with dealers and displaying the superior quotes placed by dealers in private trading venues to public in Nasdaq. It suggests a possible channel of the extreme liquidity dry-ups and liquidity improvements during uncertain market events. Corollary 2.6 establishes the impact of ambiguity on the bid-ask spread and squares with the intuition and empirical observations during uncertain market events.

The intuition that the presence of ambiguity induces an ambiguity premium is not new in the literature. Ju and Miao (2012), by three-way separation among risk aversion, ambiguity aversion and intertemporal substitution, calibrate an asset pricing model to match the mean equity premium of asset prices by adding an ambiguity premium. Epstein and Schneider (2008) show that investors require compensation for low future information quality, and therefore, expected excess returns are higher in the presence of ambiguity. The contribution of this chapter lies in delineating the roles of ambiguity and ambiguity aversion of the market maker in distorting the liquidity and generating aberrant market behavior during uncertain market events through the channel of adverse selection by providing the full picture of ambiguity premium and discount. At first, an ambiguity discount effect may seem intuitively dubious concept. One interesting aspect of this result, however, is that ambiguity can provide a unified approach in explaining the positive or negative liquidity distortions in financial markets and lead to interesting financial market phenomena. The next corollary relates the liquidity distortions to the ambiguity aversion of liquidity providers.

Corollary 2.7. For any non-zero level of ambiguity, $\delta > 0$, a higher ambiguity-aversion leads to a higher bid-ask spread.

Corollary 2.7 is consistent with the intuition that the liquidity providers' ambiguity aversion leads to an increase in the bid-ask spread. Combining Corollaries 2.6 and 2.7, we obtain, in extreme ambiguity, a fully ambiguity-averse market maker sets the maximum spread $(V_h - V_l)$. Since there is no room for exploiting private information, in that scenario, the informed traders stop trading and the trading volume collapses. This is indeed what has happened during the recent global financial crisis. Although market makers continued to post bid and ask prices on

mortgage-backed securities and collateralized debt obligations, the trading volume of these securities decreased substantially (Easley and O'Hara (2010*a*)). We conclude this subsection by examining some important special cases of our model before investigating the impacts of trading histories on the evolution of prices and spread.

Case 1 ($\delta = 0$): When there is no ambiguity, the bid and ask quotes reduce to the probabilistic bid and ask quotes. This is because the market maker's perceived probability of informed trading, $\mu_{\alpha,\delta}$, reduces to the actual probability of informed trading, μ.

Case 2 ($\alpha = 0$, $\delta > 0$): This case represents the quotes of the fully ambiguity-averse market maker in the presence of ambiguity. It follows from Eqs. (2.9) and (2.10) that, for non-zero level of the conditional probabilities, the bid is lower than the standard bid and the ask is greater than the standard ask (i.e., $B_{\alpha,\delta} < B$ and $A_{\alpha,\delta} > A$). Following Corollary 2.6, an incremental ambiguity adds an ambiguity premium over the bid-ask spread by reducing the bid and increasing the ask.

Case 3 ($\delta = 1$): In this case, the quotes of the ambiguity-averse market maker is characterized only by her degree of ambiguity aversion,

$$B_{\alpha,\delta} = \alpha \cdot V_h + (1 - \alpha) \cdot V_l, \quad \text{and} \quad A_{\alpha,\delta} = (1 - \alpha) \cdot V_h + \alpha \cdot V_l. \quad (2.18)$$

Put differently, the market maker posts quotes fully based on her optimism and pessimism toward the ambiguity irrespective of the intensity of informed trading in the market.

Case 4 ($\delta = 1$ and $\alpha = 0$): The results obtained in this case resemble what is indeed observed during the market crashes. In this case, the market maker has no confidence in her probability assessment of the final payoff (i.e., $v_l = v_h = 0$). Therefore, she sets the minimum bid, $B_{\alpha,\delta} = V_l$, and the maximum ask, $A_{\alpha,\delta} = V_h$. In probabilistic beliefs, a little evidence supporting V_l implies a large amount of evidence supporting V_h. This is not necessarily the case in our representation of the beliefs about the final payoff. In Case 4, the market maker does not have any evidence (or she has inconsistent evidence) supporting V_l or V_h (i.e., $v_l = v_h = 0$).

Case 5 ($\alpha = 0.5$): The market maker is ambiguity neutral in the sense that she posts the quotes as if she has a probabilistic belief since there is no probability gap (i.e., $\delta \cdot (1 - 2 \cdot \alpha) = 0$).

Case 6 ($\delta = 1$ and $\alpha = 0.5$): This is a combination of Cases 3 and 5, in which the market maker is ambiguity neutral in the presence of full ambiguity. In this case, the conditional beliefs about the risky payoff always correspond to $v_h^s = 0.5$ and $v_l^b = 0.5$. Therefore, the bid and ask of the market maker converge to $\frac{(V_l + V_h)}{2}$ with zero bid-ask spread.

2.4.2 Effects of the trading history on the evolution of quotes and spread

The effects described thus far focus on ambiguity and ambiguity aversion of the market maker on the initial quotes and spread, and therefore are static in the sense that they do not depend on the past trading history. We now investigate the dynamic behavior of the market maker's quotes and spread in the presence of ambiguity conditional on the trading history. To do this we assume naive initial priors at $t = 1$ and normalized security payoffs (i.e., $\pi_{l,1} = \pi_{h,1} = 0.5$ and $\hat{V} \in \{0, 1\}$).

Let D_t denote the trade direction, $D_t = -1$ for a sell, $D_t = +1$ for a buy, and P_t denote the transaction price at time t. Public information at time t consists of the sequence of past buys and sells and their transaction prices, denote by $h_t = \{D_\tau, P_\tau\}_{\tau=1}^{t-1}$ for $t > 1$. The other variables are denoted the same as in the static analysis, the only difference being a t subscript to denote the variable of interest at time t. Similar to the initial quotes (see Eqs. (2.9 and 2.10)), the bid and ask quotes of the market maker at time t are given by

$$B_{\alpha,\delta,t} = E_v[\hat{V} = 1 | h_t, D_t = -1] = \left(1 - \delta_t^s\right) \cdot B_t + \delta_t^s \cdot \alpha, \qquad (2.19)$$

$$A_{\alpha,\delta,t} = E_v[\hat{V} = 1 | h_t, D_t = +1] = \left(1 - \delta_t^b\right) \cdot A_t + \delta_t^b \cdot (1 - \alpha), \qquad (2.20)$$

and the standard bid and ask quotes are

$$B_t = E[\hat{V} = 1 | h_t, D_t = -1] = \Pr\{\hat{V} = 1 | h_t, D_t = -1\} = \frac{\pi_{h,t}}{\pi_{h,t} + \frac{1+\mu}{1-\mu} \cdot (1 - \pi_{h,t})}, \quad (2.21)$$

$$A_t = E[\hat{V} = 1 | h_t, D_t = +1] = \Pr\{\hat{V} = 1 | h_t, D_t = +1\} = \frac{\pi_{h,t}}{\pi_{h,t} + \frac{1-\mu}{1+\mu} \cdot (1 - \pi_{h,t})}, \quad (2.22)$$

where

$$\pi_{h,t} = \Pr\{\hat{V} = 1 | h_t\} = \frac{\Pr\{h_t | \hat{V} = 1\}}{\Pr\{h_t\}} \cdot \Pr\{\hat{V} = 1\} = \frac{1}{1 + \left(\frac{1+\mu}{1-\mu}\right)^{b_t - s_t}} \qquad (2.23)$$

follows from the Bayes' rule given the difference between number of buys b_t and sells s_t (i.e., order imbalance) up to time t, and δ_t^s and δ_t^b follow from the iterative applications of the generalized Bayesian updating given the past order flow.

In Figure 2.3, we plot the dynamics of the quotes and spread using Eqs. (2.19)-(2.23) during two order flow patterns — continuous sells and balanced orders. Panel (A) plots the quotes and spread during continuous sell orders when the market maker is fully ambiguity averse ($\alpha = 0$) and ambiguity neutral ($\alpha = 0.5$) and contrasts them with the standard model. Panel (B) plots the same when the market maker receives perfectly balanced order flow (we provide a spreadsheet to investigate different order flow patterns in Appendix 2.5).

Panel (A) illustrates that in the standard model, continuous sell orders reduce the ask more than the bid. This improves liquidity since selling pressure reveals private information that the fundamental value is low (informed traders all tend to sell when prices are too high), thus reduces uncertainty about the fundamental value, and thereby increases liquidity (narrows bid-ask spread) over time — an effect that drives price discovery in the standard model. These standard results are at odds with what we observe in financial markets during large selling pressure during which market practitioners, in particular algorithmic market makers, often withdraw their quotes in the face of selling pressure, making markets less liquid and more volatile (e.g., Chordia et al. (2002), Anand and Venkataraman (2016)). The experience of the U.S. financial markets on May 6, 2010, ("Flash Crash") and treasury markets on October 15, 2014, ("Flash Rally") are the recent extreme examples.

While the standard price discovery effect is also present in our model, an additional effect emerges due to the evolution of ambiguity and the market maker's ambiguity attitude. Unlike the paradoxical result in the standard model, Panel (A) in Figure 2.3 shows that, in our model, when the market maker is fully-ambiguity averse ($\alpha = 0$), she reduces the bid and increases the ask, leading to a liquidity deterioration in the face of selling pressure, similar to those observed empirically during flash crashes (e.g., CFTC-SEC (2010a, 2010b)). In contrast, when the market maker is ambiguity neutral ($\alpha = 0.5$), initially there is an ambiguity discount on the standard spread which can switch to an ambiguity premium during the selling pressure. The reason for both of these effects is illustrated in Panel (A) of Figure 2.4. When the fully ambiguity-averse market maker ($\alpha = 0$) receives a sequence of sell orders, she revises ambiguity (δ_t^s and δ_t^b) upward over time and the dynamic

(A) Continuous sell orders

(B) Perfectly balanced orders
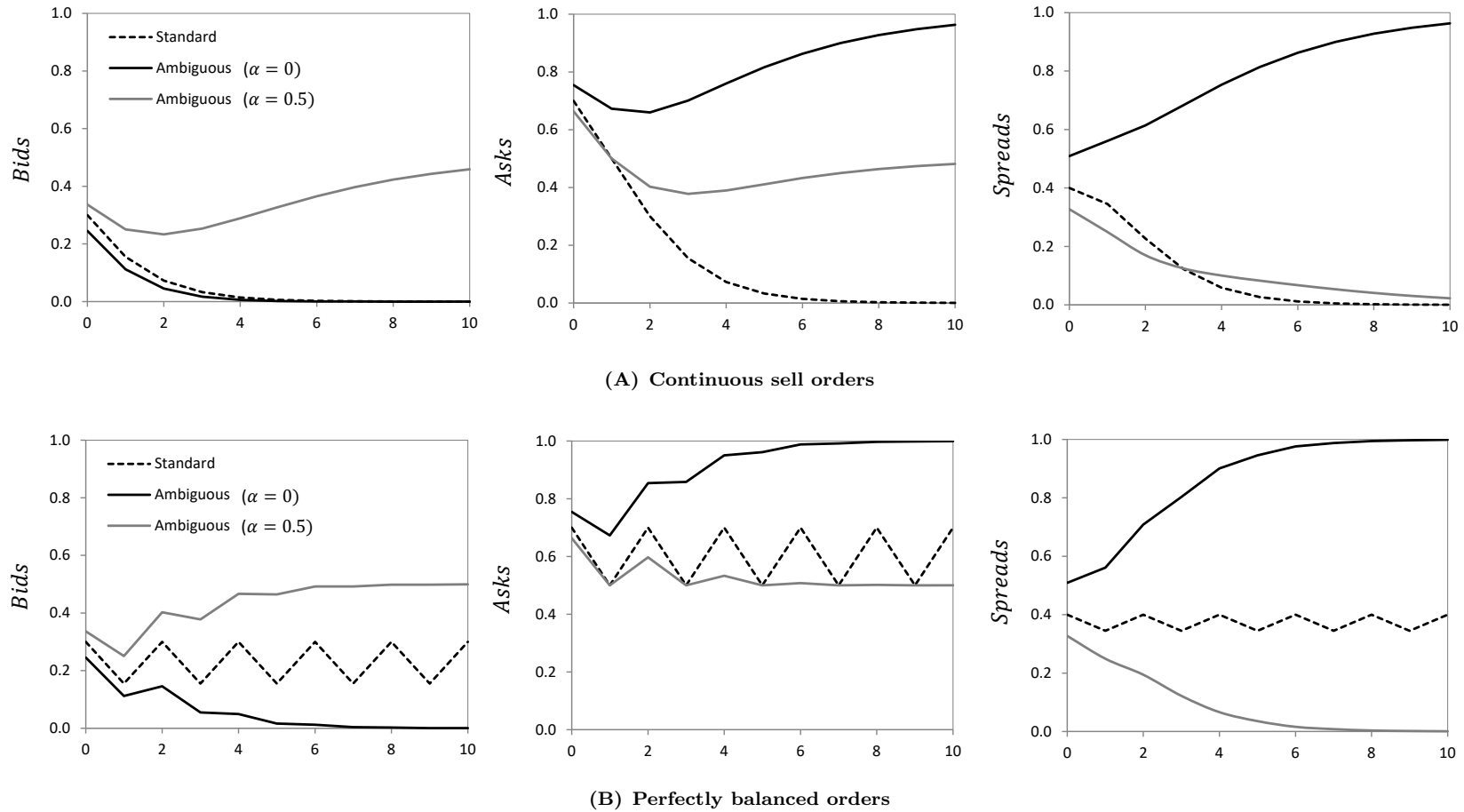
Figure 2.3: Panel (A) plots the bids, asks and spreads of the market maker that receives continuous sell orders up to time $t = 10$ in the standard model (dashed line) and in the presence of ambiguity when she is fully ambiguity averse ($\alpha = 0$) and ambiguity neutral ($\alpha = 0.5$). Panel (B) plots the same variables when the market maker receives perfectly balanced orders ($D_1 = -1$, $D_2 = 1$, $D_3 = -1$, ..., $D_{10} = 1$). The other parameter values are $V_l = 0, V_h = 1, \pi_{l,1} = \pi_{h,1} = 0.5, \delta_1 = 0.1$ and $\mu = 0.4$.

version of the sufficient ambiguity aversion condition in Corollary 2.5, i.e.,

$$\alpha < \alpha_t^* = \frac{\delta_t^b \cdot \Pr\{\hat{V} = 0 | h_t, D_t = +1\} + \delta_t^s \cdot \Pr\{\hat{V} = 1 | h_t, D_t = -1\}}{\delta_t^b + \delta_t^s} \tag{2.24}$$

is always satisfied, resulting in an increase in the bid-ask spread during the selling pressure.[7] When $\alpha = 0.5$, a selling pressure can make α_t^* to switch from $\alpha_t^* < \alpha$ to $\alpha_t^* > \alpha$, leading to a switch from an ambiguity discount to an ambiguity premium. In essence, $\hat{\alpha}_t = \alpha_t^* - \alpha$ is the "effective" time-varying ambiguity aversion of the market maker. When $\hat{\alpha}_t > 0$ (resp. $\hat{\alpha}_t < 0$), the market maker is ambiguity averse (resp. ambiguity seeking) with an ambiguity premium (resp. ambiguity discount) on the standard spread. The effective ambiguity aversion $\hat{\alpha}_t$ of the market maker increases during selling pressure, leading to a switch from an ambiguity discount to an ambiguity premium on the standard spread.

(A) Continuous sell orders

(B) Perfectly balanced orders
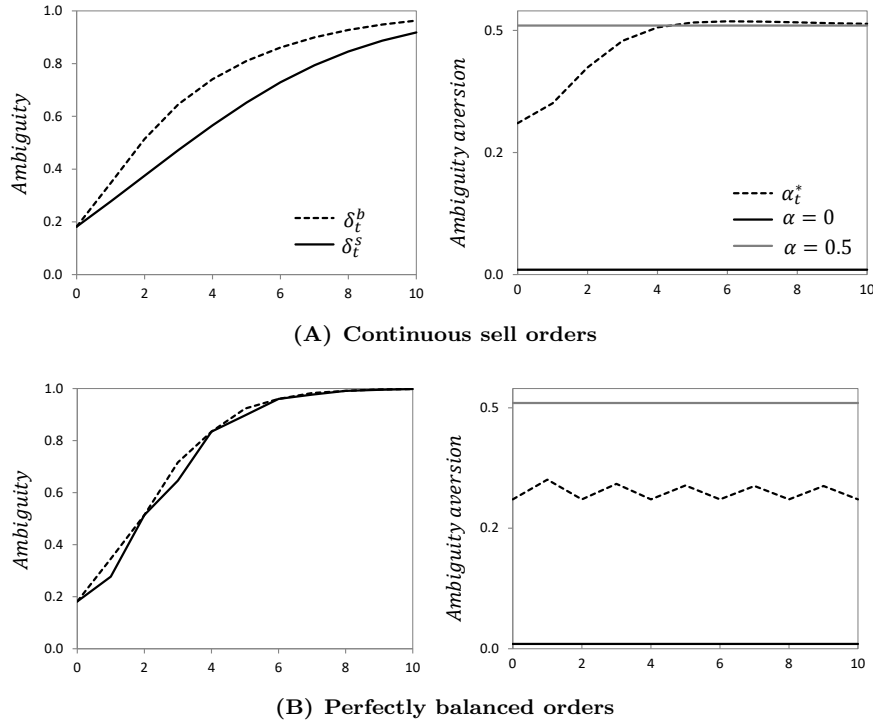
Figure 2.4: Panel (A) plots the evolution of the market maker's ambiguity conditional on a buy δ_t^b and a sell δ_t^s and the sufficient ambiguity aversion condition α_t^* over time during a sequence of sell orders up to time $t = 10$. Panel (B) plots the same during perfectly balanced order flow. The parameter values are the same as in Figure 3.

---

[7]During the sell sequence the revised ambiguity conditional on a buy δ_t^b is higher than the ambiguity conditional on a sell δ_t^s since a sequence of sell orders increases the probability of a sell and decreases the probability of a buy (recall from Lemma 2.2 that the more likely it was for an order to occur probabilistically, the updated ambiguity conditional on that order will be lower). The opposite is true when there is a buy sequence.

The prediction of our model is also different from the standard model during balanced order flow. Panel (B) in Figure 2.3 shows that, in the standard model with no ambiguity, balanced orders reveal no new information, and thus has no effect on prices or liquidity (i.e., $B_1 = B_{10}$, $A_1 = A_{10}$ and $S_1 = S_{10}$). However, balanced orders lead to a liquidity improvement when liquidity providers are optimistic about their belief assessments, whereas to a liquidity deterioration when they are pessimistic. Panel (B) in Figure 2.4 complements the quotes and spread of the market maker by plotting the dynamics of ambiguity and sufficient ambiguity aversion condition during balanced order flow. Similar to continuous sell orders, balanced orders increase the amount of ambiguity (i.e., δ_t^b and δ_t^s). In contrast to continuous sell orders, balanced orders do not change the effective ambiguity aversion $\hat{\alpha}_t$ of the market maker.

2.5 The implications for value of information and welfare

We now investigate how the market maker's ambiguity about the final payoff and her ambiguity aversion impact the value of information to the informed traders and the welfare to society. We focus on the naive initial priors and normalized security payoffs (i.e., $\pi_l = \pi_h = 0.5$ and $\hat{V} \in \{0, 1\}$) and the initial bid/ask quotes and spread of the market maker.

2.5.1 Value of information

There are two possible scenarios for the informed traders. First, when the true state of the final payoff is $\hat{V} = 1$, they buy one unit of the asset and obtain a profit of $1 - A_{\alpha,\delta}$. Second, when the true state is $\hat{V} = 0$, they sell one unit of the asset and obtain a profit of $B_{\alpha,\delta}$. Summing over the two possible scenarios with probabilities π_h and $1 - \pi_h$ obtains the value of information (or the expected profits of informed traders) as

$$V_{\alpha,\delta} = (1 - \pi_h) \cdot B_{\alpha,\delta} + \pi_h \cdot (1 - A_{\alpha,\delta}). \tag{2.25}$$

Substituting Eqs. (2.9) and (2.10) into Eq. (2.25) obtains the value of information. We decompose the value of information into the standard and ambiguous components in the following proposition.

Proposition 2.8. In the presence of the market maker's ambiguity, the value of information about the final payoff is

$$V_{\alpha,\delta} = K + K_{\alpha,\delta}, \tag{2.26}$$

where

$$K = (1 - \pi_h) \cdot B + \pi_h \cdot (1 - A) \tag{2.27}$$

and

$$K_{\alpha,\delta} = \delta^s \cdot (1 - \pi_h) \cdot (\alpha - \pi_h^s) + \delta^b \cdot \pi_h \cdot (\alpha - \pi_l^b) \tag{2.28}$$

are the standard and ambiguous components of the value of information.

The standard value of learning the realization of $\hat{V}$ (i.e., K in Eq. (2.27)) is always positive and decreasing with the probability of informed trading μ, meaning, all else equal, the greater the number of informed traders the less information is valuable for each informed trader. The ambiguous component (i.e., $K_{\alpha,\delta}$ in Eq. (2.28)) can be negative and increasing with the probability of informed trading μ. In fact, the sign of the ambiguous component is dependent on the market maker's ambiguity aversion. When the market maker is sufficiently ambiguity averse (i.e., $\alpha \leq \min(\pi_h^s, \pi_l^b)$), the ambiguous component is negative. This occurs because the sufficiently ambiguity-averse market maker widens the bid-ask spread that results in the expected profits of the informed traders to decrease. The opposite prevails when the insufficiently ambiguity-averse market maker (i.e., $\alpha \geq \max(\pi_h^s, \pi_l^b)$) narrows the bid-ask spread.

Figure 2.5 illustrates the value of information in the presence and absence of ambiguity against the amount of ambiguity δ and ambiguity aversion α of the market maker. Consistent with the changes in liquidity illustrated in Figure 2.1, in the figure, the highest and lowest value of information correspond to the ambiguity-neutral ($\alpha = 0.5$) and fully ambiguity-averse ($\alpha = 0$) cases in the presence of full ambiguity ($\delta = 1$). The value of information attains its lowest (resp. highest) value when the market is the most illiquid (resp. liquid) due to the ambiguity premium (resp. ambiguity discount) on the standard spread.
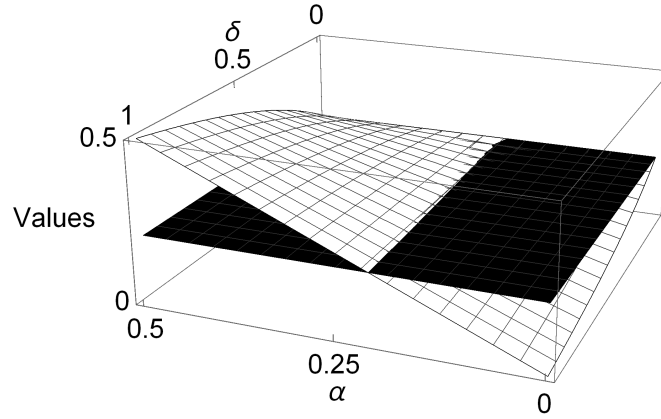
Figure 2.5: The relation between the value of information in the presence of ambiguity (curved layer), $V_{\alpha,\delta}$, in the absence of ambiguity (fixed black layer), K, with respect to ambiguity, δ, and ambiguity aversion, α, of the market maker. The parameter values are $\pi_l = 0.35$ and $\mu = 0.55$.

When $\pi_l = \pi_h = 0.5$ and $\hat{V} \in \{0,1\}$, the revised ambiguity and conditional probabilities ($\delta^b = \delta^s = \left(\frac{2\delta}{1+\delta}\right)$ and $\pi_l^b = \pi_h^s = \left(\frac{1-\mu}{2}\right)$) and the standard bid and ask quotes ($B = \left(\frac{1-\mu}{2}\right)$ and $A = \left(\frac{1+\mu}{2}\right)$) lead the standard and ambiguous components to be

$$K = \frac{1-\mu}{2} \quad \text{and} \quad K_{\alpha,\delta} = \frac{\delta}{1+\delta} \cdot (\mu - \phi), \tag{2.29}$$

where $\phi = (1 - 2 \cdot \alpha)$. When the market maker is sufficiently ambiguity averse (i.e., $\phi > \mu$), she widens the spread, leading to the negative ambiguous component. Conversely, when the market maker is not sufficiently ambiguity averse (i.e., $\phi < \mu$), she narrows the spread, leading to the positive ambiguous component. Consequently, the opposites of Corollaries 2.6 and 2.7 apply to the value of information. For completeness, we formally state them in the following corollaries.

Corollary 2.9. If the market maker is sufficiently ambiguity averse (i.e., $\alpha \leq \min\{\pi_h^s, \pi_l^b\}$);

   (i) there is a value discount on the value of information,

   (ii) incremental ambiguity increases the value discount,

   (iii) the magnitude of increase in the value discount is decreasing with ambiguity.

If the market maker is not sufficiently ambiguity averse $\alpha \geq \max\{\pi_h^s, \pi_l^b\}$;

   (i) there is a value premium on the value of information,

(ii) incremental ambiguity increases the value premium,

(iii) the magnitude of increase in the value premium is decreasing with ambiguity.

Corollary 2.10. For any non-zero level of ambiguity, $\delta > 0$, a higher ambiguity-aversion leads to a lower value of information.

2.5.2 Welfare

We now model uninformed traders as fully maximizing agents trading for noninformational motives with elastic demand sensitive to trading costs (i.e., endogenize uninformed trading) for two main reasons. First, we want to ensure that the liquidity deteriorations and improvements in the baseline model are robust to the exogenous uninformed trading assumption. Second, when the uninformed traders act as fully maximizing agents, sometimes they may choose not to trade because their valuation lies inside the bid and ask quotes, resulting in a welfare loss to society. Therefore, we are able to explicitly study the welfare implications of the ambiguity and ambiguity attitude of liquidity providers.

The standard device to endogenize uninformed trading is modeling them as hedgers with a risk exposure correlated with the security payoff, leading to a distribution of private valuation of the security (e.g., Spiegel and Subrahmanyam (1992), Glosten and Putnins (2016)). We assume that the private valuation Ω of an uninformed trader who arrives at the market with a probability of $1 - \mu$ is uniformly distributed (i.e., $F(\omega) = \omega$ for $0 \leq \omega \leq 1$).[8] Given the uniformly distributed private valuations, an uninformed trader arriving at the market buys (resp. sells) if his private valuation is greater (resp. less) than the ask (resp. bid), and otherwise does not trade. That is, he buys with a probability

$$\Pr\{\omega > A_{\alpha,\delta}\} = 1 - F(A_{\alpha,\delta}) = 1 - A_{\alpha,\delta}, \tag{2.30}$$

sells with a probability

$$\Pr\{\omega < B_{\alpha,\delta}\} = F(B_{\alpha,\delta}) = B_{\alpha,\delta}, \tag{2.31}$$

---

[8]Glosten and Putnins (2016) provide a micro foundation (a utility function and an endowment shock) for the general distribution of private valuations which reduces to the uniform distribution when $\pi_l = \pi_h = 0.5$.

and does not trade with a probability

$$\Pr\{B_{\alpha,\delta} \le \omega \le A_{\alpha,\delta}\} = F(A_{\alpha,\delta}) - F(B_{\alpha,\delta}) = A_{\alpha,\delta} - B_{\alpha,\delta}. \tag{2.32}$$

Unlike the exogenously specified uninformed trading, in this setting, the amount of informed trading determines the bid/ask quotes and spread, which in turn determine whether an uninformed trader chooses to trade. Eq. (2.32) shows that when the ask $A_{\alpha,\delta}$ is higher and the bid $B_{\alpha,\delta}$ is lower (i.e., the spread is wider), it is more likely that uninformed chooses not to trade. The equilibrium follows from the profit maximizations of informed (i.e., buy when $\Theta = H$ and sell when $\Theta = L$) and uninformed traders (i.e., buy with a probability of $1 - A_{\alpha,\delta}$, sell with a probability of $B_{\alpha,\delta}$ and no trade with a probability of $A_{\alpha,\delta} - B_{\alpha,\delta}$) and zero-expected-profit condition of the market maker (i.e., $B_{\alpha,\delta} = E_v[V|s]$ and $A_{\alpha,\delta} = E_v[V|b]$).

Proposition 2.11. (i) In the absence of ambiguity, the market maker's bid B and ask A are respectively given by

$$A = \frac{1}{2 \cdot (1 - \mu)}, \tag{2.33}$$

$$B = \frac{1 - 2\mu}{2 \cdot (1 - \mu)}, \tag{2.34}$$

and the bid-ask spread S takes the form of

$$S = \frac{\mu}{1 - \mu}. \tag{2.35}$$

(ii) In the presence of ambiguity, the market maker's bid $B_{\alpha,\delta}$ and ask $A_{\alpha,\delta}$ are respectively given by

$$A_{\alpha,\delta} = \frac{3 - \delta \cdot (1 - \mu)^2 - \mu \cdot (4 - \mu) - \sqrt{\gamma}}{4 \cdot (1 - \delta) \cdot (1 - \mu)^2}, \tag{2.36}$$

$$B_{\alpha,\delta} = \frac{1 - 3\delta \cdot (1 - \mu)^2 - \mu \cdot (4 - 3\mu) + \sqrt{\gamma}}{4 \cdot (1 - \delta) \cdot (1 - \mu)^2}, \tag{2.37}$$

and the bid-ask spread $S_{\alpha,\delta}$ takes the form of

$$S_{\alpha,\delta} = \frac{1}{2} \cdot \left(\frac{2 \cdot (1 - \mu) - \sqrt{\gamma}}{(1 - \delta) \cdot (1 - \mu)^2} - 1 \right) \tag{2.38}$$

where γ is given in Eq. (A2.1.27).

Proposition 2.11 shows that, in the absence of ambiguity, the market is open for $\mu < \frac{1}{2}$. This is because when the informed traders have perfect signals, for $\mu \geq \frac{1}{2}$ there are no quotes that will allow trade and nonnegative expected profits to the market maker.[9] In the presence of ambiguity, however, the market is open when $\mu < \mu^*$, where μ^* is given in Eq. (A2.1.28) in Appendix 2.1. When the market maker is fully ambiguity averse ($\phi = 1$), the market is open at $\mu < \frac{1}{2}$ (i.e., $\mu^* = \frac{1}{2}$) irrespective of ambiguity δ. That is, the pessimistic market maker always closes the market for the high informed trading. When the market maker is ambiguity neutral ($\phi = 0$), the market is open at $\mu < 1$ when $\delta = 1$ and $\mu < \frac{1}{2}$ when $\delta = 0$. That is, the optimistic market maker leaves the market open for the high informed trading when ambiguity is high. We focus on the open market. The following corollary ensures that the effects of ambiguity and ambiguity aversion of the market maker on the spread are robust to the exogenous uninformed trading assumption in the baseline model.

Corollary 2.12. (i) Higher ambiguity increases (resp. decreases) the spread when the market maker is sufficiently (resp. insufficiently) ambiguity averse; that is, $\frac{\partial S_{\alpha,\delta}}{\partial \delta} > 0$ when $\phi > S$ and $\frac{\partial S_{\alpha,\delta}}{\partial \delta} < 0$ when $\phi < S$. (ii) Higher ambiguity aversion always increases the spread; that is, $\frac{\partial S_{\alpha,\delta}}{\partial \phi} > 0$.

The welfare effects of ambiguity arise because ambiguity can increase or decrease the bid-ask spread, and consequently affect the uninformed trading decisions. More precisely, the wider (resp. narrower) spread compared to the standard spread leads the uninformed traders more (resp. less) likely to refrain from trading compared to the standard model (the uninformed traders refrain from trading when their private valuation ω lies inside the bid and ask quotes). The welfare effects arise due to this lack or excess of uninformed trading, constituting a loss or gain to society stemming from the market maker's ambiguity. The welfare effects of ambiguity is calculated by the difference between the total welfare of all participants in the presence and absence of ambiguity.

The welfare gain of an informed trader is $1 - A_{\alpha,\delta}$ when $\Theta = H$ and $B_{\alpha,\delta}$ when $\Theta = L$. Similarly, the welfare gain of an uninformed trader is $\omega - A_{\alpha,\delta}$ when $\omega > A_{\alpha,\delta}$ and $B_{\alpha,\delta} - \omega$ when $\omega < B_{\alpha,\delta}$. The market maker loses when trading

---

[9]When the information signal of the informed traders are not perfectly informative, the market is open as long as $\mu < 1$. Imperfect information signal produces qualitatively similar results, and therefore in the interests of simplicity, we focus on the perfect information signal in this section. We extend the baseline model to account for imperfect information signals in Appendix 2.3.

against an informed, but gains when trading against an uninformed trader. More precisely, the welfare of the market maker trading against an informed trader is $A_{\alpha,\delta} - 1$ when $\Theta = H$ and $-B_{\alpha,\delta}$ when $\Theta = L$. Similarly, the welfare of the market maker trading against an uninformed trader is $A_{\alpha,\delta} - 0.5$ when $\omega > A_{\alpha,\delta}$ and $0.5 - B_{\alpha,\delta}$ when $\omega < B_{\alpha,\delta}$ since $\pi_l = \pi_h = 0.5$. The total welfare of market participants are the sum of these components; the net welfare $W_{\alpha,\delta} = \omega - 0.5$ when $\omega > A_{\alpha,\delta}$ and $W_{\alpha,\delta} = 0.5 - \omega$ when $\omega < B_{\alpha,\delta}$, only arising when the arriving trader is an uninformed trader. Thus the total welfare in the presence of ambiguity is given by

$$W_{\alpha,\delta} = \mathbf{1}_u[(\omega - 0.5) \cdot \mathbf{1}_{\{\omega > A_{\alpha,\delta}\}} + (0.5 - \omega) \cdot \mathbf{1}_{\{\omega < B_{\alpha,\delta}\}}], \qquad (2.39)$$

where $\mathbf{1}_u$ is an indicator function for the arrival of uninformed and $\mathbf{1}_{\{\cdot\}}$ is an indicator function for $\{\cdot\}$. Similarly, the total welfare in the absence of ambiguity is

$$W = \mathbf{1}_u[(\omega - 0.5) \cdot \mathbf{1}_{\{\omega > A\}} + (0.5 - \omega) \cdot \mathbf{1}_{\{\omega < B\}}]. \qquad (2.40)$$

The welfare effect of ambiguity is given by

$$\Delta W = W_{\alpha,\delta} - W \qquad (2.41)$$

and implies a welfare gain (resp. loss) when $\Delta W > 0$ (resp. $\Delta W < 0$). The following proposition characterizes the expected welfare change stemming from the market maker's ambiguity.

Proposition 2.13. The expected welfare change due to the presence of ambiguity is given by

$$\mathrm{E}[\Delta W] = (1 - \mu) \cdot \left(\int_{A_{\alpha,\delta}}^{A} (w - 0.5) \cdot dw + \int_{B}^{B_{\alpha,\delta}} (0.5 - w) \cdot dw \right), \qquad (2.42)$$

where the expectation is conditional on $\pi_l = \pi_h = 0.5$.

Evaluating Eq. (2.42) obtains Eq. (A2.1.33) in Appendix 2.1. which shows that ambiguity of the market maker can improve or deteriorate the total welfare of the market participants. Similar to the value of information, this is also in line with the change in liquidity. Ambiguity of the liquidity providers can improve (resp. deteriorate) the liquidity in the market resulting in the excess (resp. lack) of uninformed trading compared to the standard model. It is this excess (resp. lack) of uninformed trading that constitutes a gain (resp. loss) to society due to

the liquidity providers' ambiguity. Figure 2.6 illustrates the welfare gain and loss against ambiguity and ambiguity aversion of the market maker.
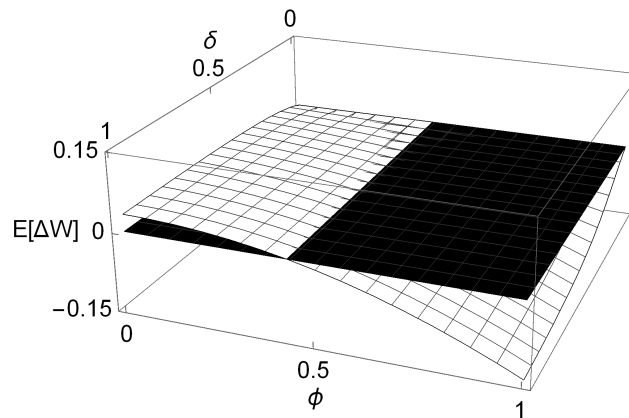
Figure 2.6: The relation between the expected welfare change, $E[\Delta W]$, and ambiguity, δ, and ambiguity aversion, $\phi = 1 - 2 \cdot \alpha$, of the market maker when uninformed traders trade endogenously. The parameter values are $V_l = 0, V_h = 1, \pi_l = \pi_h = 0.5$, and $\mu = 0.3$.

The figure shows that when the market is sufficiently (resp. insufficiently) ambiguity averse, ambiguity reduces (resp. increases) the welfare of the market participants. The intuition of this result follows from the liquidity effects of the market maker's ambiguity and ambiguity aversion. When the market maker is sufficiently ambiguity averse, the liquidity deterioration causes a welfare loss to society due to a lack of uninformed trading, whereas when the market maker is insufficiently ambiguity averse, the liquidity improvement causes a welfare gain to society due to an excessive uninformed trading compared to the standard model. Formally, we have the following corollary.

Corollary 2.14. (i) Ambiguity results in a welfare gain (resp. loss) when the market maker is insufficiently (resp. sufficiently) ambiguity averse; that is, $E[\Delta W] > 0$ when $\phi < S$ and $E[\Delta W] < 0$ when $\phi > S$. (ii) Higher ambiguity aversion of the market maker always reduces the welfare; that is, $\frac{E[\Delta W]}{\partial \phi} < 0$.

Similar to the ambiguity discount on the spread, a welfare gain of ambiguity may seem counter-intuitive. By extending Rothschild and Stiglitz (1976) model of competitive insurance market, Koufopoulos and Kozhan (2014) also present an example where an increase in ambiguity can increase the utility of low-risk insurees, and ultimately result in a strict Pareto improvement. In our model, a welfare gain occurs when the optimistic market maker narrows the spread in the presence of

ambiguity, reflecting the empirical observations during reforms in trading rules. This reduces the lack of uninformed trading and result in a gain from trade, and thus a welfare gain to society.

2.6 Ambiguous Price-Quantity Analysis

In this section, we examine how ambiguity and ambiguity aversion of the market maker affect prices and the trading strategies of informed traders when they can trade different quantities. To do this, we enrich the ambiguous price formation model described in Section 2.3 by allowing the traders to trade two different quantities for both buys with $0 < b_1 < b_2$ and sells with $0 < s_1 < s_2$. The market maker sets the quotes by taking into account that the traders can trade different quantities, the informed traders' strategy is dependent on the market maker's pricing rule and the uninformed traders trade exogenously.

We distinguish between the separating equilibrium, in which informed traders trade only large quantities and the pooling equilibrium, in which informed traders trade either small or large quantities with positive probabilities (e.g., Easley and O'Hara (1987)). We explore the separating and pooling equilibria when the market maker has ambiguous beliefs about the eventual security payoff. This allows us to provide the intuition about the behavior of the market participants (especially market makers and informed traders) during extreme market events.

2.6.1 The Separating Equilibrium

The market maker now revises her beliefs about the final payoff given the direction and size of the trade by GBU. Let γ denote the probability of large orders submitted by the informed traders. That means, when the informed traders have a high signal, they submit a large buy (b_2) and when they have a low signal, they submit a large sell (s_2) with a probability of γ. In the separating equilibrium, the informed traders refrain from trading with a probability of $1 - \gamma$. The uninformed traders submit large orders with a probability of θ and small orders with a probability of $1 - \theta$. The probabilities are divided equally among the buy and sell orders (i.e., $\Pr(b_1) = \Pr(s_1) = \frac{1-\theta}{2}$ and $\Pr(b_2) = \Pr(s_2) = \frac{\theta}{2}$).

In the separating equilibrium, small trades cannot be information based since the informed traders only submit large orders. Therefore, the conditional probabilities about the final payoff do not change with small orders (i.e., $\pi_h^{b_1} = \pi_h^{s_1} = \pi_h$). This leads the market maker's conditional beliefs about the payoff to be

$$v_h^{b_1} = (1 - \delta^{b_1}) \cdot \pi_h + \delta^{b_1} \cdot \alpha \quad \text{and} \quad v_h^{s_1} = (1 - \delta^{s_1}) \cdot \pi_h + \delta^{s_1} \cdot \alpha, \qquad (2.43)$$

where δ^{b_1} and δ^{s_1} follow from Lemma 2.2. Although the probabilities of the payoff outcomes are not revised with small orders, the ambiguity is revised upward (i.e., $\delta^{b_1} > \delta$ for $0 < \pi_{b_1} < 1$).

For the large orders, however, the probabilities as well as the ambiguity are revised, leading the market maker's revised beliefs about the low payoff to be

$$v_h^{b_2} = (1 - \delta^{b_2}) \cdot \underbrace{\left(\frac{(2 \cdot \mu \cdot \gamma + (1 - \mu) \cdot \theta) \cdot \pi_h}{2 \cdot \mu \cdot \gamma \cdot \pi_h + (1 - \mu) \cdot \theta} \right)}_{\pi_h^{b_2} > \pi_h} + \delta^{b_2} \cdot \alpha, \qquad (2.44)$$

$$v_h^{s_2} = (1 - \delta^{s_2}) \cdot \underbrace{\left(\frac{(1 - \mu) \cdot \theta \cdot \pi_h}{2 \cdot \mu \cdot \gamma \cdot (1 - \pi_h) + (1 - \mu) \cdot \theta} \right)}_{\pi_h^{s_2} < \pi_h} + \delta^{s_2} \cdot \alpha, \qquad (2.45)$$

where $\pi_h^{b_2}$ and $\pi_h^{s_2}$ follow from Bayes' rule, and δ^{b_2} and δ^{s_2} follow from Lemma 2.2.

Let $B_{\alpha,\delta}^i$, $A_{\alpha,\delta}^i$ (resp. B^i, A^i) denote the bid and ask quotes of the market maker given the order size $i = \{1, 2\}$ in the presence (resp. absence) of ambiguity. It follows from Eqs. (2.9) and (2.10) that the bid and ask quotes in the presence of ambiguity are, respectively, given by

$$B_{\alpha,\delta}^i = B^i + \delta^{s_i} \cdot (\alpha - \pi_h^{s_i}) \cdot (V_h - V_l), \qquad (2.46)$$

$$A_{\alpha,\delta}^i = A^i + \delta^{b_i} \cdot (\pi_l^{b_i} - \alpha) \cdot (V_h - V_l), \qquad (2.47)$$

for each order size i. In the standard model, the market maker reduces the bid and increases the ask since with large orders since $\pi_h^{s_2} < \pi_h$ and $\pi_h^{b_2} > \pi_h$, leading to a wider bid-ask spread. Therefore, large trades are made at less favorable prices. The intuition, in the presence of ambiguity, is the same as in our baseline model. There is an additional ambiguity premium when the market maker is sufficiently ambiguity averse (i.e., $\alpha \leq \min\{\pi_h^{s_i}, \pi_l^{b_i}\}$) and an ambiguity discount when she is not (i.e., $\alpha \geq \max\{\pi_h^{s_i}, \pi_l^{b_i}\}$).

In the separating equilibrium, the market maker sets the quotes by assuming that informed traders only submit large orders. For the separating equilibrium to exist, the profit maximization condition of the informed traders must always correspond to large orders, since their trading strategy is also dependent on the market maker's pricing rule. For the informed traders with a high signal, the separating equilibrium exists if and only if

$$b_2 \cdot (V_h - A_{\alpha,\delta}^2) \geq b_1 \cdot (V_h - A_{\alpha,\delta}^1), \tag{2.48}$$

and for the informed traders with a low signal, the separating equilibrium exists if and only if

$$s_2 \cdot (B_{\alpha,\delta}^2 - V_l) \geq s_1 \cdot (B_{\alpha,\delta}^1 - V_l). \tag{2.49}$$

The next proposition formally derives the necessary and sufficient conditions for the separating equilibria to exist on both sides of the market.

Proposition 2.15. There is a separating equilibrium on the ask side, if and only if

$$\frac{b_2}{b_1} \geq \frac{v_l^{b_1}}{v_l^{b_2}}, \tag{2.50}$$

and on the bid side if and only if

$$\frac{s_2}{s_1} \geq \frac{v_h^{s_1}}{v_h^{s_2}}. \tag{2.51}$$

Proposition 2.15 demonstrates that the order size ratios (i.e., $\frac{b_2}{b_1}$ and $\frac{s_2}{s_1}$) have lower bounds for the separating equilibrium to exist. The lower bound of the order size ratio is the ratio of the market maker's conditional beliefs. The immediate result that follows from this proposition is that, in the absence of ambiguity, the order size ratios for the separating equilibria to exist on both sides of the market reduce to

$$\frac{b_2}{b_1} \geq \frac{\pi_l}{\pi_l^{b_2}} \quad \text{and} \quad \frac{s_2}{s_1} \geq \frac{\pi_h}{\pi_h^{s_2}}. \tag{2.52}$$

To compare the conditions of the separating equilibrium in the presence and absence of ambiguity, we assume that the market maker is sufficiently ambiguity-averse. Formally we have the following corollary on the ask side. A similar result holds for the bid side.

Corollary 2.16. For the sufficiently ambiguity-averse market maker, where the sufficiency is defined as

$$\alpha < \underline{w} \cdot \pi_l + (1 - \underline{w}) \cdot \pi_l^{b_2} \quad \text{and} \quad \underline{w} = \frac{\delta^{b_1} \cdot \pi_l^{b_2}}{\delta^{b_1} \cdot \pi_l^{b_2} - \delta^{b_2} \cdot \pi_l}, \qquad (2.53)$$

the lower bound of the order size ratio $\left(\frac{b_2}{b_1}\right)$ for the existence of a separating equilibrium in the presence of ambiguity is lower than its unambiguous counterpart. In addition, under full ambiguity, the market is always in the separating equilibrium.

Corollary 2.16 implies that higher ambiguity leads to the separating equilibrium to prevail more in financial markets, and in extreme, it becomes the only equilibrium. This is because, in our model, ambiguity has a direct influence on the market maker's quotes, whereas an indirect influence on the informed traders' trading strategy since their strategy is affected by the market maker's pricing rule. Condie and Ganguli (2017) investigate the direct impact of ambiguous private information on the informed traders' trading strategies to show that asset prices may be informationally inefficient in rational expectations equilibrium. In the separating equilibrium, the ambiguity-averse market maker behaves according to her worst case scenario and the informed traders only trade large quantities with the knowledge of the market maker's pricing rule. This is supported by the recent empirical research by Gradojevic, Erdemlioglu and Gençay (2017). By analyzing the tick-by-tick foreign exchange transaction prices and the corresponding volumes for several exchange rates, they find that large currency orders are likely to be placed by informed traders during more volatile episodes, which in our model are associated with high ambiguity.

2.6.2 The Pooling Equilibrium

We now examine the pooling equilibrium when the order size ratios violate the existence of the separating equilibrium. In the pooling equilibrium, the informed traders submit a large order with a probability of γ and a small order with a probability of $1 - \gamma$. The uninformed traders trade the same as in the separating equilibrium.

For the pooling equilibrium to exist, the informed traders must be indifferent between submitting a large or a small order, meaning that they expect to have

equal profits from trading a small or large quantity. The informed traders with a high signal must be indifferent between submitting a small buy order at the ask $A^1_{\alpha,\delta}$ or a large buy order at the ask $A^2_{\alpha,\delta}$,

$$b_2 \cdot (V_h - A^2_{\alpha,\delta}) = b_1 \cdot (V_h - A^1_{\alpha,\delta}), \tag{2.54}$$

and with a low signal must be indifferent between submitting a small sell order at the bid $B^1_{\alpha,\delta}$ or a large sell order at the bid $B^2_{\alpha,\delta}$,

$$s_2 \cdot (B^2_{\alpha,\delta} - V_l) = s_1 \cdot (B^1_{\alpha,\delta} - V_l). \tag{2.55}$$

For the pooling equilibrium to exist, there must be $0 < \gamma < 1$ satisfying conditions (4.32) and (2.55) for the informed traders and the zero-expected-profit condition for the market maker. The next proposition establishes the existence of the pooling equilibrium.

Proposition 2.17. There is a pooling equilibrium on the ask side, if and only if

$$\frac{b_2}{b_1} < \frac{v_l^{b_1}}{v_l^{b_2}}, \tag{2.56}$$

and on the bid side if and only if

$$\frac{s_2}{s_1} < \frac{v_h^{s_1}}{v_h^{s_2}}. \tag{2.57}$$

The upper bound of the order size ratio in the pooling equilibrium corresponds to the lower bound of the order size ratio in the separating equilibrium. This is because the pooling equilibrium only occurs when the order size ratio to guarantee the separating equilibrium is violated. Therefore, there is always a separating or a pooling equilibrium in the market. However, with the sufficiently ambiguity-averse market maker, the pooling equilibrium becomes less prevalent as the ambiguity increases, and in extreme ambiguity, there is no pooling equilibrium.

2.7 Empirical Implications

Our analysis thus far has examined the effects of the liquidity providers' ambiguity about the final payoff on the market behavior. In this section, we discuss the empirical implications of our model. To state our predictions, we need a few measures of ambiguity. Thus we first describe some measures of ambiguity proposed in the literature and then elaborate on the predictions of our model.

2.7.1 Measuring ambiguity

A number of ambiguity measures have been developed in the literature some of which are independent of risk.

(i) Brenner and Izhakian (2018) propose to measure the degree of ambiguity (denoted $\mho^2$) by the expected volatility of uncertain probabilities across the relevant outcomes and aversion to ambiguity by the aversion to mean-preserving spreads in uncertain probabilities. The intuition of $\mho^2$ is that ambiguity and ambiguity aversion can independently be measured in the same way as the risk and risk aversion.

(ii) Hansen and Sargent (2001) propose to measure ambiguity (or model uncertainty) independent of risk by relative entropy. Relative entropy is measured by the distance of a probability distribution from a reference distribution (reference model).

(iii) Anderson, Ghysels and Juergens (2009), Ilut and Schneider (2014), and Antoniou, Harris and Zhang (2015) use the degree of disagreement among professional forecasters as a proxy for ambiguity.

(iv) Driouchi, Trigeorgis and So (2018) extract option implied ambiguity from the prices of options written on the S&P 500 Index.

Alternatively, the literature uses the variance of the variance (e.g., Faria and Correia-da Silva (2014)), the variance of the mean (e.g., Izhakian and Benninga (2011)) or the volatility index (VIX) (e.g., Williams (2015)) as measures of ambiguity.

2.7.2   Model predictions

Our model makes a number of empirical predictions. Some of these predictions
have been reported in the empirical market microstructure and asset pricing liter-
ature, others are yet to be tested.

(i) Perhaps the most fundamental prediction of our model is that the ambiguity
and ambiguity aversion of liquidity providers impact their quotes and spread
(Proposition 2.4). More precisely, when the market maker is sufficiently
ambiguity averse, she decreases the bid and increases the ask with ambiguity,
resulting in a wider bid-ask spread due to the ambiguity premium. When she
is insufficiently ambiguity averse the opposite prevails, resulting in a lower
bid-ask spread due to the ambiguity discount (Corollary 2.6). In fact, by
using $\mho^2$ measure as a proxy for ambiguity, Brenner and Izhakian (2018)
provide evidence, in an asset pricing context, that the ambiguity premium
embedded in the equity premium can be positive or negative depending on
the attitude toward ambiguity. Brenner and Izhakian (2018) also document
that even when the ambiguity premium is negative, it is smaller than the risk
premium such that the equity premium as a whole remains positive. This
is consistent with our analysis of the bid-ask spread that, even when there
is an ambiguity discount on the standard spread, the spread in our model is
always positive.

(ii) Our analysis also predicts the determinants of ambiguity aversion (Corol-
lary 2.5). In our analysis, the "effective" ambiguity aversion of liquidity
providers is determined by the difference of α^* in the bid-ask spread neutral-
ity condition and the given ambiguity aversion α (i.e., $\hat{\alpha} = \alpha^* - \alpha$). When
$\hat{\alpha} > 0$ (resp. $\hat{\alpha} < 0$), the market maker is effectively ambiguity averse (resp.
seeking) with an ambiguity premium (resp. discount) on the spread. When
$\pi_l = \pi_h$, $\hat{\alpha} = (\phi - \mu)/2$, leading the effective ambiguity aversion to decrease
with the probability of informed trading (i.e., $\frac{\partial \hat{\alpha}}{\partial \mu} < 0$). This means that
the ambiguity premium (resp. discount) is more likely to happen when the
probability of informed trading is low (resp. high). Brenner and Izhakian
(2018) also presents evidence that for a high expected probability of favor-
able returns (low informed trading in our setting), the ambiguity premium is
positive, implying an ambiguity aversion and for a high expected probability
of unfavorable returns (high informed trading in our setting), the ambiguity

premium is negative, implying an ambiguity-seeking. Moreover, the magnitude of the premium increases with the ambiguity aversion (Corollary 2.7).

(iii) Our model also has important empirical implications about the dynamics of the quotes and spread during various order flow patterns. When the ambiguity-averse market maker receives continuous selling pressure, the bid moves downward faster than the ask (or the ask moves to the opposite direction), leading the spread to continuously increase (Panel (A) in Figure 2.3). This effect is not possible in the standard model because a sell order during selling pressure always impacts the ask more than the bid, leading to a liquidity improvement due to a resolution of uncertainty about the fundamental value. This is consistent with the empirical results of Engle and Patton (2004) who find that sells impact the bid more than the ask, which stands in contrast to the standard results. Additionally, a cursory examination of the transactions data series of E-mini and SPY (S&P 500 ETF) confirms that a similar phenomenon was present during the May 2010 Flash Crash (e.g., CFTC-SEC (2010a, 2010b)). Our model also predicts that, in a completely polar case, the ambiguity-neutral market maker receiving balanced order flow will reduce the bid-ask spread as opposed to the standard result of no change in the quotes and spread (Panel (B) in Figure 2.3).

(iv) The analysis provides insights about the time-varying ambiguity aversion of liquidity providers. During continuous sell orders the effective ambiguity aversion, $\hat{\alpha}_t = \alpha_t^* - \alpha$, increases with an additional sell order (Panel (A) in Figure 2.4), whereas it is constant during balanced order flow (Panel (B) in Figure 2.4). Thus the initial ambiguity discount on the spread can switch to the ambiguity premium due to the increasing ambiguity aversion of liquidity providers during continuous selling pressure. In fact, by incorporating ambiguity into Black-Scholes option pricing, Driouchi et al. (2018) document shifts in ambiguity aversion among US SPX index put option (options written on the S&P 500 index) holders in the period leading up to the fall 2008 crash.

(v) The analysis in Section 2.5 shows that the impact of ambiguity on the value of information and welfare is ambiguous. The ambiguity can increase or decrease the value of information and welfare depending on the ambiguity aversion of liquidity providers. It is not clear whether reducing ambiguity by regulations such as a mark-to-market accounting (i.e., fair value accounting)

implemented in 2007 will increase the welfare. Thus it remains an empirical question whether reducing ambiguity has the effects that regulators desire.

(vi) The analysis in Section 2.6 reveals that when liquidity providers are sufficiently ambiguity averse the presence of ambiguity leads the separating equilibrium to become more prevalent in financial markets, increasing the possibility of informed traders submitting only large orders. This is also consistent with the recent empirical research by Gradojevic et al. (2017) who provide evidence that large currency orders are likely to be placed by informed traders during more volatile episodes, which in our model are loosely associated with high ambiguity. A direct test of this prediction is to investigate the explicit relationship between order size and ambiguity proxied for example by the degree of disagreement among professional forecasters.

To sum up, our theory of ambiguous price formation can generate empirical regularities associated with prices, liquidity and their dynamics during uncertain market events such as market crashes and trading reforms, and provides a set of new testable predictions. Pushing the boundaries of the market microstructure theory beyond the Bayesian paradigm clearly has a potential.

Appendix 2.1. Proofs

Proof of Lemma 2.2 The proof follows from applying GBU rule in Eq. (2.1) to the market maker's beliefs in Eq. (2.3).

$$
\begin{aligned}
v(\hat{V} = V_l|b) &= \frac{v(V_l \cap b)}{v(V_l \cap b) + 1 - v(V_l \cup s)} \\
&= \frac{(1-\delta) \cdot \pi(V_l \cap b) + \delta \cdot \alpha}{\left((1-\delta) \cdot \pi(V_l \cap b) + \delta \cdot \alpha\right) + 1 - \left((1-\delta) \cdot \pi(V_l \cup s) + \delta \cdot \alpha\right)} \\
&= \frac{(1-\delta) \cdot \pi(V_l \cap b) + \delta \cdot \alpha}{(1-\delta) \cdot \pi(V_l \cap b) + 1 - (1-\delta) \cdot \left(\pi_l + \pi_s - \pi(V_l \cap s)\right)} \\
&= \frac{(1-\delta) \cdot \pi(V_l \cap b) + \delta \cdot \alpha}{1 - (1-\delta) \cdot \pi_s} = \frac{(1-\delta) \cdot \pi(V_l \cap b) + \delta \cdot \alpha}{(1-\delta) \cdot \pi_b + \delta} \\
&= \underbrace{\frac{(1-\delta) \cdot \pi_b}{(1-\delta) \cdot \pi_b + \delta}}_{1-\delta^b} \cdot \underbrace{\frac{\pi(V_l \cap b)}{\pi_b}}_{\pi_l^b} + \underbrace{\frac{\delta}{(1-\delta) \cdot \pi_b + \delta}}_{\delta^b} \cdot \alpha \\
&= (1-\delta^b) \cdot \pi_l^b + \delta^b \cdot \alpha.
\end{aligned}
\tag{A2.1.1}
$$

The conditional probability $\pi_l^b = \frac{(1-\mu)\cdot\pi_l}{(1-\mu)\cdot\pi_l+(1+\mu)\cdot\pi_h}$ follows from Bayes' rule. The notation and proof of v_l^s, v_h^b and v_h^s follow similarly.

<div align="right">■</div>

Proof of Proposition 2.4. The proof follows from the definition of the Choquet expectation and zero-expected-profit condition.

$$
\begin{aligned}
B_{\alpha,\delta} &= E_v[V|s] = \left(v_l^s + \delta^s \cdot (1 - 2 \cdot \alpha)\right) \cdot V_l + v_h^s \cdot V_h = V_l + v_h^s \cdot (V_h - V_l) \\
&= V_l + \left((1-\delta^s) \cdot \pi_h^s + \delta^s \cdot \alpha\right) \cdot (V_h - V_l) \\
&= (1-\delta^s) \cdot (\pi_l^s \cdot V_l + \pi_h^s \cdot V_h) + \delta^s \cdot \left(\alpha \cdot V_h + (1-\alpha) \cdot V_l\right) \\
&= (1-\delta^s) \cdot B + \delta^s \cdot \left(\alpha \cdot V_h + (1-\alpha) \cdot V_l\right),
\end{aligned}
\tag{A2.1.2}
$$

where B is the bid quote in the absence of ambiguity and δ^s is the revised ambiguity conditional on a sell order. The only difference in the calculation of the Choquet expectation for the ask price is to recognize the relevant minimizing probability when evaluating expectation as the probability in the core of (v) that puts the most possible weight on V_h as oppose to V_l.

$$
\begin{aligned}
A_{\alpha,\delta} &= E_v[V|b] = v_l^b \cdot V_l + \left(v_h^b + \delta^b \cdot (1 - 2 \cdot \alpha)\right) \cdot V_h = V_h - v_l^b \cdot (V_h - V_l) \\
&= V_h - \left((1-\delta^b) \cdot \pi_l^b + \delta^b \cdot \alpha\right) \cdot (V_h - V_l) \\
&= (1-\delta^b) \cdot (\pi_h^b \cdot V_h + \pi_l^b \cdot V_l) + \delta^b \cdot \left((1-\alpha) \cdot V_h + \alpha \cdot V_l\right) \\
&= (1-\delta^b) \cdot A + \delta^b \cdot \left((1-\alpha) \cdot V_h + \alpha \cdot V_l\right),
\end{aligned}
\tag{A2.1.3}
$$

Finally, the bid-ask spread follows from the difference of Eqs. (A2.1.3) and (A2.1.2) as,

$$S_{\alpha,\delta} = A_{\alpha,\delta} - B_{\alpha,\delta} = S + \left(\delta^b \cdot (\pi_l^b - \alpha) - \delta^s \cdot (\alpha - \pi_h^s)\right) \cdot \left(V_h - V_l\right). \quad \text{(A2.1.4)}$$

■

Proof of Corollary 2.5. For $S_{\alpha,\delta} = S$ to hold, it follows from Eq. (A2.1.4) that

$$\delta^b \cdot (\pi_l^b - \alpha) - \delta^s \cdot (\alpha - \pi_h^s) = 0, \quad \text{(A2.1.5)}$$

must hold. Substituting the values of

$$\delta^b = \frac{\delta}{\left((1-\delta)\cdot\pi_b + \delta\right)}, \quad \text{and} \quad \delta^s = \frac{\delta}{\left((1-\delta)\cdot\pi_s + \delta\right)}, \quad \text{(A2.1.6)}$$

from Lemma 2 and rearranging yields

$$\alpha = w \cdot \pi_l^b + (1 - w) \cdot \pi_h^s, \quad \text{(A2.1.7)}$$

where we have denoted

$$w = \frac{\delta^b}{\delta^b + \delta^s} = \frac{\pi_s + \delta \cdot \pi_b}{1 + \delta}. \quad \text{(A2.1.8)}$$

■

Proof of Corollary 2.6. We only prove when the market maker is sufficiently ambiguity averse. (i) For $\alpha \leq \min\{\pi_h^s, \pi_l^b\}$, $S_{\alpha,\delta} - S > 0$ trivially follows from Eq. (A2.1.4). (ii) For $\alpha \leq \min\{\pi_h^s, \pi_l^b\}$, it follows from the direct differentiation of $S_{\alpha,\delta} - S$ with respect to (w.r.t.) δ as

$$\begin{aligned}
\frac{\partial(S_{\alpha,\delta} - S)}{\partial\delta} =& \frac{\partial}{\partial\delta}\left(\left(\frac{\delta}{(1-\delta)\cdot\pi_b + \delta}\cdot(\pi_l^b - \alpha) - \frac{\delta}{(1-\delta)\cdot\pi_s + \delta}\cdot(\alpha - \pi_h^s)\right)\cdot(V_h - V_l)\right)\\
=&\left((\pi_l^b - \alpha)\cdot\left(\frac{1}{(1-\delta)\cdot\pi_b + \delta}\cdot(1 - \frac{\delta\cdot\pi_s}{\pi_b + \delta\cdot\pi_s})\right)\right.\\
&\left.+ (\pi_h^s - \alpha)\cdot\left(\frac{1}{(1-\delta)\cdot\pi_s + \delta}\cdot(1 - \frac{\delta\cdot\pi_b}{\pi_s + \delta\cdot\pi_b})\right)\right)\cdot(V_h - V_l) > 0,
\end{aligned}$$

$$\text{(A2.1.9)}$$

leading the ambiguity premium to increase with ambiguity. (iii) For $\alpha \leq \min\{\pi_h^s, \pi_l^b\}$, it follows from Eq. (A2.1.9) that

$$\frac{\partial^2 (S_{\alpha,\delta} - S)}{\partial \delta^2} = \frac{2 \cdot \pi_s \cdot (\pi_l^b - \alpha)}{\left((1-\delta) \cdot \pi_b + \delta\right)^2} \cdot \left(\frac{\delta \cdot \pi_s}{\pi_b + \delta \cdot \pi_s} - 1\right) + \frac{2 \cdot \pi_b \cdot (\pi_h^s - \alpha)}{\left((1-\delta) \cdot \pi_s + \delta\right)^2} \cdot \left(\frac{\delta \cdot \pi_b}{\pi_s + \delta \cdot \pi_b} - 1\right)$$
$$< 0,$$

$$(A2.1.10)$$

leading to the concavity of ambiguity premium.

∎

Proof of Corollary 2.7. The proof is immediate from the direct differentiation of Eq. (A2.1.4) w.r.t. α as

$$\frac{\partial S_{\alpha,\delta}}{\partial \alpha} = -(\delta^b + \delta^s) \cdot (V_h - V_l) < 0, \qquad (A2.1.11)$$

for $\delta > 0$.

∎

Proof of Proposition 2.8. The proof follows from inserting Eqs. (9) and (10) into Eq. (25), and decomposing the standard value of information $K = \pi_l \cdot B + (1 - \pi_l) \cdot (1 - A)$ and the ambiguous component $K_{\alpha,\delta} = \delta^s \cdot \pi_l \cdot (\alpha - \pi_h^s) + \delta^b \cdot (1 - \pi_l) \cdot (\alpha - \pi_l^b)$.

∎

Proof of Corollary 2.9. We only show the first part of the corollary when the market maker is sufficiently ambiguity averse (i.e., $\alpha \leq \min\{\pi_h^s, \pi_l^b\}$). (i) trivially follows from Eq. (2.28). (ii) follows from $\frac{\partial K_{\alpha,\delta}}{\partial \delta} < 0$ for $\alpha \leq \min\{\pi_h^s, \pi_l^b\}$. We first substitute

$$\delta^b = \frac{\delta}{(1-\delta) \cdot \pi_b + \delta}, \quad \text{and} \quad \delta^s = \frac{\delta}{(1-\delta) \cdot \pi_s + \delta}, \qquad (A2.1.12)$$

into Eq. (2.28) and then differentiate $K_{\alpha,\delta}$ w.r.t δ as

$$\frac{\partial K_{\alpha,\delta}}{\partial \delta} = \frac{\partial}{\partial \delta} \left(\frac{\delta}{(1-\delta) \cdot \pi_s + \delta} \cdot \pi_l \cdot (\alpha - \pi_h^s) + \frac{\delta}{(1-\delta) \cdot \pi_b + \delta} \cdot (1 - \pi_l) \cdot (\alpha - \pi_l^b) \right)$$
$$= \frac{\pi_s \cdot \pi_l \cdot (\alpha - \pi_h^s)}{\left((1-\delta) \cdot \pi_s + \delta\right)^2} + \frac{\pi_b \cdot (1 - \pi_l) \cdot (\alpha - \pi_l^b)}{\left((1-\delta) \cdot \pi_b + \delta\right)^2} < 0,$$

$$(A2.1.13)$$

for $\alpha \leq \min\{\pi_h^s, \pi_l^b\}$. (iii) follows from $\frac{\partial^2 K_{\alpha,\delta}}{\partial \delta^2} > 0$ for $\alpha \leq \min\{\pi_h^s, \pi_l^b\}$). Differentiating Eq. (A2.1.13) w.r.t δ again

$$\frac{\partial^2 K_{\alpha,\delta}}{\partial \delta^2} = \frac{2 \cdot \pi_b \cdot \pi_s \cdot \pi_l \cdot (\pi_h^s - \alpha)}{\left((1-\delta) \cdot \pi_s + \delta\right)^3} + \frac{2 \cdot \pi_b \cdot \pi_s \cdot \pi_h \cdot (\pi_l^b - \alpha)}{\left((1-\delta) \cdot \pi_b + \delta\right)^3} > 0, \qquad (A2.1.14)$$

for $\alpha \leq \min\{\pi_h^s, \pi_l^b\}$. The second part of the corollary is analogous.

∎

Proof of Corollary 2.10. The proof is immediate from $\frac{\partial K_{\alpha,\delta}}{\partial \alpha} = \pi_l \cdot \delta^s + (1 - \pi_l) \cdot \delta^b > 0$.

∎

Proof of Proposition 2.11. Given the trading decisions of the traders, the following expressions are immediate.

$$\Pr\{\text{buy}|\hat{V} = 1\} = \mu + (1 - \mu) \cdot (1 - A_{\alpha,\delta}), \qquad (A2.1.15)$$

$$\Pr\{\text{sell}|\hat{V} = 1\} = (1 - \mu) \cdot B_{\alpha,\delta}, \qquad (A2.1.16)$$

$$\Pr\{\text{buy}\} = \frac{\mu}{2} + (1 - \mu) \cdot (1 - A_{\alpha,\delta}), \qquad (A2.1.17)$$

$$\Pr\{\text{sell}\} = \frac{\mu}{2} + (1 - \mu) \cdot B_{\alpha,\delta}, \qquad (A2.1.18)$$

leading to

$$\Pr\{\hat{V} = 1|\text{buy}\} = \pi_h^b = \frac{\Pr\{\text{buy}|\hat{V} = 1\}}{\Pr\{\text{buy}\}} \cdot \Pr\{\hat{V} = 1\} = \frac{\mu + (1 - \mu) \cdot (1 - A_{\alpha,\delta})}{\mu + 2 \cdot (1 - \mu) \cdot (1 - A_{\alpha,\delta})},$$
$$(A2.1.19)$$

$$\Pr\{\hat{V} = 1|\text{sell}\} = \pi_h^s = \frac{\Pr\{\text{sell}|\hat{V} = 1\}}{\Pr\{\text{sell}\}} \cdot \Pr\{\hat{V} = 1\} = \frac{(1 - \mu) \cdot B_{\alpha,\delta}}{\mu + 2 \cdot (1 - \mu) \cdot B_{\alpha,\delta}}. \quad (A2.1.20)$$

In the absence of ambiguity, it follows from the market maker's zero expected profit condition (i.e., $A = \pi_h^b$ and $B = \pi_h^s$) that

$$A = \frac{1}{2 \cdot (1 - \mu)} \qquad B = \frac{1 - 2 \cdot \mu}{2 \cdot (1 - \mu)}, \qquad \text{and} \qquad S = \frac{\mu}{1 - \mu}, \qquad (A2.1.21)$$

making the market open for $\mu < \frac{1}{2}$. In the presence of ambiguity, however, the ask is a solution to

$$A_{\alpha,\delta} = \left(1 - \frac{\delta}{(1-\delta)\cdot\pi_b + \delta}\right)\cdot A + \left(\frac{\delta}{(1-\delta)\cdot\pi_b + \delta}\right)\cdot(1-\alpha)$$

$$= \frac{(1-\delta)\cdot\left(\frac{\mu}{2} + (1-\mu)\cdot(1 - A_{\alpha,\delta})\right)\cdot\frac{1}{2\cdot(1-\mu)} + \delta\cdot(1-\alpha)}{(1-\delta)\cdot\left(\frac{\mu}{2} + (1-\mu)\cdot(1 - A_{\alpha,\delta})\right) + \delta} \quad \text{(A2.1.22)}$$

and the bid is a solution to

$$B_{\alpha,\delta} = \left(1 - \frac{\delta}{(1-\delta)\cdot\pi_s + \delta}\right)\cdot B + \left(\frac{\delta}{(1-\delta)\cdot\pi_s + \delta}\right)\cdot\alpha$$

$$= \frac{(1-\delta)\cdot\left(\frac{\mu}{2} + (1-\mu)\cdot B_{\alpha,\delta}\right)\cdot\frac{1-2\cdot\mu}{2\cdot(1-\mu)} + \delta\cdot\alpha}{(1-\delta)\cdot\left(\frac{\mu}{2} + (1-\mu)\cdot B_{\alpha,\delta}\right) + \delta}. \quad \text{(A2.1.23)}$$

Solving Eqs. (A2.1.22) and (A2.1.23) for $A_{\alpha,\delta}$ and $B_{\alpha,\delta}$ respectively and rearranging obtains

$$A_{\alpha,\delta} = \frac{\delta\cdot(\mu-1)^2 - \mu\cdot(\mu-4) - 3 + \sqrt{\gamma}}{4\cdot(\delta-1)\cdot(\mu-1)^2}, \quad \text{(A2.1.24)}$$

$$B_{\alpha,\delta} = \frac{3\delta\cdot(\mu-1)^2 + \mu\cdot(4-3\mu) - 1 - \sqrt{\gamma}}{4\cdot(\delta-1)\cdot(\mu-1)^2}, \quad \text{(A2.1.25)}$$

$$S_{\alpha,\delta} = \frac{1}{2}\cdot\left(\frac{2\cdot(1-\mu) - \sqrt{\gamma}}{(1-\delta)\cdot(1-\mu)^2} - 1\right), \quad \text{(A2.1.26)}$$

where

$$\gamma = (1-\mu)^2\cdot\left(\delta^2\cdot\left(1+8\phi-\mu\cdot(6+8\phi-\mu)\right)+\delta\cdot\left(2\cdot(1-\mu^2)+8\mu(1+\phi)-8\phi\right)+(1-\mu)^2\right)$$
$$\text{(A2.1.27)}$$

and $\phi = 1 - 2\alpha$. Lastly, the market is open as long as $B_{\alpha,\delta} \geq 0$ and $A_{\alpha,\delta} \leq 1$ which is guaranteed when

$$\mu \leq \mu^* = \frac{1}{4}\left(3 - 2\phi + \frac{\sqrt{1 + \delta\cdot\left(10 - 12\phi + \delta\cdot(4\phi\cdot(1+\phi)-7)\right)} - 2\cdot(1-\phi)}{1-\delta}\right).$$
$$\text{(A2.1.28)}$$

∎

Proof of Corollary 2.12. (i) Inserting Eq. (A2.1.27) into Eq. (A2.1.26) and taking a partial derivative w.r.t. δ obtains

$$\frac{\partial S_{\alpha,\delta}}{\partial \delta} = \frac{\sqrt{\gamma} + \mu \cdot \left(\mu - 2 \cdot (2 - \mu) \cdot \phi\right) - \delta \cdot (1 - \mu)^2 \cdot (2\phi + 1) + 2\phi - 1}{(1 - \delta)^2 \cdot (1 - \mu) \cdot \sqrt{\gamma}},$$
(A2.1.29)

which is positive when $\phi > \frac{\mu}{1-\mu}$ and negative when $\phi < \frac{\mu}{1-\mu}$. (ii) Differentiating γ w.r.t. ϕ obtains

$$\frac{\partial \gamma}{\partial \phi} = -8\delta \cdot (1 - \mu)^2 \cdot (1 - \mu) \cdot (1 - \delta) < 0,$$
(A2.1.30)

which leads to $\frac{\partial S_{\alpha,\delta}}{\partial \phi} > 0$.

∎

Proof of Proposition 2.13. The difference between Eqs. (2.39) and (2.40) obtains

$$\Delta W = W_{\alpha,\delta} - W = \mathbf{1}_u[(\omega - 0.5) \cdot \mathbf{1}_{\{A > \omega > A_{\alpha,\delta}\}} + (0.5 - \omega) \cdot \mathbf{1}_{\{B < \omega < B_{\alpha,\delta}\}}], \quad \text{(A2.1.31)}$$

The expectation of ΔW can be computed using the integral

$$E[\Delta W] = (1 - \mu) \cdot \left(\int_{A_{\alpha,\delta}}^{A} (w - 0.5) \cdot f(w) \cdot dw + \int_{B}^{B_{\alpha,\delta}} (0.5 - w) \cdot f(w) \cdot dw \right), \quad \text{(A2.1.32)}$$

where $f(w) = 1$ is the probability density function of the uniformly distributued private valuation between 0 and 1. Evaluating Eq. (A2.1.32) obtains

$$E[\Delta W] = \frac{\left(\sqrt{\gamma} - (1 - \mu) \cdot (1 + \delta - (1 - \delta) \cdot \mu)\right) \cdot \left((1 - \mu) \cdot (1 + \delta + 3(1 - \delta) \cdot \mu) - \sqrt{\gamma}\right)}{16 \cdot (1 - \delta)^2 (1 - \mu)^3}$$
(A2.1.33)

∎

Proof of Corollary 2.14. (i) Inserting Eq. (A2.1.27) into Eq. (A2.1.33) and solving for ϕ that satisfies $E[\Delta W] = 0$ obtains $\phi = \frac{\mu}{1-\mu}$. Since $E[\Delta W]$ monotonically decreases in ϕ (we show this next), $E[\Delta W] > 0$ when $\phi < \frac{\mu}{1-\mu}$ and $E[\Delta W] < 0$ when $\phi > \frac{\mu}{1-\mu}$. (ii) Differentiating Eq. (A2.1.33) w.r.t. ϕ obtains

$$\frac{\partial E[\Delta W]}{\phi} = \frac{\delta \cdot \left(\sqrt{\gamma} + \mu^2 - \delta \cdot (1 - \mu)^2 - 1\right)}{2 \cdot (1 - \delta) \cdot \sqrt{\gamma}} < 0 \qquad \text{for } \mu < 1. \qquad \text{(A2.1.34)}$$

∎

Proof of Proposition 2.15. The proof of the existence of the separating equilibrium on the ask side follows from substituting the quotes of the market maker given in Eq. (47) into Eq. (48) and rearranging obtains

$$
\begin{aligned}
\frac{b_2}{b_1} \geq \frac{V_h - A_{\alpha,\delta}^1}{V_h - A_{\alpha,\delta}^2} &= \frac{V_h - A^1 - \delta^{b_1} \cdot (\pi_l^{b_1} - \alpha) \cdot (V_h - V_l)}{V_h - A^2 - \delta^{b_2} \cdot (\pi_l^{b_2} - \alpha) \cdot (V_h - V_l)} \\
&= \frac{\pi_l^{b_1} \cdot (V_h - V_l) - \delta^{b_1} \cdot (\pi_l^{b_1} - \alpha) \cdot (V_h - V_l)}{\pi_l^{b_2} \cdot (V_h - V_l) - \delta^{b_2} \cdot (\pi_l^{b_2} - \alpha) \cdot (V_h - V_l)} \\
&= \frac{\pi_l - \delta^{b_1} \cdot (\pi_l - \alpha)}{\pi_l^{b_2} - \delta^{b_2} \cdot (\pi_l^{b_2} - \alpha)} = \frac{v_l^{b_1}}{v_l^{b_2}},
\end{aligned}
\tag{A2.1.35}
$$

where we have used $A^1 = V_h - \pi_l^{b_1} \cdot (V_h - V_l)$ and $A^2 = V_h - \pi_l^{b_2} \cdot (V_h - V_l)$. The condition of the existence of the separating equilibrium on the bid side follows similarly.

■

Proof of Corollary 2.16. For the separating equilibrium on the ask side to exist in the presence and absence of ambiguity,

$$
\frac{b_2}{b_1} \geq \frac{\pi_l - \delta^{b_1} \cdot (\pi_l - \alpha)}{\pi_l^{b_2} - \delta^{b_2} \cdot (\pi_l^{b_2} - \alpha)}, \quad \text{and} \quad \frac{b_2}{b_1} \geq \frac{\pi_l}{\pi_l^{b_2}},
\tag{A2.1.36}
$$

must be satisfied respectively. For,

$$
\alpha < \frac{\delta^{b_1} \cdot \pi_l^{b_2}}{\delta^{b_1} \cdot \pi_l^{b_2} - \delta^{b_2} \cdot \pi_l} \cdot \pi_l + (1 - \frac{\delta^{b_1} \cdot \pi_l^{b_2}}{\delta^{b_1} \cdot \pi_l^{b_2} - \delta^{b_2} \cdot \pi_l}) \cdot \pi_l^{b_2},
\tag{A2.1.37}
$$

$$
\frac{\pi_l - \delta^{b_1} \cdot (\pi_l - \alpha)}{\pi_l^{b_2} - \delta^{b_2} \cdot (\pi_l^{b_2} - \alpha)} < \frac{\pi_l}{\pi_l^{b_2}}
\tag{A2.1.38}
$$

is always satisfied. In addition, in the presence of full ambiguity, $\delta = 1$ leads to $\delta^{b_1} = \delta^{b_2} = 1$ and

$$
\frac{b_2}{b_1} \geq 1,
\tag{A2.1.39}
$$

which is always satisfied. Similar results hold for the bid side.

■

Proof of Proposition 2.17. Suppose the necessary and sufficient condition for the existence of the separating equilibrium is violated on the ask side of the market (i.e., $\frac{b_2}{b_1} < \frac{v_l^{b_1}}{v_l^{b_2}}$). Since the informed traders are indifferent between submitting a small buy or a large buy order, we can choose γ that satisfies both the profit

maximization of the informed traders and zero expected profit condition of the market maker. This leads to the pooling equilibrium on the ask side with the necessary and sufficient condition of

$$\frac{b_2}{b_1} < \frac{v_l^{b_1}}{v_l^{b_2}}. \tag{A2.1.40}$$

A similar argument applies for the existence of the pooling equilibrium on the bid side.

∎

Appendix 2.2. Extension to three payoff states

In this Appendix, we extend the two-point Bernoulli distribution of the final security payoff to the trinomial distribution to show that the results of ambiguous price formation model in Section 3 is robust to the distributional assumption. We represent the final security payoff by a random variable $\hat{V}$, which can take low value V_l with a probability of π_l, medium value V_m with a probability of π_m and high value V_h with a probability of $\pi_h = 1 - \pi_l - \pi_m$. The market maker's beliefs about the low, medium and high outcomes of the payoff, respectively, are

$$v_l = (1-\delta)\cdot\pi_l + \delta\cdot\alpha, \quad v_m = (1-\delta)\cdot\pi_m + \delta\cdot\alpha, \quad \text{and} \quad v_h = (1-\delta)\cdot\pi_h + \delta\cdot\alpha, \quad \text{(A2.2.1)}$$

where δ is the amount of ambiguity and α is the market maker's attitude toward the ambiguity. The probability gap due to the presence of ambiguity is

$$1 - v_l - v_m - v_h = \delta \cdot (1 - 3 \cdot \alpha). \quad \text{(A2.2.2)}$$

Similar to the baseline model, at each trading round, a trader arrives at the market to buy one unit of the security at the ask $A_{\alpha,\delta}$ or sell one unit at the bid $B_{\alpha,\delta}$. With a probability of μ the trader arriving at the market is informed and with a probability of $1 - \mu$ is uninformed. The risk-neutral informed traders receive a signal $\Theta = \{H, M, L\}$ about the final security payoff. We assume that they have a strictly increasing signal distribution, i.e.,

$$\underbrace{\Pr\{\Theta = H | \hat{V} = V_l\}}_{q_1} < \underbrace{\Pr\{\Theta = H | \hat{V} = V_m\}}_{q_2} < \underbrace{\Pr\{\Theta = H | \hat{V} = V_h\}}_{q_3}. \quad \text{(A2.2.3)}$$

To simplify the computations, we choose $q_1 = 0$, $q_3 = 1$ meaning that they always sell when $\hat{V} = V_l$ and buy when $\hat{V} = V_h$, and $q_2 = q < 1$ meaning that they buy (resp. sell) with a probability of $q < 1$ (resp. $1 - q$) when $\hat{V} = V_m$.[10]

---

[10]Park and Sabourian (2011) argue that the assumption of two payoff-states and two signals prevent herding and contrarian behavior to occur in a sequential trading model. In a similar model with 3 payoff states, they show that herding (trading against information) occurs when the information is sufficiently dispersed so that the traders consider the high and low outcomes more likely than the medium outcome (i.e., U-shaped signal). That is, a U-shaped signal is necessary for herding. They show that herding occurs with positive probability if there exists at least one U-shaped signal and a sufficient amount of noise trading. In our model, we stick to monotonically increasing signal distribution as in Eq. (A2.2.3) that rule out herding.

The uninformed traders trade with equal probabilities as in the baseline model (endogenous uninformed trading is not necessary for the analysis in this appendix). The definition of equilibrium is analogous to Definition 2.3. Hence, the bid and ask quotes follow from the zero (Choquet) expected profit condition of the market maker. The next proposition presents the equilibrium quotes and spread of the market maker with the new distributional assumption.

Proposition A2.2.1. In the presence of ambiguity, the equilibrium bid and ask are, respectively, given by

$$B_{\alpha,\delta} = (1 - \delta^s) \cdot B + \delta^s \cdot \left[\alpha \cdot (V_h - V_l) + \alpha \cdot V_m + (1 - \alpha) \cdot V_l) \right], \qquad \text{(A2.2.4)}$$

$$A_{\alpha,\delta} = (1 - \delta^b) \cdot A + \delta^b \cdot \left[(1 - \alpha) \cdot V_h + \alpha \cdot V_m - \alpha \cdot (V_h - V_l) \right], \qquad \text{(A2.2.5)}$$

and the bid-ask spread takes the form of

$$S_{\alpha,\delta} = S + \left(\delta^b \cdot \left[(\alpha - \pi_l^b) \cdot V_l + (\alpha - \pi_m^b) \cdot V_m + (1 - 2\alpha - \pi_h^b) \cdot V_h \right] \right.$$
$$\left. - \delta^s \cdot \left[(1 - 2\alpha - \pi_l^s) \cdot V_l + (\alpha - \pi_m^s) \cdot V_m + (\alpha - \pi_h^s) \cdot V_h \right] \right),$$
$$\text{(A2.2.6)}$$

where $S_{\alpha,\delta}$ and S denote the bid-ask spread in the presence and absence of ambiguity respectively.

Proof. The proof follows from the definition of the Choquet expectation (assign the most possible weight on V_l) and zero-expected-profit condition, i.e.,

$$B_{\alpha,\delta} = E_v[V|s] = \left[v_l^s + \delta^s \cdot (1 - 3 \cdot \alpha) \right] \cdot V_l + v_m^s \cdot V_m + v_h^s \cdot V_h$$
$$= (1 - \delta^s) \cdot \underbrace{\left[V_l + \pi_m^s \cdot (V_m - V_l) + \pi_h^s \cdot (V_h - V_l) \right]}_{B}$$
$$+ \delta^s \cdot \left[\alpha \cdot (V_h - V_l) + \alpha \cdot V_m + (1 - \alpha) \cdot V_l) \right] \qquad \text{(A2.2.7)}$$
$$= (1 - \delta^s) \cdot B + \delta^s \cdot \left[\alpha \cdot (V_h - V_l) + \alpha \cdot V_m + (1 - \alpha) \cdot V_l) \right],$$

where we have used that

$$v_l^s + \delta^s \cdot (1 - 3 \cdot \alpha) = 1 - v_m^s - v_h^s, \qquad \text{(A2.2.8)}$$

$$v_m^s = (1 - \delta^s) \cdot \pi_m^s + \delta^s \cdot \alpha, \qquad \text{(A2.2.9)}$$

$$v_h^s = (1 - \delta^s) \cdot \pi_h^s + \delta^s \cdot \alpha. \qquad \text{(A2.2.10)}$$

Similarly, the ask follows as from the definition of the Choquet expectation (assign the most possible weight on V_h as oppose to V_l) and zero-expected-profit condition, i.e.,

$$
\begin{aligned}
A_{\alpha,\delta} =& E_v[V|b] = v_l^b \cdot V_l + v_m^b \cdot V_m + \left[v_h^b + \delta^b \cdot (1 - 3 \cdot \alpha)\right] \cdot V_h \\
=& (1 - \delta^b) \cdot \underbrace{\left[V_h - \pi_l^b \cdot (V_h - V_l) - \pi_m^b \cdot (V_h - V_m)\right]}_{A} \\
& + \delta^b \cdot \left[(1 - \alpha) \cdot V_h + \alpha \cdot V_m - \alpha \cdot (V_h - V_l)\right] \\
=& (1 - \delta^b) \cdot A + \delta^b \cdot \left[(1 - \alpha) \cdot V_h + \alpha \cdot V_m - \alpha \cdot (V_h - V_l)\right],
\end{aligned}
\tag{A2.2.11}
$$

where we have used that

$$
v_h^b + \delta^b \cdot (1 - 3 \cdot \alpha) = 1 - v_m^b - v_l^b,
\tag{A2.2.12}
$$

$$
v_m^b = (1 - \delta^b) \cdot \pi_m^b + \delta^b \cdot \alpha,
\tag{A2.2.13}
$$

$$
v_l^b = (1 - \delta^b) \cdot \pi_l^b + \delta^b \cdot \alpha.
\tag{A2.2.14}
$$

The spread $S_{\alpha,\delta}$ follows from the difference of $A_{\alpha,\delta}$ in Eq. (A2.2.11) and $B_{\alpha,\delta}$ in Eq. (A2.2.7), where we use $B = \pi_l^s \cdot V_l + \pi_m^s \cdot V_m + \pi_h^s \cdot V_h$ and $A = \pi_l^b \cdot V_l + \pi_m^b \cdot V_m + \pi_h^b \cdot V_h$.

∎

An illustrative example

With 3 payoff states, the following expressions follow from Bayes' and generalized Bayes' theorems.

$$
\pi_l^s = \frac{\left(\frac{1+\mu}{2}\right) \cdot \pi_l}{\left(\frac{1+\mu}{2}\right) \cdot \pi_l + \left(\frac{1+\mu}{2} - \mu \cdot q\right) \cdot \pi_m + \left(\frac{1-\mu}{2}\right) \cdot \pi_h},
\tag{A2.2.15}
$$

$$
\pi_m^s = \frac{\left(\frac{1+\mu}{2} - \mu \cdot q\right) \cdot \pi_m}{\left(\frac{1+\mu}{2}\right) \cdot \pi_l + \left(\frac{1+\mu}{2} - \mu \cdot q\right) \cdot \pi_m + \left(\frac{1-\mu}{2}\right) \cdot \pi_h},
\tag{A2.2.16}
$$

$$
\pi_h^s = \frac{\left(\frac{1-\mu}{2}\right) \cdot \pi_h}{\left(\frac{1+\mu}{2}\right) \cdot \pi_l + \left(\frac{1+\mu}{2} - \mu \cdot q\right) \cdot \pi_m + \left(\frac{1-\mu}{2}\right) \cdot \pi_h},
\tag{A2.2.17}
$$

$$
\pi_l^b = \frac{\left(\frac{1-\mu}{2}\right) \cdot \pi_l}{\left(\frac{1-\mu}{2}\right) \cdot \pi_l + \left(\frac{1-\mu}{2} + \mu \cdot q\right) \cdot \pi_m + \left(\frac{1+\mu}{2}\right) \cdot \pi_h},
\tag{A2.2.18}
$$

$$\pi_m^b = \frac{\left(\frac{1-\mu}{2} + \mu \cdot q\right) \cdot \pi_m}{\left(\frac{1-\mu}{2}\right) \cdot \pi_l + \left(\frac{1-\mu}{2} + \mu \cdot q\right) \cdot \pi_m + \left(\frac{1+\mu}{2}\right) \cdot \pi_h}, \tag{A2.2.19}$$

$$\pi_h^b = \frac{\left(\frac{1+\mu}{2}\right) \cdot \pi_h}{\left(\frac{1-\mu}{2}\right) \cdot \pi_l + \left(\frac{1-\mu}{2} + \mu \cdot q\right) \cdot \pi_m + \left(\frac{1+\mu}{2}\right) \cdot \pi_h}, \tag{A2.2.20}$$

$$\delta^s = \frac{\delta}{(1-\delta) \cdot \pi_s + \delta}, \tag{A2.2.21}$$

$$\delta^b = \frac{\delta}{(1-\delta) \cdot \pi_b + \delta}, \tag{A2.2.22}$$

where

$$\pi_s = \left(\frac{1+\mu}{2}\right) \cdot \pi_l + \left(\frac{1+\mu}{2} - \mu \cdot q\right) \cdot \pi_m + \left(\frac{1-\mu}{2}\right) \cdot \pi_h, \tag{A2.2.23}$$

and

$$\pi_b = \left(\frac{1-\mu}{2}\right) \cdot \pi_l + \left(\frac{1-\mu}{2} + \mu \cdot q\right) \cdot \pi_m + \left(\frac{1+\mu}{2}\right) \cdot \pi_h. \tag{A2.2.24}$$

To illustrate the effects of the market maker's ambiguity and ambiguity aversion we again choose equal naive priors $\pi_l = \pi_m = \pi_h = 1/3$, and $q = 0.5$, leading to $\pi_s = \pi_b = 0.5$ and $\delta_s = \delta_b = \frac{2 \cdot \delta}{1+\delta}$. To simplify the computations, we set $\{V_l, V_m, V_h\} = \{0, 1, 2\}$, leading Eqs. (A2.2.7) and (A2.2.11) to reduce to

$$B_{\alpha,\delta} = (1 - \delta^s) \cdot B + \delta^s \cdot (1 - \phi), \tag{A2.2.25}$$

$$A_{\alpha,\delta} = (1 - \delta^b) \cdot A + \delta^b \cdot (1 + \phi), \tag{A2.2.26}$$

and consequently,

$$S_{\alpha,\delta} = S + \delta^b \cdot (2 \cdot \phi - S) \tag{A2.2.27}$$

where $\phi = 1 - 3 \cdot \alpha$ is a normalized ambiguity aversion of the market maker (i.e., $\phi \in [0, 1]$). When $\phi = 0$, the market maker is ambiguity neutral and when $\phi = 1$, she is fully ambiguity averse. The standard probabilistic bid-ask spread follows from the conditional probabilities as $S = \frac{4 \cdot \mu}{3}$. Then, the following corollary is immediate.

Corollary A2.2.2. If the market maker is sufficiently ambiguity averse (i.e., $\phi > \frac{2 \cdot \mu}{3}$);

(i) there is an ambiguity premium on the bid-ask spread,

(ii) incremental ambiguity increases the ambiguity premium,

(iii) the magnitude of increase in the ambiguity premium is decreasing with ambiguity.

If the market maker is not sufficiently ambiguity averse $\phi < \frac{2 \cdot \mu}{3}$;

(i) there is an ambiguity discount on the bid-ask spread,

(ii) incremental ambiguity increases the ambiguity discount,

(iii) the magnitude of increase in the ambiguity discount is decreasing with ambiguity.

Proof. (i) follows from Eq. (A2.2.27) that when the market maker is sufficiently ambiguity averse (i.e., $\phi > \frac{2 \cdot \mu}{3}$) the bid-ask spread with ambiguity is wider, i.e.,

$$S_{\alpha,\delta} - S = \frac{4\delta}{1+\delta} \cdot (\phi - \frac{2\mu}{3}) > 0, \qquad (A2.2.28)$$

and when she is not sufficiently ambiguity averse (i.e., $\phi < \frac{2 \cdot \mu}{3}$) the spread is narrower

$$S - S_{\alpha,\delta} = \frac{4\delta}{1+\delta} \cdot (\frac{2\mu}{3} - \phi) > 0. \qquad (A2.2.29)$$

(ii) follows from the fact that when $\phi > \frac{2\mu}{3}$,

$$\frac{\partial(S_{\alpha,\delta} - S)}{\partial \delta} = \frac{4}{(1+\delta)^2} \cdot (\phi - \frac{2\mu}{3}) > 0, \qquad (A2.2.30)$$

and when $\phi < \frac{2\mu}{3}$,

$$\frac{\partial(S - S_{\alpha,\delta})}{\partial \delta} = \frac{4}{(1+\delta)^2} \cdot (\frac{2\mu}{3} - \phi) > 0. \qquad (A2.2.31)$$

(iii) follows from the fact that $\frac{\partial(S_{\alpha,\delta} - S)^2}{\partial^2 \delta} < 0$ when $\phi > \frac{2 \cdot \mu}{3}$ and $\frac{\partial(S - S_{\alpha,\delta})^2}{\partial^2 \delta} < 0$ when $\phi < \frac{2 \cdot \mu}{3}$.

∎

Corollary A2.2.2 is analogous to Corollary 2.6 in the baseline model with two payoff states. In addition, the following corollary shows that, as in Corollary 2.7 of the baseline model, a higher ambiguity aversion always leads to a higher spread.

Corollary A2.2.3. For any non-zero level of ambiguity, $\delta > 0$, a higher ambiguity-aversion leads to a higher bid-ask spread. That is, $\frac{\partial S_{\alpha,\delta}}{\partial \phi} > 0$.

Figure 2.7 illustrates the spread in the presence $S_{\alpha,\delta}$ and absence S of ambiguity against ambiguity δ and ambiguity aversion ϕ of the market maker when there are 3 payoff states. All the intuitions carry from the baseline model ensuring that our results are robust to the distributional assumption.
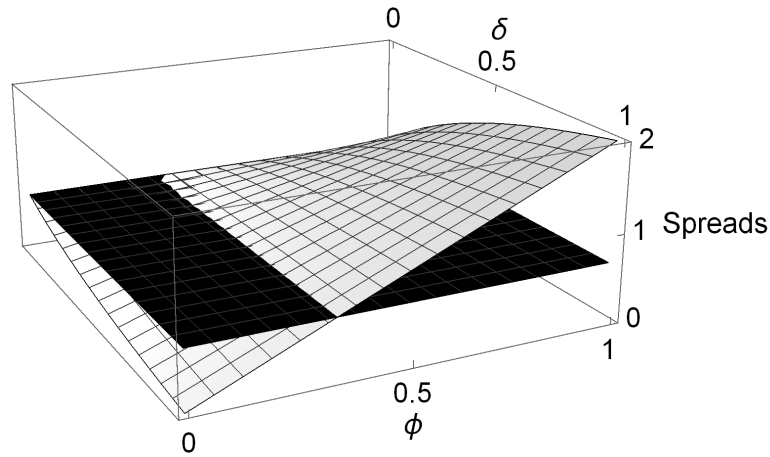
Figure 2.7: The relation between the bid-ask spread in the presence of ambiguity (curved layer), $S_{\alpha,\delta}$, in the absence of ambiguity (fixed black layer), S, with respect to ambiguity, δ, and ambiguity aversion, ϕ, of the market maker. The parameter values are $V_l = 0, V_m = 1, V_h = 2, \pi_l = \pi_m = \pi_h = 1/3, q = 0.5,$ and $\mu = 0.5$.

Appendix 2.3. Extension to imperfect information signals

In this Appendix, we extend our model to allow the information signals of informed traders to be imperfectly informative about the final security payoff. We keep all the features of the baseline model as described in Section 3. Without loss of generality, we set the final security payoff $\hat{V}$ to zero and one (i.e., $V_l = 0$ and $V_h = 1$) and assume naive priors (i.e., $\pi_l = \pi_h = 0.5$) for ease of exposition. The quality of information signals of informed traders, $\Theta \in \{H, L\}$, is measured by

$$q = \Pr\{\Theta = H | \hat{V} = 1\} = \Pr\{\Theta = L | \hat{V} = 0\}. \qquad \text{(A2.3.1)}$$

Unlike the baseline model (informed always buy when $\Theta = H$ and sell when $\Theta = L$), the informed traders with imperfect quality of signals buy (resp. sell) with a probability $q < 1$ when $\Theta = H$ (resp. $\Theta = L$). By Bayes' rule, an informed trader who receives $\Theta = H$ revises his private value to

$$\Pr\{\hat{V} = 1 | \Theta = H\} = \frac{\pi_h \cdot q}{\pi_h \cdot q + \pi_l \cdot (1 - q)}, \qquad \text{(A2.3.2)}$$

and who receives $\Theta = L$ revises his private value to

$$\Pr\{\hat{V} = 1 | \Theta = L\} = \frac{\pi_h \cdot (1 - q)}{\pi_h \cdot (1 - q) + \pi_l \cdot q}. \qquad \text{(A2.3.3)}$$

When $q = 1$, $\Pr\{\hat{V} = 1 | \Theta = H\} = 1$ and $\Pr\{\hat{V} = 0 | \Theta = H\} = 0$, leading the informed traders to always buy when $\Theta = H$ and $\Pr\{\hat{V} = 1 | \Theta = L\} = 0$ and $\Pr\{\hat{V} = 0 | \Theta = L\} = 1$, leading the informed traders to always sell when $\Theta = L$ as in the baseline model in Section 3.

In the presence of ambiguity, inserting $V_l = 0$ and $V_h = 1$ to the zero (Choquet) expected bid and ask quotes of the market maker in Proposition 4 obtains

$$B_{\alpha,\delta} = (1 - \delta^s) \cdot B + \delta^s \cdot \alpha = (1 - \delta^s) \cdot \pi_h^s + \delta^s \cdot \alpha, \qquad \text{(A2.3.4)}$$

$$A_{\alpha,\delta} = (1 - \delta^b) \cdot A + \delta^b \cdot (1 - \alpha) = (1 - \delta^b) \cdot \pi_h^b + \delta^b \cdot (1 - \alpha). \qquad \text{(A2.3.5)}$$

In this setting, an informed trader who arrives at the market with $\Theta = H$ (resp. $\Theta = L$) will buy (resp. sell) if his private valuation is higher (resp. lower) than the ask (resp. bid) quote. That is, the informed traders will always trade in the

direction of their information when

$$\Pr\{\hat{V} = 1 | \Theta = H\} > A_{\alpha,\delta}, \tag{A2.3.6}$$

$$\Pr\{\hat{V} = 1 | \Theta = L\} < B_{\alpha,\delta}, \tag{A2.3.7}$$

are satisfied.

The following proposition shows that when the quality of information signals of informed traders are sufficiently high, conditions (A2.3.6) and (A2.3.7) are satisfied, and thus the informed traders trade in the direction of their signal.

Proposition A2.3.1. In equilibrium, the informed traders always trade in the direction of their signal (i.e, buy when $\Theta = H$ and sell when $\Theta = L$) when the quality of information is sufficiently high,

$$q > q^* = 0.5 + \frac{\delta \cdot (1 - 2 \cdot \alpha)}{\left(1 + \delta\right) - \mu \cdot \left(1 - \delta\right)}. \tag{A2.3.8}$$

Proof. The following expressions follow from Bayes' and generalized Bayes' theorems.

$$\pi_h^s = \frac{\left(\mu \cdot (1 - q) + \frac{1-\mu}{2}\right) \cdot \pi_h}{\left(\mu \cdot q + \frac{1-\mu}{2}\right) \cdot \pi_l + \left(\mu \cdot (1 - q) + \frac{1-\mu}{2}\right) \cdot \pi_h} = \frac{1 - \mu \cdot (2 \cdot q - 1)}{2}, \tag{A2.3.9}$$

$$\pi_h^b = \frac{\left(\mu \cdot q + \frac{1-\mu}{2}\right) \cdot \pi_h}{\left(\mu \cdot (1 - q) + \frac{1-\mu}{2}\right) \cdot \pi_l + \left(\mu \cdot q + \frac{1-\mu}{2}\right) \cdot \pi_h} = \frac{1 + \mu \cdot (2 \cdot q - 1)}{2}, \tag{A2.3.10}$$

$$\delta^s = \frac{\delta}{(1 - \delta) \cdot \pi_s + \delta} = \frac{2 \cdot \delta}{1 + \delta}, \tag{A2.3.11}$$

$$\delta^b = \frac{\delta}{(1 - \delta) \cdot \pi_b + \delta} = \frac{2 \cdot \delta}{1 + \delta}, \tag{A2.3.12}$$

since $\pi_l = \pi_h = 0.5$, and thus , $\pi_s = \pi_b = 0.5$.

Inserting Eqs. (A2.3.9), (A2.3.10), (A2.3.11), and (A2.3.12) into the bid and ask quotes in Eqs. (A2.3.4) and (A2.3.5) obtains

$$B_{\alpha,\delta} = \left(\frac{1 - \delta}{1 + \delta}\right) \cdot \left(\frac{1 - \mu \cdot (2 \cdot q - 1)}{2}\right) + \left(\frac{2 \cdot \delta \cdot \alpha}{1 + \delta}\right), \tag{A2.3.13}$$

$$A_{\alpha,\delta} = \left(\frac{1 - \delta}{1 + \delta}\right) \cdot \left(\frac{1 + \mu \cdot (2 \cdot q - 1)}{2}\right) + \left(\frac{2 \cdot \delta \cdot (1 - \alpha)}{1 + \delta}\right). \tag{A2.3.14}$$

Inserting Eqs. (A2.3.2), (A2.3.3), (A2.3.13), and (A2.3.14) into conditions (A2.3.6) and (A2.3.7) obtains

$$\frac{\pi_h \cdot q}{\pi_h \cdot q + \pi_l \cdot (1-q)} > \left(\frac{1-\delta}{1+\delta}\right) \cdot \left(\frac{1 + \mu \cdot (2 \cdot q - 1)}{2}\right) + \left(\frac{2 \cdot \delta \cdot (1-\alpha)}{1+\delta}\right), \quad \text{(A2.3.15)}$$

$$\frac{\pi_h \cdot (1-q)}{\pi_h \cdot (1-q) + \pi_l \cdot q} < \left(\frac{1-\delta}{1+\delta}\right) \cdot \left(\frac{1 - \mu \cdot (2 \cdot q - 1)}{2}\right) + \left(\frac{2 \cdot \delta \cdot \alpha}{1+\delta}\right). \quad \text{(A2.3.16)}$$

Solving inequalities (A2.3.15) and (A2.3.16) obtains

$$q > 0.5 + \frac{\delta \cdot (1 - 2 \cdot \alpha)}{(1+\delta) - \mu \cdot (1-\delta)}. \quad \text{(A2.3.17)}$$

■

Proposition A2.3.1 shows that when the informed traders receive sufficiently informative signals they trade in the direction of their signals and thus herding and information cascade never occur. In the absence of ambiguity ($\delta = 0$), the signal is sufficiently informative when $q > 0.5$, the result which follows from Avery and Zemsky (1998). In the terminology of Avery and Zemsky (1998), the signal is called monotonic when $q > 0.5$ since it always moves the informed trader's expected value from the public information expected value in the direction of the signal (i.e., $\Pr\{\hat{V} = 1 | \Theta = H\} > \pi_h$ and $\Pr\{\hat{V} = 1 | \Theta = L\} < \pi_h$ when $q > 0.5$).

In the presence of the market maker's ambiguity ($\delta > 0$), however, the sufficient quality of signals q^* changes depending on the market maker's ambiguity and ambiguity aversion. This occurs because the market maker's ambiguity and ambiguity aversion impact the bid-ask spread, and consequently, the value of information (or the expected profits of informed traders). More precisely, as the market maker becomes more ambiguity averse, she widens the spread leading the lower bound of the sufficiency condition q^* for the informed traders to trade in the direction of their signals to increase. Similar intuition applies when the ambiguity averse market maker widens the bid-ask spread as the ambiguity increases. Figure 2.8 shows that both of these intuitions are correct. As the market maker's ambiguity δ and ambiguity aversion $\phi = 1 - 2 \cdot \alpha$ increase, the minimum quality of signal q^* for the informed traders to trade in the direction of the signal also increases. In the extreme, when $\delta = 1$ and $\phi = 1$, this occurs when the signal is perfectly informative, $q = 1$, since the spread is maximum in this scenario. Formally we have the following corollary.
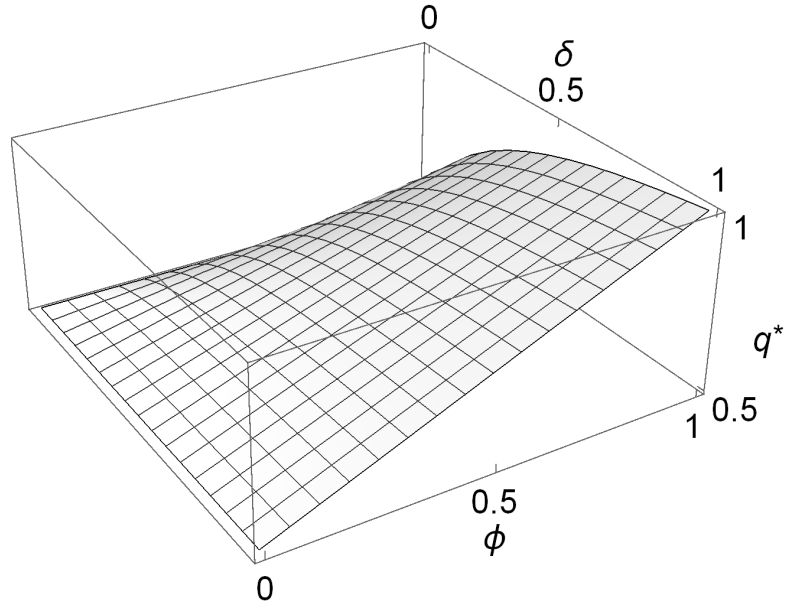
Figure 2.8: The relation between the minimum quality of signal q^* for the informed traders to trade in the direction of the signal with respect to ambiguity, δ, and ambiguity aversion, ϕ, of the market maker. The parameter values are $V_l = 0, V_h = 1, \pi_l = 0.5$ and $\mu = 0.5$.

Corollary A2.3.2. The minimum quality of signal q^* for the informed traders to trade in the direction of their signal increases with the market maker's ambiguity and ambiguity aversion; that is $\frac{\partial q^*}{\partial \delta} > 0$ and $\frac{\partial q^*}{\partial \alpha} < 0$.

Proof. The proof immediately follows from the partial derivatives of q^* in Eq. (A2.3.8) with respect to δ and α;

$$\frac{\partial q^*}{\partial \delta} = \frac{(1 - 2 \cdot \alpha) \cdot (1 - \mu)}{\left((1 + \delta) - \mu \cdot (1 - \delta)\right)^2} > 0, \tag{A2.3.18}$$

$$\frac{\partial q^*}{\partial \alpha} = \frac{-2 \cdot \delta}{(1 + \delta) - \mu \cdot (1 - \delta)} < 0, \tag{A2.3.19}$$

ensuring that the results of the baseline model hold as long as the information signals of the informed traders are sufficiently informative.

■

Appendix 2.4. Other Ways to Update Choquet Beliefs

Similar to the application of Bayesian updating in the standard probabilistic models, the results and predictions of our model are dependent on the application of the generalized Bayesian updating. In this Appendix, we contrast the results of the baseline model with the results obtained by two other extreme optimistic and pessimistic rules for updating Choquet beliefs. This allows us to see the liquidity deteriorations and improvements due to ambiguity from different angles.

First, the optimistic updating rule of Gilboa and Schmeidler (1993) is defined as

$$v^O(\hat{V} = V_l|b) = \frac{v(V_l \cap b)}{v(b)}, \qquad (A2.4.1)$$

which reduces to Bayesian updating when v is additive. When the "optimistic" market maker that revises with Gilboa and Schmeidler (1993) rule receives a buy order, she revises her belief about the low fundamental value as

$$
\begin{aligned}
v^O(\hat{V} = V_l|b) &= \frac{v(V_l \cap b)}{v(b)} = \frac{(1-\delta) \cdot \pi(V_l \cap b) + \delta \cdot \alpha}{(1-\delta) \cdot \pi_b + \delta \cdot \alpha} \\
&= \frac{(1-\delta) \cdot \pi_b}{(1-\delta) \cdot \pi_b + \delta \cdot \alpha} \cdot \pi_l^b + \frac{\delta \cdot \alpha}{(1-\delta) \cdot \pi_b + \delta \cdot \alpha} > \pi_l^b,
\end{aligned}
\qquad (A2.4.2)
$$

and when she receives a sell order, she revises her belief about the high fundamental value as

$$v^O(\hat{V} = V_h|s) = \frac{(1-\delta) \cdot \pi_s}{(1-\delta) \cdot \pi_s + \delta \cdot \alpha} \cdot \pi_h^s + \frac{\delta \cdot \alpha}{(1-\delta) \cdot \pi_s + \delta \cdot \alpha} > \pi_h^s. \qquad (A2.4.3)$$

Second, the pessimistic updating rule of Dempster (1968) and Shafer (1976) is defined as

$$v^P(\hat{V} = V_l|b) = \frac{v(V_l \cup s) - v(s)}{1 - v(s)}, \qquad (A2.4.4)$$

which similarly reduces to Bayesian updating when v is additive. Additivity implies $v(V_l \cup s) - v(s) = v(V_l \cap b)$ and $1 - v(s) = v(b)$, leading to a Bayes' rule. When the "pessimistic" market maker that revises with Dempster-Shafer rule receives a buy order, she revises her belief about the low fundamental value as

$$v^P(\hat{V} = V_l|b) = \frac{v(V_l \cup s) - v(s)}{1 - v(s)} = \frac{(1-\delta) \cdot \pi(V_l \cup s) - (1-\delta) \cdot \pi_s}{1 - \big((1-\delta) \cdot \pi_s + \delta \cdot \alpha\big)}$$
$$= \frac{(1-\delta) \cdot \pi_b}{(1-\delta) \cdot \pi_b + \delta \cdot (1-\alpha)} \cdot \pi_l^b < \pi_l^b, \tag{A2.4.5}$$

and when she receives a sell order, she revises her belief about the high fundamental value as

$$v^P(\hat{V} = V_h|s) = \frac{(1-\delta) \cdot \pi_s}{(1-\delta) \cdot \pi_s + \delta \cdot (1-\alpha)} \cdot \pi_h^s < \pi_h^s. \tag{A2.4.6}$$

Eqs (A2.4.2)-(A2.4.6) show that the "optimistic" market maker always has a revised belief higher than the conditional probability (i.e., $v^O(\hat{V} = V_l|b) > \pi_l^b$, $v^O(\hat{V} = V_h|s) > \pi_h^s$), whereas the "pessimistic" market maker always has a revised belief lower than the conditional probability (i.e., $v^P(\hat{V} = V_l|b) < \pi_l^b$, $v^P(\hat{V} = V_h|s) < \pi_h^s$). Set the final security payoff to zero and one, $\hat{V} \in \{0, 1\}$. The "optimistic" and "pessimistic" market makers' bid-ask spreads are respectively given by

$$S_{\alpha,\delta}^O = 1 - v^O(\hat{V} = V_l|b) - v^O(\hat{V} = V_h|s), \tag{A2.4.7}$$

$$S_{\alpha,\delta}^P = 1 - v^P(\hat{V} = V_l|b) - v^P(\hat{V} = V_h|s), \tag{A2.4.8}$$

whereas the spread of the market maker with probabilistic beliefs is given by

$$S = 1 - \pi_l^b - \pi_h^s. \tag{A2.4.9}$$

The following proposition is immediate.

Proposition A.2.3.1. For non-zero level of ambiguity, $\delta > 0$;

(i) the "optimistic" market maker always has an ambiguity discount on the spread (i.e., $S_{\alpha,\delta}^O < S$) and the ambiguity discount increases with ambiguity (i.e., $\frac{\partial S_{\alpha,\delta}^O}{\partial \delta} < 0$),

(ii) the "pessimistic" market maker always has an ambiguity premium on the spread (i.e., $S_{\alpha,\delta}^P > S$) and the ambiguity premium increases with ambiguity (i.e., $\frac{\partial S_{\alpha,\delta}^P}{\partial \delta} > 0$),

(iii) a higher ambiguity aversion of both the "pessimistic" and "optimistic" market makers leads to a higher bid-ask spread (i.e. $\frac{\partial S_{\alpha,\delta}^O}{\partial \alpha} < 0$ and $\frac{\partial S_{\alpha,\delta}^P}{\partial \alpha} < 0$).

Proof. The first part of the proof follows from $v^O(\hat{V} = V_l|b) > \pi_l^b$, $v^O(\hat{V} = V_h|s) > \pi_h^s$, $\frac{\partial v^O(\hat{V}=V_l|b)}{\partial \delta} > 0$, and $\frac{\partial v^O(\hat{V}=V_h|s)}{\partial \delta} > 0$. The second part of the proof follows from $v^P(\hat{V} = V_l|b) < \pi_l^b$, $v^P(\hat{V} = V_h|s) < \pi_h^s$, $\frac{\partial v^P(\hat{V}=V_l|b)}{\partial \delta} < 0$, and $\frac{\partial v^P(\hat{V}=V_h|s)}{\partial \delta} < 0$. The last part follows from $\frac{\partial v^O(\hat{V}=V_l|b)}{\partial \alpha} > 0$, $\frac{\partial v^O(\hat{V}=V_h|s)}{\partial \alpha} > 0$, $\frac{\partial v^P(\hat{V}=V_l|b)}{\partial \alpha} > 0$, and $\frac{\partial v^P(\hat{V}=V_h|s)}{\partial \alpha} > 0$.

<div align="right">■</div>

Proposition A.2.3.1 is analogous to Corollaries 2.6 and 2.7 in the baseline model. Unlike the baseline model, however, Proposition A.2.3.1 separately shows the ambiguity premium of the sufficiently ambiguity-averse and ambiguity discount of the insufficiently ambiguity-averse market maker due to the extreme nature of the pessimistic and optimistic updating. The effects of ambiguity and ambiguity aversion of the market maker on the ambiguity premium and discount are the same as the baseline model, ensuring that our results are robust. Unlike the extreme pessimistic and optimistic updating, what's interesting about the generalized Bayesian updating is that it provides a unified approach for the liquidity deteriorations and improvements in financial markets and allows us to explore the conditions under which they prevail.

Appendix 2.5. Market dynamics during various order flows

In the paper, we have provided the dynamics of the quotes and spreads as well as the ambiguity and ambiguity aversion of the market maker that receives continuous sell (i.e., $D_1 = -1$, $D_2 = -1$, $D_3 = -1$, ...) and balanced orders (i.e., $D_1 = -1$, $D_2 = +1$, $D_3 = -1$, ...). In this Appendix, we provide a spreadsheet (click here and define the parameters in the green cells and order flow in the yellow cells) to evaluate the same dynamics when the market maker receives various order flow patterns.

Chapter 3

The Pricing of Composition Uncertainty

There will always be a lot of ambiguity about who is an information trader and who is a noise trader.

Black (1986) "Noise" [p. 532].

3.1 Introduction

Financial markets have the role of aggregating the available information of individual market participants. Very often, however, the information available to the majority of market participants is uncertain (i.e., ambiguous). When investors trade with each other, it is often the case that they do not know whether or not they trade against better-informed counterparties (e.g., Banerjee and Green (2015), Aliyev et al. (2018)). The "quant meltdown" of August 2007 and subsequent unfolding of the global financial crisis highlighted the importance of this problem. In his presidential address, Stein (2009) emphasizes this as a "crowded-trade" problem — "for a broad class of quantitative trading strategies, an important consideration for each individual arbitrageur is that he cannot know in real time exactly how many others are using the same model and taking the same position as him."[11] The fundamental of the crowded-trade problem is that the actual composition

---

[11]Recognizing the importance of the crowded-trade problem, some firms started to provide tools for measuring crowdedness of strategies (e.g., the "crowding scorecard" offered by the MSCI).

of informed and uninformed traders is hard to observe. As Black (1986) points out "there will always be a lot of ambiguity about who is an information trader and who is a noise trader". This chapter examines the effects of ambiguity about the composition of traders on asset prices, information transmission of prices, and consequently the value of information.[12]

The crowded-trade problem or uncertainty about the composition of traders (composition uncertainty) is an important constituent of modern markets for various reasons. First, markets often witness unfamiliar shocks causing a sudden disruption of the composition of traders that ultimately triggers 'flash crashes' — episodes of extreme price movements accompanied by evaporation of liquidity. The recent extreme examples of these shocks include the quant meltdown of August 2007, 2007-2009 global financial crisis, and the Flash Crash of May 2010, all of which are associated with extreme price movements and liquidity evaporation. With the rise of algorithmic trading and data revolution, a financial market ecosystem has now dramatically changed. The composition of market participants has never been more complex and uncertain.[13]

Second, uncertainty about the composition of traders provides additional insights into the linkage between liquidity and asset prices in financial markets, which is often omitted in the standard asset pricing literature (e.g., CAPM). Finally, this problem — whether an uncertain composition of traders poses a threat to market participants — is of crucial interest to policymakers. Past congressional testimonies show that regulators understand the potential threat of crowded trades, but they also recognize the difficulty of tracking them.[14]

Motivated by the crowded-trade problem, we construct a rational expectations equilibrium (REE) model in the presence of composition uncertainty. This uncertainty naturally generates deviations from the "fair" price (i.e., informational

---

[12]We use ambiguity and uncertainty interchangeably. The idea of ambiguity dates back to Knight (1921) and Keynes (1921) where they distinguish between risk (when relative odds of the events are known) and uncertainty (when the degree of knowledge only allows the decision maker to work with estimates). Ellsberg (1961) provides experimental evidence to the tentative ideas of Knight and Keynes. The behavior of ambiguity aversion documented by Ellsberg (1961) has been resurrected in the decision making context by Choquet expected utility of Schmeidler (1989) and Maxmin expected utility of Gilboa and Schmeidler (1989). Since then, different approaches have been taken to model ambiguity. We refer to Machina and Siniscalchi (2014) and Gilboa and Marinacci (2013) for extensive surveys of this literature.

[13]See, for instance, "When Silicon Valley came to Wall Street" (Financial Times, October 28, 2017) and "The big changes in US markets since Black Monday" (Financial Times, October 19, 2017).

[14]See, for example, "A Risky 'Systemic' Watchdog", (Washington Post, March 2, 2009).

inefficiency). The informational inefficiency of asset prices stemming from composition uncertainty leads to several interesting phenomena. In an REE with an uncertain composition of informed and uninformed traders, the uncertainty-averse uninformed traders with maxmin preferences of Gilboa and Schmeidler (1989) reduce their risky stockholdings relative to the informationally efficient benchmark with no such uncertainty and demand a composition uncertainty premium to be compensated for this uncertainty. We show that the composition uncertainty premium of the uncertainty-averse uninformed traders is increasing and concave with respect to this uncertainty. Intuitively, uncertainty about the number of informed traders affects the uninformed more than the informed, leading the uninformed traders to be disadvantaged in the face of this uncertainty. In the presence of composition uncertainty, the uninformed (resp. informed) demand is lower (resp. higher) than the benchmark reflecting the fact that they are susceptible (resp. immune) to such uncertainty when trading. Consequently, the perceived equity premium is higher and the stock price is lower (i.e., undervalued) than that in the benchmark.

We also characterize the cost range in which the unique information market equilibrium exists and investigate the effects of composition uncertainty on the benefit of informed trading (i.e., value of information). The benefit of informed trading in our model can be decomposed into a standard and an uncertain, which we term "Knightian", component due to composition uncertainty. As in the standard REE models (e.g., Grossman (1976), Grossman and Stiglitz (1980)), the standard component decreases in the number of informed traders and increases in the idiosyncratic noise in fundamental value. For uncertainty-averse traders, the "Knightian" component reduces the value of information as opposed to what naive intuition — that more uncertainty always increases the value of information — would suggest. The Knightian component decreases in idiosyncratic and composition uncertainty, implying that the amount of uncertainty determines how much the Knightian component reduces the value of information. Overall, the benefit of informed trading increases in idiosyncratic uncertainty in fundamental value but decreases in uncertainty about the composition of traders. While this result might seem surprising at first, the intuition is that high idiosyncratic uncertainty hinders uninformed traders to learn from prices, whereas high uncertainty about the level of informed trading leads traders to refrain from becoming informed due to the potential competition.

To establish these results, we extend the standard CARA-normal REE model where market prices perfectly aggregate and communicate information (e.g., Grossman (1976), Radner (1979)) along two dimensions. First, we introduce private investment opportunities with a return correlated to idiosyncratic noise which is only available to informed traders to provide motivation for informed trading, as in Easley et al. (2014) and Wang (1994). Second, we introduce uncertainty about the composition of traders so that traders do not precisely know the number of informed and uninformed traders in the market, reflecting the practical challenges of the crowded-trade problem. The traders, however, believe the actual number of informed traders λ belongs to some interval $\hat{\lambda} \in [\lambda - \Delta\lambda, \lambda + \Delta\lambda]$, where $\Delta\lambda$ is the amount of uncertainty about the proportion of informed traders λ. To show the above results, in the baseline model, we model traders' preferences with the maxmin model of Gilboa and Schmeidler (1989). We then extend the analysis to the α−maxmin model of Marinacci (2002) to investigate the effects of uncertainty aversion of traders on the market.

In the extended α−maxmin model, we show that the demand of uninformed (resp. informed) traders decreases (resp. increases), the perceived equity premium increases, and the stock price decreases as traders become more uncertainty averse. This implies that in the baseline maxmin model, the uninformed (resp. informed) demand takes its lowest (resp. highest), the composition uncertainty premium its highest, and the stock price its lowest value. We characterize the sufficient uncertainty aversion condition for the results of the baseline model to hold and show that the opposite results can be obtained when traders are not sufficiently uncertainty averse. That is, when traders are not sufficiently uncertainty averse the perceived equity premium can be lower due to composition uncertainty discount, the price can be higher (i.e., overvalued), and consequently the value of becoming informed can be higher than that in the benchmark.

The sufficient uncertainty aversion condition for the composition uncertainty premium to prevail decreases in the proportion of informed traders. That means traders with a given uncertainty aversion can be sufficiently uncertainty averse when the proportion of informed traders is high but not as sufficient when it is low. Consequently, the value of becoming informed can be decreasing (resp. increasing) in the number of informed traders when traders are sufficiently (resp. insufficiently) uncertainty averse during the high (resp. low) informed market, leading to complementarity in information acquisition and multiple information

market equilibria. This complementarity is very different from the well-known negative relation between the value of informed trading and the number of informed traders, what is known as the "Grossman-Stiglitz free-riding effect" (i.e., as more agents acquire costly information, it becomes easier to free-ride by observing the asset price). Thus this chapter delivers predictions under both unique and multiple equilibria.

A majority of the asset pricing models focus on only uncertainty about the fundamental values of assets and fail to notice uncertainty about the market characteristics, resulting in a proliferation of anomalies. The finding that uncertainty about the composition of traders impacts the uninformed demand and hence their liquidity provision links liquidity in markets to asset prices, providing a theoretical explanation for the empirical findings of the impacts of liquidity on prices (e.g., Amihud (2002), Brennan and Subrahmanyam (1996)). The model also proposes a unified explanation for the stock undervaluation and overvaluation, which are often explained by unobservable biases in investors' beliefs (e.g., Barberis, Shleifer and Vishny (1998), Hong and Stein (1999)). In our model, when traders are sufficiently (resp. insufficiently) uncertainty averse, composition uncertainty increases (resp. decreases) the perceived equity premium, leading to a stock undervaluation (resp. overvaluation) relative to the informationally efficient benchmark with no composition uncertainty.

The model proposes two plausible explanations for the occurrences of extreme price movements in financial markets, known as mini or micro flash crashes.[15] First, a sudden increase in uncertainty about the composition of traders can cause the sufficiently (resp. insufficiently) uncertainty averse traders to significantly increase (resp. decrease) their perceived equity premium, leading the stock price to spike down (resp. up) significantly. Second, the complementarity in information acquisition and equilibrium multiplicity can also explain extreme price movements. Due to the complementarity in information acquisition, for the given cost of information, the economy can be populated by a high or low proportion of informed traders which respectively are associated with high (resp. low) and low (resp. high) price when the fundamental value is high (resp. low). When the fundamental value is high, jumping from the high (resp. low) informed trading equilibrium to the low (resp. high) informed trading equilibrium can cause the price to spike

---

[15]Micro flash crashes occur when a stock price spikes up or down in a small time frame. See, for example, http://www.nanex.net/NxResearch/ResearchPage/3/ for an exhaustive documentation of flash crashes.

down (resp. up) in financial markets. Similarly, when the fundamental value is low, jumping from the low (resp. high) informed trading equilibrium to the high (resp. low) informed trading equilibrium can cause the price to spike down (resp. up), explaining the extreme price jumps in financial markets.

The general policy message that follows from our analysis is that regulators must pay attention to uncertainty about the composition of traders and other financial market characteristics. This is of particular interest because this uncertainty impacts the traders unevenly (more precisely, uninformed more than informed) and capital market regulation is mainly concerned with maintaining a level playing field for market participants, which is important to ensure price efficiency. The complementarity of information acquisition and multiplicity of information market equilibrium stemming from the composition uncertainty suggest that reducing the cost of information decreases (resp. increases) the number of informed traders, leading the prices to be less (resp. more) informative about the fundamentals when the information market is at low (resp. high) informed trading equilibrium. Thus, a palliative approach of increasing or decreasing the cost of information to enhance market efficiency such as a mark-to-market accounting (i.e., fair value accounting) legislation implemented in 2007 may not work without creating an equal trading environment for market participants.[16]

The rest of the chapter is organized as follows. We discuss the related literature in the next section. In Section 3.3, we present a baseline model with an uncertain composition of traders to examine the effects of such uncertainty on the equilibrium demands of traders, equity premium, stock price, and the value of informed trading. In Section 3.4, we extend the baseline model to allow for the separation of uncertainty about the composition of traders and traders' uncertainty attitude. In Section 3.5, we discuss empirical implications of our model. In Section 3.6, we discuss other possible extensions and generalizations of our model. Section 3.7 concludes.

---

[16]Such disclosure requirements have also been credited with aggravating the consequences of the global financial crisis because it forced banks to disclose large losses of mortgage-based securities on their portfolios. See, for example, the testimony of former FDIC chairman W. Isaac on March 12, 2009, http://www.williamisaac.com/wp-content/uploads/2010/05/Testimony-MTM-House-Financial-Services-3-12-09-WIsaac-Final.pdf, U.S. House of Representatives Committee on Financial Services .

3.2 Related Literature

The research is related to a small number of asset-pricing and market microstructure models that study uncertainties (either in the composition of traders or the quality of informed traders' information) other than the fundamental values of assets (e.g., Romer (1993), Avery and Zemsky (1998), Banerjee and Green (2015), and Aliyev et al. (2018)). Unlike these models where uncertainty is quantified probabilistically, we model the Knightian uncertainty about the composition of traders by employing the maxmin and $\alpha-$maxmin preferences.

We contribute to the recent literature studying the implications of ambiguous information for financial markets (e.g., Easley et al. (2014), Mele and Sangiorgi (2015)). By using recursive multiple prior utility model of Epstein and Schneider (2003) and learning under ambiguity rule of Epstein and Schneider (2007), Epstein and Schneider (2008) focus on the impacts of processing the news of uncertain quality on asset prices. Similarly, by using Choquet expected utility model of Schmeidler (1989) and generalized Bayesian learning rule of Walley (1991), Aliyev and He (2018a) study the impacts of market makers' ambiguous beliefs about the fundamental values on the liquidity in financial markets and the traders' trading behavior. Learning under ambiguity is outside the scope of this chapter.

Caskey (2009), Ozsoylev and Werner (2011), and Mele and Sangiorgi (2015) construct a noisy REE with ambiguity about the fundamental values. Caskey (2009) uses "smooth" ambiguity model of Klibanoff et al. (2005) to show that persistent mispricing is consistent with ambiguity-averse investors who may choose less precise but more ambiguity-reducing information. Ozsoylev and Werner (2011) show that ambiguity of the monopolist arbitrageur can lead to decreased liquidity and market depth, and increased volatility of prices. Mele and Sangiorgi (2015) show that ambiguity about the cash flows may lead to multiple equilibria and strategic complementarities in information acquisition. We do not rely on the noisy supply to generate the mispricing in financial markets since it is less flexible to investigate our particular focus of interest. The mispricing due to uncertainty about the composition of traders provides complementary means to the noisy supply approach.

To make the mechanism clear, we follow Easley, O'Hara and Yang (2014) and Condie and Ganguli (2017), who do not consider noise traders in their model. Easley, O'Hara and Yang (2014) study an economy in which ambiguity-averse

mutual funds face ambiguity about the trading strategies (effective risk tolerance) of hedge funds. Similar to composition uncertainty, ambiguity about the trading strategies of hedge funds limits the ability of mutual funds to infer information from prices. Condie and Ganguli (2017) focus on the ambiguity-averse informed traders observing ambiguous information to generate empirically relevant return moments (ambiguity premium, negative skewness, and excess kurtosis).

This chapter contributes to this literature in various ways. First, the representation of uncertainty about the composition of traders is novel. Uncertainty (quantified by probability distributions) about the composition of traders has been studied, whereas the Knightian uncertainty about the composition of traders is new to the literature. Different representations of uncertainty about the traders' composition are important to demonstrate how uncertainty about the market or trader characteristics can affect market outcomes. Second, we investigate the asset pricing implications of this market microstructure effect when the informed and uninformed traders are both exposed to this uncertainty, but the intrinsic nature of trading disadvantages one party. This is particularly important because the capital market regulation is mainly concerned with maintaining a level playing field for the market participants. Third, our particular focus of interest is the effects of composition uncertainty on the value of informed trading and complementarities in information acquisition, and its implications for extreme price swings in financial markets. Our contribution here is to show that the presence of composition uncertainty can lead to complementarities in information acquisition and multiple information market equilibria. Finally, on a broader level, our analysis shows the effects of liquidity on asset prices, providing a theoretical explanation for the effects of liquidity on asset prices, and consequently a unified explanation for the stock market undervaluation and overvaluation.

Lastly, we model uncertainty about the composition of traders with "non-smooth" or "kinked" models rather than the "smooth" (differentiable) representations (e.g., Klibanoff et al. (2005), Hansen and Sargent (2008)) for two main reasons. First, the experimental evidence provides convincing support for the "non-smooth" models in financial markets. For instance, Bossaerts et al. (2010) find that traders who are sufficiently ambiguity averse refuse to hold securities with ambiguous payoffs, a property established in the literature as portfolio inertia in prices identified by Dow and Werlang (1992) based on the non-smoothness of the representation (see also Ahn et al. (2014) and Asparouhova et al. (2015) for similar results). Second,

the $\alpha-$maxmin model is a natural and parsimonious generalization of the maxmin model with little loss in tractability, enabling us to contrast the baseline and extended models.

3.3 Uncertain Composition of Traders: Maxmin model

This section presents an REE model that captures the impacts of composition uncertainty on the asset prices, equity premium, and consequently the value of becoming informed. We are parsimonious in the description of the model and address different modeling approaches in the footnotes.

3.3.1 Setup

The market is populated by two types of traders: informed and uninformed traders. All traders live for 3 periods and time is indexed by $t = 0, 1, 2$. There are two decision stages, information acquisition and trading decision stages. Initially identical [0,1] continuum of traders decide their types at $t = 0$ and trades take place at $t = 1$. All uncertainty is resolved and consumption occurs at $t = 2$.

There are two assets traded in the market: a risk-free bond with a constant net return normalized to zero and a risky stock with a price of $\tilde{p}$.[17] The net supply of the risky stock is exogenous and fixed at 1 for convenience. We assume that the risky stock pays off

$$\tilde{f} = \bar{f} + \tilde{\theta} + \tilde{\varepsilon}, \tag{3.1}$$

where $\bar{f} > 0$ is a constant, $\tilde{\theta} \sim N(0, \sigma_\theta^2)$ and $\tilde{\varepsilon} \sim N(0, \sigma_\varepsilon^2)$ are mutually independent with $\sigma_\theta > 0$ and $\sigma_\varepsilon > 0$, $\tilde{\theta}$ is observable at a cost c, whereas $\tilde{\varepsilon}$ is unobservable.

Moreover, there are additional investment opportunities, only available to the informed traders with a gross return of $1 + \tilde{\eta}$, where $\tilde{\eta} \sim N(0, \sigma_\eta^2)$. We let $\tilde{\varepsilon}$ and $\tilde{\eta}$ be correlated with a coefficient of $\rho^* \in (0, 1)$ so that the additional investment

---

[17]As argued by Condie and Ganguli (2017), it would be more precise to use the term "uncertainty-free bond" and "uncertain stock", but we choose to stay with the usual terminology.

opportunities can be used for hedging purposes (e.g., Easley et al. (2014)).[18] Additional investment opportunities can be interpreted as, for example, venture capital or exotic products (e.g., compound options), in which the majority of the asset holders are institutions.

We first assume that the actual proportions of informed and uninformed traders in the market are given by λ and $1 - \lambda$ respectively. The uninformed traders observe the stock price $\tilde{p}$ and the informed traders additionally observe $\tilde{\theta}$ at a cost c.[19] We depart from the standard REE framework by introducing uncertainty about the proportion of informed traders in the market. Although the traders are unable to assess the true proportion of informed traders λ, they believe that it belongs to some interval, $\hat{\lambda} \in [\lambda_1, \lambda_2]$. We choose $\Delta\lambda$ such that

$$\lambda_1 = \lambda - \Delta\lambda > 0 \quad \text{and} \quad \lambda_2 = \lambda + \Delta\lambda < 1, \tag{3.2}$$

for $\Delta\lambda > 0$ measuring the amount of uncertainty about the proportion of informed traders.[20] To separate the effects of composition uncertainty, throughout the analysis, we provide comparisons with a fully revealing benchmark model, in which there is no composition uncertainty (i.e., $\Delta\lambda = 0$).

All traders have a constant absolute risk aversion (CARA) utility u with a common risk aversion coefficient 1 defined over their final portfolio wealth $\tilde{W}$, i.e.,

$$u(\tilde{W}) = -\exp(-\tilde{W}), \tag{3.3}$$

---

[18]This assumption is crucial for our analysis. Without a noisy supply of the risky stock the market becomes fully revealing and the value of becoming informed is zero. There are two alternative approaches to investigate the impact of composition uncertainty on the value of information. First, one can adopt a noisy REE framework as in Grossman and Stiglitz (1980), where the market is not fully revealing and information has a positive value (e.g., Mele and Sangiorgi (2015)). However, due to the unmodeled noise component in this framework, implementing the comparative statics of the value of information with respect to ambiguity and ambiguity attitude becomes difficult. Second, one can adopt the standard REE framework, where the market becomes fully revealing but enforce a positive value for the superior information by the construction of the model as in Easley et al. (2014) and then implement the comparative statics of ambiguity and ambiguity attitude. We follow the second approach. The assumption that the informed traders have additional investment opportunities correlated with the idiosyncratic noise in payoff introduces an advantage to the informed traders.

[19]The analysis would not change much as long as the informed traders know more than the uninformed traders. For example, $\tilde{f} = \bar{f} + \tilde{\theta}_1 + \tilde{\theta}_2 + \tilde{\varepsilon}$ with $\tilde{\theta}_1 \sim N(0, \sigma_{\theta_1}^2)$, $\tilde{\theta}_2 \sim N(0, \sigma_{\theta_2}^2)$ and $\tilde{\varepsilon} \sim N(0, \sigma_{\varepsilon}^2)$ would yield similar results, if the uninformed traders know $\tilde{p}$ and $\tilde{\theta}_2$, and the informed traders additionallly learn $\tilde{\theta}_1$ at a cost $c > 0$.

[20]We use uncertainty about the proportion of informed traders and uncertainty about the composition of traders (i.e., composition uncertainty) interchangeably since uncertainty about the proportion of informed traders naturally leads to uncertainty about the proportion of uninformed traders.

and uncertainty-averse preferences in the sense of Gilboa and Schmeidler (1989). However, the informed traders face composition uncertainty only at the information acquisition decision stage when they choose to be informed (by acquiring the information at a cost c) and the uninformed traders face composition uncertainty both at the information acquisition and trading decision stages. This means the uninformed, not the informed, traders use the uncertain price signal to extract information about the stock payoff during the trading stage. All traders have initial wealth W_0 which is normalized to zero for convenience.[21]

3.3.2 Financial market equilibrium

We first analyze the trading decision stage at $t = 1$ and establish the financial market equilibrium given the proportion of informed traders λ and uncertainty $\Delta\lambda$. Our main focus is on the interior fraction of informed traders (i.e., $\hat{\lambda} \in (0, 1)$), where traders think that both types of traders exist in the market. Traders trade based on their information and the market clears. An REE is a price function and demand correspondences that satisfy the optimality (i.e., utility maximization) and the actual market clearing conditions.

Definition 3.1. For given proportion λ of informed traders and $\Delta\lambda$ level of uncertainty, an REE is a set of functions $\left(\tilde{p}, X_I, X_U\right)$ such that:

(i) the informed demand X_I and the uninformed demand X_U maximize the minimum expected utility of the informed and uninformed traders respectively in the market;

(ii) the price of the risky stock $\tilde{p}$ equates the supply and demand,

$$\lambda \cdot X_I + (1 - \lambda) \cdot X_U = 1. \tag{3.4}$$

3.3.3 Informed traders' demand

For the informed traders, the demand in the risky stock X_I and additional investment opportunities Z_I maximize the expected utility of their final wealth,

$$\tilde{W}_I = (\tilde{f} - \tilde{p}) \cdot X_I + Z_I \cdot \tilde{\eta} - c, \tag{3.5}$$

---

[21]Similar results follow when the traders are initially endowed with one share of stock.

where c is the cost incurred by the informed traders to know $\tilde{\theta}$. The CARA-normal structure of the model leads to

$$E\left[-\exp(-\tilde{W}_I)|\tilde{p},\tilde{\theta}\right] = -\exp\left[-\left(E[\tilde{W}_I|\tilde{p},\tilde{\theta}] - \frac{1}{2}Var[\tilde{W}_I|\tilde{p},\tilde{\theta}]\right)\right], \qquad (3.6)$$

where

$$E[\tilde{W}_I|\tilde{p},\tilde{\theta}] = (\bar{f} + \tilde{\theta} - \tilde{p}) \cdot X_I - c, \qquad (3.7)$$

and

$$Var[\tilde{W}_I|\tilde{p},\tilde{\theta}] = \sigma_\varepsilon^2 \cdot X_I^2 + \sigma_\eta^2 \cdot Z_I^2 + 2\rho^* \cdot \sigma_\varepsilon \cdot \sigma_\eta \cdot X_I \cdot Z_I, \qquad (3.8)$$

are the mean and variance of informed traders' final wealth conditional on their information set. For calculating the moments, we use the fact that the price does not contain more information than the information of informed traders. Standard utility maximization arguments for the informed traders yield the optimal demand in the risky stock as

$$X_I(\tilde{\theta},\tilde{p}) = \frac{(\bar{f} + \tilde{\theta} - \tilde{p})}{(1 - \rho^{*2}) \cdot \sigma_\varepsilon^2}, \qquad (3.9)$$

and the optimal demand in the additional investment opportunities as

$$Z_I(\tilde{\theta},\tilde{p}) = -\frac{\rho^* \cdot (\bar{f} + \tilde{\theta} - \tilde{p})}{(1 - \rho^{*2}) \cdot \sigma_\varepsilon \cdot \sigma_\eta}. \qquad (3.10)$$

Eqs. (3.9) and (3.10) show that the presence of additional investment opportunities for hedging purposes causes the informed traders to trade more aggressively in the stock by taking an opposite position in the additional investment opportunities. In what follows, we denote $\rho = (1 - \rho^{*2})$ for convenience. The demand function of informed traders given in Eq. (3.9) is equivalent to the CARA investor with a risk aversion coefficient of $0 < \rho < 1$. This means additional investment opportunities correlated with the noise $\tilde{\varepsilon}$ cause the informed traders to trade as if they are more risk tolerant, which further increases with the correlation ρ^*.[22]

---

[22]To model uncertainty about the equilibrium trading strategies of hedge funds and its implications for the aggregate welfare, Easley et al. (2014) introduce uncertainty about the "effective" risk tolerance of hedge funds which corresponds to $1/\rho$ in our notation. Instead, our focus of interest is uncertainty about the composition of traders and its implications for the stock price, equity premium, and value of information.

3.3.4 Uninformed traders' demand

To find the optimal demand of uninformed traders given their belief $\hat{\lambda} \in [\lambda_1, \lambda_2]$, we first assume that the uninformed traders rationally conjecture the price function as

$$\tilde{p} = \bar{f} + \tilde{\theta} - g(\lambda, \Delta\lambda), \tag{3.11}$$

where $g(\lambda, \Delta\lambda)$ represents the perceived equity premium since $g(\lambda, \Delta\lambda) = E_{\hat{\lambda}}[\tilde{f} - \tilde{p}]$ and $E_{\hat{\lambda}}$ denotes an expectation over the belief $\hat{\lambda} \in [\lambda_1, \lambda_2]$. The perceived equity premium is determined in equilibrium. Due to the uncertainty about the proportion of informed traders at the trading stage, the uninformed demand X_U maximizes the minimum expected utility of their final wealth, $\tilde{W}_U = (\tilde{f} - \tilde{p}) \cdot X_U$ (i.e., $\max_{X_U} \min_{\hat{\lambda} \in [\lambda_1, \lambda_2]} E_{\hat{\lambda}}[-\exp(-\tilde{W}_U)|\tilde{p}]$). Given our normal distribution structure, the optimal demand of uninformed traders is determined by

$$\max_{X_U} \min_{\hat{\lambda} \in [\lambda_1, \lambda_2]} \left(\left(E_{\hat{\lambda}}[\tilde{f}|\tilde{p}] - \tilde{p}\right) \cdot X_U - \frac{1}{2} Var_{\hat{\lambda}}[\tilde{f}|\tilde{p}] \cdot X_U^2 \right). \tag{3.12}$$

The criterion in Eq. (3.12) follows from the multiple prior model axiomatized by Gilboa and Schmeidler (1989). It characterizes the ambiguity-averse uninformed traders' trading decision problem when they try to learn about the payoff of the risky stock from the price. However, they are unable to perfectly extract the information from the price because the price itself is an uncertain signal of the payoff due to uncertainty about the proportion of informed traders. By the conjectured price function in Eq. (3.11), the uninformed traders' conditional mean and variance of the asset payoff over the particular belief $\hat{\lambda} \in [\lambda_1, \lambda_2]$ are given by

$$E_{\hat{\lambda}}[\tilde{f}|\tilde{p}] = \tilde{p} + g(\lambda, \Delta\lambda) \quad \text{and} \quad Var_{\hat{\lambda}}[\tilde{f}|\tilde{p}] = \sigma_\varepsilon^2. \tag{3.13}$$

Substituting the conditional mean and variance of the stock payoff in Eq. (3.13) into Eq. (3.12), we obtain that the optimal demand of uninformed traders is determined by

$$\max_{X_U} \min_{\lambda \in [\lambda_1, \lambda_2]} \left(g(\lambda, \Delta\lambda) \cdot X_U - \frac{1}{2} \cdot \sigma_\varepsilon^2 \cdot X_U^2 \right) = \max_{X_U} \begin{cases} g_{\min} \cdot X_U - \frac{1}{2} \cdot \sigma_\varepsilon^2 \cdot X_U^2, & \text{if } X_U \geq 0, \\ g_{\max} \cdot X_U - \frac{1}{2} \cdot \sigma_\varepsilon^2 \cdot X_U^2, & \text{if } X_U < 0, \end{cases} \tag{3.14}$$

where $g_{\min}$ and $g_{\max}$ denote the minimum and maximum value of the perceived equity premium $g(\lambda, \Delta\lambda)$ due to the composition uncertainty. In Appendix 3.1,

we show that the only feasible optimal demand of uninformed traders that follows from Eq. (3.14) is

$$X_U = \frac{g_{\min}}{\sigma_\varepsilon^2},$$ (3.15)

where $g_{\min} = g(\lambda_2, \Delta\lambda) > 0$ is the minimum of $g(\lambda, \Delta\lambda)$. Intuitively, when the uncertainty-averse uninformed traders are uncertain about the proportion of informed traders, they should restrict their demand and require a positive minimum equity premium for holding the risky stock. Based on the above analysis, we can characterize the financial market equilibrium.

Proposition 3.2. Suppose $\lambda_1 = \lambda - \Delta\lambda > 0$ and $\lambda_2 = \lambda + \Delta\lambda < 1$ for $\Delta\lambda > 0$. In the presence of composition uncertainty, there exists an REE with the equilibrium price given by

$$\tilde{p} = \bar{f} + \tilde{\theta} - g(\lambda, \Delta\lambda),$$ (3.16)

and the informed and uninformed demands are given by, respectively

$$X_I(\tilde{\theta}, \tilde{p}) = \frac{g(\lambda, \Delta\lambda)}{\rho \cdot \sigma_\varepsilon^2}, \quad \text{and} \quad X_U(\tilde{p}) = \frac{g_{\min}}{\sigma_\varepsilon^2},$$ (3.17)

where

$$g(\lambda, \Delta\lambda) = \frac{\rho \cdot \sigma_\varepsilon^2}{\lambda} \cdot \left(\frac{\lambda + \Delta\lambda \cdot (1 - \rho)}{\rho + (\lambda + \Delta\lambda) \cdot (1 - \rho)} \right),$$ (3.18)

and

$$g_{\min} = \frac{\rho \cdot \sigma_\varepsilon^2}{\rho + (\lambda + \Delta\lambda) \cdot (1 - \rho)}.$$ (3.19)

Proposition 3.2 shows that when the extra investment opportunities of informed traders is not correlated with the idiosyncratic noise (i.e., $\rho^* = 0$ or $\rho = 1$), the equilibrium with composition uncertainty reduces to the standard fully revealing REE with the equity premium $g(\lambda) = \sigma_\varepsilon^2$ and the demands $X_I = X_U = 1$ due to the disappearing advantage of informed traders. In contrast, when the extra investment opportunities and the idiosyncratic noise are perfectly correlated (i.e., $\rho^* = 1$ or $\rho = 0$), the uninformed demand and perceived equity premium reduce to zero since they rationally choose not to trade with the fully advantaged informed traders, leading the demand of informed traders to be $X_I = 1/\lambda$.

In general, the effective risk aversion of informed traders ρ plays an interesting role. As ρ decreases the informed traders trade more aggressively by demanding more, and consequently the uninformed traders reduce their demand for the risky stock, that is, $\frac{\partial X_I(\tilde{\theta}, \tilde{p})}{\partial \rho} < 0$ and $\frac{\partial X_U(\tilde{p})}{\partial \rho} > 0$. Intuitively, decreasing ρ (or increasing

ρ^*) gives more advantage (i.e., hedging benefit) to the informed traders, resulting in a lower minimum perceived equity premium $g_{\min}$ for the uninformed traders. Therefore, when the advantage of informed traders is low (high), the informed demand is low (high) while the uninformed demand is high (low). The effect of ρ on the perceived equity premium $g(\lambda, \Delta\lambda)$, however, is not monotonic because decreasing ρ means more hedging opportunities for the informed traders, but at the same time less uninformed trading in the market.[23] More precisely, $g(\lambda, \Delta\lambda)$ increases in ρ if and only if $\rho < \bar{\rho} = \frac{\lambda_2}{1-\lambda_2} \cdot (\sqrt{\frac{1-\lambda}{\Delta\lambda}} - 1)$. Accordingly, the stock price decreases in ρ if and only if $\rho < \bar{\rho}$ following the equilibrium price function in Eq. (3.16).

Proposition 3.2 also shows that unlike the standard fully revealing REE, the uninformed demand decreases with the proportion of informed traders λ since the informed traders have hedging advantage. The increasing proportion of informed traders also decreases the perceived equity premium, and therefore the informed demand due to enhanced competition. That is, $\frac{\partial X_U(\tilde{\theta},\tilde{p})}{\partial\lambda} < 0$, $\frac{\partial g(\lambda,\Delta\lambda)}{\partial\lambda} < 0$, and $\frac{\partial X_I(\tilde{\theta},\tilde{p})}{\partial\lambda} < 0$. In addition, the informed demand increases and the uninformed demand decreases in the amount of composition uncertainty $\Delta\lambda$ due to the increasing perceived equity premium. That is, $\frac{\partial X_I(\tilde{p})}{\partial\Delta\lambda} > 0$, $\frac{\partial X_U(\tilde{p})}{\partial\Delta\lambda} < 0$, and $\frac{\partial g(\lambda,\Delta\lambda)}{\partial\Delta\lambda} > 0$. Intuitively, in equilibrium, the uninformed traders must have a positive demand for the risky stock due to a positive minimum perceived equity premium. However, they do this by only restricting their risky stock holdings and requiring an additional premium to be compensated for uncertainty about the informed trading. Consequently, the equilibrium stock price decreases in the amount of uncertainty about the informed trading.

Lastly, as in the standard fully revealing REE, the idiosyncratic noise σ_ε^2 does not affect the demands of traders but increases their perceived equity premium. Intuitively, the idiosyncratic noise should not affect the demands of traders since the equilibrium price reveals all the information of informed traders, but should increase the perceived equity premium due to increased idiosyncratic uncertainty about the fundamental value. We summarize these results in the following corollary.

Corollary 3.3. In the presence of composition uncertainty;

---

[23]Figure 3.5 in Appendix 3.2 illustrates the perceived equity premium $g(\lambda, \Delta\lambda)$ against the effective risk aversion ρ and correlation ρ^*.

(i) the informed traders increase while the uninformed traders decrease their demands for the risky stock with the amount of composition uncertainty $\Delta\lambda$;

(ii) both the informed and uninformed traders decrease their demands for the risky stock with the level of informed trading λ;

(iii) the informed traders decrease (increase) while the uninformed traders increase (decrease) their demands for the risky stock with the effective risk aversion coefficient ρ (the correlation ρ^*);

(iv) the perceived equity premium increases with the amount of composition $\Delta\lambda$ and idiosyncratic σ_ε^2 uncertainty, decreases with the level of informed trading λ, and increases with the effective risk aversion coefficient ρ if and only if $\rho < \bar{\rho} = \frac{\lambda_2}{1-\lambda_2} \cdot (\sqrt{\frac{1-\lambda}{\Delta\lambda}} - 1)$;

(v) the stock price decreases with the amount of composition and idiosyncratic uncertainty, increases with the level of informed trading, and decreases with the effective risk aversion coefficient ρ if and only if $\rho < \bar{\rho}$.

These results are consistent with the presidential address of O'Hara (2003) about the implications of liquidity and price discovery for the asset prices. By comparing two risky assets, one with only public information and the other with private and public information, O'Hara (2003) shows that traders demand extra returns to hold assets in which information risk is greater (see also Easley, Hvidkjaer and O'Hara (2002), Easley and O'Hara (2004)). In our model, the uncertainty-averse uninformed traders perceive the increase in uncertainty about the informed trading similar to the increase in information risk, leading to similar results.[24]

To facilitate the interpretation, we compare the equilibrium with composition uncertainty with the benchmark without composition uncertainty (i.e., $\Delta\lambda = 0$). When $\Delta\lambda = 0$, it follows from Eq. (3.18) that the perceived equity premium $g(\lambda, \Delta\lambda)$ reduces to

$$g(\lambda) = \frac{\rho \cdot \sigma_\varepsilon^2}{\rho + \lambda \cdot (1 - \rho)}, \tag{3.20}$$

which is always less than the perceived equity premium with composition uncertainty (i.e., $g(\lambda, \Delta\lambda) > g(\lambda)$ for $\Delta\lambda > 0$). Formally, combining Eqs. (3.18) and

---

[24]The analysis of information risk in Easley et al. (2002) is limited to the exogenous population of informed traders. In our setting, it is straightforward to extend the analysis to endogenous information acquisition and investigate the effects of cost of information in equilibrium.

(3.20) yields

$$g(\lambda, \Delta\lambda) = g(\lambda) + g_u(\lambda, \Delta\lambda), \tag{3.21}$$

where $g_u(\lambda, \Delta\lambda)$ shows the composition uncertainty premium and is given by

$$g_u(\lambda, \Delta\lambda) = \frac{\rho^2 \cdot (1 - \rho) \cdot (1 - \lambda) \cdot \Delta\lambda \cdot \sigma_\varepsilon^2}{\lambda \cdot \big(\rho + (\lambda + \Delta\lambda) \cdot (1 - \rho)\big) \cdot \big(\rho + \lambda \cdot (1 - \rho)\big)}. \tag{3.22}$$

Eq. (3.22) shows that the uncertainty premium is always positive (i.e., $g_u(\lambda, \Delta\lambda) > 0$), increasing (i.e., $\frac{\partial g_u(\lambda, \Delta\lambda)}{\partial \Delta\lambda} > 0$), and concave (i.e., $\frac{\partial^2 g_u(\lambda, \Delta\lambda)}{\partial \Delta\lambda^2} < 0$) with respect to $\Delta\lambda$. Therefore, the perceived equity premium $g(\lambda, \Delta\lambda)$ with composition uncertainty is always higher than the benchmark equity premium $g(\lambda)$. Accordingly, the stock in the presence of composition uncertainty is undervalued compared to the benchmark without composition uncertainty.

Numerous authors have shown that ambiguity about the asset fundamentals modeled with the maxmin model of Gilboa and Schmeidler (1989) can induce non-participation in financial markets during the extreme market events such as market crashes (e.g., Easley and O'Hara (2009)). Additionally, ambiguity modeled by the incomplete preferences of Bewley (2002) can generate a lack of trading in financial markets (e.g., Easley and O'Hara (2010a)). These results are consistent, but not necessarily the same as the decreasing demand of uninformed traders due to composition uncertainty. Composition uncertainty is neither about the fundamental values of assets nor confined to the aggregate market events; rather it is more about the effects of exchange-specific (financial market microstructure) uncertainty on the asset prices. The following corollary contrasts the optimal demand of traders, perceived equity premium, and stock price in the presence and absence of composition uncertainty.

Corollary 3.4. In the presence of composition uncertainty;

(i) the informed traders' demand for the risky stock is higher while the uninformed traders' demand is lower than that of the benchmark;

(ii) the perceived equity premium is higher than the benchmark equity premium;

(iii) the price of the risky stock is lower than the benchmark price.

Next we characterize the information market equilibrium by endogenizing the actual proportion λ of informed traders to investigate the impacts of composition

uncertainty on the benefit of informed trading (i.e., value of information). The information market equilibrium guarantees that the results of financial market equilibrium given the proportion of informed traders hold for the given cost of information. In addition, it allows us to obtain the relevant comparative statics about the effects of the cost of information c on the perceived equity premium and stock price.

3.3.5   Information market equilibrium

During the information acquisition stage, whether an uninformed trader would like to pay the cost of information c and switch to an informed trader depends on the comparison between the ex ante expected utility of becoming informed and that of staying uninformed. An information market equilibrium follows from first, finding the utilities of informed and uninformed traders in the trading stage, second, finding the minimum ex ante expected utilities recursively in the information acquisition stage, and third, equalizing the minimum ex ante expected utilities of informed and uninformed traders. Formally, we define the information market equilibrium as follows.

Definition 3.5. For a given cost of information c and $\Delta\lambda > 0$ an information market equilibrium is λ such that: $0 < \lambda - \Delta\lambda < \lambda + \Delta\lambda < 1$ and the ex ante minimum expected utilities of informed and uninformed traders are equal,

$$\min_{\hat{\lambda} \in [\lambda_1, \lambda_2]} E_{\hat{\lambda}}\big(u(\tilde{W}_I^\lambda)\big) = \min_{\hat{\lambda} \in [\lambda_1, \lambda_2]} E_{\hat{\lambda}}\big(u(\tilde{W}_U^\lambda)\big). \tag{3.23}$$

For the informed traders, inserting the optimal demands in Eqs. (3.9) and (3.10) into the moments of their final wealth in Eqs. (3.7) and (3.8) and calculating their expected utility net of cost c during the trading stage leads to

$$E\big(u(\tilde{W}_I^\lambda)\big) = -\exp\big(-(\frac{g(\lambda, \Delta\lambda)^2}{2 \cdot \rho \cdot \sigma_\varepsilon^2} - c)\big). \tag{3.24}$$

During the trading stage, the informed traders only use the information $\tilde{\theta}$ since it is a sufficient statistic for $(\tilde{\theta}, \tilde{p})$. That is, at $t = 1$ they are immune to composition uncertainty. However, when they make the decision to be informed at $t = 0$, they face composition uncertainty since the benefit of becoming informed is affected by the proportion of informed traders λ and uncertainty about this proportion $\Delta\lambda$.

The ex ante minimum expected utility of informed traders, therefore, follows from
Eq. (3.24) as

$$\min_{\hat{\lambda} \in [\lambda_1, \lambda_2]} E_{\hat{\lambda}}\big(u(\tilde{W}_I^\lambda)\big) = -\exp\big(-\big(\frac{g_{\min}^2}{2 \cdot \rho \cdot \sigma_\varepsilon^2} - c\big)\big). \tag{3.25}$$

For the uninformed traders, however, inserting the optimal demand in Eq. (3.15)
into the expected utility of their final wealth leads to

$$E\big(u(\tilde{W}_U^\lambda)\big) = -\exp\big(-\frac{g_{\min}^2}{2 \cdot \sigma_\varepsilon^2}\big). \tag{3.26}$$

Unlike the informed traders, the minimum value of perceived equity premium ap-
pears in the expected utility of uninformed traders in the trading stage because
they use the uncertain price signal to extract information about the stock pay-
off. The ex ante minimum expected utility of staying as an uninformed trader is
therefore given by

$$\min_{\hat{\lambda} \in [\lambda_1, \lambda_2]} E_{\hat{\lambda}}[u(\tilde{W}_U^\lambda)] = -\exp\big(-\frac{g_{\min}^2}{2 \cdot \sigma_\varepsilon^2}\big). \tag{3.27}$$

Lastly, combining Eqs. (3.25) and (3.27) obtains the benefit of becoming an in-
formed trader (or the ex ante expected wealth differences between the informed
and uninformed traders) as

$$V(\lambda, \Delta\lambda) = \frac{g_{\min}^2}{2 \cdot \sigma_\varepsilon^2} \cdot \left(\frac{1-\rho}{\rho}\right) = \frac{\rho \cdot (1-\rho)}{\big(\rho + \lambda_2 \cdot (1-\rho)\big)^2} \cdot \frac{\sigma_\varepsilon^2}{2}, \tag{3.28}$$

which in equilibrium equals the cost of information c.

The benefit of informed trading in Eq. (3.28) is solely due to the benefit of
hedging through additional investment opportunities. Similar to the noisy REE
setting of Grossman and Stiglitz (1980), the benefit of informed trading increases
in the idiosyncratic noise and decreases in the proportion of informed traders (i.e.,
$\frac{\partial V(\lambda, \Delta\lambda)}{\partial \sigma_\varepsilon} > 0$ and $\frac{\partial V(\lambda, \Delta\lambda)}{\partial \lambda} < 0$).[25] Interestingly, Eq. (3.28) also shows that the
benefit of informed trading decreases in the amount of composition uncertainty

---

[25]The decreasing benefit function with the proportion of informed traders shows that the
strategic substitutability result of Grossman and Stiglitz (1980) in costly information acquisi-
tion is robust to composition uncertainty with maxmin preferences. The robustness of strategic
substitutability in costly information acquisition with maxmin preferences is consistent with the
findings of Easley et al. (2014). The finding is in contrast to Mele and Sangiorgi (2015), who
show that introducing ambiguity about the cash flow of the risky asset in the noisy REE setting

(i.e, $\frac{\partial V(\lambda, \Delta\lambda)}{\partial \Delta\lambda} < 0$). Composition uncertainty is different from fundamental value uncertainty (or idiosyncratic uncertainty) and in fact it has an opposite implication for the benefit of informed trading. While it may seem straightforward to argue that more uncertainty always leads to more benefit to the informed traders as in the case of fundamental value uncertainty, this is not true for composition uncertainty. This is because when an uncertainty-averse potential informed trader evaluates to become informed he decides based on the worst case value of becoming informed.

In addition, the benefit of informed trading is hump-shaped with respect to the effective risk aversion ρ. More precisely, $V(\lambda, \Delta\lambda)$ increases in ρ if and only if $\rho < \frac{\lambda_2}{1+\lambda_2}$. The intuition of this result is similar to that of the perceived equity premium. On the one hand, the increase in ρ increases the benefit of informed trading from trading with more uninformed trading. On the other hand, the increase in ρ is associated with low ρ^* meaning less hedging opportunities for the informed traders. Ultimately, whether $V(\lambda, \Delta\lambda)$ increases or decreases in ρ depends on the level of λ_2. For smaller λ_2, the benefit function $V(\lambda, \Delta\lambda)$ is mostly decreasing in ρ since the decrease in hedging benefit dominates the increase in uninformed demand.

To provide more intuition about the benefit of informed trading, we decompose the benefit function in Eq. (3.28) into the standard and uncertain components. It is straightforward to decompose the benefit function $V(\lambda, \Delta\lambda)$ as

$$V(\lambda, \Delta\lambda) = V(\lambda) + K(\lambda, \Delta\lambda), \tag{3.29}$$

where

$$V(\lambda) = \frac{\rho \cdot (1 - \rho)}{\left(\rho + \lambda \cdot (1 - \rho)\right)^2} \cdot \frac{\sigma_\varepsilon^2}{2}, \tag{3.30}$$

and

$$K(\lambda, \Delta\lambda) = \rho \cdot (1-\rho) \cdot \left(\frac{1}{\left(\rho + (\lambda + \Delta\lambda) \cdot (1 - \rho)\right)^2} - \frac{1}{\left(\rho + \lambda \cdot (1 - \rho)\right)^2} \right) \cdot \frac{\sigma_\varepsilon^2}{2} < 0, \tag{3.31}$$

are the standard benefit of informed trading without composition uncertainty and uncertain, which we term Knightian, component stemming from composition uncertainty, respectively. Eq. (3.30) verifies the standard results that the value of

---

of Grossman and Stiglitz (1980) may lead to multiple equilibria and strategic complementarity in information acquisition. Uncertainty about the composition of traders is absent in these papers. Moreover, by separating composition uncertainty and uncertainty attitude with $\alpha-$maxmin preferences in Section 3.4, we show that indeed composition uncertainty with different uncertainty attitudes can lead to multiple equilibria and information complementarity.

becoming informed $V(\lambda)$ increases with the idiosyncratic noise (i.e., $\frac{\partial V(\lambda)}{\partial \sigma_\varepsilon} > 0$) and decreases with the proportion of informed traders λ (i.e., $\frac{\partial V(\lambda)}{\partial \lambda} < 0$) (e.g., Grossman and Stiglitz (1980)). Unlike the value of knowing $\tilde{\theta}$ in the noisy REE setting, however, the value of becoming informed in this setting is due to the value of access to additional investment opportunities.

The Knightian component $K(\lambda, \Delta\lambda)$ introduces a variety of complications not captured by the standard fully revealing or the noisy partially revealing REE models. First, it takes a negative value and leads to a lower value of information compared to the fully revealing benchmark $V(\lambda)$.[26] Intuitively, although at different times, the uncertainty-averse informed and uninformed traders are both affected by the composition uncertainty, leading to a value reducing Knightian component. At $t = 0$, the potential informed traders assume that the proportion of informed traders in the market is its upper bound (λ_2) since it corresponds to the lower benefit of informed trading. At $t = 1$, the uninformed traders also assume that the level of informed trading in the market is its upper bound since it corresponds to the minimum value of perceived equity premium. Consequently, composition uncertainty reduces the benefit of informed trading. Second, the Knightian component decreases with composition and idiosyncratic uncertainty (i.e., $\frac{\partial K(\lambda, \Delta\lambda)}{\partial \Delta\lambda} < 0$ and $\frac{\partial K(\lambda, \Delta\lambda)}{\partial \sigma_\varepsilon} < 0$). This is also intuitive since the amount of uncertainty (either in the composition of traders or the fundamental value) determines how much the Knightian component reduces the value of information. Lastly, the Knightian component increases with the proportion of informed traders (i.e., $\frac{\partial K(\lambda, \Delta\lambda)}{\partial \lambda} > 0$). This means the same amount of composition uncertainty leads to more value reducing Knightian component for smaller proportion of informed traders. Put differently, composition uncertainty has more impact on the value of information when the proportion of informed traders is low. Formally, we have the following corollary.

Corollary 3.6. In the presence of composition uncertainty;

(i) the benefit of informed trading $V(\lambda, \Delta\lambda)$ decreases with the proportion of informed traders λ and the amount of uncertainty about the proportion of informed traders $\Delta\lambda$, and increases with the idiosyncratic noise σ_ε;

(ii) the Knightian component $K(\lambda, \Delta\lambda)$ is value reducing (i.e., $K(\lambda, \Delta\lambda) < 0$), decreases with the amount of uncertainty about the proportion of informed

---

[26]In Section 3.4, we show that this result is robust up to a certain degree of uncertainty aversion by employing a generalization of the maxmin, $\alpha-$maxmin model (e.g., Marinacci (2002), Ghirardato et al. (2004)).

traders $\Delta\lambda$ and the idiosyncratic noise σ_ε, and increases with the proportion of informed traders λ.

Figure 3.1 illustrates these results. The figure shows that the benefit of informed trading is hump-shaped and the Knightian component is U-shaped with respect to the effective risk aversion of informed traders ρ (therefore, the correlation ρ^* illustrated in Figure 3.6 in Appendix 3.2). Panels (A)-(B) of the figure illustrate that the benefit of informed trading increases and the Knightian component decreases with the idiosyncratic noise σ_ε^2. Panels (C)-(D) illustrate that the benefit of informed trading decreases and the Knightian component increases with the proportion of informed traders λ. Lastly, Panels (E)-(F) illustrate that the benefit of informed trading and the Knightian component both decrease with uncertainty about the proportion of informed traders $\Delta\lambda$. These results extend the standard fully revealing REE results along two dimensions. First, the additional investment opportunities introduce information advantages to the informed traders (recall that in the standard REE, there is no benefit to become informed, leading to Grossman-Stiglitz paradox). Second, uncertainty about the proportion of informed traders introduces an additional market friction, reflecting a practical challenge faced by real-world market participants.

As in the standard REE, for the information market equilibrium to exist, it must be the case that the uninformed traders are indifferent between paying the cost c and obtaining an additional benefit $V(\lambda, \Delta\lambda)$ of becoming an informed trader. The following proposition ensures that there exists a unique information market equilibrium for the given cost range.

Proposition 3.7. There exists a unique information market equilibrium when $c \in [\underline{c}, \bar{c}]$,

$$\underline{c} = \rho \cdot (1 - \rho) \cdot \frac{\sigma_\varepsilon^2}{2} \quad \text{and} \quad \bar{c} = \frac{\rho \cdot (1 - \rho)}{\left(\rho + 2 \cdot (1 - \rho) \cdot \Delta\lambda\right)^2} \cdot \frac{\sigma_\varepsilon^2}{2}. \tag{3.32}$$

Moreover, the cost range $(\bar{c} - \underline{c})$ decreases with uncertainty about the proportion of informed traders $\Delta\lambda$.

The existence and uniqueness of the information market equilibrium guarantee an overall equilibrium for each triple $(\lambda, \Delta\lambda, \tilde{p})$ for a fixed cost c in the cost range $c \in [\underline{c}, \bar{c}]$. The characterization of the overall equilibrium in Proposition 3.7 guarantees that the results of REE given the proportion of informed traders hold in the
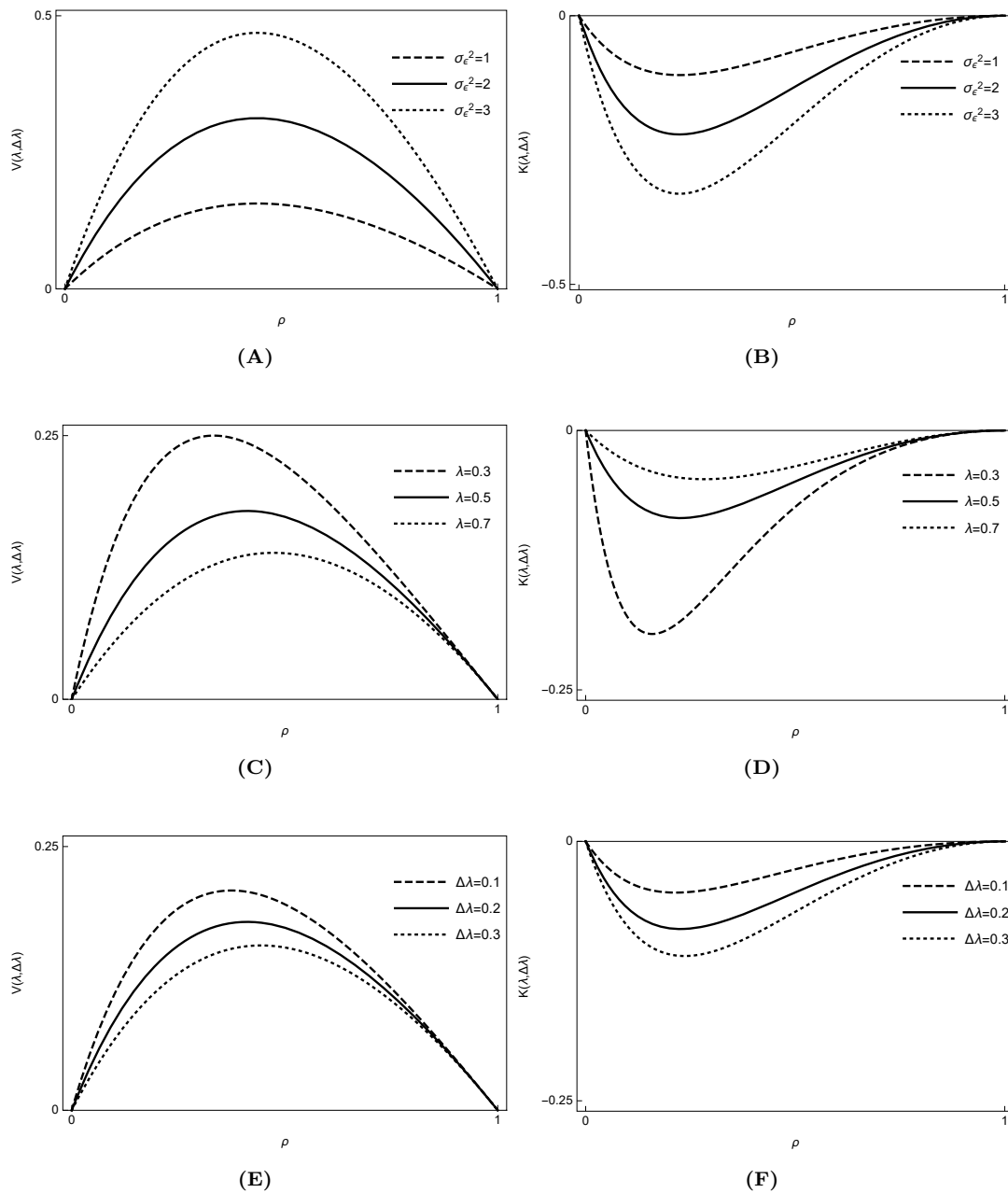
Figure 3.1: The benefit of informed trading $V(\lambda, \Delta\lambda)$ and the Knightian component $K(\lambda, \Delta\lambda)$ against the effective risk aversion ρ.
Panels (A)-(B) plot the benefit of informed trading and the Knightian component against the effective risk aversion (ρ) on the horizontal axis, for three different values of the idiosyncratic noise (σ_ε^2) when $\lambda = 0.5$ and $\Delta\lambda = 0.3$. Panels (C)-(D) plot the benefit of informed trading and the Knightian component against the effective risk aversion (ρ), for three different values of informed trading (λ) when $\Delta\lambda = 0.2$ and $\sigma_\varepsilon^2 = 1$. Panels (E)-(F) plot the benefit of informed trading against the effective risk aversion (ρ), for three different values of uncertainty about the proportion of informed traders ($\Delta\lambda$) when $\lambda = 0.5$ and $\sigma_\varepsilon^2 = 1$.

given cost range.[27] In addition, we can now investigate the effects of different levels of cost of information on the perceived equity premium and the stock price. Intuitively, a higher cost of information is associated with a lower proportion of informed traders (therefore, less aggressive trading since they are more risk tolerant). Moreover, a lower proportion of informed traders is associated with a higher perceived equity premium due to a reduced competition. The next corollary shows that this intuition is also correct in the presence of composition uncertainty.

Corollary 3.8. In the presence of composition uncertainty, increasing the cost of information c increases the perceived equity premium $g(\lambda, \Delta\lambda)$ and decreases the stock price $\tilde{p}$.

Overall, the findings thus far ensure that when the traders are uncertainty averse represented by maxmin preferences, an REE with uncertain composition of traders satisfies most of the standard results and additionally allows us to investigate the effects of composition uncertainty on the trading decisions, and consequently the equilibrium outcomes. The predictions of our model follow from explicitly incorporating the transaction cost of liquidity in the form of composition uncertainty in an otherwise standard fully revealing REE model. In our model, the market microstructure cost of not knowing the composition of traders is priced in equilibrium. This occurs because by bearing composition uncertainty at both decision stages the uncertainty averse uninformed traders are disadvantaged and thus require a compensation in equilibrium, leading to an undervaluation of the risky stock.

3.4 The Role of Uncertainty Aversion: α-maxmin Model

The baseline model assumes that the traders are fully uncertainty averse. In this section, we extend our analysis to account for different levels of the traders' uncertainty aversion. The main motivation of this extension is to separate the

---

[27]The cost range is the maximum in the benchmark model with a value of $\bar{c} - \underline{c} = \frac{(1-\rho)^2 \cdot (1+\rho) \cdot \sigma_\varepsilon^2}{2\rho}$ and decreases with the amount of composition uncertainty since the lower bound $\underline{c}$ of the cost range is fixed, whereas the upper bound $\bar{c}$ decreases with uncertainty. When $\Delta\lambda = 0.5$ (so that $\lambda_1 = 0$ and $\lambda_2 = 1$), $\underline{c} = \bar{c} = \rho \cdot (1 - \rho) \cdot \frac{\sigma_\varepsilon^2}{2}$. Figure 3.7 in Appendix 3.2 illustrates how the cost range changes with respect to ρ, $\Delta\lambda$, and σ_ε^2.

effects of composition uncertainty and uncertainty attitude of traders on the stock prices, perceived equity premium, and benefit of informed trading.

We keep all the features of the baseline model intact and only add a differential characterization of traders' uncertainty aversion. To disentangle composition uncertainty and traders' uncertainty aversion, we employ a generalization of the maxmin model, $\alpha-$maxmin model, which represents uncertainty by a set of probability distributions and uncertainty attitude by the parameter α determining the weights given to the worst and best possible expected utilities (e.g., Marinacci (2002), Ghirardato et al. (2004)). This formulation causes little loss in tractability and stays in the class of non-smooth representation of traders' preferences.

3.4.1   Traders' demands

The demands of informed traders with $\alpha-$maxmin preferences in the stock and additional investment opportunities are the same as the baseline model since they are immune to composition uncertainty during the trading stage (see Eqs. (3.9) and (3.10)).

The uninformed traders analogously conjecture the price function as in Eq. (3.11) (we denote the perceived equity premium by $g_\alpha(\lambda, \Delta\lambda)$). However, they maximize the convex combination of the minimum and maximum expected utilities over the belief $\lambda \in [\lambda_1, \lambda_2]$ as

$$\alpha \cdot \min_{\hat{\lambda} \in [\lambda_1, \lambda_2]} E_{\hat{\lambda}}(-e^{-\tilde{W}_U}|\tilde{p}) + (1 - \alpha) \cdot \max_{\hat{\lambda} \in [\lambda_1, \lambda_2]} E_{\hat{\lambda}}(-e^{-\tilde{W}_U}|\tilde{p}). \qquad (3.33)$$

Eq. (3.33) nests the baseline model with full uncertainty aversion when $\alpha = 1$ and represents uncertainty neutrality when $\alpha = 0.5$. To capture the uncertainty aversion of traders, we assume $0.5 \leq \alpha \leq 1$. Given the normal distribution structure, the optimal demand of uninformed traders is determined by

$$\max_{X_U} \left(\begin{array}{c} \alpha \cdot \min_{\hat{\lambda} \in [\lambda_1, \lambda_2]} \left(\left(E_{\hat{\lambda}}[\tilde{f}|\tilde{p}] - \tilde{p} \right) \cdot X_U - \frac{1}{2} \cdot Var_{\hat{\lambda}}[\tilde{f}|\tilde{p}] \cdot X_U^2 \right) \\ +(1 - \alpha) \cdot \max_{\hat{\lambda} \in [\lambda_1, \lambda_2]} \left(\left(E_{\hat{\lambda}}[\tilde{f}|\tilde{p}] - \tilde{p} \right) \cdot X_U - \frac{1}{2} \cdot Var_{\hat{\lambda}}[\tilde{f}|\tilde{p}] \cdot X_U^2 \right) \end{array} \right).$$
$$(3.34)$$

Substituting the conditional mean and variance of the stock payoff in Eq. (3.13) into Eq. (3.34) obtains

$$
\max_{X_U} \left(
\begin{array}{c}
\alpha \cdot \min_{\hat{\lambda} \in [\lambda_1, \lambda_2]} \left(g_\alpha(\lambda, \Delta\lambda) \cdot X_U - \frac{1}{2} \cdot \sigma_\varepsilon^2 \cdot X_U^2 \right) \\
+ (1 - \alpha) \cdot \max_{\hat{\lambda} \in [\lambda_1, \lambda_2]} \left(g_\alpha(\lambda, \Delta\lambda) \cdot X_U - \frac{1}{2} \cdot \sigma_\varepsilon^2 \cdot X_U^2 \right)
\end{array}
\right),
\tag{3.35}
$$

which is equivalent to

$$
\max_{X_U} \begin{cases}
\left(\alpha \cdot g_{\min} + (1 - \alpha) \cdot g_{\max} \right) \cdot X_U - \frac{1}{2} \cdot \sigma_\varepsilon^2 \cdot X_U^2, & \text{if} \quad X_U \geq 0, \\
\left(\alpha \cdot g_{\max} + (1 - \alpha) \cdot g_{\min} \right) \cdot X_U - \frac{1}{2} \cdot \sigma_\varepsilon^2 \cdot X_U^2, & \text{if} \quad X_U < 0.
\end{cases}
\tag{3.36}
$$

In Eq. (3.36) $g_{\min}$ and $g_{\max}$ denote the minimum and maximum of the perceived equity premium $g_\alpha(\lambda, \Delta\lambda)$. The only feasible optimal demand of uninformed traders that follows from Eq. (3.36) is

$$
X_U = \frac{\alpha \cdot g_{\min} + (1 - \alpha) \cdot g_{\max}}{\sigma_\varepsilon^2},
\tag{3.37}
$$

where $g_\alpha(\lambda_2, \Delta\lambda) = g_{\min}$, $g_\alpha(\lambda_1, \Delta\lambda) = g_{\max}$, and $\alpha \cdot g_{\min} + (1 - \alpha) \cdot g_{\max} > 0$. The next proposition formally states the REE.

Proposition 3.9. Suppose $\lambda_1 = \lambda - \Delta\lambda > 0$ and $\lambda_2 = \lambda + \Delta\lambda < 1$ for $\Delta\lambda > 0$. In the presence of composition uncertainty, there exists an REE with the equilibrium price given by

$$
\tilde{p} = \bar{f} + \tilde{\theta} - g_\alpha(\lambda, \Delta\lambda),
\tag{3.38}
$$

and the informed and uninformed demands are given by, respectively

$$
X_I(\tilde{\theta}, \tilde{p}) = \frac{g_\alpha(\lambda, \Delta\lambda)}{\rho \cdot \sigma_\varepsilon^2}, \quad \text{and} \quad X_U(\tilde{p}) = \frac{\alpha \cdot g_{\min} + (1 - \alpha) \cdot g_{\max}}{\sigma_\varepsilon^2},
\tag{3.39}
$$

where

$$
g_\alpha(\lambda, \Delta\lambda) = \frac{\rho \cdot \sigma_\varepsilon^2}{\lambda} \cdot \left(\frac{\lambda + \Delta\lambda \cdot (1 - \rho) + \frac{2 \cdot \rho \cdot \lambda \cdot \Delta\lambda}{\lambda - \Delta\lambda} \cdot (1 - \alpha)}{\rho + (\lambda + \Delta\lambda) \cdot (1 - \rho) + \frac{2 \cdot \rho \cdot \Delta\lambda}{\lambda - \Delta\lambda} \cdot (1 - \alpha)} \right),
\tag{3.40}
$$

and $g_{\min} = g_\alpha(\lambda_2, \Delta\lambda)$ and $g_{\max} = g_\alpha(\lambda_1, \Delta\lambda)$ represent the minimum and maximum perceived equity premium $g_\alpha(\lambda, \Delta\lambda)$, respectively.

Proposition 3.9 suggests that most of the intuitions of the baseline maxmin model carry forward. For example, when $\rho = 1$, $g_\alpha(\lambda, \Delta\lambda) = \sigma_\varepsilon^2$ and $X_I = X_U = 1$, and when $\rho = 0$, $g_\alpha(\lambda, \Delta\lambda) = 0$, $X_I = 1/\lambda$ and $X_U = 0$ irrespective of the uncertainty attitude of traders. Proposition 3.9 characterizes the optimal demands, perceived equity premium, and stock price as the traders become comparatively more ($\alpha \rightarrow 1$) or less ($\alpha \rightarrow 0.5$) uncertainty averse. Naturally, Proposition 3.9 reduces to Proposition 3.2 with full uncertainty aversion when $\alpha = 1$. It follows from Proposition 3.9 that the informed traders increase while the uninformed traders decrease their demand for the risky stock as they become more uncertainty averse. That is, $\frac{\partial X_I}{\partial \alpha} > 0$ and $\frac{\partial X_U}{\partial \alpha} < 0$. The increasing (resp. decreasing) demand of informed (resp. uninformed) traders with uncertainty aversion captures the fact that they are immune (resp. susceptible) to composition uncertainty during the trading stage. In addition, the perceived equity premium increases with the uncertainty aversion (i.e., $\frac{\partial g_\alpha(\lambda, \Delta\lambda)}{\partial \alpha} > 0$). Therefore, the stock price decreases with the uncertainty aversion following the REE price function in Eq. (3.38). These results suggest that in the baseline model, the demand of informed (resp. uninformed) traders takes its maximum (resp. minimum), the perceived equity premium takes its maximum, and the stock price takes its minimum value. Lastly, the results also suggest that there is a threshold uncertainty aversion below which the uncertainty premium component takes a negative value, leading to an uncertainty discount. We report the threshold uncertainty aversion and the comparative statics with respect to composition uncertainty $\Delta\lambda$ and uncertainty aversion α in the next corollary.

Corollary 3.10. In the presence of composition uncertainty;

(i) the informed traders increase while the uninformed traders decrease their demands for the risky stock with the uncertainty aversion α;

(ii) the perceived equity premium increases with the uncertainty aversion α;

(iii) there is an uncertainty aversion $\alpha^* = \frac{1}{2} + \frac{\Delta\lambda}{2\lambda}$ which equalizes the perceived equity premium in the presence and absence of composition uncertainty (i.e., $g_\alpha(\lambda, \Delta\lambda) = g(\lambda)$) and divides the perceived equity premium with composition uncertainty into uncertainty premium, $\alpha > \alpha^*$ and uncertainty discount, $\alpha < \alpha^*$, areas;

(iv) the uncertainty premium increases with composition uncertainty $\Delta\lambda$ if and only if

$$\alpha > \frac{1}{2} + \frac{\lambda \cdot \Delta\lambda}{\lambda^2 + (\Delta\lambda)^2} = \alpha^{**}. \tag{3.41}$$

These results ensure that in the presence of composition uncertainty, there exists an REE for the given proportion λ of informed traders. In this REE, the uncertainty premium prevails and increases with composition uncertainty when the traders are sufficiently uncertainty averse (i.e., $\alpha > \max(\alpha^*, \alpha^{**}) = \alpha^{**}$). On the contrary, the uncertainty discount prevails and increases with composition uncertainty when the traders are not sufficiently uncertainty averse (i.e., $\alpha < \min(\alpha^*, \alpha^{**}) = \alpha^*$)). While it is intuitive that more uncertainty reduces demand for the uncertain asset and increases the uncertainty premium (as shown in the baseline model), and consequently leads to an undervaluation of the stock, this is not always the case. These intuitive results only obtain when the uninformed traders are sufficiently uncertainty averse. When the traders are not sufficiently uncertainty averse, more uncertainty can increase demand for the uncertain asset, resulting in the uncertainty discount and stock overvaluation. The conditions for sufficient uncertainty aversion (i.e., α^* and α^{**}) decreases with the number of informed traders λ and increases with the uncertainty $\Delta\lambda$. This means a trader with a given uncertainty aversion can be sufficiently (resp. insufficiently) uncertainty averse for a high (resp. low) informed market and a low (resp. high) uncertainty.

The intuition of our results is analogous to Chapter 2 based on Aliyev and He (2018a) who show that the sufficiently pessimistic (resp. optimistic) liquidity provider or market maker facing uncertainty—quantified by non-additive probabilities with Choquet expected utility preference—about the stock payoff provides less (resp. more) liquidity by widening (resp. narrowing) the bid-ask spread. In a sequential trading model, Aliyev and He (2018a) show that this occurs because uncertainty about the payoff impacts the market maker's perceived proportion of informed traders (adverse selection risk). Unlike Aliyev and He (2018a), we model uncertainty about the proportion of informed traders explicitly in an otherwise fully revealing REE setting. In this setting, more uncertainty aversion always reduces the uninformed demand and increases the equity premium, but the same is not true for more uncertainty, leading to the undervaluation and overvaluation of the risky stock. In Figure 3.2, we contrast the benchmark (i.e., $\Delta\lambda = 0$), baseline maxmin (i.e., $\Delta\lambda > 0$, $\alpha = 1$), and extended $\alpha-$maxmin (i.e., $\Delta\lambda > 0$, $\alpha \in [0.5, 1]$) models.
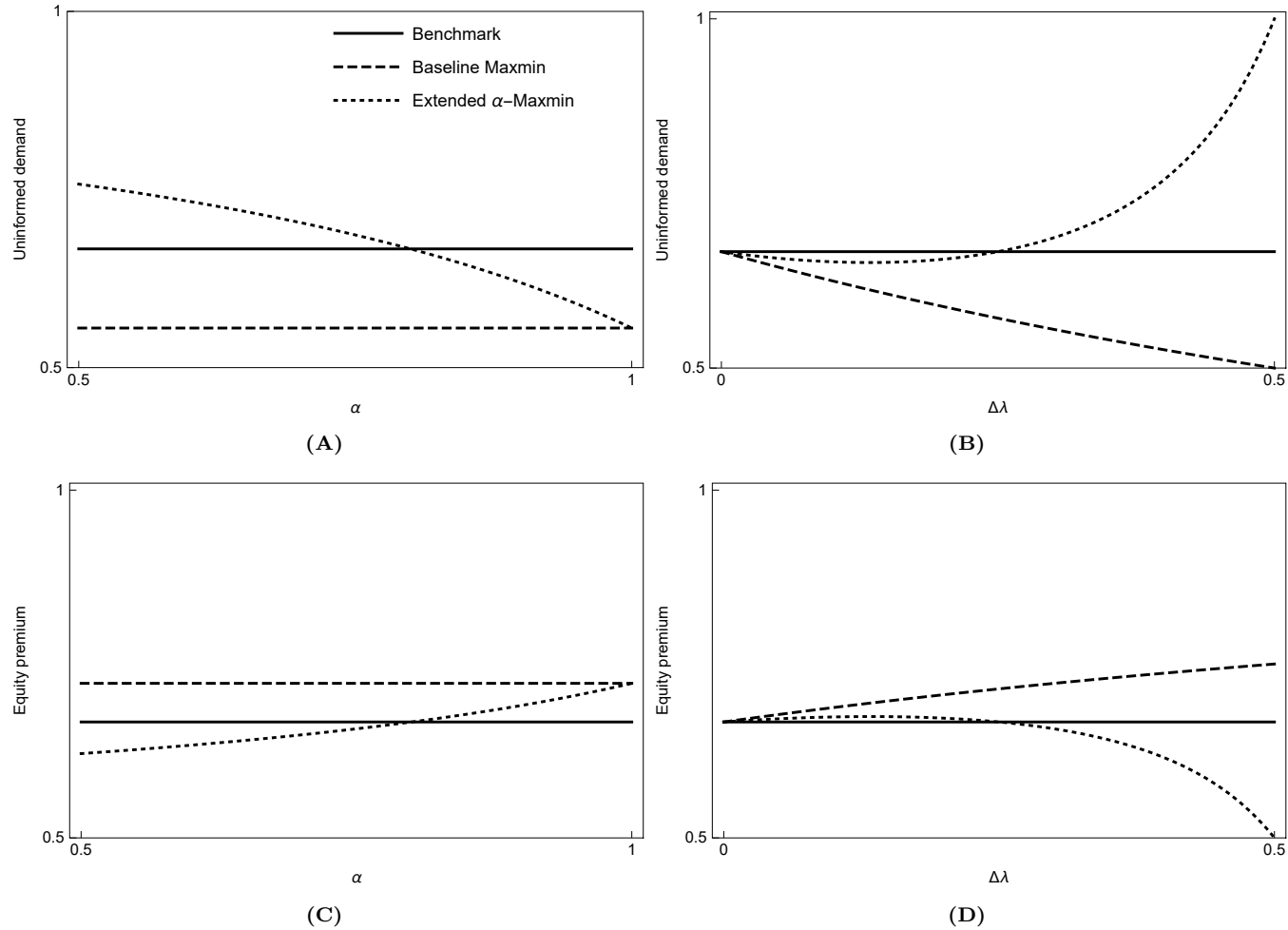
Figure 3.2: The uninformed demand X_U and the equity premium $(g(\lambda),\ g(\lambda, \Delta\lambda),\ g_\alpha(\lambda, \Delta\lambda))$ of uninformed traders against composition uncertainty $\Delta\lambda$ and uncertainty attitude α.

Panels (A) and (C) plot the demand X_U and the perceived equity premium of uninformed traders against uncertainty attitude α when uncertainty $\Delta\lambda = 0.3$. Panels (B) and (D) plot the demand X_U and the perceived equity premium of uninformed traders against composition uncertainty $\Delta\lambda$ when uncertainty attitude $\alpha = 0.75$. Other parameter values are $\lambda = 0.5$ (i.e., equally populated market), $\rho = 0.5$ (i.e., high correlation between $\tilde\eta$ and $\tilde\varepsilon$) and $\sigma_\varepsilon^2 = 1$ (i.e., standard normally distributed noise).

Panel (A) shows that in the baseline model, the uninformed traders always restrict their risky stockholdings relative to the benchmark. However, their demand increases as they become less uncertainty averse and becomes more than the benchmark when they are not sufficiently uncertainty averse. Panel (B) shows that the uninformed demand monotonically decreases with uncertainty about the proportion of informed traders in the baseline model, but it is non-monotonic in the extended model. Indeed, when the uninformed traders are not sufficiently uncertainty averse, their demand can increase with uncertainty $\Delta\lambda$. Panel (C) illustrates that there is always an uncertainty premium in the baseline model (i.e., $g_\alpha(\lambda, \Delta\lambda) > g(\lambda)$) and there is an uncertainty aversion α^* (given in Corollary 3.10) below which an uncertainty discount effect prevails (i.e., $g_\alpha(\lambda, \Delta\lambda) < g(\lambda)$). Similarly, Panel (D) illustrates that the perceived equity premium monotonically increases with uncertainty $\Delta\lambda$ about the proportion of informed traders when the traders are fully uncertainty averse, but this is not the case when $\alpha < \alpha^{**}$.

3.4.2 Information market equilibrium

In this subsection, we characterize the information market equilibrium to investigate the impacts of different uncertainty attitudes of traders on the benefit of informed trading. We first adapt Definition 3.5 of the information market equilibrium to the extended $\alpha-$maxmin model.

Definition 3.11. For a given cost of information c and $\Delta\lambda > 0$ an information market equilibrium is λ such that: $0 < \lambda - \Delta\lambda < \lambda + \Delta\lambda < 1$ and the ex ante expected utilities (in the sense of $\alpha-$maxmin) of informed and uninformed traders are equal,

$$\alpha \cdot \min E[u(\tilde{W}_I^\lambda)] + (1-\alpha) \cdot \max E[u(\tilde{W}_I^\lambda)] = \alpha \cdot \min E[u(\tilde{W}_U^\lambda)] + (1-\alpha) \cdot \max E[u(\tilde{W}_U^\lambda)],$$
(3.42)

where the min and max operators are taken with respect to $\hat{\lambda} \in [\lambda_1, \lambda_2]$.

The equilibrium follows from the same order of computing as the baseline model. For the informed traders, substituting their optimal demands into their expected utility, using the normality condition and final wealth obtain

$$E[u(\tilde{W}_I^\lambda)] = -\exp\left[-\left(\frac{g(\lambda, \Delta\lambda)^2}{2 \cdot \rho \cdot \sigma_\varepsilon^2} - c\right)\right].$$
(3.43)

The ex ante expected utility of informed traders in the extended $\alpha-$maxmin model follows from Eqs. (3.42) and (3.43) as

$$- \alpha \cdot \exp\Big[- \big(\frac{g_{\min}^2}{2 \cdot \rho \cdot \sigma_\varepsilon^2} - c\big)\Big] - (1 - \alpha) \cdot \exp\Big[- \big(\frac{g_{\max}^2}{2 \cdot \rho \cdot \sigma_\varepsilon^2} - c\big)\Big]. \qquad (3.44)$$

For the uninformed traders, the same order of calculations obtains

$$E[u(\tilde{W}_U^\lambda)] = - \exp\Big[- \frac{\big(\alpha \cdot g_{\min} + (1 - \alpha) \cdot g_{\max}\big)^2}{2 \cdot \sigma_\varepsilon^2}\Big]. \qquad (3.45)$$

The recursive utility representation of the ex ante expected utility of staying an uninformed is the same as Eq. (3.45).

Combining Eqs. (3.44) and (3.45) and solving for the cost or benefit of informed trading obtains

$$V_\alpha(\lambda, \Delta\lambda) = - \ln\Big[\alpha \cdot \exp(\frac{-g_{\min}^2}{2 \cdot \rho \cdot \sigma_\varepsilon^2}) + (1 - \alpha) \cdot \exp(\frac{-g_{\max}^2}{2 \cdot \rho \cdot \sigma_\varepsilon^2})\Big]$$
$$- \frac{\big(\alpha \cdot g_{\min} + (1 - \alpha) \cdot g_{\max}\big)^2}{2 \cdot \sigma_\varepsilon^2}, \qquad (3.46)$$

where the first and second terms show the ex ante expected wealth of informed and uninformed traders, respectively. The equilibrium proportion of informed traders similarly follows from $V_\alpha(\lambda, \Delta\lambda) = c$. When $\alpha = 1$, the benefit of informed trading $V_\alpha(\lambda, \Delta\lambda)$ reduces to that of the baseline model in Eq. (3.28) (recall that in the benchmark and baseline models $V(\lambda, \Delta\lambda)$ monotonically decreases in λ). Interestingly, Eq. (3.46) also shows that the benefit of informed trading in the extended $\alpha-$maxmin model is non-monotonic in the proportion of informed traders. Unlike the benchmark and baseline models, when the traders are not fully uncertainty averse an information purchase by one trader can increase other trader's information demand.

When the traders are sufficiently uncertainty averse (i.e., $\alpha > \alpha^{**}$), the increase in the proportion of informed traders λ decreases the equity premium and demands of both types of traders due to enhanced competition (see Corollary 3.3 (ii) and (iv)), leading to a lower benefit of informed trading $V_\alpha(\lambda, \Delta\lambda)$. When the traders are not sufficiently uncertainty averse $\alpha < \alpha^*$, however, the increase in λ can increase the perceived equity premium, and consequently the benefit of informed trading $V_\alpha(\lambda, \Delta\lambda)$. Moreover, it follows from Corollary 3.10 (iii) and

(iv) that the sufficient level of uncertainty aversion (i.e., α^* and α^{**}) for the uncertainty premium to prevail and increase with composition uncertainty decreases in the proportion of informed traders (i.e., $\frac{\partial \alpha^*}{\partial \lambda} < 0$ and $\frac{\partial \alpha^{**}}{\partial \lambda} < 0$). That means the traders with the same uncertainty aversion α can be sufficiently uncertainty averse for the high proportion of informed traders λ, but not as sufficient for the low λ. These intuitive results lead to what may seem counter-intuitive (the benefit of informed trading is non-monotonic in the proportion of informed traders). Switching between insufficiently uncertainty averse (i.e., $\alpha < \alpha^*$) to sufficiently uncertainty averse (i.e., $\alpha > \alpha^{**}$) due to an increase in the number of informed traders causes the benefit of informed trading to be non-monotonic in the number of informed traders.

Figure 3.3 illustrates the sufficient level of uncertainty aversion, perceived equity premium, and benefit of informed trading when the uncertainty aversion $\alpha = 0.75$ and uncertainty $\Delta \lambda = 0.1$ (the horizontal axis $\lambda \in (0.1, 0.9)$ so that $\lambda_1 > 0$ and $\lambda_2 < 1$). Panel (A) illustrates that traders with the given uncertainty aversion $\alpha = 0.75$ can be sufficiently uncertainty averse (i.e., $\alpha > \alpha^{**}$) for high, but not as sufficient (i.e., $\alpha < \alpha^*$) for low informed trading. By contrasting the perceived equity premium in the benchmark, baseline, and extended models, Panel (B) shows that the perceived equity premium in the benchmark and baseline models monotonically decreases in the amount of informed trading λ, whereas it can increase in the extended model when the informed trading is low. Consequently, Panel (C) shows the well-known negative relation between the benefit of informed trading and the number of informed traders in the benchmark and baseline models. In the extended α-maxmin model, however, the benefit of informed trading is no longer monotonic in the number of informed traders, leading to the information complementarity and multiple information market equilibria.

The literature has shown different mechanisms that lead to information complementarities (i.e., more agents acquire information and makes it more valuable for the uninformed agents to acquire private information) and multiple information market equilibria. These mechanisms include, for instance, a correlation between fundamentals and noisy supply (e.g., Barlevy and Veronesi (2000, 2008), a competitive information market (e.g., Veldkamp (2006)), large shocks independently affecting the fundamental value and the exogenous trading (e.g., Chamley (2010)), private information about both dividends and supply (e.g., Ganguli and Yang (2009)), relative wealth concerns of the agents (e.g., Garcia and Strobl (2011)), a departure from the normality assumption (e.g., Breon-Drish (2012)), a trade-off
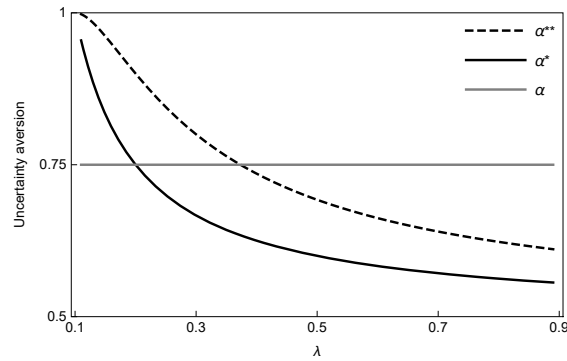
(A)
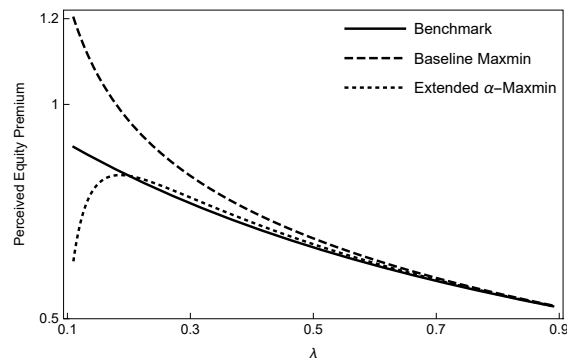
(B)
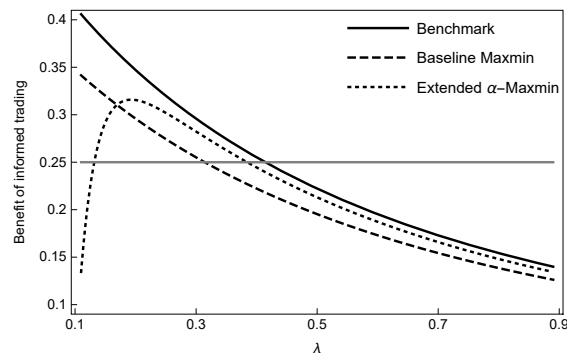
(C)

Figure 3.3: The uncertainty aversion (α, α^*, and α^{**}), perceived equity premium ($g(\lambda)$, $g(\lambda, \Delta\lambda)$, and $g_\alpha(\lambda, \Delta\lambda)$), and benefit of informed trading ($V(\lambda)$, $V(\lambda, \Delta\lambda)$, and $V_\alpha(\lambda, \Delta\lambda)$) against the proportion of informed traders λ.
Panel (A) plots the sufficient uncertainty aversion conditions α^* and α^{**}, and the traders' uncertainty aversion $\alpha = 0.75$, Panel (B) plots the perceived equity premium in the benchmark $g(\lambda)$, baseline $g(\lambda, \Delta\lambda)$, and extended $g_\alpha(\lambda, \Delta\lambda)$ models, and Panel (C) plots the benefit of informed trading in the benchmark $V(\lambda)$, baseline $V(\lambda, \Delta\lambda)$, and extended $V_\alpha(\lambda, \Delta\lambda)$ models against the proportion of informed traders λ. A fixed cost of information $c = 0.25$. Other parameter values are $\rho = 0.5$, $\Delta\lambda = 0.1$, and $\sigma_\varepsilon^2 = 1$.

between dividend information and supply information (e.g., Avdis (2016)), and ambiguity about cash flows (e.g., Mele and Sangiorgi (2015)). This chapter adds Knightian uncertainty or ambiguity about the composition of traders to this list.

What's interesting about the information complementarities stemming from composition uncertainty is that it has an ability to conceal itself in different characterizations of the traders' preferences. The composition uncertainty and uncertainty aversion as in the baseline maxmin model are not enough to reveal the non-monotonic relation between the benefit and level of informed trading. The sufficient uncertainty aversion condition inversely related to the proportion of informed traders is crucial to generate this non-monotonicity.

We conclude this section by illustrating the benefit of informed trading in the extended α-Maxmin and benchmark models. In Figure 3.4, $V(\lambda)$ (black fixed layer) and $V_\alpha(\lambda, \Delta\lambda)$ (white curved layer) plot the benefit of informed trading in the absence and presence of composition uncertainty in Eqs. (3.30) and (3.46), respectively. Similar to the baseline model, the Knightian component is determined by the difference between the two (i.e., $V_\alpha(\lambda, \Delta\lambda) = V(\lambda) + K_\alpha(\lambda, \Delta\lambda)$). The figure shows that unlike the baseline model, the Knightian component is value enhancing (i.e., $K_\alpha(\lambda, \Delta\lambda) > 0$) when the traders are sufficiently uncertainty averse and reducing (i.e., $K_\alpha(\lambda, \Delta\lambda) < 0$) when they are not. This is intuitive because when the uninformed traders are sufficiently uncertainty averse, their demand decreases with the amount of uncertainty $\Delta\lambda$, leading to a lower benefit to the informed traders and vice versa. Figure 3.4 also shows that similar to the equity premium, there is an uncertainty aversion α^{***} which equalizes the value of information in the presence and absence of composition uncertainty (i.e., $V_\alpha(\lambda, \Delta\lambda) = V(\lambda)$), leading the Knightian component to be zero. When $\alpha < \alpha^{***}$ (resp. $\alpha > \alpha^{***}$), the Knightian component is value enhancing (resp. value reducing).
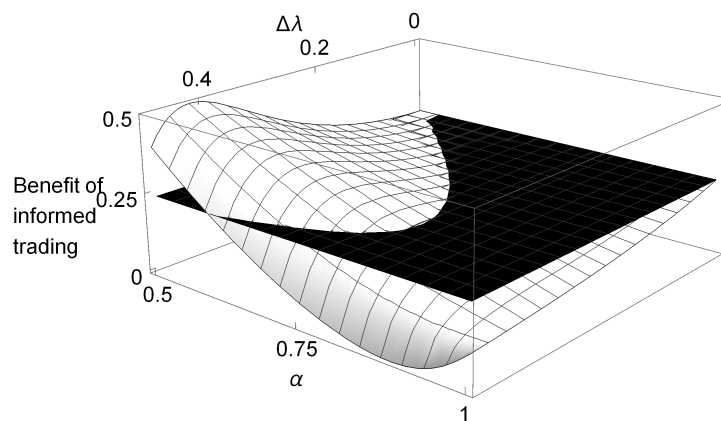
Figure 3.4: The benefit of informed trading $(V(\lambda), V_\alpha(\lambda, \Delta\lambda))$ against composition uncertainty $\Delta\lambda$ and uncertainty aversion α.
The figure plots the benefit of informed trading in the presence (white curved layer) and in the absence (black fixed layer) of composition uncertainty against the amount of composition uncertainty $\Delta\lambda$ and uncertainty attitude α of the traders. The parameter values are $\rho = 0.25$, $\lambda = 0.5$ and $\sigma_\varepsilon^2 = 1$.

3.5   Empirical Implications

In this section, we discuss the empirical implications of our model.

3.5.1   Liquidity and asset prices

A large number of empirical market microstructure studies link the asset prices
to a variety of liquidity measures such as bid-ask spreads, market depths and vol-
umes (e.g., Amihud (2002), Brennan and Subrahmanyam (1996)). For example,
Amihud and Mendelson (1986) find that market-observed expected return is in-
creasing in the spread and Amihud (2002) finds that unanticipated increases in
market illiquidity reduce the level of stock prices. Our model provides a theoretical
explanation for the empirical findings about the impacts of liquidity on prices. In
our model, when composition uncertainty increases, the sufficiently uncertainty-
averse uninformed traders reduce their demand (liquidity provision) and require
an extra premium as a compensation, leading to a reduction in the stock price,
explaining the pricing implications of liquidity.

A large volume of work has also documented a stock market overvaluation (e.g.,
dot-com bubble) and undervaluation (e.g., global financial crisis), that are hard to
explain with the standard asset pricing models. These results are often explained
by behavioral theories in which the degree of mispricing depends on unobservable
biases in investors' beliefs (e.g., Barberis et al. (1998), and Hong and Stein (1999)).
Our model with composition uncertainty offers an alternative unified explanation
for the stock undervaluation and overvaluation since the liquidity in our model has
a first order effect on the level of asset prices. Our results build upon a rational
response by investors in the presence of trading friction, reverberating practical
challenges faced by the real world market participants.

3.5.2   Extreme price movements

In modern financial markets, instantaneous and extreme price movements (i.e.,
micro flash crashes) occur nearly every day.[28] Our model proposes two plausible

---

[28]See, for example, "Flash crashes more common than thought in world's biggest market"
(Bloomberg, December 7, 2017), "The stock market has about 12 mini flash crashes a day —
and we can't prevent them" (MarketWatch, July 31, 2017).

explanations for this phenomena, even without substantial news about the fundamentals. First, a sudden increase in the uncertainty about the informed trading $\Delta\lambda$ can cause the sufficiently uncertainty averse traders to significantly increase their perceived equity premium, leading to a significant price drop. Similarly, when the traders are not sufficiently uncertainty averse, a sudden increase in the uncertainty about the informed trading can cause a significant reduction in the traders' perceived equity premium and result in a large price surge.

Second, the complementarity in information acquisition and equilibrium multiplicity in our model can also help to explain the price jumps. When the risky stock has a high fundamental value, given the cost of information, in equilibrium the economy can be populated by a high or low number of informed traders. While a high fundamental value with a high informed trading is associated with a high price, a high fundamental value with a low informed trading is associated with a low price. Jumping from the high (resp. low) informed trading equilibrium to the low (resp. high) informed trading equilibrium can cause a large price drop (resp. surge) in financial markets. A similar argument holds when the stock has a low fundamental value.

3.5.3 Policy implications

One general but important policy message that follows from our analysis is that a regulator must pay attention to uncertainty about the financial market microstructure. In our model, the uninformed traders draw correct but incomplete inferences from the price due to uncertainty about the composition of traders. Although the informed and uninformed traders are both subject to composition uncertainty, the nature of trading disadvantages the uninformed compared to the informed traders, even in the absence of noisy supply (e.g., Grossman and Stiglitz (1980), Hellwig (1980)). This is of particular interest because the capital market regulation is mainly concerned with maintaining a level playing field for the market participants. Thus the regulator must monitor uncertainty about the market characteristics and maintain an equal environment for everyone to ensure price efficiency.

Our analysis also suggests that reducing the cost of information or greater disclosure rules such as the FAS rule 157 and a mark-to-market accounting legislation implemented in 2007, may aggravate market efficiency. The complementarity of

information acquisition decisions of traders suggests that merely reducing the cost of information can reduce the number of informed traders, leading the prices to be less informative about the fundamentals. On the other hand, due to the multiplicity of equilibrium for the given cost of information, increasing this cost may also decrease the number of informed traders, leading to the same outcome of less informative prices. Thus our analysis suggests that a palliative approach of increasing or decreasing the cost of information to enhance market efficiency may not work without maintaining a fair trading environment that treats traders equally.

3.6 Model Discussion and Extensions

In this chapter, we develop an REE model to investigate the pricing implications of uncertainty about the composition of traders. A natural concern in our analysis is the simplicity of our model. In this section, we discuss some other natural extensions of our baseline model.

3.6.1 Dynamic analysis

Our model incorporates information acquisition stage and the trading stage after which all uncertainty is resolved. Allowing for multiple periods complicates the analysis but for some reasonable specifications does not change our results. As in the standard REE framework, it is straightforward to demonstrate that if new information arrives every period and information is independent across periods, then the resulting equilibrium remains the same. In addition, it seems to us that the strategic trader models (e.g., Kyle (1985)) and sequential trading models (e.g., Glosten and Milgrom (1985)) are more suitable for the dynamic analysis of composition uncertainty. Aliyev et al. (2018) focus on the dynamic analysis of composition uncertainty in a sequential trading model to show that composition uncertainty can lead to market instability and sharp price movements in financial markets in the face of order imbalance, explaining the prevalence of flash crashes in the algorithmic era.

3.6.2   Other sources of uncertainty

The other natural extension of the model is to allow the market participants to have uncertainty not only about the proportion of informed traders but also other dimensions of informed trading such as the quality of their information. In our model, the quality of informed traders' information is determined by the residual randomness σ_ε^2 in the payoff. Thus, this extension is equivalent to σ_ε^2 to be uncertain (i.e., $\sigma_\varepsilon^2 \in [\underline{\sigma}_\varepsilon^2, \bar{\sigma}_\varepsilon^2]$) (see Epstein and Schneider (2008) and Illeditsch (2011) for models characterizing uncertainty through an interval of signal variances). In the presence of uncertainty about the proportion and quality of informed traders' information, the optimal demands of informed and uninformed traders are unchanged with the same equilibrium price function given Eq. (3.16), and the perceived equity premium is given by

$$g(\lambda, \Delta\lambda) = \frac{\rho \cdot \bar{\sigma}_\varepsilon^2}{\lambda} \cdot \left(\frac{\lambda + \Delta\lambda \cdot (1 - \rho)}{\rho + (\lambda + \Delta\lambda) \cdot (1 - \rho)} \right). \tag{3.47}$$

Eq. (3.47) shows that the perceived equity premium with other source of uncertainty is higher, and consequently the stock price is lower than that of the baseline model with only uncertainty about the proportion of informed traders.

3.6.3   Multiple assets

Our model features a small economy with one risk-free bond, one risky asset available to everyone and additional investment opportunities available to only informed traders. It is also interesting to see how uncertainty about the composition of market participants would play out in a large economy with multiple risky assets. In fact, in a noisy REE with multiple assets, Admati (1985) shows that uncertainty about other assets' supplies may prevent the prices of assets in fixed supply from being fully revealing, leading to a positive value of private information. This is in line with our analysis of the risky asset in a fixed supply having a positive value of information due to a correlation ρ^* between additional investment opportunities and idiosyncratic noise. Admati (1985) also shows that the market participants have different risk-return tradeoff because of the diverse information they hold. This is also consistent with our finding that uninformed traders require an uncertainty premium as a compensation for uncertainty about the proportion of

informed traders. Further studies are needed in multi-uncertainty and multi-asset setting.

3.7 Conclusion

We construct an REE to demonstrate the effects of uncertainty about the composition of traders on the equilibrium demands of traders, perceived equity premium, stock prices, and benefit of informed trading. To capture uncertainty about the composition of traders, we employ the maxmin expected utility preference in our baseline model and extend the analysis with the $\alpha-$maxmin preference to explicitly separate uncertainty and uncertainty attitude of traders. The generalization of the standard REE to capture composition uncertainty helps to understand the crowded-trade problem and can explain empirical regularities which are difficult to explain with the standard REE.

Uncertainty about the composition of traders introduces a variety of complications faced by the real world market participants not captured by the standard models. During the trading stage, although exposed, the informed traders are immune to composition uncertainty since they don't use the uncertain price function, leading the uncertainty-averse uninformed traders to be disadvantaged in the face of composition uncertainty. Our baseline model shows that composition uncertainty induces an uncertainty premium which increases in the amount of composition uncertainty and leads to a stock undervaluation relative to the informationally efficient benchmark with no such uncertainty. The benefit of informed trading in this setting naturally incorporates the standard and uncertain Knightian components. Since the traders are fully uncertainty averse in the baseline model, the Knightian component is always value reducing and further decreases in the amount of uncertainty. As in the standard REE, the benefit of informed trading in the baseline model monotonically decreases in the number of informed traders. In the extended model, we show that the perceived equity premium increases and the stock price decreases in traders' uncertainty aversion. We characterize the sufficient level of uncertainty aversion for our results in the baseline model to hold and show that the opposite scenario (i.e., uncertainty discount and stock overvaluation) can obtain when traders are not sufficiently uncertainty averse. The condition for the sufficient level of uncertainty aversion is inversely related to the number of informed traders. Consequently, the benefit of informed trading becomes non-monotonic

in the number of informed traders, leading to complementarities in information acquisition and multiple information market equilibria.

In general, asset pricing models focus on uncertainty about the fundamental values of assets and assume that the market microstructure costs are not priced in equilibrium due to the symmetric information structure. The analysis in this chapter emphasizes the importance of uncertainty about the market characteristics in determining asset prices, and consequently shows that the liquidity and stock prices are intertwined, and suggests a unified explanation for the stock market over/undervaluation, a channel for extreme price movements, and important policy implications.

Appendix 3.1. Proofs

Proof of Proposition 3.2. We start the proof by the possible optimal demand functions of the uninformed traders which follow from the first order conditions (F.O.C.) of Eq. (3.14) as

$$
X_U = \begin{cases}
\frac{g_{\min}}{\sigma_\varepsilon^2}, & \text{if} \quad g_{\min} > 0, \\[2mm]
0, & \text{if} \quad g_{\min} < 0 < g_{\max}, \\[2mm]
\frac{g_{\max}}{\sigma_\varepsilon^2}, & \text{if} \quad g_{\max} < 0.
\end{cases}
\tag{A3.1.1}
$$

We then assess the feasibility of the possible optimal demand functions of the uninformed traders given in Eq. (A3.1.1) case by case. First, substituting

$$
X_I = \frac{(\bar{f} + \tilde{\theta} - \tilde{p})}{\rho \cdot \sigma_\varepsilon^2} \quad \text{and} \quad X_U = \frac{g_{\min}}{\sigma_\varepsilon^2} \quad \text{for} \quad g_{\min} > 0,
\tag{A3.1.2}
$$

into the market clearing condition and solving for the equilibrium price yields

$$
\tilde{p} = \bar{f} + \tilde{\theta} - \frac{\rho \cdot \left(\sigma_\varepsilon^2 - (1 - \lambda) \cdot g_{\min}\right)}{\lambda}.
\tag{A3.1.3}
$$

Comparing the equilibrium price in Eq. (A3.1.3) with the conjectured price function of the uninformed traders in Eq. (3.11), we obtain the perceived equity premium as

$$
g(\lambda, \Delta\lambda) = \frac{\rho \cdot \left(\sigma_\varepsilon^2 - (1 - \lambda) \cdot g_{\min}\right)}{\lambda}.
\tag{A3.1.4}
$$

The perceived equity premium $g(\lambda, \Delta\lambda)$ in Eq. (A3.1.4) takes its minimum at λ_1 if $g_{\min} > \sigma_\varepsilon^2$ and at λ_2 if $g_{\min} < \sigma_\varepsilon^2$. Assuming $g(\lambda, \Delta\lambda)$ takes its minimum at λ_2, we obtain

$$
g_{\min} = \frac{\rho \cdot \sigma_\varepsilon^2}{\rho + \lambda_2 \cdot (1 - \rho)} > 0,
\tag{A3.1.5}
$$

which is consistent with $g_{\min} < \sigma_\varepsilon^2$ since $0 < \rho < 1$ and $0 < \lambda_2 < 1$. Assuming $g(\lambda, \Delta\lambda)$ takes its minimum at λ_1 leads to $g_{\min} < \sigma_\varepsilon^2$ which contradicts that Eq. (A3.1.4) takes its minimum at λ_1 when $g_{\min} > \sigma_\varepsilon^2$. Second, substituting

$$
X_I = \frac{(\bar{f} + \tilde{\theta} - \tilde{p})}{\rho \cdot \sigma_\varepsilon^2} \quad \text{and} \quad X_U = 0 \quad \text{for} \quad g_{\min} < 0 < g_{\max},
\tag{A3.1.6}
$$

into the market clearing condition and rearranging for the equilibrium price function obtains the perceived equity premium $g(\lambda, \Delta\lambda) = \frac{\rho \cdot \sigma_\varepsilon^2}{\lambda}$, which is minimum at

λ_2. However, $g_{\min} = \frac{\rho \cdot \sigma_\varepsilon^2}{\lambda_2}$ is inconsistent with $g_{\min} < 0 < g_{\max}$. Lastly, substituting

$$X_I = \frac{(\bar{f} + \tilde{\theta} - \tilde{p})}{\rho \cdot \sigma_\varepsilon^2} \quad \text{and} \quad X_U = \frac{g_{\max}}{\sigma_\varepsilon^2} \quad \text{for} \quad g_{\max} < 0 \tag{A3.1.7}$$

into the market clearing condition obtains

$$g(\lambda, \Delta\lambda) = \frac{\rho \cdot \left(\sigma_\varepsilon^2 - (1-\lambda) \cdot g_{\max}\right)}{\lambda}, \tag{A3.1.8}$$

which is maximum at λ_2 if $g_{\max} > \sigma_\varepsilon^2$ and at λ_1 if $g_{\max} < \sigma_\varepsilon^2$. The former case is inconsistent with $g_{\max} > \sigma_\varepsilon^2$ and the latter is inconsistent with $g_{\max} < 0$. Hence, the optimal demand of uninformed traders is given by

$$X_U = \frac{g_{\min}}{\sigma_\varepsilon^2} = \frac{\rho}{\rho + \lambda_2 \cdot (1-\rho)}, \tag{A3.1.9}$$

where $g_{\min} > 0$ is given by Eq. (A3.1.5) and the equilibrium price function is given by

$$\tilde{p} = \bar{f} + \tilde{\theta} - g(\lambda, \Delta\lambda), \tag{A3.1.10}$$

where the last term is the perceived equity premium and is given by

$$g(\lambda, \Delta\lambda) = \frac{\rho \cdot \sigma_\varepsilon^2}{\lambda} \cdot \left(\frac{\lambda + \Delta\lambda \cdot (1-\rho)}{\rho + (\lambda + \Delta\lambda) \cdot (1-\rho)} \right), \tag{A3.1.11}$$

leading to

$$X_I = \frac{(\bar{f} + \tilde{\theta} - \tilde{p})}{\rho \cdot \sigma_\varepsilon^2} = \frac{g(\lambda, \Delta\lambda)}{\rho \cdot \sigma_\varepsilon^2} = \frac{\lambda + \Delta\lambda \cdot (1-\rho)}{\lambda \cdot \left(\rho + (\lambda + \Delta\lambda) \cdot (1-\rho)\right)}. \tag{A3.1.12}$$

∎

Proof of Corollary 3.3. (i) follows from the partial derivatives of informed and uninformed demands in Eqs. (A3.1.9) and (A3.1.12) w.r.t. $\Delta\lambda$,

$$\frac{\partial X_I}{\partial \Delta\lambda} = \frac{\rho \cdot (1-\rho) \cdot (1-\lambda)}{\lambda \cdot \left(\rho + (\lambda + \Delta\lambda) \cdot (1-\rho)\right)^2} > 0, \tag{A3.1.13}$$

$$\frac{\partial X_U}{\partial \Delta\lambda} = -\frac{\rho \cdot (1-\rho)}{\left(\rho + (\lambda + \Delta\lambda) \cdot (1-\rho)\right)^2} < 0. \tag{A3.1.14}$$

(ii) follows from the partial derivatives of informed and uninformed demands w.r.t. λ,

$$\frac{\partial X_I}{\partial \lambda} = -\frac{(1-\rho) \cdot \left(\lambda^2 + \Delta\lambda^2 \cdot (1-\rho) + \Delta\lambda \cdot (\rho + 2 \cdot \lambda \cdot (1-\rho)) \right)}{\lambda^2 \cdot \left(\rho + (\lambda + \Delta\lambda) \cdot (1-\rho) \right)^2} < 0,$$

(A3.1.15)

$$\frac{\partial X_U}{\partial \lambda} = -\frac{\lambda + \Delta\lambda}{\left(\rho + (\lambda + \Delta\lambda) \cdot (1-\rho) \right)^2} > 0.$$

(A3.1.16)

(iii) follows from the partial derivatives of informed and uninformed demands w.r.t. ρ,

$$\frac{\partial X_I}{\partial \rho} = -\frac{(1-\lambda) \cdot (\lambda + \Delta\lambda)}{\lambda \cdot \left(\rho + (\lambda + \Delta\lambda) \cdot (1-\rho) \right)^2} < 0,$$

(A3.1.17)

$$\frac{\partial X_U}{\partial \rho} = \frac{\lambda + \Delta\lambda}{\left(\rho + (\lambda + \Delta\lambda) \cdot (1-\rho) \right)^2} > 0.$$

(A3.1.18)

The opposite obtains for the correlation ρ^* since $\rho = 1 - \rho^{*2}$

(iv) follows from the partial derivatives of perceived equity premium $g(\lambda, \Delta\lambda)$ in Eq. (A3.1.11) w.r.t. $\Delta\lambda$, σ_ε^2, λ, and ρ,

$$\frac{\partial g(\lambda, \Delta\lambda)}{\partial \Delta\lambda} = \frac{\rho \cdot \sigma_\varepsilon^2}{\lambda} \cdot \frac{\rho \cdot (1-\rho) \cdot (1-\lambda)}{\left(\rho + (\lambda + \Delta\lambda) \cdot (1-\rho) \right)^2} > 0,$$

(A3.1.19)

$$\frac{\partial g(\lambda, \Delta\lambda)}{\partial \sigma_\varepsilon} = \frac{2\rho \cdot \sigma_\varepsilon}{\lambda} \cdot \left(\frac{\lambda + \Delta\lambda \cdot (1-\rho)}{\rho + (\lambda + \Delta\lambda) \cdot (1-\rho)} \right) > 0,$$

(A3.1.20)

$$\frac{\partial g(\lambda, \Delta\lambda)}{\partial \lambda} = -\frac{\rho \cdot (1-\rho) \cdot \left(\lambda^2 + \Delta\lambda^2 \cdot (1-\rho) + \Delta\lambda \cdot (\rho + 2 \cdot \lambda \cdot (1-\rho)) \right) \cdot \sigma_\varepsilon^2}{\lambda^2 \cdot \left(\rho + (\lambda + \Delta\lambda) \cdot (1-\rho) \right)^2} < 0,$$

(A3.1.21)

$$\frac{\partial g(\lambda, \Delta\lambda)}{\partial \rho} = \frac{\left(\lambda^2 + \Delta\lambda^2 \cdot (1-\rho)^2 + \Delta\lambda \cdot \left(\lambda \cdot (2 - 2\rho + \rho^2) - \rho^2 \right) \right) \sigma_\varepsilon^2}{\lambda \cdot \left(\rho + (\lambda + \Delta\lambda) \cdot (1-\rho) \right)^2} > 0,$$

(A3.1.22)

if and only if

$$\rho < \frac{\lambda_2}{1 - \lambda_2} \cdot \left(\sqrt{\frac{(1-\lambda)}{\Delta\lambda}} - 1 \right).$$

(A3.1.23)

(v) follows from (iv) and the equilibrium price function in Eq. (A3.1.10).

■

Proof of Corollary 3.4. The proofs of (i), (ii), and (iii) are immediate from the proofs of Corollary 3.3 (i), (iv), (v) respectively (or one can write explicitly the demands, perceived equity premium, and the stock price with no composition uncertainty by taking $\Delta\lambda = 0$ in Eqs. (A3.1.9), (A3.1.10), (A3.1.11), and (A3.1.12) and compare them directly to Eqs. (A3.1.9), (A3.1.10), (A3.1.11), and (A3.1.12) for $\Delta\lambda > 0$).

■

Proof of Corollary 3.6. (i) follows from the partial derivatives Eq. (3.28) w.r.t λ, $\Delta\lambda$, and σ_ε^2, respectively

$$\frac{\partial V(\lambda, \Delta\lambda)}{\partial \lambda} = -\frac{\rho \cdot (1-\rho)^2 \cdot \sigma_\varepsilon^2}{\big(\rho + (\lambda + \Delta\lambda) \cdot (1-\rho)\big)^3} < 0, \tag{A3.1.24}$$

$$\frac{\partial V(\lambda, \Delta\lambda)}{\partial \Delta\lambda} = -\frac{\rho \cdot (1-\rho)^2 \cdot \sigma_\varepsilon^2}{\big(\rho + (\lambda + \Delta\lambda) \cdot (1-\rho)\big)^3} < 0. \tag{A3.1.25}$$

$$\frac{\partial V(\lambda, \Delta\lambda)}{\partial \sigma_\varepsilon} = \frac{\rho \cdot (1-\rho) \cdot \sigma_\varepsilon}{\big(\rho + (\lambda + \Delta\lambda) \cdot (1-\rho)\big)^2} > 0. \tag{A3.1.26}$$

(ii) $K(\lambda, \lambda, \Delta\lambda) < 0$ follows from Eq. (3.31) proving that the Knightian component is value reducing. Partial derivatives of $K(\lambda, \Delta\lambda)$ w.r.t. λ, $\Delta\lambda$, and σ_ε obtain

$$\frac{\partial K(\lambda, \Delta\lambda)}{\partial \lambda} = \rho \cdot (1-\rho)^2 \cdot \left(\frac{1}{\big(\rho + \lambda \cdot (1-\rho)\big)^3} - \frac{1}{\big(\rho + (\lambda + \Delta\lambda) \cdot (1-\rho)\big)^3} \right) \cdot \sigma_\varepsilon^2 > 0, \tag{A3.1.27}$$

$$\frac{\partial K(\lambda, \Delta\lambda)}{\partial \Delta\lambda} = -\frac{\rho \cdot (1-\rho)^2 \cdot \sigma_\varepsilon^2}{\big(\rho + (\lambda + \Delta\lambda) \cdot (1-\rho)\big)^3} < 0, \tag{A3.1.28}$$

$$\frac{\partial K(\lambda, \Delta\lambda)}{\partial \sigma_\varepsilon} = \rho \cdot (1-\rho) \cdot \left(\frac{1}{\big(\rho + (\lambda + \Delta\lambda) \cdot (1-\rho)\big)^2} - \frac{1}{\big(\rho + \lambda \cdot (1-\rho)\big)^s} \right) \cdot \sigma_\varepsilon < 0. \tag{A3.1.29}$$

■

Proof of Proposition 3.7. For the information market equilibrium to exist it must be the case that $V(\lambda, \Delta\lambda)$ in Eq. (3.28) equals to the cost of information c,

i.e.,

$$\frac{\rho \cdot (1 - \rho)}{\left(\rho + (\lambda + \Delta\lambda) \cdot (1 - \rho)\right)^2} \cdot \frac{\sigma_\varepsilon^2}{2} = c. \tag{A3.1.30}$$

Since Eq. (A3.1.30) decreases with the level of informed trading λ and the composition uncertainty $\Delta\lambda$, c takes its lower bound when $\lambda + \Delta\lambda = 1$ and the upper bound when $\lambda - \Delta\lambda = 0$. Solving λ from Eq. (A3.1.30) obtains

$$\lambda = \frac{\sqrt{\frac{\rho \cdot (1-\rho) \cdot \sigma_\varepsilon^2}{2 \cdot c}} - \rho}{\left(1 - \rho\right)} - \Delta\lambda. \tag{A3.1.31}$$

It follows from Eq. (A3.1.31) that the lower and the upper bounds of the cost range for the unique λ to exist such that $\lambda_2 < 1$ and $\lambda_1 > 0$ are, respectively, given by

$$\underline{c} = \rho \cdot (1 - \rho) \cdot \frac{\sigma_\varepsilon^2}{2}, \tag{A3.1.32}$$

$$\bar{c} = \frac{\rho \cdot (1 - \rho)}{\left(\rho + 2 \cdot (1 - \rho) \cdot \Delta\lambda\right)^2} \cdot \frac{\sigma_\varepsilon^2}{2}. \tag{A3.1.33}$$

Finally, we show that the size of the cost range $(\bar{c} - \underline{c})$ decreases in $\Delta\lambda$. Following Eqs. (A3.1.32) and (A3.1.33)

$$\bar{c} - \underline{c} = \rho \cdot (1 - \rho) \cdot \left(\frac{1}{\left(\rho + 2 \cdot (1 - \rho) \cdot \Delta\lambda\right)^2} - 1\right) \cdot \frac{\sigma_\varepsilon^2}{2}, \tag{A3.1.34}$$

and the partial derivative of $(\bar{c} - \underline{c})$ w.r.t. $\Delta\lambda$ yields

$$\frac{\partial(\bar{c} - \underline{c})}{\partial\Delta\lambda} = -\frac{2 \cdot \rho \cdot (1 - \rho)^2 \cdot \sigma_\varepsilon^2}{\left(\rho + 2 \cdot (1 - \rho) \cdot \Delta\lambda\right)^3} < 0. \tag{A3.1.35}$$

$\blacksquare$

Proof of Corollary 3.8. (i) follows from

$$\underbrace{\frac{\partial g(\lambda, \Delta\lambda)}{\partial\lambda}}_{-} = \underbrace{\frac{\partial g(\lambda, \Delta\lambda)}{\partial c}}_{+} \cdot \underbrace{\frac{\partial c}{\partial\lambda}}_{-}, \tag{A3.1.36}$$

where $\frac{\partial g(\lambda, \Delta\lambda)}{\partial\lambda}$ and $\frac{\partial c}{\partial\lambda}$ are given in Eqs. (A3.1.21) and (A3.1.24) respectively. (ii) follows from (i) and the equilibrium price function in Eq. (A3.1.10).

$\blacksquare$

Proof of Proposition 3.9. The proof follows similar to the proof of Proposition 3.2. The possible optimal demand functions of uninformed traders follow from the F.O.C. of Eq. (3.36) as

$$
X_U = \begin{cases}
\dfrac{\alpha \cdot g_{\min} + (1-\alpha) \cdot g_{\max}}{\sigma_\varepsilon^2}, & \text{if} \quad \alpha \cdot g_{\min} + (1-\alpha) \cdot g_{\max} > 0, \\[2ex]
0, \quad \text{if} \quad \alpha \cdot g_{\min} + (1-\alpha) \cdot g_{\max} < 0 < \alpha \cdot g_{\max} + (1-\alpha) \cdot g_{\min}, \\[2ex]
\dfrac{\alpha \cdot g_{\max} + (1-\alpha) \cdot g_{\min}}{\sigma_\varepsilon^2}, & \text{if} \quad \alpha \cdot g_{\max} + (1-\alpha) \cdot g_{\min} < 0.
\end{cases}
$$
(A3.1.37)

We evaluate the feasibility of the possible demand functions given in Eq. (A3.1.37) case by case. First, substituting

$$
X_I = \frac{(\bar{f} + \tilde{\theta} - \tilde{p})}{\rho \cdot \sigma_\varepsilon^2} \quad \text{and} \quad X_U = \frac{\alpha \cdot g_{\min} + (1-\alpha) \cdot g_{\max}}{\sigma_\varepsilon^2}
$$
(A3.1.38)

into the market clearing condition, we obtain the stock price as

$$
\tilde{p} = \bar{f} + \tilde{\theta} - \frac{\rho \cdot \left(\sigma_\varepsilon^2 - (1-\lambda) \cdot \left(\alpha \cdot g_{\min} + (1-\alpha) \cdot g_{\max} \right) \right)}{\lambda}.
$$
(A3.1.39)

From the conjectured price function in Eq. (3.11) the perceived equity premium $g_\alpha(\lambda, \Delta\lambda)$ follows as

$$
g_\alpha(\lambda, \Delta\lambda) = \frac{\rho \cdot \left(\sigma_\varepsilon^2 - (1-\lambda) \cdot \left(\alpha \cdot g_{\min} + (1-\alpha) \cdot g_{\max} \right) \right)}{\lambda}.
$$
(A3.1.40)

Following Eq. (A3.1.40) $g_\alpha(\lambda_2, \Delta\lambda) = g_{\min}$ and $g_\alpha(\lambda_1, \Delta\lambda) = g_{\max}$ if $\alpha \cdot g_{\min} + (1-\alpha) \cdot g_{\max} < \sigma_\varepsilon^2$ and $g_\alpha(\lambda_1, \Delta\lambda) = g_{\min}$ and $g_\alpha(\lambda_2, \Delta\lambda) = g_{\max}$ if $\alpha \cdot g_{\min} + (1-\alpha) \cdot g_{\max} > \sigma_\varepsilon^2$. Suppose $g_{\min} = g_\alpha(\lambda_2, \Delta\lambda)$ and $g_{\max} = g_\alpha(\lambda_1, \Delta\lambda)$. Then,

$$
g_{\min} = \frac{\rho \cdot \left(\sigma_\varepsilon^2 - (1-\lambda_2) \cdot \left(\alpha \cdot g_{\min} + (1-\alpha) \cdot g_{\max} \right) \right)}{\lambda_2},
$$
(A3.1.41)

$$
g_{\max} = \frac{\rho \cdot \left(\sigma_\varepsilon^2 - (1-\lambda_1) \cdot \left(\alpha \cdot g_{\min} + (1-\alpha) \cdot g_{\max} \right) \right)}{\lambda_1}.
$$
(A3.1.42)

Solving Eqs. (A3.1.41) and (A3.1.42) for $g_{\min}$ and $g_{\max}$ obtains

$$g_{\min} = \frac{\rho \cdot \sigma_\varepsilon^2 \cdot \left(\lambda_1 + \rho \cdot (\lambda_2 - \lambda_1) \cdot (1 - \alpha)\right)}{\rho \cdot \lambda_1 \cdot (1 - \lambda_2) \cdot \alpha + \lambda_2 \cdot \left(\lambda_1 + \rho \cdot (1 - \lambda_1) \cdot (1 - \alpha)\right)}, \tag{A3.1.43}$$

$$g_{\max} = \frac{\rho \cdot \sigma_\varepsilon^2 \cdot \left(\lambda_2 - \rho \cdot (\lambda_2 - \lambda_1) \cdot \alpha\right)}{\rho \cdot \lambda_1 \cdot (1 - \lambda_2) \cdot \alpha + \lambda_2 \cdot \left(\lambda_1 + \rho \cdot (1 - \lambda_1) \cdot (1 - \alpha)\right)}. \tag{A3.1.44}$$

Substituting Eqs. (A3.1.43) and (A3.1.44) into $\alpha \cdot g_{\min} + (1 - \alpha) \cdot g_{\max}$ yields

$$\alpha \cdot g_{\min} + (1-\alpha) \cdot g_{\max} = \frac{\rho \cdot \left(\lambda_1 \cdot \alpha + \lambda_2 \cdot (1 - \alpha)\right) \cdot \sigma_\varepsilon^2}{\lambda_1 \cdot \lambda_2 + \rho \cdot \lambda_1 \cdot (1 - \lambda_2) \cdot \alpha + \rho \cdot \lambda_2 \cdot (1 - \lambda_1) \cdot (1 - \alpha)} < \sigma_\varepsilon^2, \tag{A3.1.45}$$

for $0 < \lambda_1 < \lambda < \lambda_2 < 1$ and $0 < \rho < 1$. Inserting Eq. (A3.1.45) into Eq. (A3.1.40) and rearranging obtain

$$g_\alpha(\lambda, \Delta\lambda) = \frac{\rho \cdot \sigma_\varepsilon^2}{\lambda} \cdot \left(\frac{\lambda + \Delta\lambda \cdot (1 - \rho) + \frac{2 \cdot \rho \cdot \lambda \cdot \Delta\lambda}{\lambda - \Delta\lambda} \cdot (1 - \alpha)}{\rho + (\lambda + \Delta\lambda) \cdot (1 - \rho) + \frac{2 \cdot \rho \cdot \Delta\lambda}{\lambda - \Delta\lambda} \cdot (1 - \alpha)} \right). \tag{A3.1.46}$$

Calculations parallel to (A3.1.41)-(A3.1.45) result in contradiction when $g_{\min} = g_\alpha(\lambda_1, \Delta\lambda)$ and $g_{\max} = g_\alpha(\lambda_2, \Delta\lambda)$.

Second, substituting

$$X_I = \frac{(\bar{f} + \tilde{\theta} - \tilde{p})}{\rho \cdot \sigma_\varepsilon^2} \quad \text{and} \quad X_U = 0, \tag{A3.1.47}$$

into the market clearing condition yields $g_\alpha(\lambda, \Delta\lambda) = \frac{\rho \cdot \sigma_\varepsilon^2}{\lambda}$, which leads to $g_{\min} = g_\alpha(\lambda_2, \Delta\lambda) > 0$ and $g_{\max} = g_\alpha(\lambda_1, \Delta\lambda) > 0$, and therefore $\alpha \cdot g_{\min} + (1-\alpha) \cdot g_{\max} > 0$, contradicting the condition $\alpha \cdot g_{\min} + (1 - \alpha) \cdot g_{\max} < 0$. Lastly, substituting

$$X_I = \frac{(\bar{f} + \tilde{\theta} - \tilde{p})}{\rho \cdot \sigma_\varepsilon^2} \quad \text{and} \quad X_U = \frac{\alpha \cdot g_{\max} + (1 - \alpha) \cdot g_{\min}}{\sigma_\varepsilon^2} \tag{A3.1.48}$$

into the market clearing condition yields

$$g_\alpha(\lambda, \Delta\lambda) = \frac{\rho \cdot \left(\sigma_\varepsilon^2 - (1 - \lambda) \cdot \left(\alpha \cdot g_{\max} + (1 - \alpha) \cdot g_{\min} \right) \right)}{\lambda} \tag{A3.1.49}$$

which leads to $g_{\min} = g_\alpha(\lambda_2, \Delta\lambda)$ and $g_{\max} = g_\alpha(\lambda_1, \Delta\lambda)$ if $\alpha \cdot g_{\max} + (1-\alpha) \cdot g_{\min} < \sigma_\varepsilon^2$ and $g_{\min} = g_\alpha(\lambda_1, \Delta\lambda)$ and $g_{\max} = g_\alpha(\lambda_2, \Delta\lambda)$ if $\alpha \cdot g_{\max} + (1 - \alpha) \cdot g_{\min} > \sigma_\varepsilon^2$, contradicting the condition $\alpha \cdot g_{\max} + (1 - \alpha) \cdot g_{\min} < 0$ in Eq. (A3.1.37),

■

Proof of Corollary 3.10. (i) and (ii) follow from the partial derivatives of X_I, X_U and $g(\lambda, \Delta\lambda)$ w.r.t. α. For the uninformed traders

$$\frac{\partial X_U}{\partial \alpha} = \frac{1}{\sigma_\varepsilon^2} \cdot \left(g_{\min} + \alpha \cdot \frac{\partial g_{\min}}{\partial \alpha} - g_{\max} + (1 - \alpha) \cdot \frac{\partial g_{\max}}{\partial \alpha}\right). \tag{A3.1.50}$$

Inserting $g_{\min}$, $g_{\max}$ in Eqs. (A3.1.43) and (A3.1.44) and

$$\frac{\partial g_{\min}}{\partial \alpha} = \frac{\rho^2 \cdot (1 - \rho) \cdot \lambda_1 \cdot (1 - \lambda_2) \cdot (\lambda_2 - \lambda_1) \cdot \sigma_\varepsilon^2}{\left(\rho \cdot \lambda_1 \cdot (1 - \lambda_2) \cdot \alpha + \lambda_2 \cdot \left(\lambda_1 + \rho \cdot (1 - \lambda_1) \cdot (1 - \alpha)\right)\right)^2} > 0$$

$$\tag{A3.1.51}$$

$$\frac{\partial g_{\max}}{\partial \alpha} = \frac{\rho^2 \cdot (1 - \rho) \cdot \lambda_1 \cdot (1 - \lambda_2) \cdot (\lambda_2 - \lambda_1) \cdot \sigma_\varepsilon^2}{\left(\rho \cdot \lambda_1 \cdot (1 - \lambda_2) \cdot \alpha + \lambda_2 \cdot \left(\lambda_1 + \rho \cdot (1 - \lambda_1) \cdot (1 - \alpha)\right)\right)^2} > 0$$

$$\tag{A3.1.52}$$

into Eq. (A3.1.50) obtains

$$\frac{\partial X_U}{\partial \alpha} = -\frac{\rho \cdot (1 - \rho) \cdot (\lambda_2 - \lambda_1) \cdot \lambda_1 \cdot \lambda_2}{\left(\rho \cdot \lambda_1 \cdot (1 - \lambda_2) \cdot \alpha + \lambda_2 \cdot \left(\lambda_1 + \rho \cdot (1 - \lambda_1) \cdot (1 - \alpha)\right)\right)^2} < 0.$$

$$\tag{A3.1.53}$$

The partial derivative of $g_\alpha(\lambda, \Delta\lambda)$ w.r.t. α is given by

$$\frac{\partial g_\alpha(\lambda, \Delta\lambda)}{\partial \alpha} = \frac{2 \cdot \rho^2 \cdot (1 - \rho) \cdot \lambda_2 \cdot (1 - \lambda) \cdot \Delta\lambda \cdot \sigma_\varepsilon^2}{\lambda \cdot \lambda_1 \cdot \left(\rho + (\lambda + \Delta\lambda) \cdot (1 - \rho) + \frac{2 \cdot \rho \cdot \Delta\lambda}{\lambda - \Delta\lambda} \cdot (1 - \alpha)\right)^2} > 0,$$

$$\tag{A3.1.54}$$

and therefore,

$$\frac{\partial X_I}{\partial \alpha} = \frac{2 \cdot \rho \cdot (1 - \rho) \cdot \lambda_2 \cdot (1 - \lambda) \cdot \Delta\lambda}{\lambda \cdot \lambda_1 \cdot \left(\rho + (\lambda + \Delta\lambda) \cdot (1 - \rho) + \frac{2 \cdot \rho \cdot \Delta\lambda}{\lambda - \Delta\lambda} \cdot (1 - \alpha)\right)^2} > 0. \tag{A3.1.55}$$

(iii) follows from solving the uncertainty aversion α^* satisfying $g_\alpha(\lambda, \Delta\lambda) = g(\lambda)$. Combining Eq. (3.20) and Eq. (3.40) obtains

$$\alpha^* = \frac{\lambda_2}{2 \cdot \lambda} = \frac{1}{2} + \frac{\Delta\lambda}{2\lambda}. \tag{A3.1.56}$$

(iv) follows from the partial derivative of Eq. (3.40) w.r.t. $\Delta\lambda$

$$\frac{\partial g_\alpha(\lambda, \Delta\lambda)}{\partial \Delta\lambda} = \frac{(1-\lambda) \cdot \left((\lambda + \Delta\lambda)^2 - 2 \cdot (\lambda^2 + \Delta\lambda^2) \cdot \alpha \right) \cdot (\rho - 1) \cdot \rho^2 \cdot \sigma_\varepsilon^2}{\lambda \cdot \left(\Delta\lambda^2 \cdot (\rho - 1) + \rho \cdot \Delta\lambda \cdot (1 - 2 \cdot \alpha) + \lambda \cdot \left(\rho + \lambda \cdot (1 - \rho) \right) \right)^2} > 0,$$

(A3.1.57)

if and only if

$$\alpha > \frac{1}{2} + \frac{\lambda \cdot \Delta\lambda}{\lambda^2 + \Delta\lambda^2} = \alpha^{**}.$$

(A3.1.58)

∎

Appendix 3.2. Additional Figures
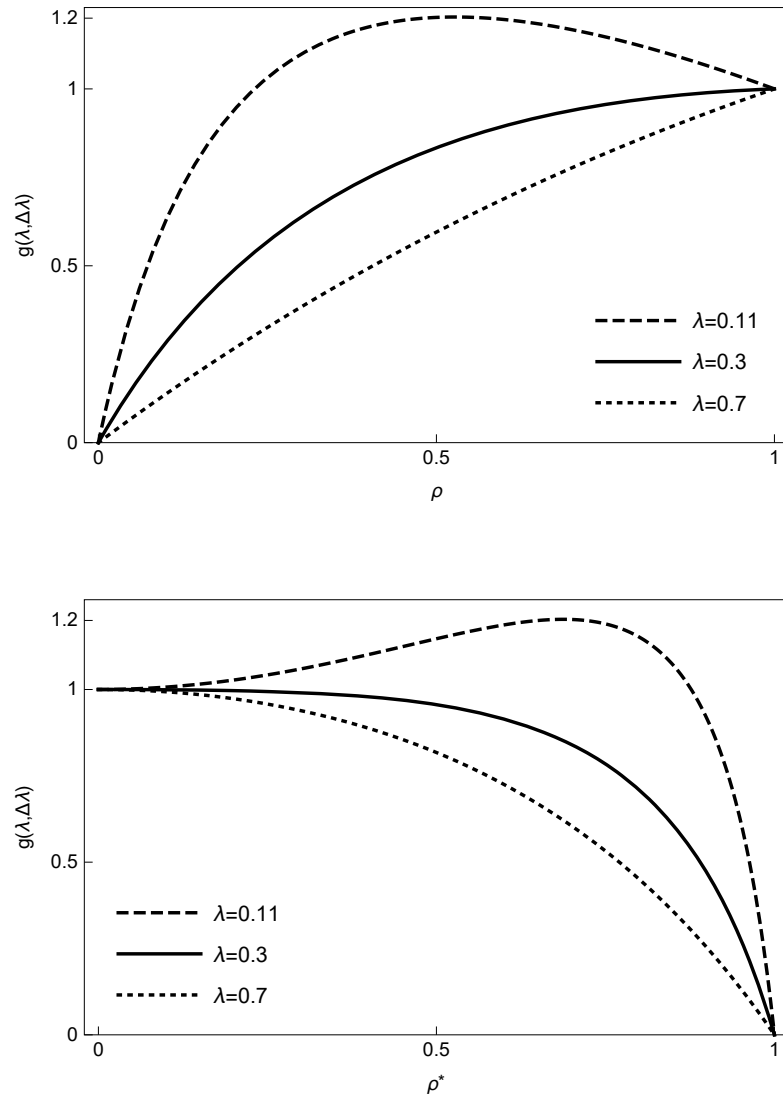
Figure 3.5: The equity premium $g(\lambda, \Delta\lambda)$ against the effective risk aversion ρ and correlation ρ^*.

Panel A plots the perceived equity premium $g(\lambda, \Delta\lambda)$ against the effective risk aversion ρ for three different values of the proportion of informed traders λ. Panel B plots the perceived equity premium $g(\lambda, \Delta\lambda)$ against the correlation ρ^* for three different values of the proportion of informed traders $\lambda = 0.11, 0.3, 0.7$. Other parameter values are $\Delta\lambda = 0.1$ and $\sigma_\varepsilon^2 = 1$.
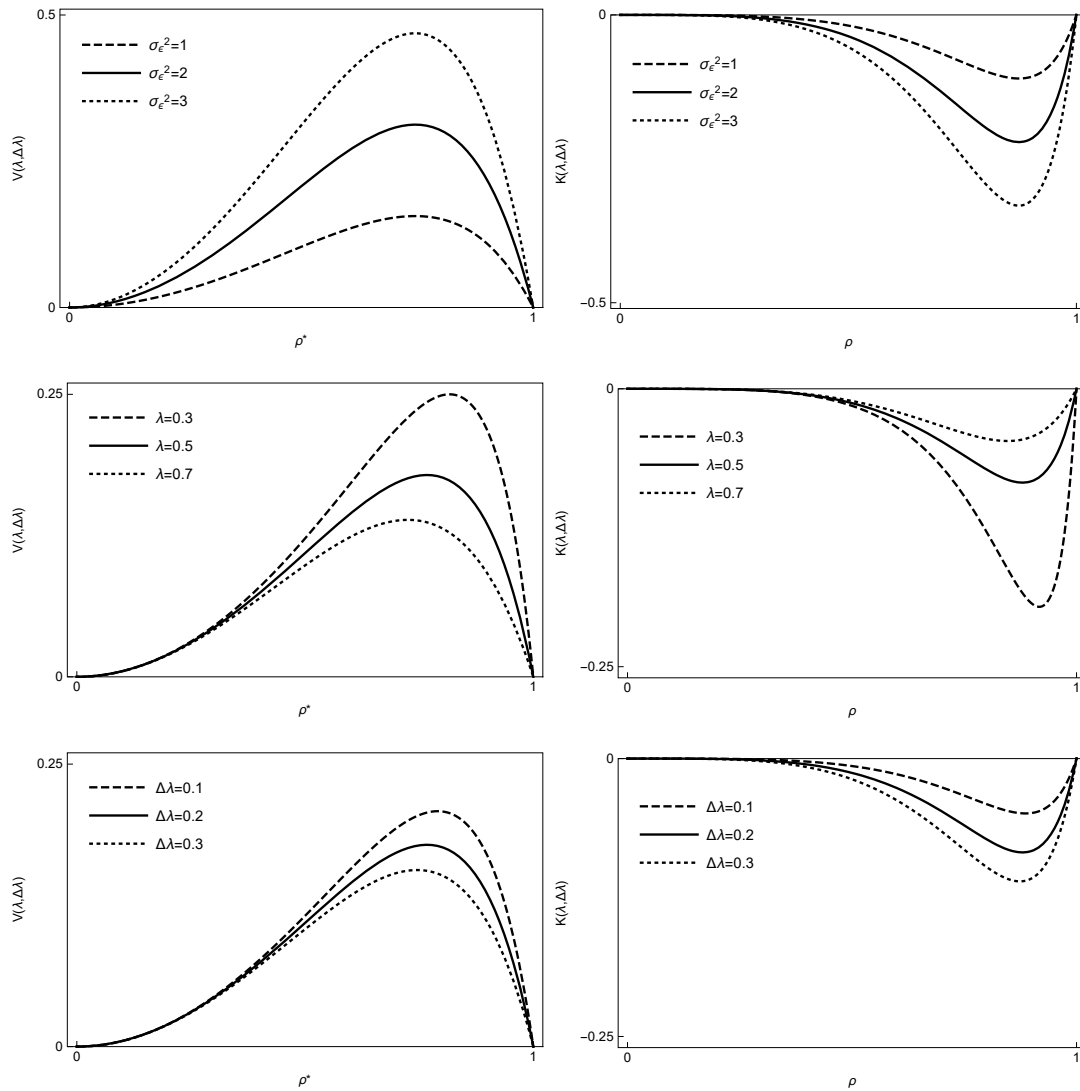
Figure 3.6: The benefit of informed trading $V(\lambda, \Delta\lambda)$ and the Knightian component $K_(\lambda, \Delta\lambda)$ against the correlation ρ^*.

Panels (A)-(B) plot the benefit of informed trading and the Knightian component against the correlation (ρ^*) on the horizontal axis, for three different values of the idiosyncratic noise (σ_ε^2) when $\lambda = 0.5$ and $\Delta\lambda = 0.3$. Panels (C)-(D) plot the benefit of informed trading and the Knightian component against the correlation (ρ^*), for three different values of informed trading (λ) when $\Delta\lambda = 0.2$ and $\sigma_\varepsilon^2 = 1$. Panel (E)-(F) plot the benefit of informed trading and the Knightian component against the correlation (ρ^*), for three different values of uncertainty about the proportion of informed traders ($\Delta\lambda$) when $\lambda = 0.5$ and $\sigma_\varepsilon^2 = 1$.
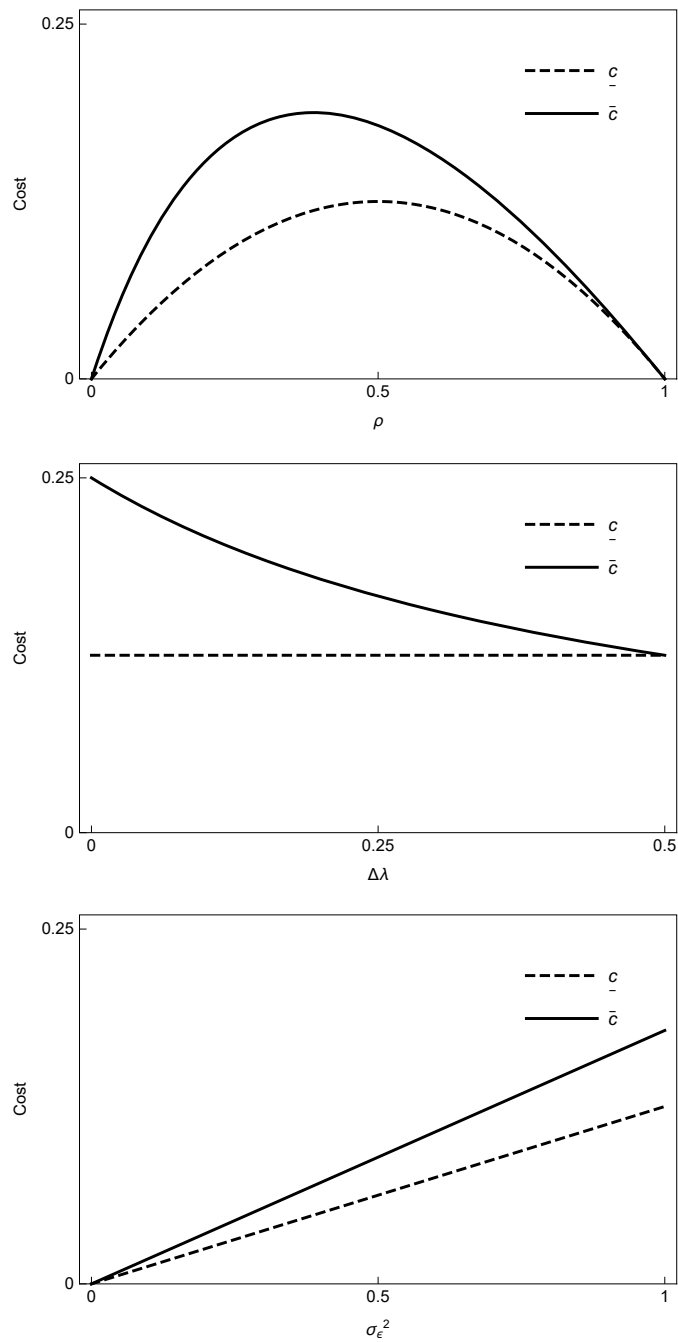
Figure 3.7: The cost range in which a unique information market equilibrium exists against the effective risk aversion ρ, composition uncertainty $\Delta\lambda$, and idiosyncratic noise σ_ε^2.

Panel (A) plots the upper $\bar{c}$ and lower $\underline{c}$ bounds of the cost range against the effective risk aversion (ρ) on the horizontal axis when $\Delta\lambda = 0.2$ and $\sigma_\varepsilon^2 = 1$. Panel (B) plots $\bar{c}$ and $\underline{c}$ against the composition uncertainty ($\Delta\lambda$) on the horizontal axis when $\rho = 0.5$ and $\sigma_\varepsilon^2 = 1$. Panel (C) plots $\bar{c}$ and $\underline{c}$ against the idiosyncratic noise (σ_ε^2) on the horizontal axis when $\rho = 0.5$ and $\Delta\lambda = 0.2$.

Chapter 4

Learning About Toxicity: Why Order Imbalance Can Destabilize Markets?

Liquidity provision is now a complex process, and levels of toxicity affect both the scale and scope of market makers' activities.
Easley, López de Prado, and O'Hara (2012) "Flow Toxicity and Liquidity in a High-frequency World" [p. 1490].

Toxic order flow is a source of financial market instability meaning it can cause evaporation of liquidity, elevated volatility, and sharp price movements. Order flow is regarded as toxic when it originates from a better-informed counterparty, causing adverse selection of market participants' orders and losses for liquidity providers. Market practitioners, in particular market makers, have long used order imbalance as an indication of order flow toxicity, adjusting their trading strategies accordingly. Liquidity providers (e.g., algorithmic market makers) often withdraw their quotes in the face of large order imbalances, making markets less liquid during and following large order imbalances (e.g., Chordia et al. (2002), Anand and Venkataraman (2016)). In the extreme, order imbalances can trigger 'flash crashes'—episodes of extreme price movements accompanied by evaporation of liquidity and elevated volatility (e.g., Easley et al. (2012), Kirilenko et al. (2017)). Given the fundamental importance of market stability in promoting economic growth, it is surprising that we know little about *why* order imbalance can destabilize markets and *when* markets are most vulnerable to destabilizing order imbalance. This chapter addresses both of these questions.

Paradoxically, standard market microstructure models with asymmetric information predict that order imbalances stabilize markets ex post, increasing liquidity and reducing volatility (e.g., Kyle (1985), Glosten and Milgrom (1985)). This prediction follows from the standard assumption that market participants are fully aware of the level of adverse selection (the probability of informed trading and/or the quality of informed traders' information). Under such an assumption, the effect of order imbalance is trivial—it reveals private information about the fundamental value, reducing uncertainty, and thereby increasing liquidity (lower price impacts in the Kyle framework and narrower bid-ask spreads in the Glosten-Milgrom framework). This prediction of standard microstructure models—that we should expect calmer and more liquid markets following periods of large order imbalances—is at odds with practice. What is missing from the standard models, we propose, is learning about adverse selection.

Our contribution to the literature is to model the process by which market participants learn about adverse selection risk ('toxicity') from order flow, in particular order imbalance, and study the implications of this learning process. To an otherwise standard sequential trade model, we add uncertainty about the proportion of informed traders (composition uncertainty) and/or the quality of their signals (signal quality uncertainty), resulting in uncertainty about the level of adverse selection. Reflecting a practical challenge faced by real-world liquidity providers, market participants in our model must learn about toxicity, rather than knowing the probability of informed trading and the quality of informed traders' information. This learning occurs from order flow. Intuitively, because informed trading tends to result in order imbalance (informed traders all tend to buy when prices are too low and sell when prices are too high), observing an episode of highly unbalanced order flow acts as a signal that there is likely to be a high proportion of informed traders or that informed traders have very precise information. This upward revision in perceived adverse selection risk can cause liquidity providers to set wider spreads to protect themselves from higher toxicity, as well as sharp price adjustments as the information contained in past order flow is reassessed. Such effects, which all follow from learning about adverse selection, oppose the standard stabilizing effect of order imbalance (learning about fundamental value). The tension between these stabilizing and destabilizing effects is what allows our model to illustrate why order imbalance can sometimes be destabilizing and offer insights about when the destabilizing effects are likely to dominate the stabilizing effects.

We use our model to explore how markets respond to three general order flow patterns — balanced orders, sequences, and reversals. Balanced orders occur when the market maker receives an equal number of buy and sell orders. Sequences are consecutive buy or sell orders. Reversals occur when a sell order follows consecutive buy orders or vice versa. Our analysis delivers four important implications for the dynamics of security prices.

First, balanced orders always stabilize the market. By receiving balanced orders, the market maker maintains her initial beliefs about the security value and revises her belief about adverse selection risk downward. This leads the information content of buy and sell orders to be time-varying and symmetric—informativeness of orders and bid-ask spreads decrease after a period of balanced orders due to lower perceived adverse selection risk. Even this basic effect is in contrast to standard microstructure models with only fundamental value uncertainty because in such models balanced order flow reveals no new information and thus has no effect on prices or liquidity.

Second, a sequence of unbalanced order flow (a series of buys or a series of sells) has two effects, with opposing impacts on liquidity. Unbalanced order flow allows the market maker to learn about the fundamental value (revising beliefs upward in response to buys and downward in response to sells), similar to standard models. This effect tends to make the market more liquid due to reduced uncertainty about the security value. Yet it also leads the market maker to revise upward her belief about the level of adverse selection risk, which tends to make the market less liquid. This means that, unlike in the standard models, order imbalances can be destabilizing. We characterize the necessary and sufficient conditions for order imbalance to be destabilizing. We show that order imbalance destabilizes the market when the initial belief about the adverse selection risk is sufficiently low. This means that financial markets are more vulnerable to order imbalances in times of low perceived toxicity, but can digest more imbalance when toxicity is believed to be high. While this result might seem surprising at first, the intuition is that a large order imbalance when it is not expected presents a larger shock than when the market expects unbalanced order flow.

Third, reversals in order flow (e.g., a sell following a string of buys) can restore liquidity. While this result is intuitive because reversals alleviate the imbalance in order flow received by liquidity providers, it in fact contrasts with standard models and highlights the important role played by learning about adverse selection. In a

standard model without learning about adverse selection, a reversal in order flow makes the market less liquid, as it increases uncertainty about the fundamental value. While this effect is also present in our model, an additional effect emerges from learning about adverse selection—a reversal leads the market maker to revise downward her belief about the level of adverse selection risk, which tends to improve liquidity.

A fourth interesting effect of learning about adverse selection, which we term "repricing history", explains accelerating price impacts, asymmetry in the information content of orders, and sharp price movements. When the market maker is uncertain about the proportion of informed traders or the quality of their information, an order has two components to how it impacts the market maker's beliefs about the fundamental value. The first is simply that buys increase the likelihood that the fundamental value is high and vice versa because informed traders tend to buy when the price is below the fundamental value—an effect that drives price discovery in standard models. But a second effect is that the market maker also updates her beliefs about the level of adverse selection or informativeness of order flow and then uses this new belief to reassess what she had learned from past order flow ("repricing history"). If an order increases the market maker's beliefs about the informativeness of order flow, she gives more credit to past orders and prices are adjusted accordingly (they move in the direction of the imbalance). For example, a market maker that receives a buy after a series of buys will revise upward her beliefs about the informativeness of order flow (due to a larger imbalance), leading her to reassess the past buy imbalance as more informed. Viewing the past buy imbalance as more informed leads to an additional upward revision in the expected fundamental value and thus a larger price increase than in the absence of learning about adverse selection. In fact, this mechanism can lead to accelerating price impacts in trade sequences, similar to those observed empirically during flash crashes. For instance, in a sequence of sells, each subsequent sell not only signals the fundamental value is likely to be low but also signals that the previous sells were more informed than initially believed, compounding the downward revision in beliefs about fundamental values.

In addition to accelerating price impacts with continuations in order flow, repricing history also implies the information content of buys and sells will be asymmetric and time-varying, depending on the past order flow. More precisely, reversals in the flow (e.g., a buy after a series of sells, or a sell after a series of buys)

decrease the market maker's beliefs about the informativeness of order flow and are more informative than continuations in the flow (buys following buys, or sells following sells). Intuitively, if an order decreases the market maker's beliefs about the informativeness of order flow, she gives less credit to past orders and adjusts prices accordingly (they move opposite to the direction of the imbalance). Rather than accelerating price impacts, this scenario can result in sharp price reversals. For example, a market maker that receives a sell after a series of buys revises downward her beliefs about the informativeness of order flow (due to a smaller imbalance), leading her to reassess the past buy imbalance as being less informative than previously believed. This leads to an additional downward revision in the likelihood of a high fundamental value and a larger price decrease than in the absence of repricing history.

The repricing history effect predicts that the price adjustments to order flow can be particularly sharp due to accelerating price impacts and more informative reversals in order flow. For example, a long string of sell orders similar to flash crashes will lead to accelerating price impact on the way down (high probability of informed trading due to the strong order imbalance), leading to sharp decline in the price. A few buy orders at such time will recover the price quickly due to the repricing history effect. The result is a sharp downward price movement and a quick recovery, amplified by learning about adverse selection.

By accounting for learning about adverse selection, our model provides a rich characterization of the dynamics of security prices in response to order flow and provides intuition about the prevalence of flash crashes with the rise of algorithmic trading. The model explains why price impacts can be asymmetric and time-varying (as has been empirically documented), without turning to frictions such as short selling constraints.[29] The results are also consistent with the empirical research by Hasbrouck (1991) that the trades that arrive when the spread is wide have a greater price impact. The analysis points out that the prevalence of the flash crashes in the algorithmic era may be related to the increased composition and signal quality uncertainty due to the increased complexity of financial markets and their participants.

---

[29]Empirical research in market microstructure finds that markets react to buy and sell orders asymmetrically (e.g., Kraus and Stoll (1972), Keim and Madhavan (1996), Chiyachantana et al. (2017)). Saar (2001) characterizes the conditions for the positive and negative price impact asymmetry between buy and sell orders by focusing on short-selling and diversification constraints of the institutional traders.

One way to view the relation between our model and early market microstructure models (e.g., Glosten and Milgrom (1985), Kyle (1985)) is that by adding learning about adverse selection, we allow the model to better reflect the current market structure. At the time of the original models, designated market makers (DMMs) with affirmative obligations to provide two-sided quotes and maintain orderly markets were integral to the functioning of US equity markets. The relative lack of competition faced by DMMs at the time meant they could cross-subsidise liquidity through time, which helped maintain orderly markets and reduce fluctuations in liquidity. They could keep the spread relatively stable, making excess profits in good times (when adverse selection is low) and using those excess profits to subsidise liquidity provision in bad times (when adverse selection is high). Thus, there was less incentive to learn about time-varying adverse selection risk and ensure spreads always reflected the level of toxicity. The abolishment of DMM monopolies and resulting competition in liquidity provision eliminated the ability to cross-subsidise liquidity through time. This is because a liquidity provider without affirmative obligations could undercut the DMM's quotes during good times when adverse selection is low to capture some of the excess profit and step away when adverse selection becomes high. Importantly, efficient learning about the time-varying level of adverse selection, or the 'toxicity' of order flow, allowing liquidity to be priced accurately at every point in time is crucial for a liquidity provider to remain competitive in today's major equity markets. Thus, we argue that our model better reflects the behavior of today's liquidity providers and therefore provides a better description of the dynamics of order flow, liquidity, and prices.

The next section relates this chapter to the literature. In Section 4.2, we introduce a benchmark model that does not require learning about adverse selection to illustrate how order imbalance stabilizes the market by reducing uncertainty about the fundamental value. In Section 4.3, we extend the model to include uncertainty about adverse selection (proportion of informed traders). In Section 4.4, we investigate the liquidity and price dynamics in the extended model, and characterize the conditions for liquidity deteriorations and sharp price movements. Section 4.5 examines the implications of our results for empirical research. Section 4.6 discusses some extensions and generalizations of our model. Section 4.7 concludes. The details of extensions and proofs are collected in the appendices.

4.1 Related Literature

Order imbalance can be caused by many factors (e.g., informed trade, macroeconomic variables, "fat-finger" trades). By focusing on traders' demand functions, much of the market crashes literature focuses on the causes of order imbalance (e.g., Gennotte and Leland (1990), Barlevy and Veronesi (2003), Hong and Stein (2003)). In this chapter, instead of the causes of order imbalance, we focus on its effects. Our model builds on Glosten and Milgrom (1985), which models financial markets as a sequential trading process with one source of uncertainty—the security payoff. This chapter is related to a subset of market microstructure literature that studies environments where market participants face multiple dimensions of uncertainty.

In environments with uncertain information quality, Romer (1993) suggests a possible rational explanation for the October 1987 crash and Blume, Easley and O'Hara (1994) investigate the informational role of volume for technical analysis in a rational expectations framework. In recent studies, Gao et al. (2013) generate multiple non-linear equilibria with strategic information complementarity and Banerjee and Green (2015) establish empirically relevant return dynamics such as asymmetric reaction to news, volatility clustering, and leverage effects in a rational expectations framework with an uncertain proportion of informed traders. The trading process in a rational expectations framework is not flexible enough to investigate our effects of interest. One reason is that the aggregation of orders in a batch-clearing system prevents them from taking different levels of informativeness at different times. The second and related reason is that in a batch-clearing system trades clear at a single price. Our focus is on how the market maker learns from order flow and when this learning stabilizes and destabilizes financial markets. Therefore, in our model, the dynamics of the quotes and the bid-ask spread play important roles in evaluating the evolution of liquidity.

In a sequential trade model, Easley and O'Hara (1992) introduce "event uncertainty" (uncertainty about whether an event that gives rise to private information about the security value has occurred) to show the relevance of time and volume in the market maker's learning process. Without an information event, the market is only populated by uninformed traders, who (unlike informed traders) sometimes choose not to trade. In this setting, the rate of trade arrivals (trades per unit time) is higher following an information event and therefore the market maker

learns from the time between trades. In our model, the market maker always faces an adverse selection problem but to an uncertain degree. Our focus is learning from the order imbalance rather than volume or the pace of trading. Order imbalance is important in how participants would learn about the presence of informed traders and the quality of their information. For example, consider an increase in the arrival intensity of uninformed traders that will increase volume per unit time but not adverse selection. In contrast, an increase in the imbalance between buyers and sellers signals high adverse selection risk and toxicity in the order flow. Our focus on learning from order imbalance rather than the time between trades produces a vastly different set of insights and empirical implications about the dynamics of prices and liquidity. Learning about adverse selection from order imbalance as in our model rather than from the clock time between trades as in Easley and O'Hara (1992) is consistent with recent empirical measures of toxicity, such as VPIN (e.g., Easley, López de Prado and O'Hara (2011)). VPIN seeks to measure toxicity (adverse selection risk) using a volume clock (thereby explicitly disregarding the clock time between trades) based on order imbalances much like how liquidity providers in our model infer the level of adverse selection.[30] Thus, a further contribution of this chapter is in providing a theoretical justification for recent empirical toxicity measures such as VPIN.

Avery and Zemsky (1998) propose multiple dimensions of uncertainty with non-monotone signals as a possible explanation for the herd behavior and market mispricing. In our model, we stick to more common monotone information structures that rule out herding and show that order imbalance can destabilize markets when there is an additional source of uncertainty.[31] The destabilizing effects of order imbalance that we analyze are quite different from those in Avery and Zemsky (1998). First, the mechanism is different. In our model, order imbalance reveals information about adverse selection, whereas in Avery and Zemsky order imbalance can be destabilizing because herding can occur and the market maker cannot distinguish between herding and trading on private signals. Second, the nature of

---

[30]Our model (unlike Easley and O'Hara (1992)) is in "event time" or uses a "volume clock" (trade arrivals index time).

[31]The non-monotone signals in Avery and Zemsky (1998) exploit a second source of uncertainty (about whether an information event has occurred or about the precision of informed traders' signals). They assume that if an information event has not occurred, the informed traders know with certainty. However, if an information event has occurred, the informed traders know that an information event has occurred, but they only have a noisy signal about whether it was good or bad news. For this reason, when an informed trader arrives and an information event has occurred, if there has been a significant price run-up, the informed might infer that it is more likely that there has been good news, not bad news even if he receives the noisy bad news signal, and thus he throws away his information and herds.

the instability is different. In our model, the second source of uncertainty causes order imbalance to move prices sharply, widen spreads, and increase volatility. Gervais (1997) also studies a sequential trade model in which the market maker is uncertain about the quality of informed traders' signal to argue that financial markets do not necessarily evolve in the direction of efficient markets. In his setting, the evolution of beliefs are path-dependent due to the independence of uncertainties and the bid-ask spread can stuck forever at a certain level in which the same equilibrium is repeated in every subsequent period leading to information cascade.

This is the first research, to our knowledge, to show how learning about the level of adverse selection from the order flow can lead to sudden liquidity dry-ups and sharp price movements in the face of large order imbalances.

4.2   The Benchmark Model

This section presents a benchmark model that mirrors the classic market microstructure models with uncertainty only about the fundamental value. In this setting, we illustrate the stabilizing effect of order imbalance. The benchmark model allows us to provide a contrast to the subsequent models with composition uncertainty in Section 4.3 and other sources of uncertainty (i.e., fundamental value, composition, and signal quality uncertainty) in Appendix 4.1.

4.2.1   Setup

We adopt a Glosten-Milgrom framework of one risky security and three types of traders; informed traders, uninformed traders, and a competitive market maker. Trade takes place in $t = 1, ..., T$ periods and the risky security pays off in period $T+1$. The payoff $\hat{V}$ takes one of two values from the set $\hat{V} \in \{0, 1\}$ with an initial prior probability $\Pr(\hat{V} = 1) = p_1$, where $0 < p_1 < 1$. For ease of exposition we assume $p_1 = 0.5$ in our analysis and address $p_1 \neq 0.5$ if relevant. Let D_t denote the trade direction, $D_t = -1$ for a sell, $D_t = +1$ for a buy, and P_t denote the transaction price at time t. Public information at time t consists of the sequence of past buys and sells and their transaction prices, denote by $h_t = \{D_\tau, P_\tau\}_{\tau=1}^{t-1}$.[32]

---

[32]For convenience, unions $\{\cdot, \cdot\}_{\tau=1}^{0}$ are taken to equal $\varnothing$, and both sums $\sum_{\tau=1}^{0}$ and products $\prod_{\tau=1}^{0}$ are taken zero.

As in Glosten-Milgrom type models, the risk-neutral, competitive market maker posts bid and ask quotes, for a fixed volume (normalized to one unit), to earn zero expected profit. At each time t, a trader arrives at the market and can buy at the ask or sell at the bid. With a probability of α the trader arriving at the market is informed and with a probability of $1 - \alpha$ he is uninformed. We focus on interior probability or intensity of informed trading, $\alpha \in (0, 1)$, as this is the empirically relevant case. After each trade, the competitive market maker updates her beliefs about the security payoff and posts new quotes before the next trader arrives.

The informed traders are risk neutral and maximize their expected profits by trading on a serially received signal $\{\theta_t\}$ about the risky security payoff. The signal takes either H (high) or L (low), $\theta_t \in \{H, L\}$, and the quality of the signal is measured by

$$q = \Pr\{\theta_t = H | \hat{V} = 1\} = \Pr\{\theta_t = L | \hat{V} = 0\}, \tag{4.1}$$

with $q \in (1/2, 1]$. When $q = 1$, the informed traders' information is perfect. When $q = 1/2$, the signal is completely uninformative. By Bayes' theorem, an informed trader who receives $\theta_t = H$ will revise his private value to

$$v_t^H = \frac{p_t \cdot q}{[p_t \cdot q + (1 - p_t) \cdot (1 - q)]} > p_t, \tag{4.2}$$

and who receives $\theta_t = L$ will revise his private value to

$$v_t^L = \frac{p_t \cdot (1 - q)}{[p_t \cdot (1 - q) + (1 - p_t) \cdot q]} < p_t, \tag{4.3}$$

where $p_t = \Pr(\hat{V} = 1 | h_t) = E_t[\hat{V} | h_t]$ is the current expected value of $\hat{V}$ conditional on the public information history h_t.

The uninformed traders trade according to their liquidity needs or hedging purposes, which are exogenous to the model. For convenience, we assume that they buy and sell with equal probabilities with perfectly inelastic demand.[33] The structure of the economy described so far is common knowledge among all participants.

---

[33]In Section 4.6, we discuss the impacts of allowing discretionary uninformed trading on our results.

4.2.2 Equilibrium

The standard Bertrand competition argument that the competitive market maker expects a zero profit implies that the market maker's bid (ask) quote is the expected future payoff of the risky security conditional on receiving a sell (buy) order. That is, the bid price $B_t = E_t[\hat{V}|h_t, D_t = -1]$ and the ask price $A_t = E_t[\hat{V}|h_t, D_t = +1]$. Now we formally define the equilibrium for the benchmark economy.

Definition 4.1. An equilibrium consists of the market maker's prices, informed traders' trading strategies, and posterior beliefs such that:

(i) the bid and ask prices satisfy the zero-expected-profit condition, given the market maker's posterior beliefs;

(ii) the informed traders at time t maximize their expected profits given the signal θ_t and the public information history h_t;

(iii) the market maker's beliefs satisfy Bayesian updating.

In the benchmark equilibrium with only fundamental value uncertainty, an informed trader who arrives at the market with $\theta_t = H$ ($\theta_t = L$) will buy (sell) if his private valuation is higher (lower) than the ask (bid) price at time t, $v_t^H > A_t$ ($v_t^L < B_t$). In equilibrium, $v_t^H > A_t$ and $v_t^L < B_t$, and therefore the informed traders always trade in the direction of their information.[34] This characterizes the equilibrium A_t and B_t in the following proposition.

Proposition 4.2. The equilibrium bid and ask prices are respectively given by:

$$B_t = \frac{p_t}{p_t + \delta \cdot (1 - p_t)}, \tag{4.4}$$

$$A_t = \frac{p_t}{p_t + \delta^{-1} \cdot (1 - p_t)}, \tag{4.5}$$

and the bid-ask spread is given by

$$S_t = \frac{p_t \cdot (1 - p_t) \cdot (\delta - \delta^{-1})}{\left[p_t + \delta \cdot (1 - p_t)\right] \cdot \left[p_t + \delta^{-1} \cdot (1 - p_t)\right]}, \tag{4.6}$$

---

[34]The reason is that if B_t and A_t are set less than v_t^L and higher than v_t^H (i.e., $B_t < v_t^L$ and $A_t > v_t^H$), then no informed traders would trade and all trades would arise from the uninformed traders. The competitive, zero expected profit B_t and A_t without any informed trading are equal to the current expected security value, $B_t = A_t = p_t$, with zero spread. Because the signals are always informative ($q > 1/2$), $v_t^H > p_t$ and $v_t^L < p_t$ and therefore in a competitive equilibrium, B_t and A_t cannot be less than v_t^L and higher than v_t^H respectively.

where $p_t = E_t[\hat{V}|h_t]$ and $\delta = \frac{1+\alpha\cdot(2\cdot q-1)}{1-\alpha\cdot(2\cdot q-1)}$. In addition, δ is always greater than unity and increases with the intensity of informed trading α and the quality of the informed traders' private information q.

In this equilibrium, the market is always open. This is because the market maker can always set a spread wide enough to recoup from the uninformed traders the losses she expects to incur from the informed traders. In addition, the market maker sets a wider spread with the intensity of informed trading and the quality of their signals. It follows from Eqs. (4.4) and (4.5) that the bid price decreases and the ask price increases with the informativeness of orders δ, leading to a wider bid-ask spread. Since δ increases with the intensity of informed trading and the quality of their signals it measures the informativeness of orders and therefore the adverse selection risk.

4.2.3 The dynamics of the quotes and the bid-ask spread

In this chapter, we are particularly interested in the dynamics of the quotes and the bid-ask spread. For this purpose, we characterize the dynamics of the risky payoff as

$$p_{t+1} = E_{t+1}[\hat{V}|h_t, D_t, P_t] = E_{t+1}[\hat{V}|h_{t+1}] = \begin{cases} B_t, & \text{if} \quad D_t = -1, \\ A_t, & \text{if} \quad D_t = +1, \end{cases} \tag{4.7}$$

and re-express in a particularly convenient form in the following lemma. Eq. (4.7) follows from the fact that, in this setting, the current expected security value is the last realized transaction price.

Lemma 4.3. Let $N_t = \sum_{\tau=1}^{\tau=t-1} D_\tau$ be the order imbalance up to (but not including) the trade at time t (number of buys minus number of sells). Then the dynamic expectations of the market maker about the risky security payoff satisfy

$$\frac{p_{t+1}}{1-p_{t+1}} = \frac{p_t}{1-p_t} \cdot \delta^{D_t}, \tag{4.8}$$

and hence

$$p_t = \frac{\delta^{N_t}}{1+\delta^{N_t}}. \tag{4.9}$$

Eq. (4.8) shows that the odds of a high future value are revised upward following a buy and downward following a sell. The amount by which the expectations are revised is determined by the informativeness of trades. More precisely, the revision in the expectation about the payoff is stronger with more informative trades (or more informative trades have higher price impacts). Eq. (4.9) shows that all of the information contained in the past trades and prices can be represented by the order imbalance, N_t, a sufficient statistic for the history of the order flows. This means that the trade sequences that do not change the order imbalance (i.e., balanced order flows) do not change the market maker's beliefs about the security payoff. Thus, the expected payoff of $\hat{V}$ and the bid-ask spread at any point in time can be expressed succinctly as a function of the order imbalance up to that point in time and the informativeness of trades.

To further facilitate interpretation, we insert Eq. (4.9) into Eq. (4.6) and re-express the bid-ask spread as a function of the informativeness of orders and order imbalance as

$$S_t = \frac{\delta^{N_t} \cdot (\delta - \delta^{-1})}{(\delta^{N_t} + \delta) \cdot (\delta^{N_t} + \delta^{-1})}. \tag{4.10}$$

Interestingly, Eq. (4.10) also shows that the bid-ask spread decreases with order imbalances in either direction, excess buy or sell orders, and takes its maximum value of

$$\bar{S} = \frac{\delta - 1}{\delta + 1}, \tag{4.11}$$

with balanced order flow (i.e., $N_t = 0$).[35]

In Figure 4.1, we illustrate the uncertainty about the payoff, p_t, and the bid-ask spread in the face of large order imbalances for three possible values of the informativeness of orders. Panel (A) illustrates that the market maker revises the expected value of the security payoff upward when she has a positive order imbalance and downward when she has a negative order imbalance. Moreover, the upward and downward revisions are larger with more informative trades. However, irrespective of the informativeness of orders, uncertainty about the payoff is highest when the market maker has balanced orders. Panel (B) shows that the spread is maximum when the market maker has balanced orders and declines in response

---

[35]In general, the maximum spread occurs when the market maker has maximum uncertainty (i.e. $p_t = 0.5$) about the payoff. With a balanced order flow the market maker learns nothing and sustains her initial maximum uncertainty (i.e., $p_1 = 0.5$). In fact, when $p_1 > 0.5$ (resp. $p_1 < 0.5$), the same maximum spread corresponds to a negative (resp. positive) order imbalance. The reason for this is that the maximum uncertainty, $p_t = 0.5$, occurs with a negative (resp. positive) order imbalance when $p_1 > 0.5$ (resp. $p_1 < 0.5$).
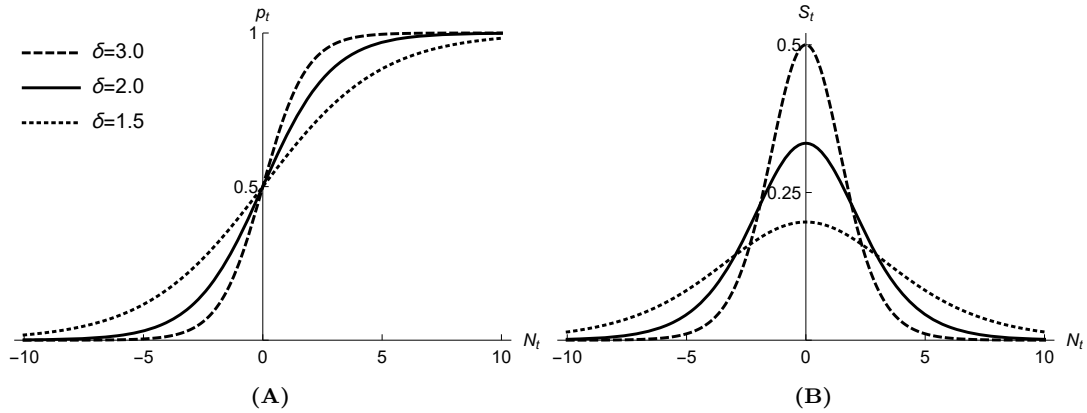
Figure 4.1: The dynamics of the belief p_t and the spread S_t with respect to order imbalance N_t.
Panel (A) plots the conditional expected value of the payoff p_t and (B) plots the spread S_t against the order imbalance for three different values of informativeness of trades, $\delta = 1.5$, 2, and 3 when $p_1 = 0.5$.

to order imbalances in either direction. This illustrates the stabilizing role of order imbalance in the benchmark model with only fundamental value uncertainty. Intuitively, order imbalance is informative about the risky payoff and therefore resolves uncertainty (either $p_t \to 0$ or $p_t \to 1$). Moreover, the bid-ask spread declines faster with more informative trades because uncertainty is resolved faster. Formally, we have the following corollary.

Corollary 4.4. In the presence of uncertainty only about the security payoff;

(i) the market maker observing balanced order flows (i.e., $N_t = 0$) learns nothing and therefore does not update her beliefs about the payoff;

(ii) the market maker with a positive (negative) order imbalance increases (decreases) the conditional expected value of the payoff and the magnitude of the increase (decrease) is larger with more informative trades;

(iii) with balanced order flows (i.e., $N_t = 0$), the bid-ask spread S_t at time t equals the initial bid-ask spread S_1;

(iv) the bid-ask spread narrows with order imbalances in either direction and converges to zero as the order imbalance goes to infinity (i.e., order imbalance stabilizes the market).

The stabilizing role of order imbalances in the benchmark model hinges upon having only uncertainty about the fundamental value of the security. These results

are at odds with what we observe in financial markets during and following large order imbalances. The experience of the U.S. financial markets on May 6, 2010, ("Flash Crash") and treasury markets on October 15, 2014, ("Flash Rally") are recent extreme examples of the destabilizing role of negative and positive order imbalances, respectively. Similar results are also observed during the global financial crisis in 2007-2009, Asian financial crisis in 1997-1998, October 1987 crash, and many other extreme events (e.g., Easley and O'Hara (2010*b*), Scholes (2000)). The destabilizing role of order imbalance is not confined to the aggregate market level extreme events. On a smaller scale, instantaneous price moves due to the destabilizing role of order imbalance are more common with the rise of algorithmic trading. In practice, order imbalance is an indication of the toxicity in order flow. Unlike in practice, however, in the benchmark model, order imbalance merely serves to convey information about the fundamental value of the security. The bid-ask spread arises entirely due to the known adverse selection risk of the competitive market maker and approaches to zero in the face of large order imbalances — order imbalance stabilizes the market when the market maker knows the true information structure of the market.

4.3   Learning about Adverse Selection

In this section, we introduce an additional source of uncertainty about the adverse selection of the market maker to explore a destabilizing role of order imbalance. Adverse selection risk is a function of the number of informed traders and the quality of their information. Adding uncertainty and learning about either of these parameters produces qualitatively similar results and therefore in the interests of simplicity, we focus on uncertainty about the number of informed traders.

Two key differences distinguish this model from the benchmark model. First, the market maker's quotes are affected not only by beliefs about the security payoff, parameters affecting the adverse selection risk (i.e., the probability of informed trading and the quality of informed traders' information), but also uncertainty about the adverse selection risk. Second, the market maker's beliefs about adverse selection risk change over time as the trading process evolves.

4.3.1   Uncertainty about the proportion of informed traders

We keep all the features of the benchmark model as described in Section 4.2 (we set $q = 1$ for notational simplicity) and incorporate uncertainty about the composition of market participants. We assume that the probability of informed trading takes either low or high values from the set $\hat{\alpha} \in \{\alpha_L, \alpha_H\}$ with an initial prior probability of $\Pr(\hat{\alpha} = \alpha_H) = \pi_1$, where $0 < \alpha_L < \alpha_H < 1$ and $0 < \pi_1 < 1$. Without perfect knowledge about the fractions of the informed and uninformed traders in the market, the market maker's beliefs about the composition of traders change depending on the evolution of the trading process.[36] In what follows, we denote the market maker's belief about the high proportion of informed traders in the market conditional on the trading history as $\pi_t = \Pr\{\hat{\alpha} = \alpha_H | h_t\}$. With two possible values for the probability of informed trading and the risky payoff (i.e., $\hat{\alpha} \in \{\alpha_L, \alpha_H\}$ and $\hat{V} \in \{0, 1\}$), there are four different combinations of the level informed trading and the payoff realization. Denote the states $S \in \{s_1, s_2, s_3, s_4\}$, where

$$
\begin{aligned}
s_1 &= \{\hat{\alpha} = \alpha_H, \hat{V} = 1\}, & s_2 &= \{\hat{\alpha} = \alpha_H, \hat{V} = 0\}, \\
s_3 &= \{\hat{\alpha} = \alpha_L, \hat{V} = 1\}, & s_4 &= \{\hat{\alpha} = \alpha_L, \hat{V} = 0\}.
\end{aligned}
\tag{4.12}
$$

Since these states are disjoint the market maker's beliefs about the payoff and the composition of traders at time t are, respectively, given by

$$
p_t = \Pr(\hat{V} = 1 | h_t) = \Pr(s_1 | h_t) + \Pr(s_3 | h_t),
\tag{4.13}
$$

$$
\pi_t = \Pr(\hat{\alpha} = \alpha_H | h_t) = \Pr(s_1 | h_t) + \Pr(s_2 | h_t).
\tag{4.14}
$$

It is very intuitive to expect different order flows with different compositions of market participants. Intuitively, a higher proportion of informed traders is more likely to result in larger positive (negative) order imbalances when $\hat{V} = 1$ ($\hat{V} = 0$). The market maker observing more unbalanced orders than expected therefore will increase her belief about the high proportion of informed traders in the market and vice versa. Before considering the market maker's learning problem in detail, we characterize the equilibrium quotes and bid-ask spread in the trading period t,

---

[36]Making other market participants (in addition to the market maker) uncertain about the composition of market participants does not affect the model since the informed traders do not use the price function to extract information about the eventual security payoff and the uninformed traders are assumed to trade exogenously. We discuss a generalization of this assumption in Section 4.6.

extending the results from the benchmark model. The concept of the equilibrium follows the same structure outlined in Definition 4.1.

Proposition 4.5. The equilibrium bid and ask prices in the presence of composition uncertainty are respectively given by

$$B_{\alpha,t} = \frac{p_t}{p_t + \delta_t^s \cdot (1 - p_t)}, \tag{4.15}$$

$$A_{\alpha,t} = \frac{p_t}{p_t + (\delta_t^b)^{-1} \cdot (1 - p_t)}, \tag{4.16}$$

and the bid-ask spread is given by

$$S_{\alpha,t} = \frac{p_t \cdot (1 - p_t) \cdot (\delta_t^s - (\delta_t^b)^{-1})}{\left[p_t + \delta_t^s \cdot (1 - p_t)\right] \cdot \left[p_t + (\delta_t^b)^{-1} \cdot (1 - p_t)\right]}, \tag{4.17}$$

where

$$\delta_t^b = \frac{(1 + \alpha_H) \cdot \Pr(s_1|h_t) + (1 + \alpha_L) \cdot \Pr(s_3|h_t)}{(1 - \alpha_H) \cdot \Pr(s_2|h_t) + (1 - \alpha_L) \cdot \Pr(s_4|h_t)} \cdot \left(\frac{1 - p_t}{p_t}\right) \tag{4.18}$$

and

$$\delta_t^s = \frac{(1 + \alpha_H) \cdot \Pr(s_2|h_t) + (1 + \alpha_L) \cdot \Pr(s_4|h_t)}{(1 - \alpha_H) \cdot \Pr(s_1|h_t) + (1 - \alpha_L) \cdot \Pr(s_3|h_t)} \cdot \left(\frac{p_t}{1 - p_t}\right) \tag{4.19}$$

show the informativeness of buy and sell orders respectively. In addition, δ_t^b and δ_t^s are always greater than unity and increase with the intensities of informed trading α_L and α_H.

The forms of the bid, ask prices and the spread are familiar from the benchmark model (see Eqs. (4.4), (4.5), and (4.6)) and most of the intuitions carry forward. There are three main differences from the benchmark model (see Proposition 4.2). First, the market maker's belief about the high informed trading π_t in Eq. (4.14) is time-varying. Second, unlike the benchmark model with constant informativeness of orders for buy and sell orders, the information content of buy and sell orders in Eqs. (4.18) and (4.19) are different and vary through time due to the time-varying beliefs about the composition of traders and the security payoff.[37] This leads to asymmetric reactions of the bid and ask quotes in response to buy and sell

---

[37]Unlike the benchmark model, the informativeness of buy and sell orders, δ_t^b and δ_t^s, also involve the market maker's belief about the risky payoff along with the parameters of the adverse selection risk. This occurs because over time the market maker's belief about the risky payoff and the composition of traders are dependent. The best way to see that δ_t^b and δ_t^s are indeed analogous to the informativeness of orders δ in the benchmark model is to assume independence between $\hat{\alpha}$ and $\hat{V}$. Then, substituting independence conditions (i.e., $\Pr(s_1|h_t) = \pi_t \cdot p_t$, $\Pr(s_2|h_t) =$

orders. Third, the dynamics of the expectations about the payoff p_t are not only affected by the order imbalance and the (constant) informativeness of orders as in the benchmark model, but also by the changing beliefs about the composition of market participants and the time-varying informativeness of orders. For example, as the orders become more informative the market maker learns the fundamental value faster. We formalize this intuition in the following lemma by characterizing the dynamics of the beliefs about the risky payoff.

Lemma 4.6. The dynamic expectations of the market maker about the risky security payoff satisfy

$$\frac{p_{t+1}}{1 - p_{t+1}} = \frac{p_t}{1 - p_t} \cdot \delta_t^b \quad \text{if} \quad D_t = +1, \tag{4.21}$$

and

$$\frac{p_{t+1}}{1 - p_{t+1}} = \frac{p_t}{1 - p_t} \cdot (\delta_t^s)^{-1} \quad \text{if} \quad D_t = -1. \tag{4.22}$$

Lemma 4.6 shows how p_{t+1} is obtained from p_t when the market maker observes a buy or a sell order. It is straightforward to check that Lemma 4.6 reduces to Lemma 4.3 in the benchmark model when the market maker knows the composition of traders. In fact, when $\alpha = \alpha_L = \alpha_H$, it follows from Eqs. (4.18) and (4.19) that $\delta_t^b = \delta_t^s = \delta = \left(\frac{1+\alpha}{1-\alpha}\right)$ leading to Eq. (4.8). Similar to Eq. (4.8) in the benchmark model, Eqs. (4.21) and (4.22) show that the odds of a high future value are revised upward following a buy and downward following a sell. Unlike the benchmark model, however, the upward (resp. downward) revisions in δ_t^b and δ_t^s lead to stronger (resp. weaker) revisions in p_t.

$\pi_t \cdot (1 - p_t)$, $\Pr(s_3|h_t) = (1 - \pi_t) \cdot p_t$ and $\Pr(s_4|h_t) = (1 - \pi_t) \cdot (1 - p_t))$ into Eqs. (4.18) and (4.19) leads to symmetric informativeness for buys and sells

$$\delta_t^b = \delta_t^s = \frac{1 + (\pi_t \cdot \alpha_H + (1 - \pi_t) \cdot \alpha_L)}{1 - (\pi_t \cdot \alpha_H + (1 - \pi_t) \cdot \alpha_L)}, \tag{4.20}$$

which is only dependent on the parameters of adverse selection as in the benchmark model. Additionally, assuming independence about the composition of traders and the fundamental value leads the sequence of orders to matter (path-dependence) in the model, consistent with the empirical findings in Hausman, Lo and MacKinlay (1992). The path-dependence is not integral for our analysis and the sufficient statistic in our setting is order imbalance and time (N_t, t), yet the destabilizing role of order imbalance with a different magnitude is also present with path-dependence. In Appendix 4.3, we illustrate the destabilizing role of order imbalance when the market maker learns about $\hat{\alpha}$ and $\hat{V}$ independently.

4.3.2 Learning about the payoff and proportion of informed traders

In this subsection, we study how the market maker learns about the proportion of informed traders. The task is to determine how π_t is updated when the market maker observes order flow up to time t. One way to understand the learning process of the market maker is to consider the probability of buy and sell orders in different states. A buy order is most likely to occur in a market with a high proportion of informed traders and high security payoff (i.e., $S = s_1$). Similarly, a sell order is most likely to occur in a market with a high proportion of informed traders and low security payoff (i.e., $S = s_2$). More precisely, it follows from the definitions of the states that the probability of an order $D_t \in \{-1, 1\}$ in each state is given by

$$
\begin{aligned}
\Pr(D_t|s_1) = \frac{1 + \alpha_H \cdot D_t}{2}, \qquad & \Pr(D_t|s_2) = \frac{1 - \alpha_H \cdot D_t}{2}, \\
\Pr(D_t|s_3) = \frac{1 + \alpha_L \cdot D_t}{2}, \qquad & \Pr(D_t|s_4) = \frac{1 - \alpha_L \cdot D_t}{2}.
\end{aligned}
\tag{4.23}
$$

The probabilities in different states imply that in the presence of uncertainty about the proportion of informed traders, the *direction* and *amount* of order imbalance are informative about the payoff and only the *amount* of order imbalance is informative about the proportion of informed traders. The *direction* is informative about the payoff because high fundamental value states (i.e. s_1 and s_3) have higher buy and lower sell probabilities than the corresponding low fundamental value states (i.e., s_2 and s_4), leading the positive (resp. negative) imbalance to increase the probabilities of s_1 and s_3 (resp. s_2 and s_4). The *amount* of imbalance is informative about the payoff since more unbalanced buy (resp. sell) orders increase s_1 more than s_3 (resp. s_2 more than s_4). The *amount* of imbalance is also informative about the proportion of informed traders because buy and sell probabilities in low informed states (i.e. s_3 and s_4) are closer to 0.5, implying that balanced orders will increase low informed states and unbalanced orders will increase high informed states (i.e., s_1 and s_2). However, the *direction* of imbalance is uninformative about the proportion of informed traders because excess buy orders increase s_1 in the same way as excess sell orders increase s_2, leading to the same belief about the proportion of informed traders.

As the market maker is Bayesian, her belief about the particular state given the trading history h_t with b_t buy and s_t sell orders follows from Bayes rule as

$$\Pr(s_1|h_t) = \frac{p_1 \cdot \pi_1 \cdot \left(1 + \alpha_H\right)^{b_t} \left(1 - \alpha_H\right)^{s_t}}{f(h_t)}, \tag{4.24}$$

where

$$
\begin{aligned}
f(h_t) = {} & p_1 \cdot \pi_1 \cdot (1 + \alpha_H)^{b_t}(1 - \alpha_H)^{s_t} + (1 - p_1) \cdot \pi_1 \cdot (1 - \alpha_H)^{b_t}(1 + \alpha_H)^{s_t} \\
& + p_1 \cdot (1 - \pi_1)(1 + \alpha_L)^{b_t}(1 - \alpha_L)^{s_t} + (1 - p_1)(1 - \pi_1)(1 - \alpha_L)^{b_t}(1 + \alpha_L)^{s_t}.
\end{aligned}
\tag{4.25}
$$

The probabilities of the other states are calculated similarly. By Eqs. (4.13) and (4.14), the revision in each state's probability after the trading history h_t is reflected in the market maker's beliefs about the security payoff and the proportion of informed traders as

$$p_t = \frac{p_1 \cdot \pi_1 \cdot \left(1 + \alpha_H\right)^{b_t} \left(1 - \alpha_H\right)^{s_t} + p_1 \cdot (1 - \pi_1) \cdot \left(1 + \alpha_L\right)^{b_t} \left(1 - \alpha_L\right)^{s_t}}{f(h_t)}, \tag{4.26}$$

$$\pi_t = \frac{p_1 \cdot \pi_1 \cdot \left(1 + \alpha_H\right)^{b_t} \left(1 - \alpha_H\right)^{s_t} + (1 - p_1) \cdot \pi_1 \cdot \left(1 - \alpha_H\right)^{b_t} \left(1 + \alpha_H\right)^{s_t}}{f(h_t)}. \tag{4.27}$$

Eqs. (4.26) and (4.27) show that the order imbalance for the given number of trades (event time) determines the market maker's beliefs about the payoff and proportion of informed traders since $b_t = \frac{(t-1)+N_t}{2}$ and $s_t = \frac{(t-1)-N_t}{2}$. It follows from Eq. (4.26) that the market maker that receives balanced order flow learns nothing about the fundamental value as in the benchmark model (i.e., $p_t = p_1$). Eq. (4.26) also show that, all else equal, the greater the positive (resp. negative) imbalance the more likely it is that the market maker believes the fundamental value of the security is high (resp. low). Additionally, it follows from Eq. (4.27) that, all else equal, the greater the imbalance the more likely it is that the market maker believes the market is highly populated by informed traders. We summarize the effects of order imbalance for the given number of trades on the market maker's beliefs in the following proposition.

Proposition 4.7. (i) The market maker's expected value of the security payoff is unchanged with zero order imbalance and increases (resp. decreases) with positive (resp. negative) order imbalance; that is $p_t = p_1$ when $N_t = 0$ and $\frac{\partial p_t}{\partial N_t} > 0$. (ii) the market maker's belief about the high informed trading increases with order imbalance in either direction; that is $\frac{\partial \pi_t}{\partial |N_t|} > 0$.

To facilitate interpretation, in Figure 4.2, we contrast the market maker's expected value of the security payoff p_t and belief about the high proportion of informed traders π_t in the benchmark and extended models. Panel (A) illustrates that, in both models, the market maker revises the expected value of the security payoff upward (resp. downward) when she has a positive (resp. negative) order imbalance. What is different in the presence of uncertainty about the adverse selection is that the upward and downward revisions are larger with more order imbalance. The reason for this effect is illustrated in Panel (B), which shows that the market maker's belief about the high proportion of informed traders π_t increases with order imbalance. The market maker learns about the fundamental value faster with more order imbalance since more order imbalance signals the presence of more informative orders. Additionally, Panel (B) shows that the market maker's beliefs about the high proportion of informed traders π_t is higher (resp. lower) than her initial belief π_1 when the order imbalance for the given time is sufficiently high (resp. low). Focusing on two extreme cases (zero order imbalance and maximum order imbalance for the given time), the following corollary formalizes these observations.

Corollary 4.8. (i) The market maker observing balanced order flows (i.e., $N_t = 0$ at time t) revises her belief about the high informed trading downward (i.e., $\pi_t < \pi_1$). (ii) The market maker observing sequences of buy or sell orders (i.e., $N_t = t - 1$ or $N_t = -(t - 1)$ at time t) revises her belief about the high informed trading upward (i.e., $\pi_t > \pi_1$).

There are two reasons why these results are of interest. First, in the presence of uncertainty about the proportion of informed traders, balanced orders will stabilize the market by reducing the bid-ask spread since the market maker retains her initial belief about the payoff but revises her belief about the high informed trading downward. This is in contrast to the benchmark model, where the market maker observing balanced orders learns nothing and maintains her initial bid-ask spread (see Corollary 4.4). Second, consecutive buy or sell orders can destabilize financial markets by widening the bid-ask spread since they signal the presence of high informed trading. This is also in contrast to the benchmark model, where the market maker that receives sequences of buy or sell orders only learns about the payoff and narrows the spread due to the resolution of uncertainty about the fundamental value. In the presence of uncertainty about the proportion of informed traders, the market maker during a period of large and temporary selling
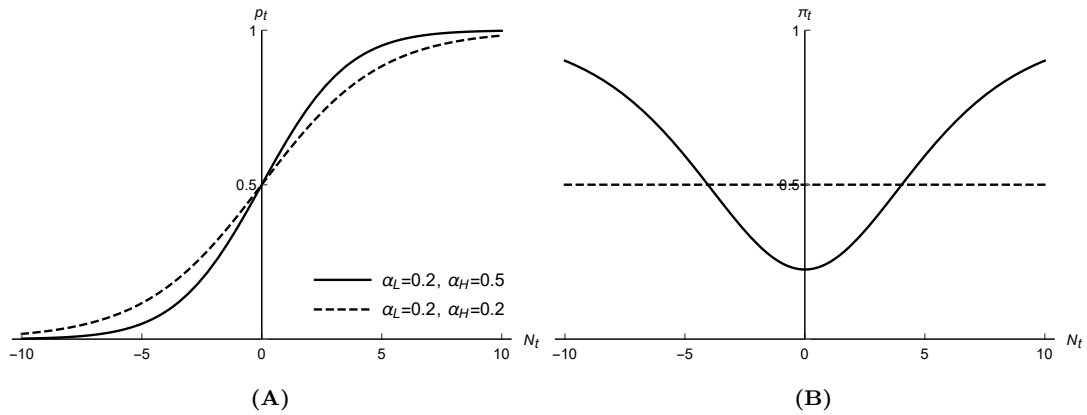
Figure 4.2: The dynamics of the beliefs p_t and π_t with respect to order imbalance N_t.

Panel (A) plots the conditional expected value of the payoff p_t and (B) plots the belief about the high proportion informed trading π_t against the order imbalance for the given time $t = 10$ for two different values of low and high informed trading $\alpha_L = \alpha_H = 0.2$ (benchmark), and $\alpha_L = 0.2$ and $\alpha_H = 0.5$ (uncertain proportion of informed traders). The initial beliefs are $p_1 = 0.5$ and $\pi_1 = 0.5$.

(resp. buying) pressure such as the Flash Crash (resp. Flash Rally) updates the expected value of security payoff downward (resp. upward). While this resolution of uncertainty about the fundamental value puts downward pressure on the bid-ask spread, there is an opposing effect on the spread from learning about the adverse selection. The market maker also updates her belief about the high informed trading in the market and widens the bid-ask spread due to the increase in the likelihood of high adverse selection risk (high informed trading). Ultimately, whether the spread widens or narrows following periods of large order imbalance depends on which effect dominates.

4.4   Liquidity and Price Dynamics

Since we are interested in the role of learning about toxicity in stabilizing and destabilizing financial markets in the face of different order flow patterns, in this section, we investigate the liquidity and price dynamics in our model.

4.4.1   Liquidity distortions

We first examine the effects of different order flows on the evolution of bid-ask spread to evaluate liquidity in the presence of uncertainty about adverse selection. For this purpose, we consider the amount of deviation of the bid-ask spread at time t from the initial spread.[38] It follows from Proposition 4.5 that the initial spread is given by

$$S_1 = \frac{\delta_1 - 1}{\delta_1 + 1} = \pi_1 \cdot \alpha_H + (1 - \pi_1) \cdot \alpha_L, \tag{4.28}$$

with equal informativeness of buy and sell orders,

$$\delta_1^b = \delta_1^s = \delta_1 = \frac{1 + (\pi_1 \cdot \alpha_H + (1 - \pi_1) \cdot \alpha_L)}{1 - (\pi_1 \cdot \alpha_H + (1 - \pi_1) \cdot \alpha_L)}. \tag{4.29}$$

Combining the bid-ask spread in the presence of uncertainty about adverse selection at time t in Eq. (4.17) and the initial spread in Eq. (4.28) yields

$$\Delta S_t = S_{\alpha,t} - S_1, \tag{4.30}$$

where ΔS_t given by Eq. (A4.2.50) in Appendix 4.2 is the net liquidity distortion at time t relative to the initial spread. The net liquidity distortion ΔS_t implies stabilizing (resp. destabilizing) order flow when $\Delta S_t < 0$ (resp. $\Delta S_t > 0$). In addition, ΔS_t includes both the effects of learning about the fundamental value and proportion of informed traders on the spread. To examine the contributions of each learning component on the net liquidity distortion, we decompose ΔS_t into two components — the distortion due to learning only about the fundamental value and the distortion due to learning about the adverse selection. Formally, the

---

[38]We choose the amount of deviation of the bid-ask spread at time t from the initial spread rather than the deviation from the benchmark spread for two reasons. First, the deviation from the initial spread shows the net liquidity distortion due to learning about the fundamental value and adverse selection, which can further be decomposed into the liquidity distortion due to each learning component, whereas the deviation from the benchmark spread measures the liquidity distortion due to learning only about adverse selection. Second, we are interested in a stricter condition to analyze whether order imbalance is destabilizing. The most strict condition to examine destabilizing order imbalance is to show that the spread is increasing in the order imbalance until a certain threshold order imbalance. The complexity of Eq. (4.17) after inserting state probabilities makes a full analytical characterization of the partial derivative $\frac{\partial S_{\alpha,t}}{\partial N_t}$ impractical. The second most strict condition is the deviation from the initial spread since the initial spread is higher than the benchmark spread in the face of large order imbalance.

net liquidity distortion is given by

$$\Delta S_t = \Delta S_t^V + \Delta S_t^A, \tag{4.31}$$

where $\Delta S_t^V = S_t - S_1$ is the liquidity distortion due to learning about the fundamental value and $\Delta S_t^A = S_{\alpha,t} - S_t$ is the liquidity distortion due to learning about the adverse selection. We investigate the net liquidity distortion ΔS_t and its contributors ΔS_t^V and ΔS_t^A during three general order flow patterns: balanced orders, consecutive buy or sell orders, and reversals. As we investigate each order flow pattern we refer to Figure 4.3, where we contrast the dynamics of belief about the proportion of informed traders, quotes, and spreads of the market makers that face this order flow in the benchmark and extended models.

For balanced order flows (i.e., $N_t = 0$ or $b_t = s_t$), the informativeness of buy and sell orders at time t are given by

$$\delta_t^b = \delta_t^s = \frac{\pi_1 \cdot (1 + \alpha_H)^{s_t+1} \cdot (1 - \alpha_H)^{s_t} + (1 - \pi_1) \cdot (1 + \alpha_L)^{s_t+1} \cdot (1 - \alpha_L)^{s_t}}{\pi_1 \cdot (1 + \alpha_H)^{s_t} \cdot (1 - \alpha_H)^{s_t+1} + (1 - \pi_1) \cdot (1 + \alpha_L)^{s_t} \cdot (1 - \alpha_L)^{s_t+1}} < \delta_1, \tag{4.32}$$

leading to

$$\Delta S_t = \Delta S_t^A = \frac{2 \cdot (\delta_t^s - \delta_1)}{\left(1 + \delta_t^s\right) \cdot \left(1 + \delta_1\right)} < 0, \tag{4.33}$$

since the market maker in the benchmark model learns nothing and retains her initial spread (i.e., $S_t = S_1$ or $\Delta S_t^V = 0$). Eq. (4.33) always takes a negative value (i.e., $\Delta S_t < 0$ for $\delta_t^s < \delta_1$), meaning that in the presence of uncertainty about adverse selection, balanced order flow is always stabilizing since it results in the narrower spread relative to the initial (also the benchmark) spread. This is because the market maker that observes balanced order flow retains her initial belief about the security payoff (see Proposition 4.7), but revises down her belief about the high informed trading (see Corollary 4.8) as illustrated in Panel (a1) of Figure 4.3. Panels (a2)-a(3) of Figure 4.3 illustrate that a balanced order flow is stabilizing in the extended model, whereas it has no effect on prices or liquidity in the benchmark model.

For consecutive sell orders (i.e., $N_t = -(t - 1)$), the informativeness of buy and sell orders at time t are respectively given by

$$\delta_t^b = \left[\frac{\pi_1 \cdot (1 + \alpha_H) \cdot (1 - \alpha_H)^{t-1} + (1 - \pi_1) \cdot (1 + \alpha_L) \cdot (1 - \alpha_L)^{t-1}}{\pi_1 \cdot (1 - \alpha_H) \cdot (1 + \alpha_H)^{t-1} + (1 - \pi_1) \cdot (1 - \alpha_L) \cdot (1 + \alpha_L)^{t-1}}\right] \cdot \left[\frac{p_t}{1 - p_t}\right]^{-1}, \tag{4.34}$$

$$\delta_t^s = \left[\frac{\pi_1 \cdot (1 + \alpha_H)^t + (1 - \pi_1) \cdot (1 + \alpha_L)^t}{\pi_1 \cdot (1 - \alpha_H)^t + (1 - \pi_1) \cdot (1 - \alpha_L)^t}\right] \cdot \left[\frac{p_t}{1 - p_t}\right], \tag{4.35}$$

where

$$\frac{p_t}{1 - p_t} = \frac{\pi_1 \cdot (1 - \alpha_H)^{t-1} + (1 - \pi_1) \cdot (1 - \alpha_L)^{t-1}}{\pi_1 \cdot (1 + \alpha_H)^{t-1} + (1 - \pi_1) \cdot (1 + \alpha_L)^{t-1}}. \tag{4.36}$$

Unlike balanced order flow, Eqs. (4.34) and (4.35) show that during unbalanced orders the informativeness of buy and sell orders are asymmetric, impacting the beliefs about the fundamental value asymmetrically. The following corollary characterizes the association between the beliefs about the fundamental value and the informativeness of orders during unbalanced order flow.

Corollary 4.9. (i) With a sequence of buy orders (i.e., $N_t = t - 1$),

$$p_t = \frac{\prod_{i=1}^{t-1} \delta_i^b}{1 + \prod_{i=1}^{t-1} \delta_i^b}, \tag{4.37}$$

(ii) With a sequence of sell orders (i.e., $N_t = -(t - 1)$),

$$p_t = \frac{\prod_{i=1}^{t-1} (\delta_i^s)^{-1}}{1 + \prod_{i=1}^{t-1} (\delta_i^s)^{-1}}. \tag{4.38}$$

Corollary 4.9 implies that when there is a consecutive buying or selling pressure in the market, the geometric mean of the informativeness of buy or sell orders up to time t (i.e., $(\prod_{i=1}^{t-1} \delta_i^b)^{\frac{1}{t-1}}$ or $(\prod_{i=1}^{t-1} \delta_i^s)^{\frac{1}{t-1}}$) play the same role as the constant informativeness orders in the benchmark model in determining p_t. Inserting the belief dynamics during sell sequences in Eq. (4.38) into the characterization of the spread in Eq. (4.17) obtains the spread as

$$S_{\alpha,t} = \frac{\prod_{i=1}^{t-1} (\delta_i^s)^{-1} \cdot \left(\delta_t^s - (\delta_t^b)^{-1}\right)}{\left(\prod_{i=1}^{t-1} (\delta_i^s)^{-1} + \delta_t^s\right) \cdot \left(\prod_{i=1}^{t-1} (\delta_i^s)^{-1} + (\delta_t^b)^{-1}\right)}. \tag{4.39}$$

A similar condition holds for buy sequences, only replacing $\prod_{i=1}^{t-1} (\delta_i^s)^{-1}$ with $\prod_{i=1}^{t-1} \delta_i^b$. Combining the initial spread and the spreads after consecutive sell orders in the benchmark and extended models (Eqs. (4.10), (4.28), and (4.39)) obtains $\Delta S_t = \Delta S_t^A + \Delta S_t^V$, where

$$\Delta S_t^V = \frac{-(\delta_1 - 1) \cdot \left(\delta_1^{-(t-1)} - 1\right)^2}{\left(\delta_1^{-(t-1)} + \delta_1\right) \cdot \left(\delta_1^{-(t-1)} + (\delta_1)^{-1}\right) \cdot (\delta_1 + 1)} < 0, \tag{4.40}$$

and

$$\Delta S_t^A = \frac{\prod_{i=1}^{t-1} (\delta_i^s)^{-1} \cdot \left(\delta_t^s - (\delta_t^b)^{-1}\right)}{\left(\prod_{i=1}^{t-1} (\delta_i^s)^{-1} + \delta_t^s\right) \cdot \left(\prod_{i=1}^{t-1} (\delta_i^s)^{-1} + (\delta_t^b)^{-1}\right)} - \frac{\delta_1^{-(t-1)} \cdot (\delta_1 - \delta_1^{-1})}{\left(\delta_1^{-(t-1)} + \delta_1\right) \cdot (\delta^{-(t-1)} + \delta_1^{-1})} > 0. \tag{4.41}$$

Eq. (4.40) shows the downward pressure on the bid-ask spread due to resolution of uncertainty about the fundamental value, whereas Eq. (4.41) shows the upward pressure due to learning about the adverse selection. Panel (b1) of Figure (4.3) illustrates that the upward pressure is due to the increase in the belief about the high proportion of informed trading π_t during consecutive sell orders. It follows from Eqs. (4.40) and (4.41) that the upward pressure due to learning about the adverse selection ΔS_t^A dominates the downward pressure due to learning about the fundamental value ΔS_t^V (i.e., $\Delta S_t^A + \Delta S_t^V > 0$) if and only if

$$\delta_1 < 1 + \frac{2 \cdot \left(\delta_t^s - (\delta_t^b)^{-1}\right)}{\left[2 \cdot (\delta_t^b)^{-1} + \prod_{i=1}^{t-1}(\delta_i^s)^{-1} + \prod_{i=1}^{t-1}(\delta_i^s) \cdot \delta_t^s \cdot (\delta_t^b)^{-1}\right]}. \qquad (4.42)$$

A similar condition holds for buy sequences. Eq. (4.42) shows that when the initial belief about the proportion of informed traders or the informativeness of orders δ_1 is sufficiently low, a continuous selling pressure leads to a wider spread (i.e., liquidity deterioration) relative to the initial spread. This is intuitive because order imbalance is not expected when the initial belief about the adverse selection (π_1 or δ_1) is sufficiently low. Thus, it presents larger shock to a market maker, leading ΔS_t^A to dominate ΔS_t^V. Panel (b2) of Figure 4.3 illustrates the effects of multidimensional learning on the quotes of a market maker and contrasts it with the quotes of a market maker learning only about the fundamental value. The quotes are consistent with the empirical observations that the bid moves downward faster than the ask during the periods of large selling pressure (e.g., CFTC-SEC (2010a, 2010b)). Consequently, the faster reaction of the bid and the delayed reaction of the ask compared to that of the benchmark model lead the spread to widen in response to order imbalance as illustrated in Panel (b3).[39] The next proposition summarizes the role of learning about toxicity in stabilizing the market during balanced and destabilizing during unbalanced order flow.

Proposition 4.10. In the presence of composition uncertainty;

---

[39]In fact, flash crashes do not only occur on the way down. The sharp price rise in the price of a 10-year US Treasury security (37 bps. trading range) on 15 Oct. 2014, also known as a "Flash Rally", is a recent example of this in which the market functioned with a strained liquidity, a high volatility, and a high trading volume in the presence of excessive buy orders (e.g., U.S. Dept. of the Treasury et al. (2014)). Consecutive buy orders obtain symmetric results. In the case of 20 consecutive buys with the same parameter values, π_t increases in the same way it does in 20 consecutive sells since only the amount, not the direction, of order imbalance is informative about π_t. As the bid and ask prices increase with each buy order, the ask price moves upward faster compared to the delayed reaction of the bid price, leading to a wider bid-ask spread.

(i) the bid-ask spread is given by

$$S_{\alpha,t} = S_1 + \Delta S_t, \tag{4.43}$$

where S_1 is the initial bid-ask spread and ΔS_t is the (stabilizing for $\Delta S_t < 0$ and destabilizing for $\Delta S_t > 0$) liquidity distortion relative to the initial spread;

(ii) balanced order flows always stabilize the market;

(iii) sequences of sell orders destabilize the market if and only if

$$\delta_1 < 1 + \frac{2 \cdot \left(\delta_t^s - (\delta_t^b)^{-1}\right)}{\left[2 \cdot (\delta_t^b)^{-1} + \prod_{i=1}^{t-1}(\delta_i^s)^{-1} + \prod_{i=1}^{t-1}(\delta_i^s) \cdot \delta_t^s \cdot (\delta_t^b)^{-1}\right]}; \tag{4.44}$$

(iv) sequences of buy orders destabilize the market if and only if

$$\delta_1 < 1 + \frac{2 \cdot \left(\delta_t^s - (\delta_t^b)^{-1}\right)}{\left[2 \cdot (\delta_t^b)^{-1} + \prod_{i=1}^{t-1} \delta_i^b + \prod_{i=1}^{t-1}(\delta_i^b)^{-1} \cdot \delta_t^s \cdot (\delta_t^b)^{-1}\right]}. \tag{4.45}$$

Lastly, Panels (c1)-(c3) of Figure 4.3 illustrate the market maker's belief about the high proportion of informed traders, quotes, and spread during consecutive sell orders with a temporary buy reversal. In the benchmark model without learning about the adverse selection, a reversal in order flow makes the market less liquid as it increases uncertainty about the fundamental value (i.e., $\Delta S_t^V > 0$). In the presence of learning about the adverse selection, however, the standard prediction is also not necessarily true since a temporary reversal can substantially decrease the market maker's belief about the adverse selection, leading to a downward pressure on the bid-ask spread (i.e., $\Delta S_t^A < 0$). When the gap between the low α_L and high α_H proportion of informed traders is sufficiently high, the downward pressure due to learning about the adverse selection ΔS_t^A dominates the upward pressure due to learning about the fundamental value ΔS_t^V, leading to a liquidity improvement. The dominating downward revision about the adverse selection (i.e., $\Delta S_t^A + \Delta S_t^V < 0$) and the resulting liquidity improvement is opposite to what is predicted by the standard models. Panel (c1) of Figure 4.3 illustrates the reduction in the market maker's belief about the high proportion of informed traders during a reversal and Panels (c2)-(c3) highlight a subsequent liquidity improvement.
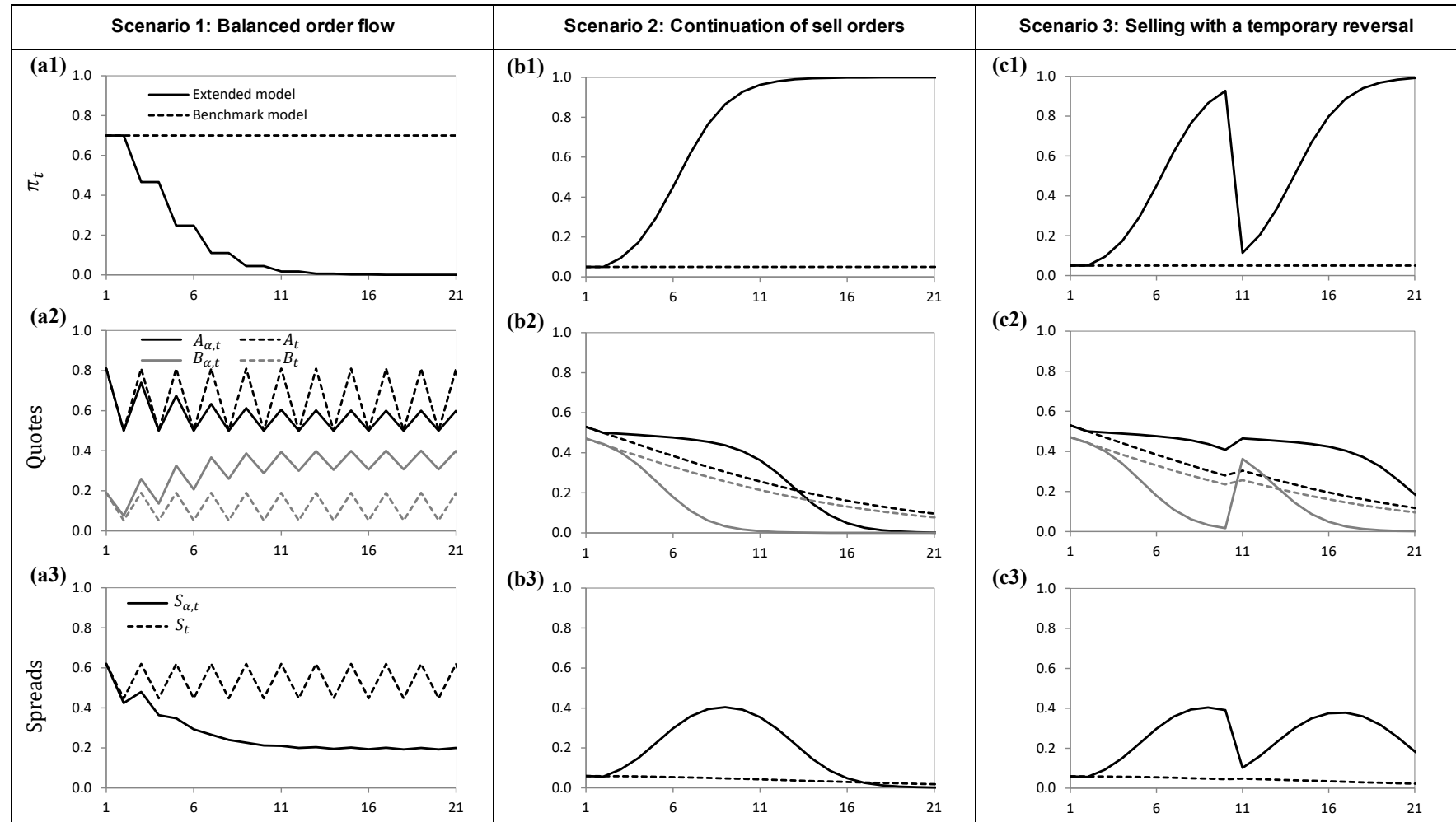
Figure 4.3: The dynamics of the market maker's belief about the adverse selection, quotes and bid-ask spreads.
Panels (a1)-(a3) plot the market maker's belief about the high informed trading, π_t, quotes, and bid-ask spread in the benchmark and extended models in the face of 20 perfectly balanced orders (i.e., a buy following a sell order). The parameter values for Panels (a1)-(a3) are $\alpha_L = 0.2$, $\alpha_H = 0.8$, and $\pi_1 = 0.7$. Panels (b1)-(b3) plot the same variables in the face of 20 consecutive sell orders (i.e., $N_t = -20$ at $t = 21$). Panels (c1)-(c3) plot the same variables in the face of 20 consecutive sell orders up to $t = 21$ with one reversal (buy) at t=10. The parameter values for Panels (b1)-(b3) and (c1)-c(3) are $\alpha_L = 0.01$, $\alpha_H = 0.99$, and $\pi_1 = 0.05$. Other parameter values are $p_1 = 0.5$ and $q = 1$.

4.4.2 Sharp price movements

In this subsection, we analyze the information content of trades in the presence of learning about adverse selection and its role in contributing to sharp price movements as another form of market instability. The main intuition that we want to rigorously characterize is that when the market maker revises her perceived informativeness of order flow, she gives more (resp. less) credit to past orders and prices are adjusted accordingly. This means when the market maker is uncertain about the proportion of informed traders, an order impacts the market maker's beliefs about the future value of the security in two ways. The first is simply that buys (resp. sells) increase (resp. decrease) the likelihood that the fundamental value is high (resp. low) (i.e., the standard price discovery effect). A second effect is that the market maker also updates her belief about the informativeness of order flow and then uses this new belief to reassess what she had learned from past order flow. We term the second effect "repricing history".[40]

To disentangle the standard price discovery and repricing history effects, we contrast the rational market maker's learning about the payoff with the learning of the myopic market maker that learns from order flow as if it is always the first order (myopically). We interpret the difference in the learning of the rational and myopic market makers about the fundamental value of the security as repricing history effect. In essence, our task is to determine the components of revision in belief about the security payoff (i.e., $\Delta p_t = p_{t+1} - p_t$) when the rational market maker learns about the payoff from p_t to p_{t+1} with an order D_t. We are interested in the contributions of the two components (standard price discovery and repricing history) of the rational market maker's learning about the payoff in the face of different order flow patterns. Various contributors to the market maker's learning about the payoff are useful in understanding large and sharp price movements in financial markets.

---

[40]The term repricing history shouldn't be confused by the fact that our notion of equilibrium requires that the market maker does not regret ex post for the trades that she is obliged to make as in the Glosten-Milgrom type models. Moreover, the stochastic process (N_t, t) is Markov, meaning that the distribution of $(N_{t+1}, t + 1)$ depends on only (N_t, t) and is independent of the history in our setting. This follows because the trades are independently and identically distributed and (N_t, t) are counting processes for trades and time.

Formally, the rational market maker's total revision in belief about the security payoff at time t follows from Eqs. (4.21) and (4.22) as

$$\Delta p_t = p_{t+1} - p_t = \begin{cases} \dfrac{p_t \cdot (1 - p_t) \cdot \left(\delta_t^b - 1\right)}{1 + p_t \cdot \left(\delta_t^b - 1\right)}, & \text{if} \quad D_t = +1, \\[3mm] \dfrac{p_t \cdot (1 - p_t) \cdot \left((\delta_t^s)^{-1} - 1\right)}{1 + p_t \cdot \left((\delta_t^s)^{-1} - 1\right)}, & \text{if} \quad D_t = -1. \end{cases} \tag{4.46}$$

To characterize the standard price discovery component, consider a myopic market maker who learns from each order as if it is always the first order (recall that initially nature independently chooses $\hat{\alpha}$ and $\hat{V}$). This means that she does not observe the history and only has posterior beliefs about the payoff $\hat{V} \in \{0, 1\}$ and the proportion of informed traders $\hat{\alpha} \in \{\alpha_L, \alpha_H\}$. Such a market maker learns from each order myopically (i.e., one step ahead) as if the nature has just independently chosen $\hat{V}$ and $\hat{\alpha}$. By one step learning about the payoff as if it is the first order, the myopic market maker does not reassess her prior learning about the payoff (i.e., the information in past order flow). The next lemma characterizes the learning of the myopic market maker without repricing history, where we carry the original notation with a superscript m describing the myopic market maker.

Lemma 4.11. Let the informativeness of orders be $\delta_t^m = \frac{1 + (\pi_t^m \cdot \alpha_H + (1 - \pi_t^m) \cdot \alpha_L)}{1 - (\pi_t^m \cdot \alpha_H + (1 - \pi_t^m) \cdot \alpha_L)}$, where π_t^m is given by

$$\pi_{t+1}^m = \frac{\left(1 + \alpha_H \cdot (2 \cdot p_t^m - 1) \cdot D_t\right) \cdot \pi_t^m}{\left(1 + \left(\pi_t^m \cdot \alpha_H + (1 - \pi_t^m) \cdot \alpha_L\right) \cdot (2 \cdot p_t^m - 1) \cdot D_t\right)}. \tag{4.47}$$

Let the geometric mean of informativeness of orders be $\bar{\delta}_t^m = \left(\prod_{i=1}^{t-1} \delta_i^m\right)^{\frac{1}{t-1}}$ and weighted order imbalance $\bar{N}_t = \sum_{\tau=1}^{t-1} D_\tau \cdot w_\tau$, where $w_\tau = \frac{(t-1) \cdot \ln \delta_\tau^m}{\sum_{i=1}^{t-1} \ln \delta_i^m}$. Then the dynamics of the expectations about the payoff satisfy

$$\frac{p_{t+1}^m}{1 - p_{t+1}^m} = \frac{p_t^m}{1 - p_t^m} \cdot (\delta_t^m)^{D_t}, \tag{4.48}$$

and hence

$$p_t^m = \frac{(\bar{\delta}_t^m)^{\bar{N}_t}}{1 + (\bar{\delta}_t^m)^{\bar{N}_t}}. \tag{4.49}$$

Lemma 4.11 shows that the myopic market maker (as in the benchmark and extended models) revises her belief upward (resp. downward) with a buy (resp. sell) order and the revision is higher with high informativeness of orders δ_t^m. Assuming

independence between $\hat{V}$ and $\hat{\alpha}$ reduces the learning of the rational market maker to that of the myopic market maker and further assuming $\alpha_L = \alpha_H$ reduces the myopic market maker's learning to the learning in the benchmark model. An interesting difference of the rational and the myopic market maker is that the myopic market maker evaluates the information content of buy and sell orders at a given point in time equally (footnote 43 shows how δ_t^b and δ_t^s reduce to symmetric δ_t^m), implying that the repricing history effect causes asymmetric informativeness of orders (δ_t^b and δ_t^s) characterized in Proposition 4.5.[41] The myopic market maker's revision in belief about the payoff, $\Delta p_t^m = p_{t+1}^m - p_t^m$, at time t follows from Eq. (4.48) as

$$\Delta p_t^m = \frac{p_t^m \cdot (1 - p_t^m) \cdot \left((\delta_t^m)^{D_t} - 1\right)}{1 + p_t^m \cdot \left((\delta_t^m)^{D_t} - 1\right)}, \tag{4.50}$$

where δ_t^m is determined by π_t^m. The repricing history effect emerges from the difference between rational belief revision about the fundamental value by considering the whole order flow history Δp_t and myopic learning with only the standard price discovery component Δp_t^m, i.e.,

$$\Delta p_t^r = \Delta p_t - \Delta p_t^m, \tag{4.51}$$

where Δp_t^r denotes the repricing history. The repricing history effect has important implications for the dynamics of informativeness and price impacts of orders, and therefore for beliefs about the fundamental value and prices, especially during highly unbalanced order flow.

First, it leads to asymmetric price reactions due to differential information content of orders (i.e., $\delta_t^b \neq \delta_t^s$). The asymmetry ($\delta_t^b - \delta_t^s$) is more pronounced during highly unbalanced order flow. More precisely, as the amount of order imbalance increases the difference between informativeness of buy and sell orders increases following Eqs. (4.18) and (4.19) ($\delta_t^b > \delta_t^s$ for negative and $\delta_t^s > \delta_t^b$ for positive imbalance).[42] This is intuitive because a buy (resp. sell) order during high sell (resp. buy) imbalance leads the market maker to learn that the past order flow may not have been as informed, resulting in a quick reassessment of the prior learning about the payoff. This means the reversal in order flow is more informative than

---

[41]Appendix 4.3 shows the details of the myopic market maker.

[42]In fact, it is straightforward to show that $\lim_{t\to\infty} \delta_t^b = \frac{1+\alpha_L}{1-\alpha_H}$ and $\lim_{t\to\infty} \delta_t^s = \frac{1+\alpha_H}{1-\alpha_L}$, leading to the maximum asymmetry $\delta_t^b - \delta_t^s = \frac{\alpha_H^2 - \alpha_L^2}{(1-\alpha_H)\cdot(1-\alpha_L)}$ when the market maker observes infinite sequences of sell orders. The results are symmetric when the market maker observes an infinite sequence of buy orders.

the continuation in order flow.[43] These results are absent in the myopic market maker since she treats each order as the first order (i.e., $\delta_t^b = \delta_t^s = \delta_t^m$) and the benchmark market maker with time-independent and symmetric informativeness of orders (i.e., $\delta_t^b = \delta_t^s = \delta$).

Second, the repricing history effect causes accelerating price impacts when the market maker receives continuation in order flow. Accelerating price impacts means the security price increases or decreases at an increasing rate. Intuitively, in the presence of uncertainty about the proportion of informed traders, a buy (resp. sell) after consecutive buys (resp. sells) not only signals that the asset value is high (resp. low), but also signals that the previous buys (sells) are more informed, leading to additional upward (resp. downward) revision in the market maker's belief about the fundamental value. The empirical literature mainly focuses on $I_t = \frac{|\Delta p_t|}{p_t}$ as a proxy for the price impact. The price impact I_t of a sell order at time t follows from Eq. (4.46) as

$$I_t = \frac{|\Delta p_t|}{p_t} = \frac{(1 - p_t) \cdot \left(1 - (\delta_t^s)^{-1}\right)}{1 + p_t \cdot \left((\delta_t^s)^{-1} - 1\right)}, \tag{4.52}$$

which reduces to

$$I_t = \frac{1 - (\delta_t^s)^{-1}}{1 + \prod_{i=1}^{t}(\delta_i^s)^{-1}}, \tag{4.53}$$

during consecutive sell orders. In the benchmark model, the price impact of an order D_t at time t is given by $I_t = \frac{|\delta^{D_t} - 1|}{\delta^{N_{t+1}} + 1}$, which always attenuates with order imbalance (i.e., $I_t < I_{t-1}$). Unlike the benchmark model, Eq. (4.53) shows that the price impact can accelerate in our model with the continuation in order flow. More precisely, the price impact accelerates $I_t > I_{t-1}$ if and only if the informativeness of order is sufficiently large, i.e.,

$$\delta_t^s > 1 + \frac{\left(\delta_{t-1}^s - 1\right) \cdot \left(\prod_{i=1}^{t-1} \delta_i^s + 1\right)}{\delta_{t-1}^s + \prod_{i=1}^{t-1} \delta_i^s}. \tag{4.54}$$

This occurs because in the presence of uncertainty about the adverse selection, the price impact decreases with the belief about the payoff p_t as in the benchmark

---

[43]Of course, there might be other explanations for why continuations and reversals in order flow have different information content. For example, one obvious explanation might be the presence of history-dependent (e.g., positively correlated with the last trade) uninformed traders in the market so that the reversal is more likely associated with informed trading (e.g., Easley, Kiefer and O'Hara (1997)). While history-dependent uninformed trading can lead to the differential information content of continuations and reversals, it does not necessarily lead to the large price swings that we explain with our model.

model (i.e., $\frac{\partial I_t}{\partial p_t} < 0$), but also increases with the informativeness of buy and sell orders (i.e., $\frac{\partial I_t}{\partial \delta_t^s} > 0$ and $\frac{\partial I_t}{\partial \delta_t^b} > 0$). Therefore, our model can explain accelerating price impacts similar to those observed during flash crashes (e.g., CFTC-SEC (2010a)), whereas the benchmark model is always associated with attenuating price impacts in response to order imbalance.

Sharp price movements in financial markets can arise as a consequence of these two effects (accelerating price impacts during continuations and more informative reversals) stemming from the repricing history effect. Compared to the myopic and benchmark market makers, the rational market maker in the presence of uncertainty about adverse selection learns faster in response to continuation in order flow (order imbalance) due to accelerating price impacts ($I_t > I_{t-1}$) and re-learns even faster in response to reversal in order flow due to more informative reversals ($\delta_t^b > \delta_t^s$ during sell and $\delta_t^s > \delta_t^b$ during buy sequences). This generates a sharp decline or rise followed by a quick reversal in price, such as the typical price path during a flash crash or rally (e.g., CFTC (2010a, 2010b) and U.S. Dept. of the Treasury et al. (2014)).

We illustrate the implications of repricing history in Figure 4.4 by considering three order flow patterns: consecutive buys in Panels (a1)-(a3), temporary sells with a subsequent reversal in Panels (b1)-(b3), and consecutive buys with a temporary reversal in Panels (c1)-(c3). Panels (a1) and (a2) of Figure 4.4 show that during the buying pressure, the rational market maker in the presence of uncertainty about the adverse selection learns faster about the payoff and the repricing history effect contributes to faster learning compared to the myopic and benchmark market makers. Panel (a3) complements these findings by showing that the informativeness of a buy order at time t, δ_t^b, slightly increases compared to the initial informativeness δ_1, which stays at its initial level in the benchmark model. Intuitively, this is because with consecutive buy orders the repricing history effect causes accelerating price impacts as the market maker reassesses what can be learned from past buy orders, leading to a faster learning. Panel (a3) additionally shows that the asymmetry between the information content of buy and sell orders increases with order imbalance. More precisely, it shows that the reversals (sells) become more informative as the market maker receives continuation in order flow (buys).
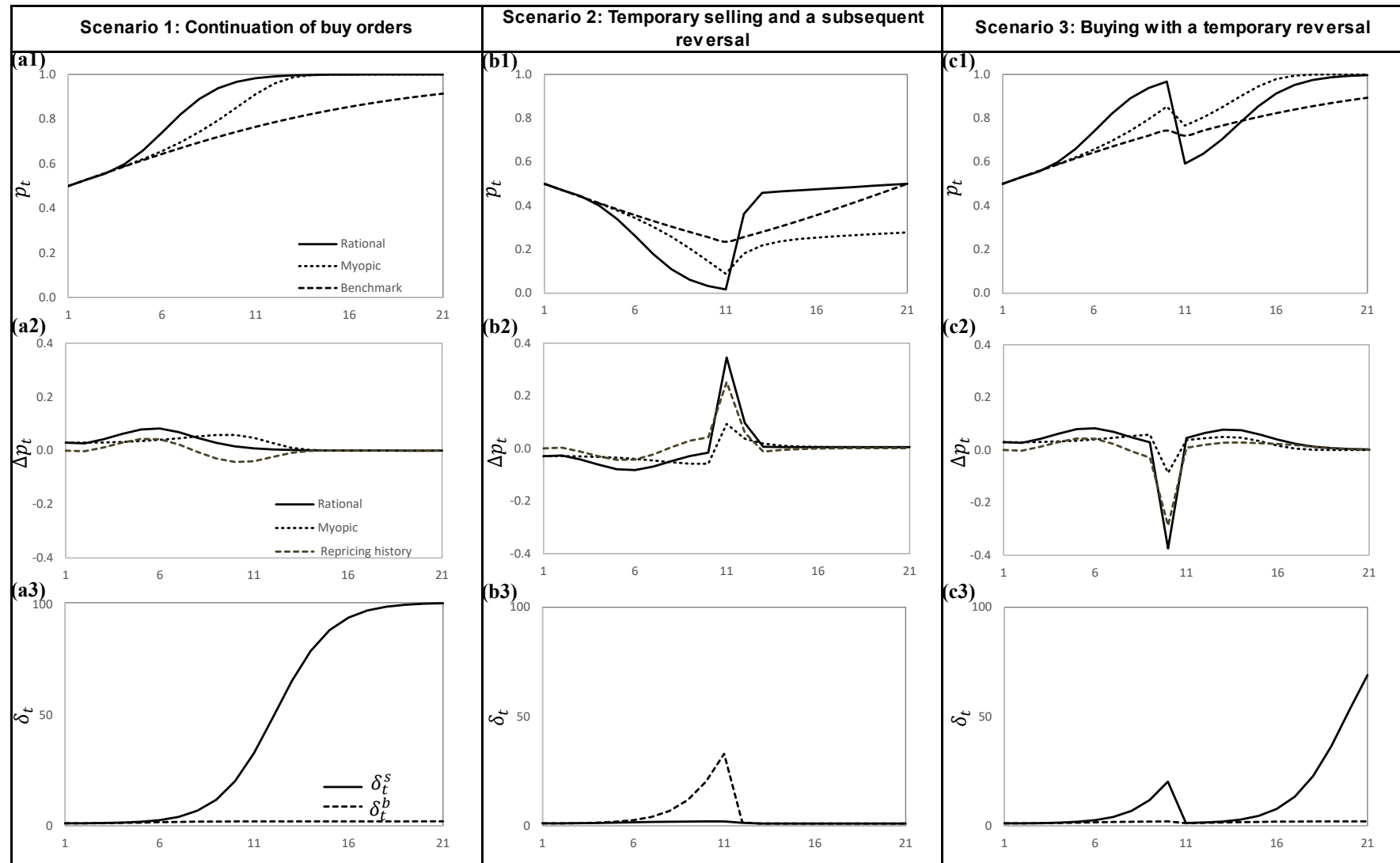
Figure 4.4: The implications of repricing history effect.

Panels (a1)-(a3) plot the three market makers' (rational, myopic and benchmark) beliefs about the payoff, p_t, the repricing history effect, Δp_t^r, and informativeness of buy and sell orders, δ_t^b and δ_t^s, in the face of consecutive buy orders up to $t = 21$. Panels (b1)-(b3) plot the same variables in the face of consecutive sells up to $t = 11$ followed by consecutive buys up to $t = 21$. Panels (c1)-(c3) plot the same variables in the face of consecutive buys up to $t = 21$ with one reversal (sell) at $t = 10$. The parameter values are $\alpha_H = 0.99$, $\alpha_L = 0.01$, $q = 1$, $p_1 = 0.5$, and $\pi_1 = 0.05$.

Panel (b1) shows that during consecutive sells followed by consecutive buys, the conditional expected payoff p_t declines faster for the same reason it occurs during consecutive buys and reverses quickly due to more informative reversals (buys) illustrated in Panel (b3). Panel (b2) highlights that the magnitude of repricing history effect in contributing the reassessment of the market maker's beliefs about the fundamental value can be substantial during the reversal in order flow. The repricing history effect leads the informativeness of buy orders δ_t^b to be greater than informativeness of sell orders δ_t^s during the sell sequence. Thus, similar to flash crashes, the repricing history effect can generate sharp crashes due to accelerating price impacts and quick recoveries due to more informative reversals.

Finally, Panels (c1) and (c2) highlight the information content of one reversal (sell) during a buy sequence. With one sell order at $t = 10$, the market maker reassesses what she had learnt up to $t = 10$ by treating the previous buy orders as less informed than previously believed, because the sell causes a downward revision in belief about the high proportion of informed traders, whereas the benchmark market maker fails to take this into account and the myopic market maker only does so without reassessment of the full order flow history. Additionally, Panel (c3) illustrates that the informativeness of sell orders increases as the market maker faces continuation in buy orders, which is symmetric to continuation in sell orders.

4.5 Empirical Implications

In this section, we discuss empirical implications of our model.

4.5.1 Prevalence of flash crashes

Anecdotal evidence suggests that "mini flash crashes" driven by large aggressive orders, e.g., intermarket sweep orders, occur nearly every day.[44] Our model offers two explanations for why this is the case. First, with the rise of algorithmic trading and availability of market data, the financial market ecosystem has now dramatically changed.[45] The composition of market participants has never been

---

[44]Mini or micro flash crashes occur when a stock price spikes up or down in a small time frame. Nanex Research offers an exhaustive documentation of "mini flash crashes": http://www.nanex.net/NxResearch/ResearchPage/3/.

[45]See, for instance, "The big changes in US markets since Black Monday" (Financial Times, October 19, 2017), "3 ways big data is changing financial trading" (Bloomberg, July 4, 2017).

more complex and uncertain.[46] The technological developments have amplified the uncertainty in asymmetric information problem of the modern liquidity providers. In our model, this corresponds to the gap between the low α_L and the high α_H probability of informed trading. Indeed, as the gap between α_L and α_H increases our model shows that the market becomes more vulnerable to order imbalance and the magnitude of the instability caused by order imbalance increases. Small order imbalances with high composition uncertainty can lead to liquidity black holes, large price swings, elevated volatility, and consequently, the prevalence of flash crashes.

A second reason is associated with the increased competition among liquidity providers as a result of the proliferation of HFT and demise of the traditional designated market makers. The current market structure incentivizes learning about the time-varying adverse selection risk to ensure spreads always reflect the level of toxicity. Therefore, efficient learning about the time-varying level of adverse selection is crucial for a liquidity provider to remain competitive in today's major equity markets. These effects can also contribute to the increased prevalence of flash crashes.

4.5.2 Model predictions

Our model makes a number of empirical predictions about the dynamics of prices, liquidity, and order flow. Some of these predictions provide a theoretical explanation for results that have been reported in the empirical market microstructure literature. Yet others are more nuanced empirical predictions that are yet to be tested, forming a foundation for future empirical analysis. The most straightforward prediction of our model is that during the selling (resp. buying) pressure the bid (resp. ask) moves faster than the ask (resp. bid), and therefore, liquidity evaporates (this effect is not possible in the benchmark model because sells during sell imbalance always impact the ask more than the bid and vice versa). This finding is consistent with the empirical results of Engle and Patton (2004), who find that sells impact the bid more than the ask, which stands in contrast to the theoretical results of Glosten and Milgrom (1985). Our model shows that this

---

[46]The new era of data revolution stimulated some of the data analytics firms to enter into a hedge fund business (e.g., Cargometrics). See, for instance, "When Silicon Valley came to Wall Street" (Financial Times, October 28, 2017), "Rise of quant: New hedge funds next year to embrace high tech" (Bloomberg, December 21, 2017).

occurs because of the market maker's learning about toxicity (adverse selection) from order flow. A cursory examination of the transactions data series of E-mini and SPY (S&P 500 ETF) confirms that the same phenomenon was present during the Flash Crash. Second, the increasing informativeness of orders and wider spreads in response to order imbalances imply that the trades that arrive when the spread is wide have a greater price impact. This is consistent with Hasbrouck (1991), who finds that trades that occur in the face of wider spreads have a larger price impact than those that occur when the spreads are narrow.

Our model also develops several other empirical implications about the dynamics of spreads, informativeness, and price impacts of trades during various order flow patterns. The model predicts that liquidity can improve during balanced orders and reversals in order flow due to learning about the adverse selection. While these results are intuitive in the presence of learning about the adverse selection, both are opposite to what is predicted by standard market microstructure models. In addition, the informativeness and price impacts of trades are time-varying and asymmetric due to learning about the adverse selection. The model predicts that the asymmetry in price impacts of buy and sell orders increases (resp. decreases) as order imbalance increases (resp. decreases).

In our model, accelerating price impacts and more informative reversals during unbalanced order flow naturally arise due to the repricing history effect. Engle and Patton (2004) find strong evidence on differential impacts of buy and sell orders on the bid and ask prices. They interpret the result with potentially multiple information events per day. Our model shows that uncertainty about the proportion of informed traders, the quality of their signals, multidimensional learning, and consequently, the repricing history effect are what drive this asymmetry. The accelerating price impact and more informative reversals due to the repricing history effect can generate a security price dynamics similar to flash crashes. Overall, our results suggest that financial markets are more susceptible to instability in response to order imbalance in times when adverse selection is believed to be low and can digest more imbalance in times when adverse selection is high. This follows because order imbalance destabilizes the market when the initial belief about the adverse selection is sufficiently low, which we also show occurs after balanced order flow.

4.6   Model Discussion and Extensions

In this chapter, we use a simple sequential trading model in the sense of Glosten and Milgrom (1985) to provide intuition about the destabilizing role of order imbalances in financial markets and the occurrences of financial crashes in the absence of the fundamental news about the security value. An interesting question is how sensitive our results are to our modeling approach. In this section, we address this by considering how some extensions and generalizations of our model affect the results.

4.6.1   Other sources of uncertainty about adverse selection

The model presented thus far incorporates uncertainty about the proportion of informed traders and the security payoff. Allowing uncertainty in the other dimensions of adverse selection problem complicates the notation, but does not affect our results. For example, in Appendix 4.1, we allow the market maker to be uncertain about the different combinations of uncertainties in the proportion of informed traders, the quality of their private information, and the security payoff. We show that while the magnitude of market instability may change, the main qualitative results of our model are robust.

4.6.2   Endogenous uninformed trading

For convenience, we assumed that the motivation for uninformed trading is exogenous. There may be other uninformed traders whose demands reflect more complex motivations resulting in distributions of private valuations that drive their trading decisions (e.g., Easley et al. (1997), Glosten and Putnins (2016)). In fact, if uninformed traders have elastic demands sensitive to trading costs, the destabilizing effects of order imbalances are also amplified. To see this, suppose there are uninformed traders who have a distribution of private values and there is occasionally a highly impatient trader that is either an informed trader or a distressed uninformed trader (one that has a private valuation very far from the current price). In normal conditions, when the spread is tight, the market maker receives some order flow from the uninformed traders (balanced) and some of the imbalance from the informed or distressed uninformed traders. If the imbalance is

sufficiently strong (that is, the desperate or highly informed trader hits the market too aggressively), the change in quotes is sufficiently large (due to updating beliefs about the probability of informed trading). This scares off most of the uninformed traders and makes the order flow even more unbalanced, causing a feedback loop that can amplify the destabilizing effects of order imbalance. Thus our modeling of uninformed traders as exogenous and insensitive to the cost of trading is conservative in that it understates the severity of the impact of order imbalance.

4.7   Conclusion

With increasing competition between liquidity providers (e.g., due to endogenous liquidity providers) and the use of algorithms in trading, market participants learn not only about the fundamental values of assets, but also other characteristics of markets that are important for extracting information from order flow, such as the degree of adverse selection. Such multidimensional learning can have very different implications for trading behavior, market liquidity, and stability compared to learning only about the fundamental values of assets. In this chapter, we explore the effects of order imbalance when liquidity providers learn not only about the fundamental value of the asset, as in the standard market microstructure models, but also about the proportion of informed traders and the quality of their information. The multidimensional learning explains a variety of empirical regularities not captured by the standard asymmetric information models of market microstructure theory.

Our theoretical model with additional learning about the toxicity of order flow shows the potentially destabilizing effects of order imbalance. We show that order imbalance can have a stabilizing effect on the market by narrowing the spread because it reduces uncertainty about the fundamental value and destabilizes the market by widening the spread because it increases belief about the high adverse selection risk. The destabilizing effect of order imbalance dominates its stabilizing effect when the initial belief about the adverse selection is sufficiently low. This means financial markets become more susceptible to imbalance-induced instability when the past order flow is more balanced. Put differently, in our setting, it is the order imbalance during stability that leads to instability.

In addition to the sudden liquidity dry-ups, order imbalance can also naturally lead to a sharp price decline and a quick recovery similar to flash crashes due to the "repricing history" effect. We show that a sharp price decline occurs due to accelerating price impacts with continuations in order flow and a quick recovery occurs due to more informative reversals in order flow, both stemming from the "repricing history" effect. Overall, our model provides a theoretical framework for further empirical work characterizing the dynamics of order flow, liquidity, and prices.

Appendix 4.1. Multidimensional Uncertainty

In this Appendix, we extend our model to allow the market maker to be uncertain about the security payoff, proportion of informed traders, and quality of informed traders' information.

Uncertainty about the proportion of informed traders and quality of their signals

Similar to Section 4.3, we assume that the probability of informed trading takes either low or high values from the set $\hat{\alpha} \in \{\alpha_L, \alpha_H\}$ with an initial prior probability of $\Pr(\hat{\alpha} = \alpha_H) = \pi_1$, where $0 < \alpha_L < \alpha_H < 1$ and $0 < \pi_1 < 1$. In addition, to extend our results to additional uncertainty about the quality of informed traders' information, we assume that the quality of their signals takes either low or high values from the set $\hat{q} \in \{q_L, q_H\}$ with an initial probability of $\Pr(\hat{q} = q_H) = \rho_1$, where $0.5 < q_L < q_H \leq 1$ and $0 < \rho_1 < 1$. We denote the market maker's belief about the high probability of informed trading conditional on the trading history as $\pi_t = \Pr\{\hat{\alpha} = \alpha_H | h_t\}$ and the informed traders having high quality private information conditional on the trading history as $\rho_t = \Pr\{\hat{q} = q_H | h_t\}$. With two possible values for the future security payoff, probability of informed trading, and quality of their private information (i.e., $\hat{\alpha} \in \{\alpha_L, \alpha_H\}$, $\hat{q} \in \{q_L, q_H\}$, $\hat{V} \in \{0, 1\}$), there are 8 possible disjoint states in this model. Denote the states $\mathcal{S} \in \{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8\}$, where

$$
\begin{aligned}
s_1 &= \{\hat{\alpha} = \alpha_H, \hat{q} = q_H, \hat{V} = 1\}, & s_2 &= \{\hat{\alpha} = \alpha_H, \hat{q} = q_L, \hat{V} = 1\}, \\
s_3 &= \{\hat{\alpha} = \alpha_L, \hat{q} = q_H, \hat{V} = 1\}, & s_4 &= \{\hat{\alpha} = \alpha_L, \hat{q} = q_L, \hat{V} = 1\}, \\
s_5 &= \{\hat{\alpha} = \alpha_H, \hat{q} = q_H, \hat{V} = 0\}, & s_6 &= \{\hat{\alpha} = \alpha_H, \hat{q} = q_L, \hat{V} = 0\}, \\
s_7 &= \{\hat{\alpha} = \alpha_L, \hat{q} = q_H, \hat{V} = 0\}, & s_8 &= \{\hat{\alpha} = \alpha_L, \hat{q} = q_L, \hat{V} = 0\}.
\end{aligned}
\tag{A4.1.1}
$$

The market maker's beliefs about the future security payoff, proportion of informed traders, and quality of their signals follow from Eq. (A4.1.1), respectively, as

$$
p_t = \Pr(\hat{V} = 1 | h_t) = \Pr(s_1 | h_t) + \Pr(s_2 | h_t) + \Pr(s_3 | h_t) + \Pr(s_4 | h_t), \tag{A4.1.2}
$$

$$
\pi_t = \Pr(\hat{\alpha} = \alpha_H | h_t) = \Pr(s_1 | h_t) + \Pr(s_2 | h_t) + \Pr(s_5 | h_t) + \Pr(s_6 | h_t), \tag{A4.1.3}
$$

$$
\rho_t = \Pr(\hat{q} = q_H | h_t) = \Pr(s_1 | h_t) + \Pr(s_3 | h_t) + \Pr(s_5 | h_t) + \Pr(s_7 | h_t). \tag{A4.1.4}
$$

The concept of equilibrium is the same as Definition 4.1. The next proposition characterizes the equilibrium quotes and bid-ask spread in the presence of multiple dimensions of uncertainty.

Proposition A4.1.1. The equilibrium bid and ask prices in the presence of multidimensional uncertainty (i.e., future security payoff, proportion of informed traders, and quality of their signals) are respectively given by

$$B_{\alpha,t} = \frac{p_t}{p_t + \delta_t^s \cdot (1 - p_t)}, \qquad and \qquad A_{\alpha,t} = \frac{p_t}{p_t + (\delta_t^b)^{-1} \cdot (1 - p_t)}, \tag{A4.1.5}$$

and the bid-ask spread is given by

$$S_{\alpha,t} = \frac{p_t \cdot (1 - p_t) \cdot (\delta_t^s - (\delta_t^b)^{-1})}{\left[p_t + \delta_t^s \cdot (1 - p_t)\right] \cdot \left[p_t + (\delta_t^b)^{-1} \cdot (1 - p_t)\right]}, \tag{A4.1.6}$$

where

$$\delta_t^b = \frac{(1 + \alpha_H(2q_H - 1)) \cdot \Pr(s_1|h_t) + (1 + \alpha_H(2q_L - 1)) \cdot \Pr(s_2|h_t) + (1 + \alpha_L(2q_H - 1)) \cdot \Pr(s_3|h_t) + (1 + \alpha_L(2q_L - 1)) \cdot \Pr(s_4|h_t)}{(1 - \alpha_H(2q_H - 1)) \cdot \Pr(s_5|h_t) + (1 - \alpha_H(2q_L - 1)) \cdot \Pr(s_6|h_t) + (1 - \alpha_L(2q_H - 1)) \cdot \Pr(s_7|h_t) + (1 - \alpha_L(2q_L - 1)) \cdot \Pr(s_8|h_t)} \cdot \left(\frac{1 - p_t}{p_t}\right), \tag{A4.1.7}$$

and

$$\delta_t^s = \frac{(1 + \alpha_H(2q_H - 1)) \cdot \Pr(s_5|h_t) + (1 + \alpha_H(2q_L - 1)) \cdot \Pr(s_6|h_t) + (1 + \alpha_L(2q_H - 1)) \cdot \Pr(s_7|h_t) + (1 + \alpha_L(2q_L - 1)) \cdot \Pr(s_8|h_t)}{(1 - \alpha_H(2q_H - 1)) \cdot \Pr(s_1|h_t) + (1 - \alpha_H(2q_L - 1)) \cdot \Pr(s_2|h_t) + (1 - \alpha_L(2q_H - 1)) \cdot \Pr(s_3|h_t) + (1 - \alpha_L(2q_L - 1)) \cdot \Pr(s_4|h_t)} \cdot \left(\frac{p_t}{1 - p_t}\right), \tag{A4.1.8}$$

show the informativeness of buy and sell orders respectively. In addition, δ_t^b and δ_t^s are always greater than unity and increase with the proportions of informed trading α_L and α_H, and the qualities of the informed traders' signals q_L and q_H.

Proof. The proof follows similar to the proof of Proposition 4.5 (see Appendix 4.2). The difference is to recognize that

$$B_{\alpha,t} = \Pr(\hat{V} = 1|h_t, D_t = -1) = \Pr(s_1|h_t, D_t = -1) + \Pr(s_2|h_t, D_t = -1) + \Pr(s_3|h_t, D_t = -1) + \Pr(s_4|h_t, D_t = -1), \tag{A4.1.9}$$

$$A_{\alpha,t} = \Pr(\hat{V} = 1|h_t, D_t = -1) = \Pr(s_1|h_t, D_t = +1) + \Pr(s_2|h_t, D_t = +1) + \Pr(s_3|h_t, D_t = +1) + \Pr(s_4|h_t, D_t = +1), \tag{A4.1.10}$$

which follow from the straightforward application of Bayes' rule, i.e., $\Pr(s_1|h_t, D_t = -1) = \frac{Pr(D_t=-1|s_1)}{\sum_{s_i \in S} Pr(D_t=-1|s_i) \cdot Pr(s_i|h_t)} \cdot \Pr(s_1|h_t)$, with the similar rules for the other states. The results follow once the probabilities of buy ($D_t = +1$) and sell ($D_t = -1$) orders in each state are calculated as

$$\Pr(D_t|s_1) = \frac{1 + \alpha_H(2q_H - 1) \cdot D_t}{2}, \qquad \Pr(D_t|s_2) = \frac{1 + \alpha_H(2q_L - 1) \cdot D_t}{2}, \qquad \Pr(D_t|s_3) = \frac{1 + \alpha_L(2q_H - 1) \cdot D_t}{2}, \qquad \Pr(D_t|s_4) = \frac{1 + \alpha_L(2q_L - 1) \cdot D_t}{2},$$

$$\Pr(D_t|s_5) = \frac{1 - \alpha_H(2q_H - 1) \cdot D_t}{2}, \qquad \Pr(D_t|s_6) = \frac{1 - \alpha_H(2q_L - 1) \cdot D_t}{2}, \qquad \Pr(D_t|s_7) = \frac{1 - \alpha_L(2q_H - 1) \cdot D_t}{2}, \qquad \Pr(D_t|s_8) = \frac{1 - \alpha_L(2q_L - 1) \cdot D_t}{2}.$$

$$(A4.1.11)$$

■

Proposition A4.1.1 maintains the forms of bid and ask in the benchmark and extended models with composition uncertainty (see Propositions 4.2 and 4.5). The only difference in this Appendix is that uncertainty about the quality of informed traders' information and the market maker's learning about it also affect the informativeness of orders, δ_t^b and δ_t^s, and consequently, the quotes and spread. One way to see the direct impact of uncertainty about the quality of traders' signals is to consider a myopic market maker, which leads to symmetric information content of a buy and a sell

$$\delta_t^b = \delta_t^s = \frac{1 + \left(\pi_t \cdot \alpha_H + (1 - \pi_t) \cdot \alpha_L \right) \cdot \left(2 \cdot \left(\rho_t \cdot q_H + (1 - \rho_t) \cdot q_L \right) - 1 \right)}{1 - \left(\pi_t \cdot \alpha_H + (1 - \pi_t) \cdot \alpha_L \right) \cdot \left(2 \cdot \left(\rho_t \cdot q_H + (1 - \rho_t) \cdot q_L \right) - 1 \right)}. \qquad (A4.1.12)$$

Eq. (A4.1.12) shows that the increase in ρ_t implies a higher adverse selection risk for the market maker, leading to a wider bid-ask spread. As Proposition A4.1.1 maintains the forms of bid and ask quotes, by the similar arguments, the dynamics of belief about the payoff, p_t, after each order D_t and for the sequences of buy or sell orders follow the same way outlined in Lemma 4.6 and Corollary 4.9, respectively. In addition, the market maker's learning about uncertainties follow analogous to Corollary 4.8.

Proposition A4.1.2. In the presence of multidimensional uncertainty (i.e., future security payoff, proportion of informed traders, and quality of their signals);

(i) the market maker observing balanced order flows (i.e., $N_t = 0$ at time t) learns nothing about the security payoff (i.e., $p_t = p_1$), but revises her beliefs about the high informed trading and the informed traders having high-quality information downward (i.e., $\pi_t < \pi_1$, $\rho_t < \rho_1$).

(ii) the market maker observing sequences of buy or sell orders (i.e., $N_t = t - 1$ or $N_t = -(t - 1)$ at time t) revises her beliefs about the high informed trading and the informed traders having high-quality information upward (i.e., $\pi_t > \pi_1$, $\rho_t > \rho_1$).

We now turn our attention to how beliefs about the future security payoff, proportion of informed traders, and quality of their signals evolve when there are sequences of buy and sell orders and their effects on the quotes and spread. Figures 4.5 and 4.6 illustrate the results in the presence of consecutive sell and buy orders, respectively. Panels (a)-(c) of the figures show that beliefs about the high informed trading and high-quality signals are revised upward stronger in the presence of multiple sources of uncertainty. The stronger upward revisions of beliefs about the high informed trading and high-quality signals result in high informativeness of orders, leading to a wider bid-ask spread first, but at the same time faster convergence due to a faster learning about the payoff. Panels (d)-(f) are consistent with the practice that the destabilizing role of order imbalances is stronger in the presence of multiple dimensions of uncertainty about the adverse selection.
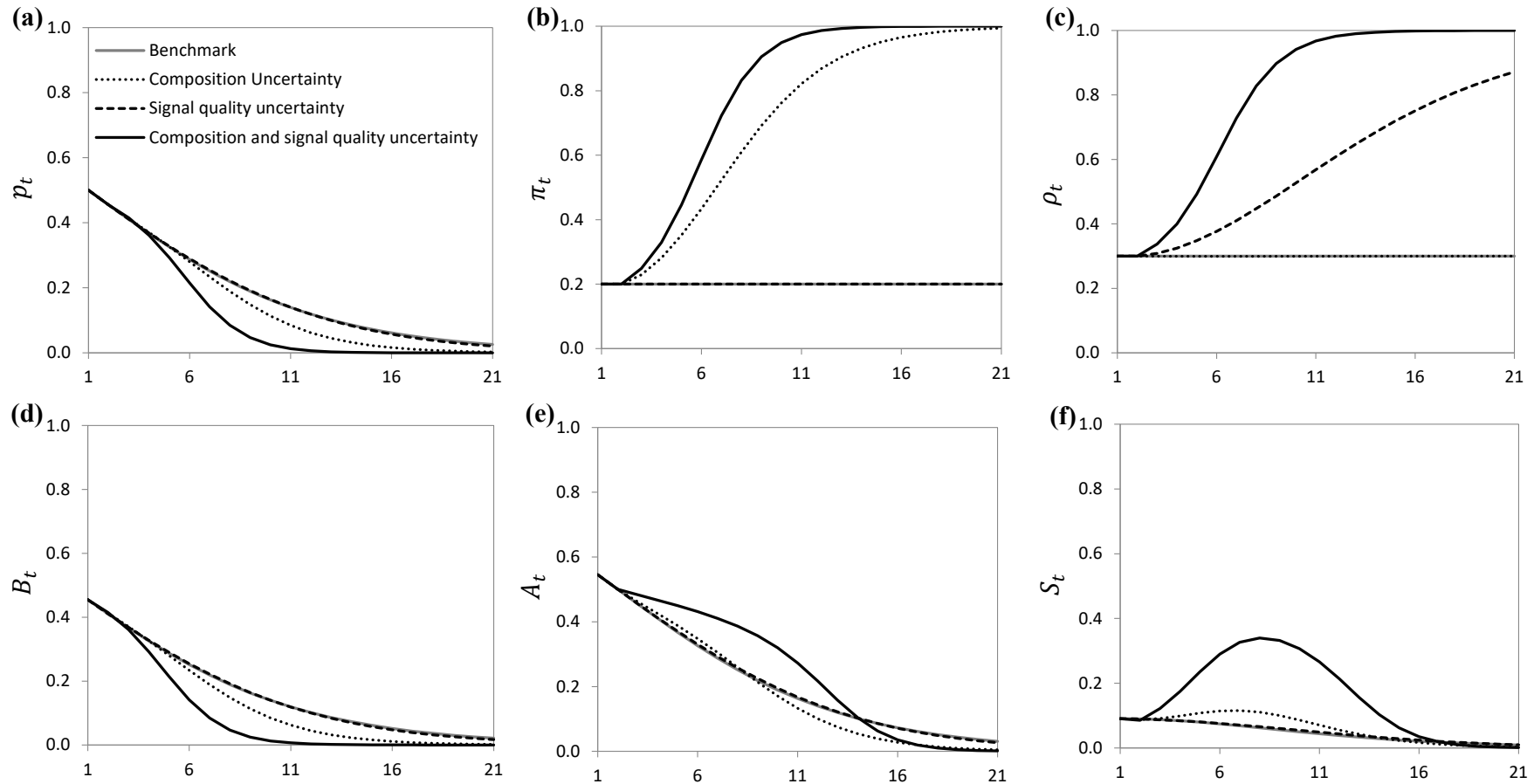
Figure 4.5: The dynamics of beliefs, quotes and bid-ask spread during the sequences of sell orders.
Panel (a) plots the market maker's belief about the payoff, p_t, (b) plots belief about the high informed trading, π_t, (c) plots belief about the informed traders having high quality information, ρ_t, (d) plots bid, B_t, (e) plots ask, A_t, and (f) plots bid-ask spread S_t in the face of 20 consecutive sell orders (i.e., $N_t = -20$ at $t = 21$). The parameter values are $\alpha_H = 0.99$, $\alpha_L = 0.01$, $q_H = 1$, $q_L = 0.6$, $p_1 = 0.5$, $\pi_1 = 0.1$, and $\rho_1 = 0.3$.
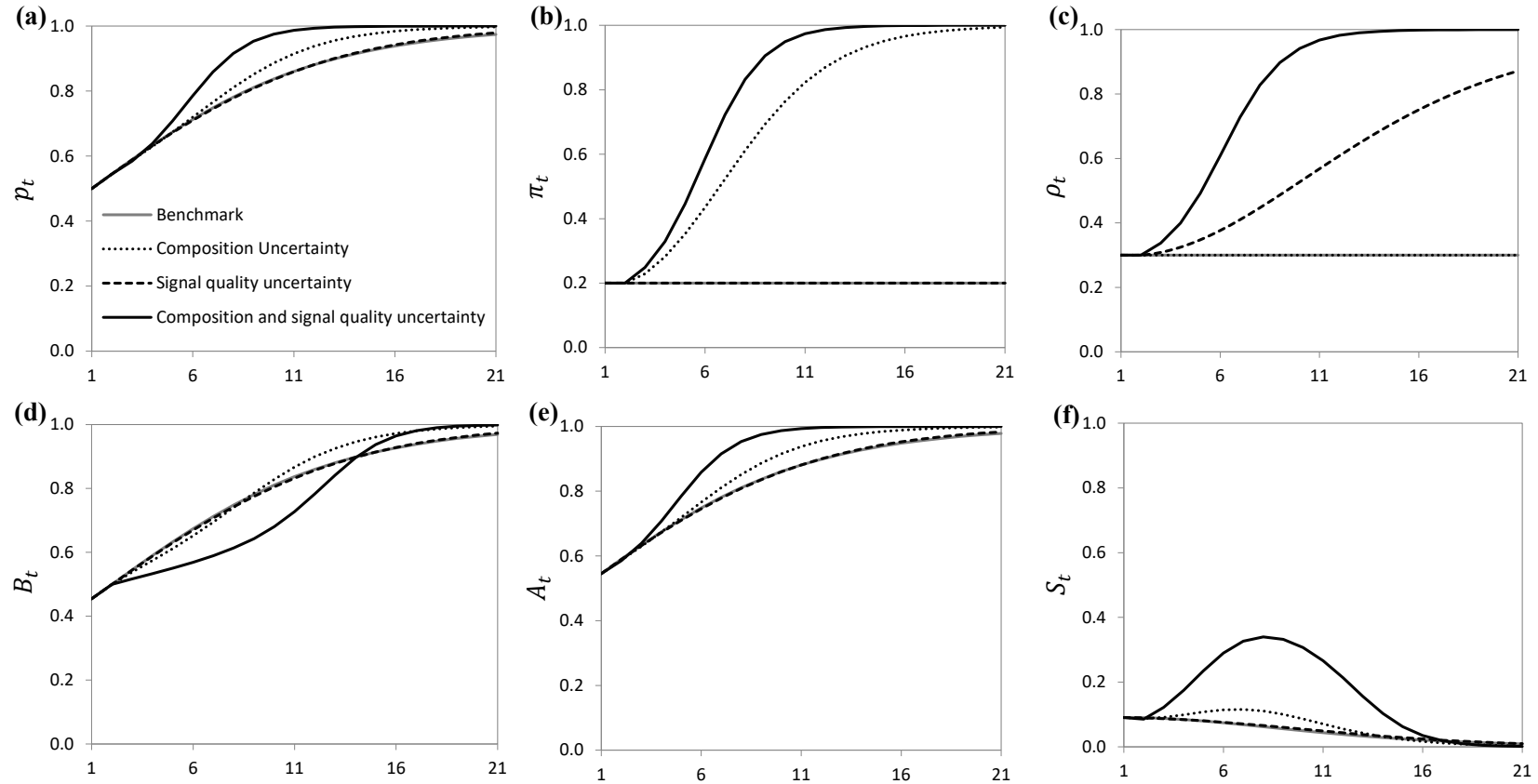
Figure 4.6: The dynamics of beliefs, quotes and bid-ask spread during the sequences of buy orders.
Panel (a) plots the market maker's belief about the payoff, p_t, (b) plots belief about the high informed trading, π_t, (c) plots belief about the informed traders having high quality information, ρ_t, (d) plots bid, B_t, (e) plots ask, A_t, and (f) plots bid-ask spread S_t in the face of 20 consecutive buy orders (i.e., $N_t = 20$ at $t = 21$). The parameter values are $\alpha_H = 0.99$, $\alpha_L = 0.01$, $q_H = 1$, $q_L = 0.6$, $p_1 = 0.5$, $\pi_1 = 0.1$, and $\rho_1 = 0.3$.

Appendix 4.2. Proofs

Proof of Proposition 4.2. The following expressions follow from Bayes' theorem.

$$\Pr\{D_t = +1 | \hat{V} = 1, h_t\} = \frac{1 + \alpha \cdot (2 \cdot q - 1)}{2}; \qquad (A4.2.1)$$

$$\Pr\{D_t = -1 | \hat{V} = 1, h_t\} = \frac{1 - \alpha \cdot (2 \cdot q - 1)}{2}; \qquad (A4.2.2)$$

$$\Pr\{D_t = +1 | h_t\} = \frac{1 + \alpha \cdot (2 \cdot p_t - 1) \cdot (2 \cdot q - 1)}{2}; \qquad (A4.2.3)$$

$$\Pr\{D_t = -1 | h_t\} = \frac{1 + \alpha \cdot (1 - 2 \cdot p_t) \cdot (2 \cdot q - 1)}{2}. \qquad (A4.2.4)$$

From conditions (1) and (3) of Definition 4.1 (i.e., the zero-expected-profit and Bayesian conditions) the bid and ask prices follow, respectively, as

$$B_t = E[\hat{V} = 1 | h_t, D_t = -1] = \underbrace{\Pr\{V = 1 | h_t\}}_{p_t} \cdot \frac{\Pr\{D_t = -1 | \hat{V} = 1, h_t\}}{\Pr\{D_t = -1 | h_t\}}, \quad (A4.2.5)$$

$$A_t = E[\hat{V} = 1 | h_t, D_t = +1] = \Pr\{V = 1 | h_t\} \cdot \frac{\Pr\{D_t = +1 | \hat{V} = 1, h_t\}}{\Pr\{D_t = +1 | h_t\}}. \quad (A4.2.6)$$

Substituting Eqs. (A4.2.2) and (A4.2.4) into Eq. (A4.2.5) and Eqs. (A4.2.1) and (A4.2.3) into Eq. (A4.2.6), defining

$$\delta = \frac{1 + \alpha \cdot (2 \cdot q - 1)}{1 - \alpha \cdot (2 \cdot q - 1)} > 1, \qquad (A4.2.7)$$

for $q \in (1/2, 1]$ and $\alpha \in (0, 1)$ and rearranging yields

$$B_t = \frac{p_t}{p_t + \delta \cdot (1 - p_t)}, \qquad (A4.2.8)$$

$$A_t = \frac{p_t}{p_t + \delta^{-1} \cdot (1 - p_t)}. \qquad (A4.2.9)$$

The bid-ask spread S_t follows from the difference of the ask in Eq. (A4.2.9) and the bid in Eq. (A4.2.8) as

$$S_t = \frac{p_t \cdot (1 - p_t) \cdot (\delta - \delta^{-1})}{\left(p_t + \delta \cdot (1 - p_t)\right) \cdot \left(p_t + \delta^{-1} \cdot (1 - p_t)\right)}. \qquad (A4.2.10)$$

Finally, differentiating Eq. (A4.2.7) with respect to (w.r.t.) α and q obtains

$$\frac{\partial \delta}{\partial \alpha} = \frac{2 \cdot (2 \cdot q - 1)}{\left(1 - \alpha \cdot (2 \cdot q - 1)\right)^2} > 0 \quad \text{and} \quad \frac{\partial \delta}{\partial q} = \frac{4 \cdot \alpha}{\left(1 - \alpha \cdot (2 \cdot q - 1)\right)^2} > 0$$
$$(A4.2.11)$$

for $q \in (1/2, 1]$ and $\alpha \in (0, 1)$.

■

Proof of Lemma 4.3. Rearranging Eqs. (A4.2.8) and (A4.2.9) obtains

$$\frac{B_t}{1 - B_t} = \delta^{-1} \cdot \frac{p_t}{1 - p_t}, \tag{A4.2.12}$$

$$\frac{A_t}{1 - A_t} = \delta \cdot \frac{p_t}{1 - p_t}. \tag{A4.2.13}$$

Since $p_{t+1} = E_{t+1}[\hat{V}|h_t, D_t]$ is B_t if $D_t = -1$ and A_t if $D_t = 1$, it follows that

$$\frac{p_{t+1}}{1 - p_{t+1}} = \begin{cases} \dfrac{B_t}{1 - B_t}, & \text{if} \quad D_t = -1, \\[2ex] \dfrac{A_t}{1 - A_t}, & \text{if} \quad D_t = +1. \end{cases} \tag{A4.2.14}$$

Therefore,

$$\frac{p_{t+1}}{1 - p_{t+1}} = \delta^{D_t} \cdot \frac{p_t}{1 - p_t}. \tag{A4.2.15}$$

Iterating from the first trade at $t = 1$ yields

$$\frac{p_t}{1 - p_t} = \left(\frac{p_1}{1 - p_1}\right) \cdot \delta^{(D_1 + \dots + D_{t-1})} = \left(\frac{p_1}{1 - p_1}\right) \cdot \delta^{N_t}, \tag{A4.2.16}$$

where p_1 is the initial prior probability and N_t is the order imbalance up to (but not including) the trade at time t. Solving Eq. (A4.2.16) for p_t obtains

$$p_t = \frac{p_1 \cdot \delta^{N_t}}{1 + p_1 \cdot (\delta^{N_t} - 1)}, \tag{A4.2.17}$$

and inserting the initial prior probability $p_1 = 0.5$ into Eq. (A4.2.17) obtains

$$p_t = \frac{\delta^{N_t}}{1 + \delta^{N_t}}. \tag{A4.2.18}$$

■

Proof of Corollary 4.4. *(i)* $p_t = p_1$ follows immediately from substituting $N_t = 0$ into Eq. (A4.2.17). *(ii)* follows from the partial derivative of Eq. (A4.2.17) w.r.t. N_t,

$$\frac{\partial p_t}{\partial N_t} = \frac{\delta^{N_t} \cdot (\ln \delta) \cdot p_1 \cdot (1 - p_1)}{\left(1 + p_1 \cdot (\delta^{N_t} - 1)\right)^2} > 0, \tag{A4.2.19}$$

which shows that p_t increases with positive order imbalance and decreases with negative order imbalance. In addition, the magnitude of increase or decrease is higher with more informative trades following

$$\frac{\partial p_t}{\partial \delta} = \frac{N_t \cdot \delta^{N_t - 1} \cdot p_1 \cdot (1 - p_1)}{\left(1 + p_1 \cdot (\delta^{N_t} - 1)\right)^2}, \tag{A4.2.20}$$

which is greater than 0 for $N_t > 0$ and less than 0 for $N_t < 0$.

(iii) Substituting Eq. (A4.2.17) into Eq. (A4.2.10), we obtain

$$S_t = \frac{\delta^{N_t} \cdot (\delta - \delta^{-1}) \cdot p_1 \cdot (1 - p_1)}{\left(p_1 \cdot \delta^{N_t} + \delta \cdot (1 - p_1)\right) \cdot \left(p_1 \cdot \delta^{N_t} + \delta^{-1} \cdot (1 - p_1)\right)}. \tag{A4.2.21}$$

It follows from Eq. (A4.2.21) that at $t = 1$, the spread is given by

$$S_1 = \frac{(\delta - \delta^{-1}) \cdot p_1 \cdot (1 - p_1)}{\left(p_1 + \delta \cdot (1 - p_1)\right) \cdot \left(p_1 + \delta^{-1} \cdot (1 - p_1)\right)}. \tag{A4.2.22}$$

Multiplying and dividing the right-hand side of Eq. (A4.2.21) with Eq. (A4.2.22) yields

$$S_t = S_1 \cdot \frac{\left(p_1 + \delta \cdot (1 - p_1)\right) \cdot \left(p_1 + \delta^{-1} \cdot (1 - p_1)\right) \cdot \delta^{N_t}}{\left(p_1 \cdot \delta^{N_t} + \delta \cdot (1 - p_1)\right) \cdot \left(p_1 \cdot \delta^{N_t} + \delta^{-1} \cdot (1 - p_1)\right)}. \tag{A4.2.23}$$

$S_t = S_1$ then follows from Eq. (A4.2.23) when $N_t = 0$. When $p_1 = 0.5$, $S_1 = \frac{\delta - 1}{\delta + 1}$, which turns out to be the maximum spread since

$$\frac{\partial S_t}{\partial N_t} = \frac{(\delta - \delta^{-1}) \cdot \delta^{N_t} \cdot (\ln \delta) \cdot (1 - \delta^{2 \cdot N_t})}{\left((\delta^{N_t} + \delta) \cdot (\delta^{N_t} + \delta^{-1})\right)^2} = 0, \tag{A4.2.24}$$

and $\frac{\partial^2 S_t}{\partial N_t^2} < 0$ when $N_t = 0$.

(iv) follows from Eq. (A4.2.24) that $\frac{\partial S_t}{\partial N_t} < 0$ if $N_t > 0$ and $\frac{\partial S_t}{\partial N_t} > 0$ if $N_t < 0$, which shows that the spread narrows with order imbalance. Finally, the spread

converges to zero as order imbalance goes to infinity following

$$\lim_{N_t \to +\infty} \frac{\delta^{N_t} \cdot (\delta - \delta^{-1})}{(\delta^{N_t} + \delta) \cdot (\delta^{N_t} + \delta^{-1})} = \lim_{N_t \to +\infty} \frac{(\delta - \delta^{-1})}{(\delta^{N_t} + \delta^{-N_t} + \delta + \delta^{-1})} = 0, \quad \text{(A4.2.25)}$$

and

$$\lim_{N_t \to -\infty} \frac{\delta^{N_t} \cdot (\delta - \delta^{-1})}{(\delta^{N_t} + \delta) \cdot (\delta^{N_t} + \delta^{-1})} = \lim_{N_t \to -\infty} \frac{\delta^{N_t} \cdot (\delta - \delta^{-1})}{(\delta^{2 \cdot N_t} + \delta^{N_t-1} + \delta^{N_t+1} + 1)} = 0,$$
$$\text{(A4.2.26)}$$

since $\delta > 1$

$\blacksquare$

Proof of Proposition 4.5. In the presence of composition uncertainty, the probability of an order D_t in each state is given by

$$\Pr(D_t|s_1) = \frac{1 + \alpha_H \cdot D_t}{2}, \qquad \Pr(D_t|s_2) = \frac{1 - \alpha_H \cdot D_t}{2},$$
$$\Pr(D_t|s_3) = \frac{1 + \alpha_L \cdot D_t}{2}, \qquad \Pr(D_t|s_4) = \frac{1 - \alpha_L \cdot D_t}{2}, \qquad \text{(A4.2.27)}$$

and the probability of an order D_t conditional on the trading history h_t is given by

$$\Pr(D_t|h_t) = \sum_{s_i \in \mathcal{S}} \Pr(D_t|s_i) \cdot \Pr(s_i|h_t), \qquad i = 1, 2, 3, 4. \qquad \text{(A4.2.28)}$$

The bid and ask quotes respectively follow as

$$\begin{aligned}
B_{\alpha,t} &= E[\hat{V} = 1|h_t, D_t = -1] = \Pr(\hat{V} = 1|h_t, D_t = -1) \\
&= \Pr\{s_1|h_t, D_t = -1\} + \Pr\{s_3|h_t, D_t = -1\} \\
&= \frac{\Pr(D_t = -1|s_1, h_t)}{\Pr(D_t = -1|h_t)} \cdot \Pr(s_1|h_t) + \frac{\Pr(D_t = -1|s_3, h_t)}{\Pr(D_t = -1|h_t)} \cdot \Pr(s_3|h_t),
\end{aligned}$$
$$\text{(A4.2.29)}$$

$$\begin{aligned}
A_{\alpha,t} &= E[\hat{V} = 1|h_t, D_t = 1] = \Pr(\hat{V} = 1|h_t, D_t = 1) \\
&= \Pr\{s_1|h_t, D_t = 1\} + \Pr\{s_3|h_t, D_t = 1\} \\
&= \frac{\Pr(D_t = 1|s_1, h_t)}{\Pr(D_t = 1|h_t)} \cdot \Pr(s_1|h_t) + \frac{\Pr(D_t = 1|s_3, h_t)}{\Pr(D_t = 1|h_t)} \cdot \Pr(s_3|h_t).
\end{aligned}$$
$$\text{(A4.2.30)}$$

Substituting Eqs. (A4.2.27) and (A4.2.28) into Eqs. (A4.2.29) and (A4.2.30), defining

$$\delta_t^b = \left(\frac{(1 + \alpha_H) \cdot \Pr(s_1|h_t) + (1 + \alpha_L) \cdot \Pr(s_3|h_t)}{(1 - \alpha_H) \cdot \Pr(s_2|h_t) + (1 - \alpha_l) \cdot \Pr(s_4|h_t)} \right) \cdot \left(\frac{1 - p_t}{p_t} \right) > 1, \qquad \text{(A4.2.31)}$$

$$\delta_t^s = \left(\frac{(1 + \alpha_H) \cdot \Pr(s_2|h_t) + (1 + \alpha_L) \cdot \Pr(s_4|h_t)}{(1 - \alpha_H) \cdot \Pr(s_1|h_t) + (1 - \alpha_l) \cdot \Pr(s_3|h_t)} \right) \cdot \left(\frac{p_t}{1 - p_t} \right) > 1, \qquad \text{(A4.2.32)}$$

for $0 < \alpha_L < \alpha_H < 1$ and rearranging yields

$$B_{\alpha,t} = \frac{p_t}{p_t + \delta_t^s \cdot (1 - p_t)}, \qquad \text{(A4.2.33)}$$

$$A_{\alpha,t} = \frac{p_t}{p_t + (\delta_t^b)^{-1} \cdot (1 - p_t)}, \qquad \text{(A4.2.34)}$$

$$S_{\alpha,t} = \frac{p_t \cdot (1 - p_t) \cdot (\delta_t^s - (\delta_t^b)^{-1})}{\left(p_t + \delta_t^s \cdot (1 - p_t) \right) \cdot \left(p_t + (\delta_t^b)^{-1} \cdot (1 - p_t) \right)}. \qquad \text{(A4.2.35)}$$

Finally, partial derivatives of δ_t^b and δ_t^s w.r.t. α_L and α_H (i.e., $\frac{\partial \delta_t^b}{\partial \alpha_L} > 0$, $\frac{\partial \delta_t^b}{\partial \alpha_H} > 0$, $\frac{\partial \delta_t^s}{\partial \alpha_L} > 0$ and $\frac{\partial \delta_t^s}{\partial \alpha_L} > 0$) complete the proof.

∎

Proof of Lemma 4.6. Rearranging Eqs. (A4.2.33) and (A4.2.34) obtains

$$\frac{B_{\alpha,t}}{1 - B_{\alpha,t}} = (\delta_t^s)^{-1} \cdot \frac{p_t}{1 - p_t}, \qquad \text{(A4.2.36)}$$

$$\frac{A_{\alpha,t}}{1 - A_{\alpha,t}} = \delta_t^b \cdot \frac{p_t}{1 - p_t}. \qquad \text{(A4.2.37)}$$

Analogous to the benchmark model, in the presence of composition uncertainty, the current belief about the payoff is the last transaction price leading to

$$\frac{p_{t+1}}{1 - p_{t+1}} = \frac{p_t}{1 - p_t} \cdot (\delta_t^b), \quad \text{if} \quad D_t = +1, \qquad \text{(A4.2.38)}$$

$$\frac{p_{t+1}}{1 - p_{t+1}} = \frac{p_t}{1 - p_t} \cdot (\delta_t^s)^{-1}, \quad \text{if} \quad D_t = -1. \qquad \text{(A4.2.39)}$$

∎

Proof of Proposition 4.7. Given the trading history h_t with b_t buy and s_t number of sell orders at time t, the market maker's belief about the payoff p_t and the proportion of high informed trading π_t respectively follow from Bayes theorem as

$$p_t = \Pr(s_1|h_t) + \Pr(s_3|h_t)$$

$$= \left(p_1 \cdot \pi_1 \cdot \left(1 + \alpha_H\right)^{b_t} \cdot \left(1 - \alpha_H\right)^{s_t} + p_1 \cdot (1 - \pi_1) \cdot \left(1 + \alpha_L\right)^{b_t} \cdot \left(1 - \alpha_L\right)^{s_t} \right) \cdot$$

$$\left(p_1 \cdot \pi_1 \cdot (1 + \alpha_H)^{b_t} \cdot (1 - \alpha_H)^{s_t} + (1 - p_1) \cdot \pi_1 \cdot (1 - \alpha_H)^{b_t} \cdot (1 + \alpha_H)^{s_t} \right.$$

$$\left. + p_1 \cdot (1 - \pi_1) \cdot (1 + \alpha_L)^{b_t} \cdot (1 - \alpha_L)^{s_t} + (1 - p_1) \cdot (1 - \pi_1) \cdot (1 - \alpha_L)^{b_t} \cdot (1 + \alpha_L)^{s_t} \right)^{-1},$$

$$(A4.2.40)$$

$$\pi_t = \Pr(s_1|h_t) + \Pr(s_2|h_t)$$

$$= \left(p_1 \cdot \pi_1 \cdot \left(1 + \alpha_H\right)^{b_t} \cdot \left(1 - \alpha_H\right)^{s_t} + (1 - p_1) \cdot \pi_1 \cdot \left(1 - \alpha_H\right)^{b_t} \cdot \left(1 + \alpha_H\right)^{s_t} \right) \cdot$$

$$\left(p_1 \cdot \pi_1 \cdot (1 + \alpha_H)^{b_t} \cdot (1 - \alpha_H)^{s_t} + (1 - p_1) \cdot \pi_1 \cdot (1 - \alpha_H)^{b_t} \cdot (1 + \alpha_H)^{s_t} \right.$$

$$\left. + p_1 \cdot (1 - \pi_1) \cdot (1 + \alpha_L)^{b_t} \cdot (1 - \alpha_L)^{s_t} + (1 - p_1) \cdot (1 - \pi_1) \cdot (1 - \alpha_L)^{b_t} \cdot (1 + \alpha_L)^{s_t} \right)^{-1},$$

$$(A4.2.41)$$

where $b_t = \frac{(t-1)+N_t}{2}$ and $s_t = \frac{(t-1)-N_t}{2}$.

(i) Substituting $b_t = s_t$ into Eq. (A4.2.40) obtains $p_t = p_1$ and taking partial derivative of Eq. (A4.2.40) w.r.t. N_t after inserting $b_t = \frac{(t-1)+N_t}{2}$ and $s_t = \frac{(t-1)-N_t}{2}$ obtains $\frac{\partial p_t}{N_t} > 0$.

(ii) Similarly, inserting $b_t = \frac{(t-1)+N_t}{2}$ and $s_t = \frac{(t-1)-N_t}{2}$ into Eq. (A4.2.41) and taking partial derivative w.r.t. N_t obtains $\frac{\partial \pi_t}{N_t} > 0$ when $N_t > 0$ and $\frac{\partial \pi_t}{N_t} < 0$ when $N_t < 0$, leading to $\frac{\partial \pi_t}{|N_t|} > 0$.

∎

Proof of Corollary 4.8. (i) Substituting $b_t = s_t$ into Eq. (A4.2.41) obtains

$$\pi_t = \left(\frac{\pi_1 \cdot (1 - \alpha_H)^{b_t} \cdot (1 + \alpha_H)^{b_t}}{\pi_1 \cdot (1 - \alpha_H)^{b_t} \cdot (1 + \alpha_H)^{b_t} + (1 - \pi_1) \cdot (1 - \alpha_L)^{b_t} \cdot (1 + \alpha_L)^{b_t}} \right) \cdot \pi_1.$$

$$(A4.2.42)$$

It follows from $(1 - \alpha_H)^{b_t} \cdot (1 + \alpha_H)^{b_t} < (1 - \alpha_L)^{b_t} \cdot (1 + \alpha_L)^{b_t}$ that the first term in Eq. (A4.2.42) is less than 1, leading to $\pi_t < \pi_1$.

(*ii*) Substituting $b_t = t - 1$ and $s_t = 0$ into Eq. (A4.2.41) obtains

$$
\begin{aligned}
\pi_t = \pi_1 \cdot \Big(&p_1 \cdot (1 + \alpha_H)^{t-1} + (1 - p_1) \cdot (1 - \alpha_H)^{t-1} \Big) \cdot \\
&\Big(\pi_1 \cdot \big(p_1 \cdot (1 + \alpha_H)^{t-1} + (1 - p_1) \cdot (1 - \alpha_H)^{t-1} \big) \\
&+ (1 - \pi_1) \cdot \big(p_1 \cdot (1 + \alpha_L)^{t-1} + (1 - p_1) \cdot (1 - \alpha_L)^{t-1} \big) \Big)^{-1}.
\end{aligned}
\tag{A4.2.43}
$$

Then $\pi_t > \pi_1$ follows from the fact that $p_1 \cdot (1 + \alpha)^{t-1} + (1 - p_1) \cdot (1 - \alpha)^{t-1}$ is increasing in α if $p_1 \geq 0.5$. The proof of the sequence of sell orders follows similarly.

$\blacksquare$

Proof of Corollary 4.9. (i) For the sequence of buy orders, iterating from the first buy order at $t = 1$, Eq. (A4.2.38) leads to

$$
\frac{p_t}{1 - p_t} = \left(\frac{p_1}{1 - p_1} \right) \cdot (\delta_1^b)^{D_1} \cdot \ldots \cdot (\delta_{t-1}^b)^{D_{t-1}} = \left(\frac{p_1}{1 - p_1} \right) \cdot \prod_{\tau=1}^{\tau=t-1} \delta_\tau^b
\tag{A4.2.44}
$$

Substituting $p_1 = 0.5$ into Eq. (A4.2.44) and solving for p_t obtains

$$
p_t = \frac{\prod_{i=1}^{t-1} \delta_i^b}{1 + \prod_{i=1}^{t-1} \delta_i^b}.
\tag{A4.2.45}
$$

Let $\bar{\delta}_t^b$ denote the geometric mean of the informativeness of the buy sequence up to time t. It follows from $N_t = t - 1$ that for the sequence of buy orders,

$$
\prod_{\tau=1}^{\tau=t-1} \delta_\tau^b = \left(\left(\prod_{\tau=1}^{\tau=t-1} \delta_\tau^b \right)^{\frac{1}{t-1}} \right)^{N_t} = (\bar{\delta}_t^b)^{N_t},
\tag{A4.2.46}
$$

leading to

$$
p_t = \frac{(\bar{\delta}_t^b)^{N_t}}{1 + (\bar{\delta}_t^b)^{N_t}},
\tag{A4.2.47}
$$

similar to the benchmark model. (ii) The proof of the sell sequence follows similarly.

$\blacksquare$

Proof of Proposition 4.10. (i) Given that initially it is equally likely that $\hat{V}$ is high or low (i.e., $p_1 = 0.5$), S_1 is given by

$$S_1 = \frac{\delta_1 - 1}{\delta_1 + 1}, \tag{A4.2.48}$$

where

$$\delta_1 = \delta_1^s = \delta_1^b = \frac{1 + (\pi_1 \cdot \alpha_H + (1 - \pi_1) \cdot \alpha_L)}{1 - (\pi_1 \cdot \alpha_H + (1 - \pi_1) \cdot \alpha_L)} \tag{A4.2.49}$$

is the initial informativeness of orders. Combining the spread in the presence of composition uncertainty at time t in Eq. (A4.2.35) and the initial spread in Eq. (A4.2.48) obtains

$$S_{\alpha,t} = S_1 +$$
$$\frac{p_t \cdot (1 - p_t) \cdot (\delta_t^s - (\delta_t^b)^{-1}) \cdot (\delta_1 + 1) - (p_t + \delta_t^s \cdot (1 - p_t)) \cdot (p_t + (\delta_t^b)^{-1} \cdot (1 - p_t)) \cdot (\delta_1 - 1)}{(p_t + \delta_t^s \cdot (1 - p_t)) \cdot (p_t + (\delta_t^b)^{-1} \cdot (1 - p_t)) \cdot (\delta_1 + 1)}, \tag{A4.2.50}$$

where the second term in Eq. (A4.2.50) shows the net liquidity distortion ΔS_t relative to the initial spread and is stabilizing when $\Delta S_t < 0$ and destabilizing when $\Delta S_t > 0$.

(ii) The market maker's perceived informativeness of orders following balanced order flow (i.e., $b_t = s_t$) follows from inserting the conditional probabilities of states into Eqs. (A4.2.31) and (A4.2.32) as

$$\delta_t^s = \delta_t^b = \frac{\pi_1 \cdot (1 + \alpha_H)^{s_t+1} \cdot (1 - \alpha_H)^{s_t} + (1 - \pi_t) \cdot (1 + \alpha_L)^{s_t+1} \cdot (1 - \alpha_L)^{s_t}}{\pi_1 \cdot (1 + \alpha_H)^{s_t} \cdot (1 - \alpha_H)^{s_t+1} + (1 - \pi_1) \cdot (1 + \alpha_L)^{s_t} \cdot (1 - \alpha_L)^{s_t+1}}. \tag{A4.2.51}$$

Since $p_t = p_1$ for balanced order flow, ΔS_t in Eq. (A4.2.50) reduces to

$$\Delta S_t = \frac{(\delta_t^s - (\delta_t^s)^{-1}) \cdot (\delta_1 + 1) - (1 + \delta_t^s) \cdot (1 + (\delta_t^s)^{-1}) \cdot (\delta_1 - 1)}{(1 + \delta_t^s) \cdot (1 + (\delta_t^s)^{-1}) \cdot (\delta_1 + 1)}, \tag{A4.2.52}$$

which takes a negative value if and only if $\delta_t^s < \delta_1$. Since δ_t^s given in Eq. (A4.2.51) is decreasing in s_t, $\frac{\partial \delta_t^s}{\partial s_t} < 0$, it follows that $\delta_t^s < \delta_1$ is always satisfied, meaning that balanced order flows always stabilize the market.

(iii) For the sequence of sell orders, substituting

$$p_t = \frac{\prod_{i=1}^{t-1}(\delta_i^s)^{-1}}{1 + \prod_{i=1}^{t-1}(\delta_i^s)^{-1}}. \tag{A4.2.53}$$

into the net liquidity distortion ΔS_t in Eq. (A4.2.50) obtains

$$\Delta S_t =$$
$$\frac{\prod_{i=1}^{t-1}(\delta_i^s)^{-1} \cdot \left(\delta_t^s - (\delta_t^b)^{-1}\right) \cdot (\delta_1 + 1) - \left(\prod_{i=1}^{t-1}(\delta_i^s)^{-1} + \delta_t^s\right) \cdot \left(\prod_{i=1}^{t-1}(\delta_i^s)^{-1} + (\delta_t^b)^{-1}\right) \cdot (\delta_1 - 1)}{\left(\prod_{i=1}^{t-1}(\delta_i^s)^{-1} + \delta_t^s\right) \cdot \left(\prod_{i=1}^{t-1}(\delta_i^s)^{-1} + (\delta_t^b)^{-1}\right) \cdot (\delta_1 + 1)},$$

$$(A4.2.54)$$

which takes a positive value if and only if

$$\delta_1 < \frac{2 \cdot \delta_t^s + \prod_{i=1}^{t-1}(\delta_i^s)^{-1} + \prod_{i=1}^{t-1}\delta_i^s \cdot \delta_t^s \cdot (\delta_t^b)^{-1}}{2 \cdot (\delta_t^b)^{-1} + \prod_{i=1}^{t-1}(\delta_i^s)^{-1} + \prod_{i=1}^{t-1}\delta_i^s \cdot \delta_t^s \cdot (\delta_t^b)^{-1}}. \qquad (A4.2.55)$$

(iv) The proof for the sequence of buy orders follows similarly.

■

Proof of Lemma 4.11. Inserting the independence conditions of the myopic market maker (i.e., $\Pr(s_1|h_t) = \pi_t^m \cdot p_t^m$, $\Pr(s_2|h_t) = \pi_t^m \cdot (1 - p_t^m)$, $\Pr(s_3|h_t) = (1 - \pi_t^m) \cdot p_t^m$ and $\Pr(s_4|h_t) = (1 - \pi_t^m) \cdot (1 - p_t^m)$) into δ_t^b and δ_t^s in Eqs. (A4.2.31) and (A4.2.32) obtains

$$\delta_t^m = \frac{1 + (\pi_t^m \cdot \alpha_H + (1 - \pi_t^m) \cdot \alpha_L)}{1 - (\pi_t^m \cdot \alpha_H + (1 - \pi_t^m) \cdot \alpha_L)}, \qquad (A4.2.56)$$

which yields

$$B_{\alpha,t}^m = \frac{p_t^m}{p_t^m + \delta_t^m \cdot (1 - p_t^m)}, \qquad (A4.2.57)$$

and

$$A_{\alpha,t}^m = \frac{p_t^m}{p_t^m + (\delta_t^m)^{-1} \cdot (1 - p_t^m)}. \qquad (A4.2.58)$$

Therefore, as in the benchmark model

$$\frac{p_{t+1}^m}{1 - p_{t+1}^m} = \frac{p_t^m}{1 - p_t^m} \cdot (\delta_t^m)^{D_t}. \qquad (A4.2.59)$$

Iterating from the first trade $t = 1$ up to time t obtains

$$\frac{p_t^m}{1 - p_t^m} = \left(\frac{p_1}{1 - p_1}\right) \cdot (\delta_1^m)^{D_1} \cdot \ldots \cdot (\delta_{t-1}^m)^{D_{t-1}} = \left(\frac{p_1}{1 - p_1}\right) \cdot \prod_{\tau=1}^{t-1}(\delta_\tau^m)^{D_\tau} = \prod_{\tau=1}^{t-1}(\delta_\tau^m)^{D_\tau},$$

$$(A4.2.60)$$

leading to

$$p_t^m = \frac{\prod_{\tau=1}^{t-1}(\delta_\tau^m)^{D_\tau}}{1 + \prod_{\tau=1}^{t-1}(\delta_\tau^m)^{D_\tau}}. \qquad (A4.2.61)$$

We now show that $\prod_{\tau=1}^{t-1}(\delta_\tau^m)^{D_\tau}$ is given by $\bar\delta_t^{\bar N_t}$. Let

$$\ln\bar\delta_t^m = \frac{1}{t-1}\cdot\sum_{i=1}^{t-1}\ln\delta_i^m = \frac{1}{t-1}\cdot\ln\prod_{i=1}^{t-1}\delta_i^m = \ln\left(\prod_{i=1}^{t-1}\delta_i^m\right)^{\frac{1}{t-1}}, \qquad \text{(A4.2.62)}$$

leading to $\bar\delta_t^m = \left(\prod_{i=1}^{t-1}\delta_i^m\right)^{\frac{1}{t-1}}$. Taking the log of $\prod_{\tau=1}^{t-1}(\delta_\tau^m)^{D_\tau}$, and multiplying and dividing with $\sum_{i=1}^{t-1}\ln\delta_i^m$ obtains

$$\ln\left(\prod_{\tau=1}^{t-1}(\delta_\tau^m)^{D_\tau}\right) = \sum_{\tau=1}^{t-1}D_\tau\cdot\ln\delta_\tau^m = \sum_{i=1}^{t-1}\ln\delta_i^m\cdot\sum_{\tau=1}^{t-1}D_\tau\cdot\frac{\ln\delta_\tau^m}{\sum_{i=1}^{t-1}\ln\delta_i^m}$$

$$= \ln\bar\delta_t^m\cdot\sum_{\tau=1}^{t-1}D_\tau\cdot\frac{(t-1)\cdot\ln\delta_\tau^m}{\sum_{i=1}^{t-1}\ln\delta_i^m} = \ln\bar\delta_t^m\cdot\sum_{\tau=1}^{t-1}D_\tau\cdot w_\tau = \ln\bar\delta_t^m\cdot\bar N_t,$$

$$\text{(A4.2.63)}$$

where $w_\tau = \frac{(t-1)\cdot\ln\delta_\tau^m}{\sum_{i=1}^{t-1}\ln\delta_i^m}$ and $\bar N_t = \sum_{\tau=1}^{t-1}D_\tau\cdot w_\tau$. Hence, $\prod_{\tau=1}^{\tau=t-1}(\delta_\tau^m)^{D_\tau} = \bar\delta_t^{\bar N_t}$.

Lastly, the myopic market maker's learning about the proportion of informed traders after a buy and a sell follows from Bayes' theorem respectively as

$$\pi_{t+1}^m = \Pr\{\hat\alpha = \alpha_H|h_t, D_t = +1\} = \frac{\Pr\{D_t = +1|h_t,\hat\alpha = \alpha_H\}\cdot\Pr\{\hat\alpha = \alpha_H|h_t\}}{\Pr\{D_t = +1|h_t\}}$$

$$= \frac{1 + \alpha_H\cdot(2\cdot p_t^m - 1)\cdot(2\cdot q - 1)}{1 + \left(\pi_t^m\cdot\alpha_H + (1 - \pi_t^m)\cdot\alpha_L\right)\cdot(2\cdot p_t^m - 1)\cdot(2\cdot q - 1)}\cdot\pi_t^m,$$

$$\text{(A4.2.64)}$$

$$\pi_{t+1}^m = \Pr\{\hat\alpha = \alpha_H|h_t, D_t = -1\} = \frac{\Pr\{D_t = -1|h_t,\hat\alpha = \alpha_H\}\cdot\Pr\{\hat\alpha = \alpha_H|h_t\}}{\Pr\{D_t = -1|h_t\}}$$

$$= \frac{1 + \alpha_H\cdot(1 - 2\cdot p_t^m)\cdot(2\cdot q - 1)}{1 + \left(\pi_t^m\cdot\alpha_H + (1 - \pi_t^m)\cdot\alpha_L\right)\cdot(1 - 2\cdot p_t^m)\cdot(2\cdot q - 1)}\cdot\pi_t^m,$$

$$\text{(A4.2.65)}$$

which combined leads to Eq. (4.47).

∎

Appendix 4.3. Myopic Market Maker

In this Appendix, we show the quotes and learning of the myopic market maker who learns about $\tilde{V}$ and $\tilde{\alpha}$ independently. Although the results of the baseline model in Section 4.4 is more general, assuming independence allows us to illustrate the learning of the market maker more clearly and provide economic intuition of the toxicity in order flow. The next proposition characterizes the quotes and spread of the myopic market maker (for brevity we drop the m superscript for the myopic market maker in this Appendix).

Proposition A4.3.1. The equilibrium bid and ask prices in the presence of composition uncertainty are respectively given by

$$B_{\alpha,t} = \frac{p_t}{p_t + \delta_t \cdot (1 - p_t)}, \tag{A4.3.1}$$

$$A_{\alpha,t} = \frac{p_t}{p_t + \delta_t^{-1} \cdot (1 - p_t)}, \tag{A4.3.2}$$

and the bid-ask spread takes the form of

$$S_{\alpha,t} = \frac{p_t \cdot (1 - p_t) \cdot (\delta_t - \delta_t^{-1})}{\left(p_t + \delta_t \cdot (1 - p_t)\right) \cdot \left(p_t + \delta_t^{-1} \cdot (1 - p_t)\right)}, \tag{A4.3.3}$$

where

$$\delta_t = \frac{1 + \left(\pi_t \cdot \alpha_H + (1 - \pi_t) \cdot \alpha_L\right) \cdot \left(2 \cdot q - 1\right)}{1 - \left(\pi_t \cdot \alpha_H + (1 - \pi_t) \cdot \alpha_L\right) \cdot \left(2 \cdot q - 1\right)}, \tag{A4.3.4}$$

is always greater than unity and increases with the belief about the high proportion of informed traders π_t.

Proof. The proof is immediate after substituting the independence conditions, i.e.,

$$\begin{aligned}
\Pr(s_1|h_t) &= \pi_t \cdot p_t, & \Pr(s_2|h_t) &= \pi_t \cdot (1 - p_t) \\
\Pr(s_3|h_t) &= (1 - \pi_t) \cdot p_t & \Pr(s_4|h_t) &= (1 - \pi_t) \cdot (1 - p_t)
\end{aligned} \tag{A4.3.5}$$

into Eqs. (4.18) and (4.19). Alternatively, the following expressions follow from Bayes' theorem:

$$\Pr\{D_t = -1|\hat{V} = 1, h_t\} = \pi_t \cdot \frac{1 - \alpha_H \cdot (2 \cdot q - 1)}{2} + (1 - \pi_t) \cdot \frac{1 - \alpha_L \cdot (2 \cdot q - 1)}{2}; \tag{A4.3.6}$$

$$\Pr\{D_t = -1|h_t\} = \pi_t \cdot \frac{1 + \alpha_H \cdot (1 - 2 \cdot p_t) \cdot (2 \cdot q - 1)}{2} + (1 - \pi_t) \cdot \frac{1 + \alpha_L \cdot (1 - 2 \cdot p_t) \cdot (2 \cdot q - 1)}{2}. \tag{A4.3.7}$$

From the zero-expected-profit and Bayesian conditions, the bid price (ask price follows similarly) follows as

$$B_t = E[\hat{V} = 1 | h_t, D_t = -1] = \Pr\{\hat{V} = 1 | h_t\} \cdot \frac{\Pr\{D_t = -1 | \hat{V} = 1, h_t\}}{\Pr\{D_t = -1 | h_t\}}$$

$$= p_t \cdot \frac{\left(\pi_t \cdot \frac{1 - \alpha_H \cdot (2 \cdot q - 1)}{2} + (1 - \pi_t) \cdot \frac{1 - \alpha_L \cdot (2 \cdot q - 1)}{2} \right)}{\left(\pi_t \cdot \frac{1 + \alpha_H \cdot (1 - 2 \cdot p_t) \cdot (2 \cdot q - 1)}{2} + (1 - \pi_t) \cdot \frac{1 + \alpha_L \cdot (1 - 2 \cdot p_t) \cdot (2 \cdot q - 1)}{2} \right)}$$

$$= p_t \cdot \frac{\left(1 - \left(\pi_t \cdot \alpha_H + (1 - \pi_t) \cdot \alpha_L \right) \cdot (2 \cdot q - 1) \right)}{\left(1 + \left(\pi_t \cdot \alpha_H + (1 - \pi_t) \cdot \alpha_L \right) \cdot (1 - 2 \cdot p_t) \cdot (2 \cdot q - 1) \right)}$$

$$= \frac{p_t}{p_t + \delta_t \cdot (1 - p_t)}, \tag{A4.3.8}$$

where

$$\delta_t = \frac{1 + \left(\pi_t \cdot \alpha_H + (1 - \pi_t) \cdot \alpha_L \right) \cdot (2 \cdot q - 1)}{1 - \left(\pi_t \cdot \alpha_H + (1 - \pi_t) \cdot \alpha_L \right) \cdot (2 \cdot q - 1)} > 1 \tag{A4.3.9}$$

for $0 < \alpha_l < \alpha_h < 1$ and $q \in (\frac{1}{2}, 1]$, and increases with π_t following,

$$\frac{\partial \delta_t}{\partial \pi_t} = \frac{2 \cdot (\alpha_H - \alpha_L) \cdot (2 \cdot q - 1)}{\left(1 - \left(\pi_t \cdot \alpha_H + (1 - \pi_t) \cdot \alpha_L \right) \cdot (2 \cdot q - 1) \right)^2} > 0. \tag{A4.3.10}$$

∎

The difference of Proposition A4.3.1 from Proposition 4.5 in the baseline model is that the myopic market maker learns about p_t and π_t as if nature has independently chosen $\hat{\alpha}$ and $\hat{V}$. However, Proposition A4.3.1 demonstrates that most of the intuitions of the baseline model carry forward (except the repricing history effect since the informativeness of buy and sell orders are symmetric, $\delta_t^b = \delta_t^s = \delta_t$). Similarly, Lemma 4.11 in Section 4.4 shows that when the market maker learns about the uncertainties about the composition of traders and fundamental value independently, the geometric mean of the informativeness of trades up to time t, $\bar{\delta}_t$, and the weighted order imbalance, $\bar{N}_t$, have similar roles as the fixed informativeness of trades, δ, and the order imbalance, N_t, in the benchmark model in determining the market maker's belief about the fundamental value. The weighted order imbalance, $\bar{N}_t$, is the sum of the weighted average of individual orders. The weighting function of the individual orders, w_τ, increases with the informativeness of the trades. Therefore, a natural interpretation of the weighting function is the market maker's belief about the toxicity of the order. An order with less informativeness is weighted less, whereas an order with high informativeness is weighted

more in counting the order imbalance $\bar{N}_t$ of the market maker. Put differently, the market maker treats a toxic unit order as more than a unit order and a non-toxic unit order as less than a unit order, yet she only receives a unit order. The following proposition follows from Lemma 4.11.

Proposition A4.3.2. In the presence of the composition uncertainty;

(i) If there is a sell order at time t, the probability of the high informed trading, and therefore, the informativeness of orders rise when $p_t < \frac{1}{2}$ and fall when $p_t > \frac{1}{2}$ (i.e., $\pi_{t+1} > \pi_t$ and $\delta_{t+1} > \delta_t$ if $p_t < \frac{1}{2}$ and $\pi_{t+1} < \pi_t$ and $\delta_{t+1} < \delta_t$ if $p_t > \frac{1}{2}$). Moreover, the risky security payoff is revised downward with a sell order (i.e., $p_{t+1} < p_t$).

(ii) If there is a buy order at time t, the probability of the high informed trading, and therefore, the informativeness of orders rise when $p_t > \frac{1}{2}$ and fall when $p_t < \frac{1}{2}$ (i.e., $\pi_{t+1} > \pi_t$ and $\delta_{t+1} > \delta_t$ if $p_t > \frac{1}{2}$, and $\pi_{t+1} < \pi_t$ and $\delta_{t+1} < \delta_t$ if $p_t < \frac{1}{2}$). Moreover, the risky security payoff is revised upward with a buy order (i.e., $p_{t+1} > p_t$).

(iii) The probability of the high informed trading and the informativeness of orders are unchanged when $p_t = \frac{1}{2}$ (i.e., $\pi_{t+1} = \pi_t$ and $\delta_{t+1} = \delta_t$ if $p_t = \frac{1}{2}$).

Proof. We only prove after a sell order. By Bayes' theorem, the market maker's belief about the high informed trading after a sell order is given by

$$\pi_{t+1} = \Pr\{\hat{\alpha} = \alpha_h | h_t, D_t = -1\} = \frac{\Pr\{D_t = -1 | h_t, \hat{\alpha} = \alpha_h\} \cdot \Pr\{\hat{\alpha} = \alpha_h | h_t\}}{\Pr\{D_t = -1 | h_t\}}$$

$$= \frac{1 + \alpha_h \cdot (1 - 2 \cdot p_t) \cdot (2 \cdot q - 1)}{1 + \left(\pi_t \cdot \alpha_h + (1 - \pi_t) \cdot \alpha_l\right) \cdot (1 - 2 \cdot p_t) \cdot (2 \cdot q - 1)} \cdot \pi_t,$$

$$(A4.3.11)$$

which is greater than π_t when $p_t < \frac{1}{2}$ and less than π_t when $p_t > \frac{1}{2}$. $\pi_{t+1} > \pi_t$ leads to $\delta_{t+1} > \delta_t$ and $\pi_{t+1} < \pi_t$ leads to $\delta_{t+1} < \delta_t$ since $\frac{\partial \delta_t}{\partial \pi_t} > 0$ following Eq. (A4.3.10). Moreover, it follows from Eq. (4.48) that $p_{t+1} < p_t$ when $D_t = -1$. The proof after a buy order follows similarly.

■

Proposition A4.3.2 demonstrates how the market maker revises her beliefs after each trade and is consistent with the learning rule of the market maker presented in Proposition 4.7 and Corollary 4.8. Now that we have characterized the quotes and the independent learning rule of the market maker, we can illustrate the destabilizing role of order imbalance by considering a scenario similar to flash crashes with a sudden selling pressure. Figure 4.7 illustrates the results.
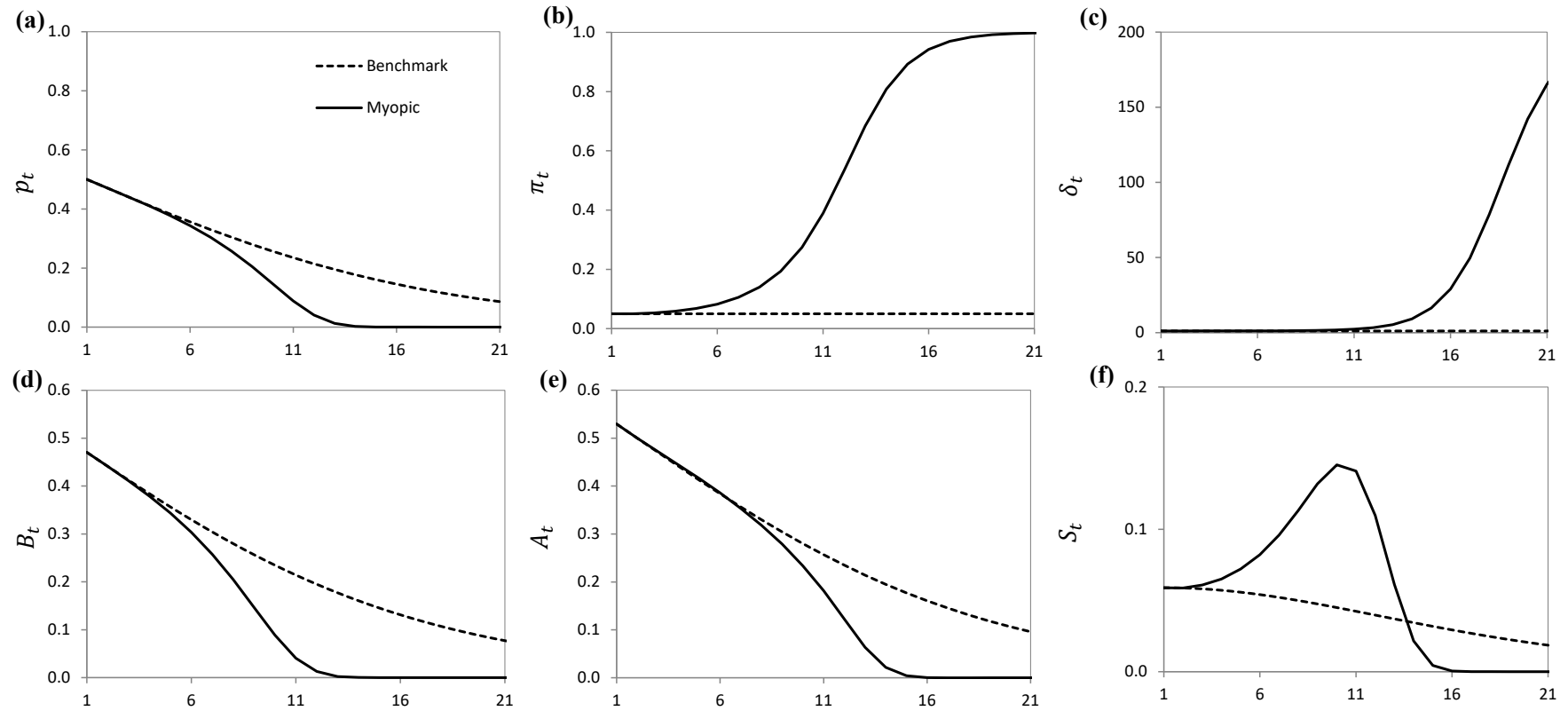
Figure 4.7: The dynamics of beliefs, quotes and bid-ask spread of the myopic market maker during the sequences of sell orders.

Panel (a) plots the belief about the payoff, p_t, (b) plots the belief about the high informed trading, π_t, (c) plots informativeness of orders, δ_t, (d) plots bid, B_t, (e) plots ask, A_t, and (f) plots bid-ask spread S_t of the myopic and benchmark market makers in the face of 20 consecutive sell orders (i.e., $N_t = -20$ at $t = 21$). The parameter values are $\alpha_H = 0.99$, $\alpha_L = 0.01$, $q = 1$, $p_1 = 0.5$, and $\pi_1 = 0.05$.

Chapter 5

Toward A General Model of Financial Markets

Is our expectation of rain, when we start out for a walk, always more likely than not, or less likely than not, or as likely as not? I am prepared to argue that on some occasions none of these alternatives hold, and that it will be an arbitrary matter to decide for or against the umbrella. If the barometer is high, but the clouds are black, it is not always rational that one should prevail over the other in our minds, or even that we should balance them, though it will be rational to allow caprice to determine us and to waste no time on the debate.

<div align="right">Keynes (1921) "A Treatise on Probability" [p. 31].</div>

5.1   Introduction

Thus far we have modeled financial markets when market participants face uncertainty other than the fundamental values of assets such as uncertainty in the beliefs about the payoffs (Chapter 2), the composition of traders (Chapters 3 and 4), and the quality of informed traders' information (Chapter 4). While these models provide useful characterizations of different dimensions of uncertainty in financial markets, they require some structure to obtain closed-form solutions. In this chapter, by using the information science and decision theories literature, we discuss *"the complexity of the real-world information and the implications for financial markets"* and raise some interesting questions.

The argument we put forth to carry the discussion takes its root from "fact" vs. "opinion". In an environment rife with heterogeneous opinions, behavioral biases naturally arise when agents deal with imprecise and partially true information. We argue that information becomes open to interpretation and leads to behavioral biases when it is not a precise fact, but rather imprecise and partially reliable. Most of the existing financial models become too simplistic to account for the real world with partially reliable and imprecise information. As the information becomes imprecise and unreliable the environment becomes too complex to be explained by the standard techniques. This chapter illustrates "*how complex the financial decision making can get?*".

In this chapter, we discuss information in the broadest possible way that lends itself to possible quantitative scrutiny. Specifically, we use Zadeh (2011) classification of information - numerical, interval-valued, second-order uncertain, fuzzy and Z information - based on its generality. We argue such that individuals are subjectively rational if they apply *"correct"* decision technique to each class of information separately rather than defining rationality based on only one decision technique such as the standard Savage's axioms of subjective expected utility for all classes of information. We present a general approximation for subjective rationality in decision making and suggest a general framework. We argue that efficient markets hypothesis (EMH) and behavioral finance (BF) become special cases of this framework with the imprecision and reliability of information approximately connecting them. That is, imprecise and partially reliable information triggers interaction between psychological factors to play a primary role in the financial decision-making process and in generating "anomalies", while precise facts approximately lead to efficient markets.[47]

We motivate the discussion by first discussing the history of EMH and BF, and their main theses. In section 5.2, we present a general framework to define subjective rationality as a broad concept. In section 5.3, consistent with the proposed framework, we suggest candidate decision theories to account for the subjectively rational behavior. In section 5.4, by using the proposed candidate theories we discuss "insurance and gambling" and "equity premium" puzzles for illustrative purposes. In section 5.5, we discuss a novel representation of market efficiency. Section 5.6 concludes. Appendix 5 contains mathematical preliminaries.

---

[47]In order to forestall needless arguments, let us also mention that we do not claim that imprecision and reliability are the only factors connecting these two seemingly opposite paradigms, but the important factors.

5.1.1 Efficient markets hypothesis

EMH argues that large price movements result from the arrival of new fundamental information into the market, in which the information is probabilistic in nature. This is first comprehensively formalized in Osborne (1959) by making a number of assumptions. One of the underlying assumptions of Osborne's world is the *"logical decision"* which means investors are assumed to form expectations probabilistically and choose the course of action with a higher expected value. That is to say, investors form objective probabilities and make rational decisions as if they know each individual outcome. Another crucial assumption made by Osborne (1959) is an *"independence of decisions"* in the sequence of transactions of a single stock which leads to independent, identically distributed successive price changes. This implies that changes in prices can only come from unexpected new information. In this setting, central limit theorem assures that daily, weekly and monthly price changes converge to Gaussian distribution which is later generalized by Mandelbrot (1963) to stable Paretian distribution to account for the empirical evidence of leptokurtic distributions of price changes.

Stable Paretian distribution hypothesis is later supported by Fama (1965). Fama also argues that an independence assumption may still hold due to the existence of sophisticated traders (the so-called smart money), even though the processes generating noise and new information are dependent. That is, an independence assumption is consistent with efficient markets where prices at every point in time represent the best estimates of intrinsic values. The combination of independence and stable Paretian distribution allows Fama to argue that the actual prices adjust instantaneously to the changes in intrinsic value due to the discontinuous nature of the stable Paretian distribution. Therefore, this version of efficient market hypothesis includes random walk theory as a special case. However, the first formal general economic argument of 'efficient markets' is given by Samuelson (1965) by focusing on the martingale property first established by Bachelier (1900). Similar to the 'logical decision' assumption of Osborne (1959), Samuelson (1965) also assumes that people in financial markets make full use of the past probability distribution.

In summary, the proponents of this paradigm essentially argued the use of a probability calculus as a foundation of the economic analysis and made substantial progress in our understanding of financial markets. This view also justified an application of probability based decision techniques to financial markets and became associated with the CAPM of Sharpe (1964), Lintner (1965) and Mossin (1966).

5.1.2   Behavioral finance

The main arguments of BF stand in sharp contradiction to the logical decisions assumption of efficient markets hypothesis in forming expectations. The early works of Kahneman and Tversky is a foundational block of this area of finance. In a series of experiments, they show that people use heuristic to decide under uncertainty and conjecture that the same heuristic plays an important role in the evaluation of uncertainty in real life. In their seminal paper, Kahneman and Tversky (1979) present a critique of the expected utility theory and develop the prospect theory as an alternative. At the same time, Shiller (1979) shows that long-term interest rates are too volatile to be justified by rational models. Inspired by Tversky and Kahneman's works, Thaler (1980) argues that consumers do not follow economic theory and proposes an alternative descriptive theory on the basis of the prospect theory. Similarly, Shiller (1981) argues that stock prices fluctuate too much to be justified by subsequent dividend changes.

All of these arguments and findings sharply contradicted the efficient markets hypothesis and shaped the emergence of a new field. Finally, Bondt and Thaler (1985) marked the birth of behavioral finance with empirical evidence of overreaction hypothesis suggested by the experimental psychology. Since then, the number and magnitude of anomalies noticed by researchers have increased (although some are the mere results of data dredging) and the focus of finance academic discussion has shifted.

In summary, the main arguments of BF is categorized by Shefrin (2000) as follows:

- Financial practitioners commit errors due to relying on rules of thumb.

- Frame of a decision problem influences financial practitioners' decisions.

- Heuristic-driven biases and framing effects lead the prices in financial markets to deviate from fundamental values.

Overall, EMH supporters criticize BF for not having any unifying principles to explain the origin of behavioral anomalies. At the same time, BF supporters criticize EMH for making unrealistic assumptions and systematic errors in predicting human behavior. We agree with both arguments. Therefore, in this chapter, we examine the possibility of a general framework where both of the paradigms coexist. An exposition of a more general view of financial markets is the main purpose of this chapter.

5.2 Broad Concept of Subjective Rationality

The main goal of this section is to attempt to answer a question of what is meant by *"rational"* behavior when the decision maker (DM) is confronted with different types of information. The proposed framework is based on the premise that the *"correct"* decision method changes when the specificity and reliability of information change. Also, the considered notion of rationality is subjective in the sense of Gilboa et al. (2010). That is to say, the decision maker cannot be convinced that he is wrong in his decision.[48]

The current interpretation of rationality in economics and finance relies heavily on the subjective expected utility (SEU) axioms of Savage (1954). Simply, you are rational if you follow axioms of SEU and irrational if you don't. This definition of rationality is too narrow to capture a real-life decision situation. Also, this definition contradicts what has been tentatively argued by many economists such as Keynes (1921), Knight (1921), Shackle (1949), Arrow (1951) to name a few.

Specifically, Knight (1921) makes a clear distinction between risk (when relative odds of the events are known) and uncertainty (when the degree of knowledge only allows us to work with estimates). Also, Arrow (1951) notes that descriptions of uncertain consequences can be classified into two major categories, those which use exclusively the language of probability distributions and those which call for some other principle, either to replace or to supplement. We agree with the need of another principle to be a supplement to a language of probability to better approximate a real-life decision situation. This is due to the fact that information that decisions are based on is not only uncertain in nature, but at the same time imprecise and partially true. Using only a probabilistic approach is not sufficient to treat uncertainty, imprecision and partial truthness of information adequately.

One of the main arguments advanced in information science literature is that imprecision of the real-life information is possibilistic rather than probabilistic in nature and a fuzzy set theory is a necessary mathematical tool to deal with the possibilistic uncertainty (e.g., Zadeh (1978)). While there has been substantial progress in modeling uncertainty probabilistically, economics as a discipline has

---

[48]Gilboa et al. (2010) specifically show how the Knightian decision theory of Bewley (2002) and the maxmin expected utility (MEU) of Gilboa and Schmeidler (1989) are complementary to each other in terms of defining objective and subjective rationality. They argue that a choice is objectively rational if the DM can convince others that he is right in making it and subjectively rational if others cannot convince the DM that he is wrong in making it.

been somewhat reluctant to account for the latter over the years. For this reason, we focus on how the subjectively rational behavior is likely to change as the imprecision and reliability of information change.

One motivation of this research comes from Gilboa et al. (2010) who propose behavioral foundation of objective and subjective rationality. The other motivation comes from Peters (2003) who defines rationality as an application of the right method to the right problem or irrationality as a mismatch of the methodology and problem. By adopting the concept of subjective rationality in Gilboa et al. (2010), we consider a decision-theoretic approach to discuss right methods for decision problems with different types of information. Finding the right decision technique is the major issue in this context. One might reasonably ask *"what is the right decision technique?"* and *"what is the right problem?"*. We can deal with these questions appropriately in certain circumstances. By classifying the problems according to the information types of a DM one can find candidate decision theories to account for the subjective rationality. For example, an application of an objective probability-based decision technique to the problem with objective probability distribution is a right decision technique, while the same technique is not available in the situation in which a DM has imprecise information. So that, the expected utility theory (EUT) of Von Neumann and Morgenstern (1944) can be regarded as a right decision technique in probability theory applicable circumstances as it builds upon objective probabilities. If the asset returns follow the random walk theory, then the application of EUT becomes acceptable.

By generous stretch of imagination, we can use a similar logic to determine the right decision techniques for more general classes of information. The higher the generality of information and corresponding decision theories, the more financial observations we can solve that we otherwise label as paradoxes (anomalies). In our framework we follow Zadeh (2011) who outlines the following classification of information based on its generality.

Numerical Information (Ground Level – 'G') - This is single-valued information with exact probability, e.g., there is 80% chance that there will be 3% growth in Australian economy next year.

Interval-valued Information (First Level – 'F') - This is the first order uncertainty in which probability and value take intervals, e.g., there is 75-85% chance that there will be 2-3.5% growth in Australian economy next year.

Information with second-order uncertainty (Second Level 'S') - This is partially reliable information with sharp boundaries, e.g., there is 70-85% chance that there will be 1.5-3.5% growth in Australian economy next year and the lower probability of the given chances being reliable is 80%.

Fuzzy Information (Third Level – 'T') - This is the information with un-sharp boundaries, e.g., there will be a moderate growth in Australian economy next year.

Z-information and visual information (Z Level – 'Z') - This is partially reliable information with un-sharp boundaries and often in natural language, e.g., it seems likely that there will be a moderate growth in Australian economy next year.

The distinction between these levels can be difficult. Sometimes we can think of one upper level as the same as one below. However, there is no doubt that the degree of informativeness or specificity of information diminishes as we move away from the ground level. The former implies the latter while it is not true for the reverse.[49] We do not wish to face here the question of whether or not the information is sufficiently informative to serve a particular purpose. However, using a logic similar to Peters (2003), though his representation is vague, we approximate the definition of rationality in Table 5.1. Individuals can be considered subjectively rational along the diagonal in this framework. That is, for each level of information class there should be a different decision theory (methodology) to account for the subjectively rational behavior.

Following the argument and representation in Table 5.1, a broad definition of rationality is given accordingly.

Definition 5.1. A subjectively rational decisions are consistent with different decision theories for different classes of information.

The representation of subjectively rational behavior in Table 5.1 also hides a philosophical subtlety in itself. Philosophically, rationality is not a 0/1 property. Then, one can modify Table 5.1 to describe a degree of irrationality of the decision maker. For example, an irrationality of applying decision theory 1 to interval-valued information and applying the same method to Z information should be different. More precisely, the latter is more irrational than the former. One can consistently

---

[49]One can think of this generalization as a subsethood relation, $G \subseteq F \subseteq S \subseteq T \subseteq Z$, but not in a strict mathematical sense of subsethood since, for example, it is not obvious to see the relation between S and T.

Information Classes

Method	Numerical Information	Interval-valued Information	Second-Order Uncertainty	Fuzzy Information	Z information
Decision Theory 1	Rational	-	-	-	-
Decision Theory 2	-	Rational	-	-	-
Decision Theory 3	-	-	Rational	-	-
Decision Theory 4	-	-	-	Rational	-
Decision Theory 5	-	-	-	-	Rational

Table 5.1: Subjectively rational behavior

apply the same logic to the upper right triangle of Table 5.1. Note that, this is different from maximally rational, rational and minimally rational classification of Rubinstein (2001). What we have in mind is to set the maximum level as rational in the underlying information class and then reduce by one unit for each level of deviation from the underlying information class to describe the degree of irrationality. Also, we do not confine ourselves to only maxmin expected utility model in defining subjective rationality as proposed by Gilboa et al. (2010). This in turn, enables us to differentiate irrationality of the decision maker. A changing degree of irrationality can provide a novel foundation on the theory of choice under uncertainty. Of course, the argument does not go strictly. Nevertheless, this or similar type of representations might be a good starting point.

Similarly, an application of the more general decision theory where the less general is sufficient to capture the given decision situation is an inefficient use of resources. One can consistently apply this logic to the lower left triangle of Table 5.1. In terms of consistency of the framework, an application of the more general decision theory should give the same result as the less general one in the corresponding information class of the less general theory, nevertheless, the latter provides computational ease. In line with consistency and computational ease, there are two fundamental reasons to move from one decision theory to another. First, a more general decision theory is needed if it solves, at least, one more paradox that the existing theory cannot solve. Second, the existing decision theory becomes inconvenient (e.g., excessively complex) at some stage and it is desirable to move to a more convenient theory. The principle of replacing the existing decision theory with a more general decision theory is similar to the principle of requisite generalization in generalized information theory (GIT) (e.g., Klir (2005)). Here, a generalization is also not optional, but requisite, imposed by the nature of the decision situation.

5.3 Subjectively Rational Decision Making

In this section, we discuss candidates for the right decision theories by considering 5 examples with different classes of information. For this purpose, we consider 5 financial practitioners who choose among three alternatives (bonds - f_1, stocks - f_2 and term deposit - f_3) for a short-term investment plan. To make the analysis more clear we make rough approximations.[50]

5.3.1 Practitioner 1 - Expected utility theory

Suppose, Practitioner 1 evaluates each alternative under strong growth (s_1), moderate growth (s_2), stable economy (s_3) and recession (s_4). He notes that the following precise utilities will be achieved under each state of the economy for different acts.

	s_1	s_2	s_3	s_4
f_1	15	9	8	4
f_2	16	9	4	0
f_3	10	10	10	10

Table 5.2: Utilities of each act under different states

He also has perfect information about the uncertainty of the states with the following (subjective) probabilities: $P(s_1) = 0.5$, $P(s_2) = 0.3$, $P(s_3) = 0.15$, and hence, $P(s_4) = 0.05$. He faces the question of what option to choose.

Clearly, the information of Practitioner 1 is numerical information and he is in the province of probability theory. For this type of simplistic information classical measure and integral are adequate tools to calculate expected utilities of each act. In this environment, Practitioner 1 can easily determine his preferences as $f_1 \succ f_2 \succ f_3$ by calculating expected utilities as

$$U(f_1) = 11.6, \quad U(f_2) = 11.3 \quad \text{and} \quad U(f_3) = 10. \tag{5.1}$$

In (subjective) expected utility theory, choice under uncertainty is perceived as the maximization of the mathematical expectation of individual utilities with respect

---

[50]This may be a right point to revise the mathematical preliminaries in Appendix 5.

to (subjective) probabilities. If preferences of Practitioner 1 coincide with what is suggested by (SEU) EUT, then his action is perfectly justifiable and can be regarded as rational based on the proposed framework. So that, the optimal solution for Practitioner 1 is the bonds. In what follows, we illustrate information classes where the classical tools are not directly applicable.

5.3.2 Practitioner 2 - Choquet expected utility

Suppose, Practitioner 2 also evaluates each alternative under strong growth (s_1), moderate growth (s_2), stable economy (s_3) and recession (s_4) and he also notes the same precise utilities shown in Table 5.2. However, he assigns the following subjective probability intervals: $P(s_1) = [0.4, 0.45]$, $P(s_2) = [0.3, 0.35]$, $P(s_3) = [0.15, 0.20]$, and hence, $P(s_4) = [0, 0.15]$. He faces the question of what option to choose.

The information of Practitioner 2 corresponds to the interval-valued information in Table 5.1. For simplicity we assumed that only his probability assessments take interval values. This can be extended to interval-valued utilities in the sense of Gul and Pesendorfer (2014).

Given a set S and its power set $\mathcal{F}(S)$, let $I =< [l(s_i), u(s_i)] \mid i \in \mathbb{N}_4 >$ denote 4-tuples of probability intervals on $s_i \in S$, where $l(s_i)$ and $u(s_i)$ denote the lower and upper probability bounds of state i. Let $\mathcal{M}$ denote a convex set of probability distribution functions p on $\mathcal{F}(S)$ satisfying

$$\mathcal{M} = \{p \mid l(s_i) \leq p(s_i) \leq u(s_i), i \in \mathbb{N}_4, \sum_{s_i \in S} p(s_i) = 1\}. \tag{5.2}$$

From the probability distributions in set $\mathcal{M}$, the lower probability measure (lower prevision) is defined for all $A \in \mathcal{F}(S)$ as $\eta(A) = \inf_{p \in \mathcal{M}} \sum_{x_i \in A} p(x_i)$. It follows from this definition that lower probabilities satisfy the conditions of capacities (i.e., monotone measures). Then, the lower probability measure η is[51]

$$\eta(A) = \max \left\{ \sum_{x_i \in A} l(x_i), 1 - \sum_{x_i \notin A} u(x_i) \right\}, \forall A \in \mathcal{F}(S). \tag{5.3}$$

---

[51]Note that, $\mathcal{M}$ is non-empty set if and only if, $\sum_{i=1}^{4} l(s_i) \leq 1$ and $\sum_{i=1}^{4} u(s_i) \geq 1$. Also, equation (5.3) is only applicable when I satisfies $\sum_{j \neq i} l(s_j) + u(s_i) \leq 1$ and $\sum_{j \neq i} u(s_j) + l(s_i) \geq 1$. These conditions are trivially satisfied when $l(s_4) = 1 - \sum_{i=1}^{3} u(s_i)$ and $u(s_4) = 1 - \sum_{i=1}^{3} l(s_i)$.

For clarity, let us exemplify,

$$\eta\big(\{s_1, s_4\}\big) = \max\{l(s_1) + l(s_4), 1 - u(s_2) - u(s_3)\} = 0.45, \tag{5.4}$$

$$\eta\big(\{s_1, s_3, s_4\}\big) = \max\{l(s_1) + l(s_3) + l(s_4), 1 - u(s_2)\} = 0.65. \tag{5.5}$$

Table 5.3 shows the values of lower probability measures that are computed similarly.

States	$\{s_1\}$	$\{s_2\}$	$\{s_3\}$	$\{s_4\}$	$\{s_1, s_2\}$
$\eta(A)$	0.4	0.3	0.15	0	0.7
States	$\{s_1, s_3\}$	$\{s_1, s_4\}$	$\{s_2, s_3\}$	$\{s_2, s_4\}$	$\{s_3, s_4\}$
$\eta(A)$	0.55	0.45	0.45	0.35	0.20
States	$\{s_1, s_2, s_3\}$	$\{s_1, s_2, s_4\}$	$\{s_1, s_3, s_4\}$	$\{s_2, s_3, s_4\}$	$\{S\}$
$\eta(A)$	0.85	0.80	0.65	0.55	1

Table 5.3: Lower probability measures

Based on his probability intervals, Practitioner 2 can determine his preferences as $f_1 \succ f_3 \succ f_2$ by first ordering utility values in a descending order and then aggregating utilities with the Choquet integral with respect to the lower probability measure η.[52] Specifically, for a given alternative, the Choquet (expected) utility is computed as

$$
\begin{aligned}
U(f_i) = & \left(u\big((f_i)(s_1)\big) - u\big((f_i)(s_2)\big) \right) \eta(\{s_1\}) \\
& + \left(u\big((f_i)(s_2)\big) - u\big((f_i)(s_3)\big) \right) \eta(\{s_1, s_2\}) \\
& + \left(\big(u(f_i)(s_3)\big) - u\big((f_i)(s_4)\big) \right) \eta(\{s_1, s_2, s_3\}) + u\big((f_i)(s_4)\big) \eta(S),
\end{aligned}
\tag{5.6}
$$

given that $u\big((f_i)(s_1)\big) \geq u\big((f_i)(s_2)\big) \geq u\big((f_i)(s_3)\big) \geq u\big((f_i)(s_4)\big)$. Following the same steps for f_1, f_2 and f_3, we obtain the expected utilities of each alternative as

$$U(f_1) = 10.5, \quad U(f_2) = 9.7 \quad \text{and} \quad U(f_3) = 10, \tag{5.7}$$

which leads to the preference order of $f_1 \succ f_3 \succ f_2$.

---

[52]Recall from Chapter 2 that the Choquet expectation is equivalent to adding the probability gap to the belief about the worst case scenario.

For Practitioner 2, due to the imprecise nature of probability intervals, the classical measure and integral become deficient to directly determine the optimal solution. Therefore, we first determine convex set of probability distribution functions from the given intervals and calculate the lower envelope of this closed convex set as a lower probability measure. As this lower probability measure satisfies the conditions of Choquet capacities (or non-additive probabilities), the Choquet integral becomes the right tool to determine the expected utilities of each act. This is essentially the Choquet expected utility (CEU) proposed by Schmeidler (1989) using the notion of non-additive probabilities. With the convex capacities it is also well-known that CEU coincides with MEU under the assumption of ambiguity aversion (see Proposition 3 of Schmeidler (1986) for proof).[53]

One point worth to note here is that the use of lower probability measure is justified with the implicit assumption of ambiguity aversion. If the degree of ambiguity aversion, $\alpha \in [0, 1]$, in the sense of Ghirardato et al. (2004) is known, Practitioner 2 can be subjectively rational if he applies α-MEU. This is because α-MEU is a natural generalization of MEU (recall that we used α−MEU to separate uncertainty and uncertainty attitude of traders in Chapter 3). Suppose, instead of being fully ambiguity averse, Practitioner 2 is 70% ambiguity averse. Then, following α-MEU, $U(f_i)$ is determined as

$$U(f_i) = \alpha \min_{P \in \mathcal{M}} \int_S u(f_i(S))dP + (1 - \alpha) \max_{P \in \mathcal{M}} \int_S u(f_i(S))dP, \qquad (5.8)$$

where α denotes the degree of ambiguity aversion. The term $\min_{P \in \mathcal{M}} \int_S u(f_i(S))dP$ is known from the previous calculation, as MEU coincides with CEU for convex capacities. We first determine $\max_{P \in \mathcal{M}} \int_S u(f_i(S))dP$ for $i = 1, 2, 3$ and then weight minimum and maximum utilities with α and $(1 - \alpha)$ respectively to determine overall utilities of Practitioner 2. The results are

$$U(f_1) = 10.8, \quad U(f_2) = 10.14 \quad \text{and} \quad U(f_3) = 10. \qquad (5.9)$$

Therefore, changing the degree of ambiguity aversion changes the preference order of Practitioner 2 from $f_1 \succ f_3 \succ f_2$ to $f_1 \succ f_2 \succ f_3$. In both situation, Practitioner 2 can be regarded as subjectively rational (he can not be convinced that he is wrong in his preference order).

---

[53]Capacity η is convex for all events $A, B \in \mathcal{F}(S)$ if it satisfies $\eta(A \cup B) + \eta(A \cap B) \geq \eta(A) + \eta(B)$.

5.3.3 Practitioner 3 - α-maxmin expected utility

Suppose Practitioner 3 also evaluates each alternative under $S = \{s_1, s_2, s_3, s_4\}$ with the same precise utilities shown in Table 5.2. He also assigns the same interval probabilities: $P(s_1) = [0.4, 0.45]$, $P(s_2) = [0.3, 0.35]$, $P(s_3) = [0.15, 0.20]$ and $P(s_4) = [0, 0.15]$. However, this time a probability interval of $[0.7, 0.8]$ is assigned to measure an imprecise degree of confidence of the assigned probabilities. This can be considered as a reliability of the assigned probabilities. The question remains the same. The traditional methods are also incapable of solving this problem due to the probability intervals and the second-order uncertainty imposed by the reliability of assigned probabilities. There are two approaches we can think of to proceed with this problem.

Approach 1. A direct way to address the given problem is to use the methodology of interval-valued information by overlooking the reliability of the assigned probabilities. With this approach, the same preference order of Practitioner 2 applies to Practitioner 3. That is, if Practitioner 3 is fully ambiguity averse $f_1 \succ f_3 \succ f_2$ holds, but with the confidence interval of $[0.7, 0.8]$. The reason for leaving the uncertainty imposed by the reliability intact can be understood by the following illustration of Shafer (1987). Suppose, we have asked Fred if the streets outside are slippery. He replies "Yes" and we know that 80% of the time he speaks truthfully and 20% of the time he speaks carelessly, saying whatever comes into his mind. With

$$p_1 = \text{"the streets are slippery"}, \tag{5.10}$$

$$p_2 = \text{""the streets are not slippery"} \tag{5.11}$$

propositions, Shafer derives a belief of 0.8 in proposition $\{p_1\}$ and 0.2 in $\{p_1, p_2\}$.

If we don't have additional information, we should not allocate the remaining 0.2 between p_1 and p_2. In our example, the Shafer's illustration suggests that there is $[0.7, 0.8]$ units of evidence supporting

$$f_1 \succ f_3 \succ f_2, \tag{5.12}$$

and $[0.2, 0.3]$ units of evidence supporting all other combinations of preference order,

$$\left\{ \begin{array}{c} \{f_1 \succ f_2 \succ f_3\}, \{f_1 \succ f_3 \succ f_2\}, \{f_2 \succ f_1 \succ f_3\}, \\ \{f_2 \succ f_3 \succ f_1\}, \{f_3 \succ f_1 \succ f_2\}, \{f_3 \succ f_2 \succ f_1\} \end{array} \right\}. \tag{5.13}$$

In line with Shafer's example, the first approach concludes that,

$$U(f_1) = 10.5, \quad U(f_2) = 9.7 \quad \text{and} \quad U(f_3) = 10 \tag{5.14}$$

with the reliability (confidence, accuracy) of $[0.7, 0.8]$ if the Practitioner 3 is fully ambiguity averse.

Approach 2. It also seems to us that reliability and ambiguity attitude of a DM are related to each other. More precisely, there is an inverse relationship between reliability and ambiguity aversion. As the information gets more and more unreliable a DM should become more ambiguity averse. In that sense, ambiguity-aversion α is a function of the reliability of information $\alpha = \psi(\underline{r}, \bar{r})$, where $\underline{r}$ and $\bar{r}$ denote lower and upper reliability of information. We have not been able to determine what confidence functional (ψ) would account for rational behavior. This problem is similar to the problem of which utility function makes sense and leads to a better outcome. For this purpose, any utility function would suffice for an agent to be rational. In that sense, any confidence functional leading to ambiguity aversion would suffice for our purposes. Suppose,

$$\psi(r) = \frac{1 - (\underline{r} + \bar{r})}{2}. \tag{5.15}$$

Then, ambiguity aversion, α, equals 0.25 and application of α-MEU results in

$$U(f_1) = 11.25, \quad U(f_2) = 10.79 \quad \text{and} \quad U(f_3) = 10, \tag{5.16}$$

which leads to the preference order of $f_1 \succ f_2 \succ f_3$.

Instead, one can also consider the smooth decision making model of Klibanoff et al. (2005),

$$U(f_i) = E_\mu[\phi(E_\pi(u(\cdot)))], \tag{5.17}$$

when an objective probability measure π and its subjective relevance μ are precise. However, if the uncertainties are imprecise in both, first and second order, one is left to use some imagination to solve similar problems. So far, α-MEU is used as the most general decision theory and it suffices to account for subjective rationality under the second-order imprecise probability. However, in the fuzzy and Z-environment, α-MEU also falls short of taking the imprecision and reliability attributes of the real-world information into account.

5.3.4 Practitioner 4 - Behavioral decision making

Consider Practitioner 4 who notes the following trends under $S = \{\tilde{s}_1, \tilde{s}_2, \tilde{s}_3, \tilde{s}_4\}$.

(i) $\tilde{f}_1$ will yield high income under $\tilde{s}_1$, medium income under $\tilde{s}_2$, less than medium income under $\tilde{s}_3$ and small income under $\tilde{s}_4$;

(ii) $\tilde{f}_2$ will yield very high income under $\tilde{s}_1$, medium income under $\tilde{s}_2$, small income under $\tilde{s}_3$ and a notable loss under $\tilde{s}_4$;

(iii) $\tilde{f}_3$ will yield approximately the same medium income in all 4 fuzzy states of economy.

Moreover, her possible set of states is $H = \{h_1, h_2\}$, where h_1 and h_2 (non-fuzzy in this example) stands for risk-aversion and risk-seeking, respectively.[54] Practitioner 4 also has information that $\tilde{s}_1$ will take place with a medium probability, $\tilde{s}_2$ will take place with a less than medium probability, $\tilde{s}_3$ with a small probability and $\tilde{s}_4$ with a very small probability. The probability of her risk-aversion is also known to be about 70% and she is assumed to be risk-seeking when she is not risk-averse. The question remains the same.

This problem is considered as the problem of decision making under possibilistic-probabilistic information and linguistic preference. At this information level, there are also two ways of dealing with the given problem.

Approach 1. The first approach is to compute a fuzzy-number-valued lower prevision and use the Choquet integral with respect to the computed lower prevision to calculate the total utility values of each act. This is essentially a generalized version of Choquet expected utility (CEU) of Schmeidler (1989) and the argument advanced by Aliev et al. (2012). This approach is also consistent with the previous decision theories used for Practitioners 1, 2 and 3.

Approach 2. The second approach is more behavioral in nature. Because of capturing interaction among behavioral determinants to account for the fundamental level dependence of human behavior, behavioral decision making with combined states under imperfect information (BDM) of Aliev, Pedrycz and Huseynov (2013) is another candidate to determine the optimal action of Practitioner 4 in this information class. Although, the first approach is also consistent with the previous decision theories, BDM captures an interaction among factors induced by the fuzzy

---

[54]Throughout the paper, we carry the notation of superimposed tilde for fuzzy values.

environment. BDM combines fuzzy states of nature and fuzzy states of the decision maker as $\Omega = S \times H$ (cartesian product of S and H) with the elements of $\tilde{\omega}_i^j = (\tilde{s}_i, \tilde{h}_j)$ to account for the fundamental level dependence of S and H as shown by Kahneman and Tversky (1979). More precisely, Kahneman and Tversky (1979) show that a DM is risk-averse in the positive domain and risk-seeking in the negative domain. Neither classical measures nor capacities is adequate to capture the given imprecision induced by the natural language and the dependence of S and H. At this information class, among the fuzzy set of actions, $A = \{\tilde{f} \in A \mid \tilde{f} : \Omega \to X\}$ where X denotes a space of fuzzy outcomes, BDM determines an optimal action $\tilde{f}^* \in A$ with $\tilde{U}(\tilde{f}^*) = max_{\tilde{f} \in A} \int_\Omega \tilde{U}(\tilde{f}(\tilde{\omega})) d\tilde{\eta}(\cdot, \cdot)$ which implies that an overall utility of an action is determined by a fuzzy number valued bi-capacity based aggregation over space Ω.

To solve the given problem with BDM, suppose the outcomes at each fuzzy states of economy are represented by triangular fuzzy numbers given in Table 5.4. In other words, the fuzzy numbers in Table 5.4 are precisiated forms of the given linguistic outcomes.

	$\tilde{s}_1$	$\tilde{s}_2$	$\tilde{s}_3$	$\tilde{s}_4$
$\tilde{f}_1$	$(8, 11, 14)$	$(5, 8, 11)$	$(3, 6, 9)$	$(1, 3, 5)$
$\tilde{f}_2$	$(11, 15, 19)$	$(5,8,11)$	$(1, 3, 5)$	$(-3, -1.5, 0)$
$\tilde{f}_3$	$(5, 8, 11)$	$(5, 8, 11)$	$(5, 8, 11)$	$(5, 8, 11)$

Table 5.4: Fuzzy outcomes of each act under different states

Then we assign fuzzy utilities $\tilde{u}(\tilde{f}_k(\tilde{\omega}_i^j))$ (utility of action $\tilde{f}_k$ under state of economy $\tilde{s}_i$ when her own state is h_j) by applying a technique of value function of Tversky and Kahneman (1992) (for demonstration purpose) as

$$\tilde{u}(\tilde{f}_k(\tilde{\omega}_i^1)) = \begin{cases} (\tilde{f}_k(\tilde{s}_i))^\alpha & \text{when } \tilde{f}_k(\tilde{s}_i) \geq 0, \\ -\lambda(-\tilde{f}_k(\tilde{s}_i))^\beta & \text{when } \tilde{f}_k(\tilde{s}_i) < 0; \end{cases} \tag{5.18}$$

$$\tilde{u}(\tilde{f}_k(\tilde{\omega}_i^2)) = \begin{cases} (\tilde{f}_k(\tilde{s}_i))^\beta & \text{when } \tilde{f}_k(\tilde{s}_i) \geq 0, \\ -\lambda(-\tilde{f}_k(\tilde{s}_i))^\alpha & \text{when } \tilde{f}_k(\tilde{s}_i) < 0; \end{cases} \tag{5.19}$$

where $\alpha = 0.88$, $\beta = 1.25$ and $\lambda = 2.25$. For instance,

$$\tilde{u}(\tilde{f}_1(\tilde{\omega}_1^1)) = (\tilde{f}_1(\tilde{s}_1))^\alpha = (8^{0.88}, 11^{0.88}, 14^{0.88}) \approx (6, 8, 10), \tag{5.20}$$

$$\tilde{u}(\tilde{f}_1(\tilde{\omega}_1^2)) = (\tilde{f}_1(\tilde{s}_1))^\beta = (8^{1.25}, 11^{1.25}, 14^{1.25}) \approx (13, 20, 27). \qquad (5.21)$$

A similar calculation follows for other utilities. Table 5.5 shows the absolute values of approximate results in a descending order.

$\tilde{u}(\tilde{f}_1(\tilde{\omega}_1^2)) \approx (13, 20, 27)$	$\tilde{u}(\tilde{f}_1(\tilde{\omega}_2^2)) \approx (7, 13, 20)$	$\tilde{u}(\tilde{f}_1(\tilde{\omega}_3^2)) \approx (4, 9, 16)$
$\tilde{u}(\tilde{f}_1(\tilde{\omega}_1^1)) \approx (6, 8, 10)$	$\tilde{u}(\tilde{f}_1(\tilde{\omega}_2^1)) \approx (4, 6, 8)$	$\tilde{u}(\tilde{f}_1(\tilde{\omega}_3^1)) \approx (3, 5, 7)$
$\tilde{u}(\tilde{f}_1(\tilde{\omega}_4^2)) \approx (1, 4, 7)$	$\tilde{u}(\tilde{f}_1(\tilde{\omega}_4^1)) \approx (1, 3, 4)$	
$\tilde{u}(\tilde{f}_2(\tilde{\omega}_1^2)) \approx (20, 30, 40)$	$\tilde{u}(\tilde{f}_2(\tilde{\omega}_2^2)) \approx (7, 13, 20)$	$\tilde{u}(\tilde{f}_2(\tilde{\omega}_1^1)) \approx (8, 11, 13)$
$\tilde{u}(\tilde{f}_2(\tilde{\omega}_2^1)) \approx (4, 6, 8)$	$|\tilde{u}(\tilde{f}_2(\tilde{\omega}_4^1))| \approx (0, 4, 9)$	$\tilde{u}(\tilde{f}_2(\tilde{\omega}_3^2)) \approx (1, 4, 7)$
$|\tilde{u}(\tilde{f}_2(\tilde{\omega}_4^2))| \approx (0, 3, 6)$	$\tilde{u}(\tilde{f}_2(\tilde{\omega}_3^1)) \approx (1, 3, 4)$	
$\tilde{u}(\tilde{f}_3(\tilde{\omega}_1^2)) \approx (7, 13, 20)$	$\tilde{u}(\tilde{f}_3(\tilde{\omega}_2^2)) \approx (7, 13, 20)$	$\tilde{u}(\tilde{f}_3(\tilde{\omega}_3^2)) \approx (7, 13, 20)$
$\tilde{u}(\tilde{f}_3(\tilde{\omega}_4^2)) \approx (7, 13, 20)$	$\tilde{u}(\tilde{f}_3(\tilde{\omega}_1^1)) \approx (4, 6, 8)$	$\tilde{u}(\tilde{f}_3(\tilde{\omega}_2^1)) \approx (4, 6, 8)$
$\tilde{u}(\tilde{f}_3(\tilde{\omega}_3^1)) \approx (4, 6, 8)$	$\tilde{u}(\tilde{f}_3(\tilde{\omega}_4^1)) \approx (4, 6, 8)$	

Table 5.5: Fuzzy utilities under different states of economy and decision maker

After assigning fuzzy utilities to each act, the next step is to construct a fuzzy joint probability distribution (FJP) $\tilde{P}$ on Ω given the fuzzy marginal probabilities of $S = \{\tilde{s}_1, \tilde{s}_2, \tilde{s}_3, \tilde{s}_4\}$ and $H = \{h_1, h_2\}$. With the given probabilities in natural language, we precisiate the fuzzy marginal probability distributions of S and H by the following triangular fuzzy numbers.[55]

$$\tilde{P}(\tilde{s}_1) = (0.45, 0.50, 0.55), \quad \tilde{P}(\tilde{s}_2) = (0.325, 0.35, 0.375),$$

$$\tilde{P}(\tilde{s}_3) = (0.1, 0.125, 0.15), \quad \tilde{P}(\tilde{s}_4) = (0, 0.025, 0.125) \text{ (computed)}, \qquad (5.22)$$

$$\tilde{P}(h_1) = (0.65, 0.70, 0.75), \quad \tilde{P}(h_2) = (0.25, 0.30, 0.35) \text{ (computed)}.$$

Given the fuzzy marginal probability distributions of S and H, we obtain the FJP distribution on the base of positive and negative dependence concept of Wise and Henrion (1985).[56] Formally, the FJP is obtained following

$$\tilde{p}(\tilde{s}_i, h_j) = \bigcup_{\alpha \in [0,1]} \alpha \left[{}^\alpha p_1(s_i) {}^\alpha p_1(h_j), \min \left({}^\alpha p_2(s_i), {}^\alpha p_2(h_j) \right) \right], \qquad (5.23)$$

$$\tilde{p}(\tilde{s}_i, h_j) = \bigcup_{\alpha \in [0,1]} \alpha \left[\max \left({}^\alpha p_1(s_i) + {}^\alpha p_1(h_j) - 1, 0 \right), {}^\alpha p_2(s_i) {}^\alpha p_2(h_j) \right] \qquad (5.24)$$

---

[55]By convention, we precisiate $(n-1)$ of the given linguistic probabilities and compute the last one in order to add up total probabilities to 1.

[56]Given the numerical probabilities $P(A)$ and $P(B)$, the joint probability of A and B is $P(A, B) = P(A) \cdot P(B)$ if A and B are independent, $P(A, B) = \min \left(P(A), P(B) \right)$ if A and B are positively dependent, and $P(A, B) = \max \left(P(A) + P(B) - 1, 0 \right)$ if A and B have negative dependence. Eqs. (5.23) and (5.24) are the extensions of these formulations to fuzzy probabilities via α-cuts.

for positive and negative dependence, respectively. For $\tilde{f}_1$ and $\tilde{f}_3$, there are positive dependences between, $(\tilde{s}_1, h_1)$, $(\tilde{s}_2, h_1)$, $(\tilde{s}_3, h_1)$, $(\tilde{s}_4, h_1)$ and negative dependences between $(\tilde{s}_1, h_2)$, $(\tilde{s}_2, h_2)$, $(\tilde{s}_3, h_2)$, $(\tilde{s}_4, h_2)$. For $\tilde{f}_2$ there are positive dependences between, $(\tilde{s}_1, h_1)$, $(\tilde{s}_2, h_1)$, $(\tilde{s}_3, h_1)$, $(\tilde{s}_4, h_2)$ and negative dependences between $(\tilde{s}_1, h_2)$, $(\tilde{s}_2, h_2)$, $(\tilde{s}_3, h_2)$, $(\tilde{s}_4, h_1)$ due to Kahneman and Tversky (1979). That is to say, people are risk-averse in the positive domain and risk-seeking in the negative domain. Then, we compute $\tilde{p}(\tilde{s}_1, h_1)$ for $\tilde{f}_1$, $\tilde{f}_2$ and $\tilde{f}_3$ given $\alpha = 0, 0.5, 1$ as

$$
\begin{aligned}
\left[{}^0p_1(\tilde{s}_1){}^0p_1(h_1), \min \left({}^0p_2(\tilde{s}_1){}^0p_2(h_1) \right) \right] &= [0.45 \cdot 0.65, \min(0.55, 0.75)] \\
&\approx [0.293, 0.55];
\end{aligned}
\tag{5.25}
$$

$$
\begin{aligned}
\left[{}^{.5}p_1(\tilde{s}_1){}^{.5}p_1(h_1), \min \left({}^{.5}p_2(\tilde{s}_1){}^{.5}p_2(h_1) \right) \right] &= [0.475 \cdot 0.675, \min(0.525, 0.725)] \\
&= [0.32, 0.525];
\end{aligned}
\tag{5.26}
$$

$$
\begin{aligned}
\left[{}^1p_1(\tilde{s}_1){}^1p_1(h_1), \min \left({}^1p_2(\tilde{s}_1){}^1p_2(h_1) \right) \right] &= [0.5 \cdot 0.7, \min(0.5, 0.7)] \\
&= [0.35, 0.5].
\end{aligned}
\tag{5.27}
$$

Hence, $\tilde{p}(\tilde{s}_1, h_1)$ can be approximated by $(0.293, 0.35, 0.5, 0.55)$ trapezoidal fuzzy number. Following these steps, the FJPs of $\tilde{s}_i$ and h_j for $\tilde{f}_1$ and $\tilde{f}_3$, respectively, are

$$
\begin{aligned}
&\tilde{p}(\tilde{s}_1, h_1) = (0.293, 0.35, 0.5, 0.55), \ \tilde{p}(\tilde{s}_2, h_1) = (0.211, 0.245, 0.350, 0.375), \\
&\tilde{p}(\tilde{s}_3, h_1) = (0.065, 0.088, 0.125, 0.15), \ \tilde{p}(\tilde{s}_4, h_1) = (0, 0.018, 0.025, 0.125),
\end{aligned}
\tag{5.28}
$$

and

$$
\begin{aligned}
&\tilde{p}(\tilde{s}_1, h_2) = (0, 0, 0.150, 0.193), \ \tilde{p}(\tilde{s}_2, h_2) = (0, 0, 0.105, 0.131), \\
&\tilde{p}(\tilde{s}_3, h_2) = (0, 0, 0.038, 0.053), \ \tilde{p}(\tilde{s}_4, h_2) = (0, 0, 0.008, 0.044).
\end{aligned}
\tag{5.29}
$$

The FJPs for $\tilde{f}_2$ are the same as $\tilde{f}_1$ and $\tilde{f}_3$ for all the combinations but two,

$$
\tilde{p}(\tilde{s}_4, h_1) = (0, 0, 0.008, 0.044), \ \tilde{p}(\tilde{s}_4, h_2) = (0, 0.018, 0.025, 0.125),
\tag{5.30}
$$

due to an inverse relationship in $\tilde{s}_4$.

The next step is to construct a fuzzy valued bi-capacity $\tilde{\eta}(\cdot, \cdot)$ based on the obtained FJPs. A fuzzy valued bi-capacity is defined, $\tilde{\eta}(\tilde{A}, \tilde{B}) = \tilde{\eta}(\tilde{A}) - \tilde{\eta}(\tilde{B})$, as a difference of fuzzy-valued lower probabilities $\tilde{\eta}(\tilde{A})$ and $\tilde{\eta}(\tilde{B})$. Given a set $\Omega = \{\omega_1^1, \omega_1^2, \omega_2^1, ..., \omega_4^1, \omega_4^2\}$ and its power set $\mathcal{F}(\Omega)$, let ${}^\alpha I = < [{}^\alpha l_i, {}^\alpha u_i] \mid i \in \mathbb{N}_8 >$

denote 8-tuples of probability intervals on $\omega_i^j \in \Omega$ where $^\alpha l_i$ and $^\alpha u_i$ denote corresponding lower and upper bounds of α-cuts of the computed FJPs, respectively. Consistent with Practitioners 2 and 3, let $\tilde{\mathcal{M}}$ denote a set of fuzzy probabilities $\tilde{p}$ on $\mathcal{F}(\Omega)$ satisfying

$$\tilde{\mathcal{M}} = \{\tilde{p} \mid {}^\alpha l(\omega_i^j) \leq {}^\alpha p(\omega_i^j) \leq {}^\alpha u(\omega_i^j), i \in \mathbb{N}_4, j \in \mathbb{N}_2, \sum_{\omega_i^j \in \Omega} \tilde{p}(\omega_i^j) = 1\}. \qquad (5.31)$$

From the fuzzy probabilities in set $\tilde{\mathcal{M}}$, the lower probability measure is defined for all $\tilde{A} \in \mathcal{F}(\Omega)$ as $\tilde{\eta}(\tilde{A}) = \inf_{\tilde{p} \in \tilde{\mathcal{M}}} \sum_{x_i \in \tilde{A}} \tilde{p}(x_i)$. An α-cut of a fuzzy lower probability measure, $^\alpha \eta$, are calculated same as Practitioner 2

$$^\alpha \eta(\tilde{A}) = \max\left\{\sum_{x_i \in \tilde{A}} {}^\alpha l(x_i), 1 - \sum_{x_i \notin \tilde{A}} {}^\alpha u(x_i)\right\}, \forall \tilde{A} \in \mathcal{F}(\Omega). \qquad (5.32)$$

For clarity, let us exemplify;

$$^\alpha \eta(\tilde{\omega}_1^2) = \max\{^\alpha l(\tilde{\omega}_1^2), 1 - \sum_{i \neq 1, j \neq 2} {}^\alpha u(\tilde{\omega}_i^j)\} = {}^\alpha l(\tilde{\omega}_1^2) = 0. \qquad (5.33)$$

Hence, $\eta(\tilde{\omega}_1^2) = (0, 0, 0)$.

$$\begin{aligned}
^\alpha \eta(\tilde{\omega}_1^2, \tilde{\omega}_2^2, \tilde{\omega}_1^1, \tilde{\omega}_2^1) = \max\Big\{ &\left({}^\alpha l(\tilde{\omega}_1^2) + {}^\alpha l(\tilde{\omega}_2^2) + {}^\alpha l(\tilde{\omega}_1^1) + {}^\alpha l(\tilde{\omega}_2^1)\right), \\
&\left(1 - {}^\alpha u(\tilde{\omega}_3^1) - {}^\alpha u(\tilde{\omega}_3^2) - {}^\alpha u(\tilde{\omega}_4^1) - {}^\alpha u(\tilde{\omega}_4^2)\right)\Big\} \\
= &\left(1 - {}^\alpha u(\tilde{\omega}_3^1) - {}^\alpha u(\tilde{\omega}_3^2) - {}^\alpha u(\tilde{\omega}_4^1) - {}^\alpha u(\tilde{\omega}_4^2)\right) \\
= &\, 0.629 + 0.176 \cdot \alpha.
\end{aligned} \qquad (5.34)$$

Hence, $\eta(\tilde{\omega}_1^2, \tilde{\omega}_2^2, \tilde{\omega}_1^1, \tilde{\omega}_2^1) \approx (0.63, 0.8, 0.8)$. Based on this formulation, we obtain Table 5.6 on the values of $\tilde{\eta}$ for $\tilde{f}_2$. As there is no loss for $\tilde{f}_1$ and $\tilde{f}_3$, $\tilde{\eta}(B)$ should be directly set to 0.[57]

Finally, we calculate fuzzy overall utilities by a fuzzy-valued bi-capacity based aggregation over space Ω using generalized version of Choquet-like aggregation

---

[57] Approaches 1 and 2 of Practitioner 4 coincide with each other when there is no loss. However, approach 2 accounts for risk-seeking when there is loss as in $\tilde{f}_2(\tilde{s}_4)$.

$A, B \subset \Omega$	$\tilde{\eta}(A)$	$\tilde{\eta}(B)$	$\tilde{\eta}(A, B)$
$\{\tilde{\omega}_1^2\}, \{\emptyset\}$	(0,0,0)	(0,0,0)	(0,0,0)
$\{\tilde{\omega}_1^2, \tilde{\omega}_2^2\}, \{\emptyset\}$	(0,0,0)	(0,0,0)	(0,0,0)
$\{\tilde{\omega}_1^2, \tilde{\omega}_2^2, \tilde{\omega}_1^1\}, \{\emptyset\}$	(0.29,0.45,0.45)	(0,0,0)	(0.29,0.45,0.45)
$\{\tilde{\omega}_1^2, \tilde{\omega}_2^2, \tilde{\omega}_1^1, \tilde{\omega}_2^1\}, \{\emptyset\}$	(0.63,0.8,0.8)	(0,0,0)	(0.63,0.8,0.8)
$\{\tilde{\omega}_1^2, \tilde{\omega}_2^2, \tilde{\omega}_1^1, \tilde{\omega}_2^1\}, \{\tilde{\omega}_4^1\}$	(0.63,0.8,0.8)	(0,0,0)	(0.63,0.8,0.8)
$\{\tilde{\omega}_1^2, \tilde{\omega}_2^2, \tilde{\omega}_1^1, \tilde{\omega}_2^1, \tilde{\omega}_3^2\}, \{\tilde{\omega}_4^1\}$	(0.68,0.84,0.84)	(0,0,0)	(0.68,0.84,0.84)
$\{\tilde{\omega}_1^2, \tilde{\omega}_2^2, \tilde{\omega}_1^1, \tilde{\omega}_2^1, \tilde{\omega}_3^2\}, \{\tilde{\omega}_4^1, \tilde{\omega}_4^2\}$	(0.68,0.84,0.84)	(0,0.02,0.02)	(0.68,0.82,0.82)
$\{\tilde{\omega}_1^2, \tilde{\omega}_2^2, \tilde{\omega}_1^1, \tilde{\omega}_2^1, \tilde{\omega}_3^2, \tilde{\omega}_3^1\}, \{\tilde{\omega}_4^1, \tilde{\omega}_4^2\}$	(0.83,0.97,0.97)	(0,0.02,0.02)	(0.83,0.95,0.95)

Table 5.6: Fuzzy-valued bi-capacities for $\tilde{f}_2$

defined in Appendix 5. For $\tilde{f}_2$ the overall utilities are given by

$$
\begin{aligned}
\tilde{U}(\tilde{f}_2) = & \big(|\tilde{u}(\tilde{f}_2(\tilde{\omega}_1^2))| -_h |\tilde{u}(\tilde{f}_2(\tilde{\omega}_2^2))|\big)\tilde{\eta}\big(\{\tilde{\omega}_1^2\}, \{\emptyset\}\big) \\
& + \big(|\tilde{u}(\tilde{f}_2(\tilde{\omega}_2^2))| -_h |\tilde{u}(\tilde{f}_2(\tilde{\omega}_1^1))|\big)\tilde{\eta}\big(\{\tilde{\omega}_1^2, \tilde{\omega}_2^2\}, \{\emptyset\}\big) \\
& + \big(|\tilde{u}(\tilde{f}_2(\tilde{\omega}_1^1))| -_h |\tilde{u}(\tilde{f}_2(\tilde{\omega}_2^1))|\big)\tilde{\eta}\big(\{\tilde{\omega}_1^2, \tilde{\omega}_2^2, \tilde{\omega}_1^1\}, \{\emptyset\}\big) \\
& + \big(|\tilde{u}(\tilde{f}_2(\tilde{\omega}_2^1))| -_h |\tilde{u}(\tilde{f}_2(\tilde{\omega}_4^1))|\big)\tilde{\eta}\big(\{\tilde{\omega}_1^2, \tilde{\omega}_2^2, \tilde{\omega}_1^1, \tilde{\omega}_2^1\}, \{\emptyset\}\big) \\
& + \big(|\tilde{u}(\tilde{f}_2(\tilde{\omega}_4^1))| -_h |\tilde{u}(\tilde{f}_2(\tilde{\omega}_3^2))|\big)\tilde{\eta}\big(\{\tilde{\omega}_1^2, \tilde{\omega}_2^2, \tilde{\omega}_1^1, \tilde{\omega}_2^1\}, \{\tilde{\omega}_4^1\}\big) \\
& + \big(|\tilde{u}(\tilde{f}_2(\tilde{\omega}_3^2))| -_h |\tilde{u}(\tilde{f}_2(\tilde{\omega}_4^2))|\big)\tilde{\eta}\big(\{\tilde{\omega}_1^2, \tilde{\omega}_2^2, \tilde{\omega}_1^1, \tilde{\omega}_2^1, \tilde{\omega}_3^2\}, \{\tilde{\omega}_4^1\}\big) \\
& + \big(|\tilde{u}(\tilde{f}_2(\tilde{\omega}_4^2))| -_h |\tilde{u}(\tilde{f}_2(\tilde{\omega}_3^1))|\big)\tilde{\eta}\big(\{\tilde{\omega}_1^2, \tilde{\omega}_2^2, \tilde{\omega}_1^1, \tilde{\omega}_2^1, \tilde{\omega}_3^2,\}, \{\tilde{\omega}_4^1, \tilde{\omega}_4^2\}\big) \\
& + \big(|\tilde{u}(\tilde{f}_2(\tilde{\omega}_3^1))|\big)\tilde{\eta}\big(\{\tilde{\omega}_1^2, \tilde{\omega}_2^2, \tilde{\omega}_1^1, \tilde{\omega}_2^1, \tilde{\omega}_3^2, \tilde{\omega}_3^1\}, \{\tilde{\omega}_4^1, \tilde{\omega}_4^2\}\big) = (3.99, 7.49, 9.61)
\end{aligned}
\tag{5.35}
$$

with $\tilde{f}_1$ and $\tilde{f}_3$ calculated similarly. The values of overall utilities follow as $\tilde{U}(\tilde{f}_1) = (4, 6.92, 8.91)$ and $\tilde{U}(\tilde{f}_3) = (4.12, 6.23, 8.25)$. Finally, we rank these fuzzy numbers as $\tilde{f}_2 \succ \tilde{f}_1 \succ \tilde{f}_3$. After obtaining the fuzzy overall utilities of each act, BDM goes further and formulates the degrees of preferences among alternatives, the concept we will not discuss here in detail. The degrees of preferences among alternatives essentially capture the vagueness of preferences of DM in the fuzzy environment. One can refer to Aliev et al. (2013) for a detailed formulation of vague preferences.

5.3.5 Practitioner 5 - General theory of decisions

Now, suppose Practitioner 5 evaluates the same alternatives under the same economic conditions $S = \{\tilde{s}_1, \tilde{s}_2, \tilde{s}_3, \tilde{s}_4\}$ and has the same information as Practitioner 4. Unlike Practitioner 4, however, he has a degree of reliability (expressed in a natural language) of the given information. Specifically, he is <u>very sure</u> that each of his three actions will yield the same results as Practitioner 4. He is also <u>sure</u> about the probability assessment of Practitioner 4. The question remains the same.

At this information class, Practitioner 5 has imprecise and at the same time partially true information (Z-information). Following the arguments set forth for Practitioner 3, there are also two approaches of solving the optimal solution for Practitioner 5.

Approach 1. One way to proceed with the given problem is to use BDM and overlook the reliability of the given information at the first stage. The argument of Practitioner 3 with the illustration of Shafer (1987) applies here with the same logic it applied for Practitioner 3. With this approach, the resulting preferences are $\tilde{f}_2 \succ \tilde{f}_1 \succ \tilde{f}_3/\underline{sure}$.

Approach 2: The second approach uses the concept of Z-number suggested by Zadeh (2011). Formally, a Z-number is defined as an ordered pair $\hat{Z} = (\tilde{A}, \tilde{B})$ of fuzzy numbers to describe a value of a variable X. Here, $\tilde{A}$ is an imprecise constraint on values of a variable X and $\tilde{B}$ is an imprecise estimation of reliability of $\tilde{A}$. One can refer to Aliev, Alizadeh and Huseynov (2015) for the arithmetic of Z-numbers and to Aliev et al. (2016) for the general theory of decisions (GTD) on the basis of a Z-number concept. The GTD uses the idea of combined states argument of BDM and develops a unified decision model which subsumes most of the well-known decision theories as its special cases including BDM. We refer to the original paper of GTD for the details.

5.4 Paradoxes and Rationality

In modern economics literature, there is a lot of evidence contradicting the preference of Savage's axioms as well as the theory itself as a valid representation of rationality. The evidence ranges from Ellsberg (1961) to Kahneman and Tversky (1979). However, over the years, the compiled evidence is regarded as an irrationality of economic agents while Savage's axioms retained its normative ground in economics and finance. Hence, different paradigms such as efficient markets hypothesis and behavioral finance are created. It is this dogmatic view that we aimed to discuss in this chapter by using the imprecision and reliability of real-world information and existing decision theory literature. An approach on the basis of imprecision and reliability of information enables us to judge when a subjectively rational belief obeys the probability calculus and when it is less structured.

This type of rationality is not the first time introduced to the economics and finance literature. For example, the famous response of Thaler (1980) to Friedman and Savage (1948) billiard player analogy essentially suggests that acting on the basis of prospect theory may be judged as rational. The flopping of a fish analogy of Lo (2004) suggests that the same motion (flopping) makes a fish rational in one environment (underwater) and makes it irrational in another environment (dry land). The analysis thus far essentially reveals the information-based picture of these environments. We next revise two of the existing well-known paradoxes of behavioral finance.

5.4.1 Insurance and gambling

Buying both insurance and lottery tickets is a norm rather than an exception and it is hard to reconcile with classical rational decision making. Buying insurance means a DM chooses a certainty in preference to uncertainty, whereas buying a lottery ticket suggests choosing uncertainty in preference to certainty. To see how both, insurance and gambling, can be rationalized by BDM, consider the following example.

Alice considers to buy fire insurance for her house. She notes *a small loss* (insurance premium) under $\tilde{s}_1$ (no fire) and *a very large gain* under $\tilde{s}_2$ (fire) if she buys the insurance ($\tilde{f}_1$). She also notes *a very large loss* under $\tilde{s}_2$, while *nothing* happens under $\tilde{s}_1$ if she does not buy the insurance ($\tilde{f}_2$). Fire occurs with a *very small* probability, $\tilde{P}(\tilde{s}_2)$. Table 5.7 summarizes the gains and losses of Alice under different circumstances.

	$\tilde{s}_1$ (no fire)	$\tilde{s}_2$ (fire)
$\tilde{f}_1$ (buy)	a small loss	a very large gain
$\tilde{f}_2$ (don't buy)	no loss/gain	a very large loss

Table 5.7: Fire Insurance

Alice also considers to buy a lottery ticket in the hope of winning the mega jackpot. She notes *a very small loss* (ticket price) under $\tilde{s}_1'$ (not win) and *a very large gain* under $\tilde{s}_2'$ (win) if she buys a ticket ($\tilde{f}_1'$). She feels *nothing* if she does not buy a ticket ($\tilde{f}_2'$). Probability of winning $\tilde{P}(\tilde{s}_2')$ is *very small*. The probability of her risk-aversion $\tilde{P}(h_1)$ is approximately 70 % and she is known to be risk-seeking (h_2) when she is not risk-averse. The bet is summarized in Table 5.8.

	$\tilde{s}_1'$ (not win)	$\tilde{s}_2'$ (win)
$\tilde{f}_1'$ (buy)	a very small loss	a very large gain
$\tilde{f}_2'$ (don't buy)	no loss/gain	no loss/gain

Table 5.8: Lottery ticket

Friedman and Savage (1948) suggest an S-shaped utility function to rationalize this behavior and the approach is criticized by Markowitz (1952). Since the value function in Eqs. (5.18) and (5.19) follow from the Prospect Theory, one can follow the steps of BDM for each case (assign utilities, find FJPs, construct fuzzy-valued bi-capacities, and aggregate with the generalized Choquet-like aggregation) and verify that, Alice should not be ashamed of buying both a lottery ticket and fire insurance at the same time.

5.4.2 Equity premium puzzle

The equity premium puzzle (first noted by Mehra and Prescott (1985)) refers to the large difference between the average equity returns and average returns of a fixed interest-bearing bonds. To see how equity premium puzzle can be rationalized consider the following simple example.

Bob is an (only) investor with an initial wealth of W_0 who can invest in two assets, a risky stock with a price of p and an uncertain payoff R_s in state $s \in S$, and a bond with a unit price and a certain payoff R. Suppose he invests a units of stock and b units of bond. Further denote $\pi(s)$ as an additive probability distribution over the state s. The end-of-period wealth is $W_s = R_s \cdot a + R \cdot b$. Using a budget constraint, $W_0 = p \cdot a + b$, we obtain the end-of-period wealth as $W_s = R \cdot W_0 + [R_s - R \cdot p] \cdot a$. First consider Bob as EU maximizer as a benchmark case,

$$U(W_1, ..., W_s) = \sum_{s \in S} \pi_s \cdot u(W_s) = \sum_{s \in S} \pi_s \cdot u(R \cdot W_0 + [R_s - R \cdot p] \cdot a). \quad (5.36)$$

Without loss of generality, for an equilibrium stock price of p_0 with a total investment in stock, $a > 0$, and bonds, $b = 0$, Bob maximizes his total utility

$$U'(a) = \sum_{s \in S} \pi_s \cdot [R_s - R \cdot p_0] \cdot u'(R_s \cdot a) = 0. \quad (5.37)$$

The stock price p_0 follows from the first order condition in Eq. (5.37) as

$$p_0 = \frac{\sum_{s \in S} \pi_s \cdot R_s \cdot u'(R_s \cdot a)}{R \cdot \sum_{s \in S} \pi_s \cdot u'(R_s \cdot a)}. \tag{5.38}$$

The benchmark equity premium is given by

$$\tau(p_0) = \frac{\sum_{s \in S} \pi(s) \cdot R_s}{p_0 \cdot R}. \tag{5.39}$$

Now suppose, the preferences of Bob are represented by α-MEU as

$$U(W_1, ..., W_s) = \alpha \cdot \min\{u(W_1), ..., u(W_s)\} + (1-\alpha) \cdot \max\{u(W_1), ..., u(W_s)\}, \tag{5.40}$$

where α denotes ambiguity aversion. Denote $\bar{R} = \max\{R_1, ..., R_s\}$ and $\underline{R} = \min\{R_1, ..., R_s\}$. Then, for $a > 0$, the total utility of investing in a amount of stock is

$$U(a) = \alpha \cdot u(R \cdot W_0 + (\underline{R} - R \cdot p) \cdot a) + (1 - \alpha) \cdot u(R \cdot W_0 + (\bar{R} - R \cdot p) \cdot a). \tag{5.41}$$

Similarly, without loss of generality, for an equilibrium stock price of p with a total investment in stock, $a > 0$, and bonds, $b = 0$, Bob maximizes his total utility

$$U'(a) = \alpha \cdot u'(\underline{R} \cdot a) \cdot (\underline{R} - R \cdot p) + (1 - \alpha) \cdot u'(\bar{R} \cdot a) \cdot (\bar{R} - R \cdot p) = 0. \tag{5.42}$$

Again, the equilibrium stock price p follows from the first order condition in Eq. (5.42) as

$$p = \frac{\alpha \cdot u'(\underline{R} \cdot a)\underline{R} + (1 - \alpha) \cdot u'(\bar{R} \cdot a)\bar{R}}{\alpha \cdot R[u'(\underline{R} \cdot a) - u'(\bar{R} \cdot a)] + u'(\bar{R} \cdot a) \cdot R}. \tag{5.43}$$

The equity premium when Bob has $\alpha-$MEU preferences is given by

$$\tau(p) = \frac{\sum_{s \in S} \pi(s) \cdot R_s}{p \cdot R}. \tag{5.44}$$

To compare the equity premium in the benchmark EU case in Eq. (5.39) and the $\alpha-$MEU case in Eq. (5.44), we first assume Bob is risk neutral (i.e., $u'(\cdot) = c$).

Case 1 (a). The equity premium increases in the ambiguity aversion of the investor.

It follows from Eqs. (5.39) and (5.44) that it is sufficient to show that the equilibrium stock price decreases with the ambiguity aversion α due to an inverse relationship between the equity premium and the stock price. For $u'(\cdot) = c$, the equilibrium stock price in Eq. (5.43) reduces to

$$p = \frac{\alpha \cdot (\underline{R} - \bar{R}) + \bar{R}}{R}. \tag{5.45}$$

Since $\underline{R} < \bar{R}$, the equilibrium stock price will be the lower and the equity premium will be the higher the more ambiguity averse the investor is.

Case 1 (b). For $\alpha > 1/2$, the equity premium $\tau(p)$ exceeds the benchmark $\tau(p_0)$ when the expected return exceeds the average of the minimum and maximum returns, i.e.,

$$\frac{(\underline{R} + \bar{R})}{2} < E_\pi[R_s]. \tag{5.46}$$

For $u'(\cdot) = c$, the benchmark stock price in Eq. (5.39) reduces to

$$p_0 = \frac{\sum_{s \in S} \pi_s \cdot R_s}{R} = \frac{E_\pi[R_s]}{R}. \tag{5.47}$$

Combining Eqs. (5.45) and (5.47) obtains

$$p = p_0 + \frac{1}{R} \cdot \left[\alpha \cdot \underline{R} + (1 - \alpha) \cdot \bar{R} - E_\pi[R_s]\right]. \tag{5.48}$$

The necessary and sufficient condition for $p < p_0$ and $\tau(p) > \tau(p_0)$ follows from Eq. (5.48) as

$$\alpha \cdot \underline{R} + (1 - \alpha) \cdot \bar{R} < E_\pi[R_s], \tag{5.49}$$

which reduces to

$$\frac{(\underline{R} + \bar{R})}{2} < E_\pi[R_s] \tag{5.50}$$

for $\alpha = 1/2$. Since $\tau(p)$ increases in α (Case 1(a)), the condition must be true for $\alpha > 1/2$.

Case 1 (c). For $\alpha = 1$ (full ambiguity aversion), the equity premium with ambiguity $\tau(p)$ always exceeds the benchmark $\tau(p_0)$.

It follows from Eq. (5.45) that for $\alpha = 1$,

$$p = \frac{\underline{R}}{R} < \frac{E_\pi[R_s]}{R} = p_0, \tag{5.51}$$

leading to $\tau(p) > \tau(p_0)$.

Now consider the second case in which Bob is a risk-averse investor with a strictly decreasing marginal utility function. Then, the following result is immediate.

Case 2. The equity premium $\tau(p)$ of a risk-averse investor with full ambiguity-aversion (i.e., $\alpha = 1$) exceeds the benchmark equity premium $\tau(p_0)$ of a risk-averse investor without ambiguity aversion.

In this scenario, the benchmark stock price is given by Eq. (5.38). For the risk- and ambiguity-averse investor with $\alpha = 1$, however, the stock price is

$$p = \frac{\underline{R}}{R} \tag{5.52}$$

Comparing Eqs. (5.38) and (5.52) obtains $p_0 > p$, and therefore, $\tau(p) > \tau(p_0)$ following

$$\frac{\sum_{s \in S} \pi_s \cdot R_s \cdot u'(R_s \cdot a)}{\sum_{s \in S} \pi_s \cdot u'(R_s \cdot a)} > \underline{R}. \tag{5.53}$$

With the example of Bob, we have only added an ambiguity attitude of the DM in the sense of Ghirardato et al. (2004) to his risk attitude and confirmed that ambiguity aversion requires an incremental equity premium. In this context, any pessimistic attitude adds an incremental requirement for the equity premium and completes the pieces of the puzzle.[58] So deeply rooted is our commitment to EUT and SEU, that we regard such patterns as paradoxical, or irrational. As we move away from the ground level of the information many of the paradoxes, it turns out, can be rationalized by more general decision theories.

---

[58]The results of this simple example with a representative investor (Bob) carry over to more general models of financial markets (e.g., Epstein and Schneider (2008), Ju and Miao (2012)). These results are also consistent with liquidity dry-ups due to ambiguity premium on the bid-ask spread in Chapter 2 and composition uncertainty premium in Chapter 3 when the traders are sufficiently uncertainty averse.

5.5 Fuzzy Representation of Market Efficiency

The key to understanding the main difference between EMH and BF comes down to the probability (classical set theory) vs. possibility (fuzzy set theory). These theories sit on the two extreme sides of the Table 5.1. Probability calculus lies in the foundation of EMH, whereas fuzzy set theory lies in the foundation of BF.[59] Since fuzzy sets are the generalized version of classical sets, the concept of market efficiency itself can become a fuzzy concept. In this section, we represent market efficiency as a fuzzy concept. In view of the presented arguments so far, we develop the following conjectures. Critical reasoning and casual empiricism are the only pillars of the proposed conjectures.

Conjecture 5.1. Imprecision and reliability of information lead to different opinions.

Conjecture 5.2. Imprecision and reliability of information lead to "behavioral biases" in financial decision making.

Conjecture 5.3. Imprecision and reliability of aggregate information in financial markets lead to more "behavioral anomalies" observed in the market.

We consider Conjecture 5.1 as an economic primitive. Conjecture 5.2 is the natural extension of Conjecture 5.1. Finally, in aggregate, Conjecture 5.2 leads to Conjecture 5.3. Note that the argument here is different from the argument of Friedman (1953) that in aggregate noise traders cancel each other and De Long, Shleifer, Summers and Waldmann (1990) response. Here we do not conjecture the survival of noise traders in financial markets. The conjecture is rather focusing on the imperfect information that is received by everyone and perceived differently.

The idea behind these conjectures is too simple to digest. Fact vs. opinion argument presented at the beginning helps to understand the intuition. If the market participants receive fully-reliable simple numerical information (pure fact) then it is no exaggeration to say there is a homogeneous belief in the market. However, if the received information is fuzzy (e.g., medium growth) and partially true (e.g., sure), different perceptions of a natural language lead to different opinions, "behavioral biases", and in aggregate, "behavioral anomalies". Also , in the current framework with a concept of subjective rationality, we are not in favor of calling

---

[59]Peters (1996) discusses how fuzzy membership functions can be used to understand some of the prominent behavioral biases.

them as "biases" or "anomalies" since we argue that when information becomes imprecise and unreliable, behavioral factors become one's only strength to play with. They are called biases and anomalies because it is odd to explain them with the probabilistic calculus.

In essence, our argument is as follows. *The room for behavioral finance increases as we move from the left to the right in Table 5.1.* This in turn illuminates a general view of market efficiency that is different from the presented arguments of the two main paradigms of finance. The beliefs represented by conceptually different theories at the two edges of the information classes separately support EMH and BF. Therefore, it seems to us efficient and inefficient markets supported by the fundamentally different beliefs have a certain truthness degree depending on the dominance of different information classes in financial markets.

Graphically, it also seems natural to us that when asset prices oscillate, for example, between P' and P illustrated in Figure 5.1 it leaves efficient and inefficient markets as its special cases without sharp boundaries.
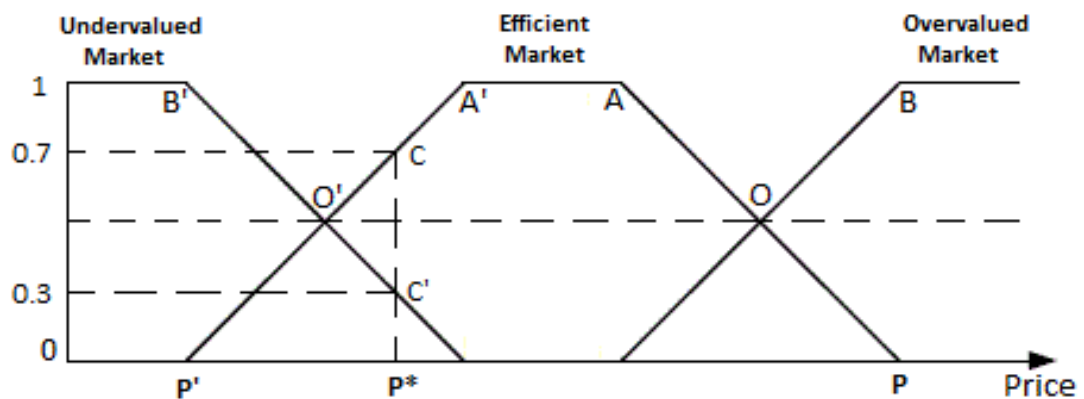
Figure 5.1: Fuzzy representation of market efficiency

Figure 5.1 subsumes 3 specific fuzzy sets, namely, "undervalued market", "efficient market" and "overvalued market" consistent with the model presented in chapter 3. The abscissa axis shows the price level of the market and the ordinate axis shows the degree of market efficiency represented by a fuzzy membership ranging between 0 and 1. When the membership to the specific set is 0 it means certain price level is not the member of the set, whereas when membership is 1 it shows certain price level fully belong to that specific set. For example, at P' and P the degree of efficiency of the market is 0 and the degree of undervaluation is 1 at P' and the degree of overvaluation is 1 at P. Moreover, at P^*, the degree of efficiency

is 0.7 and the degree of undervaluation is 0.3. It is still far from clear for us what measures to be used as a proxy for imprecision and reliability of the aggregate information in financial markets and approximate the degree of market efficiency at a given point in time.

There are two subtle reasons that this direction to be distinguished from both paradigms. Firstly, the existence of efficient markets is not entirely excluded in this framework, nevertheless, it is different from the current weak-form, semi-strong form and strong-form market efficiency concepts. Also, BF regards the probability based decision-techniques as its superior, though the subjective rationality in our argument can take behavioral factors into account when the information of a DM becomes imprecise and partially reliable.

5.6   Conclusion

The exposition of a more general view of financial markets in this chapter shows how the existing decision theory and information science literature can be used to better understand financial markets. The points which we have attempted to convey should now be clear: (i) Classical probability theory favors EMH and rules out a possible room for BF. (ii) Individual behavioral "biases" and aggregate market "anomalies" mainly originate from imprecision and reliability attributes of information. (iii) Decision theories built on the foundations of capacities, bi-capacities, set of probabilities, and in a more general setting, fuzzy set theory can be used to complement the probability based decision theories to rationalize most, if not all, of the existing behavioral "anomalies".

Appendix 5. Mathematical Preliminaries

In this Appendix, we provide mathematical preliminaries that are essential to understand candidate decision theories.

Capacities and Choquet Integral. Capacities replace the additivity requirement of classical measures with a less restrictive requirement of monotonicity. Let Ω be a universal set characterizing the *states of nature* and $\mathcal{F}(\Omega)$ its non-empty power set with appropriate algebraic structure characterizing the *events*. A capacity is a real-valued set function, $\eta(A)$, defined on the set A of events $\mathcal{F}(\Omega)$ that is normalized ($\eta(\emptyset) = 0$, $\eta(\Omega) = 1$) and monotonic (for all A, B in $\mathcal{F}(\Omega)$, $A \subseteq B \Rightarrow \eta(A) \leq \eta(B)$). Additional continuity conditions (below and above) are required when Ω is infinite. Suppose $\Omega = \{\omega_i\}_{i=1}^{n+1}$ is finite. Without loss of generality we can rank a non-negative (utility) function $f(\omega_k)$ on Ω as $f(\omega_1) \geq f(\omega_2) \geq ... \geq f(\omega_n)$ and $f(\omega_{n+1}) = 0$. Then, the expected value (i.e. Choquet integral) of f on Ω with respect to a capacity η is expressed as

$$E_\eta[f] = \sum_{k=1}^{n}(f(\omega_k) - f(\omega_{k+1}))\eta(\{\omega_1, \omega_2, ..., \omega_k\}). \qquad (A5.1)$$

The mathematical treatment of Choquet capacities and integral may be found in Choquet (1955), Dempster (1967), Shafer (1976), and Schmeidler (1986, 1989).

Bi-Capacities and Choquet-like Aggregation. Bi-capacities are natural generalization of capacities in the context of decision making where underlying scales are bipolar as in the prospect theory. Let $\mathcal{Q}(\Omega) := \{(A, B) \in \mathcal{F}(\Omega) \times \mathcal{F}(\Omega) \mid A \cap B = \emptyset\}$ denote the set of all pairs of disjoint sets. A bi-capacity is a real-valued set function, $\eta(A, B)$, defined on $\mathcal{Q}(\Omega)$ that is normalized ($\eta(\emptyset, \emptyset) = 0$, $\eta(\Omega, \emptyset) = 1 = -\eta(\emptyset, \Omega)$) and monotonic (for all A, B in $\mathcal{Q}(\Omega)$, $A \subseteq B \Rightarrow \eta(A, \cdot) \leq \eta(B, \cdot)$ and $\eta(\cdot, A) \geq \eta(\cdot, B)$).[60] Suppose $\Omega = \{\omega_i\}_{i=1}^{n+1}$ is finite. Without loss of generality we rank a real-valued function $f(\omega_k)$ on Ω as $|f(\omega_1)| \geq |f(\omega_2)| \geq ... \geq |f(\omega_n)|$ and $f(\omega_{n+1}) = 0$. Then, the expected value (i.e., Choquet-like aggregation) of f on Ω with respect to a bi-capacity η is expressed as

$$E_\eta[f] = \sum_{k=1}^{n} \left(|f(\omega_{(k)})| - |f(\omega_{(k+1)})|\right)\eta\left(\{\omega_{(1)}, ...\omega_{(k)}\} \cap N^+, \{\omega_{(1)}, ...\omega_{(k)}\} \cap N^-\right), \; (A5.2)$$

where $N^+ = \{\omega \in \Omega \mid f(\omega) \geq 0\}$ and $N^- = \{\omega \in \Omega \mid f(\omega) < 0\}$.

---

[60] The same symbol, η, is used for both, capacities and bi-capacities. This should not create any notational confusion since $\eta(\cdot)$ is a capacity and $\eta(\cdot, \cdot)$ is a bi-capacity. A fuzzy version of a bi-capacity $\eta(\cdot, \cdot)$ is further denoted by a superimposed tilde, $\tilde{\eta}(\cdot, \cdot)$.

Example A5.1. Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and the function f on Ω takes the values of $f(\omega_1) = 4$, $f(\omega_2) = 3$, and $f(\omega_3) = -2$. Then $N^+ = \{\omega_1, \omega_2\}$, $N^- = \{\omega_3\}$,

$$
\begin{aligned}
E_\eta[f] =& \big(|f(\omega_1)| - |f(\omega_2)|\big)\eta(\{\omega_1\}, \{\emptyset\}) \\
& + \big(|f(\omega_2)| - |f(\omega_3)|\big)\eta(\{\omega_1, \omega_2\}, \{\emptyset\}) + |f(\omega_3)|\eta(\{\omega_1, \omega_2\}, \{\omega_3\}) \\
=& \eta(\{\omega_1\}, \{\emptyset\}) + \eta(\{\omega_1, \omega_2\}, \{\emptyset\}) + 2\eta(\{\omega_1, \omega_2\}, \{\omega_3\}).
\end{aligned}
$$

The mathematical treatment of bi-capacities and Choquet-like aggregation can be found in Grabisch and Labreuche (2005a, 2005b) and Labreuche and Grabisch (2006).

Fuzzy Set Theory. The ideas of fuzzy sets and fuzzy logic date back to Black (1937) and it has been mathematically formalized by Zadeh (1965). The most common type of fuzzy sets is the standard fuzzy sets. Each of the standard fuzzy sets is uniquely defined by a membership function of the form $\mu_{\tilde{A}} : \Omega \to [0, 1]$, where Ω denotes a universal set and $\tilde{A}$ is a fuzzy subset of Ω. Since a characteristic function of classical sets is a special case of a membership function of fuzzy sets, $\{0, 1\} \subseteq [0, 1]$, fuzzy sets are considered a formal generalization of classical sets.

The three basic operations on sets - complementation, intersection, and union - are not unique in fuzzy sets as they are in classical sets. The standard complement of a fuzzy set $\tilde{A}$ is a fuzzy set $\tilde{A}^c$ with the membership function $\mu_{\tilde{A}^c} = 1 - \mu_{\tilde{A}}$. The standard intersection of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is a fuzzy set with the membership function $\mu_{\tilde{A} \cap \tilde{B}}(\omega) = \min\{\mu_{\tilde{A}}(\omega), \mu_{\tilde{B}}(\omega)\}$ and the standard union of two fuzzy sets is also a fuzzy set with $\mu_{\tilde{A} \cup \tilde{B}}(\omega) = \max\{\mu_{\tilde{A}}(\omega), \mu_{\tilde{B}}(\omega)\}$, where $\omega \in \Omega$ (see Bellman and Giertz (1973) for axiomatization of these standard operations). In addition, a fuzzy set $\tilde{A}$ is said to be a subset of fuzzy set $\tilde{B}$, $\tilde{A} \subseteq \tilde{B}$, if and only if $\mu_{\tilde{A}}(\omega) \leq \mu_{\tilde{B}}(\omega)$, $\forall \omega \in \Omega$ given that fuzzy sets $\tilde{A}$ and $\tilde{B}$ are defined on Ω.

One of the most important concepts of fuzzy sets is an α-cut of a fuzzy set which is one way of connecting fuzzy sets to classical sets. An α-cut of a fuzzy set $\tilde{A}$ on Ω denoted as $^\alpha A$ is a classical set that satisfies $^\alpha A = \{\omega \mid \mu_{\tilde{A}}(\omega) \geq \alpha\}$, where $\alpha \in [0, 1]$. A strong α-cut, denoted as $^{\alpha+}A$, is similar to the α-cut representation, $^{\alpha+}A = \{\omega \mid \mu_{\tilde{A}}(\omega) > \alpha\}$, but with a stronger condition. ^{0+}A and ^1A are called support and core of a fuzzy set $\tilde{A}$, respectively. When the core of a fuzzy set $\tilde{A}$ is not empty, $^1A \neq \emptyset$, $\tilde{A}$ is called normal, otherwise it is called subnormal. A fuzzy set is convex if and only if all its α-cuts are convex sets as in the classical sense.

Definition A5.2. A fuzzy set $\tilde{A}$ on $\mathcal{R}$ (a set of real numbers) is a fuzzy number if (i) $\tilde{A}$ is a normal fuzzy set, (ii) $^{\alpha}A$ is a closed interval for every $\alpha \in (0, 1]$ and (iii) the support of $\tilde{A}$ is bounded.

When fuzzy numbers are used to formulate linguistic concepts such as very small, small, and so on, the final constructs are called linguistic variables.

Definition A5.3. Let $\mathcal{E}^n$ be a space of all fuzzy subsets of $\mathcal{R}^n$ consisting of fuzzy sets which are normal, fuzzy convex, upper semi-continuous with compact support. A fuzzy function is a mapping from universal set Ω to $\mathcal{E}^n$, $\tilde{f} : \Omega \to \mathcal{E}^n$.

Definition A5.4. Let $\tilde{A}, \tilde{B} \in \mathcal{E}^n$. If there exists $\tilde{C} \in \mathcal{E}^n$ such that $\tilde{A} = \tilde{B} + \tilde{C}$, then $\tilde{C}$ is called a Hukuhara difference $(-_h)$ of $\tilde{A}$ and $\tilde{B}$.

Example A5.5. Let $\tilde{A}$ and $\tilde{B}$ be triangular fuzzy sets $\tilde{A} = (5, 7, 9)$ and $\tilde{B} = (1, 2, 3)$. Then, $\tilde{A} -_h \tilde{B} = (5, 7, 9) - (1, 2, 3) = (5 - 1, 7 - 2, 9 - 3) = (4, 5, 6)$. Hence, $\tilde{B} + (\tilde{A} -_h \tilde{B}) = (1, 2, 3) + (4, 5, 6) = (5, 7, 9) = \tilde{A}$.

Definition A5.6. Given a fuzzy number $\tilde{A}$ on Ω, absolute value $|\tilde{A}|$ is defined as

$$\mu_{|\tilde{A}|}(\omega) = \begin{cases} \max\left(\mu_{\tilde{A}}(\omega), \mu_{-\tilde{A}}(\omega)\right) & \text{for } \omega \in \mathcal{R}^+, \\ 0 & \text{for } \omega \in \mathcal{R}^-. \end{cases} \tag{A5.3}$$

The generalizations of probability theory to the theory of capacities, bi-capacities and fuzzy sets expand the classical probabilistic framework of information uncertainty substantially. The mathematical treatment of fuzzy set theory can be found in Klir and Yuan (1995), uncertainty based analysis of related topics in Klir (2005), and applications to decision theories in Aliev and Huseynov (2014).

Chapter 6

Conclusion and Future Research

It appears pretentious to have a conclusion section in this dissertation. The dissertation is only the tip of the iceberg of ways how multiple dimensions of uncertainty and various attributes of information can increase our understanding of various financial market phenomena. The models proposed in this dissertation are far from reflecting the ultimate truth about the true complexity of real-world information. However, they suggest likely qualitative behavior of financial markets and market participants.

This thesis investigates the issues related to stability and instability in financial markets and highlights the importance of different dimensions of information and uncertainty, and decision theories in addressing them. Table 6.1 illustrates the summary of the economic issues, corresponding decision theories, uncertainty types, and formalized languages of uncertainty in this thesis.

6.1 Uncertainty about the Fundamental Value

In a sequential trading model, Chapter 2 represents the preferences of liquidity providers with the Choquet expected utility of Schmeidler (1989) and their beliefs about payoffs with the neo-additive capacities of Chateauneuf et al. (2007) and investigates the effects of ambiguity on liquidity, value of information, welfare, and price-quantity relations in financial markets.

Chapter	Economic Issue	Decision Theory	Uncertainty Type & Theory
2	Liquidity & Information value & Welfare & Price-Quantity analysis	Choquet expected utility theory	Fundamental value uncertainty with non-additive probabilities
3	Liquidity and asset pricing & Benefit of informed trading & Complementarity & Sharp price movements	Maxmin and $\alpha-$maxmin expected utility theory	Fundamental value uncertainty with numerical probabilities & interval-valued composition uncertainty
4	Order imbalance & Flow toxicity & Flash crashes & Flash rallies	Expected utility theory	Fundamental value, composition & signal quality uncertainty with numerical probabilities
5	Subjective rationality & Market efficiency & Behavioral "anomalies"	Expected utility, Choquet, maxmin, $\alpha-$maxmin expected utility, behavioral decision making & general theory of decisions	Numerical, interval-valued, second-order uncertain probabilities, fuzzy & Z uncertainty

Table 6.1: Summary of the thesis

One interesting interpretation of this approach can be given by the classical discussion of Weaver (1948). In his classic paper, Weaver (1948) refers to two extreme problems in science, the problems of *"organized simplicity"* with little complexity and randomness and the problems of *"disorganized complexity"* with extreme complexity and randomness. He emphasizes that most of the real-world problems are somewhere in between these two extremes, what he calls the problems of *"organized complexity"*. In the language of Warren Weaver, the representation of the market maker's beliefs with neo-additive capacities captures the *"organized complexity"* in the modern liquidity provision.

The representation of liquidity providers' beliefs recognizes that uncertainty is too broad to be captured by the probabilistic beliefs alone and suggests a broader treatment of uncertainty with appropriate decision theory as one way of generalizing financial market models. The purpose of this chapter is to demonstrate the potentials of such generalizations in addressing the aberrant behavior of liquidity and its consequences.

6.1.1 Liquidity deteriorations and improvements

The representation of market making decisions with Choquet Expected Utility is intuitive and general enough to accommodate sudden liquidity improvements during financial crisis and deteriorations during reforms in trading rules. The ambiguous price formation model shows that ambiguity about the fundamental value and ambiguity aversion of the market maker can impact the perceived adverse selection risk (number of informed traders) of the market maker, resulting in the liquidity distortions (either improvements due to ambiguity discount or deteriorations due to ambiguity premium on the bid-ask spread) which are consistent with the historical financial market behavior.

6.1.2 Value of information and welfare

The resulting liquidity distortions can make private information more or less valuable to market participants compared to the standard probabilistic model. Chapter 2 derives the conditions under which the information is more or less valuable to market participants. Consistent with the intuition, the ambiguity premium (resp. discount) on the bid-ask spread leads to a value discount (resp. premium)

on the standard value of information. To investigate the welfare implications of ambiguity of liquidity providers, Chapter 2 introduces noninformational trading motives to uninformed traders. Consequently, it shows that uninformed traders trade more (resp. less) when there is an ambiguity discount (resp. premium), leading to a welfare gain (resp. loss) to society.

6.1.3 Price-Quantity analysis

Chapter 2 also examines the quotes and bid-ask spread of the market maker and the profit maximization strategies of the informed traders in the separating and pooling equilibria in the presence of ambiguity. When the market maker is sufficiently ambiguity averse, ambiguity of the liquidity providers leads the informed traders to separate themselves by trading only large orders, resulting in a separating equilibrium. This implies large orders are more likely to be informed orders during highly ambiguous market events.

6.2 Uncertainty about the Composition of Traders

To investigate the crowded-trade problem in financial markets, Chapter 3 develops a rational expectations equilibrium model with ambiguity about the composition of traders. This chapter represents the traders' preferences with the maxmin expected utility of Gilboa and Schmeidler (1989) and $\alpha-$maxmin model of Ghirardato et al. (2004) with the composition of traders given by interval values. An interval-valued uncertainty is a natural way of taking into account ambiguous information about the composition of traders and takes a step from the ground level to the first level in Table 5.1. This representation is general enough to investigate the asset pricing implications of the financial market microstructure.

6.2.1 Liquidity and asset prices

The analysis in Chapter 3 links the asset prices to liquidity and provides a theoretical explanation for the empirical findings about the impacts of liquidity on prices. When uncertainty about the composition of traders increases, the uncertainty-averse uninformed traders change their liquidity provision (decrease if they are

sufficiently uncertainty averse or increase if they are not), leading the stock price to deviate from the efficient benchmark price without uncertainty about the composition of traders.

6.2.2   Market overvaluation and undervaluation

Numerous authors have documented a stock market overvaluation (e.g., dot-com bubble) and undervaluation (e.g., global financial crisis) that are hard to explain with the standard asset pricing models. Uncertainty about the composition of traders offers an alternative unified explanation for the stock mispricing. In the presence of uncertainty about the composition of traders, the sufficiently (resp. insufficiently) uncertainty-averse uninformed traders perceive the equity premium more (resp. less), leading to a stock undervaluation (resp. overvaluation), explaining the pricing implications of liquidity.

6.2.3   Complementarity in information acquisition

The literature has shown different mechanisms that cause strategic complementarities (i.e., more agents acquire information and makes it more valuable for the uninformed agents to acquire private information) in information acquisition and multiple information market equilibria. The analysis in Chapter 3 also shows that ambiguity about the composition of traders can cause the complementarity in information acquisition. This occurs because the insufficiently (resp. sufficiently) uncertainty-averse traders' benefit of informed trading increases (resp. decreases) with the number of informed traders. The sufficiency condition itself, however, decreases with the number of informed traders, meaning that a trader with a given uncertainty aversion can be insufficiently (resp. sufficiently) uncertainty averse for a low (resp. high) informed trading. Consequently, the benefit of informed trading becomes non-monotonic in the number of informed traders.

6.2.4   Sharp price movements

The strategic complementarity in information acquisition and equilibrium multiplicity in information market stemming from the composition uncertainty can

explain the extreme price jumps that occur nearly every day. Jumping from one information market equilibrium to the other equilibrium can cause a large price drop or surge in financial markets. Alternatively, a sudden increase in the uncertainty about the composition of traders can also cause a significant increase (resp. decrease) in the traders' perceived equity premium which can result in a large price drop (resp. surge) in financial markets.

6.2.5 Capital market regulation

Chapter 3 shows that although the informed and uninformed traders are both subject to ambiguity about the composition of traders, the nature of trading disadvantages the uninformed compared to the informed traders. This is important because the capital market regulation is concerned with maintaining a level playing field for the market participants. The non-monotonicity of benefit of informed trading with respect to the number of informed traders shows that reducing the cost of information or greater disclosure rules such as the mark-to-market accounting legislation implemented in 2007 may counterintuitively result in lower informed trading and aggravate market efficiency. Thus the analysis suggests that decreasing the cost of fundamental information to enhance market efficiency may not work without maintaining a fair trading environment that treats traders equally about other aspects of the financial market microstructure.

6.3 Learning about the Adverse Selection

Unlike the previous approaches, Chapter 4 formalizes the uncertainty of the market participants' with numerical probabilities. Although probabilistic, the point of departure from the standard sequential trading model with only uncertainty about the fundamental value is that Chapter 4 takes different dimensions of uncertainty into account. More precisely, Chapter 4 investigates the destabilizing role of order imbalance in financial markets when the market participants are subject to the fundamental value uncertainty as well as the uncertainty about the adverse selection risk (composition and/or signal quality uncertainty). In essence, Chapter 4 incorporates the second-order uncertainty in the market maker's uncertainty (risk) about the adverse selection. The approach in this chapter takes a step from the ground level to the second level in Table 5.1 and helps to understand how markets

digest order imbalance and when they are most susceptible to imbalance-induced instability.

6.3.1 Liquidity evaporations

When the liquidity providers learn about the adverse selection risk (toxicity), order imbalance has two opposing effects on liquidity. First, it allows the market maker to learn about the fundamental value, making the market more liquid due to reduced uncertainty about the fundamental value. Second, it allows the market maker to revise her belief about the level of adverse selection risk upward, making the market less liquid due to increased perceived adverse selection risk. Chapter 4 shows that the second effect dominates the first and results in sudden liquidity evaporations in the face of order imbalance when the initial belief about the adverse selection risk is sufficiently low.

6.3.2 Extreme price movements

The model in Chapter 4 can also generate a sharp price decline (resp. surge) and a quick recovery similar to flash crashes (resp. rallies). This occurs because when the market maker is uncertain about the proportion of informed traders or the quality of their information, an order impacts the market maker's beliefs about the fundamental value in two ways: (i) a standard price discovery effect and (ii) a repricing history effect stemming from uncertainty about the adverse selection risk. By introducing the myopic market maker, Chapter 4 disentangles these two effects and shows that the repricing history effect causes accelerating price impacts during continuations in order flow resulting in sharp price changes and more informative reversals resulting in quick recoveries.

6.3.3 Prevalence of flash crashes

The model in Chapter 4 suggests two reasons for the prevalence of micro flash crashes in modern financial markets. First, the technological developments (e.g., rise of algorithmic trading, availability of market data) have amplified the uncertainty in asymmetric information problem of the modern liquidity providers

and made them vulnerable to order imbalance. Second, the increased competition in modern markets incentivizes learning about the time-varying adverse selection risk to ensure spreads always reflect the level of toxicity. Therefore, an efficient learning about the time-varying level of adverse selection is crucial for a liquidity provider to remain competitive in today's markets.

6.4 Different Languages of Uncertainty

Chapter 5 has special importance in this dissertation. The chapter breaks away from traditional approaches to market efficiency and contains a new concept and a new idea. By using different languages of uncertainty, this chapter discusses information in the broadest possible way that lends itself to possible quantitative scrutiny and develops a framework to argue that rationality is a broad and market efficiency is a fuzzy concept.

6.4.1 Subjective rationality

In the analysis of subjective rationality, Chapter 5 uses Zadeh (2011) classification of information (numerical, interval-valued, second-order uncertain, fuzzy, and Z information) based on its generality. This chapter exemplifies static decision-making scenarios in each level of information and solves them with appropriate decision theory. The specific decision-making scenarios show that as the information becomes imprecise and partially true, the decision making becomes more behavioral in nature, suggesting that the imprecision and reliability of information can connect efficient markets hypothesis and behavioral finance.

6.4.2 Fuzzy market efficiency

The efficient and inefficient markets sit on the two extreme sides of our information-theoretic framework. Probability calculus lies in the foundation of efficient markets hypothesis, whereas fuzzy set theory lies in the foundation of behavioral finance. Consequently, the concept of market efficiency itself can become a fuzzy concept, meaning that at a given point in time the market efficiency and inefficiency have a degree of truthness. Indeed, the real-world information is uncertain, imprecise

and partially true. The shift of the concept of market efficiency from the bivalent logic to fuzzy logic provides a more real and adequate approach to create a general model of financial markets. Such a shift may cause a natural concern for cautious economists. As we move from left to right in Table 5.1, some decision theories do not have an axiomatization in terms of preference over Savage acts and for some, the axiomatic backgrounds are yet to be developed. However, the models, examples, and heuristic arguments in this dissertation point in the same direction. The multiple dimensions of uncertainty formalized in different languages can explain financial market phenomena that we otherwise label as anomalies. It is probably worthwhile to formulate various financial market phenomena at different levels of information and uncertainty and work on this a little further.

6.5   Work Ahead

This dissertation raises many questions for future research. The discussions of model extensions at the end of each chapter provide natural directions to extend the models and generate a more realistic description of financial markets. Although we have tried to carry some of the extensions forward, we mainly focused on a single risky asset in this thesis. Therefore, one interesting, but at the same time necessary, direction of future research is to develop a framework with multi-dimensional uncertainty and multiple assets to explore the joint impact of complex information structures and diversification in financial markets.

It is also important to develop models where traders are uncertain about the various dimensions of uncertainty that are economically meaningful. For example, it would be interesting to investigate the effects of uncertainty about the reliability of private information of informed traders on the market efficiency. This is economically meaningful because in the era of the data revolution, stock market trading is increasingly dominated by informed trading, nevertheless the data revolution brought the risk of unreliable information by itself. Therefore, the naive intuition that the data revolution will lead to greater market efficiency can easily fail due to the risk of fake information. This intuition is consistent with the recent empirical evidence by Bai, Philippon and Savov (2016) that the revolution in information technology over the last 50 years has increased the price informativeness for the

S&P 500 firms, but decreased for the average public firms. A meaningful representation of the risk of unreliable information has a potential to justify this and associated financial market phenomena.

In a broader level, however, this dissertation emphasizes a consistent approach in developing financial market models at different information levels to explain economically interesting phenomena that escapes the theories with less complex information structures. Using broader languages to formalize uncertainty in different dimensions is challenging, but at the same time holds the key for the consistent and unified approach to explain various financial market regularities.

Bibliography

Admati, A. R. (1985), 'A noisy rational expectations equilibrium for multi-asset securities markets', *Econometrica* **53**(3), 629–658.

Ahn, D., Choi, S., Gale, D. and Kariv, S. (2014), 'Estimating ambiguity aversion in a portfolio choice experiment', *Quantitative Economics* **5**(2), 195–223.

Aliev, R., Alizadeh, A. and Huseynov, O. (2015), 'The arithmetic of discrete z-numbers', *Information Sciences* **290**, 134–155.

Aliev, R. and Huseynov, O. (2014), *Decision theory with imperfect information*, World Scientific.

Aliev, R., Pedrycz, W., Fazlollahi, B., Huseynov, O., Alizadeh, A. V. and Guirimov, B. (2012), 'Fuzzy logic-based generalized decision theory with imperfect information', *Information Sciences* **189**, 18–42.

Aliev, R., Pedrycz, W. and Huseynov, O. (2013), 'Behavioral decision making with combined states under imperfect information', *International Journal of Information Technology & Decision Making* **12**(03), 619–645.

Aliev, R., Pedrycz, W., Kreinovich, V. and Huseynov, O. (2016), 'The general theory of decisions', *Information Sciences* **327**, 125–148.

Aliyev, N. and He, X. (2018*a*), 'Ambiguous price formation', *Working Paper, UTS Business School* .

Aliyev, N. and He, X. (2018*b*), 'The pricing of composition uncertainty', *Working Paper, UTS Business School* .

Aliyev, N., He, X. and Putnins, T. (2018), 'Learning about toxicity: Why order imbalance can destabilize markets', *Working Paper, UTS Business School* .

Amihud, Y. (2002), 'Illiquidity and stock returns: cross-section and time-series effects', *Journal of Financial Markets* **5**(1), 31–56.

Amihud, Y. and Mendelson, H. (1986), 'Asset pricing and the bid-ask spread', *Journal of Financial Economics* **17**(2), 223–249.

Anand, A. and Venkataraman, K. (2016), 'Market conditions, fragility, and the economics of market making', *Journal of Financial Economics* **121**(2), 327–349.

Anderson, E. W., Ghysels, E. and Juergens, J. L. (2009), 'The impact of risk and uncertainty on expected returns', *Journal of Financial Economics* **94**(2), 233–263.

Antoniou, C., Harris, R. D. and Zhang, R. (2015), 'Ambiguity aversion and stock market participation: An empirical analysis', *Journal of Banking & Finance* **58**, 57–70.

Arrow, K. J. (1951), 'Alternative approaches to the theory of choice in risk-taking situations', *Econometrica* **19**(4), 404–437.

Asparouhova, E., Bossaerts, P., Eguia, J. and Zame, W. (2015), 'Asset pricing and asymmetric reasoning', *Journal of Political Economy* **123**(1), 66–122.

Avdis, E. (2016), 'Information tradeoffs in dynamic financial markets', *Journal of Financial Economics* **122**(3), 568–584.

Avery, C. and Zemsky, P. (1998), 'Multidimensional uncertainty and herd behavior in financial markets', *American Economic Review* **88**(4), 724–748.

Bachelier, L. (1900), *Théorie de la spéculation*, Gauthier-Villars.

Bai, J., Philippon, T. and Savov, A. (2016), 'Have financial markets become more informative?', *Journal of Financial Economics* **122**(3), 625–654.

Banerjee, S., Davis, J. and Gondhi, N. (2018), 'When transparency improves, must prices reflect fundamentals better?', *Review of Financial Studies* **31**(6), 2377–2414.

Banerjee, S. and Green, B. (2015), 'Signal or noise? uncertainty and learning about whether other traders are informed', *Journal of Financial Economics* **117**(2), 398–423.

Barberis, N., Shleifer, A. and Vishny, R. (1998), 'A model of investor sentiment', *Journal of Financial Economics* **49**(3), 307–343.

Barclay, M. J., Christie, W. G., Harris, J. H., Kandel, E. and Schultz, P. H. (1999), 'Effects of market reform on the trading costs and depths of nasdaq stocks', *Journal of Finance* **54**(1), 1–34.

Barlevy, G. and Veronesi, P. (2000), 'Information acquisition in financial markets', *Review of Economic Studies* **67**(1), 79–90.

Barlevy, G. and Veronesi, P. (2003), 'Rational panics and stock market crashes', *Journal of Economic Theory* **110**(2), 234–263.

Barlevy, G. and Veronesi, P. (2008), 'Information acquisition in financial markets: a correction', *Working Paper, Federal Reserve Bank of Chicago* .

Bellman, R. and Giertz, M. (1973), 'On the analytic formalism of the theory of fuzzy sets', *Information Sciences* **5**, 149–156.

Bewley, T. F. (2002), 'Knightian decision theory. Part I.', *Decisions in Economics and Finance* **25**(2), 79–110.

Black, F. (1986), 'Noise', *Journal of Finance* **41**(3), 528–543.

Black, M. (1937), 'Vagueness. an exercise in logical analysis', *Philosophy of Science* **4**(4), 427–455.

Blume, L., Easley, D. and O'Hara, M. (1994), 'Market statistics and technical analysis: The role of volume', *Journal of Finance* **49**(1), 153–181.

Bondt, W. F. M. and Thaler, R. (1985), 'Does the stock market overreact?', *Journal of Finance* **40**(3), 793–805.

Bossaerts, P., Ghirardato, P., Guarnaschelli, S. and Zame, W. R. (2010), 'Ambiguity in asset markets: Theory and experiment', *Review of Financial Studies* **23**(4), 1325–1359.

Brennan, M. J. and Subrahmanyam, A. (1996), 'Market microstructure and asset pricing: On the compensation for illiquidity in stock returns', *Journal of Financial Economics* **41**(3), 441–464.

Brenner, M. and Izhakian, Y. (2018), 'Asset pricing and ambiguity: Empirical evidence', *Journal of Financial Economics* **130**(3), 503–531.

Breon-Drish, B. (2012), 'Asymmetric information in financial markets: Anything goes', *Working Paper, Stanford Graduate School of Business* .

Brunnermeier, M. K. and Pedersen, L. H. (2008), 'Market liquidity and funding liquidity', *Review of Financial Studies* **22**(6), 2201–2238.

Caskey, J. A. (2009), 'Information in equity markets with ambiguity-averse investors', *Review of Financial Studies* **22**(9), 3595–3627.

Cespa, G. and Foucault, T. (2014), 'Illiquidity contagion and liquidity crashes', *Review of Financial Studies* **27**(6), 1615–1660.

CFTC-SEC (2010a), 'Preliminary findings regarding the market events of May 6, 2010', *Staff Report* .

CFTC-SEC (2010b), 'Findings regarding the market events of May 6, 2010', *Staff Report* .

Chamley, C. (2010), 'Strategic complementarity of information in financial markets with large shocks', *Annals of Finance* **6**(1), 137.

Chateauneuf, A., Eichberger, J. and Grant, S. (2007), 'Choice under uncertainty with the best and worst in mind: Neo-additive capacities', *Journal of Economic Theory* **137**(1), 538–567.

Chiyachantana, C., Jain, P. K., Jiang, C. and Sharma, V. (2017), 'Permanent price impact asymmetry of trades with institutional constraints', *Journal of Financial Markets* **36**, 1–16.

Choquet, G. (1955), 'Theory of capacities', *Ann. Inst. Fourier (Grenoble)* **5**, 131–295.

Chordia, T., Roll, R. and Subrahmanyam, A. (2002), 'Order imbalance, liquidity, and market returns', *Journal of Financial Economics* **65**(1), 111–130.

Condie, S. and Ganguli, J. (2017), 'The pricing effects of ambiguous private information', *Journal of Economic Theory* **172**, 512–557.

De Long, J. B., Shleifer, A., Summers, L. H. and Waldmann, R. J. (1990), 'Noise trader risk in financial markets', *Journal of Political Economy* **98**(4), 703–738.

Dempster, A. P. (1967), 'Upper and lower probabilities induced by a multivalued mapping', *The Annals of Mathematical Statistics* **38**(2), 325–339.

Dempster, A. P. (1968), 'A generalization of bayesian inference', *Journal of the Royal Statistical Society. Series B* **30**(2), 205–247.

Dow, J. and Werlang, S. R. (1992), 'Uncertainty aversion, risk aversion, and the optimal choice of portfolio', *Econometrica* **60**(1), 197–204.

Driouchi, T., Trigeorgis, L. and So, R. H. (2018), 'Option implied ambiguity and its information content: Evidence from the subprime crisis', *Annals of Operations Research* **262**(2), 463–491.

Duffie, D. (2010), 'Presidential address: Asset price dynamics with slow-moving capital', *Journal of Finance* **65**(4), 1237–1267.

Easley, D., Hvidkjaer, S. and O'Hara, M. (2002), 'Is information risk a determinant of asset returns?', *Journal of Finance* **57**(5), 2185–2221.

Easley, D., Kiefer, N. M. and O'Hara, M. (1997), 'The information content of the trading process', *Journal of Empirical Finance* **4**(2), 159–186.

Easley, D., López de Prado, M. M. and O'Hara, M. (2011), 'The microstructure of the "flash crash": flow toxicity, liquidity crashes, and the probability of informed trading', *The Journal of Portfolio Management* **37**(2), 118–128.

Easley, D., López de Prado, M. M. and O'Hara, M. (2012), 'Flow toxicity and liquidity in a high-frequency world', *Review of Financial Studies* **25**(5), 1457–1493.

Easley, D. and O'Hara, M. (1987), 'Price, trade size, and information in securities markets', *Journal of Financial Economics* **19**(1), 69–90.

Easley, D. and O'Hara, M. (1992), 'Time and the process of security price adjustment', *Journal of Finance* **47**(2), 577–605.

Easley, D. and O'Hara, M. (2004), 'Information and the cost of capital', *Journal of Finance* **59**(4), 1553–1583.

Easley, D. and O'Hara, M. (2009), 'Ambiguity and nonparticipation: The role of regulation', *Review of Financial Studies* **22**(5), 1817–1843.

Easley, D. and O'Hara, M. (2010*a*), 'Liquidity and valuation in an uncertain world', *Journal of Financial Economics* **97**(1), 1–11.

Easley, D. and O'Hara, M. (2010*b*), 'Microstructure and ambiguity', *Journal of Finance* **65**(5), 1817–1846.

Easley, D., O'Hara, M. and Yang, L. (2014), 'Opaque trading, disclosure, and asset prices: Implications for hedge fund regulation', *Review of Financial Studies* **27**(4), 1190–1237.

Ellsberg, D. (1961), 'Risk, ambiguity, and the Savage axioms', *Quarterly Journal of Economics* **75**(4), 643–669.

Engle, R. F. and Patton, A. J. (2004), 'Impacts of trades in an error-correction model of quote prices', *Journal of Financial Markets* **7**(1), 1–25.

Epstein, L. G. and Schneider, M. (2003), 'Recursive multiple-priors', *Journal of Economic Theory* **113**(1), 1–31.

Epstein, L. G. and Schneider, M. (2007), 'Learning under ambiguity', *Review of Economic Studies* **74**(4), 1275–1303.

Epstein, L. G. and Schneider, M. (2008), 'Ambiguity, information quality, and asset pricing', *Journal of Finance* **63**(1), 197–228.

Epstein, L. G. and Schneider, M. (2010), 'Ambiguity and asset markets', *Annual Review of Financial Economics* **2**(1), 315–346.

Fama, E. (1965), 'The behavior of stock-market prices', *Journal of Business* **38**(1), 34–105.

Fama, E. F. (1970), 'Efficient capital markets: A review of theory and empirical work', *Journal of Finance* **25**(2), 383–417.

Fama, E. F. (2014), 'Two pillars of asset pricing', *American Economic Review* **104**(6), 1467–85.

Faria, G. and Correia-da Silva, J. (2014), 'A closed-form solution for options with ambiguity about stochastic volatility', *Review of Derivatives Research* **17**(2), 125–159.

Friedman, M. (1953), *The case for flexible exchange rates*, University of Chicago Press.

Friedman, M. and Savage, L. J. (1948), 'The utility analysis of choices involving risk', *Journal of Political Economy* **56**(4), 279–304.

Ganguli, J. V. and Yang, L. (2009), 'Complementarities, multiplicity, and supply information', *Journal of the European Economic Association* **7**(1), 90–115.

Gao, F., Song, F. and Wang, J. (2013), 'Rational expectations equilibrium with uncertain proportion of informed traders', *Journal of Financial Markets* **16**(3), 387–413.

Garcia, D. and Strobl, G. (2011), 'Relative wealth concerns and complementarities in information acquisition', *Review of Financial Studies* **24**(1), 169–207.

Gennotte, G. and Leland, H. (1990), 'Market liquidity, hedging, and crashes', *American Economic Review* **80**(5), 999–1021.

Gervais, S. (1997), 'Essays on market microstructure with uncertain information precision, optimal information sales, and overconfidence.', *Ph.D. dissertation, University of California, Berkeley* .

Gervais, S. and Odean, T. (2001), 'Learning to be overconfident', *Review of Financial Studies* **14**(1), 1–27.

Ghirardato, P., Maccheroni, F. and Marinacci, M. (2004), 'Differentiating ambiguity and ambiguity attitude', *Journal of Economic Theory* **118**(2), 133–173.

Gilboa, I., Maccheroni, F., Marinacci, M. and Schmeidler, D. (2010), 'Objective and subjective rationality in a multiple prior model', *Econometrica* **78**(2), 755–770.

Gilboa, I. and Marinacci, M. (2013), 'Ambiguity and the Bayesian paradigm', *Acemoglu, D., Arellano, M., Dekel, E., (Eds.), Advances in Economics and Econometrics* **1**, 179–242.

Gilboa, I. and Schmeidler, D. (1989), 'Maxmin expected utility with non-unique prior', *Journal of Mathematical Economics* **18**(2), 141–153.

Gilboa, I. and Schmeidler, D. (1993), 'Updating ambiguous beliefs', *Journal of Economic Theory* **59**(1), 33–49.

Glosten, L. R. and Milgrom, P. R. (1985), 'Bid, ask and transaction prices in a specialist market with heterogeneously informed traders', *Journal of Financial Economics* **14**(1), 71–100.

Glosten, L. R. and Putnins, T. J. (2016), 'Welfare costs of informed trade', *Working Paper, Columbia Business School and UTS Business School* .

Goldstein, I. and Yang, L. (2015), 'Information diversity and complementarities in trading and information acquisition', *Journal of Finance* **70**(4), 1723–1765.

Grabisch, M. and Labreuche, C. (2005*a*), 'Bi-capacities i: Definition, Möbius transform and interaction', *Fuzzy Sets and Systems* **151**(2), 211–236.

Grabisch, M. and Labreuche, C. (2005*b*), 'Bi-capacities ii: The Choquet integral', *Fuzzy Sets and Systems* **151**(2), 237–259.

Gradojevic, N., Erdemlioglu, D. and Gençay, R. (2017), 'Informativeness of trade size in foreign exchange markets', *Economics Letters* **150**, 27–33.

Grossman, S. (1976), 'On the efficiency of competitive stock markets where trades have diverse information', *Journal of Finance* **31**(2), 573–585.

Grossman, S. J. and Miller, M. H. (1988), 'Liquidity and market structure', *Journal of Finance* **43**(3), 617–633.

Grossman, S. J. and Stiglitz, J. E. (1980), 'On the impossibility of informationally efficient markets', *American Economic Review* **70**(3), 393–408.

Gul, F. and Pesendorfer, W. (2014), 'Expected uncertain utility theory', *Econometrica* **82**(1), 1–39.

Hansen, L. P. and Sargent, T. J. (2001), 'Robust control and model uncertainty', *American Economic Review* **91**(2), 60–66.

Hansen, L. P. and Sargent, T. J. (2008), *Robustness*, Princeton University Press.

Hasbrouck, J. (1991), 'Measuring the information content of stock trades', *Journal of Finance* **46**(1), 179–207.

Hausman, J. A., Lo, A. W. and MacKinlay, A. C. (1992), 'An ordered probit analysis of transaction stock prices', *Journal of Financial Economics* **31**(3), 319–379.

Hellwig, M. F. (1980), 'On the aggregation of information in competitive markets', *Journal of Economic Theory* **22**(3), 477–498.

Hong, H. and Stein, J. C. (1999), 'A unified theory of underreaction, momentum trading, and overreaction in asset markets', *Journal of Finance* **54**(6), 2143–2184.

Hong, H. and Stein, J. C. (2003), 'Differences of opinion, short-sales constraints, and market crashes', *The Review of Financial Studies* **16**(2), 487–525.

Huang, J. and Wang, J. (2009), 'Liquidity and market crashes', *Review of Financial Studies* **22**(7), 2607–2643.

Illeditsch, P. K. (2011), 'Ambiguous information, portfolio inertia, and excess volatility', *Journal of Finance* **66**(6), 2213–2247.

Ilut, C. L. and Schneider, M. (2014), 'Ambiguous business cycles', *American Economic Review* **104**(8), 2368–99.

Izhakian, Y. and Benninga, S. (2011), 'The uncertainty premium in an ambiguous economy', *Quarterly Journal of Finance* **1**(02), 323–354.

Jones, C. M. (2002), 'A century of stock market liquidity and trading costs', *Working Paper, Columbia Business School* .

Ju, N. and Miao, J. (2012), 'Ambiguity, learning, and asset returns', *Econometrica* **80**(2), 559–591.

Kahneman, D. and Tversky, A. (1979), 'Prospect theory: An analysis of decision under risk', *Econometrica* **47**(2), 263–291.

Keim, D. B. and Madhavan, A. (1996), 'The upstairs market for large-block transactions: Analysis and measurement of price effects', *Review of Financial Studies* **9**(1), 1–36.

Keynes, J. M. (1921), *A Treatise on Probability*, Cambridge University Press.

Kirilenko, A. A., Kyle, A. S., Samadi, M. and Tuzun, T. (2017), 'The flash crash: High frequency trading in an electronic market', *Journal of Finance* **72**(3), 967–998.

Klibanoff, P., Marinacci, M. and Mukerji, S. (2005), 'A smooth model of decision making under ambiguity', *Econometrica* **73**(6), 1849–1892.

Klir, G. J. (2005), *Uncertainty and information: Foundations of generalized information theory*, John Wiley & Sons.

Klir, G. J. and Yuan, B. (1995), *Fuzzy sets and fuzzy logic*, Prentice Hall New Jersey.

Knight, F. H. (1921), *Risk Uncertainty and Profit*, Houghton Mifflin, Boston.

Koufopoulos, K. and Kozhan, R. (2014), 'Welfare-improving ambiguity in insurance markets with asymmetric information', *Journal of Economic Theory* **151**, 551–560.

Kraus, A. and Stoll, H. R. (1972), 'Price impacts of block trading on the New York Stock Exchange', *Journal of Finance* **27**(3), 569–588.

Kyle, A. S. (1985), 'Continuous auctions and insider trading', *Econometrica* **15**(35), 1315–1335.

Labreuche, C. and Grabisch, M. (2006), 'Generalized Choquet-like aggregation functions for handling bipolar scales', *European Journal of Operational Research* **172**(3), 931–955.

Leach, J. C. and Madhavan, A. N. (1992), 'Intertemporal price discovery by market makers: Active versus passive learning', *Journal of Financial Intermediation* **2**(2), 207–235.

Leach, J. C. and Madhavan, A. N. (1993), 'Price experimentation and security market structure', *Review of Financial Studies* **6**(2), 375–404.

Lintner, J. (1965), 'The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets', *Review of Economics and Statistics* **47**(1), 13–37.

Lo, A. W. (2004), 'The adaptive markets hypothesis: Market efficiency from an evolutionary perspective', *Journal of Portfolio Management* **30**(5), 15–29.

Maccheroni, F., Marinacci, M. and Rustichini, A. (2006), 'Dynamic variational preferences', *Journal of Economic Theory* **128**(1), 4–44.

Machina, M. J. and Siniscalchi, M. (2014), 'Ambiguity and ambiguity aversion', *Handbook of the Economics of Risk and Uncertainty* **1**, 729–807.

Mandelbrot, B. (1963), 'The variation of certain speculative prices', *Journal of Business* **36**(4), 394–419.

Marinacci, M. (2002), 'Probabilistic sophistication and multiple priors', *Econometrica* **70**(2), 755–764.

Markowitz, H. (1952), 'The utility of wealth', *Journal of Political Economy* **60**(2), 151–158.

Mehra, R. and Prescott, E. C. (1985), 'The equity premium: A puzzle', *Journal of Monetary Economics* **15**(2), 145–161.

Mele, A. and Sangiorgi, F. (2015), 'Uncertainty, information acquisition, and price swings in asset markets', *Review of Economic Studies* **82**(4), 1533–1567.

Mossin, J. (1966), 'Equilibrium in a capital asset market', *Econometrica* **34**(4), 768–783.

O'Hara, M. (2003), 'Presidential address: Liquidity and price discovery', *Journal of Finance* **58**(4), 1335–1354.

Osborne, M. F. M. (1959), 'Brownian motion in the stock market', *Operations Research* **7**(2), 145–173.

Ozsoylev, H. N. and Takayama, S. (2010), 'Price, trade size, and information revelation in multi-period securities markets', *Journal of Financial Markets* **13**(1), 49–76.

Ozsoylev, H. and Werner, J. (2011), 'Liquidity and asset prices in rational expectations equilibrium with ambiguous information', *Economic Theory* **48**(2-3), 469–491.

Park, A. and Sabourian, H. (2011), 'Herding and contrarian behavior in financial markets', *Econometrica* **79**(4), 973–1026.

Peters, E. E. (1996), *Chaos and order in the capital markets: A new view of cycles, prices, and market volatility*, John Wiley & Sons.

Peters, E. E. (2003), 'Simple and complex market inefficiencies: integrating efficient markets, behavioral finance, and complexity', *Journal of Behavioral Finance* **4**(4), 225–233.

Radner, R. (1979), 'Rational expectations equilibrium: Generic existence and the information revealed by prices', *Econometrica* **47**(3), 655–678.

Ramsey, F. P. (1931), 'Truth and Probability', *The Foundations of Mathematics and Other Logical Essays* .

Romer, D. (1993), 'Rational asset-price movements without news', *American Economic Review* **83**(5), 1112–30.

Rothschild, M. and Stiglitz, J. (1976), 'Equilibrium in competitive insurance markets: An essay on the economics of imperfect infrmation', *Quarterly Journal of Economics* **90**(4), 629–649.

Routledge, B. R. and Zin, S. E. (2009), 'Model uncertainty and liquidity', *Review of Economic Dynamics* **12**(4), 543–566.

Rubinstein, M. (2001), 'Rational markets: yes or no? the affirmative case', *Financial Analysts Journal* **57**(3), 15–29.

Saar, G. (2001), 'Price impact asymmetry of block trades: An institutional trading explanation', *Review of Financial Studies* **14**(4), 1153–1181.

Samuelson, P. A. (1965), 'Proof that properly anticipated prices fluctuate randomly', *Industrial Management Review* **6**(2), 41–49.

Savage, L. J. (1954), *The Foundations of Statistics.*, Wiley, New York [2nd ed. 1972 (Dover, New York)].

Schmeidler, D. (1986), 'Integral representation without additivity', *Proceedings of the American Mathematical Society* **97**(2), 255–261.

Schmeidler, D. (1989), 'Subjective probability and expected utility without additivity', *Econometrica* **57**(3), 571–587.

Scholes, M. S. (2000), 'Crisis and risk management', *American Economic Review* **90**(2), 17–21.

SEC (2012), 'Report to congress on decimalization', *Required by Section 106 of the JOBS Act* .

Shackle, G. (1949), *Expectation in economics*, Gibson Press, London [2nd ed. 1952 (Cambridge University Press)].

Shafer, G. (1976), *A Mathematical Theory of Evidence*, Vol. 1, Princeton university press Princeton.

Shafer, G. (1987), 'Probability judgment in artificial intelligence and expert systems', *Statistical Science* **2**(1), 3–16.

Sharpe, W. F. (1964), 'Capital asset prices: A theory of market equilibrium under conditions of risk', *Journal of Finance* **19**(3), 425–442.

Shefrin, H. (2000), *Beyond Greed and Fear: Understanding Behavioral Finance and the Psychology of Investing*, Oxford University Press.

Shiller, R. J. (1979), 'The volatility of long-term interest rates and expectations models of the term structure', *Journal of Political Economy* **87**(6), 1190–1219.

Shiller, R. J. (1981), 'Do stock prices move too much to be justified by subsequent changes in dividends?', *American Economic Review* **71**(3), 421–436.

Shiller, R. J. (2003), 'From efficient markets theory to behavioral finance', *Journal of Economic Perspectives* **17**(1), 83–104.

Shiller, R. J. (2014), 'Speculative asset prices', *American Economic Review* **104**(6), 1486–1517.

Spiegel, M. and Subrahmanyam, A. (1992), 'Informed speculation and hedging in a noncompetitive securities market', *Review of Financial Studies* **5**(2), 307–329.

Stapleton, R. C. and Subrahmanyam, M. G. (1978), 'A multiperiod equilibrium asset pricing model', *Econometrica* **46**(5), 1077–1096.

Stein, J. C. (2009), 'Presidential address: Sophisticated investors and market efficiency', *Journal of Finance* **64**(4), 1517–1548.

Stoll, H. R. (1978), 'The supply of dealer services in securities markets', *Journal of Finance* **33**(4), 1133–1151.

Thaler, R. (1980), 'Toward a positive theory of consumer choice', *Journal of Economic Behavior & Organization* **1**(1), 39–60.

Tversky, A. and Kahneman, D. (1992), 'Advances in prospect theory: Cumulative representation of uncertainty', *Journal of Risk and Uncertainty* **5**(4), 297–323.

U.S. Dept. of the Treasury, Fed. Reserve System, Fed. Reserve Bank of New York, SEC and CFTC (2014), 'The U.S. treasury market on October 15, 2014', *Staff Report* .

Veldkamp, L. L. (2006), 'Media frenzies in markets for financial information', *American Economic Review* **96**(3), 577–601.

Von Neumann, J. and Morgenstern, O. (1944), *Theory of Games and Economic Behavior*, Princeton University Press.

Walley, P. (1991), *Statistical Reasoning with Imprecise Probabilities*, Chapman and Hall, London.

Wang, J. (1994), 'A model of competitive stock trading volume', *Journal of Political Economy* **102**(1), 127–168.

Weaver, W. (1948), 'Science and complexity', *American Scientist* **36**, 536–544.

Williams, C. D. (2015), 'Asymmetric responses to earnings news: A case for ambiguity', *Accounting Review* **90**(2), 785–817.

Wise, B. P. and Henrion, M. (1985), 'A framework for comparing uncertain inference systems to probability', *Proceedings of the Annual Conference on Uncertainty in Artificial Intelligence (UAI-85)* **1**, 99–108.

Zadeh, L. A. (1965), 'Fuzzy sets', *Information and Control* **8**(3), 338–353.

Zadeh, L. A. (1978), 'Fuzzy sets as a basis for a theory of possibility', *Fuzzy sets and systems* **1**(1), 3–28.

Zadeh, L. A. (2011), 'A note on Z-numbers', *Information Sciences* **181**(14), 2923–2932.