A λ-cut and Goal Programming based Algorithm for Fuzzy Linear Multiple Objective Bi-level Optimization

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Abstract— **Bi-level programming techniques are developed to handle decentralized problems with two-level decision makers, leaders and followers, who may have more than one objective to achieve. This paper proposes a λ-cut and goal programming based algorithm to solve fuzzy linear multiple objective bi-level (FLMOB) decision problems. First, based on the definition of a distance measure between two fuzzy vectors using λ-cut, a fuzzy linear bi-level goal (FLBG) model is formatted and related theorems are proved. Then, using λ-cut for fuzzy coefficients and a goal programming strategy for multiple objectives, a λ-cut and goal programming based algorithm to solve FLMOB decision problems is presented. A case study for a newsboy problem is adopted to illustrate the application and executing procedure of this algorithm. Finally, experiments are carried out to discuss and analyze the performance of this algorithm.**

Key words: Bi-level programming, Multiple objective linear programming, Goal programming, Fuzzy sets, Optimization, Decision making

I. Introduction

Bi-level programming techniques, initiated by Von Stackelberg [40], are mainly developed for solving decentralized management problems when decision makers are in a hierarchical organization, with the upper termed the leader and the lower the follower [4]. In a bi-level problem, the control of decision factors is partitioned amongst the leader and follower who seek to optimize their individual objective functions, and the corresponding decisions do not control but affect that of the other level [1]. A leader attempts to optimize his or her objective, but he or she must anticipate all possible responses from the follower [23]. A follower observes the leader's decision and then responds to it in a way that is individually optimal. Because the set of feasible choices available to either decision makers is interdependent, a leader's decision affects both the follower's payoff and allowable actions, and vice versa. The investigation of bi-level problems is strongly motivated by real world applications, and bi-level programming techniques have been applied with remarkable success in different domains such as decentralized resource planning [42], electronic power market [17], logistics [44], civil engineering [2], and road network management [15].

A large part of the research on bi-level problems has centered on their linear version, the linear bi-level problems [41], focused on which have been proposed nearly two dozen algorithms [5], [10], [14], [26], [35]–[37]. These computation solutions can be roughly classified into three categories: the vertex enumeration based algorithmes [5], [35], which use the important characteristic that at least one global optimal solution is attained at an extreme point of the constraints set; the Kuhn-Tucker algorithmes [10], [36], [37] in which a bi-level problem is transferred into a single level problem that solves the upper level's problem while including the lower level's optimality conditions as extra constraints; and the heuristics [14], [26], which are known as global optimization techniques based on convergence analysis.

When using bi-level techniques to model real world cases, two practical issues are frequently confronted.

First, when formulating a bi-level problem, the coefficients of the objective functions and the constraints are sometimes obtained through experiments or experts' understanding of the nature of those coefficients. It has been observed that, in most situations, the possible values of these coefficients are often only imprecisely or ambiguously known to the experts and cannot be described by precise values. With this observation, it would certainly be more appropriate to interpret the experts'understanding of the coefficients as fuzzy numerical data which can be represented by means of fuzzy sets [43]. Linear bi-level programming in which the coefficients are characterized by fuzzy numbers is called fuzzy linear bi-level programming [45].

Shih et al. [39] and Lai [23] first applied a fuzzy approach to bi-level programming, although the bi-level problems addressed do not involve fuzzy coefficients. Sakawa et al. [34] have adopted the method suggested by Zimmermann [48] to make an overall satisfactory balance between both levels, and developed an interactive fuzzy algorithm. This algorithm derives a satisfactory solution and updates the satisfactory degrees of decision makers with considerations of overall satisfactory balance among all levels. In our research lab, an approximation algorithm has been developed [16], [45] based on the framework building and models formatting [27], [28]. Solutions can be reached by solving associated multiple objective bi-level decision problem under different λ-cuts.

Second, for a bi-level decision problem, the decision makers from either level may have several objectives to be considered simultaneously. Often, these objectives may be in conflict with each other, with any improvement in one achieved only at the expense of others. While multi-objective optimization has been well studied in single level decision making [3], [12], [13], [30], little research has been conducted in two levels' situations [41]. In a bi-level decision model, the selection of a solution by a leader is affected by the follower's optimal reactions at the same time. Therefore, a solution for a leader who has multiple objectives needs to consider both the solution for the leader's multiple objectives and the follower's decision.

For bi-level multi-objective problems, Shi and Xia [38] have presented an interactive algorithm. It first sets goals for a leader's objectives, then obtains many solutions those are close enough to the goals, which are set to be larger than some certain "satisfactoriness". Fixing the preferences from a leader, the follower's response will be obtained one by one. The final solution can be obtained once a follower's choice is near enough to that of the leader. In this method, to set a suitable "satisfactoriness" would be critical: if it were too big, there would be no solution at all, while a too small value would cause huge computation. However, to set a suitable "satisfactoriness" is neither easy, nor direct, which requires preliminary knowledge and profound understanding of the original problem.

When both these two practical issues are involved for bi-level decision making, the problems become fuzzy multi-objective bi-level problems for which only extremely limited research has been done. Zhang et al. [46] developed an algorithm to FLMOB problems by using a λ-cut method to defuzzify fuzzy coefficients and a weighting method to combine multiple objectives into only one. Straightforward to understand and easy to implement as the weighting method is, setting a suitable weight to every individual objective is sometimes difficult. Usually, it is more rational and feasible for decision makers to set certain goals for their objectives than allocate weighting numbers to them. In such a situation, goal programming would be a suitable technique for FLMOB problems.

Goal programming, originally proposed by Charnes and Cooper [6] in 1961 for a linear model, has been further developed by Lee [24], Ignizio [19], [20], Charnes and Cooper [7]. Recent research on goal programming can be found from [18], [25], [29], [31], [32]. Goal programming requests a decision maker to set a goal for the objective that he/she wishes to attain. A preferred solution is then defined to minimize the deviation from the goal. Therefore, goal programming seems to yield a satisfactory solution rather than an optimal one.

This research applies the idea of goal programming to FLMOB problems. Based on the formulation of a FLBG decision problem, it is proved that the solutions can be obtained by solving the corresponding linear bi-level decision problem, which can be handled easily by Kuhn-Tuchker and Simplex algorithms. Therefore, it is possible for the algorithm developed in this research to deal with FLMOB problems stably and effectively.

This paper is organized as follows. Following the introduction in Section I, Section II introduces related definitions and formulations. In Section III, after defining a λ-cut based distance measure between two fuzzy vectors and modeling a FLMOB decision problem, a λ-cut and goal programming based algorithm for FLMOB decision problems is presented. A case study for a newsboy problem is illustrated and experiments are analyzed in Section IV. Conclusions and further studies are discussed in Section V.

II. Preliminaries

In this section, some definitions and formulations used in subsequent sections are presented.

Throughout this paper, R represents the set of all real numbers, R^n is a $n-$dimensional Euclidean space, $F^*(R)$ and $(F^*(R))^n$ are the set of all finite fuzzy numbers and the set of all $n-$dimensional finite fuzzy numbers on R^n respectively. A finite fuzzy number is a fuzzy number whose 0-cut is an interval where ends are finite numbers.

In a bi-level decision problem, we suppose the leader controls the vector $x \in X \subseteq R^n$, while the follower has the control over $y \in Y \subseteq R^m$. The leader moves first by selecting an x in an attempt to minimize his or her objective function $F(x, y)$ subject to certain constraints. Then, the follower observes the leader's action and reacts by choosing a y to minimize his or her own objective function $f(x, y)$ under some constraints as well. Thus, a bi-level decision problem is formatted as follows [4]:

Definition *1:* For $x \in X \subset R^n$, $y \in Y \subset R^m$, a bi-level decision problem is defined as:

$$\min_{x \in X} F(x, y) \tag{1a}$$

$$\text{s.t. } G(x, y) \leq 0 \tag{1b}$$

$$\min_{y \in Y} f(x, y) \tag{1c}$$

$$\text{s.t. } g(x, y) \leq 0 \tag{1d}$$

where $F : R^n \times R^m \to R^s$, $G : R^n \times R^m \to R^p$, $f : R^n \times R^m \to R^t$, and $g : R^n \times R^m \to R^q$.

Definition *2:* [33] The λ-cut of a fuzzy set $\tilde{A}$ is defined as an ordinary set A_λ so that:

$$A_\lambda = \{x | \mu_{\tilde{A}}(x) \geq \lambda\}, \lambda \in [0, 1]$$

If A_λ is a non-empty bounded closed interval, it can be denoted by:

$$A_\lambda = [A_\lambda^L, A_\lambda^R]$$

where A_λ^L and A_λ^R are the lower and upper bounds of the interval respectively.

Definition *3:* [47] For any n-dimensional fuzzy vectors $\tilde{a} = (\tilde{a_1}, \cdots, \tilde{a_n})$, $\tilde{b} = (\tilde{b_1}, \cdots, \tilde{b_n})$, $\tilde{a_i}, \tilde{b_i} \in F^*(R)$, under a certain satisfactory degree $\alpha \in [0, 1]$, we define

$$\tilde{a} \preceq_\alpha \tilde{b} \text{ iff } {a_i}_\lambda^L \leq {b_i}_\lambda^L \text{ and } {a_i}_\lambda^R \leq {b_i}_\lambda^R, i = 1, \ldots, n; \forall \lambda \in [\alpha, 1]. \tag{2}$$

Definition 3 means, when comparing two fuzzy numbers, all values with membership grades smaller than α are neglected. When two fuzzy numbers can't be compared under a certain α by this ranking method, we can adjust α to a larger degree to achieve the comparison.

III. A λ-CUT AND GOAL PROGRAMMING BASED ALGORITHM FOR FLMOB PROBLEMS

A. Definitions and Theorems

Based on the fuzzy ranking method in Definition 3, a FLMOB decision problem is defined as :

Definition *4:* For $x \in X \subset R^n$, $y \in Y \subset R^m$, $F : X \times Y \to (F^*(R))^s$, and $f : X \times Y \to (F^*(R))^t$,

$$\min_{x\in X} F(x,y) = (\tilde{c}_{11}x + \tilde{d}_{11}y, \ldots, \tilde{c}_{s1}x + \tilde{d}_{s1}y)^T \quad (3a)$$

$$\text{s.t. } \tilde{A}_1 x + \tilde{B}_1 y \preceq_\alpha \tilde{b}_1 \quad (3b)$$

$$\min_{y\in Y} f(x,y) = (\tilde{c}_{12}x + \tilde{d}_{12}y, \ldots, \tilde{c}_{t2}x + \tilde{d}_{t2}y)^T \quad (3c)$$

$$\text{s.t. } \tilde{A}_2 x + \tilde{B}_2 y \preceq_\alpha \tilde{b}_2 \quad (3d)$$

where $\tilde{c}_{h1}$, $\tilde{c}_{i2} \in (F^*(R))^n$, $\tilde{d}_{h1}$, $\tilde{d}_{i2} \in (F^*(R))^m$, $h = 1, 2, \ldots, s$, $i = 1, 2, \ldots, t$, $\tilde{b}_1 \in (F^*(R))^p$, $\tilde{b}_2 \in (F^*(R))^q$, $\tilde{A}_1 = (\tilde{a}_{ij})_{p\times n}$, $\tilde{B}_1 = (\tilde{b}_{ij})_{p\times m}$, $\tilde{A}_2 = (\tilde{e}_{ij})_{q\times n}$, $\tilde{B}_2 = (\tilde{s}_{ij})_{q\times m}$, $\tilde{a}_{ij}, \tilde{b}_{ij}, \tilde{e}_{ij}, \tilde{s}_{ij} \in F^*(R)$.

To build a FLBG model, a distance measure between two fuzzy vectors is needed. There are many important measures to compare two fuzzy numbers, such as Hausdorff distance [9], Hamming distance [11], Euclidean distance [11], and maximum distance [22]. In this paper, a certain number of λ-cuts will be used to approximate a fuzzy number. A final solution is considered to be reached when solutions under two adjacent λ-cuts are near enough. To help implement this strategy, a new distance measure between two fuzzy vectors by using $\lambda-$cuts is defined below:

Definition *5:* Let $\tilde{a} = (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)$, $\tilde{b} = (\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n)$ be $n-$dimensional fuzzy vectors, $\Phi = \{\alpha \le \lambda_0 < \lambda_1 < \cdots < \lambda_l \le 1\}$ be a division of $[\alpha, 1]$, the distance between $\tilde{a}$ and $\tilde{b}$ under ϕ is defined as:

$$D(\tilde{a},\tilde{b}) \triangleq \frac{1}{l+1}\sum_{i=1}^{n}\sum_{j=0}^{l}\{|a^L_{i\lambda_j} - b^L_{i\lambda_j}| + |a^R_{i\lambda_j} - b^R_{i\lambda_j}|\} \quad (4)$$

where α is a predefined satisfactory degree.

In this fuzzy distance definition, a satisfactory degree α is used to give more flexibility to compare fuzzy vectors. It is possible that two fuzzy vectors might not be compared by Definition 5. For example, when we compare two fuzzy vectors $\tilde{a}$ and $\tilde{b}$, if some of the left λ-cuts of $\tilde{a}$ are less than those of $\tilde{b}$, while some right λ-cuts of $\tilde{a}$ are larger than those of $\tilde{b}$, there is no ranking relation between $\tilde{a}$ and $\tilde{b}$.

To solve this problem, we can enhance the aspiration levels of the attributes, i.e., we can adjust the satisfactory degree α to a point where all incomparable parts are discarded. It can be understood as a risk taken by a decision maker who neglects all values with the possibility of occurrence smaller than α. In such a situation, a solution is supposed to be reached under this aspiration level. So, normally, we take the same α for both objectives and constraints in one bi-level problem.

Lemma *1:* For any n-dimensional fuzzy vectors $\tilde{a}, \tilde{b}, \tilde{c}$, fuzzy distance D defined above satisfies the following properties:

1) $D(\tilde{a},\tilde{b}) = 0$, if $\tilde{a}_i = \tilde{b}_i$, $i = 1, 2, \ldots, n$;
2) $D(\tilde{a},\tilde{b}) = D(\tilde{b},\tilde{a})$;
3) $D(\tilde{a},\tilde{b}) \le D(\tilde{a},\tilde{c}) + D(\tilde{c},\tilde{b})$.■

Goals set for the objectives of a leader ($\tilde{g}_L$) and a follower ($\tilde{g}_F$) in (3) are defined as:

$$\tilde{g_L} = (\tilde{g}_{L1}, \tilde{g}_{L2}, \ldots, \tilde{g}_{Ls})^T, \quad (5a)$$

$$\tilde{g_F} = (\tilde{g}_{F1}, \tilde{g}_{F2}, \ldots, \tilde{g}_{Ft})^T. \quad (5b)$$

where $\tilde{g}_{Li}$, $i = 1, \ldots, s$, $\tilde{g}_{Fj}$, $j = 1, \ldots, t$ are fuzzy numbers with membership functions of $\mu_{\tilde{g}_{Li}}, \mu_{\tilde{g}_{Fj}}$.

Our concern is to make the objectives of both a leader and the follower as near to their goals as possible. Using the distance measure defined in (4), we format a FLBG problem as:

For $x \in X \subset R^n$, $y \in Y \subset R^m$, $F : X \times Y \to (F^*(R))^s$, and $f : X \times Y \to (F^*(R))^t$,

$$\min_{x\in X} D(F(x,y), \tilde{g_L}) \quad (6a)$$

$$\text{s.t. } \tilde{A}_1 x + \tilde{B}_1 y \preceq_\alpha \tilde{b}_1 \quad (6b)$$

$$\min_{y\in Y} D(f(x,y), \tilde{g_F}) \quad (6c)$$

$$\text{s.t. } \tilde{A}_2 x + \tilde{B}_2 y \preceq_\alpha \tilde{b}_2 \quad (6d)$$

where $\tilde{A}_1 = (\tilde{a}_{ij})_{p\times n}$, $\tilde{B}_1 = (\tilde{b}_{ij})_{p\times m}$, $\tilde{A}_2 = (\tilde{e}_{ij})_{q\times n}$, $\tilde{B}_2 = (\tilde{s}_{ij})_{q\times m}$, $\tilde{a}_{ij}, \tilde{b}_{ij}, \tilde{e}_{ij}, \tilde{s}_{ij} \in F^*(R)$, and α is a predefined satisfactory degree.

From Definition 3, 4, and 5, we transfer problem (6) into:

$$\min_{x\in X} \triangleq \tfrac{1}{l+1}\sum_{h=1}^{s}\sum_{j=0}^{l}\Big\{|c^L_{h1\lambda_j}x + d^L_{h1\lambda_j}y - g^L_{Lh\lambda_j}| + |c^R_{h1\lambda_j}x + d^R_{h1\lambda_j}y - g^R_{Lh\lambda_j}|\Big\}, \quad (7a)$$

$$\text{s.t. } A_1{}^L_{\lambda_j}x + B_1{}^L_{\lambda_j}y \le b_1{}^L_{\lambda_j}, \quad A_1{}^R_{\lambda_j}x + B_1{}^R_{\lambda_j}y \le b_1{}^R_{\lambda_j}, \quad j = 0, 1, \ldots, l, \quad (7b)$$

$$\min_{y\in Y} \triangleq \tfrac{1}{l+1}\sum_{i=1}^{t}\sum_{j=0}^{l}\Big\{|c^L_{i2\lambda_j}x + d^L_{i2\lambda_j}y - g^L_{Fi\lambda_j}| + |c^R_{i2\lambda_j}x + d^R_{i2\lambda_j}y - g^R_{Fi\lambda_j}|\Big\}, \quad (7c)$$

$$\text{s.t. } A_2{}^L_{\lambda_j}x + B_2{}^L_{\lambda_j}y \le b_2{}^L_{\lambda_j}, \quad A_2{}^R_{\lambda_j}x + B_2{}^R_{\lambda_j}y \le b_2{}^R_{\lambda_j} \quad j = 0, 1, \ldots, l, \quad (7d)$$

where $\Phi = \{\alpha \le \lambda_0 < \lambda_1 < \cdots < \lambda_l \le 1\}$ is a division of $[\alpha, 1]$.

For a clear understanding of the idea adopted, define:

$$v^{L-}_{h1} = \tfrac{1}{2}[|\textstyle\sum_{j=0}^{l} c^L_{h1\lambda_j}x + \sum_{j=0}^{l} d^L_{h1\lambda_j}y - \sum_{j=0}^{l} g^L_{Lh\lambda_j}| - (\sum_{j=0}^{l} c^L_{h1\lambda_j}x + \sum_{j=0}^{l} d^L_{h1\lambda_j}y - \sum_{j=0}^{l} g^L_{Lh\lambda_j})]$$

$$v^{L+}_{h1} = \frac{1}{2}\ [|\textstyle\sum_{j=0}^{l} c^L_{h1\lambda_j}x + \sum_{j=0}^{l} d^L_{h1\lambda_j}y - \sum_{j=0}^{l} g^L_{Lh\lambda_j}| + (\sum_{j=0}^{l} c^L_{h1\lambda_j}x + \sum_{j=0}^{l} d^L_{h1\lambda_j}y - \sum_{j=0}^{l} g^L_{Lh\lambda_j})]$$

$$v_{h1}^{R-} = \frac{1}{2}\ [|\sum_{j=0}^{l} c_{h1\lambda_j}^{R}x + \sum_{j=0}^{l} d_{h1\lambda_j}^{L}y - \sum_{j=0}^{l} g_{Lh\lambda_j}^{R}| - (\sum_{j=0}^{l} c_{h1\lambda_j}^{R}x + \sum_{j=0}^{l} d_{h1\lambda_j}^{L}y - \sum_{j=0}^{l} g_{Lh\lambda_j}^{R})]$$

$$v_{h1}^{R+} = \frac{1}{2}\ [|\sum_{j=0}^{l} c_{h1\lambda_j}^{R}x + \sum_{j=0}^{l} d_{h1\lambda_j}^{L}y - \sum_{j=0}^{l} g_{Lh\lambda_j}^{R}| + (\sum_{j=0}^{l} c_{h1\lambda_j}^{R}x + \sum_{j=0}^{l} d_{h1\lambda_j}^{L}y - \sum_{j=0}^{l} g_{Lh\lambda_j}^{R})]$$

$$h = 1, 2 \ldots, s,$$

$$v_{i2}^{L-} = \frac{1}{2}\ [|\sum_{j=0}^{l} c_{i2\lambda_j}^{L}x + \sum_{j=0}^{l} d_{i2\lambda_j}^{L}y - \sum_{j=0}^{l} g_{Fi\lambda_j}^{L}| - (\sum_{j=0}^{l} c_{i2\lambda_j}^{L}x + \sum_{j=0}^{l} d_{i2\lambda_j}^{L}y - \sum_{j=0}^{l} g_{Fi\lambda_j}^{L})]$$

$$v_{i2}^{L+} = \frac{1}{2}\ [|\sum_{j=0}^{l} c_{i2\lambda_j}^{L}x + \sum_{j=0}^{l} d_{i2\lambda_j}^{L}y - \sum_{j=0}^{l} g_{Fi\lambda_j}^{L}| + (\sum_{j=0}^{l} c_{i2\lambda_j}^{L}x + \sum_{j=0}^{l} d_{i2\lambda_j}^{L}y - \sum_{j=0}^{l} g_{Fi\lambda_j}^{L})]$$

$$v_{i2}^{R-} = \frac{1}{2}\ [|\sum_{j=0}^{l} c_{i2\lambda_j}^{R}x + \sum_{j=0}^{l} d_{i2\lambda_j}^{L}y - \sum_{j=0}^{l} g_{Fi\lambda_j}^{R}| - (\sum_{j=0}^{l} c_{i2\lambda_j}^{R}x + \sum_{j=0}^{l} d_{i2\lambda_j}^{L}y - \sum_{j=0}^{l} g_{Fi\lambda_j}^{R})]$$

$$v_{i2}^{R+} = \frac{1}{2}\ [|\sum_{j=0}^{l} c_{i2\lambda_j}^{R}x + \sum_{j=0}^{l} d_{i2\lambda_j}^{L}y - \sum_{j=0}^{l} g_{Fi\lambda_j}^{R}| + (\sum_{j=0}^{l} c_{i2\lambda_j}^{R}x + \sum_{j=0}^{l} d_{i2\lambda_j}^{L}y - \sum_{j=0}^{l} g_{Fi\lambda_j}^{R})]$$

$$i = 1, 2 \ldots, t,$$

where v_{h1}^{L-} and v_{h1}^{L+} are deviational variables representing the under-achievement and over-achievement of the h^{th} goal for a leader under the left λ-cut. v_{h1}^{R-} and v_{h1}^{R+} are deviational variables representing the under-achievement and over-achievement of the h^{th} goal for a leader under the right λ-cut. v_{i2}^{L-}, v_{i2}^{L+}, v_{i2}^{R-} and v_{i2}^{R+} are for a follower respectively.

Associated with the linear bi-level problem (7), we now consider the following bi-level problem:

For $(v_{11}^{L-}, v_{11}^{L+}, v_{11}^{R-}, v_{11}^{R+}, \ldots, v_{s1}^{L-}, v_{s1}^{L+}, v_{s1}^{R-}, v_{s1}^{R+}) \in R^{4s}$, $X' \subseteq X \times R^{4s}$, $(v_{12}^{L-}, v_{12}^{L+}, v_{12}^{R-}, v_{12}^{R+}, \ldots, v_{t2}^{L-}, v_{t2}^{L+}, v_{t2}^{R-}, v_{t2}^{R+}) \in R^{4t}$, $Y' \subseteq Y \times R^{4t}$, let $x = (x_1, \cdots, x_n) \in X$, $x' = (x_1, \cdots, x_n, v_{11}^{L-}, v_{11}^{L+}, v_{11}^{R-}, v_{11}^{R+}, \ldots, v_{s1}^{L-}, v_{s1}^{L+}, v_{s1}^{R-}, v_{s1}^{R+}) \in X'$, $y = (y_1, \cdots, y_m) \in Y$, $y' = (y_1, \cdots, y_m, v_{12}^{L-}, v_{12}^{L+}, v_{12}^{R-}, v_{12}^{R+}, \ldots, v_{t2}^{L-}, v_{t2}^{L+}, v_{t2}^{R-}, v_{t2}^{R+}) \in Y'$, and v_1, v_2, $: X' \times Y' \to R$.

$$\min_{x' \in X'} v_1 = \sum_{h=1}^{s}(v_{h1}^{L-} + v_{h1}^{L+} + v_{h1}^{R-} + v_{h1}^{R+}) \tag{8a}$$

$$\text{s.t. } \sum_{j=0}^{l} c_{h1\lambda_j}^{L}x + \sum_{j=0}^{l} d_{h1\lambda_j}^{L}y + v_{h1}^{L-} - v_{h1}^{L+} = \sum_{j=0}^{l} g_{Lh\lambda_j}^{L},$$
$$\sum_{j=0}^{l} c_{h1\lambda_j}^{R}x + \sum_{j=0}^{l} d_{h1\lambda_j}^{R}y + v_{h1}^{R-} - v_{h1}^{R+} = \sum_{j=0}^{l} g_{Lh\lambda_j}^{R},$$
$$v_{h1}^{L-}, v_{h1}^{L+}, v_{h1}^{R-}, v_{h1}^{R+} \geq 0,$$
$$v_{h1}^{L-} \cdot v_{h1}^{L+} = 0, v_{h1}^{R-} \cdot v_{h1}^{R+} = 0$$
$$h = 1, 2, \ldots, s,$$
$$A_1{}_{\lambda_j}^{L}x + B_1{}_{\lambda_j}^{L}y \leq b_1{}_{\lambda_j}^{L},$$
$$A_1{}_{\lambda_j}^{R}x + B_1{}_{\lambda_j}^{R}y \leq b_1{}_{\lambda_j}^{R},$$
$$j = 0, 1, \ldots, l, \tag{8b}$$

$$\min_{y' \in Y'} v_2 = \sum_{i=1}^{t}(v_{i2}^{L-} + v_{i2}^{L+} + v_{i2}^{R-} + v_{i2}^{R+}) \tag{8c}$$

$$\text{s.t. } \sum_{j=0}^{l} c_{i2\lambda_j}^{L}x + \sum_{j=0}^{l} d_{i2\lambda_j}^{L}y + v_{i2}^{L-} - v_{i2}^{L+} = \sum_{j=0}^{l} g_{Fi\lambda_j}^{L},$$
$$\sum_{j=0}^{l} c_{i2\lambda_j}^{R}x + \sum_{j=0}^{l} d_{i2\lambda_j}^{R}y + v_{i2}^{R-} - v_{i2}^{R+} = \sum_{j=0}^{l} g_{Fi\lambda_j}^{R},$$
$$v_{i2}^{L-}, v_{i2}^{L+}, v_{i2}^{R-}, v_{i2}^{R+} \geq 0,$$
$$v_{i2}^{L-} \cdot v_{i2}^{L+} = 0, v_{i2}^{R-} \cdot v_{i2}^{R+} = 0$$
$$i = 1, 2, \ldots, t,$$
$$A_2{}_{\lambda_j}^{L}x + B_2{}_{\lambda_j}^{L}y \leq b_2{}_{\lambda_j}^{L},$$
$$A_2{}_{\lambda_j}^{R}x + B_2{}_{\lambda_j}^{R}y \leq b_2{}_{\lambda_j}^{R},$$
$$j = 0, 1, \ldots, l, \tag{8d}$$

Theorem *1:* Let $(x'^*, y'^*) = (x^*, v_{11}^{L-*}, v_{11}^{L+*}, v_{11}^{R-*}, v_{11}^{R+*}, \ldots, v_{s1}^{L-*}, v_{s1}^{L+*}, v_{s1}^{R-*}, v_{s1}^{R+*}, y^*, v_{12}^{L-*}, v_{12}^{L+*}, v_{12}^{R-*}, v_{12}^{R+*}, \ldots, v_{t2}^{L-*}, v_{t2}^{L+*}, v_{t2}^{R-*}, v_{t2}^{R+*})$ be the optimal solution to bi-level problem (8), then (x^*, y^*) is the optimal solution to the bi-level problem defined by (7).

Proof: see the Proof of Theorem 1 in the Appendix.

Adopting weighting method, (8) can be further transferred into (9):

$$\min_{x' \in X'} v_1^- + v_1^+ \tag{9a}$$

$$\text{s.t. } c_1x + d_1y + v_1^- - v_1^+ = \sum_{h=1}^{s}\sum_{j=0}^{l}(g_{Lh\lambda_j}^{L} + g_{Lh\lambda_j}^{R}),$$
$$v_1^-, v_1^+ \geq 0,$$
$$v_1^- \cdot v_1^+ = 0$$
$$A_1{}_{\lambda_j}^{L}x + B_1{}_{\lambda_j}^{L}y \leq b_1{}_{\lambda_j}^{L},$$
$$A_1{}_{\lambda_j}^{R}x + B_1{}_{\lambda_j}^{R}y \leq b_1{}_{\lambda_j}^{R},$$
$$j = 0, 1, \ldots, l, \tag{9b}$$

$$\min_{y' \in Y'} v_2^- + v_2^+ \tag{9c}$$

$$\text{s.t. } c_2x + d_2y = \sum_{i=1}^{t}\sum_{j=0}^{l}(g_{Fi\lambda_j}^{L} + g_{Fi\lambda_j}^{R}),$$
$$v_2^-, v_2^+ \geq 0,$$
$$v_2^- \cdot v_2^+ = 0 \tag{9d}$$
$$A_2{}_{\lambda_j}^{L}x + B_2{}_{\lambda_j}^{L}y \leq b_2{}_{\lambda_j}^{L},$$
$$A_2{}_{\lambda_j}^{R}x + B_2{}_{\lambda_j}^{R}y \leq b_2{}_{\lambda_j}^{R},$$
$$j = 0, 1, \ldots, l,$$

where $v_1^- = \sum_{h=1}^{s}(v_{h1}^{L-} + v_{h1}^{R-})$, $v_1^+ = \sum_{h=1}^{s}(v_{h1}^{L+} + v_{h1}^{R+})$, $v_2^- = \sum_{i=1}^{t}(v_{i2}^{L-} + v_{i2}^{R-})$, $v_2^+ = \sum_{i=1}^{t}(v_{i2}^{L+} + v_{i2}^{R+})$, $c_1 = \sum_{h=1}^{s}\sum_{j=0}^{l}(c_{h1\lambda_j}^{L} + c_{h1\lambda_j}^{R})$, $d_1 = \sum_{h=1}^{s}\sum_{j=0}^{l}(d_{h1\lambda_j}^{L} + d_{h1\lambda_j}^{R})$, $c_2 = \sum_{i=1}^{t}\sum_{j=0}^{l}(c_{i2\lambda_j}^{L} + c_{i2\lambda_j}^{R})$, $d_2 = \sum_{i=1}^{t}\sum_{j=0}^{l}(d_{i2\lambda_j}^{L} + d_{i2\lambda_j}^{R})$. In this formula, v_1^- and v_1^+ are deviational variables representing the under-achievement and over-achievement of goals for a leader, and v_2^- and v_2^+

are deviational variables representing the under-achievement and over-achievement of goals for a follower respectively.

The nonlinear conditions of $v_1^- \cdot v_1^+ = 0$, and $v_2^- \cdot v_2^+ = 0$ need not be maintained if the Kuhn-Tucker algorithm [36] together with the Simplex algorithm are adopted, since only equivalence at an optimum is wanted. Further explanation can be found from [8]. Thus, problem (9) is further transformed into:

For $(v_1^-, v_1^+) \in R^2$, $\bar{X}' \subseteq X \times R^2$, $(v_2^-, v_2^+) \in R^2$, $\bar{Y}' \subseteq Y \times R^2$, let $x = (x_1, \cdots, x_n) \in X$, $\bar{x}' = (x_1, \cdots, x_n, v_1^-, v_1^+) \in \bar{X}'$, $y = (y_1, \cdots, y_m) \in Y$, $\bar{y}' = (y_1, \cdots, y_m, v_2^-, v_2^+,) \in \bar{Y}'$, and $v_1, v_2 : \bar{X}' \times \bar{Y}' \to F^*(R)$.

$$\min_{(x, v_1^-, v_1^+) \in \bar{X}'} v_1 = v_1^- + v_1^+ \tag{10a}$$

$$\begin{aligned} \text{s.t. } & c_1 x + d_1 y + v_1^- - v_1^+ = \sum_{h=1}^{s} \sum_{j=0}^{l} (g_{Lh\lambda_j}^L + g_{Lh\lambda_j}^R), \\ & A_1{}_{\lambda_j}^L x + B_1{}_{\lambda_j}^L y \le b_1{}_{\lambda_j}^L, \\ & A_1{}_{\lambda_j}^R x + B_1{}_{\lambda_j}^R y \le b_1{}_{\lambda_j}^R, \\ & j = 0, 1, \ldots, l, \end{aligned} \tag{10b}$$

$$\min_{(y, v_2^-, v_2^+) \in \bar{Y}'} v_2 = v_2^- + v_2^+ \tag{10c}$$

$$\begin{aligned} \text{s.t. } & c_2 x + d_2 y = \sum_{i=1}^{t} \sum_{j=0}^{l} (g_{Fi\lambda_j}^L + g_{Fi\lambda_j}^R), \\ & A_2{}_{\lambda_j}^L x + B_2{}_{\lambda_j}^L y \le b_2{}_{\lambda_j}^L, \\ & A_2{}_{\lambda_j}^R x + B_2{}_{\lambda_j}^R y \le b_2{}_{\lambda_j}^R, \\ & j = 0, 1, \ldots, l, \end{aligned} \tag{10d}$$

Problem (10) is a standard linear bi-level problem which can be solved by the Kuhn-Tucker algorithm [36].

B. A λ-cut and Goal Programming based Algorithm

Based on the analysis above, the λ-cut and goal programming based algorithm is detailed as:

[Step 1] (Input)

Obtain relevant coefficients which include:

1) Coefficients of (3)
2) Coefficients of (5)
3) Satisfactory degree: α
4) $\varepsilon > 0$

[Step 2] (Initialize)

Let $k = 1$, which is the counter to record current loop.

In (7), where $\lambda_j \in [\alpha, 1]$, let $\lambda_0 = \alpha$ and $\lambda_1 = 1$ respectively, then each objective will be transferred into four non-fuzzy objective functions, and each fuzzy constraint is converted into four non-fuzzy constraints.

[Step 3] (Compute)

By introducing auxiliary variables v_1^-, v_1^+, v_2^- and v_2^+, we obtain the format of (10).

The solution $(x, v_1^-, v_1^+, y, v_2^-, v_2^+)_2$ of (10) is obtained by the Kuhn-Tucker algorithm.

[Step 4] (Compare)

If $(k = 1)$

Then $(x, v_1^-, v_1^+, y, v_2^-, v_2^+)_1 = (x, v_1^-, v_1^+, y, v_2^-, v_2^+)_2$;

goto [Step 5];

Else

If $||(x, v_1^-, v_1^+, y, v_2^-, v_2^+)_2 - (x, v_1^-, v_1^+, y, v_2^-, v_2^+)_1|| < \varepsilon$

Then goto [Step 7] ;

EndIf

[Step 5] (Split)

Suppose there are $(L + 1)$ nodes λ_j, $(j = 0, 1, \ldots, L)$ in the interval $[\alpha, 1]$, insert L new nodes δ_t $(t = 1, 2 \ldots, L)$ in $[\alpha, 1]$ so that: $\delta_t = (\lambda_{t-1} + \lambda_t)/2$.

[Step 6] (Loop)

$k = k + 1$;

goto [Step 3];

[Step 7] (Output)

$(x, y)_2$ is obtained as the final solution.

IV. A Case study and Experiments

In this section, we apply the λ-cut and goal programming based algorithm proposed in this paper on a real world “newsboy problem” to illustrate its operation and application. Experiments are then carried out on some numerical examples with different scales to test the algorithm’s performance.

A. A Case Study

A classical newsboy problem is to find a newspaper’s order quantity for maximizing the profit of a newsboy (newspaper retailer) [21]. In a real world situation, both a newspaper manufacturer and a retailer have more than one concern. Using a FLMOB model, a newsboy problem is expressed as follows: the leader, a manufacturer controls the decision variable of the wholesale price (x), while the follower, a retailer, decides his or her order quantity (y). The manufacturer has two main objectives: to maximize the net profits, represented by $F_1(x, y)$, and to maximize the newspaper quality, by $F_2(x, y)$ but subject to some constraints, including the requirements of material, marketing cost and labor cost. The retailer also has two objectives to achieve: to minimise his or her purchase cost, represented by $f_1(x, y)$, and to minimise the working hours, by $f_2(x, y)$ under his own constraints. Meanwhile, both the manufacturer and the retailer will set goals $(g_{L1}, g_{L2}, g_{F1}, g_{F1})$ for each of their two objectives.

When modeling this multi-objective bi-level decision problem, the main difficulty is to establish coefficients of the objectives and constraints for both the leader and the follower. We can only estimate some values for material cost, labor cost, etc. according to our experience and previous data. For some items, the values can only be assigned by linguistic terms as about \$1000. This is a common case in any organizational decision practice. By using fuzzy numbers to describe these uncertain values in coefficients, a FLMOB model can be established for this decision problem.

To illustrate the λ-cut and goal programming based algorithm introduced in Section III, this newsboy problem will be solved step by step:

[Step 1]: (Input the relevant coefficients)

1. Coefficients of (3):

The newsboy problem is formatted as:

$$
\begin{aligned}
\text{Leader}:&\max_{x\in X} F_1(x,y)=\tilde{6}x+\tilde{3}y\\
&\max_{x\in X} F_2(x,y)=\tilde{-3}x+\tilde{6}y\\
&\text{s.t. } \tilde{-1}x+\tilde{3}y\leqslant \tilde{21}\\
\text{Follower}:&\min_{y\in Y} f_1(x,y)=\tilde{4}x+\tilde{3}y\\
&\min_{y\in Y} f_2(x,y)=\tilde{3}x+\tilde{1}y\\
&\text{s.t. } \tilde{-1}x\tilde{-3}y\leqslant \tilde{27}
\end{aligned}
$$

where $x\in R^1, y\in R^1$, and $X=x\geq 0, Y=y\geq 0$.

The membership functions for this FMOLB are as follows:

$$
\mu_{\tilde{6}}(x)=\begin{cases}0 & x<5\\ (x^2-25)/11 & 5\leq x<8\\ 1 & x=6\\ (64-x^2)/28 & 6<x\leq 8\\ 0 & x>8\end{cases},
$$

$$
\mu_{\tilde{3}}(x)=\begin{cases}0 & x<2\\ (x^2-4)/5 & 2\leq x<3\\ 1 & x=3\\ (25-x^2)/16 & 3<x\leq 5\\ 0 & x>5\end{cases},
$$

$$
\mu_{\tilde{-3}}(x)=\begin{cases}0 & x<-4\\ (16-x^2)/7 & -4\leq x<-3\\ 1 & x=-3\\ (x^2-1)/8 & -3<x\leq -1\\ 0 & x>-1\end{cases},
$$

$$
\mu_{\tilde{4}}(x)=\begin{cases}0 & x<3\\ (x^2-9)/7 & 3\leq x<4\\ 1 & x=4\\ (36-x^2)/20 & 4<x\leq 6\\ 0 & x>6\end{cases},
$$

$$
\mu_{\tilde{1}}(x)=\begin{cases}0 & x<0.5\\ (x^2-0.25)/0.75 & 0.5\leq x<1\\ 1 & x=1\\ (4-x^2)/3 & 1<x\leq 2\\ 0 & x>2\end{cases},
$$

$$
\mu_{\tilde{-1}}(x)=\begin{cases}0 & x<-2\\ (4-x^2)/3 & -2\leq x<-1\\ 1 & x=-1\\ (x^2-0.25)/0.75 & -1<x\leq -0.5\\ 0 & x>-0.5\end{cases},
$$

$$
\mu_{\tilde{21}}(x)=\begin{cases}0 & x<19\\ (x^2-361)/80 & 19\leq x<21\\ 1 & x=21\\ (625-x^2)/184 & 21<x\leq 25\\ 0 & x>25\end{cases},
$$

$$
\mu_{\tilde{27}}(x)=\begin{cases}0 & x<25\\ (x^2-625)/104 & 25\leq x<27\\ 1 & x=27\\ (961-x^2)/232 & 27<x\leq 31\\ 0 & x>31\end{cases}.
$$

2. Suppose the membership functions of the fuzzy goals set for the leader are:

$$
\mu_{\tilde{g}_{L1}}(x)=\begin{cases}0 & x<15\\ (x^2-225)/175 & 15\leq x<20\\ 1 & x=20\\ (900-x^2)/500 & 20<x\leq 30\\ 0 & x>30\end{cases},
$$

$$
\mu_{\tilde{g}_{L2}}(x)=\begin{cases}0 & x<4\\ (x^2-16)/48 & 4\leq x<8\\ 1 & x=8\\ (225-x^2)/161 & 8<x\leq 15\\ 0 & x>15\end{cases}.
$$

The membership functions of the fuzzy goals set for the follower are:

$$
\mu_{\tilde{g}_{F1}}(x)=\begin{cases}0 & x<10\\ (x^2-100)/225 & 10\leq x<15\\ 1 & x=15\\ (400-x^2)/175 & 15<x\leq 20\\ 0 & x>20\end{cases},
$$

$$
\mu_{\tilde{g}_{F2}}(x)=\begin{cases}0 & x<7\\ (x^2-49)/32 & 7\leq x<9\\ 1 & x=9\\ (121-x^2)/40 & 9<x\leq 11\\ 0 & x>11\end{cases}.
$$

3. Satisfactory degree: $\alpha=0.2$

4. $\varepsilon=0.15$

[Step 2]: (Initialize) Let k=1. Associated with this example, the corresponding MOB$_\lambda$ problem is:

$$
\begin{aligned}
&\min_{x\in X} |\sqrt{11\lambda+25}x+\sqrt{5\lambda+4}y-\sqrt{175\lambda+225}|\\
&\qquad+|\sqrt{64-28\lambda}x+25-\sqrt{25-16\lambda}y-\sqrt{900-500\lambda}|\\
&\min_{x\in X} |-\sqrt{16-7\lambda}x+\sqrt{11\lambda+25}y-\sqrt{48\lambda+16}|\\
&\qquad+|-\sqrt{8\lambda+1}+\sqrt{64-28\lambda}-\sqrt{225-161\lambda}|\\
&\text{s.t. } -\sqrt{4-2\lambda}x+\sqrt{5\lambda+4}y\leqslant\sqrt{80\lambda+361}\\
&\qquad\quad -\sqrt{-0.75\lambda+0.25}x+\sqrt{25-16\lambda}y\leqslant\sqrt{625-184\lambda}\\
&\qquad\min_{y\in Y} |\sqrt{7\lambda+9}x+\sqrt{5\lambda+4}y-\sqrt{225\lambda+100}|\\
&\qquad\quad+|\sqrt{36-20\lambda}x+25-\sqrt{25-16\lambda}y-\sqrt{400-175\lambda}|\\
&\qquad\min_{y\in Y} |-\sqrt{5\lambda+4}x+\sqrt{0.75\lambda+0.25}y-\sqrt{32\lambda+49}|\\
&\qquad\quad+|-\sqrt{25-16\lambda}x+\sqrt{4-3\lambda}y-\sqrt{121-40\lambda}|\\
&\qquad\text{s.t. } \sqrt{0.75\lambda+0.25}x+\sqrt{5\lambda+4}y\leqslant\sqrt{104\lambda+625}\\
&\qquad\quad\sqrt{4-3\lambda}x+\sqrt{25-16\lambda}y\leqslant\sqrt{901-232\lambda}
\end{aligned}
$$

where $\lambda\in[0.2,1]$.

Referring to the algorithm, only $\lambda_0=0.2$ and $\lambda_1=1$ are considered initially. Thus four non-fuzzy objective functions and four non-fuzzy constraints for the leader and follower are generated respectively:

$$\begin{aligned}
&\min_{x\in X} \tfrac{1}{4}\{|\sqrt{27.2}x+\sqrt{5}y-\sqrt{260}|+|6x+3y-20|\\
&\quad+|\sqrt{58.4}x+\sqrt{21.8}y-20\sqrt{2}|+|6x+3y-20|\\
&\quad+|-\sqrt{14.6}x+\sqrt{27.2}y-\sqrt{25.6}|+|-3x+6y-8|\\
&\quad+|-\sqrt{2.6}+\sqrt{58.4}y-\sqrt{192.8}|+|-3x+6y-8|\}\\
&\text{s.t. } -\sqrt{3.4}x+\sqrt{5}y\leqslant\sqrt{377}\\
&\quad -x+3y\leqslant 21\\
&\quad -\sqrt{0.4}+\sqrt{5}y\leqslant\sqrt{645.8}\\
&\quad -x+3y\leqslant 21\\
&\quad \min_{y\in Y} \tfrac{1}{4}\{|3x+2y-12.04|+|4x+3y-19.1|\\
&\qquad +|6x-5y-7.4|+|4x-3y-10.63|\\
&\qquad +|-2x+0.5y-18.03|+|-3x+y-15|\\
&\qquad +|-5x+2y-9|+|-3x+y-9|\}\\
&\quad \text{s.t. } \sqrt{0.4}x+\sqrt{5}y\leqslant\sqrt{645.8}\\
&\qquad x+3y\leqslant 27\\
&\qquad \sqrt{3.4}x+\sqrt{21.8}y\leqslant\sqrt{914.6}\\
&\qquad x+3y\leqslant 27
\end{aligned}$$

[Step 3]: (Compute)
By introducing auxiliary variables $v_1^-, v_1^+, v_2^-, v_2^+$, we have:

$$\begin{aligned}
&\min_{(x,v_1^-,v_1^+)\in\bar{X}'} v_1^-+v_1^+\\
&\text{s.t. } 3.083x+20.076y+v_1^--v_1^+=54.73,\\
&\quad -1.8x+2.2y\leqslant 19.4\\
&\quad -x+3y\leqslant 21\\
&\quad -0.6x+4.7y\leqslant 24.3\\
&\quad -x+3y\leqslant 21\\
&\quad \min_{(y,v_2^-,v_2^+)\in\bar{Y}'} v_2^-+v_2^+\\
&\quad \text{s.t. } 16.498x+8.205y+v_2^--v_2^+=51.337,\\
&\qquad 0.6x+2.2y\leqslant 25.4\\
&\qquad x+3y\leqslant 7\\
&\qquad 1.8x+4.7y\leqslant 30.2\\
&\qquad x+3y\leqslant 27
\end{aligned}$$

Using Branch-and-bound algorithm [5], the current solution is (1.901,0,0, 2.434,0,0).

[Step 4]: (Compare) Because k=1, goto **[Step 5]**.

[Step 5]: (Split) By inserting a new node $\lambda_1=(0.2{+}1)/2=0.6$, there are a total three nodes of $\lambda_0=0.2, \lambda_1=0.6$ and $\lambda_2=1$. Then a total six non-fuzzy objective functions for the leader and follower, together with six non-fuzzy constraints for the leader and follower respectively, are generated.

[Step 6]: (Loop) k=1+1=2, goto **[Step 3]**, and a current solution of (2.011,0,0, 2.356,0,0) is obtained. As $|2.011-1.901|+|2.356-2.434|=0.188>\varepsilon=0.15$, the algorithm continues until the solution of (1.957,0,0, 2.388,0,0) is obtained. The computing results are listed in TABLE I.

TABLE I
SUMMARY OF THE RUNNING SOLUTION

k	x	y	$v_{1\lambda}^+$	$v_{1\lambda}^-$	$v_{2\lambda}^+$	$v_{2\lambda}^-$
1	1.901	2.434	0	0	0	0
2	2.011	2.356	0	0	0	0
3	1.872	2.446	0	0	0	0
4	1.957	2.388	0	0	0	0

[Step 7]: (Output) As $|1.957-1.872|+|2.388-2.2.446|=0.14<\varepsilon=0.15$, $(x^*,y^*)=(1.957,2.388)$ is the final solution of this FLMOB problem. The objectives for the leader and follower under $(x^*,y^*)=(1.957,2.388)$ are:

$$\begin{cases}
F_1(x^*,y^*)=F(1.957,2.388)=1.957\tilde{c_{11}}+2.388\tilde{d_{11}}\\
F_2(x^*,y^*)=F(1.957,2.388)=1.957\tilde{c_{12}}+2.388\tilde{d_{12}}\\
f_1(x^*,y^*)=F(1.957,2.388)=1.957\tilde{c_{21}}+2.388\tilde{d_{21}}\\
f_2(x^*,y^*)=F(1.957,2.388)=1.957\tilde{c_{22}}+2.388\tilde{d_{22}}
\end{cases},$$

Under this solution, the membership functions for the leader's objectives are shown in Fig.1 and the membership functions for the follower's objectives are shown in Fig.2.

These fuzzy values shown in Fig. 1 and Fig. 2 describe the achievements of every objective under the solutions. From Fig. 1 we can see that if the manufacturer chooses his or her decision variable as 1.957, the most possible net profit will be 18.9025, which is very close to the goal set for this objective. The other objective values can be interpreted the same way.

B. Experiments and Evaluation

The algorithm proposed in this study was implemented by Visual Basic 6.0, and run on a desktop computer with CPU Pentium 4 2.8GHz, RAM 1G, Windows XP. To test the performance of the proposed algorithm, the following experiments are carried out.

1) To test the efficiency of the proposed algorithm, we employ ten numerical examples and enlarge the problem scales by changing the numbers of decision variables, objective functions and constraints for both leaders and followers from two to ten simultaneously. For each of these examples, the final solution has been obtained within five seconds.
2) To test the performance of the fuzzy distance measure in Definition 5, we adjust the satisfactory degree values from 0 to 0.5 on the ten numerical examples again. At the same time, we change some of the fuzzy coefficients in the constraints by moving the points whose membership values equal 0 by 10% from the left and right respectively. Experiments reveal that, when a satisfactory degree is set as 0, the average solution will change by about 6% if some of the constraint coefficients are moved as discussed above. When we increase satisfactory degrees, the average solution change decreases. At the point where satisfactory degrees are equal to 0.5, the average solution change is 0.

From Experiment 1), we can see that this algorithm is quite efficient. The reason is the fact that final solutions can be reached by solving corresponding linear bi-level programming problems, which can be handled by the Kuhn-Tucker and the Simplex algorithms.

From Experiment 2), we can see that if we change some coefficients of fuzzy numbers within a small range, solutions will be less sensitive to this change under a higher satisfactory degree. The reason is that, when the satisfactory degree is set to 0, every λ-cut of fuzzy coefficients from 0 to 1 will be considered. Thus, the decision maker can certainly be influenced by minor information.

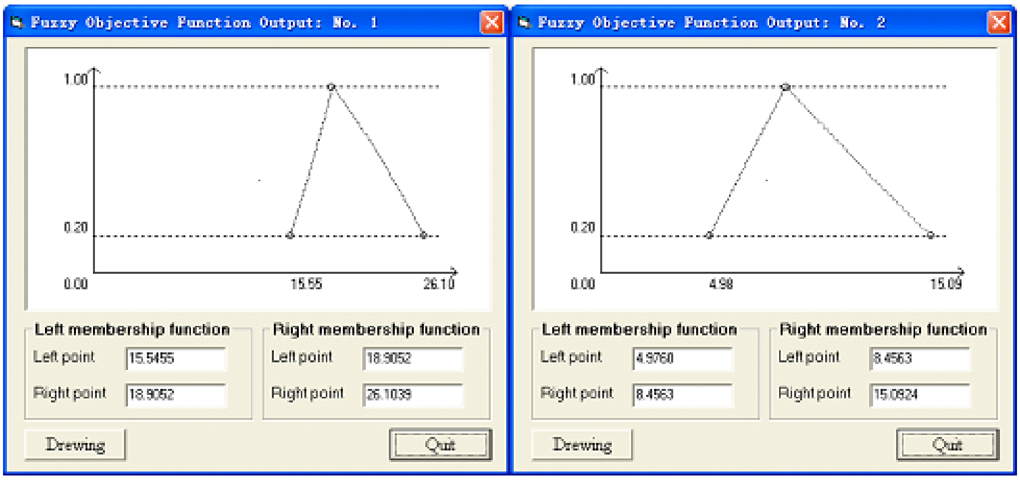

Fig. 1. Membership functions of $F_1(x^*, y^*)$ and $F_2(x^*, y^*)$

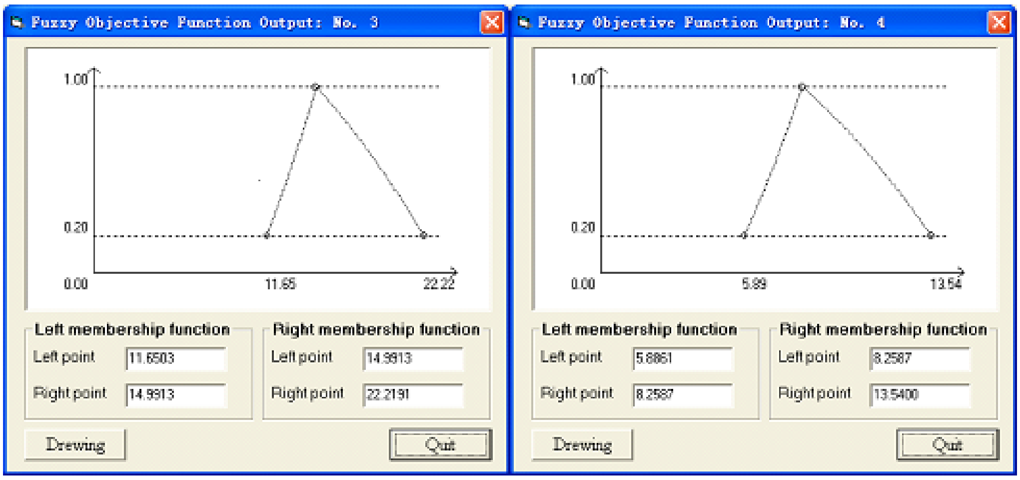

Fig. 2. Membership functions of $f_1(x^*, y^*)$ and $f_2(x^*, y^*)$

For a decision making process involved with fuzzy coefficients, decision makers may sometimes make small adjustment on the uncertain information about the preference or circumstances. If the change occurs to the minor information, i.e. with smaller satisfactory degrees, there should normally be no tremendous change to the final solution. For example, when estimating future profit, the manufacturer may adjust the possibility of five thousand dollars' profit from 2% to 3%, while the possibility of one hundred thousand dollars' profit remains 100%. In such a situation, there should be no outstanding change for his or her final decision on the device investment. Therefore, to increase the satisfactory degrees is an acceptable strategy for a feasible solution.

From the above analysis, the advantages and disadvantages of the algorithm proposed in this study are as follows:

1) This algorithm is quite efficient, as it adopts strategies to transform a non-linear bi-level problem into a linear problem;
2) When pursuing optimality, the negative effect from conflicting objectives can be avoided and a leader can finally reach his or her satisfactory solution by setting goals for the objectives;
3) The information of the original fuzzy numbers are considered adequately by using a certain number of λ-cuts to approximate the final precise solution;
4) In some situations, this algorithm might suffer from expensive calculation, as the size of λ-cuts will increase exponentially with respect to iteration counts.

V. Conclusion and Future Study

Many organizational decision problems can be formulated by bi-level decision models. In a bi-level decision model, the leader and/or the follower may have more than one objective to achieve, which is different to simple bi-level optimization problems. This kind of bi-level decision problem is studied by goal programming in this paper. Meanwhile, we take into consideration the situation where coefficients which formulate a bi-level decision model are not precisely known to us. Fuzzy set method is applied to handle these coefficients.

This paper proposed a λ-cut and goal programming based algorithm for FLMOB decision problems, and presented a real case study on a newsboy problem to explain this algorithm.

Experiments reveal that the algorithm is quite effective and efficient. In the future, we will focus on situations which involve multiple followers.

VI. Appendix

The model of a general bi-level decision problem with multiple objectives for both the leader and follower was given in [38]. It is re-formulated in this paper as:

For $x \in X \subset R^n$, $y \in Y \subset R^m$, a multiple objective bi-level (MOB) model is:

$$\min_{x \in X} F(x,y) \tag{11a}$$
$$\text{s.t. } G(x,y) \le 0 \tag{11b}$$
$$\min_{y \in Y} f(x,y) \tag{11c}$$
$$\text{s.t. } g(x,y) \le 0 \tag{11d}$$

where $F : R^n \times R^m \to R^s$, $G : R^n \times R^m \to R^p$, $f : R^n \times R^m \to R^t$, and $g : R^n \times R^m \to R^q$.

Associated with the MOB problem (11), some definitions are listed below:

Definition *6:*

1) Constraint region of the MOB (11):

$$S \triangleq \{(x,y) : x \in X, y \in Y, G(x,y) \le 0, g(x,y) \le 0\}$$

It refers to all possible combination of choices that the leader and follower may make.

2) Projection of S onto the leader's decision space:

$$S(X) \triangleq \{x \in X : \exists y \in Y, G(x,y) \le 0, g(x,y) \le 0\}$$

3) Feasible set for the follower $\forall x \in S(X)$:

$$S(x) \triangleq \{y \in Y : (x,y) \in S\}$$

4) The follower's rational reaction set for $x \in S(X)$:

$$P(x) \triangleq \{y \in Y : y \in argmin[f(x,\hat{y}) : \hat{y} \in S(x)]\}$$

where $argmin[f(x,\hat{y}) : \hat{y} \in S(x)] = \{y \in S(x) : f(x,y) \le f(x,\hat{y}), \hat{y} \in S(x)\}$.

The follower observes the leader's action and reacts by selecting y from his or her feasible set to minimize his or her objective function.

5) Inducible region:

$$IR \triangleq \{(x,y) : (x,y) \in S, y \in P(x)\}$$

which represents the set over which a leader may optimize his or her objectives.

To ensure that (11) is well posed, it is assumed that S is non-empty and compact, and that for all decisions taken by the leader, the follower has some room to respond: i.e., $P(x) \neq \emptyset$.

Thus, in terms of the above notation, the MOB can be written as:

$$\min\{F(x,y) : (x,y) \in IR\} \tag{12}$$

Proof of Theorem 1:

Proof: By Definition 6, let the notations associated with problem (7) are denoted by:

$$\begin{aligned}
S = \{(x,y) : {} & A_{k\lambda_j}^{L} x + B_{k\lambda_j}^{L} y \le b_{k\lambda_j}^{L}, \\
& A_{k\lambda_j}^{R} x + B_{k\lambda_j}^{R} y \le b_{k\lambda_j}^{R}, \\
& k = 1,2, j = 0,1,\ldots,l,\} \qquad (13a)\\
S(X) = \{x \in X : {} & \exists y \in Y, A_{k\lambda_j}^{L} x + B_{k\lambda_j}^{L} y \le b_{k\lambda_j}^{L}, \\
& A_{k\lambda_j}^{R} x + B_{k\lambda_j}^{R} y \le b_{k\lambda_j}^{R}, \\
& k = 1,2, j = 0,1\ldots,l,\} \qquad (13b)\\
S(x) = {} & \{y \in Y : (x,y) \in S\} \qquad (13c)\\
P(x) = {} & \{y \in Y : y \in argmin \Psi\} \qquad (13d)\\
\text{where } \Psi = {} & \frac{1}{l+1} \sum_{i=1}^{t} \sum_{j=0}^{l} \{|c_{i2\lambda_j}^{L} x + d_{i2\lambda_j}^{L} \hat{y} - g_{Fi\lambda_j}^{L}| \\
& + |c_{i2\lambda_j}^{R} x + d_{i2\lambda_j}^{R} \hat{y} - g_{Fi\lambda_j}^{R}|, \hat{y} \in S(x)\} \\
IR = {} & \{(x,y) : (x,y) \in S, y \in P(x)\} \qquad (13e)
\end{aligned}$$

Problem (7) can be written as

$$\begin{aligned}
\min_{x \in X} \frac{1}{l+1} & \sum_{h=1}^{s} \sum_{j=0}^{l} \{|c_{h1\lambda_j}^{L} x + d_{h1\lambda_j}^{L} y - g_{Lh\lambda_j}^{L}| \qquad (14)\\
& + |c_{h1\lambda_j}^{R} x + d_{h1\lambda_j}^{R} y - g_{Lh\lambda_j}^{R}|\} \\
\text{s.t. } & (x,y) \in IR \qquad (15)
\end{aligned}$$

And those of problem (8) are denoted by:

$$\begin{aligned}
S' = \{(x',y') : {} & A_{k\lambda_j}^{L} x + B_{k\lambda_j}^{L} y \le b_{k\lambda_j}^{L}, \qquad (16a)\\
& A_{k\lambda_j}^{R} x + B_{k\lambda_j}^{R} y \le b_{k\lambda_j}^{R}, k = 1,2, j = 0,1\ldots,l \\
& \sum_{j=0}^{l} c_{h1\lambda_j}^{L} x + \sum_{j=0}^{l} d_{h1\lambda_j}^{L} y + v_{h1}^{L-} - v_{h1}^{L+} = \sum_{j=0}^{l} g_{Lh\lambda_j}^{L}, \\
& \sum_{j=0}^{l} c_{h1\lambda_j}^{R} x + \sum_{j=0}^{l} d_{h1\lambda_j}^{R} y + v_{h1}^{R-} - v_{h1}^{R+} = \sum_{j=0}^{l} g_{Lh\lambda_j}^{R}, \\
& v_{h1}^{L-}, v_{h1}^{L+}, v_{h1}^{R-}, v_{h1}^{R+} \ge 0, \qquad (16b)\\
& v_{h1}^{L-} \cdot v_{h1}^{L+} = 0, \\
& v_{h1}^{R-} \cdot v_{h1}^{R+} = 0, \\
& h = 1,2,\ldots,s, \\
& \sum_{j=0}^{l} c_{i2\lambda_j}^{L} x + \sum_{j=0}^{l} d_{i2\lambda_j}^{L} y + v_{i2}^{L-} - v_{i2}^{L+} = \sum_{j=0}^{l} g_{Fi\lambda_j}^{L}, \\
& \sum_{j=0}^{l} c_{i2\lambda_j}^{R} x + \sum_{j=0}^{l} d_{i2\lambda_j}^{R} y + v_{i2}^{R-} - v_{i2}^{R+} = \sum_{j=0}^{l} g_{Fi\lambda_j}^{R}, \\
& v_{i2}^{L-}, v_{i2}^{L+}, v_{i2}^{R-}, v_{i2}^{R+} \ge 0, \\
& v_{i2}^{L-} \cdot v_{i2}^{L+} = 0, \\
& v_{i2}^{R-} \cdot v_{i2}^{R+} = 0, \\
& i = 1,2,\ldots,t,\}
\end{aligned}$$

$$S(X') = \{x' \in X' : \exists y' \in Y', A_{k\lambda_j}^{L} x + B_{k\lambda_j}^{L} y \le b_{k\lambda_j}^{L},$$
$$A_{k\lambda_j}^{R} x + B_{k\lambda_j}^{R} y \le b_{k\lambda_j}^{R},$$
$$k = 1, 2, j = 0, 1 \ldots, l,$$
$$\sum_{j=0}^{l} c_{h1\lambda_j}^{L} x + \sum_{j=0}^{l} d_{h1\lambda_j}^{L} y + v_{h1}^{L-} - v_{h1}^{L+} = \sum_{j=0}^{l} g_{Lh\lambda_j}^{L},$$
$$\sum_{j=0}^{l} c_{h1\lambda_j}^{R} x + \sum_{j=0}^{l} d_{h1\lambda_j}^{R} y + v_{h1}^{R-} - v_{h1}^{R+} = \sum_{j=0}^{l} g_{Lh\lambda_j}^{R},$$
$$v_{h1}^{L-}, v_{h1}^{L+}, v_{h1}^{R-}, v_{h1}^{R+} \ge 0,$$
$$v_{h1}^{L-} \cdot v_{h1}^{L+} = 0,$$
$$v_{h1}^{R-} \cdot v_{h1}^{R+} = 0,$$
$$h = 1, 2, \ldots, s,$$
$$\sum_{j=0}^{l} c_{i2\lambda_j}^{L} x + \sum_{j=0}^{l} d_{i2\lambda_j}^{L} y + v_{i2}^{L-} - v_{i2}^{L+} = \sum_{j=0}^{l} g_{Fi\lambda_j}^{L},$$
$$\sum_{j=0}^{l} c_{i2\lambda_j}^{R} x + \sum_{j=0}^{l} d_{i2\lambda_j}^{R} y + v_{i2}^{R-} - v_{i2}^{R+} = \sum_{j=0}^{l} g_{Fi\lambda_j}^{R},$$
$$v_{i2}^{L-}, v_{i2}^{L+}, v_{i2}^{R-}, v_{i2}^{R+} \ge 0,$$
$$v_{i2}^{L-} \cdot v_{i2}^{L+} = 0,$$
$$v_{i2}^{R-} \cdot v_{i2}^{R+} = 0,$$
$$i = 1, 2, \ldots, t, \} \tag{16c}$$
$$S(x') = \{y' \in Y' : (x', y') \in S'\} \tag{16d}$$
$$P(x') = \{y' \in Y' :$$
$$y' \in argmin[\sum_{i=1}^{t} (\hat{v}_{i2}^{L-} + \hat{v}_{i2}^{L+} + \hat{v}_{i2}^{R-} + \hat{v}_{i2}^{R+}) : \hat{y'} \in S(x')]\} \tag{16e}$$
$$IR' = \{(x', y') : (x', y') \in S', y' \in P(x')\} \tag{16f}$$

Problem (8) can be written as

$$\min_{x' \in X'} \{ \sum_{h=1}^{l} (v_{h1}^{L-} + v_{h1}^{L+} + v_{h1}^{R-} + v_{h1}^{R+}) : (x', y') \in IR' \} \tag{17}$$

As $(x^{'*}, y^{'*})$ is the optimal solution to problem (8), from (17), it can be seen that, $\forall (x', y') \in IR'$, we have:

$$\sum_{h=1}^{l} (v_{h1}^{L-} + v_{h1}^{L+} + v_{h1}^{R-} + v_{h1}^{R+})$$
$$\ge \sum_{h=1}^{l} (v_{h1}^{L-*} + v_{h1}^{L+*} + v_{h1}^{R-*} + v_{h1}^{R+*})$$

As: $\sum_{j=0}^{l} c_{h1\lambda_j}^{L} x + \sum_{j=0}^{l} d_{h1\lambda_j}^{L} y + v_{h1}^{L-} - v_{h1}^{L+} = \sum_{j=0}^{l} g_{Lh\lambda_j}^{L}$ and $v_{h1}^{L-} \cdot v_{h1}^{L+} = 0$, $h = 1, 2, \ldots, s$, we have:

$$v_{h1}^{-} + v_{h1}^{+} = \ |\sum_{j=0}^{l} c_{h1\lambda_j}^{L} x + \sum_{j=0}^{l} d_{h1\lambda_j}^{L} y - \sum_{j=0}^{l} g_{Lh\lambda_j}^{L}|$$
$$v_{h1}^{L-*} + v_{h1}^{L+*} = \ |\sum_{j=0}^{l} c_{h1\lambda_j}^{L} x^{*} + \sum_{j=0}^{l} d_{h1\lambda_j}^{L} y^{*} - \sum_{j=0}^{l} g_{Lh\lambda_j}^{L}|,$$

for $h = 1, 2, \ldots, s$.

Similarly, we have:

$$v_{h1}^{R-} + v_{h1}^{R+} = \ |\sum_{j=0}^{l} c_{h1\lambda_j}^{R} x + \sum_{j=0}^{l} d_{h1\lambda_j}^{R} y - \sum_{j=0}^{l} g_{Lh\lambda_j}^{R}|,$$
$$v_{h1}^{R-*} + v_{h1}^{R+*} = \ |\sum_{j=0}^{l} c_{h1\lambda_j}^{R} x^{*} + \sum_{j=0}^{l} d_{h1\lambda_j}^{R} y^{*} - \sum_{j=0}^{l} g_{Lh\lambda_j}^{R}|,$$

for $h = 1, 2, \ldots, s$.

So: $\forall (x', y') \in IR'$,

$$|\sum_{j=0}^{l} c_{h1\lambda_j}^{L} x + \sum_{j=0}^{l} d_{h1\lambda_j}^{L} y - \sum_{j=0}^{l} g_{Lh\lambda_j}^{L}|$$
$$+ \ |\sum_{j=0}^{l} c_{h1\lambda_j}^{R} x + \sum_{j=0}^{l} d_{h1\lambda_j}^{R} y - \sum_{j=0}^{l} g_{Lh\lambda_j}^{R}|$$
$$\ge \ |\sum_{j=0}^{l} c_{h1\lambda_j}^{L} x^{*} + \sum_{j=0}^{l} d_{h1\lambda_j}^{L} y^{*} - \sum_{j=0}^{l} g_{Lh\lambda_j}^{L}|$$
$$+ \ |\sum_{j=0}^{l} c_{h1\lambda_j}^{R} x^{*} + \sum_{j=0}^{l} d_{h1\lambda_j}^{R} y^{*} - \sum_{j=0}^{l} g_{Lh\lambda_j}^{R}|, \ h = 1, 2, \ldots, s \tag{18}$$

We now prove that the projection of S' onto the $X \times Y$ space, denoted by $S'|_{X,Y}$, is equal to S:

On the one hand, $\forall (x, y) \in S'|_{X,Y}$, from constraints: $A_{k\lambda_j}^{L} x + B_{k\lambda_j}^{L} y \le b_{k\lambda_j}^{L}, A_{k\lambda_j}^{R} x + B_{k\lambda_j}^{R} y \le b_{k\lambda_j}^{R}, k = 1, 2, j = 0, 1 \ldots, l$, in S', we have: $(x, y) \in S$, so $S'|_{X,Y} \subseteq S$.

On the other hand, $\forall (x, y) \in S$, by (**??**), we can always find v_{11}^{L-}, v_{11}^{L+}, v_{11}^{R-}, v_{11}^{R+},..., v_{s1}^{L-}, v_{s1}^{L+}, v_{s1}^{R-}, v_{s1}^{R+}, v_{12}^{L-}, v_{12}^{L+}, v_{12}^{R-}, v_{12}^{R+}, ..., v_{t2}^{L-}, v_{t2}^{L+}, v_{t2}^{R-}, v_{t2}^{R+} which satisfies the constraints of (8b) and (8d). Together with the inequations of $A_{k\lambda_j}^{L} x + B_{k\lambda_j}^{L} y \le b_{k\lambda_j}^{L}$, and $A_{k\lambda_j}^{R} x + B_{k\lambda_j}^{R} y \le b_{k\lambda_j}^{R}, k = 1, 2, j = 0, 1 \ldots, l$, requested by S, we have $(x, v_{11}^{L-}, v_{11}^{L+}, v_{11}^{R-}, v_{11}^{R+}, \ldots, v_{s1}^{L-}, v_{s1}^{L+}, v_{s1}^{R-}, v_{s1}^{R+}, y, v_{12}^{L-}, v_{12}^{L+}, v_{12}^{R-}, v_{12}^{R+}, \ldots, v_{t2}^{L-}, v_{t2}^{L+}, v_{t2}^{R-}, v_{t2}^{R+}) \in S'$, thus $(x, y) \in S'|_{X,Y}$, $S \subseteq S'|_{X,Y}$.

So, we can prove that:

$$S'|_{X,Y} = S \tag{19}$$

Similarly, we have

$$S(x)'|_{X,Y} = S(x) \tag{20a}$$
$$S(X)'|_{X,Y} = S(X) \tag{20b}$$

Also, from

$$\sum_{j=0}^{l} c_{i2\lambda_j}^{L} x + \sum_{j=0}^{l} d_{i2\lambda_j}^{L} y + v_{i2}^{L-} - v_{i2}^{L+} = \sum_{j=0}^{l} g_{Fi\lambda_j}^{L}$$

and

$$v_{i2}^{L-} \cdot v_{i2}^{L+} = 0,$$

for $i = 1, 2, \ldots, t$.

we have:

$$v_{i2}^{L-} + v_{i2}^{L+} = |\sum_{j=0}^{l} c_{i2\lambda_j}^{L} x + \sum_{j=0}^{l} d_{i2\lambda_j}^{L} y - \sum_{j=0}^{l} g_{Fi\lambda_j}^{L}|, \tag{21a}$$

for $i = 1, 2, \ldots, t$. Similarly, we have:

$$v_{i2}^{R-} + v_{i2}^{R+} = |\sum_{j=0}^{l} c_{i2\lambda_j}^{R} x + \sum_{j=0}^{l} d_{i2\lambda_j}^{R} y - \sum_{j=0}^{l} g_{Fi\lambda_j}^{R}|, \tag{21b}$$

for $i = 1, 2, \ldots, t$. Thus:

$$P(x') = \{y' \in Y' : y' \in argmin\Psi'\} \quad (22)$$

where $\Psi' = \sum_{i=1}^{t}\sum_{j=0}^{l}\{|c_{i2\lambda_j}^{L}x + d_{i2\lambda_j}^{L}\hat{y} - g_{Fi\lambda_j}^{L}| + |c_{i2\lambda_j}^{R}x + d_{i2\lambda_j}^{R}\hat{y} - g_{Fi\lambda_j}^{R}|, \hat{y} \in S(x')\}$.

From (19) and (22), we get:

$$P(x')|_{X\times Y} = P(x) \quad (23)$$

From (13e), (16f), (19) and (23), we get:

$$IR'|_{X\times Y} = IR \quad (24)$$

which means, in $X \times Y$ space, the leaders of problem (7) and (8) have the same optimizing space.

Thus, from (18) and (24), it can be obtained that: $\forall(x, y) \in IR$, we have:

$$\begin{aligned}&\tfrac{1}{l+1}\sum_{h=1}^{s}\sum_{j=0}^{l}\{|c_{h1\lambda_j}^{L}x + d_{h1\lambda_j}^{L}y - g_{Lh\lambda_j}^{L}| \\ &\quad + |c_{h1\lambda_j}^{R}x + d_{h1\lambda_j}^{R}y - g_{Lh\lambda_j}^{R}|\} \\ &\geq \tfrac{1}{l+1}\sum_{h=1}^{s}\sum_{j=0}^{l}\{|c_{h1\lambda_j}^{L}x^* + d_{h1\lambda_j}^{L}y^* - g_{Lh\lambda_j}^{L}| \\ &\quad + |c_{h1\lambda_j}^{R}x^* + d_{h1\lambda_j}^{R}y^* - g_{Lh\lambda_j}^{R}|\},\end{aligned}$$

So: (x^*, y^*) is the optimal solution of the problem (7). ■

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