

# Theory and Application of Model Risk Quantification

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## CERTIFICATE OF ORIGINAL AUTHORSHIP

I, Yu Feng declare that this thesis, is submitted in fulfilment of the requirements for the award of Doctor of Philosophy, in the School of Business at the University of Technology Sydney.

This thesis is wholly my own work unless otherwise reference or acknowledged. In addition, I certify that all information sources and literature used are indicated in the thesis.

This document has not been submitted for qualifications at any other academic institution.

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# DEDICATION

To my beloved family

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# Abstract

The renowned statistician George E. P. Box wrote that “essentially, all models are wrong, but some are useful.” (Box and Draper 1987) This is certainly true in finance, where many models and techniques that have been extensively empirically invalidated remain in widespread use, not just in academia, but also (perhaps especially) among practitioners. Incorrect models, and model misuse, represent a source of risk that is being increasingly recognised — this is called “model risk.”

Following on the non-parametric approach of Glasserman and Xu (2014) to model risk quantification, we develop new theory and methods for a variety of applications. The work consists of three parts. The first part focuses on the risk in applying option pricing models. In particular, there are two aspects of model risk: the inability of a chosen model to fit observed market prices at a given point in time (calibration error) and the model risk due to recalibration of model parameters (in contradiction to the model assumptions). We quantify these two sources of model risk in a common framework, and consider the trade-offs between them when choosing a model and the frequency with which to recalibrate to the market. We illustrate this approach applied to the models of Black and Scholes (1973) and Heston (1993), using option data for Apple (AAPL) and Google (GOOG).

The second part involves construction of a theory that quantifies model risk for path-dependent losses. The proposed theory generalises the relative-entropic approach of Glasserman and Xu (2014) to any  $f$ -divergence. It provides an unified treatment for all underlying dynamics and path-dependency. Three powerful tools are proposed for financial practitioners to quantify model risk. Just like derivative pricing, model risk can also be evaluated using martingale or tree approaches, or by solving partial differential equations.

The third part proposes a new approach to model risk measurement based on the Wasserstein distance between two probability measures. It formulates the theoretical motivation resulting from the interpretation of a fictitious adversary in robust risk management. The proposed approach accounts for all alternative models. It provides practically feasible results that overcome the restriction and the integrability issue imposed by the nominal model. The Wasserstein approach suits for all types of model risk problems, ranging from the single-asset hedging problem to the multi-asset allocation problem. The robust capital allocation line, accounting for the correlation risk, is not achievable with the relative-entropic approach.

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