# Boson Sampling Private-Key Quantum Cryptography

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We introduce a quantum private-key encryption protocol based on multi-photon interference in linear optics networks. The scheme builds upon Boson Sampling, and we show that it is hard to break, even for a quantum computer. We present an information-theoretic proof of the security of our protocol against an eavesdropper with unlimited (quantum) computational power but time-limited quantum storage. This protocol is shown to be optimal in the sense that it asymptotically encrypts all the information that passes through the interferometer using an exponentially smaller private key. This is the first practical application of Boson Sampling in quantum communication. Our scheme requires only moderate photon numbers and is experimentally feasible with current technology.

Private-key, or symmetric-key, encryption is the cryptographic task of obfuscating data using a secret key and sending it to a recipient who shares the key. An eavesdropper intercepting the transmission cannot decipher the message without the key. Classically there is only one such encryption protocol that provably exhibits perfect information-theoretic security, namely the one-time pad [1]. However, the one-time pad is often impractical since the secret key must be as long as the message and can never be reused, leading to significant bandwidth requirements. Quantum key distribution [2, 3] (QKD), most notably the BB84 protocol [4], enables the secret sharing of random bit-strings, which can subsequently be employed in a one-time pad. QKD therefore seems to inherit the impracticalities of the classical one-time pad. Classically, private-key protocols with keys that are short compared to the message length suffer from a lack of strong complexity proofs for their security [5, 6]. Short private-key quantum communication protocols exist, using the phenomenon of Quantum Data Locking (QDL) [7–10]. Here, we propose Boson Sampling as a practical, efficient implementation of QDL for short private-key quantum communication, and we prove its security against an eavesdropper who may have a quantum computer, but does not possess perfect quantum storage. This is the first direct practical application of Boson Sampling for a quantum information processing task.

Boson Sampling is the problem of sampling from a specific ensemble, namely, the one generated by a passive linear optical interferometer fed with single-photon

inputs [11, 12]. A strong advantage of using a Boson Sampling device is that an exponentially large Hilbert space can be arrived at using only a polynomial number of photons and optical modes. It has been shown that such devices generate a large amount of number-path entanglement [13], a fact that has inspired applications in quantum metrology [13, 14] and quantum chemistry [15]. Here we show that this entanglement can be exploited directly in quantum cryptography, since the exponential number of basis states in the number-path degrees of freedom makes it very difficult for an eavesdropper to obtain information about the state. Given that the generation and detection of single photons have become standardised and relatively straightforward to implement in the lab (see for example Refs. [16–21]), our scheme serves as a practical new protocol for efficient quantum cryptography with near-term technology.

We present the first information-theoretic proof of the security of Boson Sampling as a quantum public-key cryptography protocol under the assumption that Eve has access to a quantum computer, but not to perfect quantum storage technology. Imperfect quantum storage is a common assumption in a number of quantum cryptographic protocols, including the Bounded Quantum Storage Model, [22], the Noisy Storage Model [23] (see also Ref. [24]), and QDL [25-27]. It states that Eve cannot store unlimited amounts of quantum information for an unlimited time with unit fidelity. This assumption is valid in the near to medium term, when quantum storage technologies do not meet Eve's needs. Under this assumption, all of the above protocols are known to guarantee composable security [28]. The number of photons required for communication security in our protocol is much smaller than the number of photons that render

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Boson Sampling intractable on classical computers.

In this Letter, we first describe the task of Boson Sampling and how it leads to our quantum communication protocol. Next, we provide some intuition for the security of the protocol and introduce the QDL technique. We sketch the security proof for a noiseless communication channel and extend it to noisy channels. Finally, we show how loss and noise affect the secure communication rate and how error correction can be included without having to re-derive the security bounds.

Boson Sampling:— Boson Sampling is a non-universal model for quantum computing that involves taking samples of a multimode interferometer, where the inputs are single-photon Fock states. Let n photons be sent into m optical modes of an interferometer with at most one photon per input mode. The input modes  $\hat{a}$  evolve unitarily into output modes  $U\hat{a}U^{\dagger}$ , with U the linear optical transformation describing the interferometer. Aaronson and Arkhipov showed that when  $m \gg n^2$ , simulating a sample of the photon distribution at the output (even approximately) is computationally difficult for a classical computer given reasonable complexity-theoretic assumptions [11]. Based on the most recent simulation algorithms [29], a value of n < 100 is likely sufficient to surpass the point at which the sampling problem is tractable on classical computers. The output from the interferometer prior to photo-detection can be expanded in the photon number basis:

$$|\psi_{\text{out}}\rangle = \sum_{\boldsymbol{n}} \lambda_{\boldsymbol{n}} |n_1, ..., n_m\rangle ,$$
 (1)

where  $n=(n_1,....,n_m)$  denotes a photon-number configuration with  $n_i$  photons in the ith mode and  $\lambda_n$  its amplitude. Moreover,  $\sum_i n_i = n$ . The boson sampling task is to generate a sample from the distribution induced by  $\{\lambda_n\}$ . This can be done by a classical computer when the  $\lambda_n$  are known. However, calculating  $\{\lambda_n\}$  given U and an n-photon input state approximates calculating the permanent of a random  $n \times n$  matrix of [30], which is known to be #P-complete on a classical computer.

Protocol:— Alice and Bob aim to communicate in secret, while Eve attempts to eavesdrop on them. Our protocol consists of the following steps (see Fig. 1), and the proof of the security is based on a random coding technique (all logs are of base 2):

- 1. The code words  $|\psi_j\rangle$  are chosen randomly from the  $M=\binom{m}{n}$  possible permutations of n photons over m modes. The number of code words is  $M'=\delta M$ , where  $0<\delta<1$  is a constant. For  $\delta\ll 1$  Alice is encoding about  $\log M-O(\log{(1/\delta)})$  bits of information. Note that the overall dimension of the Hilbert space (including states with multiple photons in one mode) is  $d=\binom{m+n-1}{n}$ .
- Alice and Bob share a short unconditionally secure key of length log K bits, obtained for example using BB84.

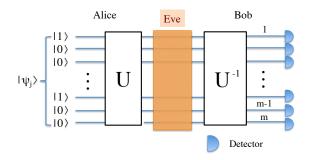


FIG. 1. Circuit layout for the encryption protocol. The code words  $|\psi_j\rangle$  are chosen from the  $\binom{m}{n}$  possible permutations of n photons in m modes. Alice and Bob share a secret key in advance, which Alice uses to encode her binary photonic input message via a random unitary U. Bob decodes using the inverse operation  $U^{-1}$ .

- 3. In advance, but not necessarily in secret, Alice and Bob agree upon a set of K Haar-random  $m \times m$  unitary matrices,  $\{U_k\}$ .
- 4. To encrypt the input code word, Alice applies the linear-optical network corresponding to the unique unitary  $U_k$  associated with the secret key.
- 5. Bob, who knows the secret key, applies  $U_k^{-1}$  and measures the output photon number distribution (it is sufficient to use on-off detectors for this task). This should match  $|\psi_j\rangle$ , the unencrypted code word sent by Alice.

The code space can be very large, leading to an efficient communication protocol. Eve does not have access to  $U_k$ , from which the security of the encryption arises.

We have presented an abstract protocol that can be implemented using any physical degrees of freedom. It is natural to implement the scheme using an interferometer with m spatial modes, however, for long-range transmission, a time-bin encoded approach [31, 32] could aid to reduce phase fluctuations.

Security:— We first provide some intuition into the security of the protocol. Eve intercepts the encrypted message and measures non-destructively to infer the code word. If she chooses a random unitary  $V \in \{U_k\}$ , the effective evolution of encryption followed by an eavesdropping attempt would be given by  $V^{\dagger}U_k$ . The probability of obtaining the guess  $|\psi_{j'}\rangle$  given the code word  $|\psi_j\rangle$  is  $P_{jj'} = |\langle \psi_{j'}|V^{\dagger}U_k|\psi_j\rangle|^2$ , which can be expressed in terms of a matrix permanent and is classically hard to compute (#P-complete).

In contrast to classical encryption techniques, where brute-force attacks are possible in principle, here the no-cloning theorem forbids the replication of arbitrary unknown states. Thus for each code word transmission Eve has only one attempt to infer the message from the cipher-text. Collapsing the state in Eq. (1) will cause errors in Bob's decoding and will reveal Eve's presence. Our claim is that even if Eve is endowed with a quantum computer, her opportunity to eavesdrop is obstructed.

Our assumptions are that the initial secret key remains secret to Eve, at least up to the point where she makes the measurement, and she does not have a quantum memory that can hold the state for indefinite time. These assumptions are identical to those of QDL, and it is therefore natural to use QDL as a theoretical tool to prove the security of our protocol.

Quantum Data Locking:— The Quantum Data Locking (QDL) effect is a uniquely quantum phenomenon that separates classical and quantum information theory [7–9]. When a classical message is encoded into a quantum state, it requires a much shorter key to completely scramble the information, such that the number of bits obtainable by Eve can be made arbitrarily small. The trusted parties, Alice and Bob, use a short shared secret key of  $\log K$  bits to select a set of M code words that they use to send information. Previous works have established a theory of a quantum enigma machine [25, 26, 33–36] that exploits QDL for key distribution and direct secret communication over a quantum communication channel. Here we show that multi-photon interference in the regime of Boson Sampling enhances the secret key efficiency.

Sketch of security proof:— Alice randomly prepares one of the d-dimensional vectors  $|\psi_j\rangle$  and scrambles it by applying one of the unitary transformations  $U_k$ . Since Eve does not have access to the secret key, her description of the state is the statistical mixture

$$\rho_{AE} = \frac{1}{M'} \sum_{j=1}^{M'} |j\rangle_A \langle j| \otimes \frac{1}{K} \sum_{k=1}^K U_k |\psi_j\rangle_E \langle \psi_j| U_k^{\dagger}, \quad (2)$$

where  $|j\rangle$  denotes Alice's record of the code word she sent to Bob (and intercepted by Eve). We assess the security via the accessible information  $I_{\rm acc}(A;E)$ , which quantifies the information that Eve can extract by measuring the quantum state [7, 8]. When A is the random variable representing the distribution of code words and E is the random variable associated with Eve's measurement outcomes, this becomes

$$I_{\rm acc}(A; E) = \max_{\mathcal{M}_E} [H(A) + H(E) - H(A, E)],$$
 (3)

where H(X) is the Shannon entropy of the random variable X, and the maximization is over Eve's measurements  $\mathcal{M}_E$ . Using the accessible information implicitly assumes that Eve must measure her share of the quantum system after a given time [27]. This is the same assumption underlying our Boson Sampling communication protocol.

Eve's best measurement for extracting information about the code words takes the form of a rank-one POVM  $\{\alpha_y | \phi_y \rangle \langle \phi_y | \}_y$  [7], with the  $|\phi_y \rangle$  possibly non-orthogonal and  $\alpha_y$  the normalisations that ensure the POVM elements sum to the identity. This POVM leads to the probability distributions  $p_E(y) = \alpha_y \langle \phi_y | \rho_E | \phi_y \rangle$  and  $p_{E|j}(y) = \alpha_y \langle \phi_y | \rho_{E|j} | \phi_y \rangle$ , where  $\rho_E$  is Eve's reduced density matrix obtained from Eq. (2), and

 $\rho_{E|j} = \sum_{k} K^{-1} U_{k} |\psi_{j}\rangle_{E} \langle \psi_{j} | U_{k}^{\dagger}$  is Eve's reduced density matrix given a specific choice for j. The accessible information can then be written as (see Supplemental Material)

$$I_{\rm acc}(A; E) \sim H(p_E) - \frac{1}{M'} \sum_{j=1}^{M'} H(p_{E|j}).$$
 (4)

To demonstrate the security of the protocol we show that  $I_{\rm acc}(A;E)$  can be made arbitrarily small for finite message size if M' and K are large enough. The proof is based on the fact that by increasing K both  $p_E$  and  $p_{E|j}$  concentrate towards their common average  $\mu$ , and the probability of a deviation larger than  $\epsilon\mu$  is exponentially suppressed. Therefore both the entropy  $H(p_E)$  and the conditional entropy  $\frac{1}{M'}\sum_{j=1}^{M'}H(p_{E|j})$  will tend to the same value. The conclusion is that for any  $\epsilon>0$  the probability of  $I_{\rm acc}(A;E)>\epsilon\log(1/\mu)$  is exponentially suppressed in K. In the Supplemental Material we determine the required size of the initial secret key K, which in turn allows us to compute the net secure communication rate per photon and per channel use.

Noiseless channels in the  $m \gg n^2$  limit:— First consider an ideal scenario in which Alice and Bob can communicate via a perfect, noiseless channel. Moreover consider for now the regime of  $m \gg n^2$  (note that this is the regime in which Boson Sampling has been first introduced). As Alice and Bob initially shared a key of  $\log K$  bits, the net information gain of the protocol is

$$R = \log M - \log K - O(\log(1/\delta)). \tag{5}$$

We find that the accessible information can be made  $\epsilon$ small if the number K of different scrambling unitaries
is chosen such that ( $\gamma$  is defined below)

$$\log K = \log \gamma + \log \frac{d}{M} + O\left(\log \frac{1}{\epsilon}\right). \tag{6}$$

The coefficient  $\gamma$  characterizes the statistics of the random variable  $X_U = |\langle \phi | U | \psi_j \rangle|^2$ . Note that, for given  $\phi$  and  $\psi$ ,  $X_U$  is a random variable because is a function of the random unitary U. We define

$$\gamma := \frac{\mathbb{E}_U[X_U^2]}{\mathbb{E}_U[X_U]^2},\tag{7}$$

where  $\mathbb{E}_U$  denotes the expectation value over the choice of the unitary U (recall that the unitaries are sampled from the uniform distribution). Note that  $\gamma$  quantifies the spread of the random variable around its expectation value. The more  $X_U$  is narrowly concentrated around its mean, the smaller the private key.

The coefficient  $\gamma$  can be computed analytically in the limit of  $m \gg n^2$ , where photon-bunching is statistically suppressed. Note that  $X_U$  is a polynomial of degree 2n in the matrix elements of U, and in this limit it takes the form of the permanent of a random  $n \times n$  sub-matrix of

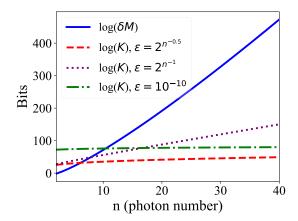


FIG. 2. The number of transmitted bits  $\log \delta M, \delta = 0.01$ , and the secret key consumption  $\log K$ . For  $m = n^3$ , code words  $\log M$  (blue solid line), secret key size  $\log K$  for different security parameters,  $\epsilon = \exp(-n^s \ln 2)$ , with s = 0.5 (red dashed), s = 1 (purple dotted), and for fixed  $\epsilon = 10^{-10}$  (green dotted dashed). If we choose the security parameter  $\epsilon \propto \exp(-n^s \ln 2)$ ,  $\epsilon \log M \to 0$  as  $n \to \infty$ .

U. The elements of the sub-matrix are effectively independent when m is sufficiently large. Using this approximation, we can analytically calculate the moments of  $X_U$  with respect to the uniform Haar measure over unitary matrices:

$$\mathbb{E}_{U}[X_{U}] = \frac{n!}{m^{n}}, \qquad \mathbb{E}_{U}[X_{U}^{2}] \le \frac{2n!(n+1)!}{m^{2n}}.$$
 (8)

Therefore  $\gamma \leq 2(n+1)$  and we can estimate explicitly how many unitaries are needed for secure encryption from Eq. (6), which only grows logarithmically with the number of photons plus a constant overhead.

Using the Stirling approximation we obtain, for  $m \gg n^2 \gg 1$ :

$$R \simeq n \log \frac{m}{n} - \log n - \log \frac{d}{M} - O\left(\log \frac{1}{\epsilon \delta}\right)$$
. (9)

This explicitly shows that the net information gain is exponentially larger than the secret key shared initially. This is the first key result of this Letter and it allows us to compute the net information gain of the protocol using Eq. (5): examples are shown in Fig. 2.

Practical protocols:— A practical communication protocol needs to be defined in the finite-size regime where the number of photons and mode are kept relatively small. Moreover, the presence of loss and noise in the communication channel necessitates the use of some form of error correction, which exploits redundancy in the protocol. This redundancy can in principle be used by Eve to learn more about the secret message, and one may expect that new security proofs have to be constructed for each new error correction code. Here we address these two issues and show that the structure of our proof en-

compasses a large class of error correcting protocols and can be also adapted to the finite-size regime.

Consider first the issue of finite photon numbers. If the number of photons and modes (per signal transmission) is kept small, and if the number of modes m is not necessarily larger than  $n^2$ , we cannot apply the boson birthday paradox [11]. This means that we need to explicitly account for photon bunching. This will have an effect on our estimation of the  $\gamma$  factor in Eq. (7). In particular, the moments of the random variable  $X_U =$  $|\langle \phi | U | \psi_j \rangle|^2$  will now be functions of the vector  $\phi$ . A generic vector  $\phi$  can be written as  $|\phi\rangle = \sum_{q,t} \alpha_{q,t} |\phi_{q,t}\rangle$ , where q identifies a subspace with given photon occupancy pattern, and t labels the computational basis vectors within. For example, given two photons and three modes, q = 1 corresponds to the single-occupancy space  $S_1$  which is generated by the computational basis vectors  $\{|110\rangle, |101\rangle, |011\rangle\}$ , and for q=2 the double-occupancy space  $S_2$  is generated by the computational basis vectors  $\{|200\rangle, |020\rangle, |002\rangle\}$ . We can show (see Supplemental Material) that the following bound holds independently of the vector  $\phi$ :

$$\gamma \le 2 \max_{q} \gamma_{q} \,, \tag{10}$$

where

$$\gamma_q = \frac{\mathbb{E}_U[|\langle \phi_{q,t} | U | \psi_j \rangle|^4]}{\mathbb{E}_U[|\langle \phi_{q,t} | U | \psi_j \rangle|^2]^2}.$$
 (11)

Note that by symmetry, the coefficients  $\gamma_q$  do not depend on the particular choice of the vectors  $\phi_{q,t}$ , nor on the code word  $\psi_j$ . Therefore the above upper bound on  $\gamma$  is feasible to be computed numerically.

Consider now using error correcting codes in the presence of loss and noise. The code words  $|\psi_j\rangle$  can be concatenated in a train of  $\nu$  signals (i.e. making  $\nu$  uses of the channel), creating a new codeword  $|\psi_j\rangle = |\psi_{j_1}\rangle \dots |\psi_{j_{\nu}}\rangle$  that incorporates error correction. The scrambling unitary transformation  $U_k$  then takes the form  $U_k = U_{k_1} \cdots U_{k_{\nu}}$ , and Eve's description of the state is

$$\rho_{AE} = \frac{1}{M^{\nu}} \sum_{j} |j\rangle_{A} \langle j| \otimes \frac{1}{K^{(\nu)}} \sum_{k} U_{k} |\psi_{j}\rangle_{E} \langle \psi_{j}| U_{k}^{\dagger}.$$
(12)

This expression is formally identical to the one in Eq. (2), therefore the general proof method holds for the noisy channel as well. The key consumption rate is then

$$\frac{1}{\nu}\log K^{(\nu)} \simeq \log(n+1) + \log\frac{d}{M}, \qquad (13)$$

where we have neglected a small term proportional to  $\frac{1}{\nu}O(\log(\epsilon^{-1}\delta^{-1}))$ .

In the case of a noisy and lossy channel, the amount of information that Alice and Bob can communicate per signal is quantified by the mutual information I(A; B) and

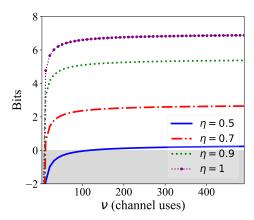


FIG. 3. Net number of bits transmitted per channel use in the presence of loss. The parameters are  $\beta=0.95, \delta=0.01,$  m,n=20,4 and  $\epsilon=\exp(-\nu^s \ln 2), s=0.43$ , where  $\nu$  is the number of channel use. The shaded region is a guide to the eye only, and data points above the region denotes net positive secret communication rate.

is in general strictly smaller than  $\log {m \choose n}$ . The amount of loss and noise in the communication channel can be experimentally determined with the standard tools of parameter estimation, a routine commonly used in quantum key distribution. This in turn allows Alice and Bob to quantify I(A;B). In general an error correcting has only finite efficiency  $\beta < 1$ , which in turn limits the amount of data transmitted to  $\beta I(A;B)$ . Putting this together with the result obtained above, we obtain our estimate for the net rate of the protocol:

$$r = \beta I(A; B) - \log(n+1) - \log\frac{d}{M}$$
. (14)

This is the second key result of this Letter: it shows that Boson Sampling cryptography is feasible, scalable, and robust to noise and loss.

We can compute explicitly I(A; B) for a pure loss channel with transmissivity  $\eta$ . The net key rate r is shown in Fig. 3, for a few values of m, n = 20, 4. For example,

for  $m \simeq n^2 \gg 1$ , we obtain  $I(A; B) \simeq \eta \log n$ . Therefore the net rate is  $r \simeq (\beta \eta n - 1) \log n$ . This implies that one can in principle achieve arbitrarily long distances (corresponding to low transmissivity  $\eta$ ) as long as  $n > (\eta \beta)^{-1}$ .

Conclusion:— Several works have attempted to apply the physical insights of Boson Sampling in a quantum information framework beyond its defining problem. In this paper we have presented the first successful quantum communication result in this direction. We have presented an information-theoretic proof that a linearoptical interferometer operating in the Boson-Sampling regime is useful for quantum cryptography. The security proof is based on the Quantum Data Locking effect and random coding techniques. Given n photons in m modes, we have shown that one can lock about  $n \log(m/n)$  bits of classical information using a much shorter secret key of only  $\sim \log n$  bits initially shared by the legitimate parties Alice and Bob. The efficiency of our protocol is therefore close to optimal. In addition, we have shown that our protocol remains secure when we use classical error correction to protect the channel against photon loss and other errors. It is therefore a scalable and efficient protocol for quantum cryptography. In fact, in contrast with Boson Sampling requiring the control of a large number of modes and photons to demonstrate a quantum advantage, our protocol is secure even for small numbers of modes and photons. This means that, given recent results in photon generation and advances in integrated linear optics, our protocol is experimentally feasible with currently available technology. In fact, this protocol design may represent a viable approach to highrate quantum secured communication beyond standard QKD.

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# Supplemental Material

#### Appendix: Preliminary observations

Before presenting the security proof of the protocol, we will have a closer look at the state

$$\bar{\rho}_E := \mathbb{E}_U[U|\psi\rangle\langle\psi|U^{\dagger}] = \int dU \, U|\psi\rangle\langle\psi|U^{\dagger}, \qquad (A.1)$$

where  $\psi$  is a vector in the single-occupancy subspace, i.e., the linear span of all vectors with at most 1 photon per mode (and a total of n photons). Note that by symmetry  $\bar{\rho}_E$  is independent of  $\psi$ .

We denote as  $P_1$  the projector into the single-occupancy space. Going beyond single-occupancy, we label with an integer q all the other possible occupancy patterns. One examples is the subspace spanned by all states having two photons in only one mode and at most one photon in the others. Another example is the subspace spanned by those vectors having two modes with three photons in each mode, and at most one photon in the others. For a given occupancy pattern q, we associate a corresponding projector  $P_q$ . By symmetry, the state  $\bar{\rho}_E$  is diagonal in the projectors  $P_q$ 's, i.e.,

$$\bar{\rho}_E = \sum_q c_q P_q \,. \tag{A.2}$$

We are particularly interested in the minimum value:

$$c_{\min} := \min_{q} c_{q} \,, \tag{A.3}$$

which can be computed numerically. Examples are shown in Table I. According to the Table, we conjecture that the minimum is always achieved for the single-occupancy subspace. In the context of eavesdropping (discussed in the next section),  $c_q$  corresponds to the expectation values of measuring certain state with occupancy pattern q.

If the number of modes is much larger than the number of photons squared,  $m \gg n^2$ , the probability of occupancy greater than 1 is suppressed. In this regime we can then write (see Section )

$$\bar{\rho}_E = \frac{n!}{m^n} P_1 \,, \tag{A.4}$$

and therefore

$$c_{\min} = c_{\max} = \frac{n!}{m^n} \,. \tag{A.5}$$

The other quantity we are interested in is

$$\gamma = \max_{\phi} \frac{\mathbb{E}_{U}[|\langle \phi | U | \psi \rangle|^{4}]}{\mathbb{E}_{U}[|\langle \phi | U | \psi \rangle|^{2}]^{2}}.$$
(A.6)

In general  $\gamma$  can be estimated numerically, see Section . In the regime of  $m \gg n^2$  it can be instead computed analytically as shown in Section .

In the following we will make repeated application of the Cauchy-Schwarz inequality to bound the coefficient  $\gamma$ . The application goes as follows. A generic state  $\phi$  is expanded as the linear combination of basis vectors,

$$|\phi\rangle = \sum_{i} \lambda_{i} |\phi_{i}\rangle. \tag{A.7}$$

The basis vectors are chosen such that

$$\mathbb{E}_{U}\left[\langle \phi_{i}|U|\psi\rangle\langle\psi|U^{\dagger}|\phi_{i'}\rangle\right] = \delta_{ii'}\mathbb{E}_{U}\left[|\langle \phi_{i}|U|\psi\rangle|^{2}\right], \tag{A.8}$$

$$\mathbb{E}_{U}\left[\langle\phi_{i}|U|\psi\rangle\langle\psi|U^{\dagger}|\phi_{i'}\rangle\langle\phi_{i''}|U|\psi\rangle\langle\psi|U^{\dagger}|\phi_{i'''}\rangle\right] = \frac{\delta_{ii'}\delta_{i''i'''} + \delta_{ii'''}\delta_{i'i''}}{2\delta_{ii''}}\mathbb{E}_{U}\left[|\langle\phi_{i}|U|\psi\rangle|^{2}|\langle\phi_{i''}|U|\psi\rangle|^{2}\right]. \tag{A.9}$$

Below we will expand our vectors in basis that, by symmetry, satisfy the above identities.

(m,n)	Photon occupancy pattern $q$	$c_q$	$n!/m^n$
(6,2)	$(1,1,0,\dots)$	0.0471	0.0556
	$(2,0,\dots)$	0.0977	
(10, 2)	$(1,1,0,\dots)$	0.0181	0.02
	$(2,0,\dots)$	0.0369	
(20, 2)	$(1,1,0,\dots)$	0.00476	0.005
	$(2,0,0,\dots)$	0.00959	
(10, 3)	$(1,1,1,0,\dots)$	0.00451	0.006
	$(1,2,0,\dots)$	0.00914	
	$(3,0,\dots)$	0.0283	
(20,3)	$(1, 1, 1, 0, \dots)$	0.000648	0.00075
	$(1,2,0,\dots)$	0.00131	
	$(3,0,\dots)$	0.00398	
(20, 4)	$(1,1,1,1,0,\dots)$	0.000114	0.00015
	$(1,1,2,0,\dots)$	0.000226	
	$(2,2,0,\dots)$	0.000456	
	$(1,3,0,\dots)$	0.00069	
	$(4,0,\dots)$	0.00285	

TABLE I. The table shows numerical estimates of the coefficient  $c_q$  for several number of photon and modes and patterns of photon occupancy. The table also show the corresponding value of the coefficients  $n!/m^n$ , which is relevant in the regime where  $n^2 \ll m$ .

Under these conditions we obtain

$$\mathbb{E}_{U}[|\langle \phi | U | \psi \rangle|^{2}] = \sum_{i} \lambda_{i}^{2} \mathbb{E}_{U}[|\langle \phi_{i} | U | \psi \rangle|^{2}], \qquad (A.10)$$

and

$$\mathbb{E}_{U}[|\langle \phi | U | \psi \rangle|^{4}] = \sum_{i} |\lambda_{i}|^{4} \mathbb{E}_{U} \left[ |\langle \phi_{i} | U | \psi \rangle|^{4} \right] + 2 \sum_{i \neq i''} |\lambda_{i}|^{2} |\lambda_{i''}|^{2} \mathbb{E}_{U} \left[ |\langle \phi_{i} | U | \psi \rangle|^{2} |\langle \phi_{i''} | U | \psi \rangle|^{2} \right] \\
\leq 2 \sum_{ii''} |\lambda_{i}|^{2} |\lambda_{i''}|^{2} \mathbb{E}_{U} \left[ |\langle \phi_{i} | U | \psi \rangle|^{2} |\langle \phi_{i''} | U | \psi \rangle|^{2} \right] \\
\leq 2 \sum_{ii''} |\lambda_{i}|^{2} |\lambda_{i''}|^{2} \sqrt{\mathbb{E}_{U} \left[ |\langle \phi_{i} | U | \psi \rangle|^{4} \right]} \mathbb{E} \left[ \langle \phi_{i''} | U | \psi \rangle|^{4} \right]} \\
= 2 \left( \sum_{i} |\lambda_{i}|^{2} \sqrt{\mathbb{E}_{U} \left[ |\langle \phi_{i} | U | \psi \rangle|^{4} \right]} \right)^{2}, \tag{A.11}$$

where the second inequality is an application of Cauchy-Schwarz.

In conclusion we obtain the upper bound

$$\gamma \leq 2 \max_{\phi} \left( \frac{\sum_{i} |\lambda_{i}|^{2} \sqrt{\mathbb{E}_{U} [|\langle \phi_{i}|U|\psi \rangle|^{4}]}}{\sum_{i} |\lambda_{i}|^{2} \mathbb{E}_{U} [|\langle \phi_{i}|U|\psi \rangle|^{2}]} \right)^{2} \\
\leq 2 \max_{i} \frac{\mathbb{E}_{U} [|\langle \phi_{i}|U|\psi \rangle|^{4}]}{\mathbb{E}_{U} [|\langle \phi_{i}|U|\psi \rangle|^{2}]^{2}}.$$
(A.12)

Compared with the definition in Eq. (A.6) we now only need to consider the basis vectors  $\phi_i$ . The expression in Eq. (A.12) is therefore suitable to be evaluated numerically. Also note that the penalty to pay for this simplification is the multiplication by a factor 2.

#### Appendix: Overview of the security proof

Consider a system of n photons over m optical modes. The dimension of the associated Hilbert space is  $d = \binom{n+m-1}{n}$ , which implies that  $\log d$  is the maximum number of bits that the system can encode. For practical reasons, we consider a communication protocol that only requires the sender Alice to prepare states with single-occupancy, i.e., states with at most one photon per mode. If m > n there are a total of  $M = \binom{m}{n}$  such states, which we denote as  $|\psi_j\rangle$ , with  $j = 1, 2, \ldots, M$ . For example, for m = 3 and n = 2 these states are  $\{|110\rangle, |101\rangle, |011\rangle\}$ .

We prove the security using a random-coding approach. This means that a code book of  $M' = \delta M$  vectors is randomly chosen sampling from the basis states  $\{|\psi_j\rangle\}_{j=1,2,\ldots,M}$ , where  $0 < \delta < 1$  is a constant. These M' code words are denoted as  $\{|\psi_x\rangle\}_{x=1,2,\ldots,M'}$ . For  $\delta \ll 1$ , we expect that the M' code words are all distinct up to terms of second order in  $\delta$ . Therefore the M' code words encode about  $\log M - O(\log(1/\delta))$  bits of information.

The sender Alice first prepares a state  $|\psi_x\rangle$ , then applies a linear optics unitary  $U_k$ . The unitary is chosen among a pool of K elements according to a secret key of  $\log K$  bits that she shares with the receiver Bob. We recall that the pool of unitaries is drawing K unitaries i.i.d. according to the uniform probability density distribution on the group of m-mode linear optics unitary transformations. For an eavesdropper who does not know the secret key the state is described by the density operator

$$\rho_E^x = \frac{1}{K} \sum_{k=1}^K U_k |\psi_x\rangle \langle \psi_x | U_k^{\dagger}. \tag{A.1}$$

Given the classical-quantum state

$$\rho_{XE} = \sum_{x=1}^{M'} p_X(x) |\psi_x\rangle \langle \psi_x| \otimes \rho_E^x , \qquad (A.2)$$

Eve attempts to extract information from her share of the state by applying a measurement  $M_{E\to Y}$ . Such a measurement is characterized by the POVM elements  $\{\alpha_y|\phi_y\rangle\langle\phi_y|\}_y$ , where  $\phi_y$ 's are unit vectors and  $\alpha_y>0$  such that  $\sum_y \alpha_y|\phi_y\rangle\langle\phi_y|=\mathbb{I}$ , with  $\mathbb{I}$  the identity operator. Without loss of generality we can consider rank-one POVM only [7]. The output of this measurement is a random variable Y such that

$$p_Y(y) = \alpha_y \left\langle \phi_y \left| \sum_{x=1}^{M'} p_X(x) \rho_E^x \right| \phi_y \right\rangle, \tag{A.3}$$

and the conditional probability is

$$p_{Y|X=x}(y) = \alpha_y \langle \phi_y | \rho_E^x | \phi_y \rangle, \tag{A.4}$$

The accessible information is the maximum mutual information between X and Y:

$$I_{\text{acc}}(X; E) = \sup_{M_{E \to Y}} I(X; Y), \qquad (A.5)$$

where

$$I(X;Y) = H(X) + H(Y) - H(XY) = H(Y) - H(Y|X)$$

$$= -\sum_{y} p_{Y}(y) \log p_{Y}(y) + \sum_{x=1}^{M'} p_{X}(x) \sum_{y} p_{Y|X=x}(y) \log \left[p_{Y|X=x}(y)\right]$$

$$= -\sum_{y} \alpha_{y} \left\langle \phi_{y} \left| \sum_{x=1}^{M'} p_{X}(x) \rho_{E}^{x} \right| \phi_{y} \right\rangle \log \left[ \alpha_{y} \left\langle \phi_{y} \right| \sum_{x=1}^{M'} p_{X}(x) \rho_{E}^{x} \right| \phi_{y} \right\rangle \right]$$

$$+ \sum_{x=1}^{M'} p_{X}(x) \sum_{x=1}^{M} \alpha_{y} \left\langle \phi_{y} \right| \rho_{E}^{x} \left| \phi_{y} \right\rangle \log \left[ \alpha_{y} \left\langle \phi_{y} \right| \rho_{E}^{x} \left| \phi_{y} \right\rangle \right]. \tag{A.6}$$

Putting for simplicity  $p_X(x) = 1/M'$  we obtain

$$I(X;Y) = \sum_{y} \alpha_{y} \left\{ -\left\langle \phi_{y} \left| \frac{1}{M'} \sum_{x=1}^{M'} \rho_{E}^{x} \right| \phi_{y} \right\rangle \log \left\langle \phi_{y} \left| \frac{1}{M'} \sum_{x=1}^{M'} \rho_{E}^{x} \right| \phi_{y} \right\rangle + \frac{1}{M'} \sum_{x=1}^{M'} \langle \phi_{y} | \rho_{E}^{x} | \phi_{y} \rangle \log \langle \phi_{y} | \rho_{E}^{x} | \phi_{y} \rangle \right\}. \quad (A.7)$$

Note that the accessible information is written as the difference of two entropy-like quantities. The rationale of the security proof is to show that for K large enough, and for random choice of the unitaries and of the code words, both the first and second terms in the curly brackets are arbitrarily close to

$$-\langle \phi_y | \bar{\rho}_E | \phi_y \rangle \log \langle \phi_y | \bar{\rho}_E | \phi_y \rangle \tag{A.8}$$

for all vectors  $\phi_y$ , where  $\bar{\rho}_E$  is as in Eq. (A.1). This implies that the accessible information can be made in turn arbitrarily small. To show this we exploit the phenomenon of concentration towards the average of the sum of i.i.d. random variables. This concentration is quantified by concentration inequalities, also called tail bounds.

We now proceed along two parallel directions:

1. We apply the matrix Chernoff bound to show that  $\frac{1}{M'}\sum_{x=1}^{M'}\rho_E^x$  is close to  $\bar{\rho}_E$ . In particular the matrix Chernoff bound implies that the inequality

$$\frac{1}{M'} \sum_{x=1}^{M'} \rho_E^x \le (1+\epsilon)\bar{\rho}_E \tag{A.9}$$

holds true with almost unit probability, and this probability can be made arbitrarily close to 1 if K is large enough.

This in turn implies

$$-\left\langle \phi \left| \frac{1}{M'} \sum_{x=1}^{M'} \rho_E^x \right| \phi \right\rangle \log \left\langle \phi \left| \frac{1}{M'} \sum_{x=1}^{M'} \rho_E^x \right| \phi \right\rangle \le -(1+\epsilon) \langle \phi | \bar{\rho}_E | \phi \rangle \log (1+\epsilon) \langle \phi | \bar{\rho}_E | \phi \rangle \tag{A.10}$$

$$\leq -(1+\epsilon)\langle\phi|\bar{\rho}_E|\phi\rangle\log\langle\phi|\bar{\rho}_E|\phi\rangle$$
 (A.11)

uniformly for all  $\phi$ .

The details are presented in Section below.

2. We apply a tail bound from A. Maurer [37] to show that

$$\langle \phi | \rho_E^x | \phi \rangle \ge (1 - \epsilon) \langle \phi | \bar{\rho}_E | \phi \rangle,$$
 (A.12)

with a probability that can be made arbitrarily close to 1 if K is large enough. The above is true for all unit vectors  $\phi$  and for almost all values of x. This in turn implies that

$$\langle \phi | \rho_E^x | \phi \rangle \log \langle \phi | \rho_E^x | \phi \rangle \le (1 - \epsilon) \langle \phi | \bar{\rho}_E | \phi \rangle \log (1 - \epsilon) \langle \phi | \bar{\rho}_E | \phi \rangle. \tag{A.13}$$

In conclusion we obtain that, up to a small probability,

$$\frac{1}{M'} \sum_{x=1}^{M'} \langle \phi | \rho_E^x | \phi \rangle \log \langle \phi | \rho_E^x | \phi \rangle \le (1 - \epsilon) \langle \phi | \bar{\rho}_E | \phi \rangle \log (1 - \epsilon) \langle \phi | \bar{\rho}_E | \phi \rangle \tag{A.14}$$

$$\leq (1 - \epsilon) \langle \phi | \bar{\rho}_E | \phi \rangle \log \langle \phi | \bar{\rho}_E | \phi \rangle. \tag{A.15}$$

The details are presented in Section below.

Putting the above results in Eq. (A.11) and (A.15) into Eq. (A.7) we finally obtain

$$I(X;Y) \le -2\epsilon \sum_{y} \alpha_{y} \langle \phi_{y} | \bar{\rho}_{E} | \phi_{y} \rangle \log \langle \phi_{y} | \bar{\rho}_{E} | \phi_{y} \rangle. \tag{A.16}$$

Recall that  $p_Y(y) = \alpha_y \langle \phi_y | \bar{\rho}_E | \phi_y \rangle$  is a probability distribution. Therefore, as the average is always smaller that the maximum, we obtain

$$I(X;Y) \le -2\epsilon \min_{\phi} \log \langle \phi | \bar{\rho}_E | \phi \rangle = 2\epsilon \log \frac{1}{c_{\min}},$$
 (A.17)

where  $c_{\min} := \min_{\phi} \langle \phi | \bar{\rho}_E | \phi \rangle$  can be computed as discussed in Section .

The above bound on the accessible information is not deterministic, but the probability that it fails can be made arbitrary small provided K is large enough. In particular we need to require (see Section below):

$$K > \max \left\{ \gamma \left[ \frac{256}{\epsilon^3 \delta} \frac{d}{M} \log \left( \frac{20}{\epsilon c_{\min}} \right) + \frac{32}{\epsilon^2} \log \delta M \right], \frac{32}{\epsilon^2 \delta} \frac{(\log 2d)^2}{M c_{\min}} \right\}.$$
 (A.18)

The size of K critically depends on the factor  $\gamma$ . This factor determines the convergence rate of the Maurer tail bound and is given by:

$$\gamma = \max_{\phi} \frac{\mathbb{E}_{U}[|\langle \phi | U | \psi_x \rangle|^4]}{\mathbb{E}_{U}[|\langle \phi | U | \psi_x \rangle|^2]^2}.$$
 (A.19)

How to estimate this coefficient is the subject of Sections and.

### Appendix: Scaling the communication protocol up

In the previous section we have described the security proof having in mind a one-shot scenario where only one signal is sent from Alice to Bob (one signal being composed of n photons in m modes).

In a practical communication scenario, not only one signal, but a large number of signals are sent from Alice to Bob. An eavesdropper tampering with the communication line may gain additional information by exploiting correlations between different signals. In this section we show how to rule out this possibility by a simple extension of the above security proof from a one-signal to a multi-signal scenario.

Consider a train of  $\nu \gg 1$  signal transmissions. Alice encodes information in code words of the form

$$|\psi_{x}\rangle = |\psi_{x_1}\rangle \otimes |\psi_{x_2}\rangle \otimes \dots |\psi_{x_n}\rangle, \tag{A.1}$$

where each component  $\psi_{x_1}$  is chosen randomly and independently as discussed in Section . In this way a code book of M' random vectors is defined. Each vector is a code word and is uniquely identified by a multi-index  $\mathbf{x} = x_1, x_2, \dots, x_{\nu}$ . We put  $M' = \delta M^{\nu}$ , where  $0 < \delta \ll 1$  is a small constant.

First Alice encodes information across  $\nu$  signal transmissions using the code words  $\psi_x$ , then she applies local unitaries to scramble it. These local unitaries are

$$U_{k} = U_{k_1} \otimes U_{k_2} \cdots \otimes U_{k_n} \,. \tag{A.2}$$

The set of possible unitaries is made of  $K^{(\nu)}=K^{\nu}$  elements. As above, these unitaries are chosen by sampling identically and independently from the distribution induced by the uniform measure on the unitary group of linear optical passive unitary transformations. Note that, whereas  $\nu$  is arbitrary large, the number of mode m in each signal transmission is constant. Also, the number of photons per signal is fixed and equal to n.

The eavesdropper's accessible information is then formally equivalent to the one in Eq. (A.7):

$$I(X^{\nu};Y) = \sum_{y} \alpha_{y} \left\{ -\left\langle \phi_{y} \left| \frac{1}{M'} \sum_{x=1}^{M'} \rho_{E}^{x} \right| \phi_{y} \right\rangle \log \left\langle \phi_{y} \left| \frac{1}{M'} \sum_{x=1}^{M'} \rho_{E}^{x} \right| \phi_{y} \right\rangle + \frac{1}{M'} \sum_{x=1}^{M'} \left\langle \phi_{y} | \rho_{E}^{x} | \phi_{y} \right\rangle \log \left\langle \phi_{y} | \rho_{E}^{x} | \phi_{y} \right\rangle \right\}, \quad (A.3)$$

where for each  $\boldsymbol{x} = x_1, x_2, \dots, x_{\nu}$ ,

$$\rho_E^{\mathbf{x}} = \sum_{k=1}^{K^{\nu}} U_k |\psi_{\mathbf{x}}\rangle \langle \psi_{\mathbf{x}}| U_k^{\dagger}$$
(A.4)

$$= \sum_{k_1,k_2,\dots,k_{\nu}=1}^{K} \left( U_{k_1} \otimes U_{k_2} \cdots \otimes U_{k_{\nu}} \right) \left( |\psi_{x_1}\rangle \langle \psi_{x_1}| \otimes |\psi_{x_2}\rangle \langle \psi_{x_2}| \cdots \otimes |\psi_{x_{\nu}}\rangle \langle \psi_{x_{\nu}}| \right) \left( U_{k_1}^{\dagger} \otimes U_{k_2}^{\dagger} \cdots \otimes U_{k_{\nu}}^{\dagger} \right)$$
(A.5)

$$= \bigotimes_{i=1}^{\nu} \sum_{k_i=1}^{K} U_{k_i} |\psi_{x_i}\rangle \langle \psi_{x_i} | U_{k_i}^{\dagger}. \tag{A.6}$$

This in particular implies

$$\bar{\rho}_E^{(\nu)} := \mathbb{E}_{U}[U_k | \psi_x \rangle \langle \psi_x | U_k^{\dagger}] = \bar{\rho}_E^{\otimes \nu}, \tag{A.7}$$

and therefore

$$c_{\min}^{(\nu)} := \min_{\phi} \langle \phi | \bar{\rho}_E^{(\nu)} | \phi \rangle = \min_{\phi} \langle \phi | \bar{\rho}_E^{\otimes \nu} | \phi \rangle = c_{\min}^{\nu} \tag{A.8}$$

$$\gamma^{(\nu)} := \max_{\phi} \frac{\mathbb{E}_{\boldsymbol{U}}[|\langle \phi | \boldsymbol{U}_{\boldsymbol{k}} | \boldsymbol{\psi}_{\boldsymbol{x}} \rangle|^{4}]}{\mathbb{E}_{\boldsymbol{U}}[|\langle \phi | \boldsymbol{U}_{\boldsymbol{k}} | \boldsymbol{\psi}_{\boldsymbol{x}} \rangle|^{2}]^{2}}. \tag{A.9}$$

To bound  $\gamma^{(\nu)}$  we can first decompose a generic vector  $\phi$  as  $|\phi\rangle = \sum_i \lambda_i |\phi_i\rangle$ , where  $\{|\phi_i\rangle\}_i$  is a basis of product vectors. Then we apply the Cauchy-Schwarz inequality as described in Section and obtain (see Eq. (A.12):

$$\gamma^{(\nu)} \le 2\gamma^{\nu} \,. \tag{A.10}$$

In conclusion, we can straightforwardly repeat the security proof of Section with this re-scaled parameter. This yields that for any arbitrarily small  $\epsilon$  the bound

$$I(X^{\nu};Y) \le 2\epsilon \log \frac{1}{c_{\min}^{\nu}}, \tag{A.11}$$

holds with almost unit probability provided that (recall that  $M' = \delta M^{\nu}$ )

$$K^{(\nu)} > \max \left\{ \gamma^{\nu} \left[ \frac{512}{\epsilon^{3} \delta} \frac{d^{\nu}}{M^{\nu}} \log \left( \frac{20}{\epsilon c_{\min}^{\nu}} \right) + \frac{64}{\epsilon^{2}} \log \delta M^{\nu} \right], \frac{2 \ln 2}{\epsilon^{2} \delta} \frac{\log d^{\nu}}{M^{\nu} c_{\min}^{\nu}} \right\}. \tag{A.12}$$

Finally, in the limit of  $\nu \gg 1$  we obtain

$$\boxed{\frac{1}{\nu}\log K^{(\nu)} \simeq \max\left\{\log\gamma + \log\frac{d}{M}, \log\frac{1}{Mc_{\min}}\right\}}.$$
(A.13)

## Appendix: Matrix Chernoff bounds

The matrix Chernoff bound states the following (this formulation can be obtained directly from Theorem 19 of Ref. [38]):

**Theorem 1** Let  $\{X_t\}_{t=1,...,T}$  be T i.i.d. d-dimensional Hermitian-matrix-valued random variables, with  $X_t \sim X$ ,  $0 \le X \le R$ , and  $c_{\min} \le \mathbb{E}[X] \le c_{\max}$ . Then, for  $\delta \ge 0$ :

$$\Pr\left\{\frac{1}{T}\sum_{t=1}^{T}X_{t} \not\leq (1+\delta)\mathbb{E}[X]\right\} \leq d\exp\left\{-TD\left[(1+\delta)\frac{c_{\min}}{R}\left\|\frac{c_{\min}}{R}\right\|\right]\right\},\tag{A.1}$$

where  $Pr\{x\}$  denotes the probability that the proposition x is true, and  $D[u||v] = u \ln (u/v) - (1-u) \ln [(1-u)/(1-v)]$ .

Note that for  $\delta > 1$  we have

$$D\left[ (1+\delta) \frac{c_{\min}}{R} \, \middle\| \, \frac{c_{\min}}{R} \, \middle] \ge \frac{\delta}{4} \frac{c_{\min}}{R} \,, \tag{A.2}$$

and for  $\delta < 1$ 

$$D\left[(1+\delta)\frac{c_{\min}}{R} \left\| \frac{c_{\min}}{R} \right\| \ge \frac{\delta^2}{4} \frac{c_{\min}}{R} \right]. \tag{A.3}$$

First consider the collection of M' code words  $\psi_x$ . We apply the Chernoff bound to the M' independent random variables  $X_x = |\psi_x\rangle\langle\psi_x|$ . Note that these operators are defined in a M-dimensional Hilbert space. For  $\tau > 1$  we then have

$$\Pr\left\{\frac{1}{M'}\sum_{x=1}^{M'}|\psi_x\rangle\langle\psi_x| \not\leq \frac{1+\tau}{M}\right\} \leq M\exp\left(-\frac{M'\tau}{4M}\right). \tag{A.4}$$

Consider now the collection of K random variables  $X_k = \frac{1}{M'} \sum_x U_k |\psi_x\rangle \langle \psi_x | U_k^{\dagger}$ . We assume that they are bounded by  $R = \frac{1+\tau}{M}$ . We apply again the Chernoff bound:

$$\Pr\left\{\frac{1}{K}\sum_{k=1}^{K}\frac{1}{M'}\sum_{x}U_{k}|\psi_{x}\rangle\langle\psi_{x}|U_{k}^{\dagger}\nleq(1+\epsilon)\mathbb{E}[X]\right\}\leq d\exp\left(-\frac{MK\epsilon^{2}c_{\min}}{4(1+\tau)}\right).\tag{A.5}$$

Thus the total probability reads

$$p_1 \le M \exp\left(-\frac{M'\tau}{4M}\right) + d \exp\left(-\frac{MK\epsilon^2 c_{\min}}{4(1+\tau)}\right)$$
 (A.6)

$$\leq M \exp\left(-\frac{M'\tau}{4M}\right) + d \exp\left(-\frac{MK\epsilon^2 c_{\min}}{8\tau}\right).$$
 (A.7)

Putting  $\tau = M\epsilon\sqrt{\frac{Kc_{\min}}{2M'}}$  we obtain

$$p_1 \le (M+d) \exp\left(-\frac{\epsilon}{4}\sqrt{\frac{M'Kc_{\min}}{2}}\right) \le 2d \exp\left(-\frac{\epsilon}{4}\sqrt{\frac{M'Kc_{\min}}{2}}\right).$$
 (A.8)

In conclusion we have obtained that, up to a probability smaller than  $p_1$ ,

$$\frac{1}{KM'} \sum_{k=1}^{K} \sum_{r=1}^{M'} U_k |x\rangle \langle x| U_k^{\dagger} = \frac{1}{M'} \sum_{r=1}^{M'} \rho_E^x \le (1+\epsilon)\bar{\rho}_E.$$
 (A.9)

## Appendix: The Maurer tail bound

We also need to apply the following concentration inequality due to A. Maurer [37]:

**Theorem 2** Let  $\{X_k\}_{k=1,\ldots,K}$  be K i.i.d. non-negative real-valued random variables, with  $X_k \sim X$  and finite first and second moments,  $\mathbb{E}[X], \mathbb{E}[X^2] < \infty$ . Then, for any  $\tau > 0$  we have that

$$\Pr\left\{\frac{1}{K}\sum_{k=1}^{K} X_k < (1-\tau)\mathbb{E}[X]\right\} \le \exp\left(-\frac{K\tau^2\mathbb{E}[X]^2}{2\mathbb{E}[X^2]}\right). \tag{A.1}$$

For any given x and  $\phi$ , we apply this bound to the random variables

$$X_k \equiv |\langle \phi | U_k | \psi_x \rangle|^2 \,. \tag{A.2}$$

Note that (see Section )

$$\frac{1}{K} \sum_{k=1}^{K} X_k = \langle \phi | \rho_E^x | \phi \rangle, \tag{A.3}$$

and

$$\mathbb{E}[X] = \langle \phi | \bar{\rho}_E | \phi \rangle. \tag{A.4}$$

The application of the Maurer tail bound then yields

$$\Pr\left\{ \langle \phi | \rho_E^x | \phi \rangle < (1 - \tau) \langle \phi | \bar{\rho}_E | \phi \rangle \right\} \le \exp\left( -\frac{K\tau^2}{2\gamma} \right). \tag{A.5}$$

where

$$\gamma = \max_{\phi} \frac{\mathbb{E}_{U}[X^{2}]}{\mathbb{E}_{U}[X]^{2}} = \max_{\phi} \frac{\mathbb{E}_{U}[|\langle \phi | U | \psi_{x} \rangle|^{4}]}{\langle \phi | \bar{\rho}_{E} | \phi \rangle^{2}}.$$
(A.6)

Note that, by symmetry,  $\gamma$  is independent of  $\psi_x$ . The calculation of  $\gamma$  is presented in Sections and .

#### 1. Extending to almost all code words

The probability bound in Eq. (A.5) is about one given value of x. Here we extend it to  $\ell$  distinct values  $x_1, x_2, \ldots, x_{\ell}$ :

$$\Pr\left\{\forall x = x_1, x_2, \dots x_\ell, \ \langle \phi | \rho_E^x | \phi \rangle < (1 - \tau) \langle \phi | \bar{\rho}_E | \phi \rangle\right\} \le \exp\left(-\frac{\ell K \tau^2}{2\gamma}\right),\tag{A.7}$$

where we have used the fact that for different values of x the variables are statistically independent (recall that the code words are chosen randomly and independently). Second, we extend to all possible choices of  $\ell$  code words. This amount to a total of  $\binom{M'}{\ell}$  events. Applying the union bound we obtain

$$\Pr\left\{\exists x_1, x_2, \dots x_\ell, \mid \forall x = x_1, x_2, \dots x_\ell, \ \langle \phi | \rho_E^x | \phi \rangle < (1 - \tau) \langle \phi | \bar{\rho}_E | \phi \rangle\right\} \le \binom{M'}{\ell} \exp\left(-\frac{\ell K \tau^2}{2\gamma}\right). \tag{A.8}$$

This implies that, up to a probability smaller than  $\binom{M'}{\ell} \exp\left(-\frac{\ell K \tau^2}{2\gamma}\right)$ ,  $\langle \phi | \rho_E^x | \phi \rangle \geq (1-\tau) \langle \phi | \bar{\rho}_E | \phi \rangle$  for at least  $M' - \ell$  values of x, which implies

$$\frac{1}{M'} \sum_{x=1}^{M'} \langle \phi | \rho_E^x | \phi \rangle \log \langle \phi | \rho_E^x | \phi \rangle \le \frac{M' - \ell}{M'} (1 - \tau) \langle \phi | \bar{\rho}_E | \phi \rangle \log \left[ (1 - \tau) \langle \phi | \bar{\rho}_E | \phi \rangle \right]. \tag{A.9}$$

Putting  $\ell = \tau M$ :

$$\frac{1}{M} \sum_{x=1}^{M'} \langle \phi | \rho_E^x | \phi \rangle \log \langle \phi | \rho_E^x | \phi \rangle \le (1 - \tau)^2 \langle \phi | \bar{\rho}_E | \phi \rangle \log \left[ (1 - \tau) \langle \phi | \bar{\rho}_E | \phi \rangle \right] \tag{A.10}$$

$$\leq (1 - \tau)^2 \langle \phi | \bar{\rho}_E | \phi \rangle \log \langle \phi | \bar{\rho}_E | \phi \rangle \tag{A.11}$$

$$= (1 - 2\tau)\langle \phi | \bar{\rho}_E | \phi \rangle \log \langle \phi | \bar{\rho}_E | \phi \rangle + O(\tau^2). \tag{A.12}$$

#### 2. Extending to all vectors $\phi$

The final step is to extend the result to all unit vectors. This can be done by exploiting the notion of  $\delta$ -net. A  $\delta$ -net is a discrete and finite set of vectors  $\{\phi_i\}$  on the unit sphere such that for any unit vector  $\phi$  there exists an element

in the  $\delta$ -net such that

$$\|\phi - \phi_i\|_1 \le \delta. \tag{A.13}$$

It is known that there exist  $\delta$ -nets with no more than  $(5/\delta)^{2d}$  elements [8], where d is the Hilbert space dimension. We put  $\delta = \tau c_{\min}$ , therefore the size of the net is  $(5/\tau/c_{\min})^{2d}$ . Applying the union bound on Eq. (A.8) we then obtain

$$\Pr\left\{\forall \phi, \exists x_1, x_2, \dots x_\ell, \mid \forall x = x_1, x_2, \dots x_\ell, \langle \phi | \rho_E^x | \phi \rangle < (1 - 2\tau) \langle \phi | \bar{\rho}_E | \phi \rangle\right\} \le \left(\frac{5}{\tau c_{\min}}\right)^{2d} \binom{M'}{\ell} \exp\left(-\frac{\ell K \tau^2}{2\gamma}\right). \tag{A.14}$$

To conclude, we put  $\epsilon = 4\tau$  and obtain that, uniformly in  $\phi$ ,

$$\frac{1}{M} \sum_{x=1}^{M'} \langle \phi | \rho_E^x | \phi \rangle \log \langle \phi | \rho_E^x | \phi \rangle \le (1 - \epsilon) \langle \phi | \bar{\rho}_E | \phi \rangle \log \langle \phi | \bar{\rho}_E | \phi \rangle + O(\epsilon^2). \tag{A.15}$$

The probability that this bound is violated is smaller than (recall that  $\ell = \tau M' = \epsilon M'/4$ )

$$p_2 = \left(\frac{5}{\tau c_{\min}}\right)^{2d} \binom{M'}{\ell} \exp\left(-\frac{\ell K \tau^2}{2\gamma}\right) \tag{A.16}$$

$$= \left(\frac{20}{\epsilon c_{\min}}\right)^{2d} \binom{M'}{\epsilon M'/4} \exp\left(-\frac{KM'\epsilon^3}{128\gamma}\right) \tag{A.17}$$

$$\leq \left(\frac{20}{\epsilon c_{\min}}\right)^{2d} M'^{\epsilon M'/4} \exp\left(-\frac{KM'\epsilon^3}{128\gamma}\right).$$
(A.18)

#### Appendix: Probability of the bad event

The above concentration inequalities allow us proving that the protocol is secure up a to certain probability. The bad event that the protocol is not secure occurs when either Eq. (A.11) or (A.15) is violated. The probability of the bad event is then smaller the sum of the corresponding probabilities, which are given in Eq. (A.8) and (A.18) respectively. We therefore have

$$P_{\text{fail}} \le p_1 + p_2 \le 2d \exp\left(-\frac{\epsilon}{4}\sqrt{\frac{M'Kc_{\text{min}}}{2}}\right) + \left(\frac{20}{\epsilon c_{\text{min}}}\right)^{2d} M'^{\epsilon M'/4} \exp\left(-\frac{KM'\epsilon^3}{128\gamma}\right)$$
(A.1)

$$= \exp\left(\log 2d - \frac{\epsilon}{4}\sqrt{\frac{M'Kc_{\min}}{2}}\right) + \exp\left[2d\log\left(\frac{20}{\epsilon c_{\min}}\right) + \frac{\epsilon M'}{4}\log M' - \frac{KM'\epsilon^3}{128\gamma}\right]. \tag{A.2}$$

This probability is bounded away from 1 if

$$K > \frac{32}{\epsilon^2} \frac{1}{M' c_{\min}} (\log 2d)^2,$$
 (A.3)

and

$$K > 128\gamma \left[ \frac{2}{\epsilon^3} \frac{d}{M'} \log \left( \frac{20}{\epsilon c_{\min}} \right) + \frac{1}{4\epsilon^2} \log M' \right]. \tag{A.4}$$

## Appendix: Asymptotic speed of convergence

The goal of this section is to estimate the factor  $\gamma$  that determines the secret key consumption rate. First we consider the regime of  $m \gg n^2$  in which we can neglect photon bunching.

The objective is therefore to evaluate the first and second moments of the random variable

$$X = |\langle \phi | U | \psi_j \rangle|^2 \,, \tag{A.1}$$

where  $\phi$  restricted to be a vector in the single-occupancy subspace  $S_1$ , which is our code space  $\mathcal{H}_n^m$ .

As a first step we compute the first and second moments of the random variable

$$X' = |\langle \psi_{j'} | U | \psi_j \rangle|^2. \tag{A.2}$$

This is a little less general than (A.1) because  $\psi_{j'}$  is not a generic vector in  $\mathcal{H}_n^m$ . In fact  $\psi_j$  and  $\psi_{j'}$  identify two sets of modes, with labels  $(i_1, i_2, \dots i_n)$  and  $(i'_1, i'_2, \dots i'_n)$ , respectively. This corresponds to photon-counting on the modes, which as we know, maps onto  $n \times n$  sub-matrix  $A^{(jj')}$  of the unitary matrix U:

$$A_{hk}^{(jj')} := U_{i_h i_k'}. \tag{A.3}$$

The random variable X' is the modulus square of the permanent of  $A^{(jj')}$ :

$$X' = |\langle \psi_{j'} | U | \psi_j \rangle|^2 = \left| \sum_{\pi} \prod_{h=1}^n A_{h\pi(h)}^{(jj')} \right|^2, \tag{A.4}$$

where the sum is over all the permutations  $\pi$ .

To further explore the statistical properties of the permanent, it is useful to recall that a given entry of a random  $m \times m$  unitary is itself distributed as a complex Gaussian variable with zero mean and variance 1/m. If instead we consider a submatrix of size  $n \times n$  the entries are with good approximation independent Gaussian variables as long as  $n \ll m$  [11]. This means that the entries  $A_{nk}^{(jj')}$  are distributed as  $n^2$  i.i.d. complex Gaussian variables with zero mean and variance 1/m. Using this fact we can compute the first and second moments of X'.

We exploit statistical independence and the fact that the forth moment of a Gaussian with variance 1/m is  $2/m^2$ .

$$X' = \left(\sum_{\tau} \prod_{j=1}^{n} a_{j\tau(j)}^{*}\right) \times \left(\sum_{\sigma} \prod_{i=1}^{n} a_{i\sigma(i)}\right) = \sum_{\sigma,\tau} \prod_{i,j=1}^{n} a_{j\tau(j)}^{*} a_{i\sigma(i)} = \frac{n!}{m^{n}}.$$
 (A.5)

since the non-zero terms are given by  $i = j, \tau = \sigma$ .

From Lemma 56 of Ref. [11], the fourth moment of the permanent can be computed as

$$\mathbb{E}_{U}[X'^{2}] = \mathbb{E}_{U}[\text{Perm}[A]^{2}\text{Perm}[A^{*}]^{2}] = \frac{n!(n+1)!}{m^{2n}}.$$
(A.6)

In conclusion we have obtained

$$\frac{\mathbb{E}_U[X'^2]}{\mathbb{E}_U[X']^2} = n + 1. \tag{A.7}$$

To extend to a generic vector  $\phi$  in the single-occupancy subspace we first note that  $|\phi\rangle = \sum_i \lambda_i |\psi_j\rangle$ . Then we apply the Cauchy-Schwarz inequality as shown in Section and obtain (see Eq. (A.12)):

$$\gamma \le 2(n+1). \tag{A.8}$$

## Appendix: Speed of convergence in the finite-size regime

The goal of this section is to estimate the factor  $\gamma$  when  $m \gg n^2$ . In this regime we cannot neglect photon bunching and states with multiple occupancy need to be considered.

A generic state can be written as

$$|\phi\rangle = \sum_{q,t} \alpha_{q,t} |\phi_{q,t}\rangle \tag{A.1}$$

where q identifies a subspace with given photon occupancy pattern, and t labels the computational basis vectors

(m,n)	Photon pattern	$2\gamma_q$	2(n+1)
(6,2)	$(1,1,0,\dots)$	3.770	6
	$(2,0,\dots)$	4.314	
(10, 2)	$(1,1,\dots)$	4.256	6
	$(2,0,\dots)$	5.136	
(20, 2)	$(1,1,0,\dots)$	4.751	6
	$(2,0,\dots)$	5.894	
(10, 3)	$(1,1,1,0,\dots)$	4.562	8
	$(1,2,0,\dots)$	5.366	
	$(3,0,\dots)$	6.968	
(20, 3)	$(1,1,1,0,\dots)$	5.482	8
	$(1,2,0,\dots)$	6.793	
	$(3,0,\dots)$	9.261	
(20,4)	$(1,1,1,1,0,\dots)$	5.427	10
	$(4,0,\dots)$	13.466	

TABLE II. The table shows numerically computed values of  $\gamma_q$  for different number of photons (n), modes (m), and photon occupancy patterns. The table also shows the upper bound  $\gamma \leq 2(n+1)$  which holds in the limit of  $n^2 \ll m$ .

within. For example, given two photons and three modes, q=1 corresponds to the single-occupancy space  $S_1$  which is generated by the computational basis vectors  $\{|110\rangle, |101\rangle, |011\rangle\}$ , and for q=2 the double-occupancy space  $S_2$  is generated by the computational basis vectors  $\{|200\rangle, |020\rangle, |002\rangle\}$ . For each value of q,  $|\phi_q\rangle$  is a vector in the linear span of  $S_q$ .

We can now then apply the Cauchy-Schwarz inequality as discussed in Section . This yields (see Eq. (A.12)):

$$\gamma \le 2 \max_{q} \frac{\mathbb{E}_{U}[|\langle \phi_{q} | U | \psi_{x} \rangle|^{4}]}{\mathbb{E}_{U}[|\langle \phi_{q} | U | \psi_{x} \rangle|^{2}]^{2}}.$$
(A.2)

By symmetry, the quantities

$$\gamma_q := \frac{\mathbb{E}_U[|\langle \phi_q | U | \psi_x \rangle|^4]}{\mathbb{E}_U[|\langle \phi_q | U | \psi_x \rangle|^2]^2} \tag{A.3}$$

do depend on q but not on the particular vector  $\phi_q$  in the subspace  $S_q$ , nor on the vector  $\psi_x$ . Therefore for each q,  $\gamma_q$  can be computed numerically and in turn obtain an estimate for the upper bound on the speed of convergence

$$\gamma \le 2 \max_{q} \gamma_{q} \,. \tag{A.4}$$

Examples are given in Table II. The Table suggests that the highest value of  $\gamma_q$  is achieved when all the photons populate only one mode.

## Appendix: Mutual information for a pure loss channel

For a lossy channel with transmissivity  $\eta$ , the mutual information I(A;B) = H(B) - H(B|A) between Alice and Bob can be computed explicitly. We assume that Bob measures by photo-detection.

If Alice sends one particular code words  $\psi_j$  containing n photons, Bob will get k photons with probability  $p(k|j) = \eta^k (1-\eta)^{n-k}$ . This is uniquely identified if Bob measures by photo-detection by k detection events. There exists

 $N_k = \binom{n}{k}$  possible measurement outputs of this kind. Therefore the conditional entropy is

$$H(B|A) = -\sum_{j} p(j) \sum_{k} N_{k} \, p(k|j) \log p(k|j)$$
(A.1)

$$= -\sum_{j} p(j) \sum_{k=0}^{n} {n \choose k} \eta^{k} (1-\eta)^{n-k} \log \left[ \eta^{k} (1-\eta)^{n-k} \right]$$
 (A.2)

$$= -\sum_{k=0}^{n} \binom{n}{k} \eta^{k} (1-\eta)^{n-k} \log \left[ \eta^{k} (1-\eta)^{n-k} \right], \tag{A.3}$$

where p(j) is the probability of code words  $\psi_j$ .

Now consider that Bob obtains a certain combination of k detection events over m modes. This output is compatible with  $M_k = \binom{m-k}{n-k}$  input code words sent by Alice. As the total number of code words is  $M = \binom{m}{n}$ , the probability of obtaining a given combination of k detection is

$$p(k) = \frac{M_k}{M} p(k|j) = \frac{\binom{m-k}{n-k}}{\binom{m}{n}} \eta^k (1-\eta)^{n-k}.$$
 (A.4)

Note that the total number of possible outputs is  $N'_k = {m \choose k}$ , therefore we have

$$H(B) = -\sum_{k=0}^{n} N'_{k} p(k) \log p(k)$$
(A.5)

$$= -\sum_{k=0}^{n} {m \choose k} \frac{{m-k \choose n-k}}{{m \choose n}} \eta^{k} (1-\eta)^{n-k} \log \left[ \frac{{m-k \choose n-k}}{{m \choose n}} \eta^{k} (1-\eta)^{n-k} \right]$$
(A.6)

$$= -\sum_{k=0}^{n} \binom{n}{k} \eta^{k} (1-\eta)^{n-k} \log \left[ \frac{\binom{m-k}{n-k}}{\binom{m}{n}} \eta^{k} (1-\eta)^{n-k} \right]. \tag{A.7}$$

Finally we obtain:

$$I(A;B) = -\sum_{k=0}^{n} \binom{n}{k} \eta^{k} (1-\eta)^{n-k} \log \frac{\binom{m-k}{n-k}}{\binom{m}{n}}$$
(A.8)

$$= \log \binom{m}{n} - \sum_{k=0}^{n} \binom{n}{k} \eta^k (1-\eta)^{n-k} \log \binom{m-k}{n-k}. \tag{A.9}$$

With the mutual information we can write the asymptotic rate

$$r = \beta I(A; B) - \log(n+1) - \log\frac{d}{M},$$
 (A.10)

where  $\beta$  is the error correction efficiency. An example of net transmission rate is given in Fig. 4. The plot includes finite size effects.

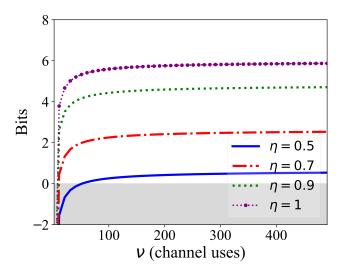


FIG. 4. Net number of bits transmitted per channel use in the presence of loss. The parameters are  $\beta=0.95, \delta=0.01,$  m,n=20,3 and  $\epsilon=\exp(-\nu^s \ln 2), s=0.43$ , where  $\nu$  is the number of channel use. The shaded region is a guide to the eye only, and data points above the region denotes net positive secret communication rate.