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Cardinal directions: A comparison of direction relation matrix and objects interaction matrix

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How to express and reason with cardinal directions between extended objects such as lines and regions is an important problem in qualitative spatial reasoning (QSR), a common subfield of geographical information science and Artificial Intelligence (AI). The *direction relation matrix* (DRM) model, proposed by Goyal and Egenhofer in 1997, is one very expressive relation model for this purpose. Unlike many other relation models in QSR, the set-theoretic *converse* of a DRM relation is not necessarily representable in DRM. Schneider et al. regard this as a serious shortcoming and propose, in their work published in ACM TODS (2012), the *objects interaction matrix* (OIM) model for modelling cardinal directions between complex regions. OIM is also a *tiling-based model* that consists of two phases: the *tiling phase* and the *interpretation phase*. Although it was claimed that OIM is a novel concept, we show that it is not so different from DRM if we represent the cardinal direction of two regions a, b by both the DRM of a to b and that of b to a. Under this natural assumption, we give methods for computing DRMs from OIMs and vice versa, and show that OIM is *almost* the same as DRM in the tiling phase, and becomes *less* precise after interpretation. Furthermore, exploiting the similarity between the two models, we prove that the consistency of a complete basic OIM network can be decided in cubic time. This answers an open problem raised by Schneider et al. regarding efficient algorithms for reasoning with OIM.

Keywords: Cardinal directions; direction relation matrix; objects interaction matrix; qualitative spatial reasoning

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1. Introduction

How to express and reason with qualitative spatial relations has been extensively studied in the recent decades by researchers in pattern recognition (Peuquet and Zhan 1987), natural language understanding (Davis 2013), robotics (Escrig and Toledo 1998), geographical information science (Egenhofer and Franzosa 1991, Frank 1996), spatial databases (Schneider *et al.* 2012), and artificial intelligence (Randell *et al.* 1992). A significant number of different qualitative relation models have been proposed, see e.g. (Frank 1996, Goyal and Egenhofer 1997, Guesgen 1989, Ligozat 1998, Peuquet and Zhan 1987, Skiadopoulos *et al.* 2007, Frank 2010) for directional models and (Randell *et al.* 1992, Egenhofer and Franzosa 1991, Li 2006, Nedas *et al.* 2007, Egenhofer and Franzosa 2010) for topological models. Among the directional models, the direction relation matrix (DRM) model proposed in (Goyal and Egenhofer 1997) is a very simple model for representing cardinal directions between extended spatial objects and has attracted interests of researchers from spatial databases, geographical information science, and artificial intelligence (see e.g. (Cicerone and di Felice 2004, Goyal and Egenhofer 2001, Liu and Li 2011, Liu *et al.* 2009, 2010, Navarrete *et al.* 2007, Skiadopoulos *et al.* 2005, Skiadopoulos and Koubarakis 2004, 2005, Zhang *et al.* 2008)).

Though simple for computing, the DRM model is very expressive. For two regions a, b, it represents the cardinal direction of a to b as a nonempty subset of

$$\mathbf{CD} = \{NW, N, NE, W, O, E, SW, S, SE\}, \tag{1}$$

where O stands for *origin*, and NW (northwest), N (north), NE (northeast), W (west), E (east), SW (southwest), S (south), SE (southeast) are the eight basic cardinal directions. Thus it defines altogether 511 direction relations between complex regions.

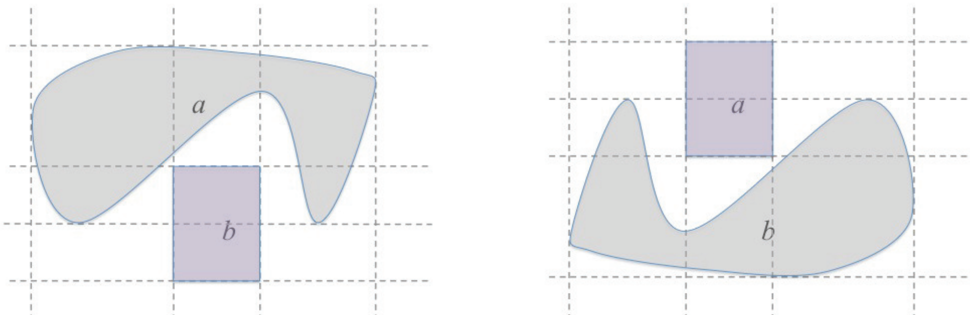

Figure 1. Two instances of *strictly_north_cap_of* defined in (Schneider *et al.* 2012)

Unlike many other relation models, DRM is not closed under converse (see Remark 2). This implies in particular that, for two regions a and b, the DRM of b to a cannot be uniquely determined by that of a to b. As a consequence, DRM does not satisfy *the converseness crieterion* of Schneider *et al.* (2012) either, which requires, for example, a is related by $\{N, NE, E\}$ to b if, and only if, b is related by $\{S, SW, W\}$ to a. Schneider *et al.* (2012) regard this violation as a serious problem and propose instead the *objects interaction matrix* (OIM) model for modelling cardinal directions between complex regions. Just like DRM, the OIM model is also a *tiling-based model* that consists of two phases: the *tiling phase* and the *interpretation phase*. Given two regions a and b, the tiling phase computes a matrix, denoted by $\mathsf{OIM}(a, b)$, as the low-level representation for the cardinal direction of a to b. The interpretation phase then interprets this matrix as a set of basic cardinal directions in **CD**, say $\{N, NE\}$. Schneider *et al.* (2012) further demonstrate how to define predicates such as *surrounds* and *strictly_north_cap_of* in spatial databases based on the interpretation. Several of these predicates are, however, counterintuitive. For example, though the OIM model can express true statements such as "Rome *surrounds* The Vatican,"[1] it also supports false statements like "The Vatican *surrounds* Rome." This is because the *surrounds* relation as defined in (Schneider *et al.* 2012) is a symmetrical relation. As another example, the OIM model also regards both configurations in Figure 1 as instances of the *strictly_north_cap_of* relation. We note that these configurations are distinguishable by the DRM model. This hence puts doubts on the rationality of the OIM interpretation and the necessity of the converseness requirement for direction relation models for regions.

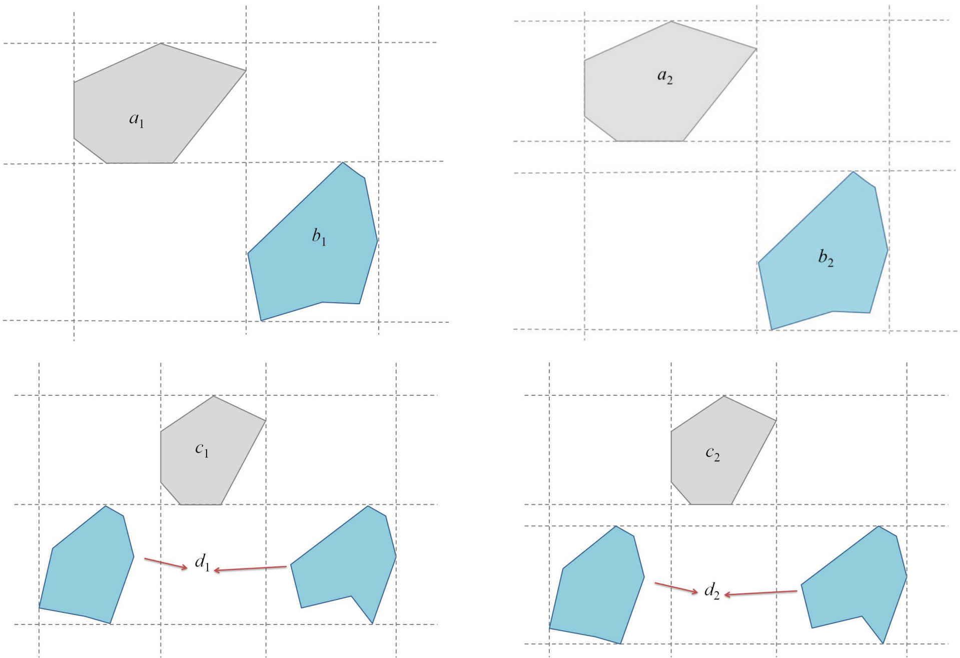

Figure 2. Two pairs of configurations that are distinguishable in the OIM model but indistinguishable in the DRM model

It was claimed in (Schneider *et al.* 2012) that the OIM model is "*a novel concept*," "*rather differs from the direction relation matrix (DRM) model and is thus not its extension*" and "*provides a much more fine-grained and complete identification of the possible valid spatial configurations between two simple regions than the DRM model.*" In this paper, however, we show that the two models are not so different.

Since DRM is not closed under converse, it is necessary to represent the cardinal direction of two regions a, b by both the DRM of a to b and that of b to a. We say a pair of DRMs (M_1, M_2) is *consistent* if there are two regions a, b such that M_1 is the DRM of a to b and M_2 is the DRM of b to a (Cicerone and di Felice 2004). Each consistent pair of DRMs defines a binary relation, which is called a *bi-DRM* relation (cf. Definition 4) in this paper. Under this natural assumption, we show that OIM is *not so different* from DRM but actually almost the same. In fact, we show that (i) every OIM relation is

[1] We assume that, as a country, The Vatican is *not* contained in Rome, the capital of Italy.

contained in a unique bi-DRM relation (cf. Proposition 3); (ii) except a few cases (104 out of 1677, or 6.2%), every relation is identical to a bi-DRM relation (cf. Proposition 4).

Figure 2 gives two pairs of configurations which have the same bi-DRM relations but different OIM relations. Consider the top two configurations. The DRM model cannot distinguish configuration $\{a_1, b_1\}$ from $\{a_2, b_2\}$, but the OIM model can (see Example 3). The two configurations differ only in that, along the y-axis, the minimal bounding rectangle (mbr, cf. Section 2 and Figure 3 for definition) of b_1 *meets* that of a_1, but there is a *gap* between the mbr of b_2 and that of a_2. Similar interpretation applies to the other two configurations in Figure 2. We will show in Proposition 4 that this actually holds for *every* OIM relation that is not identical to a bi-DRM relation.

Exploiting the similarity between the two models, we further show that the reasoning mechanisms developed for DRM (Liu and Li 2011, Liu *et al.* 2010) can also be transplanted to OIM and similar computational complexity results are obtained.

The remainder of this paper is organised as follows. Section 2 provides a *uniform* definition for objects interaction matrices and direction relation matrices. Section 3 establishes the connection between OIMs and DRMs. In Section 4 we compare the expressivity of the OIM model and the DRM model in both the tiling phase and the interpretation phase. In Section 5 we investigate the computational property of reasoning with the OIM model and Section 6 concludes the paper.

2. Definitions of DRM and OIM

This section introduces the direction relation matrix (DRM) model (Goyal and Egenhofer 1997) and the objects interaction matrix (OIM) model (Schneider *et al.* 2012) in a uniform framework. These models are designed for modelling the cardinal direction between regions in the plane.

A *region* in the plane is a nonempty, bounded, and regular closed set of the two dimensional Cartesian plane R^2. For a region a, the *minimal bounding rectangle (mbr)* of a is defined as $\mathcal{M}(a) = I_x(a) \times I_y(a)$, where $I_x(a) = [a_x^-, a_x^+]$ ($I_y(a) = [a_y^-, a_y^+]$, resp.) is the minimal closed interval that contains the x-projection (y-projection, resp.) of region a (see Figure 3). Note that when a is connected, $I_x(a)$ and $I_y(a)$ are the x- and y-projections of a.

The OIM model and the DRM model can be defined via the notion of *tiles* and *grid*. Let $L = L_x \cup L_y$ be a set of lines parallel to the coordinate axes, where

$$L_x = \{x = x_1, ..., x = x_n\} \ (x_1 < x_2 < ... < x_n)$$
$$L_y = \{y = y_1, ..., y = y_m\} \ (y_1 < y_2 < ... < y_m).$$

The *grid* generated by L is defined as

$$G_L = \{(x_i, x_{i+1}) \times (y_j, y_{j+1}) : 0 \leq i \leq n, 0 \leq j \leq m\},$$

where $x_0 = y_0 = -\infty$ and $x_{n+1} = y_{m+1} = +\infty$. Each element

$$t_{ij} = (x_i, x_{i+1}) \times (y_j, y_{j+1}) \tag{2}$$

in the grid G_L is called a *tile*. The *size* of G_L is the number of tiles contained in G_L. Here it is $(m + 1) \times (n + 1)$.

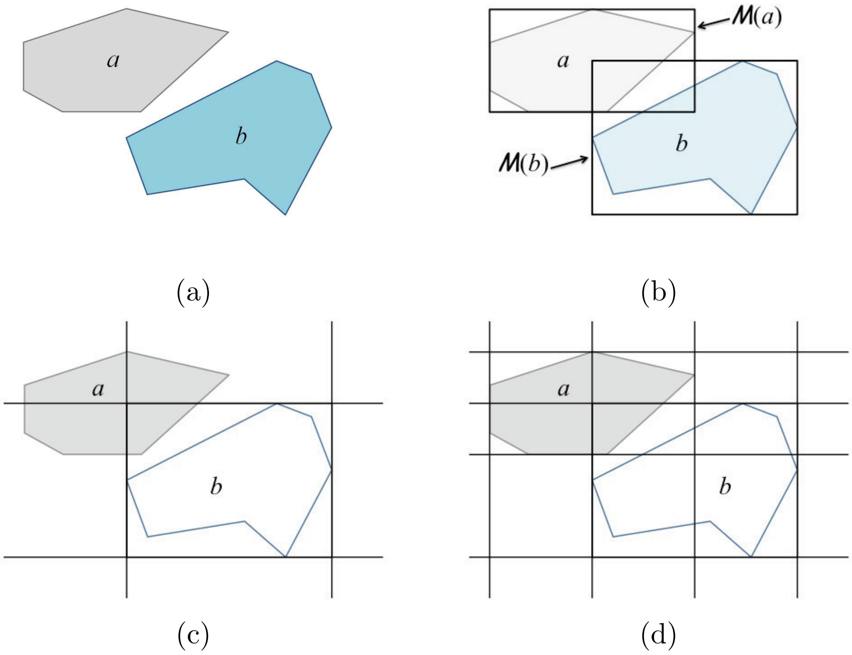

Figure 3. Illustrations of (a) regions a and b; (b) $\mathcal{M}(a)$ and $\mathcal{M}(b)$; (c) grid G^b; and (d) grid $G^{a,b}$.

Given two regions a and b, three grids can be introduced via the mbrs of a and b (cf. Figure 3). We write L^a for the set of lines $\{y = a_y^-, y = a_y^+, x = a_x^-, x = a_x^+\}$ and L^b similarly. Let G^a and G^b be the grids generated by L^a and L^b respectively, and let $G^{a,b}$ be the grid generated by $L^a \cup L^b$. Note that the sizes of G^a and G^b are both 3×3. Because lines in L^a and L^b may coincide, the size of $G^{a,b}$ could be any $s \times t$ for $3 \leq s, t \leq 5$.

We now define the *occupancy matrix* of a region a w.r.t. a grid G, which will be used later to derive the OIM and the DRM of a to b.

Definition 1. (occupancy matrix) Let G be a grid with size $(m+1) \times (n+1)$. The *occupancy matrix* of a region a w.r.t. G, denoted by $\mathsf{OM}_G(a)$, is defined as the $(m+1) \times (n+1)$ matrix (p_{ij}), where $p_{ij} = 1$ if $a \cap t_{ij} \neq \varnothing$ and $p_{ij} = 0$ otherwise, where $t_{ij} = (x_i, x_{i+1}) \times (y_i, y_{i+1})$ is the (i,j)-th tile in grid G.

Remark 1. Note that each occupancy matrix has the same size as the underlying grid. Intuitively, the occupancy matrix records those tiles in the grid that intersect the region. Given the occupancy matrix $\mathsf{OM}_G(a) = (p_{ij})$ of a region a with respect to a grid $G = (t_{ij})$, we regard the topological closure of $\bigcup\{t_{ij} \in G : p_{ij} = 1\}$ as an (upper) approximation of region a. In this sense, a grid provides a granularity to approximate a region.

For two regions a, b, we next introduce the direction relation matrix and the objects interaction matrix of a to b by using their occupancy matrices.

Definition 2. (direction relation matrix (Goyal and Egenhofer 1997)) Given two regions a and b, the direction relation matrix of the primary object a to the reference object b is defined as $\mathsf{DRM}(a, b) = \mathsf{OM}_{G^b}(a)$.

Clearly, the shape of the reference object (here b) does not affect the direction relation matrix. In particular, $\mathsf{DRM}(a, b) = \mathsf{DRM}(a, \mathcal{M}(b))$. By exchanging the role of a and b, we have that $\mathsf{DRM}(b, a) = \mathsf{OM}_{G^a}(b)$.

Definition 3. (objects interaction matrix (Schneider *et al.* 2012)) Suppose a and b are two regions, and $G^{a,b}$ is the grid generated by $L^a \cup L^b$ as above with size $(m+1)\times(n+1)$. Define the *raw objects interaction matrix* of a to b as the $(m+1)\times(n+1)$ matrix

$$\mathsf{rOIM}(a,b) = \mathsf{OM}_{G^{a,b}}(a) + 2 \times \mathsf{OM}_{G^{a,b}}(b). \tag{3}$$

The *objects interaction matrix* of a to b, written as $\mathsf{OIM}(a,b)$, is the $(m-1)\times(n-1)$ matrix obtained by removing the first and the last columns and rows from $\mathsf{rOIM}(a,b)$.

It is straightforward to see that each OIM has size $s \times t$, where $1 \le s, t \le 3$. Moreover, let $\mathsf{OIM}(a,b) = (p_{ij})_{s\times t}$. Then we can prove that

$$p_{ij} = \begin{cases} 0, \text{ if } a \cap t_{ij} = \varnothing,\ b \cap t_{ij} = \varnothing; \\ 1, \text{ if } a \cap t_{ij} \neq \varnothing,\ b \cap t_{ij} = \varnothing; \\ 2, \text{ if } a \cap t_{ij} = \varnothing,\ b \cap t_{ij} \neq \varnothing; \\ 3, \text{ if } a \cap t_{ij} \neq \varnothing,\ b \cap t_{ij} \neq \varnothing, \end{cases}$$

where t_{ij} is the (i,j)-th tile in the grid $G^{a,b}$. This shows that the above definition is equivalent to the original one in (Schneider *et al.* 2012).

Example 1. Suppose a, b are regions as shown in Figure 3(a). Figure 3(b) shows the mbrs of a, b, and Figure 3(c) and (d) show grid G^b and grid $G^{a,b}$ respectively. Matrices $\mathsf{DRM}(a,b) = \mathsf{OM}_{G^b}(a)$, $\mathsf{OM}_{G^{a,b}}(a)$, $\mathsf{OM}_{G^{a,b}}(b)$, $\mathsf{rOIM}(a,b)$ and $\mathsf{OIM}(a,b)$ are shown below.

$\mathsf{DRM}(a,b)$	$\mathsf{DRM}(b,a)$	$\mathsf{OM}_{G^{a,b}}(a)$	$\mathsf{OM}_{G^{a,b}}(b)$	$\mathsf{rOIM}(a,b)$	$\mathsf{OIM}(a,b)$
$\begin{pmatrix}1&1&0\\1&1&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&1&1\\0&1&1\end{pmatrix}$	$\begin{pmatrix}0&0&0&0&0\\0&1&1&0&0\\0&1&1&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&1&1&0\\0&0&1&1&0\\0&0&0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0&0&0\\0&1&1&0&0\\0&1&3&2&0\\0&0&2&2&0\\0&0&0&0&0\end{pmatrix}$	$\begin{pmatrix}1&1&0\\1&3&2\\0&2&2\end{pmatrix}$

It is clear that the first and the last columns and rows of $\mathsf{rOIM}(a,b)$ are all zeros. Therefore, $\mathsf{OIM}(a,b)$ and $\mathsf{rOIM}(a,b)$ can be trivially converted from one to the other. Furthermore, the two occupancy matrices $\mathsf{OM}_{G^{a,b}}(a)$ and $\mathsf{OM}_{G^{a,b}}(b)$ can be obtained from $\mathsf{OIM}(a,b)$.

Lemma 1. *Suppose a, b are two regions. Let $G^{a,b}$ be the grid generated by $L^a \cup L^b$. Then the objects interaction matrix $\mathsf{OIM}(a,b)$ can be computed from the two occupancy matrices $\mathsf{OM}_{G^{a,b}}(a)$ and $\mathsf{OM}_{G^{a,b}}(b)$ in constant time, and vice versa.*

Proof: $\mathsf{OM}_{G^{a,b}}(a)$ is obtained from $\mathsf{rOIM}(a,b)$ by replacing each occurrence of 2 with 0 and each occurrence of 3 with 1 and leaving the other entries unchanged; and $\mathsf{OM}_{G^{a,b}}(b)$ is obtained from $\mathsf{rOIM}(a,b)$ by replacing each occurrence of 1 with 0 and each occurrence of 3 with 1 and leaving the other entries unchanged. The other direction is clear from the definition of OIM. □

In this paper, we regard each DRM M and each OIM N as a binary relation between regions, which are defined as

$$\delta_M = \{(a,b) : \mathsf{DRM}(a,b) = M\}; \tag{4}$$

$$\rho_N = \{(a,b) : \mathsf{OIM}(a,b) = N\}. \tag{5}$$

For each DRM M, we call δ_M a basic DRM relation. It is easy to see that the set of basic DRM relations are jointly exhaustive and pairwise disjoint (JEPD), i.e. for any two regions a, b there exists a unique DRM M such that $\mathsf{DRM}(a, b) = M$. Similar notation and conclusion applies to basic OIM relations.

2.1. *Consistent pairs of DRMs*

Note that $\mathsf{OIM}(b, a)$ can be obtained from $\mathsf{OIM}(a, b)$ by replacing each occurrence of 1 by 2 and each occurrence of 2 by 1. This implies in particular that the OIM model meets the converseness requirement (Schneider *et al.* 2012). The following example, however, shows that the DRM model does not enjoy this property. That is, in general, the direction relation matrix of b to a cannot be *uniquely* determined by that of a to b (see also (Cicerone and di Felice 2004, Liu *et al.* 2010)).

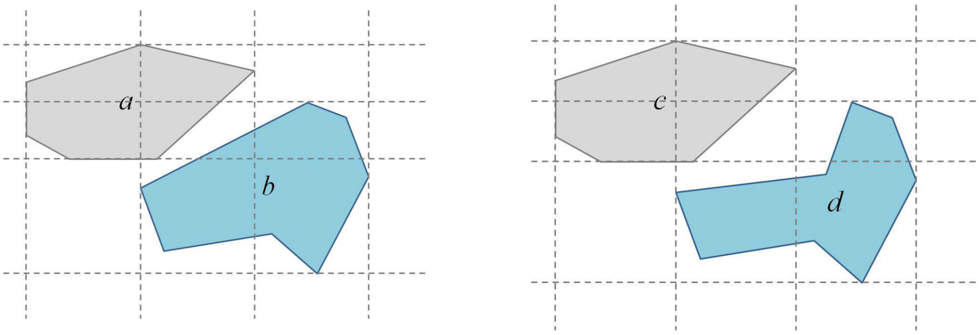

Figure 4. Illustrations of regions a, b, c and d

Example 2. Suppose regions a, b, c and d are as shown in Figure 4. We have

$$\mathsf{DRM}(a, b) = \mathsf{DRM}(c, d) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

while

$$\mathsf{DRM}(b, a) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \mathsf{DRM}(d, c).$$

For the object interaction matrices, we have

$$\mathsf{OIM}(a, b) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 2 \\ 0 & 2 & 2 \end{pmatrix} \neq \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 2 \end{pmatrix} = \mathsf{OIM}(c, d).$$

The above example illustrates that the cardinal direction between two regions a, b is *more precisely* expressed by the pair $(\mathsf{DRM}(a, b), \mathsf{DRM}(b, a))$ than by $\mathsf{DRM}(a, b)$ alone. This fact was first noticed by Cicerone and di Felice (2004), where they also call such a pair of DRMs as a *consistent pair*. In general, we say a pair of DRMs (M_1, M_2) is *consistent* if there exist two regions a, b such that $M_1 = \mathsf{DRM}(a, b)$ and $M_2 = \mathsf{DRM}(b, a)$.

By the definition of consistent DRM pairs, we have

Proposition 1. *A pair of DRMs* (M_1, M_2) *is consistent if and only if* $\delta_{M_1} \cap \delta^{-1}_{M_2}$ *is nonempty.*

For convenience, we introduce the following notion.

Definition 4. A binary relation R of regions is called a *bi-DRM relation* if there exists a consistent pair (M_1, M_2) of DRMs such that $R = \delta_{M_1} \cap \delta^{-1}_{M_2}$, i.e.

$$R = \{(a, b) : \mathsf{DRM}(a, b) = M_1, \mathsf{DRM}(b, a) = M_2\}.$$

It is clear that each bi-DRM relation corresponds to a unique consistent pair of DRMs, and vice versa. Moreover, the set of bi-DRM relations is also jointly exhaustive and pairwise disjoint. Since there are 1621 consistent pairs of DRMs (Liu *et al.* 2010), we have in total 1621 bi-DRM relations.

Remark 2. The (set-theoretical) *converse* of a binary relation R is defined as

$$R^{-1} = \{(b, a) : (a, b) \in R\}. \tag{6}$$

The DRM model is not *closed under converses*, in the sense that there exists a DRM M such that δ^{-1}_M cannot be represented as the union of several basic DRM relations. For example, let

$$M = \begin{pmatrix} 0\,0\,0 \\ 1\,0\,0 \\ 0\,0\,0 \end{pmatrix}, \quad N_1 = \begin{pmatrix} 0\,0\,0 \\ 0\,0\,1 \\ 0\,0\,0 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0\,0\,0 \\ 0\,0\,1 \\ 0\,0\,1 \end{pmatrix}, \quad N_1 = \begin{pmatrix} 0\,0\,1 \\ 0\,0\,1 \\ 0\,0\,0 \end{pmatrix}, \quad N_1 = \begin{pmatrix} 0\,0\,1 \\ 0\,0\,1 \\ 0\,0\,1 \end{pmatrix}.$$

Then (M, N_i) is a consistent pair for $1 \leq i \leq 4$, and there is no other DRM N' such that (M, N') is a consistent pair. It is clear that $\delta^{-1}_M \subseteq \bigcup_{i=1}^4 \delta_{N_i}$. Let $a = [0, 1] \times [0, 3]$ and $b = [2, 3] \times [1, 2]$. Then we have $\mathsf{DRM}(b, a) = N_1$, but $\mathsf{DRM}(a, b) \neq M$, i.e. $(b, a) \in \delta_{N_1} \subseteq \bigcup_{i=1}^4 \delta_{N_i}$ but $(b, a) \notin \delta^{-1}_M$. This shows $\delta^{-1}_M \neq \bigcup_{i=1}^4 \delta_{N_i}$ and, hence, δ^{-1}_M is not representable in DRM.

3. Connection between OIMs and DRMs

In this section we establish the connection between OIMs and DRMs. Our results show that, for two regions a, b, the DRM of a to b and that of b to a can be computed from the OIM of a to b; and the OIM of a to b can be computed from the DRM of a to b, the DRM of b to a, and the rectangle relation (see below for definition) of $\mathcal{M}(a)$ and $\mathcal{M}(b)$.

To formalise our main result, we briefly review the Interval Algebra (Allen 1983) and the Rectangle Algebra (Guesgen 1989, Balbiani *et al.* 1999). The Interval Algebra (IA) distinguishes 13 basic relations between closed intervals (see Table 1, where $x = [x^-, x^+], y = [y^-, y^+]$ are two intervals).

The Rectangle Algebra (RA) can be viewed as the product of two IAs. For two rectangles $a = [a^-_x, a^+_x] \times [a^-_y, a^+_y]$ and $b = [b^-_x, b^+_x] \times [b^-_y, b^+_y]$ with edges parallel to the coordinate axes, the RA relation between a and b is defined as $\alpha \otimes \beta$, where α and β are the IA relation between $[a^-_x, a^+_x]$ and $[b^-_x, b^+_x]$ and, respectively, the IA relation between $[a^-_y, a^+_y]$ and $[b^-_y, b^+_y]$. RA relations can also be generalised from rectangles to *regions* via their mbrs.

Table 1. Basic IA relations and their converses, where $x = [x^-, x^+], y = [y^-, y^+]$ are two intervals.

Relation	Symbol	Converse	Meaning
before	b	bi	$x^- < x^+ < y^- < y^+$
meets	m	mi	$x^- < x^+ = y^- < y^+$
overlaps	o	oi	$x^- < y^- < x^+ < y^+$
starts	s	si	$x^- = y^- < x^+ < y^+$
during	d	di	$y^- < x^- < x^+ < y^+$
finishes	f	fi	$y^- < x^- < x^+ = y^+$
equals	eq	eq	$x^- = y^- < x^+ = y^+$

Definition 5. For two regions a, b, the RA relation of a to b, written $\mathsf{RA}(a, b)$, is the RA relation of $\mathcal{M}(a)$ to $\mathcal{M}(b)$.

Our main result is stated as follows.

Theorem 1. *Suppose a, b are two regions. We can compute $\mathsf{RA}(a, b)$, $\mathsf{DRM}(a, b)$ and $\mathsf{DRM}(b, a)$ from $\mathsf{OIM}(a, b)$, and vice versa.*

3.1. *Proof of Theorem 1*

We first note that Theorem 1 is different from Lemma 1. In the latter we use the grid $G^{a,b}$ and compute $\mathsf{OM}_{G^{a,b}}(a)$ and $\mathsf{OM}_{G^{a,b}}(b)$ from $\mathsf{OIM}(a, b)$ and vice versa; while in Theorem 1, we need to compute $\mathsf{DRM}(a, b) = \mathsf{OM}_{G^b}(a)$ by using grid G^b and compute $\mathsf{DRM}(b, a) = \mathsf{OM}_{G^a}(b)$ by using grid G^a. Recall that grid $G^{a,b}$ is obtained by overlaying grid G^a on grid G^b. We need to establish the connection of the two occupancy matrices of a region w.r.t. two different grids.

Suppose G is a grid with size $(m + 1) \times (n + 1)$ generated by L, and G' is a grid with size $(m' + 1) \times (n' + 1)$ generated by $L' \subset L$. In this case we say G is *finer* than G', or G' is *coarser* than G. Note that $m' \leq m$ and $n' \leq n$, and each tile in G is contained in a tile in G'. For $n \geq 0$, we write $[n]$ for the set $\{0, 1, ..., n\}$. Define the *coarse function*

$$C : [m] \times [n] \to [m'] \times [n'] \tag{7}$$

from G to G' by $C(i, j) = (k, l)$ if tile $t_{ij} \in G$ is contained in tile $t'_{kl} \in G'$.

Lemma 2. *Suppose G is a grid generated by L, and G' a grid generated by $L' \subset L$, and C the coarse function from G to G'. For a region a, the occupancy matrix $\mathsf{OM}_{G'}(a)$ can be uniquely determined if $\mathsf{OM}_G(a)$ is given.*

Proof: Suppose $G = \{t_{ij}\}$, $G' = \{t'_{kl}\}$. Assume furthermore that $\mathsf{OM}_G(a) = (p_{ij})$ and $\mathsf{OM}_{G'}(a) = (p'_{kl})$. Then $p'_{kl} = 1$ if and only if $1 \in \{p_{ij} : C(i, j) = (k, l)\}$. □

Lemma 3. *Suppose a and b are two regions. Given the RA relation of a to b, we can uniquely determine the occupancy matrix $\mathsf{OM}_{G^{a,b}}(\mathcal{M}(a))$ and the coarse function from $G^{a,b}$ to G^b.*

Proof: This is because the RA relation $\mathsf{RA}(a, b)$ completely describes the relations of lines in L^a and lines in L^b, which further determine $\mathsf{OM}_{G^{a,b}}(\mathcal{M}(a))$ and the coarse function. The following details can be obtained by case analysis.

Suppose $\mathsf{RA}(a, b) = \alpha \otimes \beta$. Then the occupancy matrix $\mathsf{OM}_{G^{a,b}}(\mathcal{M}(a))$ is identical to $V(\beta)^T V(\alpha)$, where V is the function from basic IA relations to vectors specified in Table 2.

Table 2. Functions V and f in the proof of Lemma 3.

α	b	m	o	s	d	f	eq
$V(\alpha)$	$(0,1,0,0,0)$	$(0,1,0,0)$	$(0,1,1,0,0)$	$(0,1,0,0)$	$(0,0,1,0,0)$	$(0,0,1,0)$	$(0,1,0)$
$f_\alpha(0),...,f_\alpha(m_\alpha)$	$0,0,0,1,2$	$0,0,1,2$	$0,0,1,1,2$	$0,1,1,2$	$0,1,1,1,2$	$0,1,1,2$	$0,1,2$
α	bi	mi	oi	si	di	fi	
$V(\alpha)$	$(0,0,0,1,0)$	$(0,0,1,0)$	$(0,0,1,1,0)$	$(0,1,1,0)$	$(0,1,1,1,0)$	$(0,1,1,0)$	
$f_\alpha(0),...,f_\alpha(m_\alpha)$	$0,1,2,2,2$	$0,1,2,2$	$0,1,1,2,2$	$0,1,2,2$	$0,0,1,2,2$	$0,0,1,2$	

Suppose $V(\alpha)$ is a vector with m_α entries. Then the coarse function $C_{\alpha\otimes\beta}$ from $G^{a,b}$ to G^b can also be represented by the products of two functions f_β and f_α (defined in Table 2), i.e. $C_{\alpha\otimes\beta}(i,j) = (f_\beta(i), f_\alpha(j))$. Here for each basic IA relation α, f_α (specified in the above table) is a function with domain $\{0,1,...,m_\alpha\}$ and range $\{0,1,2\}$ (note that the size of G^b is always 3×3). □

In the following, we prove Theorem 1.

We first show that $\mathsf{RA}(a,b)$, $\mathsf{DRM}(a,b)$ and $\mathsf{DRM}(b,a)$ can be computed from $\mathsf{OIM}(a,b)$. The following lemma shows that $\mathsf{OIM}(a,b)$ determines $\mathsf{RA}(a,b)$.

Lemma 4. *Suppose a,b are two regions. Given $\mathsf{OIM}(a,b)$, we can compute $\mathsf{RA}(a,b)$ and the coarse function C from $G^{a,b}$ to G^b.*

Proof: Suppose $\mathsf{OIM}(a,b) = (p_{ij})$. Let

$$\begin{aligned} x_1 &= \min\{i : (\exists j)\ p_{ij} = 1 \text{ or } 3\},\\ x_2 &= \max\{i : (\exists j)\ p_{ij} = 1 \text{ or } 3\},\\ x_3 &= \min\{i : (\exists j)\ p_{ij} = 2 \text{ or } 3\},\\ x_4 &= \max\{i : (\exists j)\ p_{ij} = 2 \text{ or } 3\}. \end{aligned}$$

It is straightforward to prove that the IA relation between $I_x(a)$ and $I_x(b)$ is the same as that between $[x_1, x_2+1]$ and $[x_3, x_4+1]$. Similarly we compute the IA relation between $I_y(a)$ and $I_y(b)$, and thus the RA relation $\mathsf{RA}(a,b)$. The remaining part follows directly from Lemma 3. □

The following lemma shows that we can compute the DRMs from the OIM.

Lemma 5. *Suppose a,b are two regions. Then we can compute $\mathsf{DRM}(a,b)$ and $\mathsf{DRM}(b,a)$ from $\mathsf{OIM}(a,b)$.*

Proof: Given $\mathsf{OIM}(a,b)$, we have $\mathsf{OM}_{G^{a,b}}(a)$ by Lemma 1 and the coarse function C from $G^{a,b}$ to G^b by Lemma 4. By Lemma 2, we can compute $\mathsf{OM}_{G^b}(a)$, which is $\mathsf{DRM}(a,b)$ by definition. Since $\mathsf{OIM}(b,a)$ can be obtained from $\mathsf{OIM}(a,b)$, we also have $\mathsf{DRM}(b,a)$ in the same way.

□

These two lemmas show that $\mathsf{RA}(a,b)$, $\mathsf{DRM}(a,b)$ and $\mathsf{DRM}(b,a)$ can be obtained from $\mathsf{OIM}(a,b)$. To prove that $\mathsf{RA}(a,b)$, $\mathsf{DRM}(a,b)$ and $\mathsf{DRM}(b,a)$ are sufficient to decide $\mathsf{OIM}(a,b)$, we first show that $\mathsf{RA}(a,b)$ decides $\mathsf{OM}_{G^{a,b}}(\mathcal{M}(a))$, and then show that $\mathsf{OM}_{G^{a,b}}(a)$ can be computed from $\mathsf{OM}_{G^{a,b}}(\mathcal{M}(a))$ and $\mathsf{DRM}(a,b) = \mathsf{OM}_{G^b}(a)$. Note that $\mathsf{OM}_{G^{a,b}}(a)$ and $\mathsf{OM}_{G^{a,b}}(\mathcal{M}(a))$ have the same size, and each entry of $\mathsf{OM}_{G^{a,b}}(a)$ is less than or equal to the corresponding entry of $\mathsf{OM}_{G^{a,b}}(\mathcal{M}(a))$.

Lemma 6. *Suppose a is a region and G_0 a grid generated by L_0. Let G be the finer grid obtained by adding L^a (the bounding lines of a) to L_0. For a tile $t \in G$, let t_0 be the tile in G_0 that contains t. Then $t \cap a \neq \varnothing$ iff $t_0 \cap a \neq \varnothing$ and $t \cap \mathcal{M}(a) \neq \varnothing$.*

Proof: The necessity is clear because $t \subseteq t_0$ and $a \subseteq \mathcal{M}(a)$. For the sufficiency, we observe that each tile in G is contained in a unique tile in G_0 and a unique tile in grid G^a (the grid generated by L^a), and is in fact their intersection. Moreover, because t is a tile in G and $\mathcal{M}(a)$ is a tile in G^a, we have $t \cap \mathcal{M}(a) \neq \varnothing$ iff $t \subseteq \mathcal{M}(a)$. Since t is contained in t_0, we have $t = t_0 \cap \mathcal{M}(a)$. Suppose $t_0 \cap a \neq \varnothing$ and $t \cap \mathcal{M}(a) \neq \varnothing$. Then we have $t \subseteq \mathcal{M}(a)$ and hence $t = t_0 \cap \mathcal{M}(a)$. From $t \cap a = t_0 \cap \mathcal{M}(a) \cap a = t_0 \cap a$, we know $t \cap a$ is nonempty because $t_0 \cap a$ is nonempty.

□

Lemma 7. *Suppose a, b are two regions. The occupancy matrix $\mathsf{OM}_{G^{a,b}}(a)$ can be uniquely determined if $\mathsf{RA}(a, b)$ and $\mathsf{DRM}(a, b)$ are given.*

Proof: By Lemma 3, from $\mathsf{RA}(a, b)$ we have $\mathsf{OM}_{G^{a,b}}(\mathcal{M}(a))$ as well as the coarse function C from grid $G^{a,b} = \{t_{ij}\}$ to grid $G^b = \{t'_{kl}\}$. Let t_{ij} be a tile in grid $G^{a,b}$, and p_{ij} the corresponding entry in $\mathsf{OM}_{G^{a,b}}(a)$. Note that t_{ij} is contained by the tile $t'_{C(i,j)} \in G^b$. Let $\bar{p}_{ij}$ be the entry in $\mathsf{OM}_{G^{a,b}}(\mathcal{M}(a))$ corresponding to tile t_{ij} (in grid $G^{a,b}$), and $p'_{C(i,j)}$ be the entry in $\mathsf{DRM}(a, b)$ corresponding to tile $t'_{C(i,j)}$ (in grid G^b). Then we have by Lemma 6 that $p_{ij} = 1$ iff $t_{ij} \cap a \neq \varnothing$ iff $t'_{C(i,j)} \cap a \neq \varnothing$ and $t_{ij} \cap \mathcal{M}(a) \neq \varnothing$ iff $p'_{C(i,j)} = 1$ and $\bar{p}_{ij} = 1$. Therefore $\mathsf{OM}_{G^{a,b}}(a)$ can be uniquely determined by $\mathsf{RA}(a, b)$ and $\mathsf{DRM}(a, b)$.
□

Lemma 8. *Suppose a, b are two regions. The objects interaction matrix $\mathsf{OIM}(a, b)$ can be uniquely determined if $\mathsf{DRM}(a, b)$, $\mathsf{DRM}(b, a)$ and $\mathsf{RA}(a, b)$ are given.*

Proof: By Lemma 7, we compute $\mathsf{OM}_{G^{a,b}}(a)$ from $\mathsf{RA}(a, b)$ and $\mathsf{DRM}(a, b)$. As $\mathsf{RA}(b, a)$ is the converse of $\mathsf{RA}(a, b)$, we also have $\mathsf{OM}_{G^{a,b}}(b)$ from $\mathsf{RA}(a, b)$ and $\mathsf{DRM}(b, a)$. Then $\mathsf{OIM}(a, b)$ can be computed by Lemma 1.

□

Theorem 1 then follows directly from the above lemmas. We note that, from the proofs above, we can easily design algorithms with constant complexity that transform between (i) $\mathsf{OIM}(a, b)$ and (ii) $\mathsf{RA}(a, b)$, $\mathsf{DRM}(a, b)$, $\mathsf{DRM}(b, a)$. In the following subsection, we give a more intuitive method for computing $\mathsf{DRM}(a, b)$ and $\mathsf{DRM}(b, a)$ from $\mathsf{OIM}(a, b)$.

3.2. *Computing DRMs from OIM*

We note that, for regions a, b shown in Figure 3(a), we have

$$\mathsf{OIM}(a, b) = \mathsf{DRM}(a, b) + 2 \times \mathsf{DRM}(b, a).$$

In fact, this holds for all 3×3 OIMs. A more general result is given below.

Suppose a, b are two regions. Let M be $\mathsf{OIM}(a, b)$, and M^* the 3×3 matrix obtained by properly inserting rows and/or columns of zeros (see below). We show

$$M^* = \mathsf{DRM}(a, b) + 2 \times \mathsf{DRM}(b, a). \tag{8}$$

Suppose M is of size $m \times n$. There are three cases concerning the rows $r_1, ..., r_m$ of M, where r_i $(1 \leq i \leq m)$ is the i-th row of M:

(1) $m = 3$.
(2) $m = 2$, there are three exhaustive and disjoint subcases (Schneider *et al.* 2012, Lemma 4.10):

(2a) $3 \in r_1$, or $\{1,2\} \subseteq r_1$;
(2b) $3 \in r_2$, or $\{1,2\} \subseteq r_2$;
(2c) $r_1 \subseteq \{0,1\}$ and $r_2 \subseteq \{0,2\}$; or $r_1 \subseteq \{0,2\}$ and $r_2 \subseteq \{0,1\}$;
(3) $m = 1$.

For each of the above cases, we assign a 0-1 matrix U_M as below.

Case of M	(1)	(2a)	(2b)	(2c)	(3)
U_M	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}0&0\\1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\\0&0\end{pmatrix}$	$\begin{pmatrix}1&0\\0&0\\0&1\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\end{pmatrix}$

Note that U_M is a matrix with size $3 \times m$, and $U_M M$, the matrix product of U_M and M, is a matrix with size $3 \times n$. $U_M M$ is an extension of M by inserting $3 - m$ rows of zeros. For example, in case (2c), $U_M M$ is the matrix obtained by inserting a row of zeros in the middle of M.

Similar classification can be made considering the columns of M, and an $n \times 3$ matrix V_M can be defined. Note that $M^* = U_M M V_M$ is a 3×3 matrix. The following proposition has been automatically verified by a program. Here we only provide a proof sketch for the 3×3 case.

Proposition 2. *Suppose a, b are two regions and $M = \mathsf{OIM}(a,b)$. Let U_M and V_M be defined as above and $M^* = U_M M V_M$. Then $M^* = \mathsf{DRM}(a,b) + 2 \times \mathsf{DRM}(b,a)$.*

Proof: (Sketch) Take the case when M is a 3×3 matrix as an example. In this special case, U_M and V_M are the identity 3×3 matrix, and thus $M^* = M$. We need only show $\mathsf{OIM}(a,b) = \mathsf{DRM}(a,b) + 2 \times \mathsf{DRM}(b,a)$. By the definition of OIM and Equation (3) (in Definition 3), this holds if we can show that $\mathsf{DRM}(a,b)$ is the 3×3 matrix obtained by removing the first and the last rows and columns from $\mathsf{OM}_{G^{a,b}}(a)$, and that $\mathsf{DRM}(b,a)$ is the 3×3 matrix obtained by removing the first and the last rows and columns from $\mathsf{OM}_{G^{a,b}}(b)$.

We need only consider $\mathsf{DRM}(a,b)$. The argument for $\mathsf{DRM}(b,a)$ is the same. Recall that $\mathsf{DRM}(a,b) = \mathsf{OM}_{G^b}(a)$. We need to show that $\mathsf{OM}_{G^b}(a)$ can be obtained by removing the first and the last rows and columns from $\mathsf{OM}_{G^{a,b}}(a)$. Note that $G^{a,b}$ is a 5×5 grid. For a tile t in $G^{a,b}$ or in G^b, we have that $t \cap a \neq \varnothing$ only if $t \subseteq \mathcal{M}(a)$. Therefore, if the intersection of a and a tile t in G^b is nonempty then there exists a unique sub-tile t' of t in $G^{a,b}$ such that $a \cap t' \neq \varnothing$. It is then easy to see that $\mathsf{OM}_{G^b}(a)$ is obtained by removing the first and the last rows and columns from $\mathsf{OM}_{G^{a,b}}(a)$. □

The above definition and proposition provide a *more intuitive* method for computing $\mathsf{DRM}(a,b)$ and $\mathsf{DRM}(b,a)$ from $\mathsf{OIM}(a,b)$. Given M, we first compute U_M, V_M, and $U_M M V_M$, then $\mathsf{DRM}(a,b)$ can be obtained from $U_M M V_M$ by replacing each occurrence of 2 with 0, and each occurrence of 3 with 1; and $\mathsf{DRM}(b,a)$ can be obtained from $U_M M V_M$ by replacing each occurrence of 1 with 0, and each occurrence of 2 or 3 with 1.

We note that when computing $\mathsf{RA}(a,b)$, $\mathsf{DRM}(a,b)$ and $\mathsf{DRM}(b,a)$ from $\mathsf{OIM}(a,b)$, the RA relation and the two direction relation matrices depend only on the objects interaction matrix $\mathsf{OIM}(a,b)$, and not on the particular choice of regions a, b. Similarly, the objects interaction matrix $\mathsf{OIM}(a,b)$ depends only on the RA relation $\mathsf{RA}(a,b)$ and the two direction relation matrices $\mathsf{DRM}(a,b)$ and $\mathsf{DRM}(b,a)$, and not on the particular choice of regions a, b.

This observation implies the following result, where ρ_N and δ_{M_i} are binary relations associated with OIM N and DRM M_i $(i = 1,2)$ (cf. (4) and (5)).

Proposition 3. *Given an OIM N, there exist a unique DRM M_1, a unique DRM M_2,* *and a unique basic RA relation $\alpha \otimes \beta$, such that*

$$\rho_N \subseteq \delta_{M_1}, \quad \rho_N \subseteq \delta^{-1}_{M_2}, \quad \rho_N \subseteq \alpha \otimes \beta.$$

Moreover, in this case we have

$$\rho_N = \delta_{M_1} \cap \delta^{-1}_{M_2} \cap \alpha \otimes \beta. \tag{9}$$

On the other hand, suppose M_1, M_2 are two DRMs and $\alpha \otimes \beta$ a basic RA relation. Then there exists a unique OIM N which satisfies (9) *iff $\delta_{M_1} \cap \delta^{-1}_{M_2} \cap \alpha \otimes \beta$ is nonempty.*

Proof: Given an OIM N, by Theorem 1 and the observation above this proposition, we know there exist a basic RA relation $\alpha \otimes \beta$ and two DRMs M_1, M_2 that satisfies (9). In particular, we have $\rho_N \subseteq \delta_{M_1}$, $\rho_N \subseteq \delta^{-1}_{M_2}$, and $\rho_N \subseteq \alpha \otimes \beta$. The uniqueness of $M_1, M_2, \alpha \otimes \beta$ follows from the fact that both the set of basic RA relations and the set of basic DRM relations are jointly exhaustive and pairwise disjoint.

On the other hand, suppose M_1, M_2 are two DRMs and $\alpha \otimes \beta$ a basic RA relation such that $\delta_{M_1} \cap \delta^{-1}_{M_2} \cap \alpha \otimes \beta$ is nonempty. Take an instance (a, b) from $\delta_{M_1} \cap \delta^{-1}_{M_2} \cap \alpha \otimes \beta$. It is easy to see that $N = \mathsf{OIM}(a, b)$ is the unique OIM which satisfies (9). □

Recall that relations with form $\delta_{M_1} \cap \delta^{-1}_{M_2}$ are called bi-DRM relations in this paper (cf. Definition 4). This result shows in particular that OIM relations are *finer* than bi-DRM relations. In the next section, we will show that, except *a few cases* (104 out of 1677, or 6.2%), every OIM relation ρ_N is identical to a bi-DRM relation $\delta_{M_1} \cap \delta^{-1}_{M_2}$.

4. Comparison between OIM and DRM

In the previous section we have seen from Theorem 1 that the OIM model is more expressive than the DRM model in the tiling phase, even when we consider bi-DRM relations (or, but equivalently, consistent pairs of DRM relations). In this section we first make a more detailed comparison by counting the numbers of different scenarios that can be distinguished in the tiling phase, and then compare the two models in the interpretation phase.

4.1. *Comparison in the tiling phase*

Note that there are 1677 valid objects interaction matrices (Schneider *et al.* 2012), while there are only 511 valid direction relation matrices. By Proposition 3, we know that each OIM relation is contained in a unique DRM relation. This shows that the OIM model is finer than the DRM model.

As said before, it will be more precise if we use bi-DRM relations to represent the cardinal direction information between regions. There are 1621 bi-DRM relations (Liu *et al.* 2010). By Proposition 3 again, each OIM relation ρ_N is contained in a unique bi-DRM relation $\delta_{M_1} \cap \delta^{-1}_{M_2}$. It will be interesting to know when ρ_M is *identical to* a bi-DRM relation. We consider the configurations shown in Figure 2.

Example 3. For the two pairs of configurations shown in Figure 2, we have

$$\mathsf{DRM}(a_1,b_1)=\mathsf{DRM}(a_2,b_2)=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}\text{ and }\mathsf{DRM}(b_1,a_1)=\mathsf{DRM}(b_2,a_2)=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix},\text{ but}$$

$$\mathsf{OIM}(a_1,b_1)=\begin{pmatrix}1&0\\0&2\end{pmatrix}\neq\begin{pmatrix}1&0\\0&0\\0&2\end{pmatrix}=\mathsf{OIM}(a_2,b_2);\text{ and}$$

$$\mathsf{DRM}(c_1,d_1)=\mathsf{DRM}(c_2,d_2)=\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}\text{ and }\mathsf{DRM}(d_1,c_1)=\mathsf{DRM}(d_2,c_2)=\begin{pmatrix}0&0&0\\0&0&0\\1&0&1\end{pmatrix},\text{ but}$$

$$\mathsf{OIM}(c_1,d_1)=\begin{pmatrix}0&1&0\\2&0&2\end{pmatrix}\neq\begin{pmatrix}0&1&0\\0&0&0\\2&0&2\end{pmatrix}=\mathsf{OIM}(d_2,c_2).$$

This shows that DRM cannot distinguish configuration $\{a_1,b_1\}$ from $\{a_2,b_2\}$ or $\{c_1,d_1\}$ from $\{c_2,d_2\}$, but OIM can. RA can also makes such distinctions. In fact, we have

$$\mathsf{RA}(a_1,b_1)=\mathsf{m}\otimes\mathsf{mi}\neq\mathsf{m}\otimes\mathsf{bi}=\mathsf{RA}(a_2,b_2),$$
$$\mathsf{RA}(c_1,d_1)=\mathsf{d}\otimes\mathsf{mi}\neq\mathsf{d}\otimes\mathsf{bi}=\mathsf{RA}(c_2,d_2).$$

The example shows that there are bi-DRM relations that contain more than one OIM relations. Note that in these configurations the mbrs of two regions meet or disjoint along the x- or y-axis. In the following we show that this is the only case where OIM relations are finer than bi-DRM relations.

Proposition 4. *Suppose N_1,N_2 are two OIMs, M_1,M_2 are two DRMs, and $\alpha_i\otimes\beta_i$ ($i=1,2$) are basic RA relations such that*

$$\rho_{N_1}=\delta_{M_1}\cap\delta_{M_2}^{-1}\cap\alpha_1\otimes\beta_1;$$
$$\rho_{N_2}=\delta_{M_1}\cap\delta_{M_2}^{-1}\cap\alpha_2\otimes\beta_2.$$

Assume in addition that $N_1\neq N_2$. Then $\{\alpha_1,\alpha_2\}=\{\mathsf{b},\mathsf{m}\}$, or $\{\alpha_1,\alpha_2\}=\{\mathsf{bi},\mathsf{mi}\}$, or $\{\beta_1,\beta_2\}=\{\mathsf{b},\mathsf{m}\}$, or $\{\beta_1,\beta_2\}=\{\mathsf{bi},\mathsf{mi}\}$. [1]

Proof: Suppose a_i,b_i ($i=1,2$) are four regions such that $\mathsf{OIM}(a_i,b_i)=N_i$ for $i=1,2$, i.e. $\mathsf{DRM}(a_1,b_1)=\mathsf{DRM}(a_2,b_2)=M_1$, $\mathsf{DRM}(b_1,a_1)=\mathsf{DRM}(b_2,a_2)=M_2$, and $\mathsf{RA}(a_i,b_i)=\alpha_i\otimes\beta_i$ for $i=1,2$. By (Liu *et al.* 2010, Proposition 4), we know if $\alpha_1\neq\alpha_2$ then $\{\alpha_1,\alpha_2\}$ is either $\{\mathsf{b},\mathsf{m}\}$ or $\{\mathsf{bi},\mathsf{mi}\}$; and if $\beta_1\neq\beta_2$ then $\{\beta_1,\beta_2\}$ is either $\{\mathsf{b},\mathsf{m}\}$ or $\{\mathsf{bi},\mathsf{mi}\}$. □

By Proposition 3, we know each OIM relation can be uniquely represented as the intersection of a DRM relation, the converse of a DRM relation, and a basic RA relation. We use this result to analyse when a OIM relation is identical to a bi-DRM relation.

[1] Here we write for example $\{\alpha_1,\alpha_2\}=\{\mathsf{b},\mathsf{m}\}$ if α_1 and α_2 are two different basic IA relations in $\{\mathsf{b},\mathsf{m}\}$.

Proposition 5. *Let N be an OIM. Suppose M_1, M_2 are the DRMs and $\alpha \otimes \beta$ the basic RA relation such that $\rho_N = \delta_{M_1} \cap \delta_{M_2}^{-1} \cap \alpha \otimes \beta$. If α or β is a relation in $\{\mathsf{b}, \mathsf{m}, \mathsf{bi}, \mathsf{mi}\}$, then $\delta_{M_1} \cap \delta_{M_2}^{-1} \cap \alpha^\star \otimes \beta^\star$ is nonempty for any $\alpha^\star, \beta^\star$ such that*

$$\alpha^\star = \alpha,\ \ or\ \{\alpha, \alpha^\star\} = \{\mathsf{b}, \mathsf{m}\},\ \ or\ \{\alpha, \alpha^\star\} = \{\mathsf{bi}, \mathsf{mi}\}; \tag{10}$$

$$\beta^\star = \beta,\ \ or\ \{\beta, \beta^\star\} = \{\mathsf{b}, \mathsf{m}\},\ \ or\ \{\beta, \beta^\star\} = \{\mathsf{bi}, \mathsf{mi}\}. \tag{11}$$

If neither α nor β is a relation in $\{\mathsf{b}, \mathsf{m}, \mathsf{bi}, \mathsf{mi}\}$, then $\delta_{M_1} \cap \delta_{M_2}^{-1} \cap \alpha^\star \otimes \beta^\star$ is nonempty iff $\alpha^\star = \alpha$ and $\beta^\star = \beta$.

Proof: Suppose for instance $\alpha = \mathsf{m}$ and $\alpha^\star = \mathsf{b}$, $\beta^\star = \beta$. We show $\delta_{M_1} \cap \delta_{M_2}^{-1} \cap \alpha^\star \otimes \beta^\star$ is nonempty. Let a, b be two regions which satisfy ρ_N, i.e. $(a, b) \in \delta_{M_1} \cap \delta_{M_2}^{-1}$ and $(a, b) \in \alpha \otimes \beta$. If $\alpha = \mathsf{m}$, we can move a to left along the x-axis and obtain another region $a^\star$. Note that there is a gap between $a^\star$ and b along the x-axis. Note by definition of direction relation matrix, $\mathsf{DRM}(a, b) = \mathsf{DRM}(a^\star, b)$ and $\mathsf{DRM}(b, a) = \mathsf{DRM}(b, a^\star)$, but $\mathsf{RA}(a, b) = \mathsf{b} \otimes \beta$. This shows that $\delta_{M_1} \cap \delta_{M_2}^{-1} \cap \mathsf{b} \otimes \beta$ is nonempty. The other cases are entirely analogous (cf. the regions in Figure 2).

On the other hand, suppose neither α nor β is a relation in $\{\mathsf{b}, \mathsf{m}, \mathsf{bi}, \mathsf{mi}\}$, but $\delta_{M_1} \cap \delta_{M_2}^{-1} \cap \alpha^\star \otimes \beta^\star$ is nonempty. By Proposition 3, we know there exists a unique OIM $N^\star$ such that $\rho_{N^\star} = \delta_{M_1} \cap \delta_{M_2}^{-1} \cap \alpha^\star \otimes \beta^\star$. Suppose $N \neq N^\star$. By Proposition 4, this implies that $\{\alpha, \alpha^\star\} = \{\mathsf{b}, \mathsf{m}\}$, or $\{\alpha, \alpha^\star\} = \{\mathsf{bi}, \mathsf{mi}\}$, or $\{\beta, \beta^\star\} = \{\mathsf{b}, \mathsf{m}\}$, or $\{\beta, \beta^\star\} = \{\mathsf{bi}, \mathsf{mi}\}$. By our assumption that neither α nor β is a relation in $\{\mathsf{b}, \mathsf{m}, \mathsf{bi}, \mathsf{mi}\}$, none of the above four cases is possible. This is a contradiction. Therefore $N = N^\star$ and, hence, $\alpha = \alpha^\star$ and $\beta = \beta^\star$. □

As a consequence, we have the following result.

Theorem 2. *Let N be an OIM. Suppose M_1, M_2 are the DRMs and $\alpha \otimes \beta$ the basic IA relation such that $\rho_N = \delta_{M_1} \cap \delta_{M_2}^{-1} \cap \alpha \otimes \beta$. Then $\rho_N = \delta_{M_1} \cap \delta_{M_2}^{-1}$ if and only if neither α nor β is a relation in $\{\mathsf{b}, \mathsf{m}, \mathsf{bi}, \mathsf{mi}\}$.*

We next have a closer examination of those OIMs which are not bi-DRM relations.

Proposition 6. *Suppose α, β are two basic IA relations such that either α or β is a relation in $\{\mathsf{b}, \mathsf{m}, \mathsf{bi}, \mathsf{mi}\}$. If neither α nor β is a relation in $\{\mathsf{d}, \mathsf{di}\}$, then there exists a unique OIM M such that $\alpha \otimes \beta = \rho_M$; if either α or β (but not both) is a relation in $\{\mathsf{d}, \mathsf{di}\}$, then there exist exactly two OIMs M_1 and M_2 such that $\alpha \otimes \beta = \rho_{M_1} \cup \rho_{M_2}$.*

For example, $\mathsf{m} \otimes \mathsf{s} = \rho_N$ and $\mathsf{m} \otimes \mathsf{d} = \rho_{M_1} \cup \rho_{M_2}$, where

$$N = \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix},\ M_1 = \begin{pmatrix} 0 & 2 \\ 1 & 2 \\ 0 & 2 \end{pmatrix},\ M_2 = \begin{pmatrix} 0 & 2 \\ 1 & 0 \\ 0 & 2 \end{pmatrix}. \tag{12}$$

There are 36 basic RA relations $\alpha \otimes \beta$ such that $\alpha \in \{\mathsf{b}, \mathsf{m}, \mathsf{bi}, \mathsf{mi}\}$, but $\beta \notin \{\mathsf{b}, \mathsf{m}, \mathsf{bi}, \mathsf{mi}\}$; 36 basic RA relations $\alpha \otimes \beta$ such that $\alpha \notin \{\mathsf{b}, \mathsf{m}, \mathsf{bi}, \mathsf{mi}\}$, but $\beta \in \{\mathsf{b}, \mathsf{m}, \mathsf{bi}, \mathsf{mi}\}$; and 16 basic RA relations $\alpha \otimes \beta$ such that both α and β are in $\{\mathsf{b}, \mathsf{m}, \mathsf{bi}, \mathsf{mi}\}$. In total, there are 88 basic RA relations $\alpha \otimes \beta$ such that either α or β is in $\{\mathsf{b}, \mathsf{m}, \mathsf{bi}, \mathsf{mi}\}$. Among these, there are 16 basic RA relations $\alpha \otimes \beta$ such that either α or β is in $\{\mathsf{d}, \mathsf{di}\}$. By Proposition 6 we know there are 88+16=104 OIMs whose corresponding basic RA relations have the form $\alpha \otimes \beta$, where either α or β is in $\{\mathsf{b}, \mathsf{m}, \mathsf{bi}, \mathsf{mi}\}$. By Proposition 5, these OIMs merge into

36/2+36/2+16/4+16/2 = 48 bi-DRM relations. As a consequence, OIM has 104-48=56 (or 3.5%) more relations than bi-DRM.

Remark 3. When restricted to connected regions, there are 805 valid matrices in the OIM model (Schneider *et al.* 2012), while there are 757 consistent pairs of DRMs (Liu *et al.* 2010). Similar to Proposition 6, we can show that when either α or β is a relation in $\{\mathsf{b}, \mathsf{m}, \mathsf{bi}, \mathsf{mi}\}$ then there exists a unique OIM M such that $\alpha \otimes \beta = \rho_M$. This is because matrices like M_2 in (12) are not valid OIMs for connected regions. Since there are 88 valid such OIMs and only 40 valid bi-DRM relations, we have 88-40=48 (or 6.3%) more OIM relations than bi-DRM relations for connected regions.

4.2. *Comparison in the interpretation phase*

Although convenient for computation, matrices are not a convenient and comprehensible tool for our everyday communication. The DRM model and the OIM model adopt an *interpretation phase* to handle this problem. The interpretation phase translates valid matrices to sets of basic cardinal directions in

$$\mathbf{CD} = \{NW, N, NE, W, O, E, SW, S, SE\}.$$

We next show that there is significant information loss during the interpretation phase in the OIM model, while no information loss occurs in the DRM model.

To facilitate the presentation, we introduce the following notion.

Definition 6. (cardinal direction between points (Frank 1991, Ligozat 1998)) Let $P = (x_P, y_P)$ and $Q = (x_Q, y_Q)$ be two points in the plane. The cardinal direction of P to Q, written $\mathbf{dir}(P, Q)$, is uniquely determined by the signs of $x_P - x_Q$ and $y_P - y_Q$. The correspondence is given below.

$$\begin{pmatrix} (-,+) & (0,+) & (+,+) \\ (-,0) & (0,0) & (+,0) \\ (-,-) & (0,-) & (+,-) \end{pmatrix} \Rightarrow \begin{pmatrix} NW & N & NE \\ W & O & E \\ SW & S & SE \end{pmatrix} \tag{13}$$

For example, $\mathbf{dir}(P, Q) = NW$ if and only if $x_P - x_Q < 0$ and $y_P - y_Q > 0$.

Definition 7. Let $M = (p_{ij})$ be a $m \times n$ matrix. We assign for each entry (i, j) $(1 \leq i \leq m, 1 \leq j \leq n)$ an integer point $\mathbf{loc}(i, j) = (j, -i)$ in the plane as its location.

We now give the interpretations of the two models.

Definition 8. (DRM interpretation) Suppose a and b are regions, and $\mathsf{DRM}(a, b) = (p_{ij})$ where $1 \leq i, j \leq 3$. Then the cardinal direction relation of a to b in the DRM model is a subset of $\mathbf{CD}$ defined as

$$\mathbf{dir}_{\mathsf{DRM}}(a, b) = \{\mathbf{dir}(\mathbf{loc}(i, j), \mathbf{loc}(2, 2)) : p_{ij} = 1, 1 \leq i, j \leq 3\}.$$

It is clear that the above interpretation is independent of the choice of a, b. It therefore establishes a mapping from the set of DRMs to the set of nonempty subsets of $\mathbf{CD}$. This mapping is bijective and $\mathsf{DRM}(a, b)$ can be restored from $\mathbf{dir}_{\mathsf{DRM}}(a, b)$ directly. That is to say, the DRM model loses no information in the interpretation phase.

The interpretation phase for the OIM model introduced in (Schneider *et al.* 2012) can be viewed as a tile-wise synthesis. Intuitively, let a and b be two regions, and G be the grid generated by the edges of their mbrs. We say a has a cardinal direction, say NW, to b, if there are two tiles t, t' in G such that $t \cap a \neq \varnothing$, $t' \cap b \neq \varnothing$, and t is to the northwest of t'. Formally, we have

Definition 9. (OIM interpretation (Schneider *et al.* 2012)) Suppose a, b are regions, and $\mathsf{OIM}(a, b) = (p_{ij})$ is a $m \times n$ matrix. The cardinal direction relation of a to b in the OIM model, written $\mathbf{dir}_{\mathsf{OIM}}(a, b)$, is defined as

$$\{\mathbf{dir}(\mathbf{loc}(i, j), \mathbf{loc}(i', j')) : p_{ij} \in \{1, 3\}, p_{i'j'} \in \{2, 3\}, 1 \leq i, i' \leq m, 1 \leq j, j' \leq n\}.$$

It is clear that the *converseness criterion* is satisfied in above interpretation (e.g., if $\mathbf{dir}_{\mathsf{OIM}}(a, b) = \{NW, N\}$ then $\mathbf{dir}_{\mathsf{OIM}}(b, a) = \{SE, S\}$).

As $\mathbf{dir}_{\mathsf{OIM}}(a, b)$ is a nonempty subset of $\mathbf{CD}$ (which has 9 elements), there are at most $2^9 - 1 = 511$ different cardinal direction relations in the OIM model. Because there are 1677 valid OIMs, the interpretation for the OIM model is *not* injective. We here give an example.

Example 4. Suppose a, b, c, d and r, s are regions (see Figure 5). Then we have

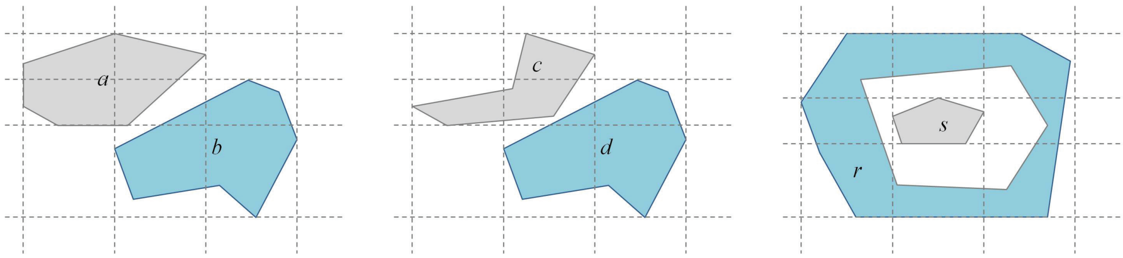

Figure 5. Illustrations of regions a, b, c, d, r and s

$$\mathbf{dir}_{\mathsf{OIM}}(a, b) = \mathbf{dir}_{\mathsf{OIM}}(c, d) = \{NW, N, W, O\},$$
$$\mathbf{dir}_{\mathsf{OIM}}(r, s) = \mathbf{dir}_{\mathsf{OIM}}(s, r) = \{NW, N, NE, E, SE, S, SW, W\},$$

but

$$\mathsf{OIM}(a, b) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 2 \\ 0 & 2 & 2 \end{pmatrix} \neq \begin{pmatrix} 0 & 1 & 0 \\ 1 & 3 & 2 \\ 0 & 2 & 2 \end{pmatrix} = \mathsf{OIM}(c, d),$$

$$\mathsf{OIM}(r, s) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \neq \begin{pmatrix} 2 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 2 \end{pmatrix} = \mathsf{OIM}(s, r).$$

That is to say, after interpretation, the cardinal direction relation of a to b is considered the same as that of c to d in the OIM model, though their objects interaction matrices are different. Meanwhile, in a ‘surrounds’ relation, the OIM model (after interpretation) cannot distinguish which region is the surrounding region and which is the surrounded region. These configurations are, however, distinguishable in the DRM model even after interpretation. Consider these configurations in the DRM model, we have

$$\mathbf{dir}_{\mathsf{DRM}}(a,b) = \{NW, N, W, O\} \neq \{N, W, O\} = \mathbf{dir}_{\mathsf{DRM}}(c,d),$$
$$\mathbf{dir}_{\mathsf{DRM}}(r,s) = \{NW, N, NE, E, SE, S, SW, W\} \neq \{O\} = \mathbf{dir}_{\mathsf{DRM}}(s,r).$$

Therefore, after interpretation, the OIM model fails to distinguish many configurations that are distinguishable before. In particular, it loses the ability to distinguish between the IA relations b and m of the projections of regions on x- and y-axes (cf. Proposition 4). More importantly, the *surrounds* relation given above suggests that the OIM model may lead to counterintuitive interpretations in some situations.

When considering consistent pairs of DRMs, the following theorem asserts that the OIM model is less expressive than the DRM model after interpretation.

Theorem 3. *Suppose a, b are two regions. Then $\boldsymbol{dir}_{\mathsf{OIM}}(a,b)$ can be derived from $\boldsymbol{dir}_{\mathsf{DRM}}(a,b)$ and $\boldsymbol{dir}_{\mathsf{DRM}}(b,a)$, but not vice versa.*

Proof: Recall that Theorem 1 has shown that $\mathsf{OIM}(a,b)$ can be derived from $\mathsf{RA}(a,b)$, $\mathsf{DRM}(a,b)$ and $\mathsf{DRM}(b,a)$. Moreover, the RA relation is needed only when the x- or y-projection of a and b related by IA relation b or m (or their converses). After interpretation, however, the difference between b and m disappears. This implies that $\mathbf{dir}_{\mathsf{OIM}}(a,b)$ can be uniquely determined by $\mathsf{DRM}(a,b)$ and $\mathsf{DRM}(b,a)$. Since the interpretation in the DRM model is bijective, we know $\mathbf{dir}_{\mathsf{OIM}}(a,b)$ can be derived from $\mathbf{dir}_{\mathsf{DRM}}(a,b)$ and $\mathbf{dir}_{\mathsf{DRM}}(b,a)$.

The opposite direction does not hold, because there are 1621 different consistent pairs of direction relation matrices but only 511 different $\mathbf{dir}_{\mathsf{OIM}}(a,b)$. □

The reduced expressivity and the counterintuitive examples therefore put doubts on the OIM interpretation given in (Schneider *et al.* 2012).

5. The consistency problem of the OIM model

The consistency problem is the central reasoning problem in qualitative spatial and temporal reasoning (QSTR). Many other reasoning problems (such as *the entailment problem* and *the minimal labelling problem*) may be reduced to the consistency problem (see e.g. (Cohn and Renz 2008)). Generally speaking, the consistency problem in QSTR is to decide whether a set of constraints on variables are satisfiable (i.e. whether the variables can be realised by spatial or temporal entities such that all the constraints are satisfied), where the constraints can only use relations from the relation model. We here concentrate on the consistency problems of the OIM model and the DRM model.

Definition 10. (constraint) Let $\mathcal{M}$ be the set of valid objects interaction matrices (direction relation matrices, resp.), and $V = \{v_1, ..., v_t\}$ be a set of spatial variables. A constraint is a formula of form $v_i \alpha v_j$, where α is a subset of $\mathcal{M}$. An assignment π is a function from V to the set of regions. Constraint $v_i \alpha v_j$ is satisfied by π if the OIM (the DRM, resp.) of $\pi(v_i)$ to $\pi(v_j)$ is in α, i.e. $\mathsf{OIM}(\pi(v_i), \pi(v_j)) \in \alpha$ ($\mathsf{DRM}(\pi(v_i), \pi(v_j)) \in \alpha$, resp.). The consistency problem is to decide, given a set of constraints, whether there exists an assignment satisfying all the constraints.

The consistency problem over many relation models is NP-hard in general, but becomes tractable when only special constraints are considered.

Definition 11. ((in)complete basic network) A constraint $v_i \alpha v_j$ is called a *basic constraint*, if $\alpha = \{M\}$ is a singleton, where M is a valid matrix in $\mathcal{M}$. In such a case, the constraint is also written as $v_i M v_j$. A complete basic network is a set of constraints that contains a basic constraint for each pair of variables; An incomplete basic network is a set of constraints that contains either a basic constraint or the universal constraint for each pair of variables.

The consistency problem of the DRM model has been investigated in (Skiadopoulos and Koubarakis 2004, 2005, Navarrete *et al.* 2007, Zhang *et al.* 2008, Liu *et al.* 2010, Liu and Li 2011) and, in particular, the following results have been obtained.

Theorem 4. *(Liu et al. 2010, Liu and Li 2011) The consistency problem of the DRM model is NP-complete. The consistency of a complete basic DRM network can be decided in cubic time, but the consistency problem becomes NP-complete for incomplete basic networks.*

We have similar conclusion for the OIM model.

Theorem 5. *The consistency problem of the OIM model is NP-complete. The consistency of a complete basic OIM network can be decided in cubic time, but the consistency problem becomes NP-complete for incomplete basic networks.*

Proof: (sketch) The cubic algorithm for the OIM model is quite analogous to that for the DRM model given in (Liu *et al.* 2010, Liu and Li 2011). Suppose $\mathcal{N} = \{v_i M_{ij} v_j\}$ is a complete basic constraint network of the OIM model, where M_{ij} is a valid OIM. The algorithm consists of three steps. The first step fixes the mbr of each variable. The edges of these mbrs generate a grid. Step 2 removes from each mbr a number of tiles in the grid that are forbidden by the constraints, and gets a candidate solution. Step 3 then verifies whether the candidate is indeed a solution.

Step 1. By Lemma 4, the RA relation between v_i and v_j is uniquely determined by M_{ij}. Therefore the OIM network entails an RA network of the variables. If the RA network is unsatisfiable, the OIM constraint network is also unsatisfiable. If the RA network is satisfiable and let $\{m_i\}$ be a solution where each m_i is a rectangle, then it can be proved that $\mathcal{N}$ is satisfiable iff it has a solution $\{a_i\}$ such that the mbr of a_i is m_i. We then try to find such a solution (if it exists) in the following steps.

Step 2. Denote by G the grid generated by all edges of these rectangles m_i. Initially we assume variable v_i is exactly m_i. Constraint $v_i M_{ij} v_j$ determines the occupancy matrices $\mathsf{OM}_{G^{i,j}}(v_i)$, where $G^{i,j}$ is the grid generated by the edges of m_i and m_j. Note that $G^{i,j}$ is coarser than G. Let t be a tile in $G^{i,j}$. Assume that the entry of $\mathsf{OM}_{G^{i,j}}(v_i)$ corresponding to t is zero. Then v_i should have empty intersection with tile t, and thus should have empty intersection with all the tiles from grid G that are contained in t. Therefore we should remove these tiles from v_i. For each constraint $v_i M_{ij} v_j$ and each zero entry of M_{ij}, we remove a number of tiles in G from v_i as above. Finally, each v_i is the union of the remaining tiles, which is contained in m_i. Denote the union of these tiles by a_i.

Step 3. It can be proved that if $\mathcal{N}$ is satisfiable, then $\{a_i\}$ obtained in Step 2 is a solution of $\mathcal{N}$. Therefore this step verifies whether each pair of a_i and a_j meets the constraint $v_i M_{ij} v_j$. If so, then $\mathcal{N}$ is satisfiable. Otherwise, $\mathcal{N}$ is unsatisfiable.

The first step requires $O(n^3)$ time, where n is the number of variables. A naive implementation of the remaining steps requires $O(n^4)$ time, which could be improved to $O(n^3)$ with a careful design of the representation of regions (see (Liu *et al.* 2010) for details). Therefore, for complete OIM basic constraint networks, the consistency problem can be solved in cubic time.

For incomplete basic networks, a polynomial reduction from 3-SAT to the consistency problem can be devised, which is similar to the one devised for the DRM model in (Liu and Li 2011).

□

The only difference between the cubic algorithm for the OIM model and that for the DRM model given in (Liu *et al.* 2010) is that the latter makes an assumption about the RA relation between a pair of variables when the DRM model fails to determine it uniquely (cf. Proposition 4). Because $\mathsf{OIM}(a, b)$ uniquely determines $\mathsf{RA}(a, b)$, the assumption is no longer necessary and the algorithm is thus slightly simpler.

As in the case of the DRM model, similar computational complexity results can be obtained for connected regions.

Remark 4. The cubic algorithm for OIM can, as in the case of DRM, be adapted to dealing with connected regions. The only extra work is to find a maximal connected component (mcc) of each a_i after Step 2. The target mcc should have m_i as its mbr, and there is at most one such mcc. If there is no such mcc for some variable v_i, then we conclude that the constraint network is unsatisfiable. Otherwise, we replace each a_i obtained in Step 3 with such a mcc and check whether these mccs forms a solution.

6. Remarks and conclusion

A number of criteria for models of cardinal directions have been proposed in (Schneider *et al.* 2012), one of which is called the *converseness* criterion. For two regions a, b, the converseness criterion requires that $\mathbf{dir}(a, b)$ should be the converse of $\mathbf{dir}(b, a)$. For example, if $\mathbf{dir}(a, b) = \{N, NW\}$, then $\mathbf{dir}(b, a)$ should be $\{S, SE\}$. Unlike the DRM model, the OIM model meets the converseness requirement and is thus claimed to be superior to the DRM model in (Schneider *et al.* 2012).

For extended objects such as regions, however, the necessity of the converseness requirement is arguable. First, though mathematically sound, the converseness criterion is not always in accordance with human cognition. For example, we agree that Rome *surrounds* The Vatican, but definitely not vice versa. Similar counterintuitive interpretations happen to several other predicates defined in (Schneider *et al.* 2012) (also see Figure 1). Second, if a model $\mathbf{M}$ of cardinal directions does not meet the converseness criterion, we can use a pair of cardinal directions $(\mathbf{dir}(a, b), \mathbf{dir}(b, a))$ in $\mathbf{M}$ to represent the directional information between two regions a, b. Write $\mathbf{Dir}(a, b) = (\mathbf{dir}(a, b), \mathbf{dir}(b, a))$. In this way, we transform $\mathbf{M}$ into a model that enjoys the converseness property. From $\mathbf{Dir}(a, b) = (\mathbf{dir}(a, b), \mathbf{dir}(b, a))$ we can directly infer that $\mathbf{Dir}(b, a) = (\mathbf{dir}(b, a), \mathbf{dir}(a, b))$.

Our Theorem 1 asserts that the objects interaction matrix $\mathsf{OIM}(a, b)$ is equivalent to the combination of the RA relation $\mathsf{RA}(a, b)$ and the direction relation matrices $\mathsf{DRM}(a, b)$ and $\mathsf{DRM}(b, a)$. This shows that the OIM model is more expressive than the DRM model in the tiling phase, but our Theorem 2 also implies that most (1573 out of 1677, or about 94%) OIM relations are exactly bi-DRM relations.

Our Theorem 3 asserts that the OIM model becomes less expressive than the DRM model after interpretation. This is mainly due to the interpretation function used in the OIM model (see Definition 9). To meet the converseness requirement, Schneider et al. treat both regions a, b as equal partners and assert that a cardinal direction, say NW, holds between a, b if there is, roughly speaking, a part of a which is to the northwest of some part of b. This limits the possible cardinal directions to 511, even if $\mathbf{dir}_{\mathsf{OIM}}(a, b)$ and $\mathbf{dir}_{\mathsf{OIM}}(b, a)$ are considered together. In contrast, the DRM model identifies 1621 different consistent pairs $(\mathbf{dir}_{\mathsf{DRM}}(a, b), \mathbf{dir}_{\mathsf{DRM}}(b, a))$. This shows that the interpretation function used in (Schneider *et al.* 2012) for OIM is inappropriate. To get a better interpretation, we suggest to use the interpretation function for DRM (see Definition 8) and consider consistent pairs of cardinal directions.

Furthermore, our Theorem 5 shows that the reasoning mechanisms developed for the DRM model can also be transplanted to the OIM model and similar computational complexity results are obtained.

In conclusion, we have shown that, if we represent the cardinal direction of two regions by a consistent pair of DRMs, then OIM is not so different from DRM but actually almost the same.

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