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Existence and Exponential Stability of Solutions for Quaternion-Valued Delayed Hopfield Neural Networks by ξ-Norms

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ABSTRACT Recently, with the development of quaternion applications, quaternion-valued neural networks (QVNNs) have been presented and studied by more and more scholars. In this paper, the existence, uniqueness and exponential stability criteria of solutions for the quaternion-valued delayed Hopfield neural networks (QVDHNNs) are mainly investigated by means of the definitions of ξ-norms. In order to construct a ξ-norm, QVDHNNs system are decomposed into four real-number systems according to Hamilton rules. Then, taking advantage of ξ-norms, inequality technique and Cauchy's test for convergence, time-invariant delays and time-varying delays are considered successively to derive ξ-exponential type sufficient conditions. Based on these, several corollaries about the existence, uniqueness and exponential stability of solutions are obtained. Finally, two numerical examples with time-invariant delays and time-varying delays are given respectively. Their simulated images illustrate the effectiveness of the main theoretical results.

INDEX TERMS Existence of solutions, exponential stability, ξ-norms, quaternion-valued neural networks, time delays.

I. INTRODUCTION

Up to now, neural networks (NNs) have attracted many more researchers' attentions on account of their potential applications after the models of Hopfield NNs (HNNs), Cohen-Grossberg NNs (CGNNs), and memristive NNs (MNNs) were constructed [1]– [5]. During the last three decades, dynamic behaviors and applications of dynamic networks systems have been deeply studied, and lots of crucial conclusions have been derived by many scholars [6]–[14]. Thereinto, dynamical behavior of NNs has been one evergreen hot topic because of its significant influence on NNs designed by VLSI [15]–[24]. For instance, synchronization problems of various NNs were studied in [16]–[18]. In [22]–[24], different stability criteria were given by means of diverse ways.

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At the beginning, the dynamical properties of NNs were mainly studied in the real number field [15]–[18], [24]. Then, with the development of NNs, these problems were generalized to CVNNs [14], [19], [23], [25]. Recently, QVNNs is starting to attract scholars attentions in the evolution of quaternion, which is a natural continuation of CVNNs. Quaternion, found by W. R. Hamilton in 1843, has been used to deal with various technical problems, such as computer graphics, array processing, 3 or 4-D data modeling, color image processing and so on [26]– [29]. Therefore, as an important application of quaternion, QVNNs has drawn more and more researchers eyes [30]–[35].

As is known to all, the implementation of NNs can be unavoidably affected by multiple time delays in reality. It is of great importance for the design of QVNNs. In general, time-invariant delays, time-varying delays, discrete delays, distributed delays and asynchronous time delays are usually considered in a dynamic system. Recently, some important conclusions about various time delays have been obtained for QVNNs [34]–[41]. For instance, in [34], [38], [41], bounded time-varying delays were considered to study the stability of QVNNs. Unbounded and asynchronous time-varying delays were considered in [35]. In [36], discrete and distributed delays were considered in the QVNNs system to investigate its stability conditions. Time-invariant delays were used to study the existence and stability of solutions for QVNNs in [37], [39]. Hence, time delays should be considered in a QVNNs system when their dynamical properties are studied.

Particularly, the existence and stability of solutions are the most fundamental dynamical property of QVNNs, which is usually studied by decomposition or direct approaches because quaternion multiplication is noncommutativity. The authors in [36], [37] researched the existence and stability criteria of multiple equilibrium points for QVNNs and impulsive QVNNs, respectively. In [38], M-matrix and matrix norm were used in quaternion-valued delayed NNs (QVDNNs) to investigate the the existence and stability of solutions. The authors in [33], [39] firstly utilized the homeomorphic mapping to get the existence criteria of solution, then they constructed a complex Lyapunov-Krasovskii functional to obtain the stable LMI conditions of QVDNNs. In [32], [35], $\{\xi, \infty\}$-norm was used in QVDNNs to get the existence and uniqueness criteria of solutions in the first step, then μ-stable criteria were obtained in step two. Although there have been these significant results, the sufficient conditions of stability are still worth further discussing in depth because of their complexity. $\{\xi\}$-norms, presented in [12], are interesting concepts to further study the existence and stability of solutions for delayed NNs. In this paper, $\{\xi, \infty\}$-norm will be used to work out these problems, the obtained results are different from [32], [35]. Furthermore, $\{\xi, 1\}$-norm is also used to study the exponential form stability criteria of QVDHNNs.

Motivated by the above analysis, in this paper, the existence, uniqueness and exponential stability of solutions are discussed for the QVHNNs with bounded time delays by ξ-norms. Firstly, since ξ-norms are defined in the real-number field, the QVDHNN system is decomposed into four real-number systems according to Hamilton rules, which avoid the noncommutativity of quaternion multiplication. Then, time-invariant delays and time-varying delays are considered successively to study the existence, uniqueness and exponential stability of solutions by constructing $\{\xi, 1\}$-norm and $\{\xi, \infty\}$-norm, respectively. In the proving process, monotone function and Cauchy convergence principle are used to obtain the exponential formal sufficient conditions. At the same time, since constructing Lyapunov type functionals is an important and complex problem, $e^{\varsigma t}$ are introduced to deal with time-varying delay term. Furthermore, in order to obtain the exponential stable criteria of QVHNN with time-varying delays, the assumptions of $\tau_{pq}(t) \leq \tau_{pq}$ and $\dot{\tau}_{pq}(t) \leq \eta_{pq} < 1$ are necessary, where $p, q = 1, 2, \cdots, n$. On the basis of the main results, several corollaries are derived in the third part. Finally, two numerical examples with time-invariant delays and time-varying delays are given to illustrate the effectiveness of obtained theoretical results, respectively.

Notation: $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Q}$ show, respectively, the set of real numbers, complex numbers and quaternion numbers. $\mathbb{R}^n$, $\mathbb{C}^n$ and $\mathbb{Q}^n$ denote, respectively, the n-dimensional Euclidean, unitary and quaternion space. $\mathbb{R}^{n\times m}$, $\mathbb{C}^{n\times m}$ and $\mathbb{Q}^{n\times m}$ are, respectively, the set of $n \times m$ real matrixes, $n \times m$ complex matrixes and $n \times m$ quaternion matrixes. $\|\cdot\|$ denotes Euclidean vector norm and $O(\cdot)$ denotes infinitesimal of the same order. If $z = (z_1, z_2, \cdots, z_n)^T \in \mathbb{Q}^n$, then $|z| = (|z_1|, |z_2|, \cdots, |z_n|)^T$.

II. PRELIMINARIES

Since the quaternion-valued system is studied in this paper, it is necessary to introduce some basic definitions and properties of quaternion. A quaternion $h \in \mathbb{Q}$ is defined as $h = h^R + \imath h^I + \jmath h^J + \kappa h^K \in \mathbb{Q}$ with $h^R, h^I, h^J, h^K \in \mathbb{R}$, which shows that the real quaternion field $\mathbb{Q}$ can be viewed as a 4-D vector space over $\mathbb{R}$. According to Hamilton rules, its imaginary units $\imath$, $\jmath$, and κ obey the following rules: $\imath\jmath = -\jmath\imath = \kappa$, $\jmath\kappa = -\kappa\jmath = \imath$, $\kappa\imath = -\imath\kappa = \jmath$, $\imath^2 = \jmath^2 = \kappa^2 = \imath\jmath\kappa = -1$, which means they are noncommutative. Its conjugate h^* or $\bar{h}$ is defined by $h^* = \bar{h} = h^R - \imath h^I - \jmath h^J - \kappa h^K$, and its modulus $|h|$ is defined by $|h| = \sqrt{h^* h} = \sqrt{(h^R)^2 + (h^I)^2 + (h^J)^2 + (h^K)^2}$. Let $s = s^R + \imath s^I + \jmath s^J + \kappa s^K \in \mathbb{Q}$, the addition $h+s$ and product hs of h and s can be defined as $h + s = (h^R + s^R) + \imath(h^I + s^I) + \jmath(h^J + s^J) + \kappa(h^K + s^K)$, and

$$\begin{aligned} hs = &(h^R s^R - h^I s^I - h^J s^J - h^K s^K) \\ &+ \imath(h^R s^I + h^I s^R + h^J s^K - h^K s^J) \\ &+ \jmath(h^R s^J + h^J s^R - h^I s^K + h^K s^I) \\ &+ \kappa(h^R s^K + h^K s^R + h^I s^J - h^J s^I), \end{aligned}$$

respectively.

In this paper, the following QVDHNNs will be considered:

$$\begin{cases} \dot{x}_p(t) = -d_p x_p(t) + \sum_{q=1}^{n} a_{pq} f_q(x_q(t)) \\ \qquad + \sum_{q=1}^{n} b_{pq} f_q(x_q(t - \tau_{pq}(t))) \\ \qquad + u_p, \qquad t \geq 0, \\ x_p(s) = \varphi_p(s), \quad s \in [-\tau, 0], \end{cases} \tag{1}$$

or, equivalently

$$\begin{cases} \dot{x}(t) = -Dx(t) + Af(x(t)) \\ \qquad + Bf(x(t - \tau(t))) + u, \quad t \geq 0, \\ x(s) = \varphi(s), \qquad s \in [-\tau, 0], \end{cases}$$

where $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T \in \mathbb{Q}^n$ with $x_p(t) = x_p^R(t) + \imath x_p^I(t) + \jmath x_p^J(t) + \kappa x_p^K(t)(p = 1, 2, \cdots, n)$ is the state vector, $D = \operatorname{diag}\{d_1, d_2, \cdots, d_n\} \in \mathbb{R}^{n\times n}$ with $d_p > 0$ is the self-inhibition matrix, $f(x(t)) = (f_1(x_1(t)), f_2(x_2(t)), \cdots, f_n(x_n(t)))^T \in \mathbb{Q}^n$ represents the neuron vector-valued activation functions, which satisfies $f_q(0) = 0$. $A = [a_{pq}]_{n\times n}, B = [b_{pq}]_{n\times n} \in \mathbb{Q}^{n\times n}$

are the connective weights matrixes, $\tau_{pq}(t) > 0$ is time delay, $u = (u_1, u_2, \cdots, u_n)^T \in \mathbb{Q}^n$ is an external input or bias vector. And the initial condition is $\varphi(s) = (\varphi_1(s), \varphi_2(s), \ldots, \varphi_n(s))^T \in C([-\tau, 0], \mathbb{Q}^n)$.

Denote $M = \{R, I, J, K\}$, then the QVDHNN model can be decomposed into four real-valued systems as follows for $L \in M$:

$$\dot{x}_p^L(t) = -d_p x_p^L(t) + \sum_{q=1}^{n}(a_{pq}f_q(x_q(t)))^L + \sum_{q=1}^{n}(b_{pq}f_q(x_q(t-\tau_{pq}(t))))^L + u_p^L. \quad (2)$$

In order to study the existence and stability of the above QVDNN model, the following assumptions should be introduced:

(H1) The activation function $f_q(x_q(t))$ can be separated into one real and three imaginary parts as

$$f_q(x_q(t)) = f_q^R(x_q^R(t)) + \imath f_q^I(x_q^I(t)) + \jmath f_q^J(x_q^J(t)) + \kappa f_q^K(x_q^K(t)),$$

where $f_q^l(x_q^l(t)) \triangleq f_q^l(t) : \mathbb{R} \to \mathbb{R}$ for every $l \in M$ satisfies the following conditions:

$$|f_q^l(x_q^l) - f_q^l(y_q^l)| \le \lambda_q^l |x_q^l - y_q^l|, \quad q = 1, 2, \cdots, n.$$

(H2) The time varying delays $\tau_{pq}(t) : \mathbb{R} \to \mathbb{R}$ are continuously differential functions and satisfy $\tau_{pq}(t) \le \tau_{pq} \le \tau$ and $\tau'_{pq}(t) \le \eta_{pq}$ for any $p, q = 1, 2, \cdots, n$ and $t > 0$, where $\tau_{pq} > 0$, $\tau > 0$ and $\eta_{pq} \in [0, 1)$ are real constants.

Remark 1: For the activation function $f_q(x_q(t))$, it may be decomposed into another form: $f_q(x_q(t)) = f_q^R(x_q^R(t), x_q^I(t), x_q^J(t), x_q^K(t)) + \imath f_q^I(x_q^R(t), x_q^I(t), x_q^J(t), x_q^K(t)) + \jmath f_q^J(x_q^R(t), x_q^I(t), x_q^J(t), x_q^K(t)) + \kappa f_q^K(x_q^R(t), x_q^I(t), x_q^J(t), x_q^K(t))$. Its every part exists continuous bounded partial derivatives in some references [32], [35], [38], which is a complex form of the above assumption (H1). In fact, the decomposed form of assumption (H1) does not affect the relative research on QVDHNN. Therefore, in order to reduce the complexity, the simple expression of $f_q(x_q(t))$ is used in this paper to study the existence and stability criteria of solutions.

Based on (H1), the systems (2) can be rewritten as

$$\dot{x}_p^L(t) = -d_p x_p^L(t) + \sum_{q=1}^{n}\sum_{(l,w)\in M^L}\psi_{lw}a_{pq}^l f_q^w(x_q^w(t)) + \sum_{q=1}^{n}\sum_{(l,w)\in M^L}\psi_{lw}b_{pq}^l f_q^w(x_q^w(t-\tau_{pq}(t))) + u_p^L, \quad (3)$$

where $M^L \in \{M^R, M^I, M^J, M^K\}$, $M^R = \{(R,R), (I,I), (J,J), (K,K)\}$, $M^I = \{(R,I), (I,R), (J,K), (K,J)\}$, $M^J = \{(R,J), (I,K), (J,R), (K,I)\}$, $M^K = \{(R,K), (I,J), (J,I), (K,R)\}$ and $\psi_{lw} \in \{\pm 1\}$ is the sign of $a_{pq}^l f_q^w(\cdot)$ and $b_{pq}^l f_q^w(\cdot)$ [36]. Then the concrete forms of $\dot{x}_p^R(t)$ $\dot{x}_p^I(t)$ $\dot{x}_p^J(t)$ and $\dot{x}_p^K(t)$ can be written as follows:

$$\begin{aligned}
\dot{x}_p^R(t) &= -d_p x_p^R(t) + \sum_{q=1}^{n}\big(a_{pq}^R f_q^R(t) - a_{pq}^I f_q^I(t) - a_{pq}^J f_q^J(t) - a_{pq}^K f_q^K(t)\big) \\
&\quad + \sum_{q=1}^{n}\big(b_{pq}^R f_q^R(t-\tau_{pq}(t)) - b_{pq}^I f_q^I(t-\tau_{pq}(t)) - b_{pq}^J f_q^J(t-\tau_{pq}(t)) - b_{pq}^K f_q^K(t-\tau_{pq}(t))\big),\\
\dot{x}_p^I(t) &= -d_p x_p^I(t) + \sum_{q=1}^{n}\big(a_{pq}^R f_q^I(t) + a_{pq}^I f_q^R(t) + a_{pq}^J f_q^K(t) - a_{pq}^K f_q^J(t)\big) \\
&\quad + \sum_{q=1}^{n}\big(b_{pq}^R f_q^I(t-\tau_{pq}(t)) + b_{pq}^I f_q^R(t-\tau_{pq}(t)) + b_{pq}^J f_q^K(t-\tau_{pq}(t)) - b_{pq}^K f_q^J(t-\tau_{pq}(t))\big),\\
\dot{x}_p^J(t) &= -d_p x_p^J(t) + \sum_{q=1}^{n}\big(a_{pq}^R f_q^J(t) - a_{pq}^I f_q^K(t) + a_{pq}^J f_q^R(t) + a_{pq}^K f_q^I(t)\big) \\
&\quad + \sum_{q=1}^{n}\big(b_{pq}^R f_q^J(t-\tau_{pq}(t)) - b_{pq}^I f_q^K(t-\tau_{pq}(t)) + b_{pq}^J f_q^R(t-\tau_{pq}(t)) + b_{pq}^K f_q^I(t-\tau_{pq}(t))\big),\\
\dot{x}_p^K(t) &= -d_p x_p^K(t) + \sum_{q=1}^{n}\big(a_{pq}^R f_q^K(t) + a_{pq}^I f_q^J(t) - a_{pq}^J f_q^I(t) + a_{pq}^K f_q^R(t)\big) \\
&\quad + \sum_{q=1}^{n}\big(b_{pq}^R f_q^K(t-\tau_{pq}(t)) + b_{pq}^I f_q^J(t-\tau_{pq}(t)) - b_{pq}^J f_q^I(t-\tau_{pq}(t)) + b_{pq}^K f_q^R(t-\tau_{pq}(t))\big). \quad (4)
\end{aligned}$$

Definition 1 [38]: A constant vector $x^* = (x_1^*, x_2^*, \cdots, x_n^*)^T \in \mathbb{Q}$ is called an equilibrium point of QVDHNN (1), if for $p, q = 1, 2, \cdots, n$,

$$-d_p x_p^* + \sum_{q=1}^{n} a_{pq} f_q(x_q^*) + \sum_{q=1}^{n} b_{pq} f_q(x_q^*) + u_p = 0,$$

Definition 2 [12]: For any vector $u(t) \in \mathbb{R}^{n\times 1}$, two generalized norms used in this paper are given as follows:

(1) $\{\xi, 1\}$-norm. $\|u(t)\|_{(\xi,1)} = \sum_i |\xi_i u_i(t)|$, where $\xi_i > 0$, $i = 1, 2, \cdots, n$.

(2) $\{\xi, \infty\}$-norm. $\|u(t)\|_{(\xi,\infty)} = \max_i |\xi_i^{-1} u_i(t)|$, where $\xi_i > 0$, $i = 1, 2, \cdots, n$.

This two type norms are called ξ-norms in this paper, denoted by $\|\cdot\|_\xi$.

Definition 3: Let x^* be an equilibrium point of QVDHNN (1), if there exists real constant $\alpha > 0$ such that $\|x(t) - x^*\|_\xi = O(e^{-\alpha t})$ holds for any solution $x(t)$, then QVDHNN (2.1) is said to be globally exponentially stable.

Remark 2: According to the above definitions of ξ-norms, they are defined in the field of real numbers. Therefore, when ξ-norms are used to study the existence and exponential stability criteria of QVDHNN (1), it is necessary for QVHNNs system to be decomposed into real-number systems. Actually, duo to the noncommutativity of quaternion multiplication, a QVHNN system can be separated into four real number systems according to the Hamilton rule, as is shown by (3) or (4).

III. MAIN RESULTS

In this section, by utilizing the definitions of ξ-norms, several theorems and their corollaries for the existence and exponential stability criteria of QVHNNs system (1) are derived throughout its decomposed systems (3) or (4). Specifically, time-invariant delays $\tau_{pq}(t) = \tau_{pq}$ are firstly considered in the QVHNNs because of its relative simplify. Two kinds of ξ-norms are used to obtain two different important theorems. Then, time-varying delays $\tau_{pq}(t) \le \tau_{pq}$ are considered in the QVHNNs.

A. CRITERIA WITH TIME-INVARIANT DELAYS τ_{PQ}

Theorem 1: Under assumption (H1), if there exist real constants $\varsigma > 0$, and $\xi_p^l > 0 (p = 1, 2, \cdots, n, l \in M)$, such that

$$(-d_q + \varsigma)\xi_q^L + \sum_{p=1}^{n} \sum_{(l,w)\in M^L} \xi_p^L(|a_{pq}^l| + e^{\varsigma\tau_{pq}}|b_{pq}^l|)\lambda_q^w \le 0$$

holds for every $L \in M$. Then, the dynamical system QVHNN (1) with time-invariant delays $\tau_{pq}(t) = \tau_{pq}$ has an unique equilibrium point x^*, which is globally exponentially stable.

Proof: From system (1) with time-invariant delays $\tau_{pq}(t) = \tau_{pq}$, we have

$$\frac{d\dot{x}_p(t)}{dt} = -d_p\dot{x}_p(t) + \sum_{q=1}^{n} a_{pq}f_q'(x_q(t))\dot{x}_q(t) + \sum_{q=1}^{n} b_{pq}f_q'(x_q(t-\tau_{pq}))\dot{x}_q(t-\tau_{pq}).$$

Define $u(t) = e^{\varsigma t}\dot{x}(t)$, then we have

$$\frac{du_p(t)}{dt} = (-d_p + \varsigma)u_p(t) + \sum_{q=1}^{n} a_{pq}f_q'(x_q(t))u_q(t) + \sum_{q=1}^{n} b_{pq}f_q'(x_q(t-\tau_{pq}))e^{\varsigma\tau_{pq}}u_q(t-\tau_{pq}).$$

By (2), we have

$$\begin{aligned}\frac{d|u_p^L(t)|}{dt} &= \mathrm{sign}\{u_p^L\}\Big((-d_p + \varsigma)u_p^L(t) + \sum_{q=1}^{n}(a_{pq}f_q'(x_q(t))u_q(t))^L \\ &\quad + \sum_{q=1}^{n}(b_{pq}f_q'(x_q(t-\tau_{pq}))e^{\varsigma\tau_{pq}}u_q(t-\tau_{pq}))^L\Big) \\ &\le (-d_p + \varsigma)|u_p^L(t)| + \sum_{q=1}^{n} \sum_{(l,w)\in M^L} |a_{pq}^l|\lambda_q^w|u_q^w(t)| \\ &\quad + \sum_{q=1}^{n} \sum_{(l,w)\in M^L} e^{\varsigma\tau_{pq}}|b_{pq}^l|\lambda_q^w|u_q^w(t-\tau_{pq})|,\end{aligned}$$

which means

$$\frac{d|u_p(t)|}{dt} \le (-d_p + \varsigma)|u_p(t)| + \sum_{q=1}^{n}\sum_{l\in M}\sum_{w\in M} |a_{pq}^l|\lambda_q^w|u_q(t)| + \sum_{q=1}^{n}\sum_{l\in M}\sum_{w\in M} e^{\varsigma\tau_{pq}}|b_{pq}^l|\lambda_q^w|u_q(t-\tau_{pq})|,$$

where $|u_p(t)| = \sum_{L\in M}|u_p^L(t)|$.

Let

$$V^L(t) = \sum_{p=1}^{n}\Big(\xi_p^L|u_p^L(t)| + \xi_p^L\sum_{q=1}^{n}\sum_{(l,w)\in M^L} e^{\varsigma\tau_{pq}}|b_{pq}^l| \times \lambda_q^w\int_{t-\tau_{pq}}^{t}|u_q(s)|ds\Big),$$

then, differentiating it, we have

$$\begin{aligned}\frac{dV^L(t)}{dt} &= \sum_{p=1}^{n}\Big(\xi_p^L\frac{d|u_p^L(t)|}{dt} + \xi_p^L\sum_{q=1}^{n}\sum_{(l,w)\in M^L} e^{\varsigma\tau_{pq}}|b_{pq}^l| \times \lambda_q^w\big(|u_q(t)| - |u_q(t-\tau_{pq})|\big)\Big), \\ &\le \sum_{p=1}^{n}\Big(\xi_p^L(-d_p+\varsigma)|u_p^L(t)| + \xi_p^L\sum_{q=1}^{n}\sum_{(l,w)\in M^L}|a_{pq}^l|\lambda_q^w|u_q(t)| \\ &\quad + \xi_p^L\sum_{q=1}^{n}\sum_{(l,w)\in M^L} e^{\varsigma\tau_{pq}}|b_{pq}^l|\lambda_q^w|u_q(t-\tau_{pq})| \\ &\quad + \xi_p^L\sum_{q=1}^{n}\sum_{(l,w)\in M^L} e^{\varsigma\tau_{pq}}|b_{pq}^l|\lambda_q^w\big(|u_q(t)| - |u_q(t-\tau_{pq})|\big)\Big) \\ &\le \sum_{p=1}^{n}\Big(\xi_p^L(-d_p+\varsigma)|u_p^L(t)| + \xi_p^L\sum_{q=1}^{n}\sum_{(l,w)\in M^L}|a_{pq}^l|\lambda_q^w|u_q(t)| \\ &\quad + \xi_p^L\sum_{q=1}^{n}\sum_{(l,w)\in M^L} e^{\varsigma\tau_{pq}}|b_{pq}^l|\lambda_q^w|u_q(t)|\Big) \\ &= \sum_{q=1}^{n}\xi_q^L(-d_q+\varsigma)|u_q^L(t)| \\ &\quad + \sum_{p=1}^{n}\sum_{q=1}^{n}\sum_{(l,w)\in M^L}\xi_p^L|a_{pq}^l|\lambda_q^w|u_q(t)| \\ &\quad + \sum_{p=1}^{n}\sum_{q=1}^{n}\sum_{(l,w)\in M^L} e^{\varsigma\tau_{pq}}\xi_p^L|b_{pq}^l|\lambda_q^w|u_q(t)| \\ &\le \sum_{q=1}^{n}\Big(\xi_q^L(-d_q+\varsigma) + \sum_{p=1}^{n}\sum_{(l,w)\in M^L}\xi_p^L|a_{pq}^l|\lambda_q^w \\ &\quad + \sum_{p=1}^{n}\sum_{(l,w)\in M^L} e^{\varsigma\tau_{pq}}\xi_p^L|b_{pq}^l|\lambda_q^w\Big)|u_q(t)| \\ &\le 0.\end{aligned}$$

Therefore, $V(t) = \sum_{L\in M} V^L(t)$ is bounded, and we have

$$\sum_{p=1}^{n} \xi_p|\dot{x}_p(t)| = \sum_{p=1}^{n}\sum_{L\in M} \xi_p^L|\dot{x}_p^L(t)| = O(e^{-\varsigma t}),$$

i.e. $\|\dot{x}(t)\|_{(\xi,1)} = O(e^{-\varsigma t})$. Consequently, for any $t_1, t_2 \in R$, $t_1 < t_2$, their exists a constant $C_1 > 0$, such that

$$\begin{aligned}\|x(t_2) - x(t_1)\|_{(\xi,1)} &= \int_{t_1}^{t_2} \|\dot{x}(t)\|_{(\xi,1)}dt \\ &\le \int_{t_1}^{t_2} C_1e^{-\varsigma t}dt \le \frac{C_1}{\varsigma}e^{-\varsigma t_1}.\end{aligned}$$

According to Cauchy's test for convergence, we can conclude that their exists an equilibrium point $x^* = (x_1^*, x_2^*, \cdots, x_n^*)^T$, such that $\|x(t) - x^*\|_{(\xi,1)} = O(e^{-\varsigma t})$.

Finally, we prove that system (1) with time-invariant delays $\tau_{pq}(t) = \tau_{pq}$ has an unique equilibrium point. Otherwise, suppose there are two equilibrium points $x^1(t)$ and $x^2(t)$, then by means of the same argument of $\dot{x}(t)$, it can be easily obtained that $\|x^1(t) - x^2(t)\|_{(\xi,1)} = O(e^{-\varsigma t})$, which means that the equilibrium point of QVDHNNs is unique and is globally exponentially stable. □

Theorem 2: Under assumption (H1), if there exist real constants $\varsigma > 0$, and $\xi_p^l > 0 (p = 1, 2, \cdots, n, l \in M)$, such that

$$(-d_p + \varsigma)\xi_p^L + \sum_{q=1}^{n}\sum_{(l,w)\in M^L} \xi_q^w(|a_{pq}^l| + e^{\varsigma\tau_{pq}}|b_{pq}^l|)\lambda_q^w \le 0$$

holds for every $L \in M$. Then, the dynamical system QVHNN (1) with time-invariant delays $\tau_{pq}(t) = \tau_{pq}$ has an unique equilibrium point x^*, which is globally exponentially stable.

Proof: According to QVHNN system (1) with time-invariant delays $\tau_{pq}(t) = \tau_{pq}$, we have

$$\begin{aligned}\frac{d\dot{x}_p(t)}{dt} &= -d_p\dot{x}_p(t) + \sum_{q=1}^{n} a_{pq}f_q'(x_q(t))\dot{x}_q(t) \\ &\quad + \sum_{q=1}^{n} b_{pq}f_q'(x_q(t-\tau_{pq}))\dot{x}_q(t-\tau_{pq}).\end{aligned}$$

Let $u(t) = e^{\varsigma t}\dot{x}(t)$, then we have

$$\begin{aligned}\frac{du_p(t)}{dt} &= (-d_p + \varsigma)u_p(t) + \sum_{q=1}^{n} a_{pq}f_q'(x_q(t))u_q(t) \\ &\quad + \sum_{q=1}^{n} b_{pq}f_q'(x_q(t-\tau_{pq}))e^{\varsigma\tau_{pq}}u_q(t-\tau_{pq}).\end{aligned}$$

Define

$$\begin{aligned}\|u(t)\|_{(\xi,\infty)} &= \max_{L\in M} \|u^L(t)\|_{(\xi,\infty)}, \\ W^L(t) &= \sup_{s\le t} \|u^L(s)\|_{(\xi,\infty)},\end{aligned}$$

if $W^L(t) = \|u^L(t)\|_{(\xi,\infty)}$ holds for some $t_0 \ge 0$ and $\|u^L(t_0)\|_{(\xi,\infty)} = |(\xi_{p_0^L}^L)^{-1}u_{p_0^L}^L(t_0)|$ holds for an index $p_0^L = p_0^L(t_0)$, then we have

$$\begin{aligned}&\xi_{p_0^L}^L \frac{d\|u^L(t)\|_{(\xi,\infty)}}{dt}\Big|_{t=t_0} \\ &= \frac{d|u_{p_0^L}^L(t)|}{dt}\Big|_{t=t_0} \\ &= \mathrm{sign}\{u_{p_0^L}^L(t_0)\}\Big\{\xi_{p_0^L}^L(-d_{p_0^L} + \varsigma)(\xi_{p_0^L}^L)^{-1}u_{p_0^L}^L(t_0) \\ &\quad + \sum_{q=1}^{n}\big(a_{p_0^Lq}f_q'(x_q(t))u_q(t_0)\big)^L \\ &\quad + \sum_{q=1}^{n}\big(b_{p_0^Lq}f_q'(x_q(t_0-\tau_{pq}))u_q(t_0-\tau_{pq})\big)^L e^{\varsigma\tau_{pq}}\Big\} \\ &\le \xi_{p_0^L}^L(-d_{p_0^L} + \varsigma)|(\xi_{p_0^L}^L)^{-1}u_{p_0^L}^L(t_0)| \\ &\quad + \sum_{q=1}^{n}\sum_{(l,w)\in M^L} \xi_q^w|a_{p_0^Lq}^l|\lambda_q^w\|u^w(t_0)\|_{(\xi,\infty)} \\ &\quad + \sum_{q=1}^{n} e^{\varsigma\tau_{p_0^Lq}}\sum_{(l,w)\in M^L} \xi_q^w|b_{p_0^Lq}^l|\lambda_q^w\|u^w(t_0-\tau_{p_0^Lq})\|_{(\xi,\infty)}.\end{aligned}$$

Let $W(t) = \max_{L\in M} W^L(t)$, we have

$$\begin{aligned}&\xi_{p_0^L}^L \frac{d\|u^L(t)\|_{(\xi,\infty)}}{dt}\Big|_{t=t_0} \\ &\le \Big\{\xi_{p_0^L}^L(-d_{p_0^L} + \varsigma) + \sum_{q=1}^{n}\sum_{(l,w)\in M^L} \xi_q^w|a_{p_0^Lq}^l|\lambda_q^w \\ &\quad + \sum_{q=1}^{n} e^{\varsigma\tau_{p_0^Lq}}\sum_{(l,w)\in M^L} \xi_q^w|b_{p_0^Lq}^l|\lambda_q^w\Big\}W(t_0) \\ &\le 0.\end{aligned}$$

Based on the above analysis, we can obtain $W(t)$ is bounded. And then, $\|u(t)\|_{(\xi,\infty)} = O(1)$ and $\|\dot{x}(t)\|_{(\xi,\infty)} = O(e^{-\varsigma t})$. As a result, for any $t_1, t_2 \in R$, $t_1 < t_2$, their exists a constant $C_2 > 0$, such that

$$\begin{aligned}\|x(t_2) - x(t_1)\|_{(\xi,\infty)} &= \int_{t_1}^{t_2} \|\dot{x}(t)\|_{(\xi,\infty)}dt \\ &\le \int_{t_1}^{t_2} C_2e^{-\varsigma t}dt \le \frac{C_2}{\varsigma}e^{-\varsigma t_1}.\end{aligned}$$

According to Cauchy's test for convergence and the analysis of Theorem 1, we can conclude that their exists an unique equilibrium point $x^* = (x_1^*, x_2^*, \cdots, x_n^*)^T$ for the system of QVHNN (2.1) with time-invariant delays $\tau_{pq}(t) = \tau_{pq}$, such that $\|x(t) - x^*\|_{(\xi,\infty)} = O(e^{-\varsigma t})$, which means that any solution $x(t)$ of QVDHNN (1) exponentially converges to its unique equilibrium point. □

From Theorem 1 and Theorem 2, we can easily derive the following two corollaries, respectively.

Corollary 1: Under assumption (H1), if there exist real constants $\xi_p^l > 0 (p = 1, 2, \cdots, n, l \in M)$, such that

$$-d_q\xi_q^L + \sum_{p=1}^{n}\sum_{(l,w)\in M^L}\xi_p^L(|a_{pq}^l| + |b_{pq}^l|)\lambda_q^w < 0$$

holds for every $L \in M$. Then, the dynamical system QVHNN (1) with time-invariant delays $\tau_{pq}(t) = \tau_{pq}$ has an unique equilibrium point x^*, which is globally exponentially stable.

Corollary 2: Under assumption (H1), if there exist real constants $\xi_p^l > 0 (p = 1, 2, \cdots, n, l \in M)$, such that

$$-d_p\xi_p^L + \sum_{q=1}^{n}\sum_{(l,w)\in M^L}\xi_q^w(|a_{pq}^l| + |b_{pq}^l|)\lambda_q^w < 0$$

holds for every $L \in M$. Then, the dynamical system QVHNN (1) with time-invariant delays $\tau_{pq}(t) = \tau_{pq}$ has an unique equilibrium point x^*, which is globally exponentially stable.

Remark 3: In this subsection, by the definitions of $\{\xi\}$-norms and Cauchy convergence principle, the existence and exponential stability criteria have been obtained for the equilibrium point of QVHNN (1) with time-invariant delays $\tau_{pq}(t) = \tau_{pq}$. When τ_{pq} changes with p, q, the time-delay is asynchronous time-delay, which is discussed in [35]. Obviously, they are also true for the same time-invariant delays $\tau_{pq}(t) = \tau$, where $\tau > 0$ is a real constant. When the time-varying delays $\tau_{pq}(t)$ is considered, its constraint conditions should be given because the derivative of $f_q(x_q(t - \tau_{pq}))$ will appear, which is discussed in the next subsection.

B. CRITERIA WITH TIME-VARYING DELAYS $\tau_{PQ}(T)$

In this subsection, time-varying delays $\tau_{pq}(t)$ will be considered in QVHNNs (1) to research the existence and exponential stability of solutions. Based on the investigation of the above subsection, if some conditions are given to constrained $\tau_{pq}(t)$, the similar results to Theorem 1 and Theorem 2 can be obtained as follows.

Theorem 3: Under assumptions (H1)and (H2), if there exist real constants $\xi_p^l > 0 (p = 1, 2, \cdots, n, l \in M)$, such that

$$(-d_q + \varsigma)\xi_q^L + \sum_{p=1}^{n}\sum_{(l,w)\in M^L}\xi_p^L(|a_{pq}^l| + \frac{e^{\varsigma\tau_{pq}}|b_{pq}^l|}{1-\eta_{pq}})\lambda_q^w \le 0$$

holds for every $L \in M$. Then, the dynamical systems QVHNN (1) with time-varying delays $\tau_{pq}(t)$ has an unique equilibrium point x^*, which is globally exponentially stable.

Proof: In the process of proving Theorem 1, if let

$$V^L(t) = \sum_{p=1}^{n}\Bigg(\xi_p^L|u_p^L(t)| + \xi_p^L\sum_{q=1}^{n}\sum_{(l,w)\in M^L} e^{\varsigma\tau_{pq}}|b_{pq}^l|\lambda_q^w\int_{t-\tau_{pq}(t)}^{t}\frac{|u_q(s)|}{1-\eta_{pq}}ds\Bigg),$$

then this theorem can be proved, the details are omitted. □

Theorem 4: Under assumptions (H1) and (H2), if there exist real constants $\xi_p^l > 0 (p = 1, 2, \cdots, n, l \in M)$, such that

$$(-d_p + \varsigma)\xi_p^L + \sum_{q=1}^{n}\sum_{(l,w)\in M^L}\xi_q^w(|a_{pq}^l| + e^{\varsigma\tau_{pq}}|b_{pq}^l|)\lambda_q^w \le 0$$

holds for every $L \in M$. Then, the dynamical systems QVHNN (1) with time-varying delays $\tau_{pq}(t)$ has an unique equilibrium point x^*, which is globally exponentially stable.

Proof: The details are omitted. □

Remark 4: For the proving of Theorem 3, the difficult is to construct a Lyapunov-type function. When this function is constructed, the process is easily. For Theorem 4, although there is no formal difference, assumption (H2) is indispensable. Actually, since the derivative of $e^{\varsigma t}f_q(x_q(t - \tau_{pq}(t))$ should be calculated in the process of Theorem 3 and Theorem 4, $\tau_{pq}(t) \le \tau_{pq}$ and $\tau'_{pq}(t) \le \eta_{pq} < 1$ are required to ensure boundedness of $e^{\varsigma\tau_{pq}(t)}$ and $1 - \tau'_{pq}(t) < 1$. Furthermore, $\tau_{pq}(t)$ is considered unbounded to study μ-stability of QVDNNs by $\{\xi, \infty\}$-norm in [32], [35]. However, $\tau_{pq}(t) \le \tau_{pq} < \tau$ are also essential when the exponential stability is studied, which means $\tau_{pq}(t)$ is bounded. Therefore, assumption (H2) is indispensable for the time-varying delays $\tau_{pq}(t)$ in this paper.

According to Corollaries 1, 2 and Theorems 3, 4, we can easily obtain another two corollaries for QVHNNs (1) with time-varying delays $\tau_{pq}(t)$ as follows.

Corollary 3: Under assumptions (H1) and (H2), if there exist real constants $\xi_p^l > 0 (p = 1, 2, \cdots, n, l \in M)$, such that

$$-d_q\xi_q^L + \sum_{p=1}^{n}\sum_{(l,w)\in M^L}\xi_p^L(|a_{pq}^l| + \frac{|b_{pq}^l|}{1-\eta_{pq}})\lambda_q^w < 0$$

holds for every $L \in M$. Then, the dynamical system QVHNN (1) with time-varying delays $\tau_{pq}(t)$ has an unique equilibrium point x^*, which is globally exponentially stable.

Corollary 4 [35]: Under assumptions (H1) and (H2), if there exist real constants $\xi_p^l > 0 (p = 1, 2, \cdots, n, l \in M)$, such that

$$-d_p\xi_p^L + \sum_{q=1}^{n}\sum_{(l,w)\in M^L}\xi_q^w(|a_{pq}^l| + |a_{pq}^l|)\lambda_q^w < 0$$

holds for every $L \in M$. Then, the dynamical system QVHNN (1) with time-varying delays $\tau_{pq}(t)$ has an unique equilibrium point x^*, which is globally exponentially stable.

Remark 5: In this section, time-invariant delays and time-varying delays have been considered successively in QVHNNs (1). Based on the definitions of $\{\xi, 1\}$-norms, the existence, uniqueness and exponential stability criteria of equilibrium point have be obtained, which is different from those results in [32], [35]–[39]. In [37]–[39], homeomorphism map was firstly used to get the existence and uniqueness criteria of QVDNNs via direct approach instead of decomposed method, then exponential stable LMI conditions are obtained by constructing Lyapunov-Krasovskii functionals. Furthermore, μ-stable criteria was considered only by $\{\xi, \infty\}$-norm in [32], [35]. In this paper, by inequality technique and Cauchy's test for convergence, $\{\xi, \infty\}$-norm and $\{\xi, 1\}$-norm are considered respectively to obtain the ξ-exponential form sufficient conditions of existence, uniqueness and exponential stability for QVDHNNs.

IV. NUMERICAL EXAMPLE

In this section, two numerical examples and their simulated images will be given to illustrate the effectiveness of the obtained theoretical results.

Example 1: Consider the following two-dimensional QVHNNs with time-invariant delays.

$$\dot{x}_p(t) = -d_p x_p(t) + \sum_{q=1}^{2} a_{pq} f_q(x_q(t)) + \sum_{q=1}^{2} b_{pq} f_q(x_q(t-\tau_{pq})) + u_p, \quad (5)$$

where $x_p(t) = x_p^R(t) + \imath x_p^I(t) + \jmath x_p^J(t) + \kappa x_p^K(t) \in \mathbb{Q}$, $f_q(x_q(t)) = \tanh(x_q^R(t)) + \imath \tanh(x_q^I(t)) + \jmath \tanh(x_q^J(t)) + \kappa \tanh(x_q^K(t))$, $\tau_{pq} = 0.6$ hold for $p, q = 1, 2$, and $d_1 = 8$, $d_2 = 8$, $a_{11} = 0.6 - 0.2\imath - 0.6\jmath + 0.3\kappa$, $a_{12} = -0.2 + 0.6\imath + 0.4\jmath - 0.5\kappa$, $a_{21} = -0.4 + 0.4\imath + 0.1\jmath - 0.7\kappa$, $a_{22} = 0.3 - 0.3\imath - 0.5\jmath + 0.3\kappa$, $b_{11} = 0.3 - 0.5\imath + 0.4\jmath - 0.5\kappa$, $b_{12} = -0.5 + 0.3\imath - 0.5\jmath + 0.3\kappa$, $b_{21} = -0.6 + 0.4\imath - 0.4\jmath + 0.4\kappa$, $b_{22} = 0.2 - 0.5\imath + 0.4\jmath - 0.4\kappa$, $u_1 = -1 + \imath + \jmath + 2\kappa$, $u_2 = 1 - 2\imath + 3\jmath - 2\kappa$.

Let $\xi_p^L = 0.1$ and $\varsigma = 0.3$, it can be calculated that $(-d_p + \varsigma)\xi_p^L + \sum_{q=1}^{2} \sum_{(l,w)\in M^L} \xi_q^w (|a_{pq}^l| + e^{\varsigma \tau_{pq}} |b_{pq}^l|)\lambda_q^w \leq 0$ holds for any $L \in M$ and $p = 1, 2$. Therefore, the conditions of Theorem 2 are satisfied, and the dynamical QVHNN systems (5) with time-invariant delays has an unique global exponential stable equilibrium point, which can be shown by Figures 1, 2 3 and 4.

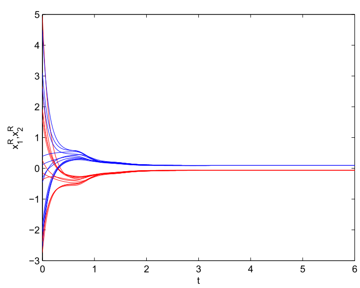

FIGURE 1. State trajectories of $x_1^R(t)$ and $x_2^R(t)$ for Example 1.

Example 2: Consider the following two-dimensional QVHNNs with time-varying delays.

$$\dot{x}_p(t) = -d_p x_p(t) + \sum_{q=1}^{2} a_{pq} f_q(x_q(t)) + \sum_{q=1}^{2} b_{pq} f_q(x_q(t-\tau_{pq}(t))) + u_p, \quad (6)$$

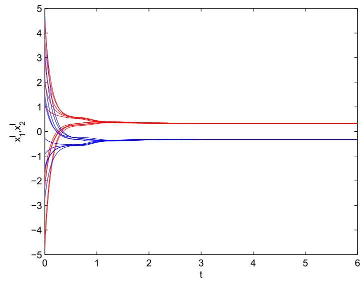

FIGURE 2. State trajectories of $x_1^I(t)$ and $x_2^I(t)$ for Example 1.

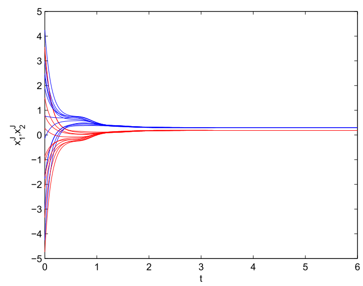

FIGURE 3. State trajectories of $x_1^J(t)$ and $x_2^J(t)$ for Example 1.

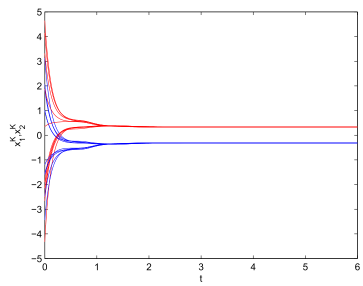

FIGURE 4. State trajectories of $x_1^K(t)$ and $x_2^K(t)$ for Example 1.

where $x_p(t) = x_p^R(t) + \imath x_p^I(t) + \jmath x_p^J(t) + \kappa x_p^K(t) \in \mathbb{Q}$, $f_q(x_q(t)) = \tanh(x_q^R(t)) + \imath \tanh(x_q^I(t)) + \jmath \tanh(x_q^J(t)) + \kappa \tanh(x_q^K(t))$, $\tau_{pq}(t) = 2 + \frac{1}{\pi}\sin(\frac{\pi}{2}t)$ hold for $p, q = 1, 2$, and $d_1 = 12$, $d_2 = 12$, $a_{11} = -0.6 + 0.2\imath - 0.6\jmath + 0.3\kappa$, $a_{12} = 0.2 - 0.6\imath + 0.4\jmath - 0.5\kappa$, $a_{21} = 0.4 - 0.4\imath + 0.1\jmath - 0.7\kappa$, $a_{22} = -0.3 + 0.3\imath - 0.5\jmath + 0.3\kappa$, $b_{11} = -0.3 + 0.5\imath - 0.4\jmath + 0.5\kappa$, $b_{12} = 0.5 - 0.3\imath + 0.5\jmath - 0.3\kappa$, $b_{21} = 0.6 - 0.4\imath + 0.4\jmath - 0.4\kappa$, $b_{22} = -0.2 + 0.5\imath - 0.4\jmath + 0.4\kappa$, $u_1 = u_2 = 0$.

Since $\tau_{pq}(t) = 2 + \frac{1}{\pi}\sin(\frac{\pi}{2}t)$, it can be obtained that $|\tau_{pq}(t)| \leq 2 + \frac{1}{\pi}$ and $|\dot{\tau}_{pq}(t)| = |\frac{1}{2}\cos(\frac{\pi}{2}t)| \leq \frac{1}{2}$ and assumption (H2) is satisfied. Let $\xi_p^L = 0.1$, $\varsigma = 0.06$, and $\eta_{pq} = 0.5$, it can be calculated that $(-d_q + \varsigma)\xi_q^L + \sum_{p=1}^{2} \sum_{(l,w)\in M^L} \xi_p^L (|a_{pq}^l| + \frac{e^{\varsigma \tau_{pq}}|b_{pq}^l|}{1-\eta_{pq}})\lambda_q^w \leq 0$ holds for any $L \in M$ and $p = 1, 2$. Therefore, the conditions of Theorem 3 are satisfied, and the dynamical QVHNN systems (6) with time-varying delays has an unique global exponential stable equilibrium point, which can be shown by Figures 5, 6, 7 and 8.

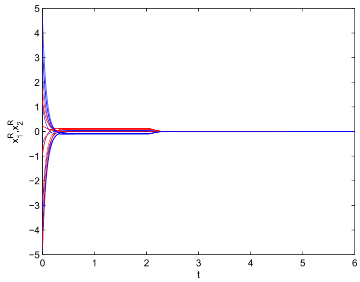

FIGURE 5. State trajectories of $x_1^R(t)$ and $x_2^R(t)$ for Example 2.

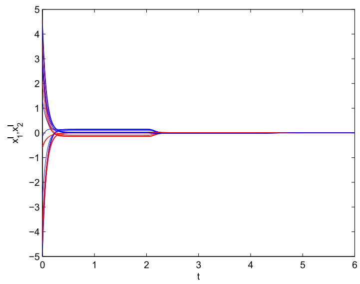

FIGURE 6. State trajectories of $x_1^I(t)$ and $x_2^I(t)$ for Example 2.

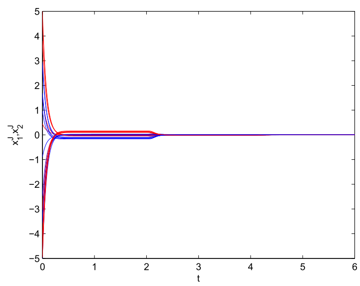

FIGURE 7. State trajectories of $x_1^J(t)$ and $x_2^J(t)$ for Example 2.

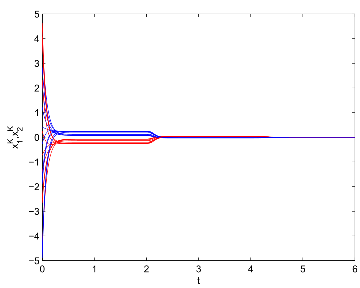

FIGURE 8. State trajectories of $x_1^K(t)$ and $x_2^K(t)$ for Example 2.

Remark 6: In the first example, we suppose that the delays of system are time-invariant delays and external input is $u \neq 0$. By calculation, we can obtain that the conditions of Theorem 2 are satisfied and the conclusions can be obtained, which can be illustrated by Figures 1–4. Similarly, we can also verify the validity of Theorem 4. In the second example, we suppose that the delays of system are time-varying delays and external input is $u = 0$. By calculation, we can obtain that the conditions of Theorem 3 are satisfied and the conclusions can be obtained, which can be illustrated by Figures 5–8. The validity of Theorem 1 can also be verified by the similar way.

V. CONCLUSION

For various NNs systems, the existence, uniqueness and exponential stability of their solutions are the evergreen topics in the past decades. Recently, with more and more extensive applications of quaternion, QVNNs have been presented and studied by many scholars. Especially, the investigation of existence, uniqueness and exponential stability criteria for QVNNs is an important content. In this paper, by utilizing inequality technique and Cauchy's test for convergence, ξ-exponential type sufficient conditions of existence, uniqueness and exponential stability for the QVDHNNs have been obtained by the definitions of ξ-norms. The QVDHNNs system has been firstly decomposed into four real-number systems according to Hamilton rules, which avoids the noncommutativity of quaternion multiplication. Then, taking advantage of ξ-norms, time-invariant delays and time-varying delays have been considered successively to derive ξ-exponential type sufficient conditions. Particularly, in order to obtain the exponential stable criteria of QVHNN with time-varying delays, $e^{\varsigma t}$ are introduced to deal with time-varying delay term, where assumption (H2) is necessary. In addition, several corollaries are derived on the basis of the main results. It is worth noting that ξ-norms can be used to deal with the synchronization and various control problems of many kinds NNs in the future. Finally, two numerical examples with time-invariant delays and time-varying delays and their simulated images have been given to illustrate the theoretical results of this paper, respectively.

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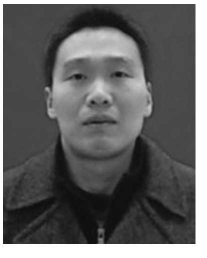

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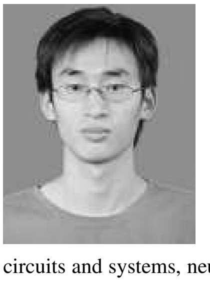

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