Refinement of Optimal Interpolation Factor for DFT Interpolated Frequency Estimator

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***Abstract*—Frequency estimation is a fundamental problem in many areas. The previously proposed q-shift estimator (QSE), which interpolates the discrete Fourier transform (DFT) coefficients by a factor of q, enables the estimation accuracy to approach the Cramér-Rao lower bound (CRLB). However, it becomes less effective when the number of samples is small. In this letter, we provide an in-depth analysis to unveil the impact of q on the convergence of QSE, and derive the bounds of a refined region of q that ensures the convergence of QSE to the CRLB even with a small number of samples. Simulations validate our analysis, showing that the refined interpolation factor is able to reduce the estimation mean squared error of QSE by up to 13.14 dB when the sample number is as small as 8.**

***Index Terms*—Frequency estimation; discrete Fourier transform (DFT) coefficients; interpolation factor.**

I. Introduction

Frequency estimation of a single-tone complex exponential signal is a fundamental research issue in many areas, including radar/medical signal processing, wireless mobile/satellite communications, power grid stability, etc. [1]–[4]. Offering a low complexity and high efficiency, frequency estimation exploiting the discrete Fourier transform (DFT) coefficients has attracted extensive attention [4]–[8]. DFT coefficients used to be directly applied for frequency estimation without interpolation, which, however, causes uneven estimation bias for different frequencies [4]. In [5], the authors introduced an iterative DFT-interpolated frequency estimator, known as A&M, which interpolates the DFT coefficients by a factor two. A&M was further extended to improve estimation bias [6], to process real sinusoidal signal [7], and to facilitate the estimation of multi-tone resolvable exponential signals [8]. Despite the improved accuracy compared to the earlier works without interpolation, the asymptotic variance of A&M (and its derivatives) is limited to 1.0147 times of the Cramér-Rao lower bound (CRLB) [5].

In [1], a q-shift estimator (QSE) was proposed by increasing the interpolation factor to $\frac{1}{q}$ (> 2) with $|q| < 0.5$, which for the first time enables QSE to asymptotically approach CRLB. Therefore, QSE is able to achieve the lowest estimation variance and outperforms, in terms of estimation accuracy, the earlier works [4]–[8]. It is noteworthy that the accuracy improvement of QSE [1] over earlier works benefits from taking q in $(-0.5, 0.5)$ (c.f. $q = 0.5$ in [4]–[8]). However, QSE only approaches CRLB when the sample number is large and it fails to converge when the sample number is small, e.g., 16 [1, Fig. 2].

This letter is motivated to uncover the reason underlying the divergence of QSE from the CRLB and proposes a simple, effective solution to substantially improve the performance of QSE, particularly when the sample number is small. A key contribution is that we carry out a comprehensive analysis to unveil the impact of q on the convergence of QSE, which was not captured in [1]. Another contribution is that we derive the bounds of a refined region of q and provide a simple refinement technique to the optimal q (specified in [1]), which ensures the convergence of QSE to the CRLB even in the presence of only small numbers of available samples.

Validated by simulation results, the new optimal q can reduce the estimation mean squared error (MSE) of QSE substantially. The MSE is reduced by up to 13.14 dB with only 8 samples (even smaller than 16 in [1]). The rest of this letter is organized as follows. Section II presents the signal model and briefly reviews QSE. Section III investigates the impact of q on the convergence of QSE, followed by simulations and conclusions in Sections IV and V, respectively.

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II. Signal Model and QSE

With reference to [1], the frequency f of the complex single-tone exponential signal $s(n) = Ae^{j\left(\frac{2\pi f n}{f_s}+\phi\right)} + z(n)$ ($n = 0, 1, \cdots, N-1$) is to be estimated, where A is the signal amplitude, f_s is the sampling rate, ϕ is the initial phase, N is the sample number and $z(n)$ is additive white Gaussian noise (AWGN) with the noise variance σ^2. Here, f can be expressed as $f = \frac{k^\star+\delta}{N} f_s,\ \delta \in [-0.5, 0.5]$, with $k^\star$ and δ denoting the closest frequency bin to f and frequency residual, respectively. It is assumed in [1] that $k^\star$ can be accurately identified via a maximum likelihood estimator, and hence the work [1] only focuses on δ estimation.

Given $k^\star$, the iteration number Q and the initial residual estimate $\hat{\delta}_0 = 0$, QSE updates the i-th ($i = 1, \ldots, Q$) frequency residual by [1, eq. 7]

$$\hat{\delta}_i = \frac{1}{c(q)} \times \mathcal{R}e\left\{\beta_i\right\} + \hat{\delta}_{i-1},\ |q| < 0.5, \tag{1}$$

where $c(q) = \frac{1-\pi q\cot(\pi q)}{q\cos^2(\pi q)}$; $\beta_i = \frac{S_{+q}-S_{-q}}{S_{+q}+S_{-q}}$; and $S_{\pm q}$ is the $\frac{1}{\pm q}$-interpolated DFT coefficients around $k^\star$, as give by $S_{\pm q} =$

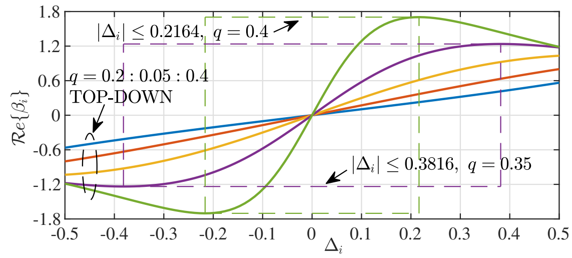

Fig. 1. Illustration of the monotonicity of $\mathcal{R}e\{\beta_i\}$ w.r.t. Δ_i, affected by q.

$\sum_{n=0}^{N-1} s(n)e^{-j\frac{2\pi(k^\star+\hat{\delta}_{i-1}\pm q)n}{N}}$. Eq.(1) is obtained based on the following linear approximation [1, eq. 19]

$$\mathcal{R}e\{\beta_i\} = c(q)\Delta_i + \mathcal{O}(\Delta_i^3),\ |q| < 0.5, \tag{2}$$

where $\Delta_i = \hat{\delta}_i - \delta$ is the estimation error in the i-th iteration. After Q iterative updates in (1), QSE produces the final frequency estimation as $\hat{f} = \frac{k^\star + \hat{\delta}_Q}{N} f_s$.

III. Impact and Refinement of q

This section analyzes the impact of q on the convergence of QSE. We start by presenting the motivation of this analysis. Then, the refined q region is derived, also leading to the refinement of the optimal q.

A. Impact of q on the Convergence of QSE

We find that the linear approximation (2) can be invalidated by some values of q. To illustrate this, Fig. 1 plots $\mathcal{R}e\{\beta_i\}$ w.r.t. $\Delta_i \in [-0.5, 0.5]$ by taking different values of q. We see that not only the linearity of $\mathcal{R}e\{\beta_i\}$ w.r.t. Δ_i degrades as q increases, but the monotonicity of $\mathcal{R}e\{\beta_i\}$ over Δ_i can also change twice for some q values. Obviously, the invalidation of (2) fails the core step of QSE, as given in (1).

Without considering the impact of q, the optimal q specified in [1, eq. 39] can make the frequency residual δ fall out of the monotonic region of $\mathcal{R}e\{\beta_i\}$ w.r.t. Δ_i, leading to the divergence of QSE from the CRLB. Take the divergence of QSE in [1, Fig. 2] for an example, where $N = 16$, $\delta = 0.25$ and the optimal $q = \frac{1}{\sqrt[3]{N}} \approx 0.4$ was taken. According to Fig. 1, $q = 0.4$ makes $\delta = 0.25$ fall out of the monotonic region of $\mathcal{R}e\{\beta_i\}$ w.r.t. Δ_i which is $|\Delta_i| \leq 0.2164$.

It is noteworthy that, although [1] states that the optimum value of q is the smallest possible value of $|q| > 0$, QSE was established based on the linear approximation in (2). Moreover, the optimal q suggested in [1, eq. (39)] can be larger than the threshold when N is small (e.g., 16 in [1, Fig. 2], which can make QSE diverge from the CRLB. Therefore, it is important to derive the upper bound of $|q|$, which specifies a condition for the convergence of QSE.

B. Refinement of q

As revealed in Section III-A, q is vital to preserving the monotonicity of $\mathcal{R}e\{\beta_i\}$ and hence the validity of the core step of QSE, i.e., (1). Therefore, we refine q through analyzing the monotonicity of $\mathcal{R}e\{\beta_i\}$ w.r.t. $\Delta_i (\in [-0.5, 0.5])$. However, it is mathematically intractable to carry out the analysis based on β_i in (1) due to the strong coupling of linear and complex exponential functions of Δ_i. In this section, we first reformulate β_i by collecting the common terms of Δ_i and q into an auxiliary function, denoted by $\mathcal{Q}(\Delta_i, q)$. Then, we apply the chain rule of nested functions [9] to separately investigate the monotonicity of $\mathcal{R}e\{\beta_i\}$ w.r.t. $\mathcal{Q}(\Delta_i, q)$ and $\mathcal{Q}(\Delta_i, q)$ w.r.t. Δ_i. Finally, the individual monotonicity is synthesized to achieve the overall monotonicity of $\mathcal{R}e\{\beta_i\}$ w.r.t. Δ_i, through which the q region of interest is specified.

β_i in (1) can be reformulated as $\beta_i = \frac{1-\mathcal{Q}(\Delta_i,q)e^{j2\pi q}}{1+\mathcal{Q}(\Delta_i,q)e^{j2\pi q}}$, where $\mathcal{Q}(\Delta_i, q)$ is given by

$$\mathcal{Q}(\Delta_i, q) = \frac{\Delta_i - q}{\Delta_i + q} \times \frac{\sin[\pi(\Delta_i + q)]}{\sin[\pi(\Delta_i - q)]}. \tag{3}$$

Taking the real part of β_i, we have $\beta_i^{\mathcal{R}} = \mathcal{R}e\{\beta_i\} = \frac{1-\mathcal{Q}^2(\Delta_i,q)}{1+2\mathcal{Q}(\Delta_i,q)\cos(2\pi q)+\mathcal{Q}^2(\Delta_i,q)}$, which is much simplified compared to [1, eq. 18]. By applying the Chain Rule of nested functions [9], the first derivative of $\beta_i^{\mathcal{R}}$ w.r.t. Δ_i is given by

$$(\beta_i^{\mathcal{R}})'_{\Delta_i} = (\beta_i^{\mathcal{R}})'_{\mathcal{Q}} \times (\mathcal{Q})'_{\Delta_i}, \tag{4}$$

where $(\beta_i^{\mathcal{R}})'_{\mathcal{Q}}$ is the first derivative of $\beta_i^{\mathcal{R}}$ w.r.t. $\mathcal{Q}(\Delta_i, q)$, and $(\mathcal{Q})'_{\Delta_i}$ is the first derivative of $\mathcal{Q}(\Delta_i, q)$ w.r.t. Δ_i. In the following, we examine the signs of $(\mathcal{Q})'_{\Delta_i}$ and $(\beta_i^{\mathcal{R}})'_{\mathcal{Q}}$ in Sections III-C and III-D, respectively. The refinement of q is accordingly achieved in Section III-D.

C. Evaluation of the Sign of $(\mathcal{Q})'_{\Delta_i}$

According to (3), $(\mathcal{Q})'_{\Delta_i}$ is expressed in (5), where the denominator is always non-negative. Therefore, the sign of its numerator, as given by $\mathcal{P}(\Delta_i, q) = 2q\sin[\pi(\Delta_i + q)]\sin[\pi(\Delta_i - q)] - \pi(\Delta_i^2 - q^2)\sin(2\pi q)$, determines the sign of $(\mathcal{Q})'_{\Delta_i}$, and hence is analyzed below.

$$(\mathcal{Q})'_{\Delta_i} = \mathcal{P}(\Delta_i, q)/(\Delta_i + q)^2 \sin^2[\pi(\Delta_i - q)] \tag{5}$$

Let $(\mathcal{P})'_{\Delta_i}$ and $(\mathcal{P})''_{\Delta_i}$ denote the first and second derivatives of $\mathcal{P}(\Delta_i, q)$ w.r.t. Δ_i, respectively. Using (5), we have

$$(\mathcal{P})'_{\Delta_i} = 2\pi q \sin(2\pi\Delta_i) - 2\pi\Delta_i \sin(2\pi q); \tag{6}$$

$$(\mathcal{P})''_{\Delta_i} = 4\pi^2 q \cos(2\pi\Delta_i) - 2\pi \sin(2\pi q). \tag{7}$$

By setting $(\mathcal{P})'_{\Delta_i} = 0$, we obtain $\Delta_i = \pm q$ at the local optima of $\mathcal{P}(\Delta_i, q)$. By taking $\Delta_i = \pm q$ in (7), $(\mathcal{P})''_{\pm q} = 4\pi^2 q \left[\cos(2\pi q) - \frac{\sin(2\pi q)}{2\pi q}\right]$, where $\cos(2\pi q) - \frac{\sin(2\pi q)}{2\pi q} < 0$ holds for $\forall |q| < 0.5$. Therefore, $(\mathcal{P})''_{\pm q}$ satisfies

$$(\mathcal{P})''_{\pm q} \begin{cases} < 0, & \text{if } q > 0 \\ > 0, & \text{if } q < 0 \end{cases}. \tag{8}$$

Eq. (8) indicates that both local optima $\Delta_i = \pm q$ of $\mathcal{P}(\Delta_i, q)$ are local minima when $q < 0$ or local maxima when $q > 0$. By taking $\Delta_i = \pm q$ in $\mathcal{P}(\Delta_i, q)$, we obtain $\mathcal{P}(\pm q, q) = 0$. This indicates that $\mathcal{P}(\Delta_i, q) \geq 0$ when $q < 0$ and $\mathcal{P}(\Delta_i, q) \leq 0$ when $q > 0$. However, $\mathcal{P}(\pm q, q) = 0$ does not guarantee $(\mathcal{Q})'_{\Delta_i} = 0$ due to the singular denominators of $(\mathcal{Q})'_{\Delta_i}$ at $\Delta_i = \pm q$; see (5). By taking $\Delta_i = \pm q$ in (5), we achieve

$$\lim_{\Delta_i \to \pm q} (\mathcal{Q})'_{\Delta_i} = \begin{cases} \frac{(\mathcal{P})''_q}{8\pi^2 q^2} & \text{when } \Delta_i \to q \\ \frac{(\mathcal{P})''_q}{2\sin^2[\pi(\Delta_i - q)]} & \text{when } \Delta_i \to -q \end{cases}. \tag{9}$$

Combining (8) and (9), we can conclude that the local minima of $(\mathcal{Q})'_{\Delta_i}$ are always positive, when $q < 0$; and the local maxima of $(\mathcal{Q})'_{\Delta_i}$ are always negative, when $q > 0$. This finally leads to

$$(\mathcal{Q})'_{\Delta_i} \begin{cases} < 0, & \text{if } \ q > 0 \\ > 0, & \text{if } \ q < 0 \end{cases}, \ \forall |\Delta_i| \le 0.5. \tag{10}$$

D. Evaluation of the Sign of $(\beta_i^{\mathcal{R}})'_{\Delta_i}$ and Refinement of q

We proceed to investigate the sign of $(\beta_i^{\mathcal{R}})'_{\mathcal{Q}}$ which, based on $\beta_i^{\mathcal{R}}$, is given in (11). Since the denominator of $(\beta_i^{\mathcal{R}})'_{\mathcal{Q}}$ is non-negative, we focus on the sign of its numerator function, as given by $\mathcal{N}(\mathcal{Q}) = -[2\mathcal{Q}^2(\Delta_i, q)+2]\cos(2\pi q)-4\mathcal{Q}(\Delta_i, q)$. It is noted that $\mathcal{N}(\mathcal{Q})$ is a quadratic function of $\mathcal{Q}(\Delta_i, q)$, and hence the sign of $\mathcal{N}(\mathcal{Q})$ only changes if the solutions to $\mathcal{N}(\mathcal{Q}) = 0$ are valid. According to (11), the solutions to $\mathcal{N}(\mathcal{Q}) = 0$ are given in (12). In the following, we examine the validity of (12) to analyze the sign of $\mathcal{N}(\mathcal{Q})$.

$$(\beta_i^{\mathcal{R}})'_{\mathcal{Q}} = \frac{\mathcal{N}(\mathcal{Q})}{[1 + 2\mathcal{Q}(\Delta_i, q)\cos(2\pi q) + \mathcal{Q}^2(\Delta_i, q)]^2} \tag{11}$$

$$\mathcal{Q}^*_{\pm} = (-1 \pm |\sin(2\pi q)|)/\cos(2\pi q) \tag{12}$$

Case 1*: In this case of* $|q| < 0.25$, $\cos(2\pi q) > 0$; and $(-1 \pm |\sin(2\pi q)|) < 0$ due to $|\sin(2\pi q)| < 1$. Thus, we have

$$\mathcal{Q}^*_{\pm} < 0, \ \forall |q| < 0.25. \tag{13}$$

However, (3) indicates that

$$\mathcal{Q}(\Delta_i, q) > 0, \ \forall |\Delta_i| \le 0.5, \ \forall |q| < 0.5, \tag{14}$$

since $\frac{\sin[\pi(\Delta_i \pm q)]}{\Delta_i \pm q} > 0$ given $|\Delta_i \pm q| < 1$. Clearly, (13) contradicts with (14), and hence $\mathcal{Q}^*_{\pm}$ invalid $\forall |q| < 0.25$.

Case 2*: In the case of* $|q| = 0.25$, $\cos(2\pi q) = 0$, and hence $\lim_{|q| \to 0.25} \mathcal{Q}^*_{-} \to -\infty$, contradicting with (14). Similarly, substituting $|q| = 0.25$ in $\mathcal{Q}^*_{+}$ leads to

$$\lim_{\substack{q \to \\ \pm 0.25}} \mathcal{Q}^*_{+} = \lim_{\substack{q \to \\ \pm 0.25}} \frac{-1 + \sin(2\pi q)}{\cos(2\pi q)} \overset{(a)}{=} \lim_{\substack{q \to \\ \pm 0.25}} \frac{2\pi\cos(2\pi q)}{-2\pi\sin(2\pi q)} = 0, \tag{15}$$

where the *L'Höpital's* rule [9] is applied to achieve the equality (a). However, by substituting $q = 0.25$ in (3), we notice that $Q = 0$ does not happen[1]. Therefore, we can conclude that both $\mathcal{Q}^*_{\pm}$ are invalid solutions in the case of $q = 0.25$. By substituting $|q| = 0.25$ into (11), we have $\mathcal{N}(\mathcal{Q}) = -4\mathcal{Q}(\Delta_i, q) < 0$ based on (14). Accordingly, by substituting (4) into (10), we can obtain

$$(\beta_i^{\mathcal{R}})'_{\Delta_i} \begin{cases} \ge 0, & q = 0.25 \\ \le 0, & q = -0.25 \end{cases}, \ \forall |\Delta_i| \le 0.5. \tag{16}$$

Case 3*: In the case of* $0.25 < |q| < 0.5$, $\cos(2\pi q) < 0$, and hence $\mathcal{Q}^*_{\pm} > 0$. To further examine the validity of $\mathcal{Q}^*_{\pm} > 0$, we check whether the solutions to (17), i.e., $\Delta^*_{i\pm}$, are valid by falling in $[-0.5, 0.5]$ in the following.

$$\mathcal{Q}(\Delta^*_{i\pm}, q) = \mathcal{Q}^*_{\pm} \tag{17}$$

[1]According to (3), $Q = 0$ can only happen when $\Delta_i \pm q = 1$; however, we have $|\Delta_i \pm q| < 1$ given $|\Delta_i| \le 0.5$ and $|q| < 0.5$.

By studying the monotonicity of $\mathcal{Q}(\Delta^*_{i\pm}, q)$ w.r.t. q, the following can be obtained; see Appendix A for details,

$$q \uparrow (\downarrow) \implies \begin{cases} \mathcal{Q}^*_{+} \uparrow (\downarrow) \implies \Delta^*_{i+} \downarrow (\uparrow) \\ \mathcal{Q}^*_{-} \downarrow (\uparrow) \implies \Delta^*_{i-} \uparrow (\downarrow) \end{cases}, \tag{18}$$

where "↑", "↓" and " $\implies$ " denote "increasing", "decreasing" and "leading to", respectively.

From (18), we assert that *there exists a* q^*_{L} *such that* $\Delta^*_i = 0.5$ *if* $|q| = q^*_{\mathrm{L}}$; otherwise, $\Delta^*_i > 0.5$ for $|q| < q^*_{\mathrm{L}}$, and $\Delta^*_i < 0.5$ for $|q| > q^*_{\mathrm{L}}$, where $\Delta^*_i = |\Delta^*_{i+}| = |\Delta^*_{i-}|$, as proved in Appendix A. To this end, when $|q| \le q^*_{\mathrm{L}}$, (17) has feasible solutions, and hence $\mathcal{Q}^*_{\pm}$ is valid, which, according to (11), lead to

$$\mathcal{N}(\mathcal{Q}) \begin{cases} \underset{(>)}{<} 0, & \text{if } \ |\Delta_i| \underset{(>)}{<} \Delta^*_i \\ = 0, & \text{if } \ |\Delta_i| = \Delta^*_i \end{cases}, \ |q| \le q^*_{\mathrm{L}}. \tag{19}$$

On the contrary, when $|q| > q^*_{\mathrm{L}}$, (17) does not have feasible solutions, hence invalidating $\mathcal{Q}^*_{\pm}$. By examining the coefficient of $\mathcal{Q}^2(\Delta_i, q)$ in $\mathcal{N}(Q)$; see (11), we obtain $-2\cos(2\pi q) > 0$ for $|q| > q^*_{\mathrm{L}}$ and, in turn,

$$\mathcal{N}(\mathcal{Q}) > 0, \ \text{for } |\Delta_i| \le 0.5, \ |q| > q^*_{\mathrm{L}}. \tag{20}$$

Finally, by substituting (10), (19) and (20) into (4), the monotonicity of $\beta_i^{\mathcal{R}}$ w.r.t. Δ_i can be revealed as follows

$$(\beta_i^{\mathcal{R}})'_{\Delta_i} \begin{cases} \ge 0, & \text{for } |\Delta_i| \le 0.5, \ 0.25 < q \le q^*_{\mathrm{L}} \approx 0.32 \\ \le 0, & \text{for } |\Delta_i| \le 0.5, \ -q^*_{\mathrm{L}} \le q < -0.25 \end{cases}; \tag{21a}$$

$$(\beta_i^{\mathcal{R}})'_{\Delta_i} \begin{cases} < 0, & \text{for } |\Delta_i| \in (\Delta^*_i, 0.5] \\ \ge 0, & \text{for } |\Delta_i| \le \Delta^*_i \end{cases}, \ q \in (q^*_{\mathrm{L}}, 0.5); \tag{21b}$$

$$(\beta_i^{\mathcal{R}})'_{\Delta_i} \begin{cases} > 0, & \text{for } |\Delta_i| \in (\Delta^*_i, 0.5] \\ \le 0, & \text{for } |\Delta_i| \le \Delta^*_i \end{cases}, \ q \in (-0.5, -q^*_{\mathrm{L}}), \tag{21c}$$

where the critical point $q^*_{\mathrm{L}} \approx 0.32$ is derived in Appendix B.

From *Cases 1∼3*, we conclude that QSE only works for a refined region of q, as given by (c.f., $|q| < 0.5$ in [1])

$$|q| \le q^*_{\mathrm{L}} \approx 0.32. \tag{22}$$

As illustrated in Section III-A, the optimal q specified in [1], denoted by $q^{\mathrm{o}}_{\mathrm{opt}}$, can invalidate QSE. Taking into account the unveiled impact of q, we propose a simple way to refine the optimal q, as given by

$$q_{\mathrm{opt}} = \min\left\{q^{\mathrm{o}}_{\mathrm{opt}}, q^*_{\mathrm{L}}\right\}. \tag{23}$$

The asymptotic convergence of QSE towards the CRLB, resulting from the new optimal q in (23), is analyzed as follows. As evident from (23), $q_{\mathrm{opt}} \le q^{\mathrm{o}}_{\mathrm{opt}}$ and therefore $\mathrm{var}(\hat{\delta}_Q)\big|_{q=q_{\mathrm{opt}}} \le \mathrm{var}(\hat{\delta}_Q)\big|_{q=q^{\mathrm{o}}_{\mathrm{opt}}}$. Here, $\mathrm{var}(\hat{\delta}_Q) = \frac{6}{2\pi^2 N\gamma} + \mathcal{O}(q^4) + o(N^{-1})$ is the estimation variance of QSE after Q iterations [1, eq. (38)]. $\gamma = \frac{A^2}{\sigma^2}$ is the received SNR. Since the convergence of $\mathrm{var}\left(\hat{\delta}_Q\right)\big|_{q=q^{\mathrm{o}}_{\mathrm{opt}}}$ has been confirmed by [1, Thrm. 3], we conclude that $\mathrm{var}(\hat{\delta}_Q)\big|_{q=q_{\mathrm{opt}}}$ asymptotically converges towards the CRLB.

The advantages of QSEr over the original QSE [1] are summarized as follows. QSEr refers to the refined QSE using the new optimal q given in (23). *First*, QSEr is able to converge

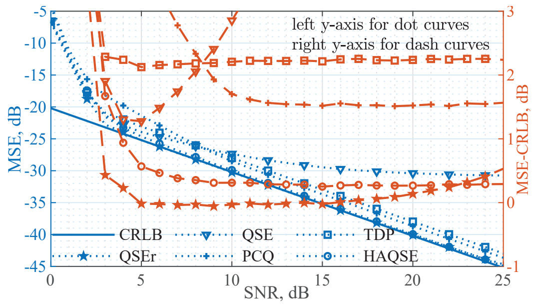

Fig. 2. MSE of frequency estimates vs SNR, where QSE and HAQSE [1], PCQ [6] and TDP [11] are simulated as benchmarks.

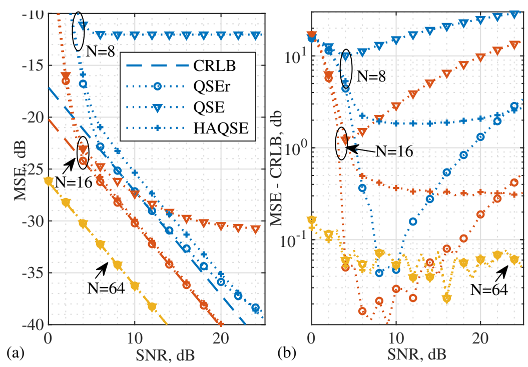

Fig. 3. MSE of frequency estimates against SNR.

to the CRLB even for small values of N, e.g., 8 and 16, while QSE diverges from the CRLB for small N. *Second*, validated by the convergence analysis in Section III-D, QSEr converges faster than QSE if $N \leq 30$. *Third*, QSEr has more uniform convergence performance across the whole region of δ ($|\delta| \leq 0.5$), as compared to QSE.

IV. Numerical Validation

In this section, we exploit the new optimal q refined in (23) to re-evaluate QSE [1]. The same simulation parameters are set as in [1, Sec. V]. The simulation codes provided by the authors of [1] on the web page [10] are modified to simulate QSE-related estimators QSE based on (23), referred to as "QSEr". By taking $N = 8$ in [1, eq. 39], the value $q^{\mathrm{o}}_{\mathrm{opt}} = 0.5$ would be used by the original QSE. However, $q = q^{\mathrm{o}}_{\mathrm{opt}} = 0.5$ leads to the singularity of $c(q)$; see (1). Thus, $q = (0.5 - 10^{-8})$ is taken to simulate the original QSE [1].

Fig. 2 plots the MSE of frequency estimates against the estimation SNR, where a hybrid algorithm of A&M [5] and QSE, referred to as HAQSE [1], is simulated as a benchmark; and the phased-corrected Quinn estimator (PCQ) [6] and the estimator using three DFT points (TDP) [11] are also simulated. In the figure, we see that only the proposed QSEr and HAQSE [1] can approach CRLB asymptotically. From the right y-axis, QSEr outperforms HAQSE mostly with the MSE improvement up to 0.5 dB. We also see that, while QSEr can reach the CRLB in a wide SNR region, PCQ [6] and TDP [11] remain 1.5 dB and 2.2 dB away from the CRLB, respectively, even after convergence.

Fig. 3 compares the MSEs of QSEr, QSE and HAQSE, as the estimation SNR increases. We see that the revised optimal

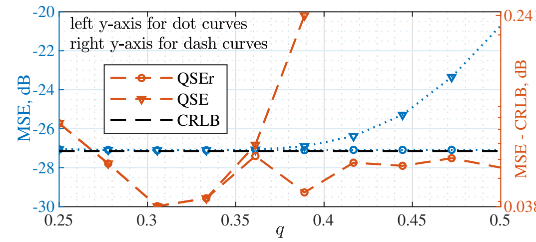

Fig. 4. MSE of frequency estimates vs q, where $N = 8$ and SNR $= 10$ dB.

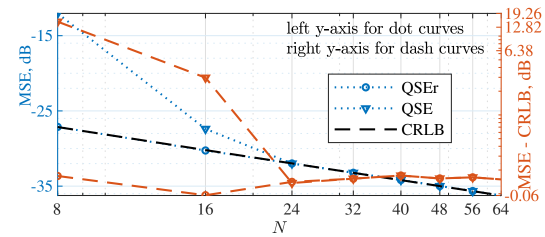

Fig. 5. MSE of frequency estimates vs N, where SNR $= 10$ dB.

q can improve the estimation accuracy of QSEr substantially at $N = 8$ and 16, as compared to the original QSE [1]. In specific, the MSE improvement of QSEr over QSE can be as large as 13.14 dB. The reason is because the refined q_{opt} can guarantee the monotonicity of $\beta_i^{\mathcal{R}}$ w.r.t. Δ_i. In contrast, the original $q^{\mathrm{o}}_{\mathrm{opt}}$ cannot guarantee the monotonicity. We also see that QSEr and QSE have the same performance at $N = 64$, which is because $q_{\mathrm{opt}} = q^{\mathrm{o}}_{\mathrm{opt}}$ according to (23).

In Fig. 3, we also see that QSEr outperforms HAQSE for a large SNR range at $N = 8$ and 16, with an MSE improvement of up to 1.897 dB. This is different to what was shown in [1, Fig. 2], where HAQSE outperformed QSE substantially. The reason underlying the superiority of HAQSE over the original QSE [1] is that HAQSE runs A&M estimator [5] prior to QSE. Unlike QSE, A&M does not require the monotonicity as QSE does. Thus, the initial frequency bias for QSE can be reduced after running A&M, and is very likely to fall within the monotonic region of $\beta_i^{\mathcal{R}}$ w.r.t. Δ_i even at $q^{\mathrm{o}}_{\mathrm{opt}} = N^{-1/3} > q^*_{\mathrm{L}}$. To this end, running A&M makes QSE valid. However, it cannot be guaranteed that A&M always reduces the frequency bias to within the monotonic region of $\beta_i^{\mathcal{R}}$ w.r.t. Δ_i. In contrast, the refined q_{opt} guarantees that $\beta_i^{\mathcal{R}}$ is always monotonic against Δ_i.

Fig. 4 compares the MSEs of QSEr and QSE [1] as q varies. We see that a smaller q produces a better frequency estimation with the MSEs of both QSEr and QSE overlapping with the CRLB. However, we see that the MSE of QSE starts to increase, as q becomes larger than $q^*_{\mathrm{L}} = 0.32$. In contrast, the proposed QSEr is able to converge to the CRLB in the whole region of q, since the new optimal q in (23) is always confined below 0.32. It is noteworthy that the asymptotic performance of QSE is supposed to be consistent for any $|q| < q_{\mathrm{opt}} = 0.5$; see [1, Sec. II-G]. However, QSE diverges from the CRLB due to the discussed impact of q, which is adequately addressed in our proposed QSEr.

Fig. 5 compares the MSEs of QSEr and QSE [1] as N increases, where, for fair comparison, q_{opt} and $q^{\mathrm{o}}_{\mathrm{opt}}$ are applied

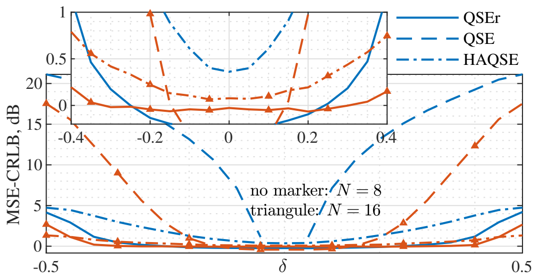

Fig. 6. MSE of frequency estimates vs δ, where SNR $= 10$ dB.

for QSEr and QSE, respectively. We see that QSEr based on the new optimal q ensures the convergence of the MSE to the CRLB even for small values of N, e.g., 8 and 16. From Fig. 5, we also see that the MSE improvement of QSEr is about 15 dB and 3 dB for $N = 8$ and 16, respectively, as compared to QSE. Moreover, Fig. 5 also validates the convergence analysis of QSEr in Section III-D.

Fig. 6 plots MSE against δ. We see that only QSEr with the proposed optimal q can achieve a relatively uniform estimation performance (by approaching the CRLB), across the whole region of δ. We also see that, as expected, the original QSE diverges from the CRLB substantially for large values of δ, which is caused by overlooking the impact of q on the monotonicity of $\beta_i^{\mathcal{R}}$ w.r.t. Δ_i. From the zoomed in sub-figure, we see that the proposed QSEr also outperforms HAQSE obviously, with an MSE improvement of 0.5 dB at $\delta = \pm 0.4$. The superiority of QSEr over QSE and HAQSE validates the significance our finding on the overlooked impact of q to the convergence of the state-of-the-art QSE [1]; and also validates the efficacy of the proposed optimal q in (23).

V. Conclusion

We investigate the impact of q on QSE and refine the region of q to ensure the asymptotic convergence of QSE to the CRLB even with a small number of small samples. Validated by simulations, the refined (optimal) q can substantially improve the accuracy of QSE. The trade-off, caused by q, between the estimation variance and bias of the frequency is also unveiled by our simulations.

Appendix

A. Derivations of (18)

Consider the positive value of q first, i.e., taking $0.25 < q < 0.5$. $\mathcal{Q}_{\pm}^*$ in (12) becomes $\frac{-1 \pm \sin(2\pi q)}{\cos(2\pi q)}$ which, based on the basic properties of trigonometric functions, can be simplified into (24). Given $0.25 < q < 0.5$, we have $\cos(\pi q) > -\cos(\pi q)$, and hence $\cos(\pi q) + \sin(\pi q) > \sin(\pi q) - \cos(\pi q)$.

$$\mathcal{Q}_+^* = \frac{\sin(\pi q) - \cos(\pi q)}{\sin(\pi q) + \cos(\pi q)}, \ \mathcal{Q}_-^* = \frac{\sin(\pi q) + \cos(\pi q)}{\sin(\pi q) - \cos(\pi q)}. \quad (24)$$

Substituting this into (24) leads to

$$\mathcal{Q}_+^* < 1 \text{ and } \mathcal{Q}_-^* > 1. \quad (25)$$

By letting $\mathcal{Q}(\Delta_i, q)$ in (3) equal to 1, $\Delta_i = 0$ is the solution. Combining this with (10) and (25), we obtain

$$\Delta_{i+}^* > 0 \text{ and } \Delta_{i-}^* < 0, \quad (26)$$

where $\Delta_{i\pm}^*$ ($\in [-0.5, 0.5]$) is the valid solution to equation (17). From (24), we have $\mathcal{Q}_+^* \mathcal{Q}_-^* = 1$. Replacing $\mathcal{Q}_\pm^*$ with $\mathcal{Q}(\Delta_{i\pm}^*, q)$ and then employing (3), we have

$$\Delta_{i+}^* = -\Delta_{i-}^*. \quad (27)$$

By substituting (25)∼(27) into (11), the sign of $\mathcal{N}(\mathcal{Q})$ satisfies (19), where $\Delta_i^* = |\Delta_{i\pm}^*|$. Given (12), the first derivative of $\mathcal{Q}_\pm^*$ w.r.t. q can be examined, leading to

$$\begin{cases} (\mathcal{Q}_+^*)_q' > 0 \\ (\mathcal{Q}_-^*)_q' < 0 \end{cases}. \quad (28)$$

Combining (10) and (28), (18) can be achieved.

B. Derivation of $q_{\mathrm{L}}^* \approx 0.32$

By substituting $\Delta_i = \Delta_i^* = 0.5$ into (3) and applying the basic manipulations, we obtain $\mathcal{Q}(0.5, q_{\mathrm{L}}^*) = \frac{0.5 - q_{\mathrm{L}}^*}{0.5 + q_{\mathrm{L}}^*} \times \frac{\sin[\pi(0.5 + q_{\mathrm{L}}^*)]}{\sin[\pi(0.5 - q_{\mathrm{L}}^*)]} = \frac{0.5 - q_{\mathrm{L}}^*}{0.5 + q_{\mathrm{L}}^*}$. Similarly, $\mathcal{Q}_+^*$ can be simplified into $\mathcal{Q}_+^* = \frac{-1 + \sin(2\pi q_{\mathrm{L}}^*)}{\cos(2\pi q_{\mathrm{L}}^*)} = \frac{\sin(\pi q_{\mathrm{L}}^*) - \cos(\pi q_{\mathrm{L}}^*)}{\cos(\pi q_{\mathrm{L}}^*) + \sin(\pi q_{\mathrm{L}}^*)}$. Setting $\mathcal{Q}(0.5, q_{\mathrm{L}}^*) = \mathcal{Q}_+^*$ and collecting terms, we have

$$\cot(\pi q_{\mathrm{L}}^*) = 2 q_{\mathrm{L}}^*. \quad (29)$$

An accurate analytical solution is intractable mathematically. To solve (29), we replace $\cot(\pi q_{\mathrm{L}}^*)$ with its Taylor series, i.e., $\cot(\pi q_{\mathrm{L}}^*) = \frac{1}{\pi q_{\mathrm{L}}^*} - \frac{\pi q_{\mathrm{L}}^*}{3} + \mathcal{O}\{(q_{\mathrm{L}}^*)^3\}$, and obtain $\frac{1}{\pi q_{\mathrm{L}}^*} - \frac{\pi q_{\mathrm{L}}^*}{3} \approx 2 q_{\mathrm{L}}^*$. This finally yields $q_{\mathrm{L}}^* \approx \left(\frac{\pi^2}{3} \pm 2\pi\right)^{-\frac{1}{2}} = 0.3232$, where the invalid solution is suppressed. Similarly, we can prove that the solution to $\mathcal{Q}(-0.5, q^*) = \mathcal{Q}_-^*$ is $q^* = -q_{\mathrm{L}}^*$.

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