

Received August 21, 2020, accepted August 29, 2020, date of publication September 2, 2020, date of current version September 17, 2020.

Digital Object Identifier 10.1109/ACCESS.2020.3021278

Consensus of Second-Order Multi-Agent Systems Without a Spanning Tree: A Sequence-Based Topology-Dependent Method

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This work was supported in part by the National Natural Science Foundation of China under Grant 61971100.

ABSTRACT This paper investigates the consensus of second-order multi-agent systems under switched topologies. Previous studies indicate that a consensus cannot be reached if the topology is fixed and has no spanning tree, but it is possible to reach a consensus for the multi-agent systems under switched topologies even if every topology has no spanning tree. However, in this paper, we show that some second-order multi-agent systems cannot reach a consensus even if the union of the directed interaction graphs has a spanning tree, and even if the union has a spanning tree frequently enough. It is often complex to judge whether the second-order multi-agent systems can reach a consensus or not under switched topologies if every topology has no spanning tree. This paper proposes a sequence-based topology-dependent method to determine whether a consensus can be reached in this circumstance. Our results are supported by examples and counterexamples.

INDEX TERMS Consensus, multi-agent systems, spanning tree, switched topologies.

I. INTRODUCTION

As a beneficial action to a group, cooperation widely exists in the nature [1]. The main target of cooperation is to reach a global goal with limited exchange of information among adjacent agents following a local control protocol [2]. There have been strong research interests in the last decade on coordination [3]–[10].

As the basic problem of cooperation, the consensus has been widely studied [11]–[13]. It refers to the states of a group of autonomous agents reaching a common value under an appropriately distributed control proposal. When the network or topology switches, the consensus problem becomes more complex. In the past years, some results were reported for the consensus of multi-agent systems under switched topologies in, e.g., [14]–[20]. The event-triggered consensus of multi-agent systems was introduced in [21]–[23]. According

to [24]–[26], consensus-based power dispatch in the smart grid is an important application of consensus computation by multi-agent systems. A lemma was given in [27] to solve the consensus of multi-agent systems with unknown control directions.

For the first-order multi-agent systems under switched topologies in [14], [15], a sufficient and necessary condition of reaching a consensus is that the union of the directed interaction graphs having a spanning tree frequently enough or with jointly connected topologies. In [14], it is shown that a consensus could be reached asymptotically when the union of the directed interaction graphs has a spanning tree frequently enough as the system evolves. A consensus algorithm was proposed for the first-order multi-agent systems with jointly connected topologies in [15].

For the consensus of second-order multi-agent systems under switched topologies, it is often assumed that every topology has a spanning tree [16]–[20]. A distributed consensus control method was developed for the second-order

The associate editor coordinating the review of this manuscript and approving it for publication was Fei Chen.

multi-agent systems under directed spanning trees in [16]. A directed spanning tree-based adaptive control protocol was developed in [17]. The consensus of second-order discrete-time multi-agent systems was studied in [18] when all switching topologies were strongly connected. A distributed impulsive algorithm for second-order multi-agent systems with a spanning tree was discussed in [19]. A novel consensus protocol designed based on synchronous intermittent local information feedback was proposed in [20] when the communication topology contains a directed spanning tree.

Some papers also studied the consensus of second-order systems under switched topologies when part topologies have no spanning tree [28]–[30]. The case that only one topology has a spanning tree and other topologies do not have was studied in [28], [30]. A distributed event-triggered scheme was presented in [29] to solve the problem of leader-following consensus under switching topologies including both graphs that have directed spanning tree and graphs that do not.

A few works designed control protocols for the consensus of second-order multi-agent systems when all topologies have no spanning tree under some special conditions in [31]–[39]. They mainly considered the condition that a consensus can be determinately reached when the topologies have jointly connected topologies. Almost all of them have to need the displacement or velocity information of their own. That is to say, they cannot run only with relative velocity or displacement. Their main work is to design protocols rather than to provide a checking rule.

The motivation of this paper is to provide a rule to judge whether a consensus can be reached for the second-order multi-agent systems if every topology has no spanning tree, because many multi-agent systems can reach a consensus under jointly connected topologies, while others cannot achieve with the same protocol. We remove the limitation of having to need agents' own displacement or velocity information.

Based on the above motivation, we propose a sequence-based topology-dependent method to study the consensus of second-order multi-agent systems when every topology has no spanning tree.

Our main contributions are summarized as follows: (1) We propose a novel sequence-based topology-dependent method for the second-order multi-agent systems under switched topologies to analyze the consensus if every topology has no spanning tree; (2) On the basis of the proposed method, we obtain a new theorem which can be seen as a rule to judge whether a consensus can be reached for the second-order multi-agent systems if every topology has no spanning tree; (3) We present counterexamples to show that agents may reach scattering status rather than a consensus even if the topologies satisfy conventional conditions. That is, having jointly connected topologies or maintaining a spanning tree frequently enough is not a sufficient condition to ensure the second-order multi-agent systems to reach a consensus.

The rest of this paper is organized as follows. In Section 2, some basic concepts and the system model are introduced.

In Section 3, we study the conditions of reaching a consensus and present the method for consensus analysis. Examples are shown in Section 4 to verify the results.

Notation 1: Throughout this paper, \mathbb{N}^+ stands for the set of positive integers. The symbol “ \times ” represents the multiplication operation or Cartesian product of sets. A matrix $P >$ (or $<$) 0 implies that matrix P is symmetric and positive (or negative) definite. The time $t_1, t_2, t_3, \dots, t_l, t_{l+1}, \dots$ are the switching times of the topologies of the multi-agent systems. The function $\sigma(t) : [0, +\infty) \rightarrow \mathcal{M} = \{1, 2, \dots, m\}$ is the switching signal of the switching topologies and m is the total amount of topologies.

II. SYSTEM DESCRIPTIONS AND PRELIMINARIES

For a network with n agents, its topology digraph can be denoted by $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ with a set of nodes $\mathcal{V} = \{v_1, v_2, \dots, v_n\}$, a set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and an adjacency matrix $\mathcal{A} = [a_{ij}]$, where the set of node indexes is $\mathcal{I} = \{1, 2, \dots, n\}$. We assume $i \neq j$ for any edge, which can be denoted by $e_{ij} = (v_i, v_j)$. The set of neighbors of node v_i is denoted by $\mathcal{N}_i = \{v_j \in \mathcal{V} : (v_i, v_j) \in \mathcal{E}, j \neq i\}$. The Laplacian matrix L is defined as: $l_{ij} = \sum_{k=1, k \neq i}^n a_{ik}$ for $i = j$, and $l_{ij} = -a_{ij}$, for $i \neq j, i, j \in \mathcal{I}$. In [28], [30], a Laplacian-like matrix is defined as $H = [h_{ij}]$, where $h_{ij} = l_{ij} - l_{nj}$.

For the network changing among m topologies, each agent is modeled by

$$\begin{aligned} \dot{x}_i(t) &= v_i(t), \\ \dot{v}_i(t) &= u_i(t), \end{aligned} \tag{1}$$

where $x_i(t)$ is the position of the i^{th} agent, u_i is the i^{th} control input.

Various consensus protocols have been discussed in the literature. In this paper, we consider

$$\begin{aligned} u_i(t) &= \beta_0 \sum_{j \in \mathcal{N}_i(t)} a_{ij\sigma(t)} (x_j(t) - x_i(t)) \\ &\quad + \beta_1 \sum_{j \in \mathcal{N}_i(t)} a_{ij\sigma(t)} (v_j(t) - v_i(t)). \end{aligned} \tag{2}$$

Remark 1: In [31]–[39], the protocols have to need the agents' own displacement or velocity information if every topology has no spanning tree. In the protocol (2), we only need relative velocity and displacement.

Remark 2: For the high-order protocol, the proposed idea is also suitable.

In general, the multi-agent systems including m subsystems (1) are said to reach a consensus if $\lim_{t \rightarrow +\infty} [x_i - x_j] = 0, \lim_{t \rightarrow +\infty} [v_i - v_j] = 0, \forall i, j \in \mathcal{I}, i \neq j$, under all initial conditions.

For a network \mathcal{G} , if the network is undirected, $j \in \mathcal{N}_i \Leftrightarrow i \in \mathcal{N}_j$, otherwise, $j \in \mathcal{N}_i$ has no fixed relationship with $i \in \mathcal{N}_j$. An undirected network can be seen as a special case of the directed network. System (2) applies to both undirected networks and directed networks. In this paper, we mainly focus on directed networks.

For the topologies considered in this paper, we make the following assumption:

Assumption 1: The union of the directed interaction graphs has a spanning tree, but every topology has no spanning tree.

Definition 1: For the multi-agent systems network switching among m topologies, we use $T_{[p|q]}(t_0, t)$ to denote the total running time of the p^{th} topology when it is instantly activated after the q^{th} topology in time section $[t_0, t)$, and $N_{[p|q]}(t_0, t)$ to indicate the total switching times in this case. Its sequence-based topology-dependent average dwell time is

$$\tau_{a([p|q])} \triangleq \frac{T_{[p|q]}(t_0, t)}{N_{[p|q]}(t_0, t)}. \quad (3)$$

To have a better understanding of this paper, we revisit the following lemma.

Lemma 1 ([28], [30]): For the directed network, if the system

$$\dot{z}(t) = \tilde{H}(t)z(t) \quad (4)$$

is globally uniformly asymptotically stable, then the multi-agent systems including m subsystems (1) with the protocol (2) reach a consensus, where

$$\tilde{H}(t) = \begin{bmatrix} 0 & I \\ -\beta_0 H & -\beta_1 H \end{bmatrix}_{\sigma(t)}, \quad (5)$$

and

$$z(t) = \begin{bmatrix} x_1 - x_n \\ \dots \\ x_{n-1} - x_n \\ v_1 - v_n \\ \dots \\ v_{n-1} - v_n \end{bmatrix} (t). \quad (6)$$

III. CONDITIONS FOR REACHING A CONSENSUS

In this section, we propose a novel method to judge whether the multi-agent system with a specific protocol can reach a consensus when every topology has no spanning tree.

Theorem 1: For the multi-agent systems including m subsystems (1) with the control protocol (2), assume that the p^{th} topology is instantly activated after the q^{th} topology. Given the constants $l_p \in \mathbb{N}^+$, $l_q \in \mathbb{N}^+$, $0 < \mu_{[p|q]} < 1$, $\alpha_p > 0$, $\alpha_q > 0$, $\tau_{min,p} = \min_{\sigma(t_i)=p} (t_i - t_{i-1})$, $\tau_{min,q} = \min_{\sigma(t_i)=q} (t_i - t_{i-1})$, if there exist a set of positive definite matrices $P_{p,j_p} > 0$, $P_{q,j_q} > 0$, $j_p = 0, 1, \dots, l_p$, $j_q = 0, 1, \dots, l_q$, such that $\forall (\sigma(t_i) = p, \sigma(t_{i-1}) = q) \in \mathcal{M} \times \mathcal{M}$, $p \neq q$, $\forall j_p = 0, 1, \dots, l_p - 1$,

$$\begin{cases} \tilde{H}_p^T P_{p,j_p} + P_{p,j_p} \tilde{H}_p + \frac{l_p(P_{p,j_p+1} - P_{p,j_p})}{\tau_{min,p}} \leq \alpha_p P_{p,j_p}, \\ \tilde{H}_p^T P_{p,j_p+1} + P_{p,j_p+1} \tilde{H}_p + \frac{l_p(P_{p,j_p+1} - P_{p,j_p})}{\tau_{min,p}} \leq \alpha_p P_{p,j_p+1}, \\ \tilde{H}_p^T P_{p,l_p} + P_{p,l_p} \tilde{H}_p \leq \alpha_p P_{p,l_p}, \end{cases} \quad (7)$$

$$P_{p,0} \leq \mu_{[p|q]} P_{q,l_q}, \quad (8)$$

$$\tau_{a([p|q])} < -\ln \mu_{[p|q]} / \alpha_p, \quad (9)$$

then the multi-agent systems including m subsystems (1) with the protocol (2) will reach a consensus.

Proof: For $\forall t \in [t_i, t_{i+1})$, $\sigma(t_i) = p$, $i > 0$, we divide the time section $[t_i, t_i + \tau_{min,p})$ into l_p equal length segments. Every segment is described by $S_{i,r} = [t_i + r \times \tau_{min,p}/l_p, t_i + (r+1) \times \tau_{min,p}/l_p)$, $r = 0, \dots, l_p - 1$.

We first design a matrix $P_p(t)$ with piecewise-linear elements over time. Let $P_{p,r}$ denote $P_p(t_i + r \times \tau_{min,p}/l_p)$. For any segment, r takes its value from $\{0, \dots, l_p - 1\}$, and $P_p(t) = (1 - \theta)P_{p,r} + \theta P_{p,r+1} = P_p^{(r)}(\theta)$, $\theta = (t - t_i - r \times \tau_{min,p}/l_p) \times l_p / \tau_{min,p}$, $0 \leq \theta \leq 1$. In the time section $[t_i + \tau_{min,p}, t_{i+1})$, we use the time invariant matrix P_{p,l_p} .

According to above discussion, the matrix $P_p(t)$ is constructed as

$$P_p(t) = \begin{cases} P_p^{(r)}(\theta), & t \in S_{i,r}. \\ P_{p,l_p}, & t \in [t_i + \tau_{min,p}, t_{i+1}). \end{cases} \quad (10)$$

We construct the multiple Lyapunov functions

$$V_p(t) = z^T(t) P_p(t) z(t), \quad p \in \mathcal{M}. \quad (11)$$

Because (10), the (11) can be rewritten as

$$V_p(t) = \begin{cases} z^T(t) P_p^{(r)}(\theta) z(t), & t \in S_{i,r}. \\ z^T(t) P_{p,l_p} z(t), & t \in [t_i + \tau_{min,p}, t_{i+1}). \end{cases} \quad (12)$$

We use $\lambda_{min} > 0$ (or $\lambda_{max} > 0$) to denote the minimal (or maximal) eigenvalue of all matrices $P_{p,j}$, $p \in \mathcal{M}$, $j \in \{1, \dots, l_p\}$, i.e.,

$$\lambda_{min} z^T(t) z(t) \leq V(t) \leq \lambda_{max} z^T(t) z(t). \quad (13)$$

According to (7) and (12), we have

$$\dot{V}_p(t) \leq \alpha_p V_p(t). \quad (14)$$

Because of (10) and (11), for $\forall t \in t_1, t_2, \dots$,

$$\begin{aligned} V_p(t) &= z^T(t) P_p(t) z(t) \\ &= z^T(t) P_p^{(0)}(0) z(t) \\ &= z^T(t) P_{p,0} z(t), \end{aligned} \quad (15)$$

and

$$\begin{aligned} V_q(t) &= z^T(t) P_q(t) z(t) \\ &= z^T(t) P_{q,l_q} z(t). \end{aligned} \quad (16)$$

According to (8), (15), and (16), we obtain

$$V_p(t) \leq \mu_{[p|q]} V_q(t^-), \quad t \in t_1, \dots, \quad (17)$$

when the network is switching from the q^{th} topology to the p^{th} topology. That is, the Lyapunov function drops at all topologies switching time t_1, \dots

According to (13)-(17), we have

$$\begin{aligned} V(z(t)) &\leq e^{\{\alpha_{\sigma(t_i)}(t-t_i)\}} V_{\sigma(t_i)}(z(t_i)) \\ &\leq e^{\{\alpha_{\sigma(t_i)}(t-t_i)\}} \mu_{\sigma(t_i)} V_{\sigma(t_{i-1})}(z(t_{i-1}^-)) \\ &\leq e^{\{\alpha_{\sigma(t_i)}(t-t_i) \mu_{\sigma(t_i)} + \alpha_{\sigma(t_{i-1})}(t_i - t_{i-1})\}} \\ &\quad \times V_{\sigma(t_{i-1})}(z(t_{i-1})) \end{aligned}$$

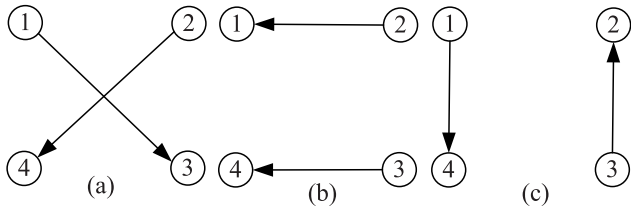


FIGURE 1. Topologies of multi-agent systems in case 1.

$$= \mu_{\sigma(t_i)} e^{\{\alpha_{\sigma(t_i)}(t-t_i) + \alpha_{\sigma(t_{i-1})}(t_i-t_{i-1})\}} \times V_{\sigma(t_{i-1})}(z(t_{i-1})). \quad (18)$$

Recursively, we have

$$V(z(t)) \leq \mu_{\sigma(t_i)} \dots \mu_{\sigma(t_1)} e^{\{\alpha_{\sigma(t_i)}(t-t_i) + \dots + \alpha_{\sigma(t_0)}(t_1-t_0)\}} V_{\sigma(t_0)}(z(t_0)). \quad (19)$$

We assume there are s'' different sequences in the time interval $[t_0, t]$. If the sequence $[p|q]$ corresponds to the g^{th} sequence, it is written as $[p|q]_{(g)}$, $g = 1, 2, \dots, s''$; and α_p is rewritten as $\alpha_{p(g)}$ for the g^{th} sequence.

Therefore,

$$\mu_{\sigma(t_i)} \dots \mu_{\sigma(t_1)} = \mu_{[\sigma(t_i)|\sigma(t_{i-1})]} \dots \mu_{[\sigma(t_1)|\sigma(t_0)]}$$

$$= \prod_{g=1}^{s''} \mu_{[p|q]_{(g)}}^{N_{[p|q]_{(g)}}(t_0, t)}, \quad (20)$$

and

$$\alpha_{\sigma(t_i)}(t-t_i) + \dots + \alpha_{\sigma(t_0)}(t_1-t_0) = \sum_{g=1}^{s''} \alpha_{p(g)} T_{[p|q]_{(g)}}(t_0, t) + t_1 - t_0. \quad (21)$$

Let $t_0 = 0$ and we obtain

$$V(z(t)) \leq \left\{ \prod_{g=1}^{s''} \mu_{[p|q]_{(g)}}^{N_{[p|q]_{(g)}}(0, t)} \right\} e^{\left\{ \sum_{g=1}^{s''} \alpha_{p(g)} T_{[p|q]_{(g)}}(0, t) + t_1 \right\}} \times V_{\sigma(0)}(z(0)) = e^{\left\{ \sum_{g=1}^{s''} \{\alpha_{p(g)} T_{[p|q]_{(g)}}(0, t) + N_{[p|q]_{(g)}}(0, t) \ln \mu_{[p|q]_{(g)}}\} + t_1 \right\}} \times V_{\sigma(0)}(z(0)). \quad (22)$$

Let $K_1 = e^{t_1}$ and one can get

$$V(z(t)) \leq K_1 e^{\left\{ \sum_{g=1}^{s''} \{\alpha_{p(g)} T_{[p|q]_{(g)}}(0, t) + N_{[p|q]_{(g)}}(0, t) \ln \mu_{[p|q]_{(g)}}\} \right\}} \times V_{\sigma(0)}(z(0))$$

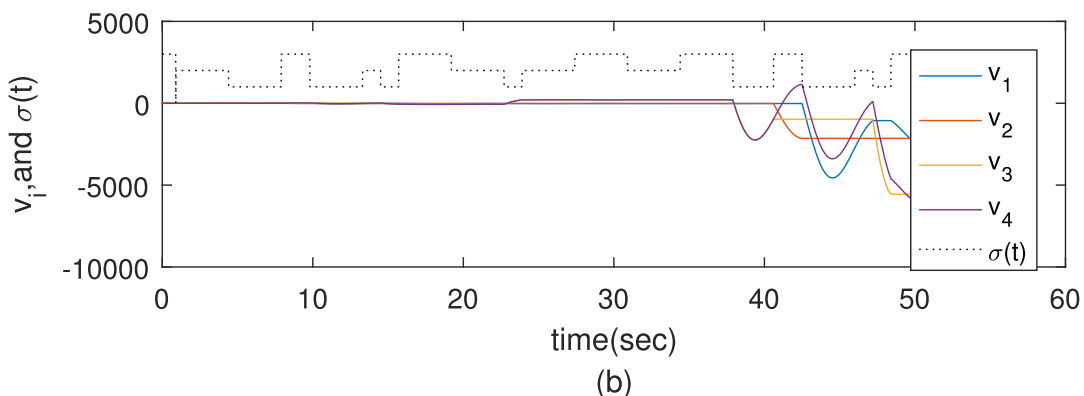
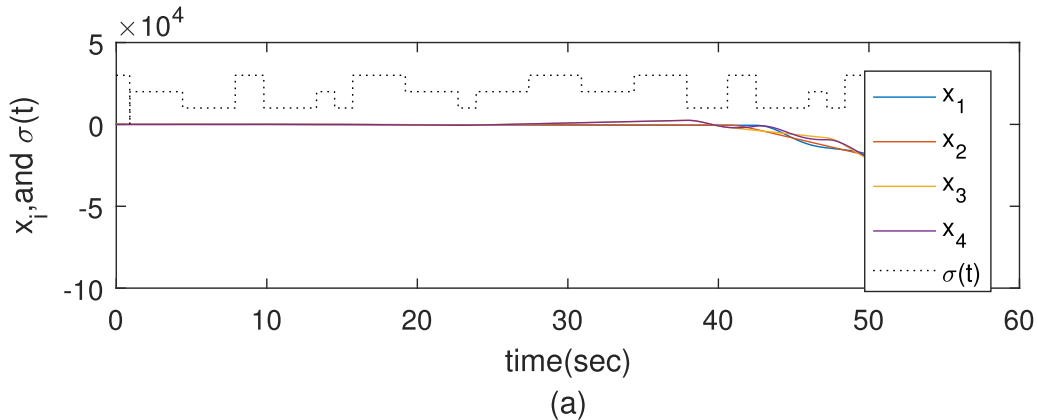


FIGURE 2. Positions and speed trajectories of agents in case 1 when conditions in Theorem 1 and Corollary 1 are not satisfied.

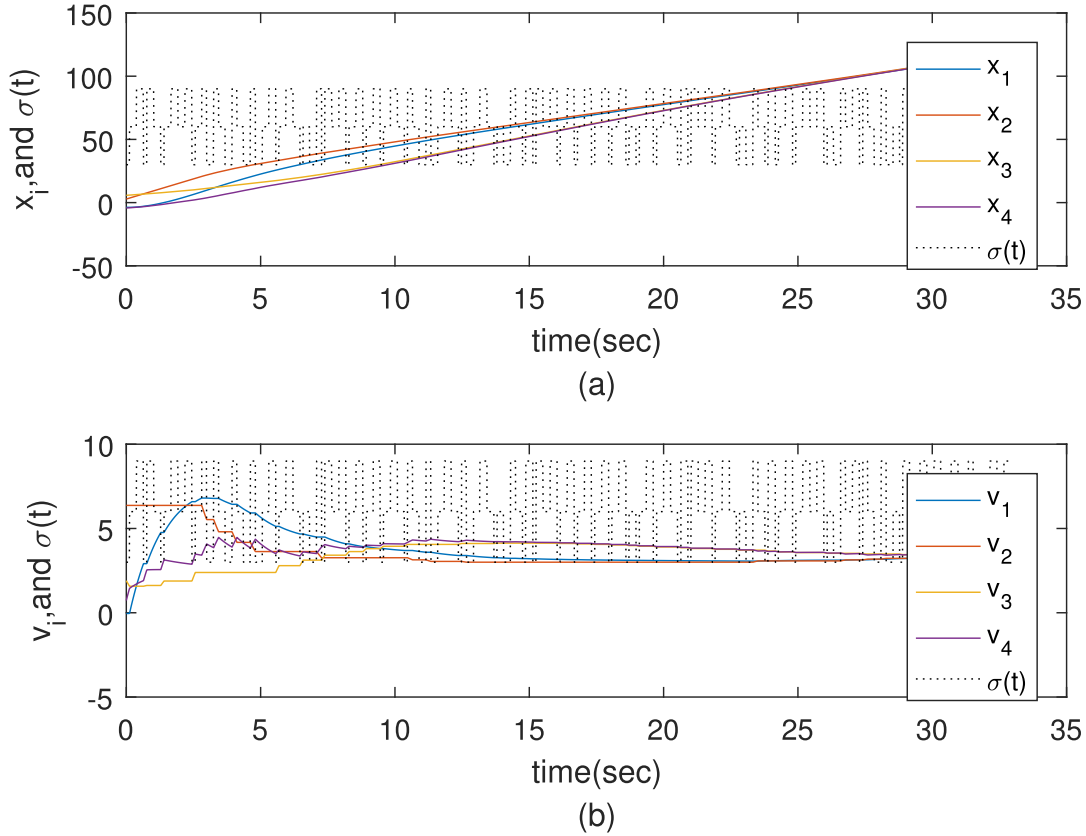


FIGURE 3. Positions and speed trajectories of agents in case 1 when all conditions in Theorem 1 and Corollary 1 are satisfied.

$$= K_1 e^{\left\{ \sum_{g=1}^{s''} T_{[p|q](g)}(0,t) \left(\alpha_{p(g)} + \frac{1}{\tau_{a([p|q](g))}} \ln \mu_{[p|q](g)} \right) \right\}} \times V_{\sigma(0)}(z(0)). \quad (23)$$

We define

$$\alpha_{p(g)} + \frac{1}{\tau_{a([p|q](g))}} \ln \mu_{[p|q](g)} \triangleq -\gamma_{(g)}^*. \quad (24)$$

According to (3) and (9), we can know that for every g , $-\gamma_{(g)}^* < 0$.

Let

$$-\gamma^* \triangleq \max_{g=1}^{s''} \{-\gamma_{(g)}^*\}.$$

We obtain,

$$V(z(t)) < K_1 e^{\{-\gamma^* t\}} V_{\sigma(0)}(z(0)). \quad (25)$$

According to (25), $V(z(t)) \rightarrow 0$ as $t \rightarrow +\infty$. According to Lemma 1 or formulation (13), we can know $z(t) \rightarrow 0$ as $t \rightarrow +\infty$, which means a consensus can be reached according to (6).

Remark 3: In Theorem 1, l_p denotes how many segments we divide the time section $[t_i, t_i + \tau_{min,p})$ into. The parameter $\mu_{[p|q]}$ stands for the variation degree of Lyapunov functions at the topology changing instance. The parameter α_p represents the increasing degree of Lyapunov functions.

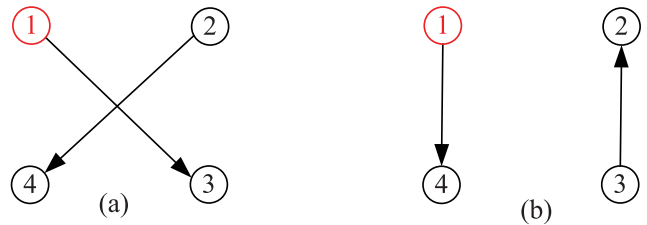


FIGURE 4. Topologies of multi-agent systems in case 2.

Remark 4: If every topology has no spanning tree, the parameter α_q is set as $\alpha_q > 0$. This implies that the value of the Lyapunov functions (11) probably increase when the q^{th} topology is available. When the p^{th} topology is after the q^{th} topology, the inequality $0 < \mu_{[p|q]} < 1$ implies that the value of the Lyapunov functions (11) must decrease in the switching instant to counterbalance the preceding increasing resulting from switching to the q^{th} topology.

If we do not consider the differences of sequences, and let

$$T_p = \sum_q T_{[p|q]}, \quad (26)$$

$$N_p = \sum_q N_{[p|q]}, \quad (27)$$

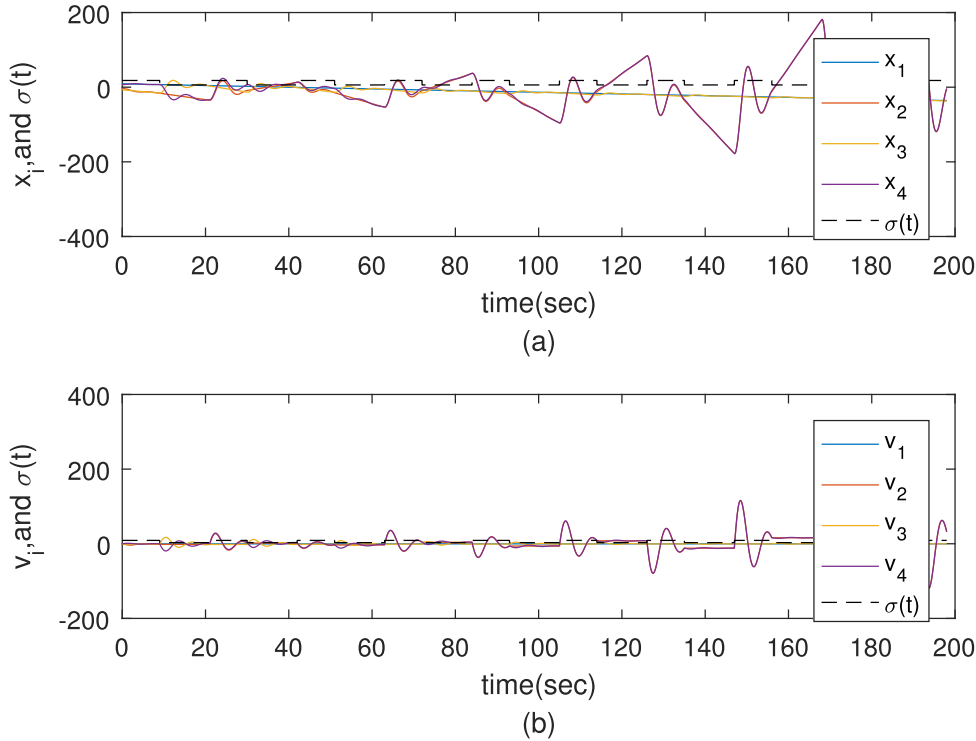


FIGURE 5. Positions and speed trajectories of agents in case 2 when conditions in Theorem 1 and Corollary 1 are satisfied.

we can get the topology-dependent average dwell time

$$\tau_{a(p)} \triangleq \frac{T_p(t_1, t)}{N_p(t_1, t)},$$

and the following corollary.

Corollary 1: For the system (4), let $0 < \mu_p < 1$, $\alpha_p > 0$, $\tau_{min,p} = \min_{\sigma(t_i)=p} (t_i - t_{i-1})$ and $l_p, p \in \mathcal{M}$, be the given constants. If there exist a set of positive definite matrices $P_{p,j} > 0, j = 0, 1, \dots, l_p$, such that $\forall j = 0, 1, \dots, l_p - 1$,

$$\begin{cases} \tilde{H}_p^T P_{p,j} + P_{p,j} \tilde{H}_p + \frac{l_p(P_{p,j+1} - P_{p,j})}{\tau_{min,p}} \leq \alpha_p P_{p,j}, \\ \tilde{H}_p^T P_{p,j+1} + P_{p,j+1} \tilde{H}_p + \frac{l_p(P_{p,j+1} - P_{p,j})}{\tau_{min,p}} \leq \alpha_p P_{p,j+1}, \\ \tilde{H}_p^T P_{p,l_p} + P_{p,l_p} \tilde{H}_p \leq \alpha_p P_{p,l_p}, \end{cases} \quad (28)$$

$$P_{p,0} \leq \mu_p P_{q,l_q}, p \neq q, \quad (29)$$

$$\tau_{a(p)} < -\ln \mu_p / \alpha_p, \quad (30)$$

then the multi-agent system (1) with the protocol (2) will reach a consensus.

Although there is no communication between different topologies, the agents are common in them. Agents can get information from neighbors in different topologies after the switching signals are added. Under proper conditions, a consensus can be reached.

IV. NUMERICAL EXAMPLES

For multi-agent systems, there are two classes of consensus problems in general: (a) there is no leader; (b) there is a leader.

In this section, we simulate both cases to verify the validity of our results. We consider multi-agent systems with 4 agents switching among three (or two) topologies. Every topology has no spanning tree. We use the values of the three-level (or two-level) signal $\sigma(t)$ to represent these topologies.

A. CASE 1: NO FIXED LEADER

We consider the topologies in Fig. 1 in this case. For the union of the three graphs (a), (b), and (c) in Fig. 1, there exists a spanning tree, for example, $\textcircled{3} \rightarrow \textcircled{2} \rightarrow \textcircled{1} \rightarrow \textcircled{4}$, that can connect all agents.

Example 1: We consider these three topologies with time parameters $\tau_{min,1} = 1.2$, $\tau_{min,2} = 1.2$, $\tau_{min,3} = 0.90$, $\tau_{a(\{1|3\})} = 2.7$, $\tau_{a(\{2|3\})} = 3.5$, $\tau_{a(\{3|1\})} = 1.9$, $\tau_{a(\{1|2\})} = 1.2$, $\tau_{a(\{2|1\})} = 1.2$, and $\tau_{a(\{3|2\})} = 0.90$. We set the coupling strength parameters as $\beta_0 = 1.0$ and $\beta_1 = 0.40$. If we set $l_1 = 3$, $l_2 = 3$, $l_3 = 3$, $\mu_{[3|1]} = 0.041$, $\mu_{[1|3]} = 0.045$, $\mu_{[2|1]} = 0.047$, $\mu_{[1|2]} = 0.048$, $\mu_{[3|2]} = 0.049$, $\mu_{[2|3]} = 0.057$, $\alpha_1 = 0.75$, $\alpha_2 = 0.85$, and $\alpha_3 = 0.90$, Matlab LMI tool boxes cannot find a proper set of positive definite matrices satisfying conditions (7)-(8) except for the condition (9). The positions and speed trajectories of all agents are shown in Fig.2. We can see that a consensus cannot be reached.

Example 2: We consider the dwell time of three topologies with time parameters $\tau_{min,1} = 0.120$, $\tau_{min,2} = 0.240$, $\tau_{min,3} = 0.180$, $\tau_{a(\{1|3\})} = 0.124$, $\tau_{a(\{2|3\})} = 0.260$, $\tau_{a(\{3|1\})} = 0.184$, $\tau_{a(\{1|2\})} = 0.120$,

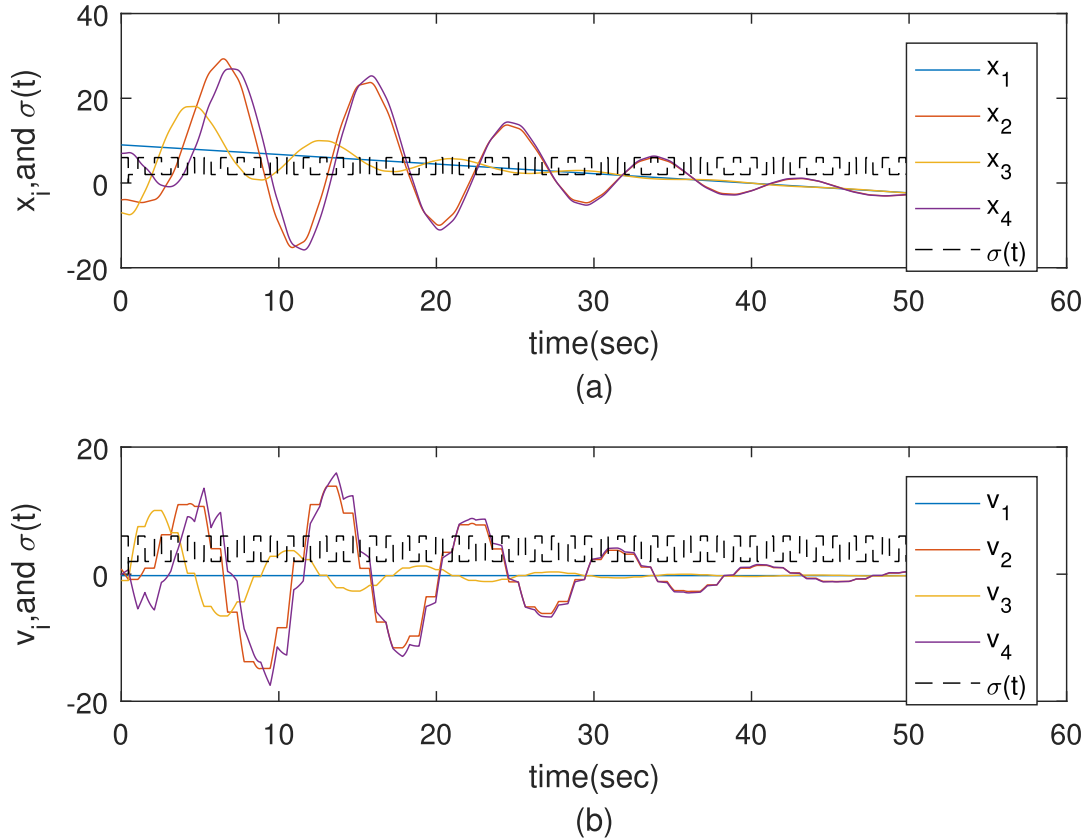


FIGURE 6. Positions and speed trajectories of agents in case 2 when all conditions in Theorem 1 and Corollary 1 are satisfied.

$\tau_{a([2|1])} = 0.240$, and $\tau_{a([3|2])} = 0.180$. We set the coupling strength parameters as $\beta_0 = 0.10$ and $\beta_1 = 1.0$. If we set $l_1 = 3$, $l_2 = 3$, $l_3 = 3$, $\mu_{[3|1]} = 0.790$, $\mu_{[1|3]} = 0.794$, $\mu_{[2|1]} = 0.780$, $\mu_{[1|2]} = 0.800$, $\mu_{[3|2]} = 0.800$, $\mu_{[2|3]} = 0.760$, $\alpha_1 = 1.748$, $\alpha_2 = 1.030$, and $\alpha_3 = 1.200$, Matlab LMI tool boxes can find a proper set of positive definite matrices satisfying conditions (7)-(9). Now all conditions in Theorem 1 are satisfied. The positions and speed trajectories of all agents are shown in Fig. 3. We can see that they reach a consensus.

B. CASE 2: THERE IS A FIXED LEADER

We consider the topologies as shown in Fig. 4 in this case. Obviously, the first agent is a fixed leader. We set $\beta_0 = 1.0$, $\beta_1 = 0.40$, $l_1 = 3$, $l_2 = 3$, $\mu_1 = 0.89$, $\mu_2 = 0.89$, $\alpha_1 = 0.89$ and $\alpha_2 = 1.1$.

Remark 5: According to (3), (26), (27) and the relationship between Theorem 1 and Corollary 1, when the switching topologies are shown in Fig. 4, we have

$$\begin{cases} \mu_{[1|2]} = \mu_1, \\ \mu_{[2|1]} = \mu_2, \\ \tau_{a([1|2])} = \tau_{a(1)}, \\ \tau_{a([2|1])} = \tau_{a(2)}. \end{cases} \quad (31)$$

Example 3: We consider the dwell time of two topologies with time parameters $\tau_{min,1} = \tau_{a(1)} = 12$ and

$\tau_{min,2} = \tau_{a(2)} = 9.0$.¹ We can find a proper set of positive definite matrices satisfying conditions (28)-(29) but not (30). The positions and speed trajectories of all agents are shown in Fig. 5. We can see that they cannot reach a consensus. The failure results from the use of an improper sequence-based topology-dependent average dwell time.

Example 4: We consider the dwell time of two topologies with time parameters $\tau_{min,1} = \tau_{a(1)} = 0.12$ and $\tau_{min,2} = \tau_{a(2)} = 0.090$. We can find a proper set of positive definite matrices satisfying conditions (28)-(30). Now all conditions in Corollary 1 are satisfied. The positions and speed trajectories of all agents are shown in Fig. 6. We can see that a consensus can be reached.

Remark 6: In Figs. 2 and 5 (Examples 1 and 3), it can be seen that the second-order multi-agent systems 1 with control protocols 2 can reach scattering status rather than a consensus when all switching topologies have no spanning tree even if they have jointly connected topologies.

V. CONCLUSION

This paper presents a novel method for analyzing the consensus of second-order multi-agent systems without a spanning tree. It is known that, although the second-order multi-agent system cannot reach a consensus under a fixed topology

¹Here, although we fixed the dwell time, it is also valid for *average* dwell time.

without a spanning tree, a consensus can be achieved under switched topologies even if every topology has no spanning tree. However, when sequence-based topology-dependent average dwell time changes, even the same systems with the same topologies may reach scattering status. Our proposed method in this paper can be used to determine whether a consensus can be reached under switched topologies if every topology has no spanning tree. The effectiveness of the proposed method is validated by examples and counterexamples.

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