LMI relaxation to Riccati equations in structured $\mathcal{H}_2$ control *

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Abstract

In this paper we discuss structured $\mathcal{H}_2$ control methods for large-scale interconnected systems. Based on a relaxation of Riccati equations, we derive some linear matrix inequality (LMI) conditions for sub-optimal controllers in which information structure can be imposed. In particular, we derive controllers by solving low-dimensional LMIs, which are decentralized except for the sharing information between neighbors, as determined by the plant interconnection; also we optimize a performance bound for each of the derived controllers.

1 Introduction

The increasing complexity of large-scale systems has stimulated extensive research in recent years, in particular for those made up of spatially interconnected components. For example, the large-scale applications, such as power grids [22], communication networks [20], and arrays of micro-sensors/actuators [12] would fall into this category. Although centralized control could achieve the optimal performance by using standard control design techniques, it requires a high level of connectivity, computational complexity, communication costs and raises reliability concerns. Therefore, there is a clear motivation to decentralize as much as possible the control process in such distributed systems. More generally, other information structures can be imposed on the control design, to allow for a tractable implementation.

The analysis and design of structured controllers has received considerable attention since the 1970s. A typical example of structure is decentralized control which has been exploited extensively and can be seen in [8, 21, 16, 19] and the references therein. Localized control, in which any sub-controller only has information from a small amount of neighbors, has been considered recently in [4, 6] for spatially invariant systems. [15] discusses distributed controller design and analysis for distributed system with arbitrary discrete symmetry groups. In [13], a class of specific structures covering nested, chained, hierarchical, delayed interaction and communications, and symmetric systems is studied. More general structured controls are presented in [23, 3, 10, 14].

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In this paper, a structured $\mathcal{H}_2$ controller design problem is addressed, that is, we seek to determine the class of structured controllers which produce a stabilized closed-loop transfer function satisfying a pre-specified $\mathcal{H}_2$ norm bound. Recently, a general linear matrix inequality (LMI) solution to the $\mathcal{H}_2$-control problem has been presented by C. Scherer, et al in [18] in the context of multi-objective output-feedback control. We study the traditional optimal state-feedback and output-feedback problems by applying this LMI method and the technique in [3] to impose general structure. In the absence of structure, we show explicitly how these solutions relate to the algebraic Riccati equation (ARE) approach. Therefore, with structure constraints a heuristic approach might be to impose structure directly on the LMI relaxation to Riccati equations rather than the LMIs in [18], which leads to a class of controllers by solving lower dimensional LMIs. In particular, three structured controllers are derived in the output-feedback problem: one will preserve arbitrary structures of the original system, while the other two work with symmetric structures; we compare these controllers to those obtained by Scherer's LMI method and derive a bound on the $\mathcal{H}_2$ norm for each of the controllers.

This paper is organized as follows. In Section 2 we briefly review the LMI method derived in [18] and derive its dual form based on an observability-gramian manipulation; also the LMI version of Riccati inequalities is presented. We explore the relationship between LMI approach and ARE approach in Section 3, leading to the explicit solutions of the optimal state-feedback and output-feedback problems. In Section 4, by imposing the structure on the LMI relaxation of Riccati equations, structured controllers are derived in both cases. In Section 5 we illustrate the method by a set of interconnected systems. Conclusions are given in Section 6.

2 Preliminary and background

For convenience, we use $He\{M\}$ to denote $M + M^*$, where M^* is the complex conjugate transpose of M.

2.1 $\mathcal{H}_2$ output feedback control via LMI

Considering the following LTI plant,

$$P\begin{cases} \dot{x} = Ax + B_\omega \omega + Bu, \\ z = C_z x + D_{z\omega}\omega + D_z u, \\ y = Cx + D_w \omega, \end{cases} \tag{1}$$

we want to find a dynamic output-feedback controller

$$K\begin{cases} \dot{\varsigma} = A_K\varsigma + B_K y, \\ u = C_K\varsigma + D_K y, \end{cases}$$

which optimizes the $\mathcal{H}_2$ performance of the closed-loop system denoted by T admitting the realization

$$T\begin{cases} \dot{x}_{cl} = \mathcal{A}x_{cl} + \mathcal{B}w, \\ z = \mathcal{C}x_{cl} + \mathcal{D}w, \end{cases} \tag{2}$$

where

$$\left(\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right) = \left(\begin{array}{cc|c} A + BD_KC & BC_K & B_\omega + BD_KD_\omega \\ B_KC & A_K & B_KD_\omega \\ \hline C_z + D_zD_KC & D_zC_K & D_{z\omega} + D_zD_KD_\omega \end{array}\right).$$

Assuming $\mathcal{A}$ stable and $\mathcal{D} = 0$, the $\mathcal{H}_2$ norm of $T_{z\omega}$ is

$$\|T_{z\omega}\|_{\mathcal{H}_2}^2 := \frac{1}{2\pi} \int_{-\infty}^{+\infty} Tr(T_{z\omega}^*(j\omega)T_{z\omega}(j\omega))d\omega.$$

An LMI approach for $\mathcal{H}_2$ controller synthesis based on a controllability-gramian manipulation is proposed in [18], stated in the following Lemma.

Lemma 1. *Given $\gamma > 0$, there exists a controller K_c that internally stabilizes the closed-loop system (2) and satisfies $\|T_{z\omega}\|_{\mathcal{H}_2}^2 < \gamma$ if and only if there exist X_c, Y_c, $\hat{A}_c$, $\hat{B}_c$, $\hat{C}_c$, $\hat{D}_c$, Q_c satisfying*

$$\begin{aligned} &\begin{bmatrix} He\{AX_c + B\hat{C}_c\} & \hat{A}_c^* + (A + B\hat{D}_c C) & B_\omega + B\hat{D}_c D_\omega \\ \star & He\{A^* Y_c + \hat{B}_c C\} & Y_c B_\omega + \hat{B}_c D_\omega \\ \star & \star & -I \end{bmatrix} < 0, \\ &\begin{bmatrix} X_c & I & (C_z X_c + D_z \hat{C}_c)^* \\ \star & Y_c & (C_z + D_z \hat{D}_c C)^* \\ \star & \star & Q_c \end{bmatrix} > 0, \\ &Tr(Q_c) < \gamma, \qquad D_{z\omega} + D_z \hat{D}_c D_\omega = 0. \end{aligned} \tag{3}$$

The controller K_c is given by

$$\begin{cases} D_{Kc} = \hat{D}_c, \\ C_{Kc} = (\hat{C}_c - D_{Kc} C X_c) M_c^{-*}, \\ B_{Kc} = N_c^{-1}(\hat{B}_c - Y_c B D_{Kc}), \\ A_{Kc} = N_c^{-1}[\hat{A}_c - N_c B_{Kc} C X_c - Y_c B C_{Kc} M_c^* \\ \qquad\qquad - Y_c(A + B D_{Kc} C) X_c] M_c^{-*}, \end{cases} \tag{4}$$

where N_c, M_c are nonsingular matrices satisfying

$$N_c M_c^* = I - Y_c X_c \ . \tag{5}$$

Similarly, a dual of (3) can be obtained by an observability-gramian-based manipulation. Alternatively, $\|T_{z\omega}\|_{\mathcal{H}_2}^2 < \gamma$ if and only if there exists $\mathcal{S}_o > 0$, such that

$$\mathcal{A}^* \mathcal{S}_o + \mathcal{S}_o \mathcal{A} + \mathcal{C}^* \mathcal{C} < 0, \tag{6a}$$

$$Tr(\mathcal{B}^* \mathcal{S}_o \mathcal{B}) < \gamma, \tag{6b}$$

which is equivalent to the following with $\mathcal{P}_o = \mathcal{S}_o^{-1}$ and an auxiliary parameter Q_o

$$\begin{aligned} &\begin{bmatrix} \mathcal{A}\mathcal{P}_o + \mathcal{P}_o \mathcal{A}^* & \mathcal{P}_o \mathcal{C}^* \\ \mathcal{C}\mathcal{P}_o & -I \end{bmatrix} < 0, \\ &\begin{bmatrix} \mathcal{P}_o & \mathcal{B} \\ \mathcal{B}^* & Q_o \end{bmatrix} > 0, \\ &Tr(Q_o) < \gamma, \quad \mathcal{D} = 0. \end{aligned} \tag{7}$$

Partition $\mathcal{P}_o$ and $\mathcal{P}_o^{-1}$ as

$$\mathcal{P}_o = \begin{pmatrix} Y_o & N_o \\ N_o^* & J_o \end{pmatrix}, \quad \mathcal{P}_o^{-1} = \begin{pmatrix} X_o & M_o \\ M_o^* & H_o \end{pmatrix}, \tag{8}$$

and define the change of controller variables as follows:

$$\begin{cases} \hat{A}_o = M_o A_K N_o^* + M_o B_K C Y_o + X_o B C_K N_o^* \\ \qquad +X_o(A + BD_K C)Y_o, \\ \hat{B}_o = M_o B_K + X_o B D_K, \\ \hat{C}_o = C_K N_o^* + D_K C Y_o, \\ \hat{D}_o = D_K. \end{cases} \tag{9}$$

By performing a congruence transformation with $\mathrm{diag}(\Pi_1, I)$ on the first two inequalities of (7), where Π_1 is defined as

$$\Pi_1 = \begin{pmatrix} X_o & I \\ M_o^* & 0 \end{pmatrix}, \tag{10}$$

then (7) is turned into the following LMIs with variables X_o, Y_o, $\hat{A}_o$, $\hat{B}_o$, $\hat{C}_o$, $\hat{D}_o$, Q_o,

$$\begin{aligned} &\begin{bmatrix} He\{X_o A + \hat{B}_o C\} & \hat{A}_o + (A + B\hat{D}_o C)^* & C_z^* + C^* \hat{D}_o^* D_z^* \\ \star & He\{AY_o + B\hat{C}_o\} & Y_o C_z^* + \hat{C}_o^* D_z^* \\ \star & \star & -I \end{bmatrix} < 0, \\ &\begin{bmatrix} X_o & I & X_o B_\omega + \hat{B}_o D_\omega \\ \star & Y_o & B_\omega + B\hat{D}_o D_\omega \\ \star & \star & Q_o \end{bmatrix} > 0, \\ &Tr(Q_o) < \gamma, \qquad D_{z\omega} + D_z \hat{D}_o D_\omega = 0. \end{aligned} \tag{11}$$

The controller K_o is given by

$$\begin{cases} D_{Ko} = \hat{D}_o, \\ C_{Ko} = (\hat{C}_o - D_{Ko} C Y_o) N_o^{-*}, \\ B_{Ko} = M_o^{-1}(\hat{B}_o - X_o B D_{Ko}), \\ A_{Ko} = M_o^{-1}[\hat{A}_o - M_o B_{Ko} C Y_o - X_o B C_{Ko} N_o^* \\ \qquad -X_o(A + BD_{Ko}C)Y_o] N_o^{-*}, \end{cases} \tag{12}$$

where N_o, M_o are nonsingular matrices satisfying

$$N_o M_o^* = I - Y_o X_o \ . \tag{13}$$

2.2 Two classic results of the optimal $\mathcal{H}_2$ control

The following two lemmas are well-known results, and can be found in many books, such as [2, 24, 7].

a. State feedback

Given a state-feedback system

$$\begin{cases} \dot{x} = Ax + B_\omega \omega + Bu, \\ z = \begin{bmatrix} \bar{C}x \\ u \end{bmatrix}, \\ y = x, \end{cases} \tag{14}$$

it is well-known that the optimal controller is given in the following lemma.

Lemma 2. *Assuming (A, B) stabilizable and $(\bar{C}, A)$ detectable, the optimal $\mathcal{H}_2$ controller for the system (14) is given by $u = -B^* P_s x$, where $P_s > 0$ is the stabilizing solution satisfying*

$$A^* P_s + P_s A + \bar{C}^* \bar{C} - P_s B B^* P_s = 0, \tag{15}$$

and $\|T_{z\omega}\|_{\mathcal{H}_2,opt}^2 = Tr(B_\omega^* P_s B_\omega)$.

b. Output feedback

Given an output-feedback system

$$\begin{cases} \dot{x} = Ax + \bar{B}d + Bu, \\ z = \begin{bmatrix} \bar{C}x \\ u \end{bmatrix}, \\ y = Cx + n, \end{cases} \tag{16}$$

the optimal controller is stated below.

Lemma 3. *Assuming* (A, B) *and* $(A, \bar{B})$ *stabilizable,* (C, A) *and* $(\bar{C}, A)$ *detectable, the optimal* $\mathcal{H}_2$ *controller for the system* (16) *has the realization*

$$\begin{cases} \dot{\varsigma} = (A + BF + LC)\varsigma - Ly, \\ u = F\varsigma, \end{cases}$$

where $L = -S_sC^*, F = -B^*P_s$, $P_s > 0$ *and* $S_s > 0$ *are stabilizing solutions of*

$$A^*P_s + P_sA + \bar{C}^*\bar{C} - P_sBB^*P_s = 0, \tag{17a}$$

$$AS_s + S_sA^* + \bar{B}\bar{B}^* - S_sC^*CS_s = 0, \tag{17b}$$

and $\|T_{z\omega}\|^2_{\mathcal{H}_2,opt} = Tr(\bar{B}^*P_s\bar{B}) + Tr(B^*P_sS_sP_sB) = Tr(\bar{C}S_s\bar{C}^*) + Tr(CS_sP_sS_sC^*)$.

2.3 Riccati inequality

As seen already, Riccati equation plays an important role in $\mathcal{H}_2$ feedback control. For more detailed topics on Riccati equations, readers are referred to [9, 1]. Here we introduce an important property of Riccati inequality and its LMI version which will be used extensively in the following sections, and in those sections we will discuss how to use this LMI relaxation of Riccati equation to impose structure on the synthesis problem in a convex fashion.

Lemma 4. *Assuming* (A, B) *stabilizable and* (C, A) *detectable, the following statements hold.*

(i). There exists $X > 0$ *satisfying*

$$XA + A^*X - XBB^*X + C^*C < 0. \tag{18}$$

(ii). For all $X > 0$ *satisfying* (18), $A - BB^*X$ *is Hurwitz and* $X > X_s$, *where* $X_s > 0$ *is the stabilizing solution of Riccati equation*

$$X_sA + A^*X_s - X_sBB^*X_s + C^*C = 0.$$

(iii). There exists a strict positive definite sequence $\{X^{(i)}\}$ *satisfying* (18) *which converges to* X_s.

Proof of claim *(ii)* is in [11], and proofs of claims *(i)* and *(iii)* are similar to the routine in [7, 17]. Based on this lemma, X_s could be obtained by minimizing X subject to (18), or equivalently the following semi-definite programming (SDP) problem by letting $Y = X^{-1}$ (please refer to [11] for details):

$$\begin{array}{r} \min \gamma \ \text{ subject to:} \\ \begin{bmatrix} AY + YA^* - BB^* & YC^* \\ CY & -I \end{bmatrix} < 0, \\ \begin{bmatrix} Y & I \\ I & Q \end{bmatrix} > 0, \\ Tr(Q) < \gamma. \end{array} \tag{19}$$

3 Relationship between LMI and ARE approach to $\mathcal{H}_2$ control

Here we work on the state-feedback and output-feedback problems in Section 2.2 via LMI method. The main purpose of this section is to show that the LMIs derived in [18] are related to Riccati inequalities, and give a new proof of Lemmas 2 and 3, that is, γ in (3) and (11) can be chosen arbitrarily close to the optimal norm and there exists a sequence of controllers convergent to the optimal controller. In Section 4, as an alternative way, we will use this relationship to derive some structured controllers via Riccati inequalities rather than imposing structures directly on Scherer's LMI method or its dual.

3.1 State feedback

Consider the system (14), that is, $C = I$, $D_\omega = 0$, $C_z = \begin{bmatrix} \bar{C} \\ 0 \end{bmatrix}$, $D_z = \begin{bmatrix} 0 \\ I \end{bmatrix}$, $D_{z\omega} = 0$ in (1). By substituting them into (11), we state the following result.

Proposition 1. *Given $\gamma > 0$, there exists a controller K_o that internally stabilizes the closed-loop system (2) and satisfies $\|T_{z\omega}\|^2_{\mathcal{H}_2} < \gamma$ if and only if there exist X_o, Y_o, $\hat{A}_o$, $\hat{B}_o$, $\hat{C}_o$, $\hat{D}_o$, Q_o that satisfy*

$$\begin{bmatrix} He\{X_oA + \hat{B}_o\} & \hat{A}_o + (A + B\hat{D}_o)^* & (\bar{C}^* \;\; \hat{D}_o^*) \\ \star & He\{AY_o + B\hat{C}_o\} & (Y_o\bar{C}^* \;\; \hat{C}_o^*) \\ \star & \star & -I \end{bmatrix} < 0, \tag{20a}$$

$$\begin{bmatrix} X_o & I & X_oB_\omega \\ \star & Y_o & B_\omega \\ \star & \star & Q_o \end{bmatrix} > 0, \tag{20b}$$

$$Tr(Q_o) < \gamma. \tag{20c}$$

The following theorem shows that γ in (20) has a lower bound, and can be chosen arbitrarily close to it.

Theorem 1. *Assuming (A, B) stabilizable and $(\bar{C}, A)$ detectable,* (20) *is feasible if and only if $\gamma > Tr(B_\omega^* P_s B_\omega)$, where P_s is the stabilizing solution of* (15)*.*

Proof. If (20) holds, by Schur complement, (20a) is equivalent to

$$\begin{bmatrix} \mathcal{M}_{11} & \hat{A}_o + (A + B\hat{D}_o)^* + \bar{C}^*\bar{C}Y_o + \hat{D}_o^*\hat{C}_o \\ \star & \mathcal{M}_{22} \end{bmatrix} < 0, \tag{21}$$

where

$$\mathcal{M}_{11} = X_oA + A^*X_o + \hat{B}_o + \hat{B}_o^* + \bar{C}^*\bar{C} + \hat{D}_o^*\hat{D}_o,$$
$$\mathcal{M}_{22} = AY_o + Y_oA^* + Y_o\bar{C}^*\bar{C}Y_o - BB^* + (B + \hat{C}_o^*)(B^* + \hat{C}_o).$$

From the (2,2) block of (21), we have

$$AY_o + Y_oA^* + Y_o\bar{C}^*\bar{C}Y_o - BB^* < 0, \tag{22}$$

which is equivalent to

$$Y_o^{-1}A + A^*Y_o^{-1} - Y_o^{-1}BB^*Y_o^{-1} + \bar{C}^*\bar{C} < 0. \tag{23}$$

Then $Y_o^{-1} > P_s$ by Lemma 4.

By Schur complement, (20b) is equivalent to

$$X_o > 0 \text{ and } \begin{bmatrix} Y_o - X_o^{-1} & 0 \\ \star & Q_o - B_\omega^*X_oB_\omega \end{bmatrix} > 0,$$

which is further equivalent to

$$Y_o > X_o^{-1} > 0 \text{ and } Q_o > B_\omega^* X_o B_\omega. \tag{24}$$

Then

$$Y_o > 0 \text{ and } Q_o > B_\omega^* Y_o^{-1} B_\omega. \tag{25}$$

Therefore $\gamma > Tr(Q_o) > Tr(B_\omega^* Y_o^{-1} B_\omega) > Tr(B_\omega^* P_s B_\omega)$.

Now the other direction. If $\gamma > Tr(B_\omega^* P_s B_\omega)$, by Lemma 4, there exists $Y_o > 0$ satisfying (23) such that $\gamma > Tr(B_\omega^* Y_o^{-1} B_\omega) > Tr(B_\omega^* P_s B_\omega)$. Set $Q_o = B_\omega^* Y_o^{-1} B_\omega + \epsilon_1 I$ with small enough $\epsilon_1 > 0$ such that $Q_o > B_\omega^* Y_o^{-1} B_\omega$ and $\gamma > Tr(Q_o)$; set $X_o = (1+\epsilon_2) Y_o^{-1}$ with small enough $\epsilon_2 > 0$ to satisfy (24), therefore satisfy (20b); set $\hat{D}_o$ arbitrarily and $-(\hat{B}_o + \hat{B}_o^*)$ big enough such that the (1,1) block of (21) is strictly negative, that is,

$$X_o A + A^* X_o + \hat{B}_o + \hat{B}_o^* + \bar{C}^* \bar{C} + \hat{D}_o^* \hat{D}_o < 0;$$

also set $\hat{C}_o = -B^*$, $\hat{A}_o = -(A + B\hat{D}_o)^* - \bar{C}^* \bar{C} Y_o - \hat{D}_o^* \hat{C}_o$ to satisfy (21), therefore satisfy (20a). We have thus found a solution to (20), this completes the proof. □

Note that the above theorem gives the optimal closed-loop norm $\|T_{z\omega}\|^2_{\mathcal{H}_2,opt} = Tr(B_\omega^* P_s B_\omega)$. However, the optimal controller is not available from Proposition 1. To see this, going back to the proof of Theorem 1, we can choose $\hat{D}_o$ arbitrarily, and $D_{Ko} = \hat{D}_o$ from (12), which is not the case for the optimal controller $u = -B^* P_s x$ in Lemma 2 with the fixed D_{Ko}. This is because $M_o N_o^* = I - X_o Y_o$ is becoming singular if γ approaches the optimum, resulting in the singularity of Π_1 in (10); consequently, the sufficiency of Proposition 1 will be destroyed due to the singular congruence transformation matrix diag(Π_1, I).

Among the family of all the feasible controllers, there always exists a sequence of stabilizing controllers convergent to the optimal static controller, as stated in the following theorem.

Theorem 2. *Assuming* (A, B) *stabilizable and* $(\bar{C}, A)$ *detectable, for any non-increasing sequence* $\{\gamma^{(i)}\}$ *convergent to* $Tr(B_\omega^* P_s B_\omega)$*, there exists a sequence of controllers satisfying* (20) *convergent to the optimal controller in Lemma 2.*

Proof. If γ converges to $Tr(B_\omega^* P_s B_\omega)$, we choose the same parameters as those in the above proof (sufficiency part) of Theorem 1, that is, $Y_o \to P_s^{-1}$, $Q_o = B_\omega^* Y_o^{-1} B_\omega + \epsilon_1 I$ with small enough $\epsilon_1 > 0$, $X_o = (1+\epsilon_2) Y_o^{-1}$ with small enough $\epsilon_2 > 0$, $\hat{D}_o = -B^* Y_o^{-1}$, $-(\hat{B}_o + \hat{B}_o^*)$ big enough, $\hat{C}_o = -B^*$, $\hat{A}_o = -(A + B\hat{D}_o)^* - \bar{C}^* \bar{C} Y_o - \hat{D}_o^* \hat{C}_o$.

It has already been shown in the above proof of Theorem 1 that those parameters satisfy (20). Now we show that the resulting controller (12) converges to the optimal controller. Indeed, let $N_o = I, M_o = -\epsilon_2 I$ satisfying (13); from (12) we have

$$D_{Ko} = \hat{D}_o = -B^* Y_o^{-1}, \quad C_{Ko} = (\hat{C}_o - D_{Ko} Y_o) N_o^{-*} = 0.$$

Therefore the resulting controller $u = -B^* Y_o^{-1} x$ is static, which is convergent to the optimal controller $u = -B^* P_s x$. □

3.2 Output feedback

Now consider the output-feedback case, linear quadratic regulator problem with the realization (16), that is, $\omega = \begin{bmatrix} d \\ n \end{bmatrix}$, $D_\omega = [0\ I]$, $C_z = \begin{bmatrix} \bar{C} \\ 0 \end{bmatrix}$, $D_z = \begin{bmatrix} 0 \\ I \end{bmatrix}$, $D_{z\omega} = 0$, $B_\omega = [\bar{B}\ 0]$ in (1). By substituting them into (3) and (11), we state the following results.

Proposition 2. *There exists a controller K_c that renders $\|T_{z\omega}\|^2_{\mathcal{H}_2} < \gamma$ if and only if there exist X_c, Y_c, $\hat{A}_c$, $\hat{B}_c$, $\hat{C}_c$, $\hat{D}_c$, Q_c that satisfy*

$$\begin{bmatrix} He\{AX_c + B\hat{C}_c\} & \hat{A}_c^* + A & (\bar{B} \;\; 0) \\ \star & He\{Y_cA + \hat{B}_cC\} & (Y_c\bar{B} \;\; \hat{B}_c) \\ \star & \star & -I \end{bmatrix} < 0, \tag{26a}$$

$$\begin{bmatrix} X_c & I & (X_c\bar{C}^* \;\; \hat{C}_c^*) \\ \star & Y_c & (\bar{C}^* \;\; 0) \\ \star & \star & Q_c \end{bmatrix} > 0, \tag{26b}$$

$$Tr(Q_c) < \gamma, \quad \hat{D}_c = 0. \tag{26c}$$

Proposition 3. *There exists a controller K_o that renders $\|T_{z\omega}\|^2_{\mathcal{H}_2} < \gamma$ if and only if there exist X_o, Y_o, $\hat{A}_o$, $\hat{B}_o$, $\hat{C}_o$, $\hat{D}_o$, Q_o that satisfy*

$$\begin{bmatrix} He\{X_oA + \hat{B}_oC\} & \hat{A}_o + A^* & (\bar{C}^* \;\; 0) \\ \star & He\{AY_o + B\hat{C}_o\} & (Y_o\bar{C}^* \;\; \hat{C}_o^*) \\ \star & \star & -I \end{bmatrix} < 0, \tag{27a}$$

$$\begin{bmatrix} X_o & I & (X_o\bar{B} \;\; \hat{B}_o) \\ \star & Y_o & (\bar{B} \;\; 0) \\ \star & \star & Q_o \end{bmatrix} > 0, \tag{27b}$$

$$Tr(Q_o) < \gamma, \quad \hat{D}_o = 0. \tag{27c}$$

The following theorems show that γ in (27) can be chosen arbitrarily close to the optimal norm, and the controller has a similar property as well.

Theorem 3. *Assuming (A, B) and $(A, \bar{B})$ stabilizable, (C, A) and $(\bar{C}, A)$ detectable,* (27) *is feasible if and only if $\gamma > \gamma_{opt} = Tr(\bar{B}^*P_s\bar{B}) + Tr(B^*P_sS_sP_sB)$, where P_s and S_s are the stabilizing solutions of* (17).

Theorem 4. *Assuming (A, B) and $(A, \bar{B})$ stabilizable, (C, A) and $(\bar{C}, A)$ detectable, for any non-increasing sequence $\{\gamma^{(i)}\}$ convergent to γ_{opt}, there exists a sequence of controllers satisfying* (27) *convergent to the optimal controller in Lemma 3.*

The proofs are given in the Appendix. Similar results also apply to Proposition 2 which is the dual of Proposition 3. So far we have provided a new proof for the optimal $\mathcal{H}_2$ control problem (optimal norm and optimal controller) in both state-feedback and output-feedback cases based on LMI approach. As seen above, Riccati inequalities, which are derived from the LMIs (20), (26) and (27), play an important role in the proofs. We will use this connection between these LMIs and Riccati inequalities in the next section for structured $\mathcal{H}_2$ control, that is, instead of imposing structures on the LMIs (20), (26) and (27), we directly apply structures to the Riccati inequalities, leading to lower-order LMIs and upper bounds as well.

4 Structured $\mathcal{H}_2$ control via Riccati inequality

We consider a system Σ composed of N interconnected subsystems, where each subsystem Σ_i is assumed to have the following state space description:

$$\dot{x}_i = A_{ii}x_i + B_{\omega i}\omega_i + B_iu_i + \sum_{j\neq i} A_{ij}x_j. \tag{28}$$

It is assumed each subsystem Σ_i has a local control input u_i and a local disturbance ω_i, which is quite common in the practical networked systems. As proposed in [3], the system (28) may have some predefined structure $\mathcal{S}$ within the states which is defined as follows.

Definition 1 (The structure of an interconnected system)**.** *Given a system Σ with N subsystems of the form* (28)*, the structure of Σ, denoted by $\mathcal{S}$, is defined by an $N \times N$ symbolic matrix in the following way:*

(i). $\mathcal{S}_{ij} = \star$, if $i = j$ or $A_{ij} \neq 0$;

(ii). $\mathcal{S}_{ij} = 0$, otherwise.

It is obvious in the above definition that $\mathcal{S}_{ij} = 0$ indicates $A_{ij} = 0$, i.e., no information is sent from Σ_j to Σ_i. The following 4×4 structure matrices characterize four simple cases:

$$\begin{aligned} \mathcal{S}^{(1)} &= \begin{pmatrix} \star & 0 & 0 & 0 \\ 0 & \star & 0 & 0 \\ 0 & 0 & \star & 0 \\ 0 & 0 & 0 & \star \end{pmatrix}, \quad \mathcal{S}^{(2)} = \begin{pmatrix} \star & \star & 0 & 0 \\ \star & \star & \star & 0 \\ 0 & \star & \star & \star \\ 0 & 0 & \star & \star \end{pmatrix}, \\ \mathcal{S}^{(3)} &= \begin{pmatrix} \star & \star & \star & \star \\ 0 & \star & \star & \star \\ 0 & 0 & \star & \star \\ 0 & 0 & 0 & \star \end{pmatrix}, \quad \mathcal{S}^{(4)} = \begin{pmatrix} \star & 0 & 0 & 0 \\ \star & \star & 0 & 0 \\ \star & \star & \star & 0 \\ \star & \star & \star & \star \end{pmatrix}. \end{aligned} \tag{29}$$

In (29), $\mathcal{S}^{(1)}$ represents the decentralized case; $\mathcal{S}^{(2)}$ represents the localized case where each subsystem Σ_i only receives information from its direct preceding and succeeding neighbors; $\mathcal{S}^{(3)}$ shows the situation where Σ_i receives information from all of its preceding neighbors; $\mathcal{S}^{(4)}$ indicates Σ_i receives information from all of its succeeding neighbors.

We will associate structures with matrices in the next definition.

Definition 2. *Given a matrix M with a predefined $N \times N$ partition: $M_{ij} \in R^{m_i \times n_j}, i, j \in [1, \cdots, N]$. We say that M satisfies a structure $\mathcal{S}$ if $M_{ij} = 0$ whenever $\mathcal{S}_{ij} = 0$. This relation is denoted by $M \in \mathcal{S}(m_1, \cdots, m_N, n_1, \cdots, n_N)$, or shortly $M \in \mathcal{S}$ when no confusion arises.*

Associated with a structure $\mathcal{S}$, the decentralized structure $\mathcal{S}_D$ is defined next.

Definition 3. *Given a structure $\mathcal{S}$, the decentralized structure $\mathcal{S}_D$ associated to it is defined by an $N \times N$ symbolic matrix in the following way:*

(i). $\mathcal{S}_{D_{ij}} = \star$, if $i = j$;

(ii). $\mathcal{S}_{D_{ij}} = 0$, otherwise.

Remark 1. *Decentralized structure $\mathcal{S}_D$ is actually a set of block diagonal matrices with conformal dimension to $\mathcal{S}$, such as $\mathcal{S}^{(1)}$ in* (29)*. For example, for a given partitioned structure $\mathcal{S}(m_1, \cdots, m_N, n_1, \cdots, n_N)$, matrices X, Y, Z are said to satisfy the associated decentralized structure $\mathcal{S}_D$ if $X = \mathrm{diag}(X_1, \ldots, X_N)$, $Y = \mathrm{diag}(Y_1, \ldots, Y_N)$, $Z = \mathrm{diag}(Z_1, \ldots, Z_N)$ where $X_i \in R^{m_i \times q_i}$, $Y_i \in R^{p_i \times n_i}$, $Z_i \in R^{p_i \times q_i}$ for all $i \in [1, \cdots, N]$, for some appropriate dimensions $p_1, \cdots, p_N, q_1, \cdots, q_N$.*

Given these definitions, we are ready to describe the system (28) as

$$A = [A_{ij}]_{i,j=1}^N \in \mathcal{S}, \quad B_\omega = \mathrm{diag}(B_{\omega 1}, \ldots, B_{\omega N}) \in \mathcal{S}_D, \quad B = \mathrm{diag}(B_1, \ldots, B_N) \in \mathcal{S}_D.$$

For more details and discussion, the readers are referred to [3].

In the following subsections, similar to the system (28), the systems under consideration satisfy the following assumptions.

Assumption 1. *Given a preexisting structure $\mathcal{S}$, we assume:*

(i). In the system (14), $A \in \mathcal{S}$, $B, B_\omega, \bar{C} \in \mathcal{S}_D$.

(ii). In the system (16), $A \in \mathcal{S}$, $B, \bar{B}, C, \bar{C} \in \mathcal{S}_D$.

Our objective is to develop some structured controllers which stabilize the system (14) or (16). Here we refer to "*structured controller*" as stabilizing controller inheriting the same structure of the original system, i.e. $A_K \in \mathcal{S}$, $B_K, C_K, D_K \in \mathcal{S}_D$.

One way to do this is proposed in [3], that is, to impose structures directly on the variables in Propositions 1, 2-3, with $\hat{A} \in \mathcal{S}$ and $\hat{B}, \hat{C}, \hat{D}, X, Y \in \mathcal{S}_D$. From (4) and (12), the controller matrices are then enforced such that $A_K \in \mathcal{S}$ and $B_K, C_K, D_K \in \mathcal{S}_D$. The resulting controller thus inherits the same structure as the original plant. We refer to this as *structured LMI method* and will use this terminology throughout this section. There are however other alternative methods to preserve the structure. For example, based on the relationship between Propositions 1, 2-3 and Riccati equations as derived in the last section, alternatively, we could work on the Riccati inequalities with structure constraints, which is referred to as *structured Riccati method*. Indeed, without structure constraints, some redundant variables can be eliminated, such as $X_o, \hat{A}_o, \hat{B}_o, \hat{C}_o, \hat{D}_o$ in Proposition 1, leading to lower dimensional LMIs. This is also true for structured cases, as to be further discussed in the following subsections, due to the special structure of the system under consideration; see Assumption 1. In the presence of structure on the Riccati inequalities, taking (18) for example, a possible heuristic is to impose block-diagonal structure constraints on the variable Y in (19). In a sense, we are seeking the "most stabilizing" solution (or the "most optimal" controller in the following sections) consistent with the structure constraints. Similar idea has already been pursued in the literature on model reduction, such as in [25] for a plant-controller interconnection, in [5] for multi-dimensional systems, and in [11] for coprime factor methods. Such a block-diagonal structure on Lyapunov variables or Riccati variables automatically forces the resulting system to respect the subsystem boundaries, and thus maintain a topological association.

In the state-feedback case, it is shown that a static decentralized controller is obtained. In the output-feedback case, we derive three structured controllers, each with an explicit bound on the resulting $\mathcal{H}_2$ norm.

4.1 State feedback

We have seen in Lemma 2 that the optimal state-feedback controller relies on the Riccati equation (15). In order to obtain structured controller, one heuristic is to impose the structure $\mathcal{S}_D$ on the Riccati inequality

$$A^*P + PA + \bar{C}^*\bar{C} - PBB^*P < 0,$$

and minimize $Tr(B_\omega^* P B_\omega)$ which are equivalent to the LMIs in the following result with $Y = P^{-1}$.

Theorem 5. *Given the system (14) under Assumption 1 and* $\gamma > 0$*, there exists a controller* K *that internally stabilizes the closed-loop system (2) and satisfies* $\|T_{z\omega}\|_{\mathcal{H}_2}^2 < \gamma$ *if and only if there exist* Y *and* Q *that satisfy*

$$\begin{bmatrix} AY + YA^* - BB^* & Y\bar{C}^* \\ \star & -I \end{bmatrix} < 0, \tag{30a}$$

$$\begin{bmatrix} Y & B_\omega \\ \star & Q \end{bmatrix} > 0, \tag{30b}$$

$$Tr(Q) < \gamma. \tag{30c}$$

Moreover, if there exist $Y \in \mathcal{S}_D$ *and* Q *satisfying* (30)*, one structured* $\mathcal{H}_2$ *controller is given by* $u = -B^*Y^{-1}x$ *with* $\|T_{zw}\|_{\mathcal{H}_2}^2 < Tr(B_\omega^* Y^{-1} B_\omega)$.

Proof. Regardless of the subscript, (30a) is equivalent to (22), and (30b) is equivalent to (25). We will prove that the feasibility of (20a) and (20b) in variables $X_o, Y_o, \hat{A}_o, \hat{B}_o, \hat{C}_o, \hat{D}_o, Q_o$ is equivalent to the feasibility of (22) and (25) in fewer variables Y_o, Q_o respectively.

The fact that (20a) implies (22), and (20b) implies (25) is already shown in the proof of Theorem 1. To see the feasibility of (22) implies that of (20a), set $\hat{D}_o$ arbitrarily and $-(\hat{B}_o + \hat{B}_o^*)$ big enough such that the (1,1) block of (21) strictly negative, also set $\hat{C}_o = -B^*$, $\hat{A}_o = -(A + B\hat{D}_o)^* - \bar{C}^*\bar{C}Y_o - \hat{D}_o^*\hat{C}_o$ to satisfy (21), therefore satisfy (20a). To see the feasibility of (25) implies that of (20b), set $X_o = (1+\epsilon_2)Y_o^{-1}$ with small enough $\epsilon_2 > 0$ to satisfy (24), and then satisfy (20b). Therefore the feasibility of (30) is equivalent to that of (20), which proves the first part.

The second part follows similar lines to the proof of Theorem 2. □

Actually we get a decentralized controller here. Note that (30) and (20) are equivalent regarding to the feasibility, either in the absence of structure or with structure constraint. By eliminating the redundant variables $X_o, \hat{A}_o, \hat{B}_o, \hat{C}_o, \hat{D}_o$ in (20), the new result does not bring any conservatism in performance. Moreover, if the structure is directly imposed on (20), in terms of controller reconstruction (12), we would encounter numerical difficulty in getting such a decentralized controller because, as claimed before, $I - X_oY_o$ is becoming singular when γ is near the optimum. Although a remedy is proposed in [18] to include some additional LMI and variable to avoid such difficulty, it will increase the size of LMIs and result in the deviation from the optimal solution as well.

4.2 Output feedback

We follow the same heuristic to relax the Riccati equations (17) to the following inequalities,

$$A^*P + PA + \bar{C}^*\bar{C} - PBB^*P < 0, \tag{31a}$$

$$AS + SA^* + \bar{B}\bar{B}^* - SC^*CS < 0, \tag{31b}$$

and denote the left-hand side of (31) by R_o, R_c, such that

$$R_o = A^*P + PA + \bar{C}^*\bar{C} - PBB^*P, \tag{32a}$$

$$R_c = AS + SA^* + \bar{B}\bar{B}^* - SC^*CS. \tag{32b}$$

We will discuss different structures in the following subsections, where the corresponding controllers are derived.

4.2.1 General structure

Assuming A has a preexisting structure $\mathcal{S}$, under Assumption 1, the controller derived below will inherit the same structure as the plant, which is applicable to an arbitrary structure $\mathcal{S}$.

Theorem 6. *Given the system (16) under Assumption 1, if there exist $P > 0$ in $\mathcal{S}_D$ and $S > 0$ in $\mathcal{S}_D$ satisfying* (31)*, one structured controller K_{RI} that internally stabilizes the closed-loop system (2) is given by*

$$K_{RI} \begin{cases} \dot{\varsigma} = (A + BF + LC)\varsigma - Ly \\ u = F\varsigma \end{cases} \tag{33}$$

with $\|T_{z\omega}\|^2_{\mathcal{H}_2} \le \min\{\gamma_o, \gamma_c\}$*, where*

$$\begin{aligned} L = -SC^*,\ F = -B^*P, \\ \gamma_o = Tr(\bar{B}^*P\bar{B}) + Tr(B^*PSPB) + Tr(W_{os}R_c), \\ \gamma_c = Tr(\bar{C}S\bar{C}^*) + Tr(CSPSC^*) + Tr(W_{cs}R_o), \end{aligned}$$

and $W_{os} \ge 0, W_{cs} \ge 0$ *satisfy*

$$W_{os}(A - SC^*C) + (A - SC^*C)^*W_{os} + PBB^*P = 0, \tag{34a}$$

$$(A - BB^*P)W_{cs} + W_{cs}(A - BB^*P)^* + SC^*CS = 0. \tag{34b}$$

Proof. Let $W_o > 0$ be any positive definite solution of

$$W_o(A - SC^*C) + (A - SC^*C)^*W_o + PBB^*P < 0, \tag{35}$$

which is feasible, guaranteed by the fact that $A - SC^*C$ is Hurwitz from Lemma 4. In a similar fashion to Theorem 3, it can be proved that (27) will admit the following parameters:

$$\begin{aligned} X_o = P + W_o,\ \ Y_o = P^{-1},\ \ M_o = -W_o, \\ N_o = P^{-1},\ \ \hat{A}_o = P(A - BB^*P)P^{-1}, \\ \hat{B}_o = -W_oSC^*,\ \ \hat{C}_o = -B^*,\ \hat{D}_o = 0, \end{aligned} \tag{36}$$

which lead to the controller K_{RI} in (33) with

$$\|T_{z\omega}\|^2_{\mathcal{H}_2} \le Tr\{\bar{B}^*(P + W_o)\bar{B}\} + Tr(CSW_oSC^*).$$

Note that this holds for every $W_o > 0$ satisfying (35); thus the least W_o will result in a better upper bound. Let $W_{os} \ge 0$ satisfy (34a), then

$$\|T_{z\omega}\|^2_{\mathcal{H}_2} \le Tr\{\bar{B}^*(P + W_{os})\bar{B}\} + Tr(CSW_{os}SC^*).$$

Right multiplying both sides of equation (34a) with S and taking trace on both sides with the identity $trace(AB) = trace(BA)$, we have

$$\begin{aligned} &Tr(W_{os}AS + W_{os}SA^* - W_{os}SC^*CS) \\ &\qquad - Tr(CSW_{os}SC^*) + Tr(B^*PSPB) = 0. \end{aligned}$$

Use equation (32b) to get the following

$$Tr(W_{os}R_c - W_{os}\bar{B}\bar{B}^*) - Tr(CSW_{os}SC^*) + Tr(B^*PSPB) = 0,$$

from which we obtain $Tr(\bar{B}^*W_{os}\bar{B}) + Tr(CSW_{os}SC^*) = Tr(B^*PSPB) + Tr(W_{os}R_c)$. Then

$$\begin{aligned} &Tr\{\bar{B}^*(P + W_{os})\bar{B}\} + Tr(CSW_{os}SC^*) \\ =&Tr(\bar{B}^*P\bar{B}) + Tr(B^*PSPB) + Tr(W_{os}R_c) = \gamma_o, \end{aligned} \tag{37}$$

leading to $\|T_{z\omega}\|^2_{\mathcal{H}_2} \le \gamma_o$.

Similarly, letting $W_c > 0$ be any positive definite solution of

$$(A - BB^*P)W_c + W_c(A - BB^*P)^* + SC^*CS < 0,$$

(26) admits the following parameters:

$$\begin{aligned} X_c = S + W_c, \quad Y_c = S^{-1}, \quad M_c = W_c, \\ N_c = -S^{-1}, \quad \hat{A}_c = S^{-1}(A - SC^*C)S, \\ \hat{B}_c = -C^*, \ \hat{C}_c = -B^*PW_c, \ \hat{D}_c = 0, \end{aligned}$$

which result in the same controller (33) and another upper bound γ_c. This completes the proof. □

Unfortunately, the two conditions (31) and (34) are not jointly convex in P, S, W_{os}, W_{cs}; in fact, they are a bilinear matrix inequality (BMI). One might think of using some standard approaches, such as coordinate decent, cone complementarity linearization, etc. to solve this BMI, however, for simplicity and convenience, here we use a three-step procedure to seek solutions for it:

- Use the heuristic in (19) to solve (31a), (31b) respectively to obtain $P, S \in \mathcal{S}_D$, such that the solution P and S are made as close as possible to the stabilizing solutions of the corresponding Riccati equations.
- Construct *structured controller* K_{RI} as in (33).
- Compute the bound, $\min\{\gamma_o, \gamma_c\}$, by solving (34).

The behavior of this heuristic method will be demonstrated in the examples in Section 5.

One question that arises is how this structured controller K_{RI} is comparable with the one from *structured LMI method* or its dual? If we denote the feasible controller set of *structured LMI method* and its dual by $\mathcal{K}_{S_c}$ and $\mathcal{K}_{S_o}$ respectively, we will have $K_{RI} \in \mathcal{K}_{S_c} \cap \mathcal{K}_{S_o}$. To see this, taking $K_{RI} \in \mathcal{K}_{S_o}$ for example, in the proof of Theorem 6, we can choose $0 < W_o = \beta S^{-1} \in \mathcal{S}_D$ for some big enough β satisfying (35), which is guaranteed by

$$S^{-1}(A - SC^*C) + (A - SC^*C)^* S^{-1} < 0 \tag{38}$$

from (31b); therefore the dual of *structured LMI method* admits (36), then $K_{RI} \in \mathcal{K}_{S_o}$. However, the particular parameters used in the proof of Theorem 6 to derive the bound γ_o lead to $X_o = P + W_o \notin \mathcal{S}_D$, thus do not fall into the scope of *structured LMI methods* in which $X_o \in \mathcal{S}_D$ is required. Consequently, if we denote the minimal γ achieved from *structured LMI method* and its dual by $\gamma^*_{S_c}$ and $\gamma^*_{S_o}$ respectively, we remark here there is no clear comparison between $\min\{\gamma^*_{S_c}, \gamma^*_{S_o}\}$ and $\min\{\gamma_o, \gamma_c\}$, or between the actual norm from the above method and LMI methods; either one could be better.

4.2.2 Symmetric structure

Symmetric structures, for example full matrices and $\mathcal{S}^{(2)}$ in (29), are very common in the actual networks, representing the situations where the state information flow between any two connected subsystems is bidirectional. The following two controllers are for such structures.

Theorem 7. *Given the system (16) under Assumption 1, if there exist $P > 0$ in $\mathcal{S}_D$ and $S > 0$ in $\mathcal{S}_D$ satisfying* (31)*, two structured controllers K_{ro} and K_{rc} that internally stabilize the closed-loop system (2) are given by*

$$K_{ro} \begin{cases} \dot{\varsigma} = (A + BF + LC + \bar{W}_o^{-1} R_o)\varsigma - Ly \\ u = F\varsigma \end{cases}$$

$$K_{rc} \begin{cases} \dot{\varsigma} = (A + BF + LC + R_c \bar{W}_c^{-1})\varsigma - Ly \\ u = F\varsigma \end{cases}$$

with $\left\| T_{z\omega}^{(K_{ro})} \right\|_{\mathcal{H}_2}^2 \leq \gamma_{ro}$ *and* $\left\| T_{z\omega}^{(K_{rc})} \right\|_{\mathcal{H}_2}^2 \leq \gamma_{rc}$*, where*

$$\gamma_{ro} = Tr\{\bar{B}^*(P + \bar{W}_o)\bar{B}\} + Tr(CS\bar{W}_oSC^*),$$
$$\gamma_{rc} = Tr\{\bar{C}(S + \bar{W}_c)\bar{C}^*\} + Tr(B^*P\bar{W}_cPB),$$

and $\bar{W}_o > 0$ *in* $\mathcal{S}_D$*,* $\bar{W}_c > 0$ *in* $\mathcal{S}_D$ *are any strict positive definite solutions satisfying*

$$\bar{W}_o(A - SC^*C) + (A - SC^*C)^*\bar{W}_o + PBB^*P + R_o < 0, \tag{39}$$

$$(A - BB^*P)\bar{W}_c + \bar{W}_c(A - BB^*P)^* + SC^*CS + R_c < 0, \tag{40}$$

and R_o, R_c *are defined in* (32)*.*

Proof. Since K_{rc} is the dual of K_{ro}, we only prove the case of K_{ro}.

(39) is always feasible for $\bar{W}_o > 0$ in $\mathcal{S}_D$ guaranteed by the fact of (38). It is easy to check that (27) admits the following parameters:

$$\begin{array}{c} X_o = P + \bar{W}_o, \quad Y_o = P^{-1}, \quad M_o = -\bar{W}_o, \\ N_o = P^{-1}, \quad \hat{A}_o = -A^* - \bar{C}^*\bar{C}P^{-1}, \\ \hat{B}_o = -\bar{W}_oSC^*, \ \hat{C}_o = -B^*, \ \hat{D}_o = 0. \end{array} \tag{41}$$

This leads to the controller K_{ro} with norm bound γ_{ro}. □

Based on the above theorem, to obtain the best upper bound on the $\mathcal{H}_2$ norm, we could minimize $Tr(\bar{B}^*\bar{W}_o\bar{B})+Tr(CS\bar{W}_oSC^*)$ subject to (39) for controller K_{ro} and minimize $Tr(\bar{C}\bar{W}_c\bar{C}^*)+Tr(B^*P\bar{W}_cPB)$ subject to (40) for controller K_{rc}, which are actually SDP problems. The corresponding three-step algorithm can be developed similarly to that in Section 4.2.1, thus is omitted here.

As seen in Theorem 7, the additional symmetric terms R_c and R_o will only allow preserving the symmetric structure $\mathcal{S}$, provided that we impose the structure $\mathcal{S}_D$ on $\bar{W}_o$ in (39) and $\bar{W}_c$ in (40). According to the proof of Theorem 7, $X_o = P + \bar{W}_o$ ($X_c = S + \bar{W}_c$ for controller K_{rc}) are also in $\mathcal{S}_D$. This brings the fact that the two controllers from Theorem 7 are special cases of *structured LMI methods*, and consequently the corresponding norm bounds are always worse than those of *structured LMI methods*. If no structure constraint is imposed, they tend to be the optimal controller in Lemma 3. As claimed before, there is no quantitative evidence which demonstrates that one method would achieve better performance than another.

Remark 2. *If A is a full matrix, $\bar{W}_o$ in* (39) *and $\bar{W}_c$ in* (40) *need not to be restricted in $\mathcal{S}_D$ any more, then K_{rc} or K_{ro} in Theorem 7 would give better norm bound than K_{RI} in Theorem 6 because W_{cs} and W_{os} are one of the solutions of* (40) *and* (39) *respectively. In this particular case, although we can impose structure constraints directly on the variables of Propositions 2-3, it brings some conservatism; actually the controllers obtained from Theorem 7 do not fall into this category because of $X_o = P + \bar{W}_o \notin \mathcal{S}_D$ and $X_c = S + \bar{W}_c \notin \mathcal{S}_D$ in the proof of Theorem 7.*

5 Illustrative Example

5.1 Example 1

To better illustrate our approach, we provide a set of examples for the output-feedback problem. The following four cases are explored: (EX1) full matrix $A^{(1)}$, (EX2) localized $A^{(2)} \in \mathcal{S}^{(2)}$, (EX3) upper triangular

Table 1: $\mathcal{H}_2$ norm and norm bound

Method ↓	EX1		EX2		EX3		EX4	
	$\|T_{z\omega}\|_{\mathcal{H}_2}$	Bound	$\|T_{z\omega}\|_{\mathcal{H}_2}$	Bound	$\|T_{z\omega}\|_{\mathcal{H}_2}$	Bound	$\|T_{z\omega}\|_{\mathcal{H}_2}$	Bound
RIo	37.58	39.94	37.12	38.92	36.95	39.59	35.72	36.61
RIRo	37.20	44.61	36.77	40.75	-	-	-	-
RIRfo	37.58	38.26	-	-	-	-	-	-
So	37.65	43.59	36.68	40.30	37.14	42.46	35.59	37.04
RIc	37.58	42.39	37.12	39.84	36.95	41.46	35.72	37.56
RIRc	37.20	45.78	36.93	42.73	-	-	-	-
RIRfc	38.74	39.56	-	-	-	-	-	-
Sc	37.28	44.37	37.07	41.67	37.16	44.19	35.66	37.50

$A^{(3)} \in \mathcal{S}^{(3)}$ and (EX4) lower triangular $A^{(4)} \in \mathcal{S}^{(4)}$ in (29), where

$$A^{(1)} = \begin{pmatrix} 9 & 4 & -17 & -28 \\ -17 & -2 & 25 & 15 \\ 24 & 30 & 14 & 5 \\ 10 & 10 & -6 & 28 \end{pmatrix}, \quad A^{(2)} = \begin{pmatrix} 9 & 4 & 0 & 0 \\ -17 & -2 & 25 & 0 \\ 0 & 30 & 14 & 5 \\ 0 & 0 & -6 & 28 \end{pmatrix},$$
$$A^{(3)} = \begin{pmatrix} 9 & 4 & -17 & -28 \\ 0 & -2 & 25 & 15 \\ 0 & 0 & 14 & 5 \\ 0 & 0 & 0 & 28 \end{pmatrix}, \quad A^{(4)} = \begin{pmatrix} 9 & 0 & 0 & 0 \\ -17 & -2 & 0 & 0 \\ 24 & 30 & 14 & 0 \\ 10 & 10 & -6 & 28 \end{pmatrix}.$$

The system under consideration consists of four subsystems with dimension equal to one, and we use the following $B, \bar{B}, C, \bar{C}$ for all four cases,

$$B = \text{diag}(26, 16, 11, -23), \qquad \bar{B} = \text{diag}(4, 3, 20, 23),$$
$$C = \text{diag}(9, -19, -16, -22), \quad \bar{C} = \text{diag}(-11, 27, 4, -17).$$

Numerical solutions were found using the LMI control toolbox of Matlab. We compare the following methods: Riccati inequality approach without residue from Theorem 6, denoted by **RI** (We use **RIo**, **RIc** to represent the same approach with different norm bound γ_o, γ_c); Riccati inequality approach with residue from Theorem 7, denoted by **RIRo** and **RIRc** respectively; no structure constraint on $\bar{W}_o$ and $\bar{W}_c$ in Theorem 7, denoted by **RIRfo** and **RIRfc** respectively; LMI approach from Proposition 2 and Proposition 3 with structure constraints, denoted by **Sc** and **So** respectively.

Table 1 shows the $\mathcal{H}_2$ norm and norm bound achieved by each method in the four cases. Note that **RIRo** and **RIRc** are only applicable to EX1-EX2, and **RIRfo** and **RIRfc** are only applicable to EX1 (see Remark 2). We divide the methods into two groups: controllability and observability gramian based methods. We can not claim which group is better, either in terms of the norm bound or the actual norm, therefore comparison is within the group. For example, in the observability gramian based group (**RIo**, **RIRo**, **RIRfo**, **So**), from the results in Table 1, we confirm the following claims which are stated before:

(i). **RI** may give a better norm bound and an actual norm than **So** and **Sc**; see EX3.

(ii). **RIRfo** and **So** have a smaller norm bound than **RIRo** because **RIRo** is a special case of those two; see EX1.

Table 2: Random experiment: n=1, 1000 experiments

Method ↓	Better norm	Better bound	Number of variables	Average time
RIc, **RIo**	45	614	8	5.9312
Sc, **So**	955	386	26	9.1854

Table 3: Random experiment: n=2, 100 experiments

Method ↓	Better norm	Better bound	Number of variables	Average time
RIc, **RIo**	7	33	24	10.2053
Sc, **So**	93	67	80	22.6539

(iii). The bound of **RIRfo** is smaller than that of **RIo** as expected in Section 4.2; see EX1.

(iv). Although **RIRo** always has a worse norm bound than **So**, **RIRo** could give a better actual norm instead; see **RIRo** and **So** in EX1.

5.2 Example 2

In this example, a set of localized systems are randomly generated to test the performance of two approaches: *structured Riccati method* **RI** (**RIc** and **RIo**) and *structured LMI methods* **Sc**, **So**, as illustrated in Example 1. These random systems consist of 4 subsystems with the following structure:

$$\begin{gathered} A = [A_{ij}]_{i,j=1}^{4} \in \mathcal{S}^{(2)}, \\ B = \mathrm{diag}(B_1, \ldots, B_4), \quad \bar{B} = \mathrm{diag}(\bar{B}_1, \ldots, \bar{B}_4), \\ C = \mathrm{diag}(C_1, \ldots, C_4), \quad \bar{C} = \mathrm{diag}(\bar{C}_1, \ldots, \bar{C}_4), \\ A_{ij} \in R^{n\times n}, \quad B_i \in R^{n\times 1}, \quad \bar{B}_i \in R^{n\times 1}, \\ C_i \in R^{1\times n}, \quad \bar{C}_i \in R^{1\times n}, \quad i,j = 1,2,3,4, \end{gathered}$$

where $\mathcal{S}^{(2)}$ is defined in (29) and n is an integer to be chosen for different systems.

Given $n = 1, 2, 3$, corresponding random systems are generated. We compare the performance in terms of four criterions: *number of experiments with better norm*, *number of experiments with better bound*, *number of variables used in LMIs*, and *average computation time* (in second). The results given in Tables 2-4 show that, as mentioned before, our methods employ lower dimensional LMIs with less variables, and consume less computation time.

6 Conclusion

We gave a new proof for the optimal $\mathcal{H}_2$ control problem in the state-feedback and output-feedback cases, derived from the LMI approach. Based on this observation, the LMI relaxation of the Riccati equations was used to propose new structured $\mathcal{H}_2$ control algorithms, aimed at preserving topological structure of the plant states. A class of structured controllers with explicit bounds on the $\mathcal{H}_2$ norm are derived in this context.

Table 4: Random experiment: n=3, 10 experiments

Method ↓	Better norm	Better bound	Number of variables	Average time
RIc, **RIo**	1	3	48	16.1375
Sc, **So**	9	7	162	38.0547

Appendix

Proof of Theorem 3.

If (27) holds, by Schur complement, (27a) is equivalent to

$$\begin{bmatrix} \mathcal{N}_{11} & \hat{A}_o + A^* + \bar{C}^*\bar{C}Y_o \\ \star & \mathcal{N}_{22} \end{bmatrix} < 0, \tag{42}$$

where

$$\begin{aligned} \mathcal{N}_{11} &= X_oA + A^*X_o + \hat{B}_oC + (\hat{B}_oC)^* + \bar{C}^*\bar{C}, \\ \mathcal{N}_{22} &= AY_o + Y_oA^* + Y_o\bar{C}^*\bar{C}Y_o - BB^* + (B + \hat{C}_o^*)(B^* + \hat{C}_o). \end{aligned}$$

From the (2,2) block of (42), we have

$$AY_o + Y_oA^* + Y_o\bar{C}^*\bar{C}Y_o - BB^* < 0. \tag{43}$$

Then $Y_o^{-1} > P_s$ by Lemma 4.

By Schur complement, (27b) is equivalent to $X_o > 0$ and

$$\begin{bmatrix} Y_o - X_o^{-1} & (0 \quad -X_o^{-1}\hat{B}_o) \\ \star & Q_o - \begin{bmatrix} \bar{B}^*X_o\bar{B} & \bar{B}^*\hat{B}_o \\ \star & \hat{B}_o^*X_o^{-1}\hat{B}_o \end{bmatrix} \end{bmatrix} > 0. \tag{44}$$

Using Schur complement again with respect to the (1,1) block of (44), (44) is equivalent to $Y_o > X_o^{-1}$ and

$$\begin{aligned} Q_o &> \begin{bmatrix} \bar{B}^*X_o\bar{B} & \bar{B}^*\hat{B}_o \\ \star & \hat{B}_o^*[X_o^{-1} + X_o^{-1}(Y_o - X_o^{-1})^{-1}X_o^{-1}]\hat{B}_o \end{bmatrix} \\ &= \begin{bmatrix} \bar{B}^*X_o\bar{B} & \bar{B}^*\hat{B}_o \\ \star & \hat{B}_o^*(X_o - Y_o^{-1})^{-1}\hat{B}_o \end{bmatrix}. \end{aligned}$$

Then (27b) is equivalent to

$$X_o > Y_o^{-1} > 0 \text{ and } Q_o > \begin{bmatrix} \bar{B}^*X_o\bar{B} & \bar{B}^*\hat{B}_o \\ \star & \hat{B}_o^*(X_o - Y_o^{-1})^{-1}\hat{B}_o \end{bmatrix}. \tag{45}$$

Let $Y_o^{-1} = P_s + W_{o1}$, $X_o = Y_o^{-1} + W_{o2}$ for some $W_{o1} > 0, W_{o2} > 0$; define $W_o = W_{o1} + W_{o2}$, then $X_o = P_s + W_o$, and $N_o = (P_s + W_{o1})^{-1}, M_o = -W_{o2}$ satisfying (13); also $\hat{B}_o = M_oB_{Ko} = -W_{o2}B_{Ko}$ from (9).

Substituting above parameters into (45) and using (27c), we have

$$\gamma > Tr(Q_o) > Tr\{\bar{B}^*(P_s + W_o)\bar{B}\} + Tr(B_{Ko}^*W_{o2}B_{Ko}). \tag{46}$$

Also substitute them into the (1,1) block of (42),

$$W_oA + A^*W_o - W_{o2}B_{Ko}C - C^*B_{Ko}^*W_{o2} + A^*P_s + P_sA + \bar{C}^*\bar{C} < 0,$$

which by (17a) is same as

$$W_oA + A^*W_o - W_{o2}B_{Ko}C - C^*B_{Ko}^*W_{o2} + P_sBB^*P_s < 0. \tag{47}$$

Define the left-hand side of (47) to be $-V_o$ with $V_o > 0$, then

$$\begin{aligned} &W_{o2}(A - B_{Ko}C) + (A - B_{Ko}C)^*W_{o2} \\ &\quad + P_sBB^*P_s + W_{o1}A + A^*W_{o1} + V_o = 0. \end{aligned} \tag{48}$$

Perform congruence transformation with $S_s^{1/2}$ on (48), and take trace of both sides. By using the identity $trace(AB) = trace(BA)$, we have

$$\begin{aligned} &Tr(W_oAS_s + W_oS_sA^*) - Tr(CS_sW_{o2}B_{Ko} + B_{Ko}^*W_{o2}S_sC^*) \\ &\qquad + Tr(B^*P_sS_sP_sB) + Tr(S_s^{1/2}V_oS_s^{1/2}) = 0. \end{aligned}$$

Use (17b) to get the following

$$\begin{aligned} &Tr(W_oS_sC^*CS_s - W_o\bar{B}\bar{B}^*) - Tr(CS_sW_{o2}B_{Ko} + B_{Ko}^*W_{o2}S_sC^*) \\ &\qquad + Tr(B^*P_sS_sP_sB) + Tr(S_s^{1/2}V_oS_s^{1/2}) = 0, \end{aligned}$$

which is

$$\begin{aligned} &Tr(CS_sW_oS_sC^*) - Tr(CS_sW_{o2}B_{Ko} + B_{Ko}^*W_{o2}S_sC^*) \\ &\quad - Tr(\bar{B}^*W_o\bar{B}) + Tr(B^*P_sS_sP_sB) + Tr(S_s^{1/2}V_oS_s^{1/2}) = 0. \end{aligned}$$

By $W_o = W_{o1} + W_{o2}$, we have

$$\begin{aligned} &Tr(\bar{B}^*W_o\bar{B}) + Tr(B_{Ko}^*W_{o2}B_{Ko}) \\ =&Tr(B^*P_sS_sP_sB) + Tr(CS_sW_{o1}S_sC^*) + Tr(S_s^{1/2}V_oS_s^{1/2}) \\ &\qquad + Tr\{(S_sC^* - B_{Ko})^*W_{o2}(S_sC^* - B_{Ko})\}. \end{aligned} \tag{49}$$

Then substitute (49) into (46),

$$\begin{aligned} \gamma >&Tr(\bar{B}^*P_s\bar{B}) + Tr(B^*P_sS_sP_sB) + Tr(CS_sW_{o1}S_sC^*) \\ &+ Tr(S_s^{1/2}V_oS_s^{1/2}) + Tr\{(S_sC^* - B_{Ko})^*W_{o2}(S_sC^* - B_{Ko})\}. \end{aligned} \tag{50}$$

Therefore

$$\gamma > \gamma_{opt} = Tr(\bar{B}^*P_s\bar{B}) + Tr(B^*P_sS_sP_sB).$$

Now prove the sufficiency part. If given any $\gamma > \gamma_{opt}$, by Lemma 4, $W_{o1} = Y_o^{-1} - P_s > 0$ can be chosen arbitrarily small, where Y_o satisfies (43); set $B_{Ko} = S_sC^*$, $V_o > 0$ small enough, W_{o1} small enough such that (50) and $W_{o1}A+A^*W_{o1}+V_o > 0$ hold; then (48) admits some solution $W_{o2} > 0$ since $A-B_{Ko}C = A-S_sC^*C$ is Hurwitz, therefore (47) holds; let $X_o = Y_o^{-1} + W_{o2}$, $\hat{B}_o = -W_{o2}B_{Ko}$, $\hat{C}_o = -B^*$, $\hat{A}_o = -A^* - \bar{C}^*\bar{C}Y_o$; notice that the (1,2) and (2,1) blocks of (42) become zero, the (1,1) block becomes the left-hand side of (47), and the (2,2) block becomes the left-hand side of (43), then those parameters satisfy (42), thus satisfy (27a); we already show that (48) holds, then (49) holds; from (49) and (50), (45) and (46) are satisfied with some Q_o, then (27b) holds. Therefore (27) admits those parameters. This completes the proof. □

Proof of Theorem 4.

If γ converges to γ_{opt}, we choose the same parameters as those in the above proof (sufficiency part) of Theorem 3, that is, $Y_o \to P_s^{-1}$, $W_{o1} = Y_o^{-1} - P_s \to 0$, $V_o \to 0$, $B_{Ko} = S_s C^*$, $X_o = Y_o^{-1} + W_{o2}$, $\hat{B}_o = -W_{o2} B_{Ko}$, $\hat{C}_o = -B^*$ and $\hat{A}_o = -(A^* + \bar{C}^* \bar{C} Y_o)$.

It has already been shown in the above proof of Theorem 3 that those parameters satisfy (27). Now we show that the resulting controller (12) converges to the optimal controller. Indeed, letting $N_o = Y_o, M_o = -W_{o2}$ satisfying (13), and substituting $B_{Ko} = S_s C^*$, $\hat{C}_o = -B^*$, $\hat{A}_o = -A^* - \bar{C}^* \bar{C} Y_o$, $X_o = Y_o^{-1} + W_{o2}$ into (12), we have

$$
\begin{aligned}
D_{Ko} &= 0,\ C_{Ko} = -B^* Y_o^{-1}, \\
A_{Ko} &= W_{o2}^{-1}[A^* Y_o^{-1} + \bar{C}^* \bar{C} - Y_o^{-1} B B^* Y_o^{-1} + Y_o^{-1} A] \\
&\quad - S_s C^* C - B B^* Y_o^{-1} + A.
\end{aligned}
$$

Notice that by (48) $W_{o2} \to W_{o2}^{(*)} > 0$, where $W_{o2}^{(*)}$ satisfies

$$
W_{o2}^{(*)}(A - S_s C^* C) + (A - S_s C^* C)^* W_{o2}^{(*)} + P_s B B^* P_s = 0.
$$

Since $Y_o^{-1} \to P_s$, then $A^* Y_o^{-1} + \bar{C}^* \bar{C} - Y_o^{-1} B B^* Y_o^{-1} + Y_o^{-1} A \to 0$. Therefore $C_{Ko} \to -B^* P_s$ and $A_{Ko} \to A - B B^* P_s - S_s C^* C$. This completes the proof. $\square$

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