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EXPLICIT ANALYTICAL SOLUTIONS FOR THE AVERAGE RUN LENGTH OF CUSUM AND EWMA CHARTS

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Abstract

We use the Fredholm type integral equations method to derive explicit formulas for the Average Run Length (ARL) in some special cases. In particular, we derive a closed form representation for the ARL of Cumulative Sum (CUSUM) chart when the random observations have hyperexponential distribution. For Exponentially Weighed Moving Average (EWMA) chart we solve the corresponding ARL integral equation when the observations have the Laplace distribution. The explicit formulas obviously takes less computational time than the other methods, e.g. Monte Carlo simulation or numerical integration.

1. Introduction

Cumulative Sum (CUSUM) chart was first proposed by Page (1954) in quality control in order to detect a small shift in the mean of a production process as

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soon as it occurs, as an extension to Shewhart's (1931) charts.

In practice, CUSUM charts are widely used in statistical control, to detect changes in the characteristics of a stochastic system e.g., mean or variance, see Brodsky and Darkhovsky [1], or Basseville and Nikiforov [2] for an introduction to CUSUM charts and their applications.

The recursive equation for CUSUM chart designed to detect an increase in the mean of observed sequence of nonnegative independent and identically distributed (i.i.d.) random variables ξ_n, is defined as

$$X_n = (X_{n-1} + \xi_n - a)^+, \ n = 1, 2, ..., \quad X_0 = x, \tag{1}$$

where $y^+ = \max(0, y)$. Several cases which lead to this representation are presented in [1], [2], and [4]. Denote by

$$\tau_b = \inf\{k \geq 0 : X_k \geq b\} \tag{2}$$

the first exit time of a random sequence X_n over the positive level b, with $b > x - a$ (otherwise $\tau_b = 1$).

Let $\mathbb{P}_x$ and $\mathbb{E}_x$, denotes the probability measure and the induced expectation corresponding to the initial value $X_0 = x \geq 0$.

The problem studied in here is to find the Average Run Length (ARL) of the CUSUM procedure defined as a function $j(x) = \mathbb{E}_x\tau_b$.

The Exponentially Weighted Moving Average (EWMA) control chart was first proposed by Roberts (1959) in quality control, in order to detect a shift in the mean of a process. We consider here the EWMA as an AR(1), i.e., an Autoregressive Process of order one which is a simple generalization of a random walk (see [10]). We consider the AR(1) process described by the equation

$$X_t = \rho X_{t-1} + \eta_t \tag{3}$$

where $\rho \in (0, 1)$ and $\{\eta_t\}_{t\geq 1}$ is a sequence of independent identically distributed random variables, with $X_0 = x$. As a particularly case of the AR(1) one can obtain the EWMA chart in a standard form, by setting in Eq. (3), $\rho = 1 - \lambda$ and $\eta_t = \lambda\varepsilon_t$, with $\lambda \in (0, 1)$.

The problem is to find the expectation of the stopping time

$$\nu_b = \inf\{t \geq 0 : X_t \geq b\}, \ b > x. \tag{4}$$

One could use Monte Carlo simulation or numerical integration for both charts CUSUM and EWMA to find the ARL, but it is always desirable to have a closed form analytical solution to check the accuracy of the results.

The paper is organized as follow. In Section we obtain the ARL for CUSUM chart in the case of hyperexponential distribution (see Theorem 2.1).

It is well-known that any completely monotone probability density function can be approximated, by hyperexponential distributions, sometime also called mixture of exponentials. For example, Pareto and Weibull are completely monotone distributions, and so they can be approximated by mixture of exponentials (e.g., see [7] and [8]). Therefore, one can use the closed form representation given in Theorem 2.1 as an approximation for the cases when the random variables ξ_n in Eq. (1) have Pareto or Weibull distributions.

In Section we discuss a closed form representation for $\mathbb{E}_x\nu_b$ and present the results in Theorem 2.2 for the case when the random variables η_t in Eq. (3) has Laplace distribution. Our result generalize the result of Larralde [9], who use a different technique and obtain $\mathbb{E}_x\nu_b$ only in the particularly case $x = 0$ and $b = 0$. In Section we present several numerical examples.

2. The ARL Integral Equations for CUSUM and EWMA Procedures

It can be shown, see [5] and [6], that the ARL of the CUSUM chart, $j(x) = \mathbb{E}_x\tau_b$, is a solution of the integral equation

$$j(x) = 1 + \mathbb{E}_x\{I(0 < X_1 < b)j(X_1)\} + \mathbb{P}_x\{X_1 = 0\}j(0). \tag{5}$$

If ξ_n are continuous distributed i.i.d random variables with a given d.f. $F(x)$, and density $f(x) = \frac{dF(x)}{dx}$, then we can write equation Eq. (5) as a Fredholm-type integral equation of the form

$$j(x) = 1 + j(0)F(a - x) + \int_0^b j(y)f(y + a - x)dy. \tag{6}$$

When ξ_n are continuous i.i.d random variables with exponential distribution, then $F(x) = 1 - \exp(-x), x \geq 0$, Eq. (6) becomes

$$j(x) = 1 + \int_0^b j(y)e^{x-a-y}dy + (1 - e^{-(a-x)^+})j(0). \tag{7}$$

and it's solution for $x \in [0, a]$ has the form (see e.g.[6])

$$j(x) = e^b(1 + e^a - b) - e^x, \quad \text{for} \quad x \in [0, a] \tag{8}$$

2.1 The ARL Fredholm Integral Equation for CUSUM chart with hyperexponential distributions

In this section we consider the case when the observed random variables ξ_n have hyperexponential distribution with the d.f.

$$F(x) = 1 - \sum_{i=1}^{n} \lambda_i e^{-\alpha_i x} \tag{9}$$

with $\lambda_i \in \mathbb{R}_+$ subject to the condition that $\sum_{i=1}^{n} \lambda_i e^{-\alpha_i x}$ is a distribution function on $\mathbb{R}_+$, that is $\sum_{i=1}^{n} \lambda_i = 1$. Now, the Fredholm integral equation Eq. (6) can be written as follows

$$j(x) = 1 + \int_0^b j(y) \sum_{i=1}^{n} \lambda_i \alpha_i e^{\alpha_i(x-a-y)} dy + (1 - \sum_{i=1}^{n} \lambda_i e^{-\alpha_i(a-x)}) j(0), \quad 0 \le x \le a \tag{10}$$

Theorem 1.1 *The solution of the integral equation* Eq. 10 *is*

$$j(x) = 1 + j(0) + \sum_{i=1}^{n} [d_i - \lambda_i j(0)] e^{\alpha_i(x-a)}, \quad \textit{for} \quad x \in [0, a], \ b < a \tag{11}$$

where

$$j(0) = \frac{1 + \sum_{i=1}^{n} d_i e^{-\alpha_i a}}{\sum_{i=1}^{n} \lambda_i e^{-\alpha_i a}} \tag{12}$$

and the coefficients d_i, $i = 1, \ldots, n$, are solutions of the linear system

$$\Delta\, \mathbf{d} = \mathbf{M} \tag{13}$$

where

$$\mathbf{d} = [d_1, d_2, \ldots, d_n]^T,$$

Δ is the the non-singular matrix

$$\begin{bmatrix} D - M_{1,n} e^{-\alpha_1 a} - \lambda_1 \alpha_1 b e^{-\alpha_1 a} D & -M_{1,n} e^{-\alpha_2 a} - \lambda_1 \alpha_1 e^{-\alpha_2 a} A_{2,1} D & \cdots & -M_{1,n} e^{-\alpha_n a} - \lambda_1 \alpha_1 e^{-\alpha_n a} A_{n,1} D \\ -M_{2,n} e^{-\alpha_1 a} - \lambda_2 \alpha_2 e^{-\alpha_1 a} A_{1,2} D & D - M_{2,n} e^{-\alpha_2 a} - \lambda_2 \alpha_2 b e^{-\alpha_2 a} D & \cdots & -M_{2,n} e^{-\alpha_n a} - \lambda_2 \alpha_2 e^{-\alpha_n a} A_{n,2} D \\ \vdots & \vdots & \vdots & \vdots \\ -M_{n,n} e^{-\alpha_1 a} - \lambda_n \alpha_n e^{-\alpha_1 a} A_{1,n} D & -M_{n,n} e^{-\alpha_n a} - \lambda_n \alpha_n e^{-\alpha_2 a} A_{2,n} D & \cdots & D - M_{n,n} e^{-\alpha_n a} - \lambda_n \alpha_n b e^{-\alpha_n a} D \end{bmatrix} \tag{14}$$

and

$$\mathbf{M} = \left[\lambda_1 (1 - e^{-b\alpha_1}) D + M_{1,n}, \lambda_2 (1 - e^{-b\alpha_2}) D + M_{2,n}, \ldots \lambda_n (1 - e^{-b\alpha_n}) D + M_{n,n}\right]^T$$

with $D = \sum_{i=1}^{n} \lambda_i e^{-\alpha_i a}$,

$$M_{k,n} = (1 - e^{-b\alpha_k}) \lambda_k - \lambda_k \alpha_k \sum_{i=1}^{n} \lambda_i e^{-\alpha_i a} A_{i,k} \tag{15}$$

and

$$A_{i,k} = \begin{cases} \frac{e^{(\alpha_i - \alpha_k)b} - 1}{(\alpha_i - \alpha_k)}, & i \ne k \\ b, & i = k \end{cases}$$

Proof. For $0 \le x \le a$ and $b < a$ Eq. (10) can be written as

$$j(x) = 1 + \sum_{i=1}^{n} d_i e^{\alpha_i(x-a)} + (1 - \sum_{i=1}^{n} \lambda_i e^{-\alpha_i(a-x)}) j(0), \quad 0 \le x \le a \tag{16}$$

where

$$d_i = \int_0^b j(y) \lambda_i \alpha_i e^{-\alpha_i y} dy, \quad i = 1, \ldots, n. \tag{17}$$

or

$$j(x) = 1 + j(0) + \sum_{i=1}^{n} (d_i - \lambda_i j(0)) e^{\alpha_i(x-a)}, \quad 0 \le x \le a. \tag{18}$$

From Eq. (16) at $x = 0$ we obtain $j(0)$ given by Eq. (12)

To evaluate the coefficients d_k for $k = 1, 2, \ldots, n$ we substitute Eq. (18) in Eq. (17) and obtain

$$d_k = [1 + j(0)] \lambda_k \left(1 - e^{-b\alpha_k}\right) + \lambda_k \alpha_k \sum_{i=1}^{n} d_i e^{-\alpha_i a} A_{i,k} - j(0) \lambda_k \alpha_k \sum_{i=1}^{n} \lambda_i e^{-\alpha_i a} A_{i,k} \tag{19}$$

for $k = 1, 2, \ldots, n$, where

$$A_{i,k} = \int_0^b e^{(\alpha_i - \alpha_k) y} dy = \begin{cases} \frac{e^{(\alpha_i - \alpha_k)b} - 1}{(\alpha_i - \alpha_k)}, & i \ne k \\ b, & i = k \end{cases}$$

Inserting the expression for $j(0)$ given by Eq. (16) in Eq. (19) we obtain a linear system of n equations with d_k unknowns $k = 1, 2, \ldots, n$:

$$\begin{aligned} d_k \sum_{i=1}^{n} \lambda_i e^{-\alpha_i a} &= \lambda_k \left(1 - e^{-b\alpha_k}\right) \sum_{i=1}^{n} \lambda_i e^{-\alpha_i a} + \left(1 + \sum_{i=1}^{n} d_i e^{-\alpha_i a}\right) \Big[\left(1 - e^{-b\alpha_k}\right) \lambda_k \\ &\qquad - \lambda_k \alpha_k \sum_{i=1}^{n} \lambda_i e^{-\alpha_i a} A_{i,k}\Big] \\ &\quad + \lambda_k \alpha_k \left(\sum_{i=1}^{n} \lambda_i e^{-\alpha_i a}\right) \left(\sum_{i=1}^{n} d_i e^{-\alpha_i a} A_{i,k}\right) = \lambda_k \left(1 - e^{-b\alpha_k}\right) \sum_{i=1}^{n} \lambda_i e^{-\alpha_i a} \\ &\quad + \left(1 + \sum_{i=1}^{n} d_i e^{-\alpha_i a}\right) M_{k,n} + \lambda_k \alpha_k \left(\sum_{i=1}^{n} \lambda_i e^{-\alpha_i a}\right) \left(\sum_{i=1}^{n} d_i e^{-\alpha_i a} A_{i,k}\right) \end{aligned}$$

$k = 1, 2, \ldots, n$

where

$$M_{k,n} = \left(1 - e^{-b\alpha_k}\right)\lambda_k - \lambda_k\alpha_k \sum_{i=1}^{n} \lambda_i e^{-\alpha_i a} A_{i,k}$$

or

$$\begin{cases} d_1\left(D - M_{1,n}e^{-\alpha_1 a} - \lambda_1\alpha_1 b e^{-\alpha_1 a} D\right) + d_2\left(-M_{1,n}e^{-\alpha_2 a} - \lambda_1\alpha_1 e^{-\alpha_2 a} A_{2,1} D\right) + \dots \\ +d_n\left(-M_{1,n}e^{-\alpha_n a} - \lambda_1\alpha_1 e^{-\alpha_n a} A_{n,1} D\right) = \lambda_1\left(1 - e^{-b\alpha_1}\right) D + M_{1,n} \\ d_1\left(-M_{2,n}e^{-\alpha_1 a} - \lambda_2\alpha_2 e^{-\alpha_1 a} A_{1,2} D\right) + d_2\left(D - M_{2,n}e^{-\alpha_2 a} - \lambda_2\alpha_2 b e^{-\alpha_2 a} D\right) + \dots \\ +d_n\left(-M_{2,n}e^{-\alpha_n a} - \lambda_2\alpha_2 e^{-\alpha_n a} A_{n,2} D\right) = \lambda_2\left(1 - e^{-b\alpha_2}\right) D + M_{2,n} \\ \vdots \\ d_1\left(-M_{n,n}e^{-\alpha_1 a} - \lambda_n\alpha_n e^{-\alpha_1 a} A_{1,n} D\right) + d_2\left(-M_{n,n}e^{-\alpha_n a} - \lambda_n\alpha_n e^{-\alpha_2 a} A_{2,n} D\right) + \dots \\ +d_n\left(D - M_{n,n}e^{-\alpha_n a} - \lambda_n\alpha_n b e^{-\alpha_n a} D\right) = \lambda_n\left(1 - e^{-b\alpha_n}\right) D + M_{n,n} \end{cases} \tag{20}$$

with $D = \sum_{i=1}^{n} \lambda_i e^{-\alpha_i a}$, and the proof of Theorem 1.1 is completed. □

2.2 Solution for the ARL Fredholm Integral Equation of EWMA chart with symmetric Laplace distributions

We analyze now the case of EWMA chart (see Eq. (3)) where $\eta_t \sim \text{Laplace}(0,1)$. Recall that the density function of η_t is given by $f(x) = \frac{1}{2}e^{-|x|}$.

It is well known that (e.g., see [5]) the function $h(x) = \mathbb{E}_x\nu_b$ is a solution of the following integral equation

$$h(x) = 1 + \mathbb{E}_x\left[I\{X_1 \le b\}\, h(X_1)\right]. \tag{21}$$

Then it can be shown that Eq. (21) becomes the following integral equation

$$h(x) = 1 + \frac{1}{2}\int_{\rho x}^{b} h(u)e^{\rho x - u}du + \frac{1}{2}\int_{0}^{+\infty} h(\rho x - y)e^{-y}dy. \tag{22}$$

The main result in this section is the following

Theorem 2.1 *For any $0 < \rho < 1$ and $0 \le x < b$ the solution of the integral equation Eq. (22) is*

$$\begin{aligned} h(x) = E_x\nu_b &= 2e^b - c_1(1+b) - \rho^2\left[2e^b - \left(1 + b + \frac{b^2}{2}\right)\right] \\ &- c_1 e^b \sum_{k=3,5,7,..}^{\infty} \rho^{k-1}P_1(k)\left[\frac{\Gamma(k+1,b)}{k!}\right] - e^b \sum_{k=4,6,8,..}^{\infty} \rho^k P_2(k)\left[2 - \frac{\Gamma(k+1,b)}{k!}\right] \\ &+ c_1\left[x + \sum_{k=3,5,7,..}^{\infty} \rho^{k-1}P_1(k)\frac{x^k}{k!}\right] + \left[-\rho^2\frac{x^2}{2!} - \sum_{k=4,6,8,..}^{\infty} \rho^k P_2(k)\frac{x^k}{k!}\right] \end{aligned} \tag{23}$$

with the constant c_1 given by

$$c_1 = \frac{\rho(1-\rho^2) - \sum_{k=4,6,8,..}^{\infty} \rho^{k+1}P_2(k)}{(\rho - 1) + \sum_{k=3,5,7,..}^{\infty} \rho^k P_1(k)} \tag{24}$$

where $\Gamma(a,z) = \int_z^{+\infty} t^{a-1}e^{-t}dt$ denotes the Incomplete Gamma function, and

$$P_1(k) = \prod_{m=1}^{\left(\frac{k-1}{2}\right)} \left(1 - \rho^{2m-1}\right), \qquad P_2(k) = \prod_{m=2}^{\left(\frac{k}{2}\right)} \left(1 - \rho^{2m-2}\right) \tag{25}$$

Proof. It can be shown that Eq. (22) can be reduced to the following second order differential equation

$$h''(x) = \rho^2 h(x) - \rho^2 h(\rho x) - \rho^2 \tag{26}$$

We try to find a series solution for Eq. (26) of the form

$$h(x) = \sum_{k=0}^{\infty} \frac{c_k x^k}{k!} = c_0 + \sum_{k=1}^{\infty} \frac{c_k x^k}{k!} \tag{27}$$

Then from Eq. (27) $j(0) = c_0$ and from Eq. (26) at $x = 0$, $h''(0) = -\rho^2$. It can be shown also that the coefficients c_k satisfied the non-linear recurrent equation

$$c_{k+2} = \rho^2(1 - \rho^k)c_k \quad \text{for} \quad k \ge 1 \tag{28}$$

and using the recurrence Eq. (28) we may find that

$$\begin{aligned} c_k &= -\rho^k \prod_{m=2}^{\left(\frac{k}{2}\right)} \left(1 - \rho^{2m-2}\right) \quad \text{for } k = 4, 6, 8, \dots \\ c_k &= c_1\rho^{k-1} \prod_{m=1}^{\left(\frac{k-1}{2}\right)} \left(1 - \rho^{2m-1}\right) \quad \text{for } k = 3, 5, 7, \dots \end{aligned} \tag{29}$$

With the coefficients c_1 and c_0 given by

$$c_1 = \frac{\rho(1-\rho^2) - \sum_{k=4,6,8,..}^{\infty} \rho^{k+1}P_2(k)}{(\rho - 1) + \sum_{k=3,5,7,..}^{\infty} \rho^k P_1(k)} \quad \text{with } 0 < \rho < 1 \tag{30}$$

$$
\begin{aligned}
c_0 = 2e^b - c_1(1+b) - \rho^2 \left[2e^b - \left(1+b+\frac{b^2}{2}\right)\right] \\
- c_1 e^b \sum_{k=3,5,7,..}^{\infty} \rho^{k-1} P_1(k) \left[\frac{\Gamma(k+1,b)}{k!}\right] - e^b \sum_{k=4,6,8,..}^{\infty} \rho^k P_2(k) \left[2 - \frac{\Gamma(k+1,b)}{k!}\right]
\end{aligned} \tag{31}
$$

The solution for the integral equation Eq. (22) is

$$
h(x) = E_x \nu_b = c_0 + c_1 \left[x + \sum_{k=3,5,7,..}^{\infty} \rho^{k-1} P_1(k) \frac{x^k}{k!}\right] + \left[-\rho^2 \frac{x^2}{2!} - \sum_{k=4,6,8,..}^{\infty} \rho^k P_2(k) \frac{x^k}{k!}\right] \tag{32}
$$

where the constant c_1 is given by Eq. (30), and the proof of Theorem 2.1 is completed.

3. Comparisons of the results with numerical integration and Monte Carlo simulations

In this section we present the scheme to evaluate numerically the solutions of the integral equation Eq. (10) (see also Eq. (6)).

By elementary quadrature rule we can approximate, in general, the integral $\int_0^b f(y)dy$ by a sum of areas of rectangles with bases b/m with heights chosen as the values of f at the midpoints of intervals of length b/m beginning at zero, i.e. on the interval $[0, b]$ with the division points $0 \le a_1 \le a_2 \le \ldots \le a_m \le b$ and weights $w_k = b/m \ge 0$, we can writing

$$
\int_0^b f(y)dy \approx \sum_{k=1}^{m} w_k f(a_k) \quad \text{with} \quad a_k = \frac{b}{m}\left(k - \frac{1}{2}\right), \quad k = 1, 2, \ldots m.
$$

If $j^*(x)$ denotes the approximated solution of $j(x)$ then the last term in the Eq. (6) can be expressed as

$$
\sum_{k=1}^{m} w_k j^*(a_k) f(a_k + a - a_i), \quad i = 1, 2, \ldots m. \tag{33}
$$

and the integral equation Eq. (6) becomes the following system of m linear equations in the m unknowns $j^*(a_1), j^*(a_2), \ldots, j^*(a_m)$

$$
\begin{cases}
j^*(a_1) = 1 + j^*(a_1)\left[F(a-a_1) + w_1 f(a)\right] + \sum_{k=2}^{m} w_k j^*(a_k) f(a_k + a - a_1) \\
j^*(a_2) = 1 + j^*(a_1)\left[F(a-a_2) + w_1 f(a_1 + a - a_2)\right] + \sum_{k=2}^{m} w_k j^*(a_k) f(a_k + a - a_2) \\
\vdots \\
j^*(a_m) = 1 + j^*(a_1)\left[F(a-a_m) + w_1 f(a_1 + a - a_m)\right] + \sum_{k=2}^{m} w_k j^*(a_k) f(a_k + a - a_m)
\end{cases} \tag{34}
$$

For numerical implementation is preferable to writing the linear system Eq. (34) in matrix form as

$$
\mathbf{J}_{m\times 1} = \mathbf{1}_{m\times 1} + \mathbf{R}_{m\times m}\mathbf{J}_{m\times 1} \quad \text{or} \quad (\mathbf{I}_m - \mathbf{R}_{m\times m})\,\mathbf{J}_{m\times 1} = \mathbf{1}_{m\times 1} \tag{35}
$$

where

$$
\mathbf{J}_{m\times 1} = \begin{pmatrix} j^*(a_1) \\ j^*(a_2) \\ \vdots \\ j^*(a_m) \end{pmatrix}, \mathbf{1}_{m\times 1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \tag{36}
$$

$$
\mathbf{R}_{m\times m} = \begin{pmatrix}
F(a-a_1) + w_1 f(a) & w_2 f(a_2 + a - a_1) & \ldots & w_m f(a_m + a - a_1) \\
F(a-a_1) + w_1 f(a_1 + a - a_2) & w_2 f(a) & \ldots & w_m f(a_m + a - a_2) \\
\vdots & \vdots & & \vdots \\
F(a-a_m) + w_1 f(a_1 + a - a_m) & w_2 f(a_2 + a - a_m) & \ldots & w_m f(a)
\end{pmatrix} \tag{37}
$$

and $\mathbf{I}_m = \operatorname{diag}(1, 1, \ldots, 1)$ is the unit matrix of order m. If there exists $(\mathbf{I}_m - \mathbf{R}_{m\times m})^{-1}$, then the solution of the matrix equation Eq. (35) is

$$
\mathbf{J}_{m\times 1} = (\mathbf{I}_m - \mathbf{R}_{m\times m})^{-1}\, \mathbf{1}_{m\times 1}
$$

Solving this set of equations for the approximate values of $j^*(a_1), j^*(a_2), \ldots, j^*(a_m)$, we may approximate the function $j(x)$ as

$$
j(x) \approx 1 + j^*(a_1)F(a-x) + \sum_{k=1}^{m} w_k j^*(a_k) f(a_k + a - x) \text{ with } w_k = \frac{b}{m} \text{ and } a_k = \frac{b}{m}\left(k - \frac{1}{2}\right). \tag{38}
$$

Numerical Examples We denotes by $j^*(x)$ the approximated solution, respectively by $j(x)$ the exact solutions, and define here the relative errors as $\epsilon_r = | j(x) - j^*(x) | / j(x)$.

For some values of a, b and the number of divisions m specified several numerical examples are presented in Table 1 and Table 2.

Mixture of two exponentials

For mixture of two exponentials the integral equation Eq. (6) is

$$
j(x) = 1 + \int_0^b j(y) \sum_{i=1}^{2} \lambda_i \alpha_i e^{\alpha_i (x-a-y)} dy + \left(1 - \sum_{i=1}^{2} \lambda_i e^{-\alpha_i (a-x)}\right) j(0) \tag{39}
$$

The coefficients $M_{k,n}$ for $n=2$ and $k=1,2$ are

$$
\begin{aligned}
M_{1,2} &= \lambda_1\left(1-e^{-b\alpha_1}\right)-\lambda_1\alpha_1\sum_{i=1}^{2}\lambda_i e^{-\alpha_i a}A_{i,1}\\
&= \lambda_1\left(1-e^{-b\alpha_1}\right)-\lambda_1\alpha_1\left[\lambda_1 e^{-\alpha_1 a}b+\lambda_2 e^{-\alpha_2 a}\left(\frac{e^{(\alpha_2-\alpha_1)b}-1}{\alpha_2-\alpha_1}\right)\right] \qquad (10)\\
M_{2,2} &= \lambda_2\left(1-e^{-b\alpha_2}\right)-\lambda_2\alpha_2\sum_{i=1}^{2}\lambda_i e^{-\alpha_i a}A_{i,2}\\
&= \lambda_2\left(1-e^{-b\alpha_2}\right)-\lambda_2\alpha_2\left[\lambda_1 e^{-\alpha_1 a}\left(\frac{e^{(\alpha_1-\alpha_2)b}-1}{\alpha_1-\alpha_2}\right)+\lambda_2 e^{-\alpha_2 a}b\right]
\end{aligned}
$$

with

$$A_{1,1}=A_{2,2}=b \text{ and } A_{1,2}=\left(\frac{e^{(\alpha_1-\alpha_2)b}-1}{\alpha_1-\alpha_2}\right),\quad A_{2,1}=\left(\frac{e^{(\alpha_2-\alpha_1)b}-1}{\alpha_2-\alpha_1}\right)$$

The solution is

$$j(x)=1+\sum_{i=1}^{2}d_i e^{\alpha_i(x-a)}+\left(1-\sum_{i=1}^{2}\lambda_i e^{-\alpha_i(a-x)}\right)j(0) \qquad (11)$$

with

$$j(0)=\frac{\left(1+\sum_{i=1}^{2}d_i e^{-\alpha_i a}\right)}{\sum_{i=1}^{2}\lambda_i e^{-\alpha_i a}} \qquad (12)$$

and the coefficients d_1, and d_2 are obtained as the solutions of the linear system Eq. (13).

$$
\begin{cases}
d_1\left[D-M_{1,2}e^{-\alpha_1 a}-\lambda_1\alpha_1 b e^{-\alpha_1 a}D\right]+d_2\left[-M_{1,2}e^{-\alpha_2 a}-\lambda_1\alpha_1 e^{-\alpha_2 a}A_{2,1}D\right]\\
=\lambda_1\left(1-e^{-b\alpha_1}\right)D+M_{1,2}\\
d_1\left[-M_{2,2}e^{-\alpha_1 a}-\lambda_2\alpha_2 e^{-\alpha_1 a}A_{1,2}D\right]+d_2\left[D-M_{2,2}e^{-\alpha_2 a}-\lambda_2\alpha_2 b e^{-\alpha_2 a}D\right]\\
=\lambda_2\left(1-e^{-b\alpha_2}\right)D+M_{2,2}
\end{cases} \qquad (13)
$$

with $D=\sum_{i=1}^{2}\lambda_i e^{-\alpha_i a}$.

Numerical results are presented in Table 1, for the case $m=800$ division points.

Mixture of four exponentials

Table 1: **Comparisons with the numerical results for mixture of two exponentials**

$x=0$		$\lambda_1=\lambda_2=0.5$, $\alpha_1=1.5$, $\alpha_2=2.8$,			$\lambda_1=0.3$, $\lambda_2=0.7$, $\alpha_1=1.1$, $\alpha_2=3.5$		
a	b	$j(x)$	$j^*(x)$	$\epsilon_r(\%)$	$j(x)$	$j^*(x)$	$\epsilon_r(\%)$
2.5	0.5	175.965	175.745	0.12	89.995	89.914	0.09
3.0	1.0	799.111	797.115	0.24	270.156	269.666	0.18
3.5	1.5	3597.65	3584.14	0.37	811.241	809.023	0.27
4.0	2.0	16158.2	16076.3	0.5	2438.48	2429.55	0.36
4.5	2.5	72504.7	72033.5	0.64	7332.76	7298.95	0.46
5.0	3.0	325183	325095	0.02	22050.8	21927.1	0.56
5.5	3.5	1.45801×10^6	1.45782×10^6	0.01	66299.6	658551	0.67

In the case of mixture of four exponentials, the integral equation Eq. (6) becomes

$$j(x)=1+\int_0^b j(y)\sum_{i=1}^{4}\lambda_i\alpha_i e^{\alpha_i(x-a-y)}dy+\left(1-\sum_{i=1}^{4}\lambda_i e^{-\alpha_i(a-x)}\right)j(0) \qquad (44)$$

and the coefficients $M_{k,n}$ for $n=4$ and $k=1,2,3,4$ given by Eq. (15) and $A_{i,i}=b$, $i=1,..,4$.

Table 2: **Comparisons with the numerical results for mixture of four exponentials**

	$a=2.3$, $b=1.5$, $\lambda_1=\lambda_2=\lambda_3=\lambda_4=1/3$, $\alpha_1=0.5$, $\alpha_2=0.7$, $\alpha_3=1.1$, $\alpha_4=1.3$		
x	$j(x)$	$j^*(x)$	$\epsilon_r(\%)$
0.0	15.614	15.594	0.129
0.5	15.240	15.221	0.128
1.0	14.702	14.683	0.127
1.5	13.912	13.895	0.125
2.0	12.729	12.714	0.121
2.5	10.914	10.902	0.113
3.0	8.061	8.053	0.092

The solution is

$$j(x) = 1 + \sum_{i=1}^{4} d_i e^{\alpha_i(x-a)} + \left(1 - \sum_{i=1}^{4} \lambda_i e^{-\alpha_i(a-x)}\right) j(0)$$

with $j(0) = \frac{\left(1+\sum_{i=1}^{4} d_i e^{-\alpha_i a}\right)}{\sum_{i=1}^{4} \lambda_i e^{-\alpha_i a}}$; d_1, d_2, d_3 and d_4 are the solutions of the linear algebraic system Eq. (13)

For the number of division points $m = 600$, we present in Table 2 several numerical results in the case of mixture of four exponentials. In Table 3 we present the results of Monte Carlo simulations for ARL when the EWMA chart is used with symmetric Laplace distributed random variables, and compare it with the closed-form expression given by the Eq. (23) in Theorem 2.1.

Table 3: $E_x\nu_b$ for $x = 0.3$ and different b and ρ with $b > x$. Comparisons with MC simulations

ρ	$b = 0.4$		$b = 0.6$		$b = 0.8$		$b = 1.0$	
	ARL	MCsim	ARL	MCsim	ARL	MCsim	ARL	MCsim
0.1	3.090	3.090	3.769	3.771	4.594	4.591	5.596	5.598
0.2	3.197	3.192	3.893	3.898	4.732	4.730	5.745	5.745
0.3	3.311	3.313	4.025	4.027	4.877	4.878	5.896	5.898
0.4	3.442	3.444	4.176	4.174	5.041	5.042	6.065	6.060
0.5	3.604	3.602	4.361	4.363	5.243	5.245	6.272	6.281
0.6	3.819	3.823	4.609	4.606	5.513	5.523	6.552	6.568
0.7	4.134	4.133	4.973	4.970	5.914	5.924	6.975	6.982
0.8	4.668	4.668	5.594	5.599	6.610	6.625	7.726	7.716
0.9	5.901	5.901	7.038	7.052	8.248	8.240	9.537	9.536

4. Conclusions

We have used the integral equations method to obtain closed form analytical expressions for the ARL of the CUSUM and EWMA control charts, when the observed random variables have hyperexponential distribution for CUSUM chart, respectively symmetric Laplace distribution for EWMA chart. We compare our analytical results with the numerical one and the Monte Carlo simulations. The methods are consistent with a high level of accuracy up to 98%.

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