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On a Compositeness Test for $(2^p + 1)/3$

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Abstract

In this note, we give a necessary condition for the primality of $(2^p + 1)/3$.

1 Introduction

Let p be an odd prime and $M_p := 2^p - 1$. For $n \geq 0$ define the sequence $\{S_n\}_{n \geq 0}$ by

$$S_0 = 4,$$
$$S_{k+1} = S_k^2 - 2, \qquad k \geq 0.$$

The celebrated Lucas-Lehmer test states:

Theorem 1. *M_p is prime if and only if $S_{p-2} \equiv 0 \pmod{M_p}$.*

The numbers M_p have interested experts and non-experts throughout history. See [7] for an interesting mathematical and historical account. These numbers have been a popular focus among those searching for large primes because of their unique set of convenient properties for primality testing, the most important of these being the Lucas-Lehmer test, given in Theorem 1. Indeed, via Lucas-Lehmer test, the determination of the primality of M_p is achieved through the calculation of $p-2$ ($< \log M_p$) squares modulo M_p. Furthermore, the reduction of a $2p$-bit integer modulo M_p is very fast compared to the reduction modulo any other number of a similar size.

Observe that $M_p = \phi_p(2)$, where $\phi_p(X)$ is the p-th cyclotomic polynomial. In this paper, we look at primes of the form

$$N_p := \phi_p(-2) = \frac{2^p+1}{3}.$$

For p a prime, the family of numbers $\{N_p\}_{p\geq 3}$ shares some of the properties that make the numbers $\{M_p\}_{p\geq 3}$ interesting to searchers of large primes. For instance, if N_p is prime, then p must be a prime. Additionaly, divisors of N_p are congruent to 1 modulo $2p$, an observation that helps in the search for small prime divisors of N_p. Furthermore, Melham proved the following theorem (see Theorem 7 in [5]), to which we will refer as Melham's probable prime test for N_p.

Theorem 2. *Let p be an odd prime. Define the sequence $\{S_n\}_{n\geq 0}$ by*

$$\begin{aligned} S_0 &= 6, \\ S_{k+1} &= S_k^2 - 2, \qquad k \geq 0. \end{aligned}$$

If N_p is prime then $S_{p-1} \equiv -34 \pmod{N_p}$.

Similar congruences involving Fibonacci numbers and more general Lucas sequences instead of only Mersenne numbers appear in [1] and [3].

It is easy to see that the reduction of a $2p$-bit number modulo N_p is also very fast. However, it is not known whether the numbers $\{N_p\}_{p\geq 3}$ have a very important property enjoyed by the numbers $\{M_p\}_{p\geq 3}$. Specifically, it is not known if $S_{p-1} \equiv -34 \pmod{N_p}$ implies that N_p is prime.

The numbers $\{N_p\}_{p\geq 3}$ were studied by Bateman, Selfridge, and Wagstaff, Jr. [2] who proposed the following conjecture.

Conjecture 3. *If two of the following statements about an odd positive integer p are true, then the third one is also true.*

- *$p = 2^k \pm 1$ or $p = 4^k \pm 3$;*
- *M_p is prime;*
- *N_p is prime.*

Observe that $2(i-1) = -2\sqrt{2}\omega$, where $\omega = (1-i)/\sqrt{2}$ is a root of unity of order 8. Since $p \geq 5$, it follows that $q \equiv 3^{-1} \equiv 11 \pmod{32}$, which implies easily that $(q^2-1)/4 \equiv -2 \pmod 8$. Thus, the left side of formula (4) is

$$(\gamma\sigma)^{(q^2-1)/4} = (-2\sqrt{2})^{(q^2-1)/4}\omega^{(q^2-1)/4} = (-1)^{(q^2-1)/4}2^{3(q^2-1)/8}\omega^{-2} = -i. \tag{6}$$

Next, observe that

$$(\tau^2)^{(q^2-1)/4} = \left(\tau^{q+1}\right)^{(q-1)/2}.$$

By Frobenius, we have that $\tau^{q+1} = \tau^q\tau = \sigma\tau = 2i\sqrt{2}$. Thus,

$$(\tau^2)^{(q^2-1)/4} = (2i\sqrt{2})^{(q-1)/2} = i^{(q-1)/2}2^{(q-1)/2}(\sqrt{2})^{(q-1)/2} = -i(\sqrt{2})^{(q-1)/2}, \tag{7}$$

where we have used the fact that $(q-1)/2 \equiv 1 \pmod 4$, which follows easily from the fact that $q \equiv 11 \pmod{32}$. Inserting (6) and (7) into (4), and using also (5), we obtain

$$(2\alpha)^{(q^2-1)/4} = (-i)(-i)(\sqrt{2})^{(q-1)/2} = -(\sqrt{2})^{(q-1)/2}.$$

Using now $2^{(q^2-1)/4} = (2^{q-1})^{(q+1)/4} = 1$, and $\alpha^{q-1} = \alpha^q\alpha^{-1} = \beta/\alpha$, we deduce that

$$\left(\frac{\beta}{\alpha}\right)^{(q+1)/4} = \alpha^{(q^2-1)/4} = (2\alpha)^{(q^2-1)/4} = -(\sqrt{2})^{(q-1)/2}.$$

Now, $(q+1)/4 = (2^p+4)/12 = (2^{p-2}+1)/3$. Thus,

$$\left(\frac{\beta}{\alpha}\right)^{2^{p-2}} = -(\sqrt{2})^{3(q-1)/2}\left(\frac{\alpha}{\beta}\right).$$

Applying the Frobenius automorphism, and summing the resulting relations, we arrive at

$$\left(\frac{\beta}{\alpha}\right)^{2^{p-2}} + \left(\frac{\alpha}{\beta}\right)^{2^{p-2}} = -(\sqrt{2})^{3(q-1)/2}\left(\frac{\alpha}{\beta} - \frac{\beta}{\alpha}\right).$$

In the line immediately above, the left side is $R_{p-1}/(\alpha\beta)^{2^{p-2}} = R_{p-1}/2^{2^{p-2}}$. The right side is

$$-(\sqrt{2})^{3(q-1)/2}\left(\frac{\alpha^2-\beta^2}{\alpha\beta}\right) = -(\sqrt{2})^{3(q-1)/2}4\sqrt{2} = -2^{(3q+7)/4}.$$

Since $(3q+7)/4 = 2^{p-2}+2$, we obtain

$$\frac{R_{p-1}}{2^{2^{p-2}}} = -2^{2^{p-2}+2},$$

which finally leads to $R_{p-1} = -2^{2^{p-1}+2}$. Using (3), we obtain the desired result.

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