General theory of three-dimensional radiance measurements with optical microprobes

N. Fukshansky-Kazarinova, L. Fukshansky, M. Kühl, and B. B. Jørgensen

Measurements of the radiance distribution and fluence rate within turbid samples with fiber-optic radiance microprobes contain a large variable instrumental error caused by the nonuniform directional sensitivity of the microprobes. A general theory of three-dimensional radiance measurements is presented that provides correction for this error by using the independently obtained function of the angular sensitivity of the microprobes. © 1997 Optical Society of America

Key words: Optics of living tissue; radiance measurements; fiber-optic microprobe; three-dimensional picture of a light field in turbid media.

1. Introduction

Studies of light-controlled processes in photomedicine, photobiology, and ecology require a detailed knowledge of the light microenvironment within absorbing turbid media such as living tissue and sediments. Direct three-dimensional measurements of radiation fields in such samples with optical fiber microprobes have been increasingly employed over the past several years.1–9

The most universal type of optical fiber probe is the radiance microprobe, which has a tip diameter as small as 10 μm and a directional sensitivity mainly concentrated around the axis of the probe within a solid angle as small as 10°. In contrast to other types that sense the entire spherical or hemispherical flux,8 it is the only probe that provides the angular distribution of radiance. A radiance distribution at a given depth is found on the basis of successive measurements with a radiance microprobe advanced to this depth in different zenithal directions, \( \theta \), relative to the light source as shown in Fig. 1. The unit sphere containing all directions (Fig. 1) is subdivided into \( K \) spherical bands, and within each band \( i \) one measurement under the angle \( \theta_i \) is carried out. This measurement is representative for the radiance in any direction within the band number \( i \), i.e., the radiance within each band is assumed to be constant. We designate this radiance value as \( L_i \). To the accuracy of this discretization the solid radiance distribution is given by the sequence of measured values \( L_i \):

\[
L(\theta) = \{L_1, L_2, \ldots, L_K\}.
\]

The fluence rate \( I(P) \) in a point \( P \) (see Fig. 1) is then obtained as a sum of \( L_i \) weighted with the fractional areas of the corresponding spherical bands, \( w_i \):

\[
I(P) = 2\pi \int_0^\pi L(\theta) d\theta \approx 4\pi \sum_{i=1}^{K} L_i w_i
\]

The weighting factors \( w_i \) are easily derived from the geometrical considerations as

\[
w_i = \frac{\cos \tilde{\theta}_i - \cos \tilde{\theta}_{i-1}}{2},
\]

\[
\tilde{\theta}_0 = 0, \tilde{\theta}_K = \pi, \quad \sum_i w_i = 1,
\]

where \( \tilde{\theta}_{i-1}, \tilde{\theta}_i \) are the zenithal angles for the boundary circles on the unit sphere delimiting the band \( i \) (see Fig. 1). Importantly, a measurement in this scheme in principle yields not the radiance \( L_i \) but the radiant flux \( L_i S \), where \( S \) is the reference area of the microprobe on the unit sphere (\( S \) is shown as a bright circle in Fig. 1). Thus the reading of a measurement should be divided by \( S \) to obtain \( L_i \).

However, as shown elsewhere,10 the described measurements produce erroneous values of \( L_i \). This intrinsic instrumental error results from the nonuniform angular sensitivity of the microprobe.

Radiance microprobes (for technical details see Refs. 2, 4, and 6) have well-defined light-collecting
properties. The directional sensitivity of a radiance microprobe is specified by a numerical aperture $n_0 \sin(\theta_0)$, where $n_0$ is the refraction index of the medium and $\theta_0$ is the acceptance half-angle of the optical fiber. The meaning of $\theta_0$ is clear from Fig. 2, which shows examples of the angular sensitivity distribution of a probe $h$ as a function of the deviation $\nu$ from the optical axis of the probe. The sensitivity $h(\nu)$, being maximal along the axis of the probe, decreases monotonously with $\nu$. The angle $\theta_0$ specifies such deviation $\nu$ for which $h(\nu) = 0.5h_{\text{max}}$. The bell-shaped distribution $h(\nu)$ is directly measurable. It can be approximated by a slightly modified Gaussian formula:

$$h(\nu) = \cos(\nu) \exp[-m \sin^2(\nu)],$$

(2)

where the fitting coefficient $m$ adjusts the function to the individual curve of the microprobe under consideration. Parameters $m$ and $\theta_0$ are related as

$$m = \ln(2 \cos \theta_0)/\sin^2 \theta_0.$$
In this paper the general theory for arbitrarily spaced measurements is presented. In Subsection 2.A the problem is formulated in mathematical terms and the basic system of linear equations connecting \( L_i \) and \( M_i \) is derived. The coefficients of the equations are surface integrals of \( h(\nu) \), with the complicated domains of integration arising from the geometry of measurements. Subsection 2.B contains an elucidation of these domains that is the prerequisite for the solution of the integrals given in Subsection 2.C. In Subsection 2.D we consider an application of the theory to our measurements in a costal sediment with diatoms. This treatment yields the instrumental error for the measured radiance distribution and fluence rate. It also shows that the divergent measured values obtained on the same sample with different probes converge very well after they were treated by the correcting procedure.

2. Results

A. Mathematical Formulation of the Problem

Let us consider radiation in an arbitrary point \( P \) within a turbid sample illuminated from above as shown in Fig. 1. Introducing spherical coordinates \( \theta \) (zenith angle) and \( \Psi \) (azimuth angle) associated with the unit sphere circumscribing \( P \), we assume that at any point \((\theta, \Psi)\) the radiance depends on \( \theta \) but not on \( \Psi \). As described in Section 1, \( K \) measurements with a microprobe having the directional sensitivity distribution \( h(\nu) \) are performed in the point \( P \) in the directions \( \theta_i \):

\[
0 \leq \theta_1 < \theta_2 \cdots < \theta_K \leq \pi.
\]

The results of these measurements are designated \( M_i(i = 1, 2, \ldots, K) \). The unit sphere is subdivided into \( K \) spherical bands (see Section 1), and radiance within each band is assumed to have a constant value \( L_i(i = 1, 2, \ldots, K, K - 1) \), where \( \theta_0 \) can be chosen arbitrarily within a set that obeys the inequality

\[
\theta_i < \tilde{\theta}_i < \theta_{i+1}, \quad \tilde{\theta}_0 = 0, \quad \tilde{\theta}_K = \pi.
\]

The zenithal distribution of radiance thus appears as a stepwise constant function:

\[
L(\theta) = L_i \quad \text{for } \tilde{\theta}_{i-1} \leq \theta < \tilde{\theta}_i \quad (i = 1, 2, \ldots, K)
\]

(compare Fig. 1). The problem is to calculate all \( L_i \) on the basis of the measured quantities \( M_i(i = 1, 2, \ldots, K) \) and known function \( h(\nu) \).

For an individual measurement under zenith angle \( \theta = \theta_i \leq \pi/2 \), we introduce local Cartesian coordinates \( x, y, z \) and also corresponding spherical coordinates \( \nu, \psi \) associated with the point \( P \) and direction \( \theta_i \), so that \( \nu = \theta - \theta_i \) (see Fig. 3). The angular sensitivity of the measurement, \( h(\nu) \), is defined on the whole hemisphere

\[
\nu, \psi: (0 \leq \nu \leq \pi/2, \quad 0 \leq \psi \leq 2\pi)
\]

visible from the direction \( \theta_i \). As described above, the hemisphere \( \Sigma_i \) is subdivided into a set of spherical bands \( \Sigma_{ij} \) with constant radiance values \( L_{ij} \). Thus the magnitude \( M_i \) obtained in this measurement contains contributions from different parts of the hemisphere \( \Sigma_i \):

\[
M_i = \int_{\Sigma_i} L(\theta)h(\nu)d\Sigma = \sum_{j=1}^{n_i} L_{ij}\int_{\Sigma_{ij}} h(\nu)d\Sigma = \sum_{j=1}^{n_i} L_{ij}J_{ij}
\]

with

\[
J_{ij} = \int_{\Sigma_{ij}} h(\nu)d\Sigma,
\]

where \( n_i \) is the number of spherical bands contributing to the hemisphere \( \Sigma_i \).

The system of linear equations, Eq. (3), contains \( K \) equations with \( K \) unknown variables \( L_{ij} \). Its coefficients \( J_{ij} \) can be estimated because \( h(\nu) \) is a known function. Thus the system (3) provides a solution to our problem. However, the estimation of \( J_{ij} \) requires a major effort because the domains of integrations in Eq. (4) are complicated functions of all \( \theta_i \) and \( \theta_j \), i.e., of the entire geometry of measurements.
In this case we have performed for the measurement in the \( \theta = \theta_i \) direction. Circumference \( S_i \) is the projection of the boundary of the hemisphere \( \Sigma_i \) onto the plane \( xPy \). Ellipses \( S_{ij} \) are the projections of circles \( \theta = \theta_j \) onto plane \( xPy \). The plane areas \( \sigma_i \) are projections of spherical bands \( \Sigma_i \) onto plane \( xPy \); the index \( i \) means only that the mapping is performed within the geometry induced by the \( i \)th measurement. \( L_i \) is the radiance within the band \( \Sigma_i \). \( \varphi_{i2} \) is the polar coordinate of the point of tangency for a tangent drawn from point \( P \) to ellipse \( S_{ij} \); \( \varphi_{i5} \) is the polar coordinate of the point of contact of ellipse \( S_{i5} \) with the circumference \( S_i \).

In the Subsection 2.B we analyze this geometry in order to facilitate the solution of the integrals \( J_{ij} \).

B. Geometry of the Surface Integrals \( J_{ij} \)

Our starting point is the general relation between the integral of a function \( f \) over an arbitrary convex surface \( \Sigma \) and that over the plane \( \sigma \):

\[
\iint_{\Sigma} f(M) d\Sigma = \iint_{\sigma} \frac{f(N)}{\cos(n, n_0)} d\sigma, \tag{5}
\]

where \( (n, n_0) \) is the angle between the normal to \( \Sigma \) in a point \( M \) and the normal to the plane \( \sigma \). \( N \) is the projection of point \( M \) onto the plane, and \( f(M) = f(N) \).

To calculate the integrals in Eq. (4) we map the spherical surface \( \Sigma_i \) onto the plane \( xPy \) (see Fig. 3). In this case we have \( (n, n_0) = v_i \), the zenithal spherical coordinate of the local coordinate system associated with the \( i \)th measurement; \( \Sigma = \Sigma_i \), the visible hemisphere corresponding to the \( i \)th measurement; and \( \sigma = \sigma_i \), the circular disk bounded by the circumference \( S_i \) of radius 1 centered at point \( P \).

The mapping transfers the delimiting circles \( \theta = \bar{\theta}_j \) into ellipses \( S_{ij} \) and the surface areas \( \Sigma_{ij} \) into the plane areas \( \sigma_{ij} \) (compare Figs. 3 and 4). On the basis of formulas (4) and (5) we obtain

\[
J_{ij} = \iint_{\sigma_{ij}} \frac{h(v)}{\cos(v)} d\sigma. \tag{6}
\]

Let us consider projections of the areas \( \Sigma_{ij} \) and their boundaries \( \theta = \bar{\theta}_j \) onto the plane \( xPy \). We are not interested in the circles \( \theta = \theta_j (\theta_i \leq \theta_j + \pi/2) \) located completely outside the hemisphere \( \Sigma_i \) and consider only circles that belong to \( \Sigma_i \) either completely or partially. In the first case the corresponding ellipses \( S_{ij} \) are located completely inside the circular disk \( \sigma_i \). In the second case they contact the circumference \( S_i \) in the points \( Q_{ij} \) (cf. Figs. 3 and 4), and we are interested only in the parts corresponding to those parts of the circles \( \theta = \theta_j \) which are located on the hemisphere \( \Sigma_i \). We can also restrict the following consideration to the values \( \theta_i \leq \pi/2 \) because the case \( \pi/2 < \theta_j \leq \pi \) is transferred to the case \( \theta_i \leq \pi/2 \) by the coordinate transformation \( \theta' = \pi - \theta \). Thus the admissible domain of values of \( (\theta_i, \theta_j) \) under consideration is \( (\theta_i < \pi/2; \theta_j < \theta_i + \pi/2) \), which is illustrated in Fig. 5. Also, the sum limit \( n_i \) in formula (3) can be easily estimated: it is equal to the maximal value of \( j \), satisfying the inequality (see Fig. 3)

\[
\tilde{\theta}_j \geq \theta_i + \pi/2.
\]

It is more convenient to consider within the circular disk \( \sigma_i \) a set of nested domains \( \sigma_{ij} \),

\[
\sigma_{i1} \subset \sigma_{i2} \subset \ldots \subset \sigma_{i n_i} = \sigma_i,
\]

instead of the set of nonoverlapping domains \( \sigma_{ij} \).

Each domain \( \sigma_{ij} \) is the internal area of the ellipse \( S_{ij} \) if this ellipse has no points of contact to \( S_i \). Otherwise the domain \( \sigma_{ij} \) is circumscribed by the part of the ellipse \( S_{ij} \) contained between its points of contact \( Q_{ij} \), \( Q_{ij'} \) to the circumference \( S_i \) and that part of \( S_j \) located between \( Q_{ij} \) and \( Q_{ij'} \) that is convex toward the positive direction of the axis \( x \) (see Fig. 4).

Let us introduce the integrals over the nested domains:

\[
\tilde{J}_{ij} = \iint_{\sigma_{ij}} \frac{h(v)}{\cos(v)} d\sigma. \tag{7}
\]
Obviously

\[ J_{ij} = J_{ij} - J_{ij-1} \quad (j = 1, 2, \ldots, n_i), \quad (8) \]

where (cf. Fig. 4)

\[ J_{i0} = 0, \quad J_{in_i} = 2\pi \int_0^{\pi} \frac{h(\psi)}{\cos \psi} r dr. \]

Thus the integrals \( J_{ij} \) can be easily calculated from integrals \( J_{ij} \) that have much more convenient integration domains. However, prior to this the limits of the surface integrals should be analytically derived. This derivation and the subsequent calculation of \( J_{ij} \) are given in Subsection 2.C.

C. Calculation of Integrals \( J_{ij} \)

For the integration limits to be derived, the boundaries of the domains \( \sigma_{ij} \) should be described. We introduce polar coordinates \( r, \phi \) (note that \( r = \sin \psi \) on the plane \( xPy \) and also the double-valued function \( f_{ij}^- (\phi) \) and \( r = f_{ij}^+ (\phi) \). In this case the point of tangency is located on the boundary of a domain \( \sigma_{ij} \) and

\[ \cos \phi_{ij} = \frac{\sin(\theta_i + \theta_j)\sin(\theta_i - \theta_j)}{\cos^2 \theta_i + \sin^2 \theta_j \cos^2 \phi} \quad (10) \]

If \( \tilde{\theta}_i > \theta_i \) then the ellipse \( S_{ij} \) is located within the negative semiaxis \( x \). In this case only the branch \( r = f_{ij}^+ (\phi) \) makes a part of the boundary of \( \sigma_{ij} \), and the point of tangency is not located on the boundary.

The important boundary point for integration to be performed is the point of contact for an ellipse \( S_{ij} \) with the circumference \( S_{ij} \). If such a point exists it is always located, as one can easily see, on a positive branch \( r = f_{ij}^+ (\phi) \). The \( \phi \) coordinate of this point we designate as \( \phi_{ij} \). The value of \( \phi_{ij} \) is determined through the parameters \( \theta_i, \theta_j \) as (see Appendix A)

\[ \cos \phi_{ij} = \cos \tilde{\theta}_i / \sin \theta_i. \quad (11) \]

\[ f_{ij}^- (\phi) = \frac{\cos \theta_i \sin \theta_j \cos \phi \pm \cos \theta_i \sin(\theta_i + \theta_j) \sin(\theta_i - \theta_j)}{\cos^2 \theta_i + \sin^2 \theta_j \cos^2 \phi} \quad (9) \]

which will be useful for the description of the ellipses \( S_{ij} \).

An ellipse \( S_{ij} \) contains the coordinates origin \( P \) when the parameters \( \theta_i, \theta_j \) satisfy the inequality

\[ \sin(\theta_i + \tilde{\theta}_j) \sin(\theta_i - \tilde{\theta}_j) < 0, \]

i.e., \( \theta_i < \tilde{\theta}_j < \pi - \theta_i \). In this case its equation is presented by the single-valued function \( r = f_{ij}^+ (\phi) \) (for the proof see Appendix A). Otherwise point \( P \) is located outside the ellipses \( S_{ij} \). In such a case a tangent line from point \( P \) to the ellipse is always feasible. We designate the \( \phi \) coordinate of the point of tangency as \( \phi_{ij} \) (cf. Fig. 4).

If the ellipse is located completely within the positive semiaxis \( x \) (which takes place for \( \theta_i < \tilde{\theta}_j \)), then to any value \( 0 < \phi < \phi_{ij} \) correspond two values of \( r: r = f_{ij}^- (\phi) \) and \( r = f_{ij}^+ (\phi) \).

On the basis of the above considerations, four different cases appear with respect to the subdivision of the integration domain for an integral \( J_{ij} \) and elucidation of the integration limits in each subdomain. Accordingly, the admissible domain of parameters \( \theta_i, \theta_j \) is subdivided into four subdomains, shown in Fig. 5. Below we consider all four cases and derive explicit formulas for \( J_{ij} \) in each case.

Case I. \( \theta_i < \theta_j, \theta_j \leq \pi/2 - \theta_i \). Ellipse \( S_{ij} \) is located completely on the positive semiaxis \( x \) and has no points of contact with \( S_i \) (or the unique point of contact has coordinate \( \phi_{ij} = 0 \)). This case is depicted in Fig. 6(a).

\[ J_{ij} = 2 \int_0^{\phi_{ij}} d\phi \int_{f_{ij}^- (\phi)}^{f_{ij}^+ (\phi)} \frac{h(\psi)}{\cos(\psi)} r dr. \quad (12a) \]
Case II. $\bar{\theta}_i < \theta_i$, $\bar{\theta}_i > \pi/2 - \theta_i$. The ellipse $S_{ij}$ is located on the positive semiaxis $x$ and has the point of contact $\varphi_{ij} \neq 0$ with $S_i$ [see Fig. 6(b)].

$$J_{ij} = 2 \int_0^{\delta_{ij}} \int_{\varphi_{ij}}^{\delta_{ij} + \varphi} \frac{h(v)}{\cos(v)} r \, dr \, d\varphi + 2 \int_{\varphi_{ij}}^{\delta_{ij}} \int_{\varphi_{ij}}^{\delta_{ij} + \varphi} \frac{h(v)}{\cos(v)} r \, dr. \quad (12b)$$

Case III. $\bar{\theta}_i > \theta_i$, $\bar{\theta}_i \leq \pi/2 - \theta_i$. Point P is located inside the ellipse $S_{ij}$, which has no point of contact with $S_i$ [or the unique point of contact has coordinate $\varphi_{ij} = 0$; see Fig. 6(c)].

$$J_{ij} = 2 \int_0^{\delta_{ij}} \int_{\varphi_{ij}}^{\delta_{ij} + \varphi} \frac{h(v)}{\cos(v)} r \, dr. \quad (12c)$$

Case IV. $\bar{\theta}_i > \theta_i$, $\bar{\theta}_i > \pi/2 - \theta_i$. Ellipse $S_{ij}$ is located completely or partially on the negative semiaxis $x$ and has points of contact $\pm \varphi_{ij} \neq 0$ [see Fig. 6(d)].

$$J_{ij} = 2 \int_0^{\delta_{ij}} \int_{\varphi_{ij}}^{\delta_{ij} + \varphi} \frac{h(v)}{\cos(v)} r \, dr - 2 \int_{\varphi_{ij}}^{\delta_{ij}} \int_{\varphi_{ij}}^{\delta_{ij} - \varphi} \frac{h(v)}{\cos(v)} r \, dr. \quad (12d)$$

Substituting Eq. (2) for $h(v)$ into Eqs. (12) and noting that $r = \sin v$, we can easily obtain a general solution for the internal integral in formulas (12):

$$\int_{\varphi_{ij}}^{\delta_{ij}} \int_{\varphi_{ij}}^{\delta_{ij} + \varphi} \exp(-mr^2) r \, dr \, d\varphi = \frac{1}{2m} \left[ \exp[-mf_1^2(\varphi)] - \exp[-mf_2^2(\varphi)] \right].$$

Thus for any set of $\theta_i$, $\bar{\theta}_i$ presenting a concrete geometry of measurements and for a given $m$ presenting the directional sensitivity of the concrete microprobe, integrals (12) and then [using Eq. (8)] integrals (4) can be calculated. Substituting integrals (4) into Eq. (3) and solving these equations with respect to $L_i(i = 1, 2, \ldots, K)$, one obtains the real radiances from the measured quantities $M_i$.

Note that when the measured radiation field contains a strong collimated component, for example, under illumination with a laser, the calculation of integrals $J_{11}$ can be developed in a special way involving the description of the collimated component as a singularity by means of $\delta$ functions. This appears to be necessary when the measurements are planed and discussed in the framework of the theory of radiative transfer, in which the collimated component is always separated as a singularity. The appropriate procedure for the calculation of $J_{11}$ is given in Appendix B.

D. Example of Application: Correction of the Measurements and Stability of the Correcting Procedure

Here the theory is applied to our radiance measurements in costal sediments with diatoms. Proceeding from a true radiance distribution, solid $L(\theta)$, and corresponding fluence rate $I(P) = \int L(\theta) \, d\theta$, we find estimates of $L(\theta)$ and $I(P)$ (a) as the raw experimental data and (b) as the data processed by the theory, and then we compare estimates (a) and (b) one to another and to the true magnitudes. Furthermore, we repeat this procedure for probes with four different acceptance angles $\theta_a$ as well as for different angular distances between the measurements in order to reveal the effect of these characteristics on both processed and nonprocessed data.

We start with the true radiance distribution $L(\theta)$ shown in Fig. 7. Its diffuse part is a continuous function; the collimated part we interpret as a constant level radiance concentrated in the narrow angular range $(0, 2.5^\circ)$. The fluence rate of this radiation $I(\theta) = 349.9$ relative units. On the basis of known $L(\theta)$, the measurable quantity $M(\theta)$ can be calculated for any direction $\theta_i$ and any microprobe as

$$\int_{\Omega} L(\theta) h(\theta - \theta_i) \, d\Sigma = M_i. \quad (13)$$

This quantity calculated for four different $m$ values is shown in Fig. 8. Because of a very extended range of values of $M(\theta)$, the angular domain near $\theta = 0^\circ$ is

1 September 1997 / Vol. 36, No. 25 / APPLIED OPTICS 6525
Fig. 8. Zenithal distributions of the measurable photon fluxes \( M(\theta) \) based on the true radiance distribution \( L(\theta) \) as they can be obtained by four microprobes with a different parameter \( m \). Specified points of \( \theta = 0^\circ, 20^\circ, 40^\circ, \ldots, 180^\circ \) were used to perform the correction procedure. The initial part of the curves \( M(\theta) \) for the lower \( \theta \) range is shown in the inset.

The results for the four probes are given in columns 4 to 7 of Table 1. These results should be compared with the true radiance values \( L(\theta) \) averaged over each band, presented in column 3 of Table 1 as \( L_i \). Figure 7 shows the averaged true values \( L_i \) on the background of the radiance distribution \( L(\theta) \). As one can see from Table 1, the correcting procedure recovers the radiance distribution solid with extremely high accuracy. The relative deviation of \( L_i \) from \( L_i \) is in almost all cases within \( \pm 5\% \).

Let us address the estimates of the fluence rate \( I(P) \) that are presented in Table 2. Column 1 contains the parameters of the four probes under consideration. Columns 2–4 contain the characteristics of the equidistant measurements constructed in accordance with each probe. For example, the probe with \( \theta_a = 7.2^\circ \) implies the step \( \Delta = 15^\circ \) between measurements. Correspondingly, the number of measurements is \( K = 2\pi/\Delta + 1 = 13 \). The reference area of a microprobe (see Section 1) for this band size is \( S = 0.0538 \). Note that the rather artificial magnitude \( S \) is necessary only for nonprocessed calculations to proceed from the measured flux \( M_i \) to the radiance \( L_i \). In the framework of the theory, there is no place for the notion of \( S \) because the reference area is the entire hemisphere \( \Sigma_i \) and the transmission from the measured fluxes \( M_i \) to the radiance \( L_i \) is accomplished automatically through the integration and solution of system (3). Column 5 contains the results provided by the measurements characterized in columns 2–4. All these numbers strongly overestimate the true value \( I(P) = 349.9 \) presented above. In order to see the effect of deviations of the band size from the probe acceptance angle, we performed calculations with the fixed band size \( (20^\circ) \) and corresponding number of measurements (ten) for different probes (column 6). The probe with \( \theta_a = 7.2^\circ \) used under these conditions underestimates the fluence rate; the probes with \( \theta_a = 15.1^\circ \) and \( \theta_a = 21.2^\circ \) that have acceptance angles exceeding the band size

### Table 1. Radiance Values \( L_i \) Recovered by Means of the Correcting Procedure Applied to the Measured Fluxes \( M_i \) from the Curves in Fig. 8

<table>
<thead>
<tr>
<th>Band No.</th>
<th>Range (deg)</th>
<th>( L_i ) Values</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>( m = 44.23 )</td>
</tr>
<tr>
<td>1</td>
<td>0–2.5</td>
<td>7800</td>
</tr>
<tr>
<td>2</td>
<td>2.5–30</td>
<td>74.5</td>
</tr>
<tr>
<td>3</td>
<td>30–50</td>
<td>36.1</td>
</tr>
<tr>
<td>4</td>
<td>50–70</td>
<td>33.5</td>
</tr>
<tr>
<td>5</td>
<td>70–90</td>
<td>25.4</td>
</tr>
<tr>
<td>6</td>
<td>90–110</td>
<td>16.8</td>
</tr>
<tr>
<td>7</td>
<td>110–130</td>
<td>8.5</td>
</tr>
<tr>
<td>8</td>
<td>130–150</td>
<td>8.0</td>
</tr>
<tr>
<td>9</td>
<td>150–170</td>
<td>10.7</td>
</tr>
<tr>
<td>10</td>
<td>170–180</td>
<td>9.6</td>
</tr>
</tbody>
</table>

\(^a\)For each of four curves representing different microprobes, the \( M_i \) values corresponding to the ten equidistant points \( \theta \) with the \( 20^\circ \) step \((0^\circ, 20^\circ, \ldots, 180^\circ)\) were used. \( L_i \) is the true mean value as shown in Fig. 7.
highly overestimate it. Column 7 of Table 2 describes the same measurements as column 6, with the difference that the data were processed by the theory. The estimates in column 7 are very precise, with deviations from the true value within 1%, independent of whether the band size is adjusted to the acceptance angle of the probe (line 2) or not (lines 1, 3, and 4).

Appendix A: Geometry of Ellipses \( S_y \)

We consider the measurement \( \theta_i \) and the corresponding circular disk \( S_i \) normal to the direction \( \theta = \theta_i \). The circumference \( \theta = \theta_i \) on the unit sphere is mapped onto the plane \( S_i \) as the ellipse \( S_{ij} \). Let us designate the half-axes of \( S_{ij} \) as \( a_{ij}, b_{ij} \) and the \( x \) coordinate of the center as \( q_{ij} \) (see Fig. 9). As we can easily see from Fig. 9,

\[
\begin{align*}
    a_{ij} &= \sin \theta_i \cos \theta_j, \\
    b_{ij} &= \sin \theta_j, \\
    q_{ij} &= \cos \theta_j \sin \theta_i, \\
    c_{ij}^2 &= b_{ij}^2 - a_{ij}^2 = \sin^2 \theta_j \sin^2 \theta_i. \\
\end{align*}
\]  

(A1)

The equation of the ellipse \( S_{ij} \) in the polar coordinates \((r, \varphi)\) is given by expressions (see the appendix in Ref. 10) \( r = f_{ij}^*(\varphi) \) when point P is inside \( S_{ij} \) and \( r = f_{ij}^-(\varphi) \) when point P is outside \( S_{ij} \), where Point P is located inside the ellipse \( S_{ij} \) when \( a_{ij}^2 - q_{ij}^2 > 0 \), which yields the condition \( \sin(\theta_i + \theta_j) \sin(\theta_i - \theta_j) > 0 \), i.e., \( \pi - \theta_j > \theta_j > \theta_i \). Outside the range where this condition holds the inequality \( \sin(\theta_i + \theta_j) \sin(\theta_j - \theta_i) > 0 \) is valid.

Let us consider the area \( \theta_j > \theta_i \), i.e., the case in which the ellipse \( S_{ij} \) is completely on the positive semiaxis \( x \). We draw a tangent to the ellipse \( S_{ij} \) from point P. The coordinate of the point of tangency \( \tilde{\varphi}_{ij} \) is found from the condition for vanishing of the radical in Eq. (A3): \( \sin(\theta_i + \theta_j) \sin(\theta_j - \theta_i) + \sin^2 \theta_j \cos^2 \varphi = 0 \), which yields \( \cos \tilde{\varphi}_{ij} = \frac{\sin(\theta_i + \theta_j) \sin(\theta_j - \theta_i)}{\sin \theta_j} \).

If the ellipse \( S_{ij} \) has a point of contact with disk \( S_i \), its coordinate \( \varphi = \tilde{\varphi}_{ij} \) is found from the condition \( f_{ij}^- = 1 \), which holds when \( \cos \tilde{\varphi}_{ij} = \cos \theta_j / \sin \theta_i \).

Appendix B: Calculation of integrals \( J_{ij} \)

This calculation is of integrals \( J_{ij} \) when the collimated component is presented as a singularity by means of a \( \delta \) function. The collimated component of radianse \( F_c \) can be presented as \( F_c = F_p \delta(\mu - 1) \), where

\[
F_p \text{ is the collimated flux at point P,}  \\
\mu = \cos \theta,  \\
f_0 \delta(\mu - 1) d\mu = 1,
\]

\[
f_{ij}^- = \frac{g_{ij} b_{ij}^2 \cos(\varphi) \pm a_{ij} b_{ij} [a_{ij}^2 - q_{ij}^2 + (q_{ij}^2 + c_{ij}^2) \cos^2(\varphi)]^{1/2}}{a_{ij}^2 + c_{ij}^2 \cos^2(\varphi)}. 
\]  

(A2)

Substituting Eq. (A1) into Eq. (A2), we obtain

\[
f_0^1 f(\mu) \delta(\mu - 1) d\mu = f(1)
\]

\[
f_{ij}^-(\varphi) = \frac{\cos \theta_j \sin \theta_i \cos \varphi \pm \cos \theta_i [\sin(\theta_i + \theta_j) \sin(\theta_j - \theta_i) + \sin^2 \theta_j \cos^2 \varphi]^{1/2}}{\cos^2 \theta_i + \sin^2 \theta_j \cos^2 \varphi}. 
\]  

(A3)
For a measurement in the direction $\theta_i \leq 90^\circ$, one should specify

$$L_1 = F_{P^1},$$
$$J_{i1} = \int_0^1 h(\nu)\delta(\mu - 1)d\mu.$$ 

For $h(\nu) = \cos \nu \exp(-m \sin^2 \nu)$ we obtain

$$\nu = \theta - \theta_i,$$
$$\cos \nu = \mu \cos \theta_i + \sqrt{1 - \mu^2} \sin \theta_i,$$
$$\sin \nu = \sqrt{1 - \mu^2} \cos \theta_i - \mu \sin \theta_i,$$
$$J_{i1} = \cos \theta_i \exp(-m \sin^2 \theta_i).$$

This research was supported by grant Fu 152/6-2 from Deutsche Forschungsgemeinschaft, the Max-Planck Gesellschaft, and the Carlsberg foundation, Denmark (for B. Jørgensen). This paper is dedicated to P-S. Song on the occasion of his 60th birthday.

References and Notes

12. By calibrating the microprobe against a standard source, we can express the measured values in absolute units of $W \text{m}^{-2} \text{s}^{-1} \text{sr}^{-1}$ for $L_i$ and $W \text{m}^{-2} \text{s}^{-1}$ for $I(P)$. However, this is not necessary for our purposes.